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Research Article

Adomian Decomposition Method for a Class of Nonlinear Problems

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The Adomian decomposition method together with some properties of nested integrals is used to provide a solution to a class of nonlinear ordinary differential equations and a coupled system.

1. Introduction

Most scientific problems and phenomena such as heat transfer occur nonlinearly. We know that only a limited number of these problems have a precise analytical solution [1–5]. In the 1980t's, George Adomian (1923–1996) introduced a powerful method for solving nonlinear functional equations. His method is known as the *Adomian decomposition method* (ADM) [6]. This technique is based on the representation of a solution to a functional equation as series of functions. Each term of the series is obtained from a polynomial generated by a power series expansion of an analytic function. Although the abstract formulation of the Adomian method is very simple, the calculations of the polynomials and the verification of convergence of the function series in specific situations are usually a difficult task [7, 8].

We will see that if the nested integral properties are used properly in the Adomian decomposition method, the analytical solution to the initial value problem is easily obtained.

Nested integrals integrals which are evaluated several times *on the same variable*. In contrast, multiple integrals consist of a number of integrals evaluated with respect to different variables. Concretely, if f is a continuous function defined on a (open) interval $\mathbb{I} \subset \mathbb{R}$ and $x_0 \in \mathbb{I}$,

$$\int_{x_0}^{x}\int_{x_0}^{x_1}\int_{x_0}^{x_2}\cdots\int_{x_0}^{x_{k-1}} f(x_1)f(x_2)f(x_2)\cdots f(x_k)dx_k dx_{k-1}\cdots dx_2 dx_1 = \frac{1}{k!}\left(\int_{x_0}^{x} f(x_1)dx_1\right)^k. \tag{1.1}$$

Also, (see [9]),

$$\int_{x_0}^{x}\int_{x_0}^{x_n}\cdots\int_{x_0}^{x_2}\int_{x_0}^{x_1} f(x_1)dx_1dx_2\cdots dx_{n-1}dx_n = \frac{1}{\Gamma(n+1)}\int_{x_0}^{x}(x-u)^n f(u)du. \tag{1.2}$$

2. Solution Method

Consider the IVP

$$\frac{dy}{dx} = f(x) + e^{-y(x)}; \qquad y(x_0) = 0, \tag{2.1}$$

where f is a continuous function defined on an (open) interval $\mathbb{I} \subset \mathbb{R}$, and $x_0 \in \mathbb{I}$.

In operator form, (2.1) becomes

$$Ly = f(x) + e^{-y(x)}, \tag{2.2}$$

where $L(\cdot) = (d/dx)(\cdot)$. Then inverse of L is, therefore, $L^{-1}(\cdot) = \int_{x_0}^{x}(\cdot)ds$. Applying l^{-1} to both sides of 4 we find that

$$y(x) = F(x) + \int_{x_0}^{x} e^{-y(s)}ds, \tag{2.3}$$

where $F(x) = \int_{x_0}^{x} f(s)ds$.

Adomiant's technique consists in writing the solution of (1.2) as an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{2.4}$$

and decomposing the nonlinear term $N(y) = e^{-y}$ as

$$e^{-y} = \sum_{n=0}^{\infty} A_n, \tag{2.5}$$

where each A_n is an Adomian polynomial depending on $y_0, y_1, \ldots, y_n$, which is given by

$$A_n(y_0, y_1, \ldots, y_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{n} y_t\lambda^t\right)\right]_{\lambda=0}. \tag{2.6}$$

(see [6, 7]).

Substituting (2.4) and (2.5) into (2.3), we obtain

$$\sum_{n=0}^{\infty} y_n(x) = F(x) + \int_{x_0}^{x}\left(\sum_{n=0}^{\infty} A_n(s)\right)ds. \tag{2.7}$$

This leads to the following recurrence scheme

$$y_0(x) = F(x)y_{k+1}(x) = \int_{x_0}^{x} A_k(s)ds \quad k = 0, 1, 2, \ldots. \tag{2.8}$$

We then define the solution $y(x)$ as

$$y(x) = \lim_{n\to\infty}\sum_{k=1}^{n} y_k(x). \tag{2.9}$$

The following algorithm will be used in order to calculate the Adomian polynomials (see [10])

$$\begin{aligned}
A_0(x) &= N(y_0)\\
A_1(x) &= \frac{d}{d\lambda}N(y_0+\lambda y_1)\Big|_{\lambda=0} = \frac{d}{d\lambda}e^{-(y_0+\lambda y_1)}\Big|_{\lambda=0} = y_1(x)N'(y_0),\\
A_2(x) &= \frac{1}{2!}\frac{d^2}{d\lambda^2}N\left(y_0+\lambda y_1+\lambda^2 y_2\right)\Big|_{\lambda=0} = y_2(x)N'(y_0) + \frac{1}{2}y_1^2 N''(y_0),\\
A_3(x) &= \frac{1}{3!}\frac{d^3}{d\lambda^3}N\left(\sum_{n=0}^{3}\lambda^n y_n(x)\right)\Big|_{\lambda=0} = y_3(x)N'(y_0) + y_1(x)y_3(x) + \frac{1}{3!}y_1^3 N'''(y_0),\\
A_4(x) &= \frac{1}{4!}\frac{d^4}{d\lambda^4}N\left(\sum_{n=0}^{4}\lambda^n y_n(x)\right)\Big|_{\lambda=0} = y_4(x)N'(y_0) + \left(y_1(x)y_3(x) + \frac{1}{2}y_2^2(x)\right)N''(y_0)\\
&\quad + \frac{1}{2}y_1^2(x)y_2(x)N'''(y_0) + \frac{1}{4!}y_1^4(x)N^{(4)}y_0\\
&\vdots
\end{aligned} \tag{2.10}$$

Combining this with (2.8), one obtains

$$\begin{aligned}
y_0(x) &= F(x),\\
y_1(x) &= \int_{x_0}^{x} A_0(s)ds = \int_{x_0}^{x} e^{-y_0(s)}ds = \int_{x_0}^{x} e^{-F(s)}ds,
\end{aligned} \tag{2.11}$$

where $A_0(x) = N(y_0) = e^{-y_0} = e^{-F(x)}$.

By using

$$A_1(x) = \frac{d}{d\lambda}N(y_0+\lambda y_1)\Big|_{\lambda=0} = \frac{d}{d\lambda}e^{-(y_0+\lambda y_1)}\Big|_{\lambda=0} = -e^{-y_0(x)}y_1(x), \tag{2.12}$$

we find that

$$y_2(x) = \int_{x_0}^{x} A_1(s)ds = -\int_{x_0}^{x} e^{-F(s)} y_1(s)ds = -\int_{x_0}^{x} e^{-F(s)} \left(\int_{x_0}^{s} e^{-F(u)}du\right)ds. \tag{2.13}$$

Now, using property (1.1) in (2.13) yields

$$-\int_{x_0}^{x} e^{-F(s)}\left(\int_{x_0}^{s} e^{-F(u)}du\right)ds = \int_{x_0}^{x}\left(\int_{x_0}^{s} e^{-F(s)}e^{-F(u)}du\right)ds = \frac{1}{2!}\left(\int_{x_0}^{x} e^{-F(s)}ds\right)^2. \tag{2.14}$$

Since

$$A_2(x) = y_2(x)N'(y_0) + \frac{1}{2}y_1^2 N''(y_0) = \left[-\frac{1}{2}\left(\int_{x_0}^{x} e^{-F(s)}ds\right)\right]\left(-e^{-F(x)}\right) + \frac{1}{2}\left(\int_{x_0}^{x} e^{-F(s)}ds\right)^2 e^{-F(x)}. \tag{2.15}$$

One finally obtains

$$A_2(x) = \left(\int_{x_0}^{x} e^{-F(s)}ds\right)^2 e^{-F(x)}. \tag{2.16}$$

In order to obtain y_3, we again use (1.1) and (2.16)

$$y_3(x) = \int_{x_0}^{x} A_2(s)ds = \int_{x_0}^{x} e^{-F(s)}\left(\int_{x_0}^{x} e^{-F(u)}du\right)^2 ds = \frac{1}{3!}\left(\int_{x_0}^{x} e^{-F(s)}ds\right)^3; \tag{2.17}$$

continuing in this fashion, we obtain

$$y_k(x) = (-1)^{k+1}\left[\frac{1}{k!}\left(\int_{x_0}^{x} e^{-F(s)}ds\right)^k\right], \quad k = 1, 2, 3, \ldots. \tag{2.18}$$

The solution is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \cdots + y_n(x) + \ldots. \tag{2.19}$$

By replacing (2.18) into (2.19), one obtains

$$y(x) = F(x) + \int_{x_0}^{x} e^{-F(s)}ds - \frac{1}{2!}\left(\int_{x_0}^{x} e^{-F(s)}ds\right)^2 + \cdots + (-1)^{k+1}\left[\frac{1}{k!}\left(\int_{x_0}^{x} e^{-F(s)}ds\right)^k\right] + \ldots. \tag{2.20}$$

Or, in a more compact form,

$$y(x) = F(x) + \sum_{n=1}^{\infty} (-1)^{k+1} \left[\frac{1}{k!} \left(\int_{x_0}^{x} e^{-F(s)} ds \right)^k \right]. \tag{2.21}$$

The latter equation can be written as

$$y(x) = F(x) + \ln\left(1 + \int_{x_0}^{x} e^{-F(s)} ds \right). \tag{2.22}$$

Observe that in this case, Adomiant's method yields an exact analytical solution. The analytical solution to this probleme can be obtained by performing the substitution $u = \exp(-y(x))$, which leads to a Bernoulli differential equation whose solution is a straightforward exercise.

3. Examples

3.1. Example 1

Consider the nonlinear initial value problem

$$\begin{gathered} \frac{dy}{dx} = \frac{1}{x} + e^{-y}, \qquad x > 0, \\ y(1) = 0. \end{gathered} \tag{3.1}$$

In this case, $f(x) = 1/x$, $x > 0$ and consequently $F(x) = \int_1^x (1/s) ds = \ln x$.

Thus, the analytical solution is given by

$$\begin{aligned} y(x) &= \ln x + \ln\left(1 + \int_0^x e^{-\ln s} ds \right) \\ &= \ln x + \ln\left(1 + \int_1^x \frac{1}{s} ds \right) \\ &= \ln x + \ln(1 + \ln x), \quad x > e^{-1} \cong 0.36787944. \end{aligned} \tag{3.2}$$

3.2. Example 2

Consider the nonlinear initial value problem

$$\frac{dy}{dx} = x + e^{-y}, \qquad y(0) = 0. \tag{3.3}$$

In this case, $f(x) = x$, and therefore $F(x) = x^2/2$. The analytical solution is

$$y(x) = \frac{x^2}{2} + \ln\left(1 + \sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right), \quad x > -1.2755, \tag{3.4}$$

where $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2} dt$.

4. Application of the Method to Coupled Systems of ODE's

Consider the coupled system

$$\frac{dx}{dt} - ay(t) = f(t), \qquad \frac{dy}{dt} + ax(t) = 0 \quad (a \in \mathbb{R},\ a \neq 0). \tag{4.1}$$

Together with the initial conditions

$$x(0) = \alpha, \qquad y(0) = \beta, \tag{4.2}$$

we shall obtain its solution by using the Dirichlet's integral formula (1.2) and the Adomian decomposition method.

Equation (4.1) in operator form takes the form

$$Lx(t) = f(t) + ay(t), \qquad Ly(t) = -ax(t), \tag{4.3}$$

where $L(\cdot) = (d/dx)(\cdot)$. The inverse of L is $L^{-1}(\cdot) = \int_0^t (\cdot) ds$. Applying L^{-1} to both sides of (4.3) and using (4.2), we find that

$$x(t) = \alpha + \int_0^t f(s) ds + a\int_0^t y(s) ds, \tag{4.4}$$

$$y(t) = \beta - a\int_0^t x(s) ds. \tag{4.5}$$

In order to obtain $x(t)$, we apply the Adomian iterative scheme

$$x_0(t) = \alpha + \int_0^t f(s) ds, \qquad x_{k+1}(t) = a\int_0^t y_k(s) ds, \quad k = 0, 1, 2, \ldots \tag{4.6}$$

in (4.4).

Similarly, $y(t)$ is obtained by applying the scheme

$$y_0(t) = \beta, \qquad y_{k+1}(t) = -a\int_0^t x_k(s) ds, \quad k = 0, 1, 2, \ldots \tag{4.7}$$

to (4.5).

Replacing $y_0(t) = \beta$ in (4.6), we find $x_1(t)$. In fact,

$$x_1(t) = a\int_0^t y_0(s)ds = a\int_0^t \beta ds = a\beta t. \tag{4.8}$$

Also, by replacing $x_0(t) = \alpha + \int_0^t f(s)$ and (1.2) in (4.5), we obtain

$$\begin{aligned} y_1(t) &= -a\int_0^t x_0(s)ds \\ &= -a\int_0^t \left(\alpha + \int_0^s f(u)du\right)\beta\, ds \\ &= -a\alpha t - a\int_0^t\int_0^s f(u)du\, ds \\ &= -a\alpha t - a\frac{1}{\Gamma(2)}\int_0^t (t-u)f(u)du. \end{aligned} \tag{4.9}$$

Applying (1.2) to the right hand side of the last equation, one finds that

$$\begin{aligned} x_2(t) &= a\int_0^t y_1(s)ds \\ &= a\int_0^t \left(-a\alpha s - a\frac{1}{\Gamma(2)}\int_0^s (s-u)f(u)du\right)ds \\ &= -a^2\alpha\frac{t^2}{2!} - a^2\frac{1}{\Gamma(2)}\int_0^t\left(\int_0^s (s-u)f(u)du\right)ds \\ &= -a^2\alpha\frac{t^2}{2!} - a^2\frac{1}{\Gamma(3)}\int_0^t (t-u)^2 f(u)du. \end{aligned} \tag{4.10}$$

To obtain $y_2(t)$, (4.8) into (4.7), and (1.2), we have

$$\begin{aligned} y_2(t) &= -a\int_0^t x_1(s)ds \\ &= -a\int_0^t \left(-a^2\alpha\frac{t^2}{2!} - a^2\frac{1}{\Gamma(3)}\int_0^s (s-u)^2 f(u)du\right)ds \\ &= a^3\alpha\frac{t^3}{3!} + a^3\frac{1}{\Gamma(3)}\int_0^t\left(\int_0^s (s-u)^2 f(u)du\right)ds \\ &= a^3\alpha\frac{t^3}{3!} + a^2\frac{1}{\Gamma(4)}\int_0^t (t-u)^3 f(u)du. \end{aligned} \tag{4.11}$$

Continuing in this fashion, one arrives at the formula

$$\begin{aligned}
x_n(t) &= (-1)^n\left[a^n\alpha\frac{t^n}{n!} + a^n\frac{1}{\Gamma(n+1)}\int_0^t (t-u)^n f(u)du\right],\\
y_n(t) &= (-1)^{n+1}\left[a^n\alpha\frac{t^n}{n!} + a^n\frac{1}{\Gamma(n+1)}\int_0^t (t-u)^n f(u)du\right].
\end{aligned} \tag{4.12}$$

The solution $x(t)$ is the given by

$$\begin{aligned}
x(t) &= x_1(t) + x_2(t) + x_3(t) + \cdots + x_n(t) + \cdots\\
&= \alpha + \int_0^t f(u)du + a\beta t + \left[-a^2\alpha\frac{t^2}{2!} - a^2\frac{1}{\Gamma(3)}\int_0^t (t-u)^2 f(u)du\right] - \alpha a^3\frac{t^3}{3!}\\
&\quad + \left[a^4\alpha\frac{t^4}{4!} - a^4\frac{1}{\Gamma(5)}\int_0^t (t-u)^4 f(u)du\right] + \cdots.
\end{aligned} \tag{4.13}$$

Rearranging terms and writing as a single integral, we have

$$\begin{aligned}
x(t) &= \alpha\left[1 - \frac{(at)^2}{2!} + \frac{(at)^4}{4!} - \cdots + (-1)^n\frac{(at)^{2n}}{(2n)!} + \cdots\right]\\
&\quad + \beta\left[(at) - \frac{(at)^2}{3!} + \frac{(at)^4}{5!} - \cdots + (-1)^n\frac{(at)^{2n+1}}{(2n+1)!} + \cdots\right]\\
&\quad + \int_0^t\left[\frac{(a(t-u))^0}{\Gamma(1)} - \frac{(a(t-u))^2}{\Gamma(3)} + \frac{(a(t-u))^4}{\Gamma(5)} - \cdots + (-1)^n\frac{(a(t-u))^{2n}}{\Gamma(2n+1)} + \cdots\right]f(u)du.
\end{aligned} \tag{4.14}$$

This is easily recognized as

$$\begin{aligned}
x(t) &= \alpha\cos(at) + \beta\sin(at) + \int_0^t \cos(a(t-u))f(u)du\\
&= \alpha\cos(at) + \beta\sin(at) + f(t) * \cos(at),
\end{aligned} \tag{4.15}$$

where $*$ denotes convolution.

The analogous process gives

$$\begin{aligned} y(t) &= y_1(t) + y_2(t) + y_3(t) + \cdots + y_n(t) + \cdots \\ &= \beta - \alpha a t - a\frac{1}{\Gamma(2)}\int_0^t (t-u) f(u)du - a^2\beta\frac{t^2}{2!} + \alpha a^3\frac{t^3}{3!} + \frac{1}{\Gamma(4)}\int_0^t (t-u)^3 f(u)du \\ &\quad + \cdots + (-1)^n\left[a^n\alpha\frac{t^n}{n!} + a^n\frac{1}{\Gamma(2n+2)}\int_0^t (t-u)^{2n+1} f(u)du\right] + \cdots . \end{aligned} \tag{4.16}$$

Rearranging, we obtain

$$\begin{aligned} y(t) = \beta&\left[1 - \frac{(at)^2}{2!} + \frac{(at)^4}{4!} - \cdots + (-1)^n\frac{(at)^{2n}}{(2n)!} + \cdots\right] \\ &- \alpha\left[(at) - \frac{(at)^2}{3!} + \frac{(at)^4}{5!} - \cdots + (-1)^n\frac{(at)^{2n+1}}{(2n+1)!} + \cdots\right] \\ &+ \int_0^t f(u)du - \frac{a^2}{\Gamma(3)}\int_0^t (t-u)^2 f(u)du + \frac{a^4}{\Gamma(5)}\int (t-u)^4 f(u)du \\ &+ \cdots + (-1)^n\frac{a^{2n}}{\Gamma(2n+2)}\int_0^t (t-u)^{2n} f(u)du + \cdots . \end{aligned} \tag{4.17}$$

Writting this as a single integral, we have

$$y(t) = \beta\cos(at) - \alpha\sin(at) + \int_0^t \sin(a(t-s)) f(s)ds. \tag{4.18}$$

And then,

$$y(t) = \beta\cos(at) - \alpha\sin(at) + f(t) * \sin(at). \tag{4.19}$$

It is important to observe that the analytical solution of the IVP given by (4.1) and (4.2) is precisely

$$\begin{aligned} x(t) &= \alpha\cos(at) + \beta\sin(at) + f(t) * \cos(at), \\ y(t) &= \beta\cos(at) - \alpha\sin(at) + f(t) * \sin(at). \end{aligned} \tag{4.20}$$

In particular, let us consider the forced undamped system given by

$$\frac{d^2y}{dt^2} + a^2 y(t) = \cos(t), \qquad y(0) = \alpha, \qquad y'(0) = \beta. \tag{4.21}$$

(Note that (4.21) is equivalent to the system formed by (4.1) and (4.2) with $f(t) = \cos(t)$).

There are two cases.

Case 1. $a \neq 1$. In this case, we obtain the solutions

$$x(t) = \alpha \cos(at) + \beta \sin(at) + \frac{1}{1-a^2}\sin(t),$$
$$y(t) = \beta \cos(at) - \alpha \sin(at) + \frac{a}{1+a^2}\cos(t). \tag{4.22}$$

Let us observe that the solutions are bounded in this case

Case 2. $a = 1$. In this case, one obtain

$$x(t) = \alpha \cos(t) + \beta \sin(t) + \frac{1}{2}t\cos(t), \qquad y(t) = \left(\alpha - \frac{1}{2}\right)\cos(t) + \beta \sin(t) - \frac{1}{2}t\sin(t). \tag{4.23}$$

Observe that the solutions are unbounded in this case, and we have resonance.

5. Conclusion

The results obtained in this paper show that the Adomian decomposition method is a powerful technique for finding the theoretical solutions of nonlinear initial value problems and coupled systems if properties of nested integrals are used properly. If a solution in closed form is not found, the method always provides a convergent series which solve the problem, see [11].

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