Construction of Time-Consistent Preferences

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Outline

- Primer and motivation: Independence, time-inconsistency, and consequentialism
- Review of classical, time-consistent control
- Examples of preferences leading to time-inconsistency and discussion of different approaches
- Dynamic utility approach: Mean-variance and rank-dependent utility
- Forward performance criteria and forward rank-dependent utility

Independence axiom and time-consistency

Recall the key axiom for the von Neumann-Morgenstern Theorem:

Independence: For all lotteries X, Y, Z, and every $r \in (0,1]$,

$$X \succsim Y \quad \text{if and only if} \quad rX + (1-r)Z \succsim rY + (1-r)Z.$$

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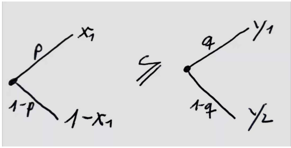

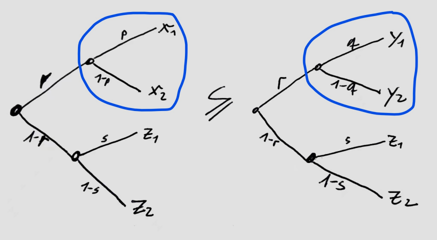

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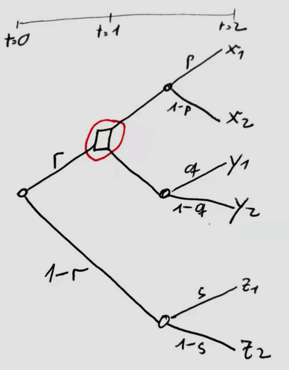

Violation of independence: Allais' paradox

Question 1: Choose between

1A) Win USD 1 Million with certainty.

1B) Win USD 1 Million with probability 89%, win nothing with probability 1%, win USD 5 Million with probability 10%.

Question 2: Choose between

2A) Win nothing with probability 89% and win USD 1 Million with probability 11%.

2B) Win nothing with probability 90% and win USD 5 Million with probability 10%.

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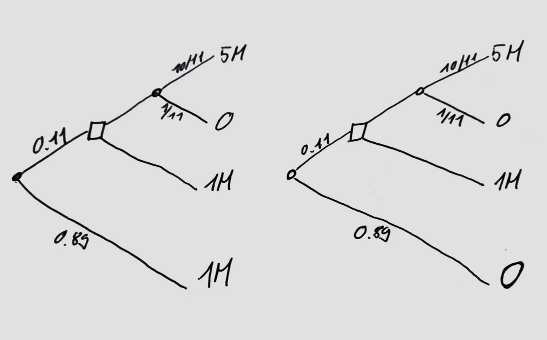

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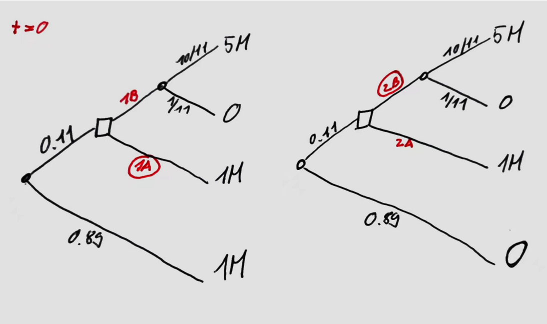

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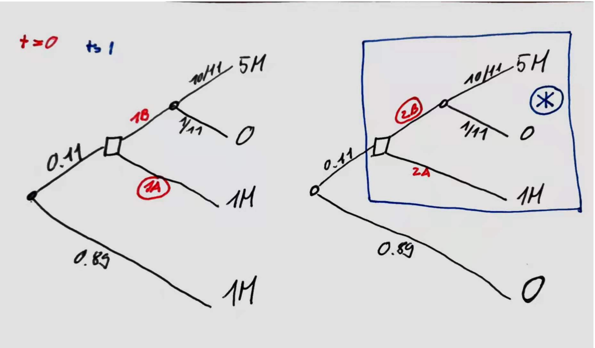

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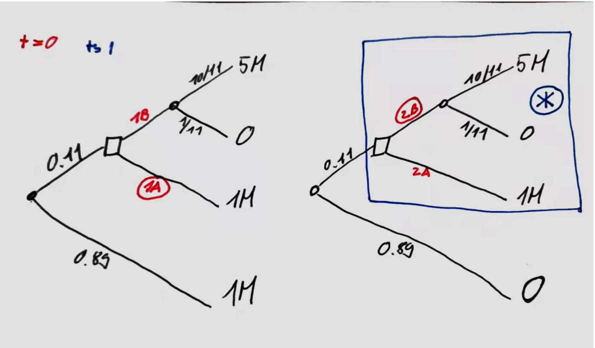

Violation of independence leads to time-inconsistent policies.

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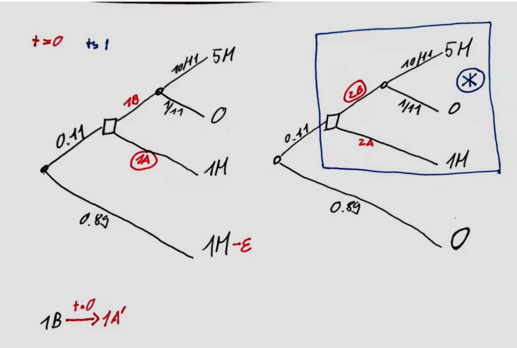

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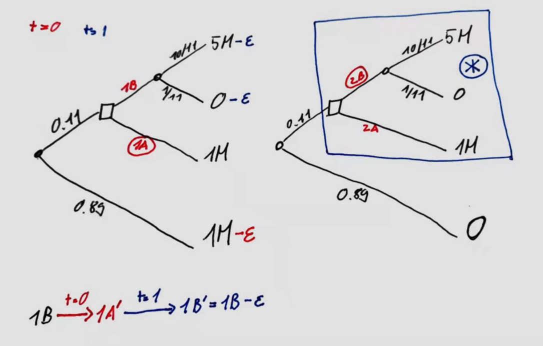

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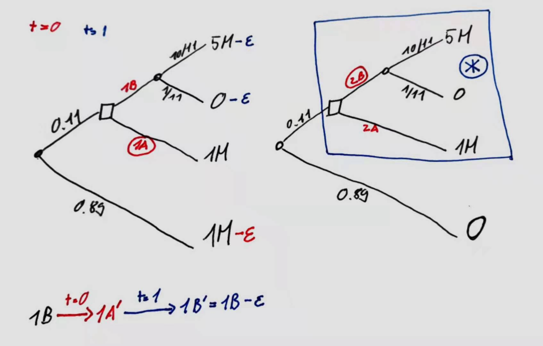

Time-inconsistent policies can be exploited.

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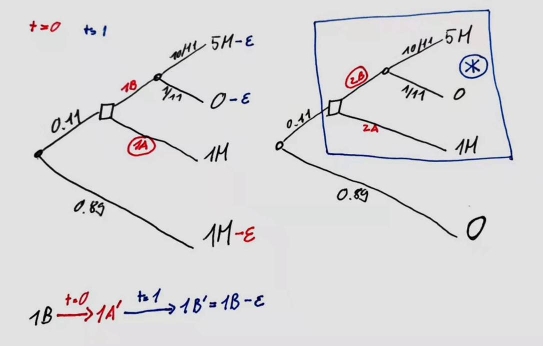

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Hidden assumption: **Consequentialism** - The only determinants of how decisions should be made in the continuation of a decision tree are the original preference ordering over probability distributions and the attributes of the continuation of the tree.

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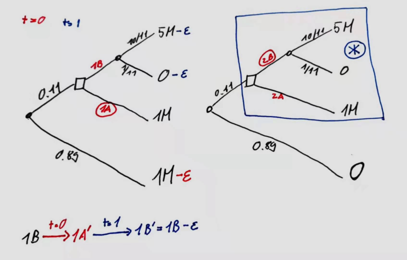

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Possible alternative: Take past uncertainty into account in a manner consistent with original preferences.

Consequentialism can be inappropriate

- A mother has a single item which she can give to one of her two children, Alice and Bob.
- In violation of EUT, she strictly prefers a coin flip over either of the sure outcomes.
- Smart Alice gets her Mom to put her preferences in writing: ‘I prefer a coin flip over giving the item to Bob.’
- If Alice wins the coin flip, she claims the item. If she looses, she reminds her Mom of her preferences for flipping the coin over giving the item to Bob.

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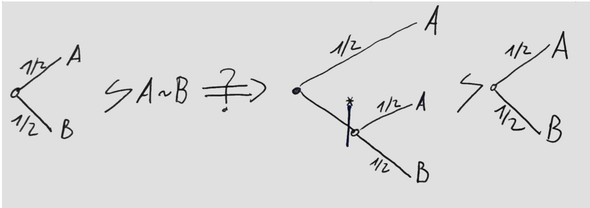

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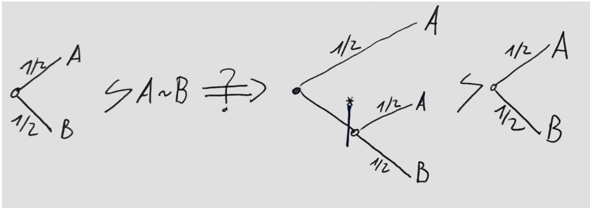

Consequentialism: 1) Snip the tree at $*$; 2) Throw the rest of the tree away; 3) Apply original preferences to continuation of the tree.

Dynamically consistent preferences: Conditional on having borne, but not realized, a $1/2$ probability of giving the item to Alice, mom strictly prefers giving the item to Bob than flipping another coin.

Consequentialism can be inappropriate

‘*The intertemporal analogue of consequentialism, which states that agents would neglect their consumption histories when recalculating in the middle of an intertemporal choice situation, is clearly inappropriate to impose on an individual who has intertemporally non-separable preferences.*

[...] if an individual informs you from the start that his preferences are non-separable (over time, over events, or over any other economic dimension), then it is inappropriate to impose separability ex post by explicitly or implicitly invoking consequentialism, and it is hardly surprising that doing so would lead to predictions of nonsensical behavior.’

Machina (1989)

Review of classical stochastic control

Recall the ingredients of classical stochastic control, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t), \mathbb{P})$ for a time horizon T:

1. set $\mathcal{A}$ of **admissible controls**
2. controlled **state process** $X = X^\vartheta$ for $\vartheta \in \mathcal{A}$:

   $$dX_s^\vartheta = \mu\left(s, X_s^\vartheta, \vartheta(s, X_s^\vartheta\right) ds + \sigma\left(s, X_s^\vartheta, u\left(s, X_s^\vartheta\right)\right) dW(s),$$

   $s \in [0, T]$, with $X_t^\vartheta = x$.
3. **performance criterion**

   $$J(\vartheta) = \mathbb{E}\left[\int_0^T C\left(s, X_s^\vartheta, \vartheta(s, X_s^\vartheta)\right) ds + F(X_T^\vartheta)\right]$$

   and

   $$J(t, x; \vartheta) = \mathbb{E}_{t,x}\left[\int_t^T C\left(s, X_s^\vartheta, \vartheta(s, X_s^\vartheta)\right) ds + F(X_T^\vartheta)\right].$$

Time-consistent policies

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." Bellman (1957).

Bellman equation

Define the value function $v(t,x) = \sup_{\vartheta \in \mathcal{A}} J(t,x;\vartheta)$. Then

$$v(t,x) = \sup_{\vartheta \in \mathcal{A}} \mathbb{E}_{t,x} \left[\int_t^\theta C\left(s, X_s^\vartheta, \vartheta(s, X_s^\vartheta)\right) ds + v(\theta, X_\theta^\vartheta) \right]$$

Martingale optimality principle

For $\vartheta \in \mathcal{A}$, define $V_t^\vartheta = v(t, X_t^\vartheta) + \int_0^t C(s, X_s^\vartheta, \vartheta(s, X_s^\vartheta)) ds$. Then

- V^ϑ is a supermartingale for any $\vartheta \in \mathcal{A}$
- $V^{\hat\vartheta}$ is a martingale for the optimal $\hat\vartheta \in \mathcal{A}$

Examples of preferences leading to time-inconsistency

Mean-variance optimization: $\vartheta^* = \arg\max_{\vartheta \in \mathcal{A}} \mathbb{E}[X_T^\vartheta] - \gamma Var(X_T^\vartheta)$ is in general not optimal for $J(t, x; \vartheta) = \mathbb{E}_{t,x}[X_T^\vartheta] - \gamma Var_{t,x}(X_T^\vartheta)$.

Hyperbolic discounting: $\vartheta^* = \arg\max_{\vartheta \in \mathcal{A}} \mathbb{E}\left[\int_0^T h(s) C(X_s^\vartheta) ds\right]$, where $h(s) = (1 + \alpha s)^{-\beta/\alpha}$ for parameters $\alpha, \beta > 0$, is in general not optimal for $J(t, x; \vartheta) = \mathbb{E}_{t,x}\left[\int_t^T h(s-t) C(X_s^\vartheta) ds\right]$.

Rank-dependent utility:
$\vartheta^* = \arg\max_{\vartheta \in \mathcal{A}} \int_0^\infty u(\xi) d\left(-w\left(1 - F_{X_T^\vartheta}(\xi)\right)\right)$ is in general not optimal for $J(t, x, ; \vartheta) = \int_0^\infty u(\xi) d\left(-w\left(1 - F_{X_T^\vartheta | X_t^\vartheta = x}(\xi)\right)\right)$.

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Observation: Time-inconsistency is partially a consequence of consequentialism: $J(t,x;\cdot)$ is essentially the same as $J(0,x;\cdot)$ modulo conditioning.

Approaches to time-inconsistency

- **Pre-commitment / static optimality**: Solve the optimization problem only once, at time zero, and commit to the resulting strategy without ever reconsidering the problem.
- **Naïve / spendthrift / dynamic optimality / β-people**: Non-expected utility maximizers that are consequentialist and revise their policies at each point in time.
- **Consistent planning / thrift / intra-personal equilibrium / δ-people**: Optimize by taking the future disobedience as a constraint. The agent's selves at different times are considered to be the players of a game, and a consistent plan chosen by the agent becomes an equilibrium of the game from which no selves are willing to deviate.
- **Dynamically consistent preferences / γ-people**: Non-expected utility maximizers that are not consequentialist, but instead have time-consistent dynamic preferences.
- References: Strotz (1955), Hammond (1976), Machina (1989), Pedersen and Peskir (2016), Karnam, Ma, and Zhang (2017)

Model for continuous-time mean-variance example

Consider a Black-Scholes market: There is a risk-free bond with price process B solving

$$dB_t = rB_t dt, \quad t \in [0, T],$$

with $B_0 = b > 0$, and a risky asset with price process S solving

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad t \in [0, T],$$

with $S_0 = s > 0$.

Denoting the amount invested in the risky asset at time t by ϑ_t, the wealth process of the agent has dynamics given by

$$dX_t = (rX_t + (\mu - r)\vartheta_t)\, d_t + \sigma\vartheta_t dW_t, \quad t \in [0, T],$$

with $X_0 = x_0 > 0$.

Under **pre-commitment**, the agent maximizes

$$\mathbb{E}[X_T] - \gamma Var(X_T)$$

once at time zero and then never reconsiders the problem. The parameter γ represents risk-aversion.

The optimal strategy is given by

$$\vartheta_t^{pc} = \frac{\lambda}{\sigma}\left(x_0 e^{rt} + \frac{1}{2\gamma} e^{\lambda^2 T - r(T-t)} - X_t^{pc}\right),$$

where $\lambda = \frac{\mu - r}{\sigma}$ denotes the market price of risk.

References: Zhou and Li (2000), Li and Ng (2000), ...

A **naïve** agent with initial mean-variance preferences revises her policies at each point in time by re-optimizing the consequentialist preferences

$$\mathbb{E}_{t,x}[X_T^{\vartheta}] - \gamma Var_{t,x}(X_T^{\vartheta})$$

and then myopically implementing this policy. This results in the time-inconsistent strategy

$$\vartheta_t^n = \frac{\lambda}{2\gamma\sigma} e^{(\lambda^2 - r)(T-t)}.$$

References: Pedersen and Peskir (2016), Vigna (2018).

Mean-variance: Consistent planning

For a fixed point (t, x) in the time-wealth space, the preferences are again given by $J(t, x; \vartheta) = \mathbb{E}_{t,x}[X_T^{\vartheta}] - \gamma Var_{t,x}(X_T^{\vartheta})$. The agent seeks to determine an **equilibrium strategy**, i.e., a strategy $\hat{\vartheta}$ such that

$$\liminf_{h \to 0} \frac{J(t, x; \hat{\vartheta}) - J(t, x; \vartheta_h)}{h} \geq 0$$

for any ϑ_h constructed, with an arbitrary $\theta \in \mathbb{R}$, as

$$\vartheta_h(s, y) = \begin{cases} \theta, & t \leq s < t + h, \\ \hat{\vartheta}(s, y), & t + h \leq s \leq T. \end{cases}$$

The equilibrium strategy is given by $\vartheta^c = \frac{\lambda}{2\gamma\sigma} e^{-r(T-t)}$.

References: Basak and Chabakauri (2010), Czichowsky (2013), Björk, Murgoci, and Zhou (2014), He and Jiang (2020),...

Mean-variance: Dynamically consistent preferences

We want to construct a **dynamic preference functional** $J(t, x; \vartheta)$, $t \in [0, T]$, $x \in \mathbb{X}$, such that

$$J(0, x; \vartheta) = \mathbb{E}_{0,x}[X_T^{\vartheta}] - \gamma Var_{0,x}(X_T^{\vartheta})$$

and such that the preferences lead to time-consistent strategies, i.e.,

$$\vartheta_t^{pc} = \frac{\lambda}{\sigma}\left(x_0 e^{rt} + \frac{1}{2\gamma} e^{\lambda^2 T - r(T-t)} - X_t^{pc}\right),$$

is optimal for the problem with objective $J(t, x; \cdot)$ at any $t \in [0, T]$, $x \in \mathbb{X}$,

$$\vartheta^{pc} \in \arg\max_{\vartheta \in \mathcal{A}} J(t, x; \vartheta).$$

Mean-variance: Dynamically consistent preferences

Proposition

The dynamic preference functional consistent with mean-variance is given by $J(t,x;\vartheta) = \mathbb{E}_{t,x}[X_T^\vartheta] - \gamma_t(x) Var_{t,x}(X_T^\vartheta)$, $t \in [0,T], x > 0$, *where*

$$\gamma_t(x) = \frac{1}{2} \frac{e^{\lambda^2(T-t)}}{x_0 e^{rT} + \frac{1}{2\gamma} e^{\lambda^2 T} - e^{r(T-t)} x}.$$

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Proof.

Under dynamically consistent preferences, the pre-committed and naïve strategies must coincide, i.e., $\vartheta_t^{pc}(x) = \vartheta_t^{n}$. This leads to

$$\frac{\lambda}{\sigma}\left(x_0 e^{rt} + \frac{1}{2\gamma} e^{\lambda^2 T - r(T-t)} - x\right) = \frac{\lambda}{2\gamma_t \sigma} e^{(\lambda^2 - r)(T-t)}.$$

□

References: Cui et al. (2012), Cui, Li, and Li (2017), Karnam, Ma, and Zhang (2017).

Mean-variance: Dynamically consistent preferences

- The dynamic risk aversion $\gamma_t(x)$ is always non-negative. This is termed **time-consistency in efficiency** by Cui et al. (2012).
- Cui et al. (2012) show that, in discrete-time, one can still obtain a dynamic risk-aversion $\gamma_t(x)$ leading to time-consistent strategies. However, it can happen that $\gamma_t(x) < 0$, and this in turn leads to free cash flow streams. These cannot exist when the market is complete (Bäuerle and Grether (2015)) or asset prices are continuous (Strub and Li (2020)).
- Dynamic risk preferences have also been obtained when risk is measured by the CVaR, see Pflug and Pichler (2016), Godin (2016), and Strub et al. (2019). In this case, the confidence level of the CVaR needs to be updated as well.
- A general theory which does not rely on the existence of an optimal strategy has been developed in Karnam, Ma, and Zhang (2017).

Model for rank-dependent utility example

The financial market consists of one risk-free and N risky assets. The price of the i^{th} risky asset solves

$$dS_t^i = S_t^i\Big(\mu_t^i dt + \sum_{j=1}^N \sigma_t^{ij} dW_t^j\Big), \quad t \geqslant 0,$$

with $S_0^i = s_0^i > 0$, $i = 1, \ldots, N$.

The drift and volatility coefficients are assumed to be deterministic functions. We also define the market price of risk process, $\lambda_t := \sigma_t^{-1}\mu_t$ and assume that $\lambda_t > 0$, $t \geqslant 0$.

Given a self-financing trading policy $(\vartheta_t)_{t\geqslant 0}$, the wealth process $X = (X_t^{x,\vartheta})_{t\geqslant 0}$ solves the stochastic differential equation

$$dX_t^{x,\vartheta} = \vartheta'_t \mu_t dt + \vartheta'_t \sigma_t dW_t, \quad t \geqslant 0,$$

with $X_0^{x,\vartheta} = x$.

Model for rank-dependent utility example

For each $t > 0$, we consider the (unique) pricing kernel

$$\rho_t = \exp\left(- \int_0^t \frac{1}{2} \|\lambda_r\|^2 dr - \int_0^T \lambda_r' dW_r \right).$$

For $0 < s \leqslant t$, we further define

$$\rho_{s,t} := \frac{\rho_t}{\rho_s} = \exp\left(- \int_s^t \frac{1}{2} \|\lambda_r\|^2 dr - \int_s^t \lambda_r' dW_r \right).$$

We also denote the cumulative distribution function of ρ_t and $\rho_{s,t}$ by F_t^ρ and $F_{s,t}^\rho$ respectively.

Key features of the model: **Complete financial market** with **deterministic coefficients**.

Rank-dependent utility preferences

The rank-dependent utility value of a strategy ϑ is defined as

$$J(\vartheta) = \int_0^\infty u(\xi) d\left(-w\left(1 - F_{X^\vartheta}(\xi)\right)\right),$$

where u is a **utility function** and w is a **probability distortion function**.

Definition

Let $\mathcal{U}$ be the set of all **utility functions** $u : [0, \infty) \to \mathbb{R}$, with u being strictly increasing, strictly concave, twice continuously differentiable in $(0, \infty)$, and satisfying the Inada conditions $\lim_{x \downarrow 0} u'(x) = \infty$ and $\lim_{x \uparrow \infty} u'(x) = 0$.
Let $\mathcal{W}$ be the set of **probability distortion functions** $w : [0, 1] \to [0, 1]$ that are continuously differentiable, strictly increasing and satisfying $w(0) = 0$ and $w(1) = 1$.

Under **pre-commitment**, the agent maximizes

$$\sup_{\vartheta \in \mathcal{A}} \int_0^\infty u(\xi) d\left(-w\left(1 - F_{X^\vartheta}(\xi)\right)\right).$$

Because the market is complete, any $\mathcal{F}_T$-measurable prospect X that satisfies the budget constraint $\mathbb{E}\left[\rho_T X\right] = x$ can be replicated by a self-financing policy. In turn, the problem reduces to

$$\begin{aligned} &\sup_X \int_0^\infty u(\xi) d\left(-w\left(1 - F_X(\xi)\right)\right) \\ &\text{with} \quad \mathbb{E}\left[\rho_T X\right] \leqslant x, \; X \geqslant 0, \; X \in \mathcal{F}_T. \end{aligned} \tag{1}$$

Theorem (Xia and Zhou (2016), Xu (2016))

Let $u \in \mathcal{U}$ and $w \in \mathcal{W}$. If there exists an optimal wealth to (1), it is given by

$$X_T^* = (u')^{-1}\left(\eta^* \widehat{N}'\left(1 - w\left(F_T^\rho(\rho_T)\right)\right)\right),$$

where $\widehat{N}$ is the concave envelope of $N(z) := -\int_0^{w^{-1}(1-z)} (F_T^\rho)^{-1}(t)dt$, $z \in [0,1]$. and the Lagrangian multiplier $\eta^* > 0$ is determined by $\mathbb{E}[\rho_T X_T^*] = x$.

RDU: Pre-commitment

Theorem (Xia and Zhou (2016), Xu (2016))

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If the **Jin-Zhou monotonicity condition** holds, namely, if the function $f(p) := \frac{(F_T^\rho)^{-1}(p)}{w'(p)}$ is nondecreasing, then the solution simplifies to

$$X_T^* = \left(u'\right)^{-1}\left(\eta^* \frac{\rho_T}{w'\left(F_T^\rho(\rho_T)\right)}\right).$$

Dynamic rank-dependent utility process

Consider the classical RDU maximization problem of the form

$$\max_{\pi\in\mathcal{A}} \quad \int_0^\infty u_{0,T}(\xi)d\left(-w_{0,T}\left(1-F_{X_T^\vartheta}(\xi)\right)\right) \tag{2}$$

with $dX_r^{x,\vartheta} = \vartheta'_r\mu_r dr + \vartheta'_r\sigma_r dW_r$, $r\in[0,T]$ and $X_0^{x,\pi}=x>0$ for a fixed time horizon $T>0$.

Definition

A family of utility functions $u_{t,T}\in\mathcal{U}$, $t\in(0,T)$, and probability distortion functions $w_{t,T}\in\mathcal{W}$, $t\in(0,T)$ is called a **dynamic rank-dependent utility process** for $u_{0,T}\in\mathcal{U}$ and $w_{0,T}\in\mathcal{W}$ over the time horizon $T>0$ if $\lim_{t\searrow 0} u_{t,T}=u_{0,T}$, $\lim_{t\searrow 0} w_{t,T}=w_{0,T}$ and the optimal policy ϑ^* for (2) also solves, for any $t\in(0,T)$,

$$\max_{\vartheta\in\mathcal{A}(\vartheta^*,t)} \quad \int_0^\infty u_{t,T}(\xi)d\left(-w_{t,T}\left(1-F_{X_T^{x,\vartheta}|\mathcal{F}_t}(\xi)\right)\right).$$

Theorem (He, Strub, and Zariphopoulou (2021))

Let $T > 0$, $u_{0,T} \in \mathcal{U}$ and $w_{0,T} \in \mathcal{W}$, and suppose that the optimal solution to (2) exists. We have the following two cases:
i) If

$$w_{0,T}(p) \leqslant \mathbb{E}\left[\rho_{0,T} \mathbf{1}_{\left\{\rho_{0,T} \leqslant \left(F^{\rho}_{0,T}\right)^{-1}(p)\right\}}\right], \quad p \in [0,1], \tag{3}$$

then a family of utility functions $u_{t,T} \in \mathcal{U}$, $t \in (0, T)$, together with a family of probability distortion functions $w_{t,T} \in \mathcal{W}$, $t \in (0, T)$, is a dynamic rank-dependent utility process for $u_{0,T}$ and $w_{0,T}$ if and only if the family of probability distortion functions satisfies

$$w_{t,T}(p) \leqslant \mathbb{E}\left[\rho_{t,T} \mathbf{1}_{\left\{\rho_{t,T} \leqslant \left(F^{\rho}_{t,T}\right)^{-1}(p)\right\}}\right], \quad p \in [0,1],$$

for any $t \in [0, T)$.

Dynamic rank-dependent utility process

Theorem (Continued)

ii) Otherwise, a family of utility functions $u_{t,T} \in \mathcal{U}$, $t \in (0, T)$ *and probability distortion functions* $w_{t,T} \in \mathcal{W}$, $t \in (0, T)$ *with* $\lim_{t \searrow 0} u_{t,T} = u_{0,T}$ *and* $\lim_{t \searrow 0} w_{t,T} = w_{0,T}$ *is a dynamic rank-dependent utility process for* $u_{0,T}$ *and* $w_{0,T}$ *if and only if there is a deterministic process* $\gamma_t \geqslant 0$, $t \in [0, T)$, *continuous at zero and such that*

$$w_{t,T}(p) = \frac{1}{\mathbb{E}\left[\rho_{t,T}^{1-\gamma_t}\right]} \int_0^p \left(\left(F_{t,T}^{\rho}\right)^{-1}(q)\right)^{1-\gamma_t} dq$$

for any $t \in [0, T)$, *and the dynamic measure of risk-aversion satisfies*

$$-\frac{1}{\gamma_t}\frac{u''_{t,T}(x)}{u'_{t,T}(x)} = -\frac{1}{\gamma_0}\frac{u''_{0,T}(x)}{u'_{0,T}(x)} \tag{4}$$

for any $t \in [0, T)$ *and* $x > 0$.

Sketch of the proof

- We first show: the family probability distortion functions either satisfies the Jin-Zhou monotonicity condition ($\frac{(F^{\rho}_{t,T})^{-1}(\cdot)}{w'_{t,T}(\cdot)}$ is nondecreasing), or inequality (3) holds.
- Denote by X^* the optimal wealth at time T that maximizes (2).
- The optimal wealth for

$$\begin{aligned} \max_{X} \quad & \int_0^{\infty} u_{t,T}(\xi) d\left(-w_{t,T}\left(1 - F_{X|\mathcal{F}_t}(\xi)\right)\right) \\ \text{s.t.} \quad & \mathbb{E}\left[\rho_{t,T} X | \mathcal{F}_t\right] = \mathbb{E}\left[\rho_{t,T} X^* | \mathcal{F}_t\right], \;\; X \geqslant 0, \;\; X \text{ is } \mathcal{F}_T - \text{meas.} \end{aligned}$$

is given by

$$X^{*,t} = (u'_{t,T})^{-1}\left(\eta^*_{t,T}\left(\mathbb{E}\left[\rho_{t,T} X^* | \mathcal{F}_t\right]\right) \hat{N}'_{t,T}\left(1 - w_{t,T}\left(F^{\rho}_{t,T}(\rho_{t,T})\right)\right)\right).$$

- Optimality of the initial optimal solution $X^* = X^{*,0}$ is maintained if and only if

$$\begin{aligned} &\eta^*_{t,T}\left(\mathbb{E}\left[\rho_{t,T}X^*|\mathcal{F}_t\right]\right)\hat{N}'_{t,T}\left(1-w_{t,T}\left(F^{\rho}_{t,T}(\rho_{t,T})\right)\right) \\ &\quad = u'_{t,T}\left((u'_{0,T})^{-1}\left(\eta^*_{0,T}(x)\,\hat{N}'_{0,T}\left(1-w_{0,T}\left(F^{\rho}_{0,T}(\rho_{0,T})\right)\right)\right)\right). \end{aligned} \tag{5}$$

- Define functions $g_t^{1,x}, g_t^2, g_t^{3,x} : (0,\infty) \to (0,\infty)$ by

$$\begin{aligned} g_t^{1,x}(y) &= \eta^*_{t,T}\left(\mathbb{E}\left[\rho_{t,T}(u'_{0,T})^{-1}\left(\eta^*_{0,T}(x)\,\hat{N}'_{0,T}\left(1-w_{0,T}\left(F^{\rho}_{0,T}(y\rho_{t,T})\right)\right)\right)\right]\right) \\ g_t^2(y) &= \hat{N}'_{t,T}\left(1-w_{t,T}\left(F^{\rho}_{t,T}(y)\right)\right), \\ g_t^{3,x}(y) &= u'_{t,T}\left((u'_{0,T})^{-1}\left(\eta^*_{0,T}(x)\,\hat{N}'_{0,T}\left(1-w_{0,T}\left(F^{\rho}_{0,T}(y)\right)\right)\right)\right). \end{aligned} \tag{6}$$

- Because $\rho_{0,t}$ and $\rho_{t,T}$ are independent with $\rho_{0,t}\rho_{t,T} = \rho_{0,T}$, we can deduce that for all y, z

$$g_t^{1,x}(y)g_t^2(z) = g_t^{3,x}(yz), \tag{7}$$

Sketch of the proof

- Next, we show that either $g_t^{1,x}, g_t^2, g_t^{3,x}$ are all strictly increasing, or constant. Then conclude that there are two cases: either the probability distortion $w_{s,t}$ satisfies the Jin-Zhou monotonicity condition or the family of probability distortion functions satisfies the Case i) of the Theorem.
- In the first case, $g_t^{1,x}, g_t^2, g_t^{3,x}$ simplify to

$$g_t^{1,x}(y) = \eta_{t,T}^* \left(\mathbb{E}\left[\rho_{t,T}(u'_{0,T})^{-1} \left(\eta_{0,T}^*(x) \frac{y\rho_{t,T}}{w'_{0,T}\left(F_{0,T}^{\rho}(y\rho_{t,T})\right)} \right) \right] \right),$$

$$g_t^2(y) = \frac{y}{w'_{t,T}\left(F_{t,T}^{\rho}(y)\right)}, \quad g_t^{3,x}(y) = u'_{t,T}\left((u'_{0,T})^{-1}\left(\eta_{0,T}^*(x) \frac{y}{w'_{0,T}\left(F_{0,T}^{\rho}(y)\right)} \right) \right).$$

- Taking $y = 1$ in (7) yields $g_t^{3,x}(z) = g_t^{1,x}(1) g_t^2(z)$ while taking $z = 1$ gives $g_t^{3,x}(y) = g_t^{1,x}(y) g_t^2(1)$. Combining the two gives

$$g_t^{3,x}(yz) = g_t^{1,x}(y) g_t^2(z) = \frac{g_t^{3,x}(y) g_t^{3,x}(z)}{g_t^{1,x}(1) g_t^2(1)} = \frac{g_t^{3,x}(y) g_t^{3,x}(z)}{g_t^{3,x}(1)}.$$

- Define $g:\mathbb{R}\to\mathbb{R}$ by $g(z):=\log\left(g_t^{3,x}(e^z)\right)-\log g_t^{3,x}(1)$.
- Deduce that g satisfies Cauchy's functional equation $g(y+z)=g(y)+g(z)$ and conclude that there must be a $\gamma_t\in\mathbb{R}$ such that $g(z)=\gamma_t z$.
- This yields that for $z>0$, $g_t^{3,x}(z)=g_t^{3,x}(1)z^{\gamma_t}$.
- Because $g_t^{3,x}$ is strictly increasing, it must be that γ is positive.
- From (7), we deduce that

$$\frac{g_t^{1,x}(y)}{y^{\gamma_t}}=g_t^{3,x}(1)\frac{z^{\gamma_t}}{g_t^2(z)}=C_{x,t}.$$

  for some constant $C_{x,t}$.
- From the definition of g_t^2 we obtain that, for any $t\in(0,T)$,

$$w_{t,T}(p)=\frac{1}{\mathbb{E}\left[\rho_{t,T}^{1-\gamma_t}\right]}\int_0^p\left(\left(F_{t,T}^{\rho}\right)^{-1}(q)\right)^{1-\gamma_t}dq.$$

- By continuity of $w_{t,T}$ in t at zero,

$$w_{0,T}(p) = \frac{1}{\mathbb{E}\left[\rho_{0,T}^{1-\gamma_0}\right]} \int_0^p \left(\left(F_{0,T}^{\rho}\right)^{-1}(q)\right)^{1-\gamma_0} dq$$

where $\gamma_0 = \lim_{t \searrow 0} \gamma_t$.

- The equality $g_t^{3,x}(y) = g_t^{3,x}(1) y^{\gamma_t}$ yields

$$u'_{t,T}\left(\left(u'_{0,T}\right)^{-1}\left(\eta^*_{0,T}(x)\,\mathbb{E}\left[\rho_{0,T}^{1-\gamma_0}\right] y^{\gamma_0}\right)\right) = g_t^{3,x}(1) y^{\gamma_t}.$$

- With the substitution $z = \left(u'_{0,T}\right)^{-1}\left(\eta^*_{0,T}(x)\mathbb{E}\left[\rho_{0,T}^{1-\gamma_0}\right] y^{\gamma_0}\right)$ the above becomes

$$u'_{t,T}(z) = \frac{g_t^{3,x}(1)}{\left(\eta^*_{0,T}(x)\mathbb{E}\left[\rho_{0,T}^{1-\gamma_0}\right]\right)^{\gamma_t/\gamma_0}} \left(u'_{0,T}(z)\right)^{\gamma_t/\gamma_0}.$$

- Some further calculations lead to $-\frac{1}{\gamma_t}\frac{u''_{t,T}(x)}{u'_{t,T}(x)} = -\frac{1}{\gamma_0}\frac{u''_{0,T}(x)}{u'_{0,T}(x)}$.

Discussion of dynamic rank-dependent utility process

- When there is some non-zero investment in the risky asset, constructing a dynamic rank-dependent utility process is possible only for a narrow class of distortion functions.
- It remains an open problem to find a dynamic preference functional $J(t, x; \vartheta)$ consistent with initial rank-dependent utility preferences for general distortion functions.
- The relationship

$$-\frac{1}{\gamma_t}\frac{u''_{t,T}(x)}{u'_{t,T}(x)} = -\frac{1}{\gamma_0}\frac{u''_{0,T}(x)}{u'_{0,T}(x)}$$

can be interpreted as: In order to be time-consistent, the investor must become more risk-averse if she becomes less pessimistic and less risk-averse if she becomes more pessimistic.

The dynamic distortion function of Ma-Wong-Zhang

- Ma, Wong and Zhang (2021) introduce the notion of a **dynamic distortion function**.
- They consider an Itô process described by the SDE

$$Y_t = y_0 + \int_0^t b(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dW_s, \quad 0 \leqslant t \leqslant T$$

  and a given family of probability distortion functions $(w_{0,t})_{0\leqslant t\leqslant T}$, where $w_{0,t}$ applies over $[0, t]$.
- They are able to derive a family of (random) probability distortion functions $(w_{s,t})_{0\leqslant s\leqslant t\leqslant T}$ such that $w_{s,t}$ is $\mathcal{F}_s$-measurable, for any $0 \leqslant s \leqslant t \leqslant T$, and the tower-property

$$\mathcal{E}_{r,t}[g(Y_t)] = \mathcal{E}_{r,s}\left[\mathcal{E}_{s,t}\left[g(Y_t)\right]\right], \quad 0 \leqslant r \leqslant s \leqslant t \leqslant T,$$

  holds for any continuous, bounded, increasing and nonnegative function g, where $\mathcal{E}_{s,t}$ denotes the nonlinear conditional expectation $\mathcal{E}_{s,t}[\xi] = \int_0^\infty w_{s,t}\left(\mathbb{P}\left[\xi \geqslant x\right]\right) dx$.

Review of classical stochastic control (again)

Recall (again) the classical problem in stochastic control

$$
\begin{aligned}
\max_{\vartheta \in \mathcal{A}} \quad & \mathbb{E}\left[U(X_T^{\vartheta}) \right] \\
\text{s.t.} \quad & dX_s^{\vartheta} = \mu\left(s, X_s^{\vartheta}, \vartheta(s, X_s^{\vartheta}\right) ds + \sigma\left(s, X_s^{\vartheta}, u\left(s, X_s^{\vartheta}\right)\right) dW(s), \\
& X_0^{\vartheta} = x_0.
\end{aligned}
$$

Three ingredients are required:

1) A time horizon T.
2) A model for the dynamics of the controlled state process.
3) A utility function U applying at the end of the time horizon.

All model ingredients have to be specified at initial time, $t = 0$.

Forward performance criteria: Motivation

- In the classical setting, the Dynamic Programming Principle holds, policies are time-consistent, and we have the martingale optimality principle.
- There are however several drawbacks:
  - What happens at time T?
  - Requirement to today specify preferences which apply in the future.
  - Relies on model selection for the entire horizon.
- Under forward performance criteria on the other hand, the investor starts with specifying her preferences for today.
- The agent then updates her preferences under the guidance of the martingale optimality principle. This imposes the DPP forward, and not backwards, in time.
- Consequently, forward performance criteria adapt to the changing market conditions and accommodate dynamically changing horizons.

The financial market consists of one risk-free and k risky assets. The price of the i^{th} risky asset solves

$$dS_t^i = S_t^i\left(\mu_t^i dt + \sum_{j=1}^{N} \sigma_t^{ij} dW_t^j\right), \quad t \geqslant 0,$$

with $S_0^i = s_0^i > 0$, $i = 1, \ldots, N$.

The market price of risk process is defined as $\lambda_t := (\sigma_t^{-1})^+ \ mu_t$.

Given a self-financing trading policy $(\vartheta_t)_{t\geqslant 0}$, the wealth process $X = (X_t^{x,\vartheta})_{t\geqslant 0}$ solves the stochastic differential equation

$$dX_t^{x,\vartheta} = \vartheta'_t \mu_t dt + \vartheta'_t \sigma_t dW_t, \quad t \geqslant 0,$$

with $X_0^{x,\vartheta} = x$.

Forward performance criteria have been introduced by by Musiela and Zariphopoulou (2006, 2008, 2009, 2010a,b,2011), see also Henderson and Hobson (2007).

Definition

An $\mathbb{F}$-adapted process $(U_t)_{t\geqslant 0}$ is a **forward performance criterion** if

i) for any $t \geqslant 0$ and fixed $\omega \in \Omega$, $U_t \in \mathcal{U}$,

ii) for any $\vartheta \in \mathcal{A}$, $0 \leqslant s \leqslant t$ and $x > 0$

$$\mathbb{E}\left[U_t\left(X_t^{x,\vartheta}\right) \middle| \mathcal{F}_s \right] \leqslant U_s\left(X_s^{x,\vartheta}\right),$$

iii) there exists $\vartheta^* \in \mathcal{A}$ such that, for any $0 \leqslant s \leqslant t$, and $x > 0$,

$$\mathbb{E}\left[U_t\left(X_t^{x,\vartheta^*}\right) \middle| \mathcal{F}_s \right] = U_s\left(X_s^{x,\vartheta^*}\right).$$

Time-monotone forward performance criteria

Musiela and Zariphopoulou (2010) characterize **time-monotone forward performance criteria** $U_t(x)$. They are given by

$$U_t(x) = v(x, A_t) \quad \text{with} \quad A_t := \int_0^t \|\lambda_s\|^2 ds.$$

The function $v(x,t)$ solves, for $x \geqslant 0$, $t \geqslant 0$,

$$v_t = \frac{1}{2}\frac{v_x^2}{v_{xx}},$$

and $v(x,0)$ must be of the form $(v')^{-1}(x,0) = \int_{0^+}^{\infty} x^{-y}\mu(dy)$, where μ is a positive finite Borel measure.

Time-monotone forward performance criteria

Musiela and Zariphopoulou (2010) find that the solutions to the PDE

$$v_t = \frac{1}{2}\frac{v_x^2}{v_{xx}},$$

are of the following structure

$$v\left(x,t\right) = -\frac{1}{2}\int_0^t e^{-h^{-1}(x,s)+\frac{s}{2}} h_x\left(h^{-1}(x,s),s\right) ds + \int_0^x e^{-h^{-1}(z,0)} dz$$

with $h(z,t)$, $z \in \mathbb{R}$, $t \geqslant 0$, given by

$$h\left(z,t\right) := \int_{(0,+\infty)} e^{zy-\frac{1}{2}y^2 t}\mu(dy).$$

Can we build forward rank-dependent performances?

- Can we build forward criteria for preferences which, from a classical perspective, would be time-inconsistent, for example forward rank-dependent performance criteria?
- The definition of forward performance criteria is motivated by the **Martingale Optimality Principle**, a consequence of the existence of an optimal control.
- Directly embedded is the fundamental connection between DPP and time-consistency of the optimal policies.
- For RDU however, none of these features exist in the classical, backward case: No DPP, no time-consistency of optimal policies and not even a notion of a (super-)martingale.
- We propose two distinct definitions: The first directly imitates the definition without probability distortion, the second is based on time-consistency of optimal strategies. We then establish there equivalence.

Forward rank-dependent performance criteria

For forward rank-dependent performance criteria, we additionally assume that the coefficients μ and σ are deterministic and that the market is complete.

Definition

A pair of deterministic processes $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s<t}\right)$ is a **forward rank-dependent performance criterion** if the following properties hold:

i) for any $t \geqslant 0$, $u_t(\cdot) \in \mathcal{U}$ and for any $0 \leqslant s < t$, $w_{s,t}(\cdot) \in \mathcal{W}$.

ii) for any $\pi \in \mathcal{A}$, $0 \leqslant s < t$ and $x > 0$,

$$\int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\pi}|\mathcal{F}_s}(\xi)\right)\right) \leqslant u_s\left(X_s^{x,\pi}\right).$$

iii) there exists $\pi^* \in \mathcal{A}$, such that for any $0 \leqslant s < t$ and $x > 0$,

$$\int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\pi^*}|\mathcal{F}_s}(\xi)\right)\right) = u_s\left(X_s^{x,\pi^*}\right).$$

Forward rank-dependent performance criteria

Definition

Let $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s<t}\right)$ be deterministic with $u_t \in \mathcal{U}$ and $w_{s,t} \in \mathcal{W}$.

i) The pair is called a **time-consistent rank-dependent performance criterion**, if there exists $\pi^* \in \mathcal{A}$, such that π^* solves

$$\max_{\pi\in\mathcal{A}(\pi^*,s)} \int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\pi}|\mathcal{F}_s}(\xi)\right)\right)$$

with $dX_r^{x,\pi} = \pi'_r\mu_r dr + \pi'_r\sigma_r dW_r$, $r \in [0,t]$ and $X_0^{x,\pi} = x$, for any $0 \leqslant s < t$ and $x > 0$.

ii) A pair satisfying (i) is called a **time-consistent rank-dependent performance criterion preserving the performance value** if, for any optimal policy π^* as in (i), we have for any $0 \leqslant s < t$, and $x > 0$

$$\int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\pi^*}|\mathcal{F}_s}(\xi)\right)\right) = u_s\left(X_s^{x,\pi^*}\right).$$

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Forward rank-dependent performance criteria

The following proposition shows that the two definitions are in fact equivalent and thereby builds a direct connection between time-consistency and forward rank-dependent criteria.

Proposition (He, Strub, and Zariphopoulou (2021))

A pair of deterministic functions $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s<t}\right)$ *is a forward rank-dependent performance criterion if and only if it is a time-consistent rank-dependent performance criterion preserving the performance value.*

Construction method for forward RDPPs

Theorem (He, Strub, and Zariphopoulou (2021))

Forward rank-dependent performance criteria can thus be constructed as follows. Let $\gamma \geqslant 0$ *and consider an initial datum of the form* $(u')^{-1}(x,0) = \int_{0^+}^{\infty} x^{-y}\mu(dy)$*. Let* $h(z,t)$ *and* $v(x,t)$ *be given by as before. Then, the pair* $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s<t}\right)$*, with*

$$u_t(x) := v\left(x, \gamma^2 \int_0^t \|\lambda_s\|^2 ds\right)$$

and

$$w_{s,t}(p) := \frac{1}{\mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right]} \int_0^p \left(\left(F_{s,t}^{\rho}\right)^{-1}(q)\right)^{1-\gamma} dq$$

is a forward rank-dependent performance criterion.

Economic interpretation

To give an economic interpretation of the distortion parameter γ we recall the notion of **pessimism** introduced in Quiggin (1993). For a RDU representation V with utility function u, distortion function w and a prospect X, one can define the **certainty equivalent** $CE(X)$ of X, by $CE(X) := u^{-1}(V(X))$ and **risk-premium** $\Delta(X)$ of X by $\Delta(X) := \mathbb{E}[X] - CE(X)$ exactly as in expected utility theory. Under RDU however, the risk premium of X can be decomposed into the **pessimism premium** $\Delta_w(X)$ of X, defined by

$$\Delta_w(X) := \mathbb{E}[X] - \int_0^\infty \xi d\left(-w(1 - F_X(\xi)\right),$$

and the **outcome premium** $\Delta_{u,w}(X)$ of X, defined by

$$\Delta_{u,w}(X) := \int_0^\infty \xi d\left(-w(1 - F_X(\xi)\right) - CE(X).$$

Definition

Let V, V_1, V_2 be RDU representations with utility functions u, u_1, u_2 and distortion functions w, w_1, w_2, respectively. Then,

i) V is called pessimistic if for any X with bdd support, $\Delta_w(X) \geq 0$.

ii) V_1 is said to be more pessimistic than V_2 if for any X with bounded support, $\Delta_{w_1}(X) \geqslant \Delta_{w_2}(X)$.

Definition

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Proposition (He, Strub, and Zariphopoulou (2021))

Let V, V_1, V_2 be RDU representations with utility functions u, u_1, u_2 and distortion functions w, w_1, w_2 given by (2) with distortion parameters $\gamma, \gamma_1, \gamma_2$ respectively. Then the following holds:

i) *V is pessimistic if and only if $\gamma \leqslant 1$.*

ii) *V_1 is more pessimistic than V_2 if and only if $\gamma_1 \leqslant \gamma_2$.*

Summary

- Time-inconsistency is a consequence of preferences violating the independence axiom and applying the original preferences at all times in a consequentialist manner.
- Alternative to consequentialism: Dynamic utility approach. Under this approach, dynamic preferences are constructed which are consistent with the original preferences at time zero and lead to time-consistent policies.
- Forward performance criteria: Starting from initial preferences, preferences, strategies, and wealth evolve together endogenously and forward in time under the guidance of the martingale optimality principle.