Backward Stochastic Differential Equations

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(Forward) SDE

The equation:

$$X_0 = x, \quad dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t,\ t \in [0, T].$$

Causality principle(s):

$$X_t = F(t, \{B_s\}_{s\in[0,t]}) \qquad \text{(strong)}$$

$$\{X_s\}_{s\in[0,t]} \perp\!\!\!\perp \{B_s - B_t\}_{s\in[t,T]} \qquad \text{(weak)}$$

Solution by simulation (Euler scheme):

$$1)\ X_0 = x, \quad 2)\ X_{t+\Delta t} \approx X_t + \mu(X_t)\, \Delta t + \sigma(X_t)\, \Delta\zeta,$$

where we draw $\Delta\zeta = B_{t+\Delta t} - B_t$ from $N(0, \sqrt{\Delta t})$.

BSDE ARE NOT BACKWARD SDE

The equation:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t,\ t \in [0,T], \qquad X_T = \xi.$$

Backwards solution by simulation:

1) $X_T = \xi$, 2) $X_{t-\Delta t} \approx X_t - \mu(X_{t-\Delta t})\,\Delta t - \sigma(X_{t-\Delta t})(B_t - B_{t-\Delta t})$

The solution is no longer defined, or, at best, no longer adapted:

$$\text{e.g., if} \quad dX_t = dB_t,\ X_T = 0 \quad \text{then} \quad X_t = B_t - B_T.$$

Fix: to restore adaptivity, make $Z_t = \sigma(X_t)$ a part of the solution

$$dX_t = \mu(X_t)\,dt + Z_t\,dB_t, \quad X_T = \xi.$$

MRT: for $\mu \equiv 0$ we get the martingale representation problem:

$$dX_t = Z_t\,dB_t, \quad X_T = \xi.$$

Backward SDE

A change of notation:

$$dY_t = -f(Y_t, Z_t)\,dt + Z_t\,dB_t,\ t \in [0, T], \qquad Y_T = \xi.$$

A solution is a pair (Y, Z). The function f is called the driver.

Time- and uncertainty-dependence is often added:

$$dY_t = -f(t, \omega, Y_t, Z_t)\,dt + Z_t\,dB_t,\ t \in [0, T], \qquad Y_T = \xi(\omega),$$

and the ω-dependence factored through a (forward) diffusion

$$\begin{aligned} X_0 = x,\ dX_t &= \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t \\ dY_t &= -f(t, X_t, Y_t, Z_t)\,dt + Z_t\,dB_t,\ Y_T = g(X_t). \end{aligned}$$

Existing theory - dimension 1

Linear: Bismut '73, (or even Wentzel, Kunita-Watanabe or Itô)

Lipschitz: Pardoux-Peng '90

Linear-growth: Lepeltier-San Martin '97

Quadratic: Kobylanski '00

Superquadratic: Delbaen-Hu-Bao '11 - mostly negative

Option replication

Conside the Samuelson's model for the risky asset (stock, etc.):

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t).$$

and the riskless asset $S_t^0 = e^{rt}$. Invest the proportion α_t of your total wealth Y (*notation change*!) in the risky asset and the rest in the riskless asset.The dynamics of Y is

$$dY_t = \frac{\alpha_t Y_t}{S_t}\, dS_t + \frac{(1-\alpha_t)Y_t}{S_t^0}\, dS_t^0,$$

or, when simplified,

$$dY_t = rY_t + \sigma\alpha_t Y_t\Big(\theta\, dt + dW_t\Big) \text{ where } \theta = \frac{\mu - r}{\sigma}.$$

Instead of utility-maximization, we need to replicate an option with payoff ξ (say $\xi = (S_T - K)_+$).

In other words, we are asked to solve the BSDE

$$dY_t = (rY_t + \theta Z_t)\,dt + Z_t\,dW_t,\ Y_T = \xi,$$

and then report $\alpha_t = \frac{Z_t}{\sigma Y_t}$. Here, $f(Y, Z) = -(rY + \theta Z)$.

An example in stochastic control

Going back to a utility-maximization problem in the same framework, but with $r = 0$ and $\sigma = 1$, for simplicity. On the other hand, we allow μ to be a stochastic process. We are asked to find

$$\sup_{\alpha} \mathbb{E}[-\exp(-(X_T^\alpha + \xi))]$$

where $U(x) = -\exp(-x)$, is the exponential utility, and $\xi \in \mathbb{L}^\infty$ is the random endowment.

Insipred by the verification approach to stochastic control, for each $\alpha \in \mathcal{A}$ we define the process V^α as follows:

$$V_t^\alpha := -\exp\Big(-(X_t^\alpha + Y_t)\Big),$$

for some, yet unspecified, process Y with dynamics

$$dY_t = F_t\,dt + Z_t\,dW_t.$$

Itô's formula yields the following dynamics for V^α:

$$dV_t^\alpha = V_t^\alpha\Big(\big(-F_t + \tfrac{1}{2}Z_t^2 + \tfrac{1}{2}\alpha_t^2 (X_t^\alpha)^2 + \alpha_t X_t^\alpha (Z_t - \mu_t)\big)\,dt - (Z_t + \alpha_t X_t^\alpha)\,dW_t\Big).$$

We would like to determine F so that V^α is always a supermartingale, and a martingale for one particular α. herefore, we are looking for F with the following property:

$$\sup_a \left(F_t - \tfrac{1}{2} Z_t^2 - \tfrac{1}{2} a^2 (X_t^\alpha)^2 - a X_t^\alpha (Z_t - \mu_t) \right) = 0,$$

i.e., $F_t = \frac{1}{2} Z_t^2 + \inf_a \left(\frac{1}{2} a^2 (X_t^\alpha)^2 + a X_t^\alpha (Z_t - \mu_t) \right) = Z_t \mu_t - \frac{1}{2} \mu_t^2$, i.e., we need Y to satisfy the BSDE

$$dY_t = \left(Z_t \mu_t - \tfrac{1}{2} \mu_t^2 \right) dt + Z_t \, dW_t.$$

In order to complete the verification procedure, we also need the terminal condition $Y_T = \xi$: in that case, for each α,

$$\mathbb{E}[-\exp(-(X_T^\alpha + \xi))] = \mathbb{E}[-\exp(X_T^\alpha + Y_T)] \leq V_0^\alpha = -\exp(-X_0 - Y_0),$$

with the equality if and only if $\alpha_t X_t^\alpha = \mu_t - Z_t$.

Existence and Uniqueness in the Lipschitz case

For a pair (Y, Z), define

$$||(Y,Z)||_\beta^2 = \mathbb{E}[\int_0^T e^{\beta t}(|Y_t|^2 + |Z_t|^2)\,dt].$$

Lemma (Fundamental estimate) For each $F \in \mathbb{L}^{2,1}$ and $\xi \in \mathbb{L}^2$ there exists a unique continuous semimartingale Y and a process $Z \in \mathbb{L}^{2,1}$ such that

$$dY_t = -F_t\,dt + Z_t\,dW_t, Y_T = \xi.$$

Moreover, there exists a universal constant $C > 0$ such that for each $\beta > 0$ we have

$$||(Y,Z)||_\beta \leq C\Big(e^{\beta T}||\xi||_{\mathbb{L}^2} + \frac{1}{\beta}\mathbb{E}[\int_0^T |F_t|^2\,dt]\Big)$$

Proof: Set $Y_t = \mathbb{E}[\xi + \int_t^T F_t\,dt|\mathcal{F}_t]$ to obtain a solution. Apply Itô's formula to $e^{\beta t}\,|Y_t|^2$ for the estimate.

Theorem (Pardoux and Peng, '90). Suppose that $f(t, y, z)$ is Lipschitz in (y, z), uniformly in t. Then the BSDE

$$dY_t = -f(t, Y_t, Z_t)\, dt + Z_t\, dW_t,\ Y_T = \xi \in \mathbb{L}^2.$$

has a unique solution in $L^{2,1} \times L^{2,1}$.

Proof. Consider the map $\Phi(y, z) = (Y, Z)$, where (Y, Z) is the unique solution to

$$dY_t = -f(t, y_t, z_t)\, dt + Z_t\, dW_t, Y_t = \xi.$$

The fundamental estimate shows that Φ is a contraction in $|| \cdot ||_\beta$ for β large enough.

Remark.

1. This works for systems, too.
2. In dimension 1, moreover, if f is continuous (but not necessarily Lipschitz) and grows at most linearly as $|y|, |z| \to \infty$ there exists a *minimal* solution (Lepeltier and San Martin, '97).

Quadratic BSDEs

Consider the following *quadratic* BSDE with $f(y, z) = -\frac{1}{2}az^2$:

$$dY_t = \tfrac{1}{2}aZ_t^2\,dt + Z_t\,dW_t,\ \ Y_T = \xi.$$

If $\tilde{Y}_t = \exp(-aY_t)$, then

$$\begin{aligned} d\tilde{Y}_t &= b\tilde{Y}_t\,dY_t + \tfrac{1}{2}b^2\tilde{Y}_t\,d\langle Y\rangle_t \\ &= -a\tilde{Y}_t(\tfrac{1}{2}aZ_t^2\,dt + Z\,dW_t) + \tfrac{1}{2}a^2\tilde{Y}_tZ_t^2\,dt \\ &= -a\tilde{Y}_tZ_t\,dW_t = \tilde{Z}_t\,dW_t \end{aligned}$$

where $\tilde{Z}_t = -a\tilde{Y}_tZ_t$, so that

$$Y_t = -\frac{1}{a}\log\mathbb{E}[\exp(a\xi)|\mathcal{F}_t].$$

Theorem (Kobylanski '00, etc.). Suppose that $d = 1, f$ is continuous, there exists a constant M such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq M\Big(|y_2 - y_1| + (1 + |z_1| + |z_2|)\, |z_2 - z_1| \Big)$$

and $\xi \in \mathbb{L}^\infty$. Then there exists a unique solution (Y, Z) to

$$dY_t = -f(t, Y_t, Z_t)\, dt + Z_t\, dW_t,\ Y_T = \xi.$$

Proof. Highly nontrivial. Based on the comparison principle and monotone stability.

Connections with PDEs

Suppose that $\xi = g(W_T)$ for some function g. We could try $Y_t = v(t, W_t)$, for some v. By Itô's formula

$$dY_t = (v_t + \tfrac{1}{2} v_{xx})(t, W_t)\, dt + v_x(t, W_t)\, dW_t.$$

Matching terms yields $v_x(t, W_t) = Z_t$ and as well as

$$\begin{cases} v_t + \frac{1}{2} v_{xx} + f(t, v, v_x) = 0, \\ v(T, \cdot) = g. \end{cases}$$

Remark.

1. Works as written if v has enough regularity, in any dimension.
2. Always a viscosity solution when $d = 1$. A stochastic representation for quasilinear equations.