Probabilistic methods for elliptic and parabolic PDEs: from linear equations to free-boundary problems

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Outline

1. Linear PDEs and SDEs
    - Feynman-Kac formula
    - From SDE to PDE
    - Krylov-Safonov method
2. HJB equation, Stochastic Control and BSDEs
    - Stochastic control and HJB equation
    - Utility maximization with random endowment in a price impact model
3. Probabilistic representations for free-boundary problems
    - Obstacle problems and optimal stopping
    - Stefan equation and mean-field systems

Section 1

Linear PDEs and SDEs

Linear parabolic PDEs and SDEs

- Consider a linear parabolic Partial Differential Equation (PDE):

$$
\begin{aligned}
&u_t + \mathcal{L}(u_{xx}, u_x, x, t) = 0, \quad x \in D(t) \subset \mathbb{R}^d, \quad t < T,\\
&u(x,t) = \phi(x,t),\ (x,t) \in \partial D_T,\\
&\partial D_T := \partial\{(x,t) : x \in D(t), t \in (0,T)\} \setminus (\bar{D}(0) \times \{0\}),
\end{aligned}
$$

where $\phi(x,t)$ is a Cauchy-Dirichlet boundary condition, and

$$
\begin{aligned}
&\mathcal{L}(u_{xx}, u_x, x, t) = \sum_{i,j=1}^{d} a^{ij}(x,t) u_{x^i x^j} + \sum_{i=1}^{d} b^i(x,t) u_{x^i},\\
&\lambda^\top (a^{ij}) \lambda \geq 0, \quad \forall \lambda \in \mathbb{R}^d.
\end{aligned}
$$

- A Stochastic Differential Equation (SDE) driven by Brownian motion W is given by

$$
dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad X_0 = \xi : \Omega \to \mathbb{R}^d.
$$

See e.g. Karaztas-Shreve 1991 for more on the existence and uniqueness of such X.

Feynman-Kac formula

$$u_t + \sum_{i,j=1}^{d} a^{ij}(x,t) u_{x^i x^j} + \sum_{i=1}^{d} b^i(x,t) u_{x^i} = 0, \tag{1}$$

$$u(x,t) = \phi(x,t), \; (x,t) \in \partial D_T,$$

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \tag{2}$$

Feynman-Kac formula (Karaztas-Shreve 1991): if there exists a classical solution u to (1), then, under additional technical assumptions, we have

$$u(x,t) = \mathbb{E}\left[\phi(X_\tau, \tau) \,|\, X_{t\wedge\tau} = x\right], \quad (x,t) \in D \times [0,T],$$

where X is a solution to (2) and

$$(a^{ij}) = \frac{1}{2}\sigma\sigma^\top, \quad (b^i) = \mu,$$

$$\tau := \inf\{s \geq 0 : \; X_s \notin D(s)\} \wedge T.$$

Proof

$$u_t + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^\top)^{ij} u_{x^i x^j} + \sum_{i=1}^{d} \mu^i u_{x^i} = 0, \quad u = \phi \text{ on } \partial D_T,$$
$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t,$$
$$u(x,t) = \mathbb{E}\left[\phi(X_\tau, \tau) \,|\, X_{t\wedge\tau} = x\right], \quad \tau = \inf\{s \geq 0 : \; X_s \notin D(s)\} \wedge T$$

- Itô's formula yields that $u(X_t, t)$ is a local martingale on $[0, \tau)$:

$$du(X_t, t) = \left[u_t + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^\top)^{ij} u_{x^i x^j} + \sum_{i=1}^{d} \mu^i u_{x^i}\right] dt + [\ldots]\, dW_t = [\sigma u_x]\, dW_t$$

- Under additional assumptions, we verify that $u(X_{t\wedge\tau}, t\wedge\tau)$ is a true martingale. Then, using continuity of ϕ,

$$u(X_{t\wedge\tau}, t\wedge\tau) = \mathbb{E}\left[\phi(X_\tau, \tau) \,|\, \mathcal{F}^W_{t\wedge\tau}\right]$$

- Markov property of X yields $u(X_{t\wedge\tau}, t\wedge\tau) = \mathbb{E}\left[\phi(X_\tau, \tau) \,|\, X_{t\wedge\tau}\right]$.

Applications of Feynman-Kac

$$u_t + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^\top)^{ij}u_{x^ix^j} + \sum_{i=1}^{d}\mu^i u_{x^i} = 0, \quad u = \phi \text{ on } \partial D_T, \tag{3}$$

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t,$$

$$u(x,t) = \mathbb{E}\left[\phi(X_\tau,\tau)\,|\,X_{t\wedge\tau} = x\right], \quad \tau = \inf\{s \geq 0:\ X_s \notin D(s)\} \wedge T \tag{4}$$

- The stochastic representation (4) leads to Monte-Carlo methods for computing the solution to (3), which is more efficient than traditional PDE methods if d is large.
- Representation (4) may help us deduce additional (useful) properties of u via the (known) properties of X (see upcoming Krylov-Safonov method),
- or deduce additional (useful) properties of X via the (known) properties of u (e.g. possibility of hitting a particular area of ∂D via maximum principle for (3)).

Extensions: elliptic PDEs

$$\frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^{\top})^{ij}u_{x^ix^j}+\sum_{i=1}^{d}\mu^i u_{x^i}=0,\quad u=\phi \text{ on } \partial D, \tag{5}$$

$$dX_t=\mu(X_t,t)dt+\sigma(X_t,t)dW_t, \tag{6}$$

- **Feynman-Kac formula**: if there exists a classical solution u to (5), then, under additional technical assumptions, we have

  $$u(x)=\mathbb{E}\left[\phi(X_\tau)\,|\,X_0=x\right],\quad x\in D,\quad \tau:=\inf\{s\geq 0:\ X_s\notin D\},$$

  where X is a solution to (6).
- **Exercise**: prove the above.

More extensions

- **Exercise**: derive a Feynman-Kac formula for

$$u_t + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x,t)u_{x^i x^j} + \sum_{i=1}^{d} b^i(x,t)u_{x^i} + c(x,t)u + f(x,t) = 0,$$
$$u = \phi \text{ on } \partial D_T.$$

- **Question**: how about a problem with VonNeumann boundary condition:

$$u_t + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x,t)u_{x^i x^j} + \sum_{i=1}^{d} b^i(x,t)u_{x^i} = 0,$$
$$u(x,T) = \phi(x), \quad (\nabla u(x,t)\cdot\nu)|_{x\in\partial D} = \psi(x,t),$$

  where ν is the unit outer normal to ∂D and ϕ, ψ are given?

- **Exercise**: find a Feynman-Kac representation for a solution to the VonNeumann problem, assuming $d = 1$, $D = (0,\infty)$, $\psi \equiv 0$, $\phi \in C_0^1$, and smooth a^{ij}, b^i with bounded derivatives of all order.

Viscosity solution

- Feynman-Kac allows you to go from a PDE solution to a probabilistic object.
- If the PDE theory does not yield a sufficiently regular solution, one can try to construct a solution to target PDE via probabilistic methods directly.
- Consider a solution X to the SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

- **Exercise**: show that, for unifomrly continuous μ, σ, ϕ, the function

$$u(x,t) := \mathbb{E}\left[\phi(X_T) \,|\, X_t = x\right], \quad (x,t) \in \mathbb{R}^d \times (0,T),$$

is a viscosity solution (see e.g. Crandall et al 1992) to the Cauchy problem:

$$u_t + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^\top)^{ij} u_{x^i x^j} + \sum_{i=1}^{d} \mu^i u_{x^i} = 0, \quad (x,t) \in \mathbb{R}^d \times (0,T),$$
$$u(x,T) = \phi(x),\ x \in \mathbb{R}^d.$$

Adjoint equation

- Denote by $v(\cdot, t)$ the density of X_t that solves $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$.
- Itô's formula yields for any $f \in C_0^\infty(\mathbb{R}^d \times (0, T))$:

$$\frac{d}{dt}\mathbb{E}f(X_t, t) = \mathbb{E}\left[f_t + \sum_{i=1}^d f_{x^i}\mu^i + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^\top)^{ij} f_{x^i x^j}\right],$$

$$0 = \mathbb{E}f(X_t, t)\big|_{t=0}^T = \int_0^T \int_{\mathbb{R}^d} \left[f_t + \sum_{i=1}^d f_{x^i}\mu^i + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^\top)^{ij} f_{x^i x^j}\right] v(x, t)dxdt,$$

$$\int_0^T \int_{\mathbb{R}^d} \left[-v_t - \sum_{i=1}^d (\mu^i v)_{x^i} + \frac{1}{2}\sum_{i,j=1}^d ((\sigma\sigma^\top)^{ij} v)_{x^i x^j}\right] f(x, t)dxdt = 0$$

- Thus, v is a weak solution of the adjoint equation

$$v_t + \sum_{i=1}^d (\mu^i v)_{x^i} - \frac{1}{2}\sum_{i,j=1}^d ((\sigma\sigma^\top)^{ij} v)_{x^i x^j} = 0$$

Adjoint equation: boundary conditions

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad X_0 = \xi, \tag{7}$$

$$v_t + \sum_{i=1}^{d} (\mu^i v)_{x^i} - \frac{1}{2} \sum_{i,j=1}^{d} ((\sigma\sigma^\top)^{ij} v)_{x^i x^j} = 0 \tag{8}$$

- **Question**: what are the boundary conditions for (8)?
- It is easy to see that $v(\cdot, 0)$ is the density of ξ.
- Given a (constant) domain D, with ∂D of class C^1, and assuming that X is absorbed (stopped) at the first exit time from D, one can show (under additional assumptions on μ, σ) that

$$v(x,t)|_{x \in \partial D} = 0.$$

- **Exercise**: using conformal invariance of planar Brownian motion (see LeGall 1992), show that for $d = 2$, $D = \mathbb{R}^2 \setminus \mathbb{R}^2_+$, $\mu \equiv 0$, $\sigma = I$ (so that X is a 2-dim. Brownian motion), for any ξ supported in D and any $t > 0$, there exists a sequence $D \ni x_n \to 0$, s.t. $\lim_{n\to\infty} v(x_n, t) \neq 0$.

Adjoint equation: VonNeumann condition

$$v_t + \sum_{i=1}^{d} (\mu^i v)_{x^i} - \frac{1}{2} \sum_{i,j=1}^{d} ((\sigma\sigma^\top)^{ij} v)_{x^i x^j} = 0, \quad (x,t) \in D \times (0,T), \tag{9}$$

$$v(x,0) = \phi(x), \quad x \in D \tag{10}$$

- **Question**: what if we equip (10) with a VonNeumann boundary condition

  $$(\nabla v(x,t) \cdot \nu)|_{x \in \partial D} = \psi(x,t),$$

  where ν is the unit outer normal to ∂D and ψ is a given function?
- **Exercise**: find a probabilistic representation of v for $\psi \equiv 0$, assuming $\phi \geq 0$ and bounded smooth D.
- **Exercise**: find a probabilistic representation of v for $\psi \equiv \text{const} > 0$, assuming $\phi \geq 0$ and bounded smooth D.

Krylov-Safonov method for improving regularity

- This method allows one to deduce Hölder regularity of a solution to heat equation with zero boundary condition, eve if the boundary is not smooth.
- **Theorem** (Krylov-Safonov 1979). Assume $d = 1$, ϕ is bounded, and $\Lambda(\cdot)$ is a $1/2$-Hölder function, and u is a classical solution to

$$u_t + \frac{1}{2} u_{xx} = 0, \quad t \in (0, T), \quad x > \Lambda(t),$$
$$u(x, T) = \phi(x), \quad u(\Lambda(t), t) = 0.$$

Then, for any $\epsilon > 0$, there exist $\chi, C > 0$, s.t.

$$|u(x,t)| \le C(x - \Lambda(t))^{\chi}, \quad t \in (0, T - \epsilon], \quad x > \Lambda(t).$$

Proof

$$u_t + \frac{1}{2} u_{xx} = 0, \quad t \in (0, T), \quad x > \Lambda(t)$$

- Let us fix $t \in (0, T - \epsilon/2]$, $\delta^2 \in (0, \epsilon/2)$, and $x \in (\Lambda(t), \Lambda(t) + \delta]$.
- Using Feynman-Kac (or its proof), we deduce that

$$u(x, t) = \mathbb{E} u(x + W_\tau, t + \tau),$$

where W is a Brownian motion and τ is any stopping time with values in $[0, \tau_0]$, with

$$\tau_0 := \inf\{s \geq 0 : \ x + W_s \leq \Lambda(t + s)\} \wedge \delta^2$$

- Fix $L \geq 1$ and set

$$\tau := \inf\{s \geq 0 : \ x + W_s \geq \Lambda(t + s) + L\delta\} \wedge \tau_0$$

Proof

$$\tau = \inf\{s \geq 0 : \ x + W_s \geq \Lambda(t+s) + L\delta\} \wedge \inf\{s \geq 0 : \ x + W_s \leq \Lambda(t+s)\} \wedge \delta^2,$$
$$u(x,t) = \mathbb{E} u(x + W_\tau, t + \tau)$$

- From the above, we deduce
$$|u(x,t)| \leq [1 - \mathbb{P}\left(x + W_\tau = \Lambda(t+\tau)\right)] \sup_{(y,s)\in Q(t,\delta^2,L\delta)} |u(y,s)|,$$
$$Q(t,\delta^2,L\delta) = \{(y,s) : \ s \in [t, t+\delta^2], \, y \in [\Lambda(t+s), \Lambda(t+s) + L\delta]\}$$
- The $1/2$-Hölder property of Λ yields constant κ s.t. $\sup_{s\in[0,\tau]} |\Lambda(t+s) - \Lambda(t)| \leq \kappa\delta$.
- Therefore,
$$\mathbb{P}(x + W_\tau = \Lambda(t+\tau))$$
$$\geq \mathbb{P}\left(\inf_{s\in[0,\delta^2]} W_s < -(1+\kappa)\delta, \ \sup_{s\in[0,\delta^2]} W_s < (L - (1+\kappa))\delta\right)$$

Proof

$$|u(x,t)| \leq [1 - \mathbb{P}(x + W_\tau = \Lambda(t+\tau))] \sup_{(y,s)\in Q(t,\delta^2,L\delta)} |u(y,s)|,$$

$$\mathbb{P}(x + W_\tau = \Lambda(t+\tau))$$
$$\geq \mathbb{P}\left(\inf_{s\in[0,\delta^2]} W_s < -(1+\kappa)\delta, \sup_{s\in[0,\delta^2]} W_s < (L-(1+\kappa))\delta \right)$$

- Choosing $L = 2(1+\kappa)$ and using the scaling properties of W, we obtain a constant $c \in (0,1)$, depending only on κ, s.t.

$$\mathbb{P}(x + W_\tau = \Lambda(t+\tau)) \geq c,$$
$$|u(x,t)| \leq (1-c) \sup_{(y,s)\in Q(t,\delta^2,L\delta)} |u(y,s)|,$$

Proof

$$Q(t,\delta^2,L\delta)=\{(y,s):\ s\in[t,t+\delta^2],\ y\in[\Lambda(t+s),\Lambda(t+s)+L\delta]\},$$
$$|u(x,t)|\leq(1-c)\sup_{(y,s)\in Q(t,\delta^2,L\delta)}|u(y,s)|, \quad (11)$$
$$t\in(0,T-\epsilon/2],\ \ \delta^2\in(0,\epsilon/2),\ \ x\in(\Lambda(t),\Lambda(t)+\delta]$$

- Note that we can apply (11) with (x,t) replaced by any $(y,s)\in Q(t,\delta^2,L\delta)$ and with δ replaced by $L\delta$:

$$|u(y,s)|\leq(1-c)\sup_{(z,r)\in Q(s,L^2\delta^2,L^2\delta)}|u(z,r)|$$
$$\leq(1-c)\sup_{(z,r)\in Q(t,(1+L^2)\delta^2,L^2\delta)}|u(z,r)|,$$

  provided $t\in(0,T-\epsilon]$ and $(1+L^2)\delta^2<\epsilon/2$.
- This gives us

$$|u(x,t)|\leq(1-c)^2\sup_{(y,s)\in Q(t,(1+L^2)\delta^2,L^2\delta)}|u(y,s)|$$

Proof

$$Q(t, \delta^2, L\delta) = \{(y, s) : \ s \in [t, t + \delta^2], \, y \in [\Lambda(t + s), \Lambda(t + s) + L\delta]\},$$
$$|u(x, t)| \leq (1 - c) \sup_{(y,s) \in Q(t, \delta^2, L\delta)} |u(y, s)|,$$
$$t \in (0, T - \epsilon/2], \ \delta^2 \in (0, \epsilon/2), \ x \in (\Lambda(t), \Lambda(t) + \delta]$$

- Iterating the above estimate, we obtain, for $t \in (0, T - \epsilon]$:

$$|u(x, t)| \leq (1 - c)^{k+1} \sup_{(y,s) \in Q(t, (1 + \sum_{i=1}^{k} L^{2k})\delta^2, L^{k+1}\delta)} |u(y, s)|,$$

  for any k s.t. $\delta^2 L^{2(k+1)} \leq \epsilon/2$.

- Taking $k + 1 = \lfloor \left(\log \frac{\epsilon}{2\delta^2}\right) / \log L^2 \rfloor$ and $x = \Lambda(t) + \delta$, we obtain

$$|u(\Lambda(t) + \delta, t)| \leq \delta^{-2 \log(1-c)/ \log L^2} C(c, L, \epsilon) \sup_{(y,s) \in Q(0, T, \sqrt{\epsilon/2})} |u(y, s)|,$$

$$\forall \, \delta \in (0, \sqrt{\epsilon/2}), \quad t \in (0, T - \epsilon/2]$$

Section 2

HJB equation, Stochastic Control and BSDEs

Stochastic Control Problem and HJB equation

- Consider a controlled diffusion process

$$dX_t^\nu = \mu(X_t^\nu, t, \nu_t)dt + \sigma(X_t^\nu, t, \nu_t)dW_t,$$

where ν is a stochastic process from a given class $\mathcal{A}$, referred to as the control.

- The goal is to solve

$$\sup_{\nu \in \mathcal{A}} \mathbb{E}\left[U(X_T^\nu) + \int_0^T g(X_t^\nu, t, \nu_t)dt\right]$$

- Hamilton-Jacobi-Bellman (HJB) equation for the value function $V(x,t)$

$$V_t + \sup_\nu \left[V_x \cdot \mu(\nu) + \frac{1}{2}\mathrm{Tr}(V_{xx}\sigma(\nu)\sigma^\top(\nu)) + g(\nu)\right] = 0,$$
$$V(x, T) = U(x),$$

where $\mathrm{Tr}(V_{xx}\sigma(\nu)\sigma^\top(\nu)) = \sum_{i,j=1}^d (\sigma(\nu)\sigma^\top(\nu))^{ij} V_{x^i x^j}$.

Verification

$$dX_t^\nu = \mu(X_t^\nu, t, \nu_t)dt + \sigma(X_t^\nu, t, \nu_t)dW_t,$$

$$V_t + \sup_\nu \left[V_x \cdot \mu(\nu) + \frac{1}{2}\mathrm{Tr}(V_{xx}\sigma(\nu)\sigma^\top(\nu)) + g(\nu) \right] = 0, \quad V(x,T) = U(x)$$

- If we can find a sufficiently smooth V (with appropriate bounds on its derivatives), we can solve the stochastic control problem:

$$d\left[V(X_t^\nu, t) + \int_0^t g(X_s^\nu, s, \nu_s)ds \right]$$
$$= \left(V_t + V_x \cdot \mu(\nu) + \frac{1}{2}\mathrm{Tr}(V_{xx}\sigma(\nu)\sigma^\top(\nu)) + g(\nu) \right) dt + (\ldots)dW_t$$

- We deduce that $Y_t^\nu := V(X_t^\nu, t) + \int_0^t g(X_s^\nu, s, \nu_s)ds$ is a supermartingale for any $\nu \in \mathcal{A}$. Hence,

$$V(X_0, 0) = Y_0^\nu \geq \mathbb{E}Y_T^\nu = \mathbb{E}\left[U(X_T^\nu) + \int_0^T g(X_t^\nu, t, \nu_t)dt \right], \quad \forall \nu \in \mathcal{A}$$

Verification

$$V_t + \sup_{\nu} \left[V_x \cdot \mu(\nu) + \frac{1}{2}\mathrm{Tr}(V_{xx}\sigma(\nu)\sigma^\top(\nu)) + g(\nu) \right] = 0, \tag{12}$$

$$V(X_0, 0) \geq \mathbb{E}\left[U(X_T^\nu) + \int_0^T g(X_t^\nu, t, \nu_t)dt \right], \quad \forall \nu \in \mathcal{A}$$

- Denote by $\nu^*(x,t)$ the value of ν that attains the supremum in (12).
- Assume that the SDE

  $$dX_t^* = \mu(X_t^*, t, \nu^*(X_t^*, t))dt + \sigma(X_t^*, t, \nu^*(X_t^*, t)dW_t$$

  has a solution.
- Then $Y_t^* := V(X_t^*, t) + \int_0^t g(X_s^*, s, \nu^*(X_s^*, s))ds$ is a martingale and

  $$V(X_0, 0) = Y_0^* = \mathbb{E}Y_T^* = \mathbb{E}\left[U(X_T^*) + \int_0^T g(X_t^*, t, \nu^*(X_t^*, t))dt \right]$$

Beyond verification

$$V_t + \sup_{\nu} \left[V_x \cdot \mu(\nu) + \frac{1}{2}\mathrm{Tr}(V_{xx}\sigma(\nu)\sigma^\top(\nu)) + g(\nu) \right] = 0, \;\; V(x,T) = U(x) \quad (13)$$

- The verification approach presents multiple technical challenges, which mainly stem from the fact that we need to know the existence of a classical solution to the fully nonlinear PDE (13), with additional bounds on its derivatives.
- To avoid verification, one can define V directly via the control problem,

$$V(x,t) := \sup_{\nu \in \mathcal{A}} \mathbb{E}\left[U(X_T^\nu) + \int_t^T g(X_s^\nu, s, \nu_s) ds \,\middle|\, X_t^\nu = x \right],$$

  and show that such V is a viscosity solution to (13) (see Bouchard-Touzi 2011, Bayraktar-Sirbu 2012). This does not yield optimal control $\nu*$.
- We may **combine the above with direct analysis of the control problem and with probabilistic methods, including Backward Stochastic Differential Equations (BSDEs)**, to find a classical solution to (13).

Utility maximization with random endowment in Almgren-Chriss model (Ekren-N. 2019)

- Consider an unperturbed price process $S_t := s + \sigma W_t$.
- An agent trades at rate ν_t at time t, so that she holds π_t shares at time t, with

$$\pi_t^\nu = \pi_0 + \int_0^t \nu_u du$$

- Trading affects the price via linear temporary price impact with coefficient η, so that trading at time t occurs at price $S_t + \eta\nu_t$.
- Then, the agent's cash position at time t is

$$X_t^\nu = x - \int_0^t (\eta\nu_u + S_u)\nu_u du$$

- The agent holds a derivative with payoff $H(S_T)$ and solves

$$\sup_{\nu \in L^4} \mathbb{E} U \left[X_T^\nu + \pi_T^\nu S_T + H(S_T) \right], \quad U(x) := -e^{-\gamma x}$$

HJB equation

$$d\pi_t^\nu = \nu_t dt, \quad dS_t := \sigma dW_t, \quad dX_t^\nu = -(\eta\nu_t + S_t)\nu_t dt,$$
$$\sup_{\nu\in L^4} \mathbb{E}U\left[X_T^\nu + \pi_T^\nu S_T + H(S_T)\right]$$

- The HJB equation for the value function $V(s,\pi,x,t)$ control problem is

$$V_t + \frac{\sigma^2}{2}V_{ss} + \sup_{\nu\in\mathbb{R}}\left[\nu V_\pi - \nu(s+\eta\nu)V_x\right] = 0, \quad V(s,\pi,x,T) = -e^{-\gamma[x+\pi s+H(s)]},$$

$$\sup_{\nu\in\mathbb{R}}\left[\nu V_\pi - \nu(s+\eta\nu)V_x\right] = \frac{1}{4\eta}\left(\frac{V_\pi}{V_x} - s\right)^2 V_x$$

- The above PDE is degenerate and has quadratic growth in the gradient.
- **Question**: how to deduce the existence of its classical solution?

Bounded controls

- Recall $U(x) = -e^{-\gamma x}$.
- Assume that the set of admissible controls is given by $\mathcal{A}_\epsilon$, s.t. each control $\nu \in \mathcal{A}_\epsilon$ satisfies $|\nu| \leq 1/\epsilon$, for $\epsilon > 0$.
- Then, the value function can be written as

$$V(s,\pi,x,t) := \sup_{\nu \in \mathcal{A}_\epsilon} J(s,\pi,x,t;\nu) :=$$

$$\sup_{\nu \in \mathcal{A}_\epsilon} \mathbb{E} U \left[x - \int_t^T (\eta \nu_u + S_u)\nu_u du + (\pi + \int_t^T \nu_u du) S_T + H(S_T) \right]$$

- Using dominated convergence, we can show directly (from the above) that $V_x, 1/V_x, V_\pi$ are well defined and continuous: e.g.,

$$J(s,\pi,x,t;\nu) - J(s,\pi',x,t;\nu) = \mathbb{E}\left[\left(1 - e^{-\gamma(\pi' - \pi) S_T}\right) \cdot \right.$$

$$\left. \cdot U \left(x - \int_t^T (\eta \nu_u + S_u)\nu_u du + (\pi + \int_t^T \nu_u du) S_T + H(S_T) \right) \right]$$

Solution

$$V_t + \frac{\sigma^2}{2} V_{ss} + \frac{1}{4\eta}\left(\frac{V_\pi}{V_x} - s\right)^2 V_x = 0, \quad V(s,\pi,x,T) = -e^{-\gamma[x+\pi s+H(s)]} \tag{14}$$

- The weak dynamic programming principle of Bouchard-Touzi 2011 yields that the value function V is a viscosity solution to (14).
- As we know that the blue part in (14) is well defined, we can treat (14) as a linear equation

$$V_t + \frac{\sigma^2}{2} V_{ss} + g = 0, \tag{15}$$

  with the source term $g(s,\pi,x,t) := \frac{1}{4\eta}\left(\frac{V_\pi}{V_x} - s\right)^2 V_x$.
- There exists a classical solution $\hat{V} \in C^{2,1,1,1}$ to (15). As the classical solution is a viscosity solution and the viscosity solution to (15) is unique, we conclude that $V = \hat{V}$.

How to deduce the boundedness of controls?

$$J(s,\pi,x,t;\nu) = \mathbb{E}U\left[x - \int_t^T (\eta\nu_u + S_u)\nu_u du + (\pi + \int_t^T \nu_u du)S_T + H(S_T)\right],$$

$$U(x) = -e^{-\gamma x}$$

- We can deduce that

$$V^\epsilon(s,\pi,x,t) := \sup_{\nu\in\mathcal{A}_\epsilon} J(s,\pi,x,t;\nu)$$

  is a classical solution to the ϵ-HJB:

$$V_t + \frac{\sigma^2}{2}V_{ss} + \sup_{|\nu|\leq 1/\epsilon}\left[\nu V_\pi - \nu(s+\eta\nu)V_x\right] = 0.$$

- **Question**: how to obtain a classical solution to the desired HJB:

$$V_t + \frac{\sigma^2}{2}V_{ss} + \sup_{\nu\in\mathbb{R}}\left[\nu V_\pi - \nu(s+\eta\nu)V_x\right] = 0?$$

The forward-backward system

$$J(s,\pi,x,t;\nu) = \mathbb{E}U\left[x - \int_t^T (\eta\nu_u + S_u)\nu_u du + (\pi + \int_t^T \nu_u du)S_T + H(S_T)\right],$$

$$V^\epsilon(s,\pi,x,t) := \sup_{\nu\in\mathcal{A}_\epsilon} J(s,\pi,x,t;\nu)$$

- We compute the directional derivative $\frac{d}{d\lambda}J(s,\pi,x,t;\nu+\lambda\nu^0)$ from the definition of J and set this derivative to zero, for any test-process ν^0, to obtain the (necessary) first-order condition of optimality.
- The first-order condition takes the form of a forward-backward system of SDEs (FBSDE):

$$dY_t^1 = \left(G^\epsilon(S_t, Y_t^2, t) - \sigma^2\gamma^2 Y_t^2\right) dt + Z_t dW_t, \quad Y_T^1 = 0,$$
$$dY_t^2 = -\phi^\epsilon\left(Y_t^1/(2\eta\gamma)\right) dt, \quad Y_0^2 = \pi_0,$$

where G^ϵ is expressed via V^ϵ, and

$$|G^\epsilon| \leq \text{const}, \ \forall\epsilon>0, \ \ \phi^\epsilon(x) := (x\wedge 1/\epsilon)\vee(-1/\epsilon), \ \ \nu_t^{*,\epsilon} = -\phi^\epsilon\left(Y_t^1/(2\eta\gamma)\right)$$

Boundedness of Y^1

$$V^\epsilon(s,\pi,x,t) := \sup_{\nu\in\mathcal{A}_\epsilon} J(s,\pi,x,t;\nu),$$
$$dY_t^1 = \left(G^\epsilon(S_t,Y_t^2,t) - \sigma^2\gamma^2 Y_t^2\right)dt + Z_t dW_t, \quad Y_T^1 = 0,$$
$$dY_t^2 = -\phi^\epsilon\left(Y_t^1/(2\eta\gamma)\right)dt, \quad Y_0^2 = \pi_0$$

- **Proposition** (Ekren-N. 2019). There exists a constant C, s.t. $\sup_{t\in[0,T]}|Y^1| \le C$ a.s. for all $\epsilon > 0$.
- The above implies that $|\nu_t^{*,\epsilon}| = |\phi^\epsilon\left(Y_t^1/(2\eta\gamma)\right)| \le C_1$ for all $\epsilon > 0$, $t\in[0,T]$.
- Then, for small enough $\epsilon > 0$, V^ϵ is a classical solution to the desired HJB:

$$V_t + \frac{\sigma^2}{2}V_{ss} + \frac{1}{4\eta}\left(\frac{V_\pi}{V_x} - s\right)^2 V_x = 0,$$

$$\frac{1}{4\eta}\left(\frac{V_\pi}{V_x} - s\right)^2 V_x = \sup_{\nu\in\mathbb{R}}\left[\nu V_\pi - \nu(s+\eta\nu)V_x\right] = \sup_{|\nu|\le 1/\epsilon}\left[\nu V_\pi - \nu(s+\eta\nu)V_x\right]$$

Proof of the boundedness of Y^1

$$
\begin{aligned}
-\,dY_t^1 &= \left(-G^{\epsilon}(S_t, Y_t^2, t) + \sigma^2\gamma^2 Y_t^2\right) dt - Z_t dW_t, \quad Y_T^1 = 0,\\
dY_t^2 &= -\phi^{\epsilon}\left(Y_t^1/(2\eta\gamma)\right) dt, \quad Y_0^2 = \pi_0
\end{aligned}
$$

- Apply Itô's formula to $Y^1 Y^2$ on time interval $[0, T]$:

$$
\begin{aligned}
0 = \mathbb{E} Y_T^1 Y_T^2 = Y_0^1 \pi_0 + \mathbb{E} \int_0^T & \left[-\sigma^2\gamma^2 (Y_u^2)^2 \right.\\
& \left. + Y_u^2 G^{\epsilon}(S_u, Y_u^2, u) - Y_u^1 \phi^{\epsilon}(Y^1/(2\eta\gamma))\right] du
\end{aligned}
$$

- Noticing that, uniformly over $\epsilon > 0$,

$$
\begin{aligned}
|Y_u^2 G^{\epsilon}(S_u, Y_u^2, u)| &\leq \frac{\sigma^2\gamma^2}{2} (Y_u^2)^2 + C_2,\\
Y_u^1 \phi^{\epsilon}(Y^1/(2\eta\gamma)) &\geq C_3 (Y_u^1)^2, \quad C_3 > 0,
\end{aligned}
$$

we obtain

$$
\mathbb{E} \int_0^T \left[\frac{\sigma^2\gamma^2}{2} (Y_u^2)^2 + C_3 (Y_u^1)^2\right] du \leq C_4 + Y_0^1 \pi_0
$$

Proof of the boundedness of Y^1

$$-\,dY^1_t = \left(-G^{\epsilon}(S_t, Y^2_t, t) + \sigma^2\gamma^2 Y^2_t\right) dt - Z_t dW_t, \quad Y^1_T = 0,$$

$$dY^2_t = -\phi^{\epsilon}\left(Y^1_t/(2\eta\gamma)\right) dt, \quad Y^2_0 = \pi_0,$$

$$\mathbb{E}\int_0^T \left[(Y^2_u)^2 + (Y^1_u)^2\right] du \leq C_6\left(1 + Y^1_0\right)$$

- Apply Itô's formula to $(Y^1)^2$ on time interval $[0, T]$:

$$0 = \mathbb{E}(Y^1_T)^2 = (Y^1_0)^2 + \mathbb{E}\int_0^T \left[-2\sigma^2\gamma^2 Y^1_u Y^2_u + 2Y^1_u G^{\epsilon}(S_u, Y^2_u, u) + Z^2_u\right] du,$$

$$(Y^1_0)^2 \leq C_7 \mathbb{E}\int_0^T \left[1 + (Y^1_u)^2 + (Y^2_u)^2\right] du$$

Proof of the boundedness of Y^1

$$-\,dY_t^1 = \left(-G^\epsilon(S_t, Y_t^2, t) + \sigma^2\gamma^2 Y_t^2\right) dt - Z_t dW_t, \quad Y_T^1 = 0,$$

$$dY_t^2 = -\phi^\epsilon\left(Y_t^1/(2\eta\gamma)\right) dt, \quad Y_0^2 = \pi_0,$$

$$\mathbb{E}\int_0^T \left[(Y_u^2)^2 + (Y_u^1)^2\right] du \leq C_6\left(1 + Y_0^1\right) \leq C_8 + \frac{1}{2C_7}(Y_0^1)^2$$

- Apply Itô's formula to $(Y^1)^2$ on time interval $[0, T]$:

$$0 = \mathbb{E}(Y_T^1)^2 = (Y_0^1)^2 + \mathbb{E}\int_0^T \left[-2\sigma^2\gamma^2 Y_u^1 Y_u^2 + 2Y_u^1 G^\epsilon(S_u, Y_u^2, u) + Z_u^2\right] du,$$

$$(Y_0^1)^2 \leq C_7\mathbb{E}\int_0^T \left[1 + (Y_u^1)^2 + (Y_u^2)^2\right] du$$

Proof of the boundedness of Y^1

$$-dY^1_t = \left(-G^\epsilon(S_t, Y^2_t, t) + \sigma^2\gamma^2 Y^2_t\right) dt - Z_t dW_t, \quad Y^1_T = 0,$$

$$dY^2_t = -\phi^\epsilon\left(Y^1_t/(2\eta\gamma)\right) dt, \quad Y^2_0 = \pi_0,$$

$$\mathbb{E}\int_0^T \left[(Y^2_u)^2 + (Y^1_u)^2\right] du \le C_6\left(1 + Y^1_0\right) \le C_8 + \frac{1}{2C_7}(Y^1_0)^2$$

- Apply Itô's formula to $(Y^1)^2$ on time interval $[0, T]$:

$$0 = \mathbb{E}(Y^1_T)^2 = (Y^1_0)^2 + \mathbb{E}\int_0^T \left[-2\sigma^2\gamma^2 Y^1_u Y^2_u + 2Y^1_u G^\epsilon(S_u, Y^2_u, u) + Z^2_u\right] du,$$

$$(Y^1_0)^2 \le C_7\mathbb{E}\int_0^T \left[1 + (Y^1_u)^2 + (Y^2_u)^2\right] du,$$

$$\mathbb{E}\int_0^T \left[(Y^2_u)^2 + (Y^1_u)^2\right] du \le C_8 + \frac{1}{2}\mathbb{E}\int_0^T \left[1 + (Y^1_u)^2 + (Y^2_u)^2\right] du,$$

$$\mathbb{E}\int_0^T \left[(Y^2_u)^2 + (Y^1_u)^2\right] du \le C_9, \quad (Y^1_0)^2 \le C_{10}$$

Section 3

Probabilistic representations for free-boundary problems

Obstacle problems

- Consider a parabolic PDE

$$u_t + \mathcal{L}(u_{xx}, u_x, x, t) = 0, \quad x \in D(t) \subset \mathbb{R}^d, \quad t < T,$$
$$u = \phi \text{ on } D_T,$$

where the domain D is **not given but is a part of the solution**!

- To find D, we need an additional free-boundary condition: e.g., for a given (**obstacle**) function Φ,

$$u(x,t) \geq \Phi(x,t), \quad \forall\,(x,t),$$
$$u(x,t) > \Phi(x,t) \text{ if and only if } x \in D(t).$$

Of course, Φ must coincide with ϕ on D_T.

- Such obstacle problem can be formulated as a variational inequality:

$$\max\left[u_t + \mathcal{L}(u_{xx}, u_x, x, t), \Phi - u\right] = 0,$$

and is tightly connected to the probabilistic problem of optimal stopping.

- See Bensoussan-Lions 2000 for more on this connection; ElKaroui et al 1997 for representing u via reflected BSDEs; Chassagneux-N.-Richou 2021 for systems of reflected BSDEs.

Stefan equation (Visintin 1998)

$$u_t - \mathcal{L}(u_{xx}, u_x, x, t) = 0, \quad x \in \mathbb{R}^d \setminus \partial D(t), \quad t > 0, \quad u(x,0) = \phi(x)$$

- The most popular elliptic and parabolic free-boundary problems in Physics are not of obstacle type. Instead of defining D via the value of u, we define the growth rate of $D(\cdot)$ by writing an equation for the normal velocity V of $\partial D(\cdot)$:

$$V(x,t) = \frac{1}{2}\left[\lim_{y\uparrow x} u_x(y,t)\cdot\nu(t,x) - \lim_{y\downarrow x} u_x(y,t)\cdot\nu(t,x)\right], \quad x \in \partial D(t),$$

  where ν is the outer unit normal to $\partial D(t)$, the limit $y \uparrow x$ is taken over y converging x from inside D, and the limit $y \uparrow x$ is taken over y converging x from outside D.

- If we equip (38) with the boundary condition

$$u = aV + bH \text{ on } \partial D(t), \quad H \text{ is the curvature of } \partial D(t),$$

  we obtain a general form of **Stefan equation**, which is used to model the processes of melting, solidification, and crystal growth.

- Elliptic version of this problem is known as the Hele-Shaw equation.

Single-phase Stefan equation in $\mathbb{R}$

$$
\begin{aligned}
& u_t - \frac{1}{2} u_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \\
& u(x,0) = \phi(x), \quad u(\Lambda(t), t) = 0, \\
& \dot{\Lambda}(t) = -\frac{1}{2} u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0
\end{aligned}
$$

- Assume that the area $x > \Lambda$ is occupied by a liquid (water) and $x < \Lambda$ by a solid (ice).
- u denotes the temperature, which equals zero at the melting/freezing point, and which is kept at zero in the solid phase (**single-phase problem**).
- Then, the amount of heat in the liquid close to the boundary is proportional to $u_x(\Lambda(t)^+, t)$, which determine the melting speed (i.e., the speed at which Λ decreases).
- If $u_x(\Lambda(t)^+, t) < 0$, the liquid freezes and the boundary Λ increases (at the rate proportional to $|u_x(\Lambda(t)^+, t)|$).

Probabilistic representation: nonnegative init. value

$$u_t - \frac{1}{2}u_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \tag{16}$$
$$u(x,0) = \phi(x), \quad u(\Lambda(t),t) = 0,$$
$$\dot{\Lambda}(t) = -\frac{1}{2}u_x(\Lambda(t)^+,t), \quad \Lambda(0) = 0$$

- Assume $\phi \geq 0$, $\int_0^\infty \phi = 1$, and let $\varphi(\cdot,t)$ be Gaussian kernel with variance t.
- Feynman-Kac formula and time reversal imply

$$\sigma := \inf\{s \geq 0 : \ x + W_s \leq \Lambda(t-s)\} \wedge t, \quad u(x,t) = \mathbb{E}\left[\phi(x+W_t)\mathbf{1}_{\sigma>t}\right]$$
$$= \int_0^\infty \phi(y)\varphi(x-y,t)\mathbb{P}\left(\inf_{s\in[0,t]}(x + W_s - \Lambda(t-s)) > 0 \,|\, x + W_t = y\right) dy$$
$$= \int_0^\infty \phi(y)\varphi(x-y,t)\mathbb{P}\left(\inf_{s\in[0,t]}(y + W_s - \Lambda(s)) > 0 \,|\, y + W_t = x\right) dy$$
$$= \mathbb{P}(X_t \in dx, \tau > t)/dx, \quad X_t = \xi + W_t, \quad \tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\},$$

with independent r.v. ξ having density ϕ

Probabilistic representation: growth condition

$$u_t - \frac{1}{2}u_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \quad u(x,0) = \phi(x), \quad u(\Lambda(t),t) = 0,$$

$$\dot{\Lambda}(t) = -\frac{1}{2}u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0 \tag{17}$$

$$u(x,t) = \mathbb{P}(X_t \in dx)/dx, \quad X_t = (\xi + W_t)\mathbf{1}_{\tau > t} - \infty \mathbf{1}_{\tau \le t},$$

$$\tau := \inf\{s \ge 0 : \xi + W_s \le \Lambda(s)\}$$

- We have established that $u(\cdot, t)$ is the marginal density of Brownian motion killed at hitting Λ.
- To derive a probabilistic representation for Λ (to replace (17)), we notice that

$$\frac{d}{dt}\mathbb{P}(\tau > t) = \frac{d}{dt}\int_{\Lambda(t)} u(x,t)dx = -\dot{\Lambda}(t)u(\Lambda(t),t) + \frac{1}{2}\int_{\Lambda(t)} u_{xx}(x,t)dx$$

$$= -\frac{1}{2}u_x(\Lambda(t),t) = \dot{\Lambda}(t), \quad \Lambda(t) = 1 - \mathbb{P}(\tau \le t)$$

Stefan problem as McKean-Vlasov equation

$$u(x,t) = \mathbb{P}(X_t \in dx)/dx, \quad X_t = (\xi + W_t)\mathbf{1}_{\tau > t} - \infty \mathbf{1}_{\tau \le t},$$
$$\tau := \inf\{s \ge 0 : \xi + W_s \le \Lambda(s)\}, \quad \Lambda(t) = 1 - \mathbb{P}(\tau \le t) = \mathbb{P}(X_t \in \mathbb{R})$$

- The above system is a McKean-Vlasov equation, as the dynamics of X depend explicitly on its distribution.
- Such systems/equations are also called "of mean-field type", because they arise as large-population limits of particle systems that interact with each other through their empirical measure.
- Levine-Peres 2010 showed that the Stefan (and Hele-Shaw) equation can be obtained as limits of internal Diffusion Limited Aggregation (internal DLA) models.
- Our probabilistic connection is interesting but does not contribute to the PDE theory (besides a new numerical method). This is because, in the case of **nonnegative initial condition**, the Stefan problem was solved a long time ago (e.g., Kamenomostskaja 1961).

Supercooled Stefan problem

$$u_t - \frac{1}{2}u_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \quad u(x,0) = \phi(x), \quad u(\Lambda(t),t) = 0,$$
$$\dot{\Lambda}(t) = -\frac{1}{2}u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0$$

- If $\phi \leq 0$ (and w.l.o.g. $\int_0^\infty \phi = -1$), we consider $v := -u \geq 0$, $\psi := -\phi \geq 0$:

$$v_t - \frac{1}{2}v_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \quad v(x,0) = \psi(x), \quad v(\Lambda(t),t) = 0,$$
$$\dot{\Lambda}(t) = +\frac{1}{2}v_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0$$

- Such supercooled version of Stefan equation is known to have **singular** $\Lambda(\cdot)$.
- Indeed, if the value of $v(\cdot,t)$ close to $\Lambda(t)$ is large (i.e., $v_x(\Lambda(t)^+,t)$ is large), the frontier moves faster. But the faster it moves the higher is the value of $v(\cdot,t)$ close to $\Lambda(t)$ (as $v(\cdot,t)$ is locally increasing close to $\Lambda(t)^+$). Such positive reinforcement creates blow-ups in $\dot{\Lambda}$ and even jumps in $\Lambda(\cdot)$ (see Sherman 1970).

Probabilistic representation

$$X_t = \xi + W_{t\wedge\tau},$$
$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t)$$

- Despite this singularity, one can still derive a probabilistic representation for the supercooled Stefan problem (above).
- This representation is almost identical to the one for regular Stefan, except the growth condition for Λ.
- Due to singularities, we have to add a minimal-jump condition to this representation: $\Lambda(\cdot)$ is cádlág and

$$\Lambda(t) - \Lambda(t^-) = \inf\{x > 0 : \mathbb{P}\left(\tau \geq t,\, X_{t-} \in (\Lambda(t^-), \Lambda(t^-) + x]\right) < x\}$$

- In this case, the **probabilistic representation is actually useful**, as there is no existence or uniqueness theory for the supercooled Stefan problem via PDE methods (with the exception of small initial data, $\psi \leq 1$, treated e.g. in Fasano-Primicerio 1981).

Equivalence of probabilistic and PDE solutions

$$X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim \psi(x)dx,$$
$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),$$
$$\Lambda(t) - \Lambda(t^-) = \inf\{x > 0 : \mathbb{P}\left(\tau \geq t,\, X_{t-} \in (\Lambda(t^-), \Lambda(t^-) + x]\right) < x\},$$

$$v_t - \frac{1}{2} v_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \quad v(x,0) = \psi(x),$$
$$\lim_{s\to t, x\downarrow\Lambda(t)} v(x,s) = 0 \text{ except for at most countably many } t,$$
$$d\Lambda(t) = -d\int_{\Lambda(t)}^{\infty} v(x,t)dx, \quad t \geq 0,$$
$$\Lambda(t) - \Lambda(t^-) = \inf\{x > 0 : \lim_{s\uparrow t} \int_{\Lambda(t^-)}^{\Lambda(t^-)+x} v(y,s)dy < x\}$$

- **Proposition** (Delarue-N.-Shkolnikov 2019) If (X, Λ) solve the blue system, then $v(x,t) := \mathbb{P}(X_t \in dx)/dx$ and Λ solve the red system. If (v, Λ) solve the red system, then the Brownian motion X absorbed at Λ solves the blue system, and $v(x,t) := \mathbb{P}(X_t \in dx)/dx$.

Existence and uniqueness of probabilistic solution

$$
\begin{aligned}
&X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim \psi(x)dx,\\
&\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),\\
&\Lambda(t) - \Lambda(t^-) = \inf\{x > 0 : \mathbb{P}\left(\tau \geq t,\, X_{t-} \in (\Lambda(t^-), \Lambda(t^-) + x]\right) < x\},
\end{aligned}
$$

- **Question**: does a solution to the probabilistic Stefan problem exist? is it unique?
- It is shown in Ledger-Sojmark 2018, Cuchiero et al 2020 that a probabilistic solution (X, Λ) exists for any initial distribution $\psi \geq 0$, $\int_0^\infty \psi < \infty$.
- **Proposition** (Delarue-N.-Shkolnikov 2019) Assume that ψ is bounded and changes monotonicity finitely many times on any compact. Then, the probabilistic solution (X, Λ) is unique.

Sketch of the proof: heuristics

$$X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim \psi(x)dx,$$
$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),$$
$$\Lambda(t) - \Lambda(t^-) = \inf\{x > 0 : \mathbb{P}\left(\tau \geq t,\, X_{t-} \in (\Lambda(t^-), \Lambda(t^-) + x]\right) < x\},$$

- Consider the mapping $\Gamma \mapsto \Lambda$:

$$\Lambda(t) := \mathbb{P}(\inf_{s\in[0,t]}(\xi + W_s - \Gamma(s)) \leq 0)$$
$$= \int_0^\infty \psi(x)\mathbb{P}(\inf_{s\in[0,t]}(W_s - \Gamma(s)) \leq -x)dx,$$
$$|\Lambda(t) - \tilde{\Lambda}(t)| \leq \int_0^\infty \psi(x)\mathbb{P}(\inf_{s\in[0,t]}(W_s - \Gamma(s)) \in [-x, -x - \sup_{[0,T]}|\Gamma - \tilde{\Gamma}|])dx$$
$$\leq \sup_{[0,T]}|\Gamma - \tilde{\Gamma}| \int_0^\infty \psi(x)\mathbb{P}(\inf_{s\in[0,t]}(W_s - \Gamma(s)) \in dx) \leq \sup_{\mathbb{R}_+}\psi \sup_{[0,T]}|\Gamma - \tilde{\Gamma}|$$

- Idea: show that $v(x,t) < 1$ for small x and "most t".

Sketch of the proof: first proposition

$$
\begin{aligned}
&X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim \psi(x)dx,\\
&\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),\\
&\Lambda(t) - \Lambda(t^-) = \inf\{x > 0 : \mathbb{P}\left(\tau \geq t,\, X_{t-} \in (\Lambda(t^-), \Lambda(t^-) + x]\right) < x\},
\end{aligned}
$$

- Recall that $v(\cdot, t)$ is the density of X_t. Denote by $\rho(\cdot, t)$ the density of X_{t-}.
- **Proposition** (2.1 in Delarue-N.-Shkolnikov 2019). Assume that $\rho(\cdot, t)$ has a finite number of changes of monotonicity on any compact (and hence has a right-continuous modification). Then, there exist $\delta, \epsilon > 0$ and $C(z) < 1$, s.t. $\sup_{s\in[t+z,t+\epsilon]} \int_{\Lambda(s^-)}^{\Lambda(s^-)+x} \rho(y, s)dy \leq Cx$ for any $z \in (0, \epsilon]$, $x \in (0, \delta]$.
- **Proof**. First, we resolve the jump at t, if needed, and replace $\rho(\cdot, t)$ by $v(\cdot, t)$.
- Right-continuity of $\rho(\cdot, t)$ and the definition of $\Lambda(t) - \Lambda(t^-)$ imply that $v(\Lambda(t)^+, t) \leq 1$. For simplicity, let us only treat the case $v(\Lambda(t)^+, t) < 1$.

Proof of the proposition

$$X_t = \xi + W_{t\wedge\tau}, \quad \tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}$$

- Assume for convenience that $t = 0$ and recall that $v(x, 0) < 1$ for $x \in (\Lambda(t), \Lambda(t) + \delta']$. Denote by F the cdf of $v(\cdot, 0)$ and recall that $\varphi(\cdot, s)$ is Gaussian kernel with variance s.
- Denoting by F the cdf of $v(\cdot, 0)$ (supported in $\mathbb{R}_+$),

$$\mathbb{P}(X_{s-} \leq \Lambda(s^-) + x,\, \tau \geq s) \leq \mathbb{P}\left(X_{s-} \in [\Lambda(s^-), \Lambda(s^-) + x]\right)$$
$$= \int_{-\infty}^{\Lambda(s^-)+x} \left[F(\Lambda(s^-) + x - y) - F(\Lambda(s^-) - y)\right] \varphi(y, s) dy$$
$$= \int_{\Lambda(s^-)}^{\Lambda(s^-)+x} (\ldots) dy + \int_{-\varepsilon}^{\Lambda(s^-)} (\ldots) dy + \int_{-\infty}^{-\varepsilon} (\ldots) dy$$

Estimating the first two terms

$$t = 0, \quad v(x,0) < C_1 < 1, \quad x \in (\Lambda(t), \Lambda(t) + \delta'], \quad F(x) = \int_{-\infty}^{x} v(x,0)dx$$

- For $x \in (\Lambda(t), \Lambda(t) + \delta']$,

$$\int_{\Lambda(s^-)}^{\Lambda(s^-)+x} \left[F(\Lambda(s^-) + x - y) - F(\Lambda(s^-) - y)\right] \varphi(y,s)dy$$
$$\leq F(x) \int_{\Lambda(s^-)}^{\Lambda(s^-)+x} \varphi(y,s)dy \leq C_1 x \int_{\Lambda(s^-)}^{\Lambda(s^-)+x} \varphi(y,s)dy$$

- Making sure that s is small enough,

$$\int_{-\varepsilon}^{\Lambda(s^-)} \left[F(\Lambda(s^-) + x - y) - F(\Lambda(s^-) - y)\right] \varphi(y,s)dy$$
$$\leq C_1 x \int_{-\varepsilon}^{\Lambda(s^-)} \varphi(y,s)dy$$

Estimating the third term

$$t = 0, \quad v(x,0) < C_1 < 1, \quad x \in (\Lambda(t), \Lambda(t) + \delta'], \quad F(x) = \int_{-\infty}^{x} v(x,0)dx$$

- Note that the tails of $\varphi(\cdot, s)$ decay fast:

$$\int_{-\infty}^{-2\varepsilon} \varphi(y,s)dy \leq e^{-\varepsilon^2/(2s)} \int_{-\infty}^{-\varepsilon} \varphi(y,s)dy.$$

- Choosing small enough $\gamma > 0$ s.t. $\gamma \|v\|_{L^\infty} + C_1 < 1$, we decrease s if needed, to obtain

$$\int_{-\infty}^{-2\varepsilon} \varphi(y,s)dy \leq \gamma \int_{-\infty}^{-\varepsilon} \varphi(y,s)dy.$$

- Then, for small enough $s, \varepsilon > 0$,

$$\int_{-\infty}^{-\varepsilon} \left[F(\Lambda(s^-) + x - y) - F(\Lambda(s^-) - y)\right] \varphi(y,s)dy \leq x\|v\|_{L^\infty} \int_{-\infty}^{-2\varepsilon} \varphi(y,s)dy$$

$$+ C_1 x \int_{-2\varepsilon}^{-\varepsilon} \varphi(y,s)dy \leq (\gamma\|v\|_{L^\infty} + C_1)x \int_{-\infty}^{-\varepsilon} \varphi(y,s)dy$$

Proof of the proposition

$$t=0, \quad v(x,0) < C_1 < 1, \quad x \in (\Lambda(t), \Lambda(t)+\delta'], \quad F(x) = \int_{-\infty}^{x} v(x,0)dx$$

- Collecting the results form three previous slides, we obtain for small enough $s > t$ and $x > 0$

$$\int_{\Lambda(s^-)}^{\Lambda(s^-)+x} \rho(y,s)dy$$

$$= \mathbb{P}(X_{s-} \leq \Lambda(s^-)+x,\ \tau \geq s) \leq C_1 x \int_{\Lambda(s^-)}^{\Lambda(s^-)+x} \varphi(y,s)dy$$

$$C_1 x \int_{-\varepsilon}^{\Lambda(s^-)} \varphi(y,s)dy + (\gamma\|v\|_{L^\infty} + C_1)x \int_{-\infty}^{-\varepsilon} \varphi(y,s)dy \leq C_2 x,$$

with $C_2 < 1$.

Proof of the theorem

$$X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim \psi(x)dx,$$
$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),$$
$$\Lambda(t) - \Lambda(t^-) = \inf\{x > 0 : \mathbb{P}\left(\tau \geq t,\, X_{t-} \in (\Lambda(t^-), \Lambda(t^-) + x]\right) < x\},$$

- **Proposition** (2.1 in Delarue-N.-Shkolnikov 2019). If there exist $\delta > 0$ and $C < 1$, s.t. $\int_{\Lambda(t^-)}^{\Lambda(t^-)+x} \rho(y,t)dy \leq Cx$ for $x \in (0,\delta]$, then $\Lambda(\cdot)$ is 1/2-Hölder at t.
  - Proven by stochastic dominance.
- **Proposition** (2.5 in Delarue-N.-Shkolnikov 2019). If $\Lambda(\cdot)$ is 1/2-Hölder in $(t, t+\epsilon)$, then $v(x,s) \leq C(x - \Lambda(s))^\beta$ for all $x > \Lambda(s)$ and $s \in (t, t+\epsilon)$, with constants $C, \beta > 0$.
  - Proven via Krylov-Safonov.
- Comparison principle for the PDE allows us to improve the above result and deduce that $\Lambda \in C^1$ in $(t, t+\epsilon)$ and $v_x(\cdot, s)$ is continuous up to and including the boundary for $s \in (t, t+\epsilon)$.

Proof of the theorem

$$X_t = \xi + W_{t\wedge\tau}, \quad \xi \sim \psi(x)dx,$$
$$\tau := \inf\{s \geq 0 : \xi + W_s \leq \Lambda(s)\}, \quad \Lambda(t) = \mathbb{P}(\tau \leq t),$$
$$\Lambda(t) - \Lambda(t^-) = \inf\{x > 0 : \mathbb{P}\left(\tau \geq t,\, X_{t-} \in (\Lambda(t^-), \Lambda(t^-) + x]\right) < x\},$$

- **Lemma** (4.1 in Delarue-N.-Shkolnikov 2019). If ψ changes monotonicity finitely many times on any compact, then the same is true for $\rho(\cdot, t)$ for all $t \geq 0$.
  - Proven via analysis of zero curves of a solution to heat equation: as time increases (for a forward equation), new zeros cannot appear in the interior of the domain.

Open problems

$$u_t - \frac{1}{2}u_{xx} = 0, \quad x > \Lambda(t), \quad t > 0, \quad u(x,0) = \phi(x), \quad u(\Lambda(t),t) = 0,$$
$$\dot{\Lambda}(t) = -\frac{1}{2}u_x(\Lambda(t)^+, t), \quad \Lambda(0) = 0$$

- What about the two-phase problem with $\phi \leq 0$?
- What if ϕ has varying sign?
- What about multiple dimension? (Probabilistic representation is developed in N.-Shkolnikov-Zhang 2021, but the natural particle system no longer converges to the solution.)
- What about more general condition $u(\Lambda(t), t) = aV + bH$? (See Baker-Shkolnikov 2020 for the case $a = 1, b = 0$ in dimension one)