Foundations of Stochastic Control for Controlled Diffusions

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Defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, a stochastic control problems comes with the following features:

1. the set $\mathcal{A}$ of $(\mathcal{F}_t)$-adapted[1] processes called controls
2. the controlled state process $X = X^\alpha$, $\alpha \in \mathcal{A}$:

$$dX_t^\alpha = b(X_t^\alpha, \alpha_t)\,dt + \sigma(X_t^\alpha, \alpha_t)\,dW_t, \quad X_0^\alpha = x_0$$

3. the performance criterion (objective) $J = J(\alpha)$:

$$J(\alpha) = \mathbb{E}\left[\int_0^T f(X_t^\alpha, \alpha_t)\,dt + g(X_T^\alpha)\right] \qquad g = \begin{cases} 1 & X_T^\alpha = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^T |\alpha_u|\,du$$

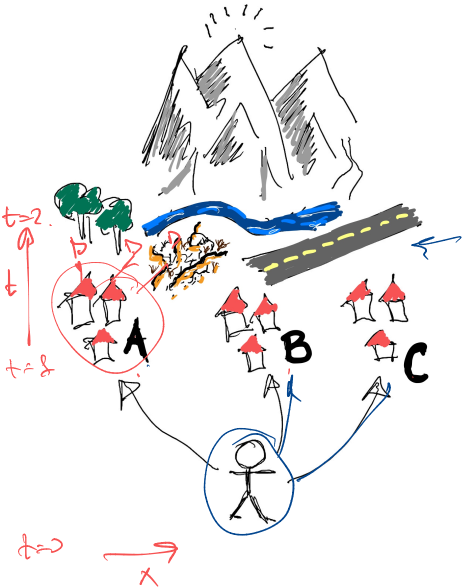
[1]Many subtle measurability issues will be swept under a rug!

"Solving" a stochastic control problem can mean many things:

1. Computing/characterizing the value of the problem:

$$v = \sup_{\alpha \in \mathcal{A}} J(\alpha).$$

2. Finding an explicit expression for v or the optimal control $\hat{\alpha}$
3. Describing an efficient numerical method for approximating v or $\hat{\alpha}$
4. Writing down an equation whose solution leads to v or $\hat{\alpha}$
5. Proving, abstractly, that an optimal control $\hat{\alpha}$ exists
6. Relating the solutions of two control problems without solving either one of them.
7. …



Bellman's principle:

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision"

(Bellman, 1957)

Let the value function v be given by

$$v(t,x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big[\int_t^T f(\alpha_s, X_s^{\alpha,t,x})\,ds + g(X_T^{\alpha})\Big].$$

t,x

Then

Theorem[2] **(DPP):** For each stopping time $\theta \in [t,T]$ we have:

$$v(t,x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{t,x}\Big[\int_t^{\theta} f(\alpha_s, X_s^{\alpha,t,x})\,dt + v(\theta, X_\theta^{\alpha,t,x})\Big]$$

Proof.

$$J(\alpha) = \mathbb{E}\Big[\int_t^\theta f(\ldots)\,du + \int_\theta^T f(..)\,du + g(X_T)\Big]$$

$$= \mathbb{E}\Big[\int_t^\theta f(\,)\,du + \mathbb{E}\big[\quad \mid \mathcal{F}_\theta\big]\Big]$$

$J(\alpha)$ at (θ, X_θ^t)

[2] Terms and conditions apply.

$v(\theta, X_\theta^\alpha)$

$$J(\alpha) \leq \mathbb{E}\left[\int_t^\theta f(\dots)\,du + v(\theta, X_\theta^\alpha)\right]$$

$\alpha \longrightarrow$ α up to θ

$\hat{\alpha}$ after θ.

Apply the DPP with $\theta = t + h$, for h "small" to get the following HJB equation (H = Hamilton, J=Jacobi, B=Bellman):

$$v_t + \sup_a \left[\mathcal{L}^a v + f(\cdot, a)\right] = 0$$

$$\cancel{v(t,x)} = \sup_{\alpha} \mathbb{E}_{x,t}\left[\int_t^{t+h} f(\ )\,du + v(t+h, X^{\alpha}_{t+h})\right]$$

$$\approx h \cdot f(\alpha, x) + \underbrace{\mathbb{E}\left[v(t+h, X^{\alpha}_{t+h}) \mid \mathcal{F}_t\right]}_{\cancel{v(t,x)} + \int_t^{t+h} v_t + \mathcal{L}^{\alpha}_x \; + \; \cancel{\int_t^{t+h} v_x^{\rhd} dW}}$$

$$0 = \cancel{h} \cdot f(\alpha, x) + \underbrace{\int_t^{t+h} (v_t + \mathcal{L}^a v)\,du}_{\cancel{h} \cdot (v_t + \mathcal{L}^a v)} \qquad \mathbb{E}[\qquad \mid \mathcal{F}_t]$$

The verification approach: the logic is usually run backwards when the HJB equation (with the terminal condition $v(T,\cdot) = g$) has a smooth solution $v \in \cancel{C^{2,1}}$ $C^{1,2}$: $0 = v_t + \sup_a \left(\mathcal{L}^a v + f(a,\cdot) \right)$

1. Pick an admissible control α and apply Itô's formula to the process $V_t = v(t, X_t^\alpha) + \int_0^t f(\alpha_s, X_s^\alpha)\, ds$

$$dV_t = \underbrace{\left(v_t + \mathcal{L}^\alpha v + f \right)}\, dt + \underbrace{\square\, dW}$$

$$0 = \sup_a \Big(\underbrace{v_t + \mathcal{L}^a v + f(a,\cdot)}_{\le 0} \Big)$$

2. Conclude that V is a supermartingale for ANY α.

V is a martingale for $\alpha = \hat{\alpha}$

3. Hence

$$V_0 \underset{=}{\geq} \mathbb{E}[V_T] = \underbrace{\mathbb{E}\left[\int_0^T f(X,\alpha)\,du + g(X_T)\right]}_{J_t(\alpha)}$$

$$\| \qquad\qquad V(t,x) \underset{=}{\geq} \sup_{\alpha} J_t(\alpha) =$$

$$V(0,x)$$

Example (Merton's problem): Use Samuelson's model for the risky asset (stock, etc.):

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t).$$

and the following dynamics for the riskless asset (bank account, etc.)

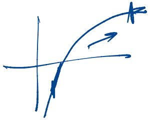
$$dS_t^0 = S_t^0 r\, dt.$$

Invest the proportion α_t of your total wealth X in the risky asset and the rest in the riskless asset. The dynamics of X is

$$dX_t = \frac{\alpha_t X_t}{S_t} dS_t + \frac{(1-\alpha_t)X_t}{S_t^0} dS_t^0, \quad X_0 = x,$$

or, when simplified,

$$dX_t = X_t\Big(r + \alpha_t(\mu - r)\Big)\, dt + X_t \alpha_t \sigma\, dW_t.$$



Use expected utility as the performance measure:

$$J(\alpha, X) = \mathbb{E}[U_p(X_T^\alpha)], \text{ where } U_p(x) = \frac{1}{p}x^p \text{ for } p < 1, p \neq 0.$$

$$\mathcal{L}^a v = x(r + a(\mu - r))v_x + \tfrac{1}{2}x^2 a^2 \sigma^2 v_{xx} \qquad f = 0. \quad g = U.$$

$$\operatorname*{argmax}_a \mathcal{L}^a v = -\frac{v_x}{x v_{xx}} \frac{\mu - r}{\sigma^2}$$

$$v_t + \sup_a \mathcal{L}^a v = v_t + r x v_x - \frac{1}{2} \frac{v_x^2}{v_{xx}} \cdot \left(\frac{\mu - r}{\sigma}\right)^2$$

Ansatz: $v(t,x) = \varphi(t) U(x)$

HJB: $\varphi' U + r x \varphi U' - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \varphi \frac{(U')^2}{U''} = 0.$

$$U(x) = \frac{1}{p} x^p \longrightarrow \begin{cases} \varphi'(t) = \varphi(t) \cdot C \\ \varphi(T) = 1 \end{cases}, \text{ where } C = \frac{1}{2}\frac{p}{p-1}\left(\frac{\mu - r}{\sigma}\right)^2 - pr.$$

$$\varphi(t) = e^{-C(T-t)}, \qquad \hat{a} = \frac{\mu - r}{\sigma^2(1-p)}$$

Value functions do not need to be smooth, though:

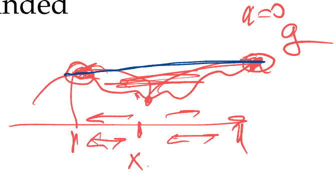
$$dX_t = \alpha_t X_t \, dt, \; f \equiv 0, \; g(x) = x \text{ where } \mathcal{A} = \{\alpha : |\alpha_t| \leq 1\}.$$

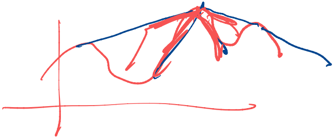
$$X_t = X_0 e^{\int_0^T \alpha_u du} \longrightarrow \max.$$

$$v(t,x) = \begin{cases} x \cdot e^{T-t} & x > 0 \\ 0 & \\ x \cdot e^{-(T-t)} & x < 0. \end{cases}$$

And worse things can happen:

$$dX_t = \alpha_t \, dW_t, \; f \equiv 0, g \text{ bounded}$$





Viscosity solutions. Consider the HJB equation

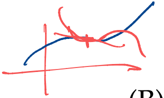
$$v_t + H(v_x, v_{xx}) = 0 \text{ where } H = \sup_a \left(\mathcal{L}^a v + f(a, \cdot) \right) \tag{HJB}$$

The Hamiltonian H has the following key feature (ellipticity):

$$M_1 \leq M_2 \Rightarrow H(u, p, M_1) \leq H(u, p, M_2).$$

Therefore, if v and φ are two *smooth* functions such that

$$v(t, x) = \varphi(t, x), v_x(t, x) = \varphi_x(t, x) \text{ and } v_t(t, x) = \varphi_t(t, x) \tag{A}$$



but

$$v_{xx}(t, x) \leq \varphi_{xx}(t, x). \tag{B}$$

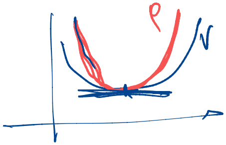
Then, at (t, x), we have

$$v_t + H(v_x, v_{xx}) \leq \varphi_t + H(\varphi_x, \varphi_{xx})$$

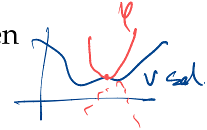
The central idea is that we can check whether (A) and (B) hold without evoking derivatives at all:

Definition. We say that φ touches v from above (below) at (t_0, x_0) if (t_0, x_0) is point of local minimum (maximum) of $\varphi - v$.

min $= 0$.



Suppose, now, that v is smooth and *solves* (HJB). Then



$$\varphi_t + H(\varphi_x, \varphi_{xx}) \geq 0 \ (\leq 0)$$

at (t_0, x_0) for every smooth φ that touches v from above (below).

Definition. We say that a (*not-necessarily smooth*) function v is a viscosity subsolution (supersolution) of (HJB) if

$$\varphi_t + H(\varphi_x, \varphi_{xx}) \geq 0 \ (\leq 0)$$

at (t_0, x_0) for every smooth φ that touches v from above (below).

Viscosity solution = subsolution + supersolution.

A generic theorem looks like this:

Theorem template: The value function v of the control problem is the unique viscosity solution to the corresponding HJB equation with the appropriate terminal condition.

Proof template. Write down the dynamic programming principle (DPP) for the value function v. Replace v by a test function that touches from above (below) and use Itô's formula to get the supersolution (subsolution) property. Uniqueness is much harder.

The Pontryagin maximum principle. Consider the (deterministic) control problem

$$X_0 = 0,\ dX_t = b(X_t^\alpha, \alpha_t)\,dt.\ g(X_T) \to \max.$$

Think of the problem in the following way: maximize $g(X_T)$ over the set of all pairs of functions (X, α) on $[0, T]$ subject to the following *constraint*

$$X_t = \int_0^t b(X_s, \alpha_s)\,ds \text{ for all } t$$

Rewrite the constraint using the Lagrange multiplier Z

$$\int_0^T Z_u \Big(X_t' - b(X_t, \alpha_t) \Big)\,du,\ X_0 = 0. \qquad \forall Z.$$

$$J(\alpha, X) = g(X_T) + \int_0^T Z_u \underline{(X_u' - b(X_u, \alpha_u))}\, du$$

$$= g(X_T) + Z_T X_T - \int_0^T Z_u' X_u - \int_0^T Z_u\, b(X_u, \alpha_u)\, du.$$

$\hat{X}$ $\qquad J(\hat{\alpha}, \hat{X}) \geq J(\hat{\alpha}, \underline{\hat{X} + \varepsilon h})$ $\qquad h(0) = 0.$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(J(\hat{\alpha}, \hat{X} + \varepsilon h) - J(\hat{\alpha}, \hat{X})\right) \leq 0$$

$$"\frac{\partial}{\partial h}" = g'(\hat{X}_T) \cdot h + Z_T \cdot h + \int_0^T Z_u' \cdot h$$

$$- \int_0^T Z\, b_x(\hat{X}, \hat{\alpha}) \cdot h. \quad \leq 0 \quad \forall h.$$

$$\rightarrow \boxed{Z_T = -g'(\hat{X}_T)} \qquad \underline{Z_t' = -Z_t \cdot b_x(\hat{X}_t, \hat{\alpha}_t)}$$

Convex duality in utility maximization. Consider the Merton problem, but now with a general utility function U. We set $r = 0$ for simplicity and define

$$Z_0 = 1,\ dZ_t = -Z_t \frac{\mu}{\sigma} dW_t.$$

Using the form of X (simplifying thanks to $r = 0$) we get

$$X_t^{\alpha} = \sigma \alpha_t X_t \left(\frac{\mu}{\sigma} dt + dW_t \right)$$

α

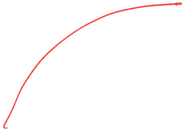
$\mathbb{E}[X_T^{\alpha} \cdot Z_T] = X_0$

and Itô's formula implies that XZ is a local martingale. The converse is more interesting:

Theorem (budget constraint). A random variable $\xi \geq 0$ is of the form $\xi = X_T^{\alpha}$ for some α if and only if $\mathbb{E}[Z_T \xi] = X_0$.

$$\sup \mathbb{E}[U(\xi)]$$

$$\{\xi : \mathbb{E}[Z\cdot\xi] = X_0\}$$



$$\sup_{\xi} \mathbb{E}[U(\xi)] - \lambda \mathbb{E}[Z\cdot\xi]$$

$$= \sup_{\xi} \mathbb{E}\{\underbrace{U(\xi) - \lambda Z_T \xi}\}$$

$$\sup_{x \geq 0} [U(x) - \lambda Z_T(\omega) x]$$

$$U'(x) = \lambda Z_T(\omega)$$

$$x = (U')^{-1}(\lambda Z_T(\omega))$$

$$\xi(\omega) = \underline{(U')^{-1}(\lambda \cdot Z_T(\omega))}.$$

Find λ s.t.

$$\mathbb{E}[(U')^{-1}(\lambda Z_T)\cdot Z_T] = X_0$$

Some other formulations:

1. optimal stopping,
2. impulse control,
3. singular control,
4. ergodic control,
5. risk sensitive control,
6. …