Application of Adomian Decomposition Method to Solve a Class of Diusion Problem Arises During MRI

3 authors:

Kazem Nouri
Semnan University

Morteza Garshasbi
Iran University of Science and Technology

Javad Damirchi
Iran University of Science and Technology and Department of Mathematics, Semnan…

Mathematical Sciences *Vol. 2, No. 2 (2008) 207-218*

Application of Adomian Decomposition Method to Solve a Class of Diffusion Problem Arises During MRI

K. Nouri[a,b], **M.Garshasbi**[a,b,1], **J. Damirchi**[a,b]

[a]Young Research Club, Islamic Azad University, Karaj Branch, Iran.

[b]Department of Mathematics, Iran University of Science and Technology, Tehran, Iran.

Abstract

This paper deals with a class of nonlinear diffusion equation subject to initial and boundary conditions which arises during Magnetic Resonance Imaging (MRI) process. Our interest problem is investigated by using Adomian decomposition method. Some examples are considered to illustrate the ability of this algorithm.

Keywords: MRI, Adomian decomposition method, diffusion equation.

1 Introduction

The recent introduction of DT-MRI (Diffusion Tensor Magnetic Resonance Imaging) has raised a strong interest in the medical imaging community . This non-invasive modality consists in measuring the water molecule motion within the tissues, using magnetic resonance techniques. The success of diffusion MRI is deeply rooted in the powerful concept that during their random, diffusion-driven displacements molecules probe tissue structure at a microscopic scale well beyond the usual image resolution. The diffusion MRI can reveal orientation-dependent behavior of water molecules for localization of specific organs and pathologies, and for functionality assessment. As diffusion is truly a three dimensional process, molecular mobility in tissues may be

[1]Corresponding Author. E-mail Address: m_garshasbi@iust.ac.ir

anisotropic, as in brain white matter. With diffusion tensor imaging (DTI), diffusion anisotropy effects can be fully extracted, characterized, and exploited, providing even more exquisite details on tissue microstructure. On the other hand, measuring diffusion in the real space, for instance water diffusion in a living body, is not a simple task though it provides useful and important information. Indeed, incoherent motion of water molecules has certain anisotropy in living bodies relating to normal and abnormal structures [1-5].

The diffusion may be one of the most important and attractive notions in mathematical methods for image analysis. Diffusion equation is one of the most important models which appears in the MRI, and often is nonlinear. Nonlinear partial differential equations are encountered in such various fields as physics, mathematics and engineering. Most nonlinear models of real life problems are still very difficult to solve either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient methods for determining a solution, approximate or exact, analytical or numerical, to the nonlinear models [1-3,6-8,20].

In this paper we consider a class of diffusion problems which arises in MRI frequently as follows

$$
\begin{aligned}
u_t &= (D(x)u_x)_x + \mu(x)F(u), \quad (x,t) \in [0,1] \times [0,T] && (1)\\
u(x,0) &= f(x), \quad 0 < x < 1 && (2)\\
u(0,t) &= p(t), \quad 0 < t < T && (3)\\
u_x(1,t) &= q(t), \quad 0 < t < T && (4)\\
|u(x,t)| &< K, \quad (x,t) \in [0,1] \times [0,T], && (5)
\end{aligned}
$$

where $D(x) > 0$, $\mu(x)$, $F(u)$, $f(x)$, $p(t)$, and $q(t)$ are known functions and K is a known constant. In addition we suppose that $\mu(x)$, $f(x)$, $p(t)$, and $q(t)$ are point-wise continuous functions, $D(x) \in C^1(0,1)$, and $F(u)$ is a continues Lipschitz function; i.e. there exist a positive constant $l \in R$ such that for each u_1 and u_2

$$|F(u_1) - F(u_2)| \leq l|u_1 - u_2|. \tag{6}$$

In (1), $D(x)$ represents the diffusive coefficient and $\mu(x)F(u)$ represents the source term. Here we represent an algorithm based on Adomian decomposition method (ADM) for solving the problem (1)-(5). The ADM has been proved to be effective and reliable for handling differential equations, linear or nonlinear. Unlike the traditional methods, The ADM needs no discritization, linearization, spatial transformation or perturbation. The ADM provide an analytical solution in the form of an infinite convergent power series. A large amount of research works has been devoted to the application of the ADM to a wide class of linear and nonlinear, ordinary or partial differential equations [7-13].

2 Basic Method

Let us first recall the basic principles of the ADM for solving differential equations. Consider the general equation: $\Upsilon u = g$, where Υ represents a general nonlinear differential operator involving both linear and nonlinear terms. The linear term is decomposed into $L + R$, where L is easily invertible and R is the remainder of the linear operator. For convenience, L may be taken as the highest order derivation. Thus the equation may be written as

$$Lu + Ru + Nu = q, \tag{7}$$

where Nu represents the nonlinear terms. Solving Lu from (7), we have

$$Lu = g - Ru - Nu. \tag{8}$$

Since L is invertible, the equivalent expression is

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{9}$$

Therefor, u can be expressed as following series

$$u = \sum_{n=0}^{\infty} u_n, \tag{10}$$

with reasonable u_0 which may be identified with respect to the definition of L^{-1} and g, and $u_n,\ n>0$ is to be determined. The nonlinear term Nu will be decomposed by the infinite series of Adomian polynomials

$$Nu=\sum_{n=0}^{\infty}A_n, \tag{11}$$

where A_n's are obtained by writing

$$v(\lambda) = \sum_{n=0}^{\infty}\lambda^n u_n, \tag{12}$$

$$N(v(\lambda)) = \sum_{n=0}^{\infty}\lambda^n A_n. \tag{13}$$

Here λ is a parameter introduced for convenience. From (12) and (13), we have

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{k=0}^{\infty}\lambda^k u_k)]_{\lambda=0},\quad n\geq 0. \tag{14}$$

The A_n's are given as

$$\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1\frac{d}{du_0}F(u_0),\\
A_2 &= u_2\frac{d}{du_0}F(u_0)+\frac{u_1^2}{2!}\frac{d^2}{du_0^2}F(u_0),\\
A_3 &= u_3\frac{d}{du_0}F(u_0)+u_1u_2\frac{d^2}{du_0^2}F(u_0)+\frac{u_1^3}{3!}\frac{d^3}{du_0^3}F(u_0),\\
&\vdots
\end{aligned}$$

Now, substituting (10) and (11) into (9) yields

$$\sum_{n=0}^{\infty}u_n=u_0+L^{-1}R(\sum_{n=0}^{\infty}u_n)-L^{-1}\sum_{n=0}^{\infty}A_n. \tag{15}$$

Consequently, with a suitable u_0 we can write

$$\begin{aligned}
u_1 &= -L^{-1}Ru_0-L^{-1}A_0,\\
&\vdots\\
u_{n+1} &= -L^{-1}Ru_n-L^{-1}A_n.
\end{aligned}$$

All of u_n are calculable, and $u = \sum_{n=0}^{\infty} u_n$. Since the series converges and does so very rapidly, the n-term partial sum $s_n = \sum_{k=0}^{n-1} \lambda^k u_k$ can serve as a practical solution.

For the convergence of the decomposition method, the readers are referred to [13-19].

3 Method of solution

In this section, consider following linear operators

$$L_{xx} = \frac{\partial^2}{\partial x^2}, \quad L_x = \frac{\partial}{\partial x}, \quad L_t = \frac{\partial}{\partial t}. \tag{16}$$

Using this notation, the equation (1) becomes

$$L_t(u) = L_x(D(x)L_x u) + \mu(x)F(u). \tag{17}$$

By defining the inverse operators L_t^{-1} one may formally obtain from (9) that

$$u(x,t) = f(x) + L_t^{-1}[L_x(D(x)L_x u) + \mu(x)F(u)], \tag{18}$$

where $L_t^{-1}(.) = \int_0^t (.)d\tau$. Now if we define

$$u_0(x,t) = f(x), \tag{19}$$

we can seek the solution $u(x,t)$ of the problem (1)-(5) based on Adomian decomposition approach as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{20}$$

The nonlinear terms are decomposed as

$$F(u) = \sum_{n=0}^{\infty} A_n, \tag{21}$$

where A_n, may be found as follows [5-8]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[F(\sum_{k=0}^{\infty} \lambda^k u_k)]_{\lambda=0}, \quad n \geq 0. \tag{22}$$

Substituting (20) and (21) into (18) gives

$$\sum_{n=0}^{\infty} u_n = f(x) + L_t^{-1}[L_x(D(x)L_x \sum_{n=0}^{\infty} u_n) + \mu(x) \sum_{n=0}^{\infty} A_n]. \tag{23}$$

Using above decomposition analysis, the following recurrence relation can be derived

$$u_0 = f(x) \tag{24}$$

$$u_{n+1} = L_t^{-1}[L_x(D(x)L_x u_n) - \mu(x)A_n], \quad n \geq 0. \tag{25}$$

On the other hand, we may use the operator L_{xx} and it's inverse to introduce the solution. Conditions (3) and (4) suggest defining the operator L_{xx}^{-1} and the starting term as

$$L_{xx}^{-1} = \int_0^x dx' \int_1^{x'} dx'', \tag{26}$$

$$u_0(x,t) = p(t) + xq(t). \tag{27}$$

Then the solution of problem (1)-(5) can be derived as follows

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{28}$$

where

$$u_{n+1} = L_{xx}^{-1}[\frac{1}{D(x)}\{L_t u_n - D^{'}(x)L_x u_n - \mu(x)A_n\}], \quad n \geq 0. \tag{29}$$

In (29), $D^{'}(x)$ shows the derivation of $D(x)$ with respect to variable x and the Adomian polynomial A_n is obtained subject to u_n, $n \geq 0$ calculated using L_{xx}^{-1}. The decomposition series (25) and (29) are generally convergent very rapidly in real physical problems. The convergence of the decomposition series have been investigated by several authors [5-8, 12, 13]. One can use each one of the decomposition series (25) or (29) for constructing the solution of the problem (1)-(5). In addition if one wants to introduce the solution with respect to the initial and boundary conditions (2)-(4), the average of the relations (25) or (29) can be used.

4 Numerical Experiments

In this section, for illustrations purpose we consider some problems and we show that how the ADM presented in the preceding section is computationally efficient.

Example 1. Consider the following nonlinear diffusion problem

$$\begin{cases} u_t = u_{xx} + u(1 - \frac{u}{3}), \quad 0 < x < \pi, \ 0 < t < 1 \\ u(x, 0) = sinx \\ u(0, t) = 0, \quad u_x(\pi, t) = 0. \end{cases} \tag{30}$$

Using the recursive relation (25) yields

$$\begin{aligned} u_0 &= \sin x \\ u_1 &= \int_0^t (u_{0xx}(x, \tau) + A_0(x, \tau))d\tau = \frac{\sin^2 x}{3} t \\ u_2 &= \int_0^t (u_{1xx}(x, \tau) + A_1(x, \tau))d\tau = [(\frac{2}{3}(\cos^2 x - \sin^2 x) \\ &- \frac{1}{9} \sin^3 x] \frac{t^2}{2} \\ u_3 &= \int_0^t (u_{2xx}(x, \tau) + A_2(x, \tau))d\tau = [\frac{8}{3}(1 - 2\sin^2 x) \\ &+ \frac{1}{3} \sin^2 x - \frac{1}{3} \sin x \cos^2 x + \frac{1}{9} \sin^3 x + \frac{2}{27} \sin^4 x] \frac{t^3}{3!}, \end{aligned}$$

and so on, the solution u(x,t) is in the following form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t).$$

Example 2. Consider following initial- boundary value problem

$$\begin{cases} u_t = \frac{1}{x} u_{xx} + \frac{1}{x} u^2, \quad 0 < x < L, \ 0 < t < 1 \\ u(x, 0) = x \\ u(0, t) = 0, \quad u_x(L, t) = \frac{1}{1+t}. \end{cases} \tag{31}$$

Using the recursive relation (25) yields

$$u_0 = x$$

$$\begin{aligned}
u_1 &= \frac{1}{x}\int_0^t (u_{0xx}(x,\tau)+A_0(x,\tau))d\tau = xt,\\
u_2 &= \frac{1}{x}\int_0^t (u_{1xx}(x,\tau)+A_1(x,\tau))d\tau = xt^2\\
u_3 &= \frac{1}{x}\int_0^t (u_{2xx}(x,\tau)+A_2(x,\tau))d\tau = xt^3\\
u_4 &= \frac{1}{x}\int_0^t (u_{3xx}(x,\tau)+A_3(x,\tau))d\tau = xt^4
\end{aligned}$$

and so on, the solution u(x,t) in a series form is given by

$$u(x,t) = x\sum_{n=0}^{\infty} t^n.$$

Since $0 < t < 1$, then we have

$$u(x,t) = \frac{x}{1-t}.$$

Example 3. Consider following nonlinear initial-boundary value problem

$$\begin{cases} u_t = u_{xx} + u(1-u)(u-10^{-3}), \quad 0<x<1,\ 0<t<1\\ u(x,0) = \frac{1}{2}(1+\tanh(x))\\ u(0,t) = \frac{1}{2}(1+\tanh(\frac{\sqrt{2}}{2}(2-10^{-3})t))\\ u(1,t) = \frac{1}{2}(1+\tanh\{1+\frac{\sqrt{2}}{2}(2-10^{-3})t\}). \end{cases} \tag{32}$$

The exact solution of this problem can be derived as [14]

$$u(x,t) = \frac{1}{2}(1+\tanh\{x+\frac{\sqrt{2}}{2}(2-10^{-3})t\}). \tag{33}$$

Table 1 shows the decomposition solution using the average of (25) or (29) using 3 terms, exact solution $u(x,t)$, and the absolute errors between them at some points.

5 Conclusion

In this paper a class of diffusion equations subject to initial and boundary conditions is solved by using ADM. This class of problem appear during the modeling of a lot

Table 1: Decomposition solutions using 3 terms.

x	t	Exact	S_3	Absolute errors
0.2	0.3	5.00242026E-4	5.00133504E-4	1.08521533E-7
	0.6	5.00484052E-4	5.00334801E-4	1.49250462E-7
	0.9	5.00726077E-4	5.00536098E-4	1.89979346E-7
0.4	0.3	5.00272026E-4	5.00128692E-4	1.43334091E-7
	0.6	5.00544052E-4	5.00356134E-4	1.87917588E-7
	0.9	5.00816077E-4	5.00583576E-4	2.32501015E-7
0.6	0.3	5.00302026E-4	5.00164199E-4	1.37826021E-7
	0.6	5.00604057E-4	5.00417787E-4	1.86264116E-7
	0.9	5.00906077E-4	5.00671375E-4	2.34702119E-7

of physical phenomena specially in MRI. Using this method has this advantages that it needs no discritization, linearization, spatial transformation or perturbation and it seems that ADM is a reasonable method for solving the nonlinear problems.

References

[1] Le Bihan D., Breton E. (1985) "Imagerie de diffusion in vivo par resonance magnetique nucleaire," CR. Acad. Sci. Paris, 301, 1109-1112.

[2] Merboldt K.D., Hanicke W., Frahm J. (1985) "Self-diffusion NMR imaging using stimulated echoes," J. Magn. Reson., 64, 479-486.

[3] Taylor D.G., Bushell M.C. (1985) "The spatial mapping of translational diffusion coefficients by the NMR imaging technique," Phys. Med. Biol., 30, 345-349.

[4] Stejskal E.O., Tanner J.E. (1965) "Spin diffusion measurements: spin echoes in the presence of a time-dependent field gradient," J. Chem. Phys. 42, 288-292.

[5] Le Bihan D., Breton E., Lallemand D. (1986) "MR imaging of intravoxel incoherent motions: application to diffusion and perfusion in neurologic disorders," Radiology, 161, 401-407.

[6] Logan J.D., Nonlinear partial differential equations, Wiley/Interscience, New York, 1994.

[7] Shidfar A., Djalalvand M., Garshasbi M. (2005) "A numerical scheme for solving special class of nonlinear diffusion-convection equation," Appl. Math. Copmut., 167, 1080-1089.

[8] Shidfar A., Garshasbi M. (in press) "A Weighted Algorithm Based on Adomian Decomposition Method for Solving an Special Class of Evolution Equations," Commun. Nonlinear Sci. Num. Simul.

[9] Lesnic D., Elliott L. (1999) "The decomposition approach to inverse heat conduction," J. Math. Anal. Appl., 232 82-98.

[10] Abbaoui K., Cherruault Y. (1995) "New ideas for proving convergence of decomposition method," Comput. Math. Appl., 29, 103-108.

[11] Adomian G. (1992) "A review of the decomposition method and some recent results for nonlinear equation," Math. Comput. Model., 13, 7-17.

[12] Adomian G. (1984) "Convergent series solution of nonlinear equations," J. Comput. Appl. Math., 2, 225-230.

[13] Adomian G. (1991) "Solving frontier problems modelled by nonlinear partial differential equations," Comput. Math. Anal., 22, 91-94.

[14] Adomian G. Meyers R. (1993) "Nonlinear transport in moving fluids," Appl. Math. Lett., 6, 35-38.

[15] Wazwaz A.M. (1999) "A reliable modification of Adomian decomposition method," Appl. Math. Comput., 175, 77-86.

[16] Wazwaz A.M., EI-Sayed S.M. (2001) "A new modification of the Adomian decomposition method for linear and nonlinear operators," Appl. Math. Comput., 122, 393-405.

[17] Kaya D., Yakus A. (2002) "A numerical comparison of partial solutions in the decomposition method for linear and nonlinear partial differential equations," Math. Comput. Simul., 60, 507-512.

[18] Kaya D., Inan I.E. (2005) "A convergence analysis of the ADM and an application," Appl. Math. Copmut., 161, 1015-1025.

[19] polyanin A.D., Zaistev V.F., Handbook of nonlinear partial differential equations, Chapman & Hall/CRC Press, Boca Raton, 2004.

.