IMSI Tutorial on Games with Ambiguity

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Introduction

- Game Theory main tool for modeling strategic situations
- Optimal decisions for a player may depend on what others do (or will do or are believed to do)
- Major influence across many fields: Economics, Political Science, Computer Science, Mathematics, International Relations, Biology, ...
- Connection with Decision Theory: need to model players' preferences under uncertainty
    - von Neumann & Morgenstern's famous axiomatization of expected utility under risk as a model of preferences over probability distributions (lotteries) was part of their seminal *Theory of Games and Economic Behavior* (1944)
    - motivated in part by their need to evaluate mixed strategies in analysis of zero-sum games

Introduction

- Despite this history, game theory has been relatively slow to incorporate modern advances in decision theory under uncertainty
- Today (and also Sujoy Mukerji's talk tomorrow on Mechanism Design): Discuss some progress in incorporating ambiguity and ambiguity aversion models and concerns into game theory.
- Not aiming for full generality/full literature coverage

Multi-stage Gameforms with Perfect Recall

- A **(finite) extensive-form multistage gameform with (possibly) incomplete information and perfect recall** consists of:
  - N a finite set of players
  - H a finite set of (terminal) histories $h = (h_{-1}, (h_{0,i})_{i \in N}, \ldots, (h_{T,i})_{i \in N})$
    - $\Theta \equiv \{h_{-1} \mid h \in H\}$ is a space of "parameters" or "types"
    - Includes games without a parameter space by taking Θ to be a singleton
  - $\mathcal{I}_i \equiv \bigcup_{0 \leq t \leq T} \mathcal{I}_i^t$ are the information sets for player i
    - Perfect Recall: i's information sets reflect all her previously visited information sets.
    - When all information sets are singletons, there is *perfect observability* of previous stages (parameters and actions)

Extensive Form Strategies

- **A (behavior) strategy** for player i is a function σ_i specifying the *distribution* over i's actions conditional on each possible information set in $\mathcal{I}_i$.
- A strategy profile, $\sigma \equiv (\sigma_i)_{i \in N}$, is a strategy for each player.
- Example: Gas-Station game
- Example: Sealed-bid first-price auction

Extensive Form Strategies

- To go from a gameform to a *game* need players' *preferences* over strategy profiles σ
- Standard approach: assume each player i has Subjective Expected Utility (SEU) preferences
- Add to the gameform:
  - $u_i : H \to \mathbb{R}$ is the utility payoff of player i given the history
  - π is a probability over Θ
  - Use these to represent i's (ex-ante) preferences over i's strategies σ'_i when the others play σ_{-i}:

$$V_i(\sigma'_i, \sigma_{-i}) \equiv \sum_{h \in H} u_i(h) p_{(\sigma'_i, \sigma_{-i})}(h|h^0)\pi(h^0),$$

    where $p_\sigma(h|h^0)$ is the probability of reaching terminal history of play h from $h^0 \in \Theta$ according to strategies σ
- Example: Gas-Station game
- Example: Sealed-bid first-price auction

Nash Equilibrium

- By far the most common solution concept for games
- John Nash won Nobel in Economics for the concept and existence proof
- Nash equilibrium: Strategy profile σ such that

$$V_i(\sigma) \geq V_i(\sigma'_i, \sigma_{-i}) \quad \forall i, \sigma'_i \in \Sigma_i$$

- Each player is playing a strategy that maximizes her preferences given the strategies of the other players
- Dual role of strategies here: i's behavior and $-i$'s beliefs about i's behavior
- In NE these beliefs are correct and behavior is optimizing given beliefs

Strategic Issues

- How to model ambiguity and ambiguity-averse players in games?
- We focus on equilibrium analysis (and not e.g. rationalizability)
- Apparent conflict with Nash equlibrium
  - Dual role of strategies in any NE: i's behavior and $-i$'s beliefs about i
  - Equilibrium: beliefs are correct
  - So no scope for ambiguity about others' strategies!
  - Often, beliefs space and strategy space are not even the same!
    - E.g. MEU set of measures vs. a mixture over actions
- How has literature dealt with this?

First Wave

- Relax what it means for beliefs to be “correct”
- Examples: Dow-Werlang (1994 JET), Klibanoff (1996 wp), Lo (1996 JET), Eichberger-Kelsey (2000 GEB), Marinacci (2000 GEB), Mukerji-Shin (2002 BEJTE), Eichberger-Kelsey (2014 IER)
- For example: with MEU, might say set C is “correct” if *contains* others’ equilibrium mixed strategies
- Initially attracted interest, but less recent impact

Current Approaches

1. Expand mixtures to include ambiguous mixtures
   - enlarge strategy space to match belief space; direct strategic ambiguity
2. Relax Nash to self-confirming equilibrium
   - correctness only on path; scope for ambiguity off-path
3. Add incomplete information, about which there may be ambiguity
   - correctness about information-contingent strategies; allows for strategic ambiguity

We will be brief on 1 and 2, and focus on 3

Ambiguous Mixtures

- In addition to objective mixing devices, players have access to ambiguous devices
- Mathematically, represented as sets of mixed strategies
- Strategic ambiguity: in some games, when at least some players care about ambiguity, can be strictly optimal to play an ambiguous mixture because of the strategic impact it has on others
- Reidel-Sass (2014 Th. and Dec.)
- Issue: assumes all possible ambiguous mixtures always available
- Issue: assumes that all players agree on perceived ambiguity of each ambiguous device. No way to model disagreement

Self-Confirming Equilibrium with Ambiguity

- Fudenberg-Levine (1993 ECMA) propose SCE motivated by learning foundations for equilibria
- Since off-path behavior is not observed, it cannot be (fully) learned
- SCE requires correct beliefs only on the equilibrium path of play

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- They thus allow for ambiguous beliefs about off-path play, but no on-path ambiguity
- Off-path beliefs can influence on-path play, so off-path ambiguity can induce new SCE outcomes when players are ambiguity-averse

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- Off-path beliefs can influence on-path play, so off-path ambiguity can induce new SCE outcomes when players are ambiguity-averse
- Main result: conditions under which more ambiguity aversion enlarges the set of SCE (with pure-strategy deviations only)
- Issue: SCE dynamic but analysis in AER 2015 entirely ex-ante. Battigalli, Catonini, Lanzani, Marinacci (2019 GEB) explore similar issues in a truly dynamic setting

Incomplete Information with Ambiguity

- Parameter space Θ
- Players may receive information about it and payoffs may depend on it
- Ambiguity only through players' possibly ambiguous beliefs about Θ
- Strategies are now maps from information to (mixtures over) actions
- Correct beliefs about strategies
- Ambiguity about Θ can nevertheless induce strategic ambiguity: example shortly
- Azrieli-Teper (2011 GEB), Bade (2011 GEB) for 1-stage games; extensive-form: Hanany, Klibanoff, Mukerji (2020 AEJ:Micro); Pahlke (2021 dissertation)

Multi-stage *Games* with Perfect Recall

- A **(finite) extensive-form multistage game with incomplete information, perfect recall and (weakly) ambiguity averse smooth ambiguity preferences** Γ consists of:
  - N a finite set of players
  - H a finite set of (terminal) histories $h = (h_{-1}, (h_{0,i})_{i \in N}, \ldots, (h_{T,i})_{i \in N})$
    - $\Theta \equiv \{h_{-1} \mid h \in H\}$ is a space of "parameters" or "types"
  - $\mathcal{I}_i \equiv \bigcup_{0 \le t \le T} \mathcal{I}_i^t$ are the information sets for player i
    - Perfect Recall: i's information sets reflect all her previously visited information sets
  - $u_i : H \to \mathbb{R}$ is the utility payoff of player i given the history
  - μ_i is a (finite-support) probability over $\Delta(\Theta)$, where $\Delta(\Theta)$ is the set of all probability measures over Θ and $\sum_{\pi \in \Delta(\Theta)} \mu_i(\pi)\pi(\theta) > 0$ for all $\theta \in \Theta$
  - $\phi_i : \mathrm{co}(u_i(H)) \to \mathbb{R}$ is a continuously differentiable, concave and strictly increasing function.

Preferences and Ex-ante Equilibrium

Ex-ante Preferences:

$$V_i(\sigma_i', \sigma_{-i}) \equiv \sum_{\pi \in \Delta(\Theta)} \phi_i \left(\sum_{h \in H} u_i(h) p_{(\sigma_i', \sigma_{-i})}(h|h^0) \pi(h^0) \right) \mu_i(\pi),$$

where $p_\sigma(h|h^0)$ is the probability of reaching terminal history of play h from $h^0 \in \Theta$ according to strategies σ

An Example

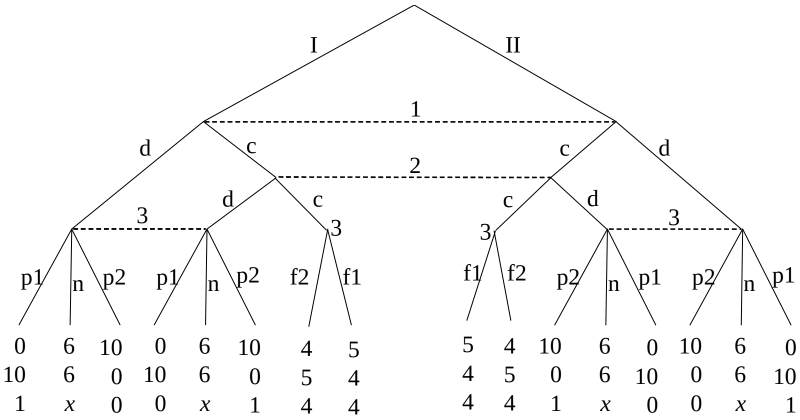

Figure: Modified Greenberg (2000) Game from HKM 2020

Example: Notation and Preferences

- Assume 1 and 2 view Θ as ambiguous
- Smooth ambiguity preferences, shared $\mu \in \Delta(\Delta(\Theta))$
- For concreteness, μ equal weight on $\delta_{\theta=I}$ and $\delta_{\theta=II}$.
- Behavior strategy σ_i is a map from information sets of i to mixtures over actions available there; set Σ_i
- $U_i(\sigma, \theta)$: EU of i when profile σ is played and $\theta \in \Theta$ is realized
- i's ex-ante preference: $V_i(\sigma) = \mathrm{E}_\mu[\phi_i(\mathrm{E}_\pi[U_i(\sigma, \cdot)])]$

Example: Ex-Ante Equilibrium

- If 1, 2 ambiguity neutral, no (Bayesian) Nash equilibrium yields (c, c) with positive probability
  - Not true for SCE: 1 and 2 each incorrectly sure they would be the ones punished if they deviate
- What if 1 and/or 2 are ambiguity-averse?
- Ex-ante (Nash) equilibrium:

$$V_i(\sigma) \geq V_i(\sigma'_i, \sigma_{-i}) \quad \forall i, \sigma'_i \in \Sigma_i$$

Example: Analysis

- If 1 and/or 2 are sufficiently ambiguity-averse, there is an ex-ante equilibrium where (c, c) is played with probability 1
- 1 and 2 willing to play c only if each sufficiently worried they will be punished if they play d
  - with ambiguity neutrality and any shared μ, any strategy of 3 that incentivizes one of them will lead the other to play d
  - with ambiguity aversion:
    - suppose 3 punishes 1 after d and favors 2 after (c, c) when $\theta = I$, and does the opposite when $\theta = II$
    - creates ambiguity in players 1 and 2 about who will be punished / favored
    - ambiguity aversion leads to effective beliefs for each that are "more pessimistic" than μ-average belief
    - since 1 does worse under $\theta = I$ and 2 under $\theta = II$, their effective beliefs are pushed apart by ambiguity aversion
    - hence the single strategy of player 3 can enforce cooperation for both players

Example: Strategic Ambiguity

- 3 knows that conditioning play on the ambiguous θ induces strategic ambiguity in the minds of 1 and 2
- Equilibrium requires that such ambiguity-inducing play is permitted only when it is a best response
- This is what makes this ambiguity "strategic"
- Given (c, c), 3 is indifferent among who is favored, and thus is willing to play the given strategy
- As punishment is off-path here, ex-ante equilibrium imposes no restrictions on the punishment part of 3's strategy
- This leads us to consider "perfection" concerns...

Example: Sequential Optimality

- Suppose we require optimality at all information sets, including those off-path
- Would player 3 be willing to punish according to their strategy?
- Depends on the payoff x that 3 gets from staying neutral
- Also depends on 3's beliefs after observing d
- The "purest" expression of perfection is to place no restriction on these beliefs other than consistency with what 3 knows—the value of θ and the fact that someone chose d
- Adding this perfection requirement to ex-ante equilibrium is called sequential optimality by HKM (2020)
- Under ambiguity neutrality, this is Kreps-Wilson (1982 ECMA) sequential rationality plus Nash equilibrium

Preferences and Ex-ante Equilibrium

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$$V_i(\sigma_i', \sigma_{-i}) \equiv \sum_{\pi \in \Delta(\Theta)} \phi_i \left(\sum_{h \in H} u_i(h) p_{(\sigma_i', \sigma_{-i})}(h|h^0) \pi(h^0) \right) \mu_i(\pi),$$

where $p_\sigma(h|h^0)$ is the probability of reaching terminal history of play h from $h^0 \in \Theta$ according to strategies σ

Preferences at an information set:

$$V_{i,I_i}(\sigma_i', \sigma_{-i}) \equiv \sum_{\pi \in \Delta(I_i)} \phi_i \left(\sum_{h | h^t \in I_i} u_i(h) p_{(\sigma_i', \sigma_{-i})}(h|h^t) \pi(h^t) \right) \nu_{i,I_i}(\pi)$$

Sequential Optimality

Definition

Fix a game Γ. A pair (σ, ν) consisting of a strategy profile and interim belief system is *sequentially optimal* if, for all players i, all information sets I_i and all $\sigma_i' \in \Sigma_i$,

$$V_i(\sigma) \geq V_i(\sigma_i', \sigma_{-i})$$

and

$$V_{i,I_i}(\sigma) \geq V_{i,I_i}(\sigma_i', \sigma_{-i}).$$

Example: Sequential Optimality

- If $x > 1$, never sequentially optimal for 3 to punish either 1 or 2. Thus (c, c) never played
- If $x \leq 1$, if 1 and/or 2 sufficiently ambiguity-averse, there is a sequential optimum with (c, c) always played
- If $0.5 < x \leq 1$, 3 more ambiguity aversion needed for (c, c) than in ex-ante equilibrium

Updating

- A strategy profile σ is sequentially optimal if (σ, ν) is sequentially optimal for *some* interim belief system ν

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- A strategy profile σ is sequentially optimal if (σ, ν) is sequentially optimal for *some* interim belief system ν
- We show that any sequentially optimal profile is sequentially optimal with respect to an interim belief system generated by one particular update rule.

Smooth Rule Updating (Hanany & Klibanoff 2009)

An interim belief system ν satisfies the *smooth rule* using σ if the following holds for each player i and information set I_i: except immediately following a deviation, for all $\pi \in \Delta(I_i)$,

$$\nu_{i,I_i}(\pi) \propto \sum_{\hat{\pi} \in \Delta(I_i^{-1}) | \hat{\pi}_{I_i} = \pi} \frac{\phi_i'\left(\sum_{h | h^{t-1} \in I_i^{-1}} u_i(h) p_\sigma(h | h^{t-1}) \hat{\pi}(h^{t-1})\right)}{\phi_i'\left(\sum_{h | h^t \in I_i} u_i(h) p_\sigma(h | h^t) \pi(h^t)\right)} \cdot \left(\sum_{h^t \in I_i} p_{-i,\sigma_{-i}}(h^t | h^{t-1}) \hat{\pi}(h^{t-1})\right) \nu_{i,I_i^{-1}}(\hat{\pi}),$$

where

$$\hat{\pi}_{I_i}(h^t) = \frac{p_{-i,\sigma_{-i}}(h^t | h^{t-1}) \hat{\pi}(h^{t-1})}{\sum_{\hat{h}^t \in I_i} p_{-i,\sigma_{-i}}(\hat{h}^t | \hat{h}^{t-1}) \hat{\pi}(\hat{h}^{t-1})}$$

and similar requirements hold at initial information sets $I_i \subseteq \Theta$.

Theorem

Fix a game Γ. A strategy profile σ is sequentially optimal if and only if there exists an interim belief system $\hat{\nu}$ satisfying the smooth rule using σ such that $(\sigma, \hat{\nu})$ is sequentially optimal.

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Theorem

Fix a game Γ. A strategy profile σ is sequentially optimal if and only if there exists an interim belief system $\hat{v}$ satisfying the smooth rule using σ such that $(\sigma, \hat{v})$ is sequentially optimal.

- Analysis of sequential optima of a game may be undertaken under the "as if" assumption that all players use smooth rule updating
- Under ambiguity neutrality, identifies the same strategy profiles as Kreps and Wilson (1982)'s sequential rationality plus the assumption of Bayesian updating given σ, which are, in turn, the same as weak PBE.

Sequential optimality and one-stage deviations

Under the smooth rule strengthened to apply also immediately following *own* deviations, checking for one stage deviations is sufficient to determine sequential optimality:

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Theorem

Fix a game Γ *and a pair* (σ, ν) *such that* ν *satisfies the strong smooth rule using* σ. *Then* (σ, ν) *is sequentially optimal if and only if* (σ, ν) *has no profitable one-stage deviations.*

Sequential optimality and one-stage deviations

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Theorem

Fix a game Γ and a pair (σ, ν) such that ν satisfies the strong smooth rule using σ. Then (σ, ν) is sequentially optimal if and only if (σ, ν) has no profitable one-stage deviations.

- Important implications shared with the standard ambiguity neutral case:
  - Allows the use of "folding back" to check whether some $\hat{\sigma}$ is sequentially optimal given beliefs.
  - Beliefs not determined by "folding back", but by updating given $\hat{\sigma}$

Sequential Equilibrium under Ambiguity

- What are "reasonable" off-path beliefs?
- One answer: Kreps-Wilson consistency embodied in sequential equilibrium

Definition (Kreps-Wilson Consistency)

There exists a sequence of completely mixed strategy profiles (σ^k) converging to σ such that the corresponding sequence (ν^k) converges to ν, where each ν^k are the beliefs derived from σ^k and the prior beliefs μ via Bayesian updating

- To generalize this so as to allow for ambiguity-averse players, one needs to take a stand on the form of beliefs and of belief updating.
- HKM (2020) use the second-order beliefs in smooth ambiguity preferences and update via the smooth rule we saw previously.
- They define a sequential equilibrium under ambiguity (SEA) as a (σ, ν) satisfying both sequential optimality and this smooth-rule consistency.

Sequential Equilibrium under Ambiguity

- HKM show that, for a pair (σ, ν) satisfying smooth-rule consistency, sequential optimality is equivalent to no profitable one-stage deviations (in general of course no profitable one-stage deviations is a weaker requirement).
- One could imagine generalizing sequential equilibrium for other classes of ambiguity-averse preferences and/or other generalizations of Bayesian updating by pairing no-profitable one-stage deviations with consistency defined using the chosen update rule.
- To the extent that the generalized updating was not dynamically consistent, such a generalized definition of sequential equilibrium would automatically incorporate the sophistication approach discussed above.

Example: SEA

- Only if $x \leq 0.5$ can there be an SEA in which (c, c) is played with probability 1
- Smooth-rule consistency forces beliefs of 3 about which player played d to be the same at both of 3's information sets

Comparative Statics in Ambiguity Aversion

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    - Taking the union over all equilibria generated by *any* beliefs, the set of ex-ante or sequentially optimal or SEA strategies is the same for any ambiguity aversions.

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  - Taking the union over all equilibria generated by *any* beliefs, the set of ex-ante or sequentially optimal or SEA strategies is the same for any ambiguity aversions.
  - This last result sensitive to whether mixtures that may help hedge against ambiguity are considered – if limit attention to pure strategies and pure-strategy deviations then last set expands with ambiguity aversion as well (similar to Battigalli et al. (2015) and Battigalli et al. (2019) on self-confirming equilibria with ambiguity aversion).

Connections with Dynamic Decision Making

- Our next example illustrates how different approaches to dynamic decision making under ambiguity relate to different notions of perfection in extensive-form games
  - Sequential optimality relates to dynamically consistent updating
  - No-profitable one-stage deviations (No OSD) relates to sophistication

Connections with Dynamic Decision Making

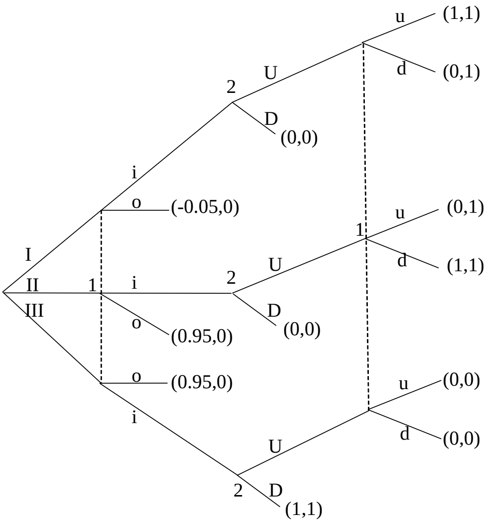

Figure: Illustrating different approaches to “sequential rationality”

Connections with Dynamic Decision Making

- Unlike the previous game, here θ affects payoffs directly
- Player 1 potentially moves twice
- Only Player 2 learns θ before moving
- Straightforward for 2: U if $\theta = I, II$ and D if $\theta = III$ is strictly dominant
- If 1 expects 2 to behave rationally (as in equilibrium), she effectively faces the following pure-strategy payoffs:

	I	II	III
iu	1	0	1
id	0	1	1
o	-0.05	0.95	0.95

Connections with Dynamic Decision Making

- Can think of *o* as "committing to *id* at a cost of 0.05"
- Under sequential optimality, no value for commitment, so *o* is never played
- If 1 ambiguity averse and uses dynamically inconsistent update rule (e.g. Bayes' Rule), then under sophistication, commitment may be valuable
- For example, 1 smooth ambiguity with $\phi(x) = -e^{-10x}$ and μ is $\frac{1}{2}$–$\frac{1}{2}$ on $\pi_1 = (1/3, 1/9, 5/9)$ and $\pi_2 = (1/3, 5/9, 1/9)$
  - Unique sequential optimum has 1 play *id*
  - Under Bayesian updating and No OSD, unique strategy for 1 plays *o* with positive probability
    - Under Bayesian updating and No Pure OSD, unique pure strategy for 1 is to play *o*

Ambiguity Aversion and Robustness

- **Example:**

θ_1	A	B
A	$0, 0$	$1, -8$
B	$-8, 1$	$-6, -6$

θ_2	A	B
A	$6, 6$	$1, 16$
B	$16, 1$	$12, 12$

Figure: Robustness Example

- $\mu(\pi_1) = \mu(\pi_2) = \frac{1}{2}$, where $\pi_1(\theta_1) = \frac{2}{3}$ and $\pi_2(\theta_1) = \frac{1}{2}$, do not learn anything about θ before choosing their action. $\phi_1 = \phi_2 = \phi$.
- Under ambiguity neutrality, i.e. ϕ affine, both (A, A) and (B, B) are equilibrium strategy profiles.
- (A, A) is robust to increased ambiguity aversion (i.e., remains an equilibrium when ϕ becomes more concave), but (B, B) is not.
- To see that (A, A) is robust, note that, assuming her opponent plays A, a player evaluates the mixed strategy $\lambda A + (1 - \lambda)B$ according to $\frac{1}{2}\phi(2\lambda) + \frac{1}{2}\phi(4 - \lambda)$, which is maximized at $\lambda = 1$ for any concave ϕ.

Ambiguity Aversion and Robustness

- To see that (B, B) is not robust, note that for example, if $\phi(x) = -e^{-\alpha x}$ with $\alpha > \ln(\frac{1+\sqrt{5}}{2}) \approx 0.48$, it is profitable to deviate to A.
- Another sense of robustness is that an equilibrium supported for a wider range of beliefs is more robust.
- Set of weights μ on π_1 and π_2 that support (A, A) as an equilibrium: $\mu(\pi_1) \geq \frac{\phi'(3)}{2\phi'(2)+\phi'(3)}$.
- As ϕ becomes more concave, $\frac{\phi'(3)}{2\phi'(2)+\phi'(3)}$ decreases, approaching 0, so sufficient ambiguity aversion results in a large set of weights μ supporting (A, A).
- The fact that ambiguity aversion leads to such a large set of beliefs supporting (A, A) is not special to this example.
- HKM show that equilibria that are robust to increased ambiguity aversion *must* be supported by a large set of beliefs for sufficient ambiguity aversion, and this supporting set of beliefs may be made as large as desired

Ambiguity Aversion and Robustness

Two robustness notions:

- Robust to increased ambiguity aversion:

Definition

An equilibrium σ is *robust to increased ambiguity aversion* if it remains an equilibrium whenever one or more players becomes more ambiguity averse.

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Definition

An equilibrium σ is *robust to increased ambiguity aversion* if it remains an equilibrium whenever one or more players becomes more ambiguity averse.

- Belief robust (in words, formal definition in HKM 2020):

Definition

Ambiguity aversion makes an equilibrium σ *belief robust* if sufficient increases in players' ambiguity aversion, holding the π's in the supports of players' beliefs $(\mu_i)_{i \in N}$ fixed, make all beliefs placing sufficient weight on each such π support σ as an equilibrium.

Ambiguity Aversion and Robustness

Assumption (1)

For each player i, $\sum_{h \in H} u_i(h) p_\sigma(h|h^0)\pi(h^0)$ can be strictly ordered across the π in the support of μ_i

Robustness to increased ambiguity aversion implies belief robustness:

Theorem

If an equilibrium σ is robust to increased ambiguity aversion and Assumption (1) holds, then ambiguity aversion makes σ belief robust.

- Holds for both ex-ante equilibrium and sequential optimum notions.
- Holds with an additional condition for SEA.

Ambiguity Aversion and Robustness

Intuition for the result:

- Robustness to increased ambiguity aversion implies that σ_i must be a best response given the minimizing π
- If one were infinitely ambiguity averse (i.e., all effective weight placed on the minimizing π), then the beliefs over the π cease to matter and all beliefs with the same support make σ_i a best response.
- Along the way, as ambiguity aversion is increased, the proof uses concave transformations tailored to generate specific shifts in effective beliefs that maintain optimality for all beliefs placing sufficient weight on each π in the support.
- Assumption 1 ensures enough flexibility in the manner in which more ambiguity aversion can shift the effective weight placed on expected payoffs for the various π.

Ambiguity Aversion and Robustness

- Consider a population having heterogeneous beliefs. Equilibria that, under ambiguity neutrality, are not supported by many beliefs might not be expected to occur often. Our robustness result offers ambiguity aversion as a possible explanation for unexpected prevalence of such equilibria. Specifically, if such an equilibrium is robust to increased ambiguity aversion, ambiguity aversion can make it an equilibrium for more of the population (i.e., for more beliefs).

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- Consider a population having heterogeneous beliefs. Equilibria that, under ambiguity neutrality, are not supported by many beliefs might not be expected to occur often. Our robustness result offers ambiguity aversion as a possible explanation for unexpected prevalence of such equilibria. Specifically, if such an equilibrium is robust to increased ambiguity aversion, ambiguity aversion can make it an equilibrium for more of the population (i.e., for more beliefs).
- HKM apply to show conditions under which entrant ambiguity aversion makes Limit Pricing equilibria more robust

Related Literature

- Extensive form games with ambiguity and incomplete information
  - Battigalli et al. (2019) – SCE, smooth ambiguity prefs, Bayesian updating, not sequential opt even on path, focus on comparative static in ambiguity aversion
  - Pahlke (2021) – sequential equil., Recursive MEU, ex-ante preference not a primitive – analyze different game for each σ
- Specific extensive form applications with ambiguity and incomplete information – all work by violating sequential optimality
  - Eichberger and Kelsey (1999, 2004), Dominiak and Lee (2017) – signaling games
  - Bose and Daripa (2009), Bose and Renou (2014) – dynamic mechanism design
  - Kellner and Le Quement (2017, 2018) – sender-receiver cheap talk games
  - Beauchêne, Li and Li (2019) – persuasion
  - Auster and Kellner (2018) – descending price auctions

Existing and Potential Applications

- To date, most strategic applications of ambiguity-averse preferences assume Bayesian or prior-by-prior Bayesian updating together with No OSD or consistent planning
  - Bose and Daripa (JET 2009) on dynamic auctions
  - Bose and Renou (ECMA 2014) on mechanism design
  - Kellner and Le Quement (JET 2018) on cheap talk
  - Beauchêne, Li, and Li (2019 JET) on persuasion
- Many interesting open questions on implications of sequential optimality in these and other applications

Concluding Remarks

- Ambiguity models include EU as a special case
- So equilibrium under ambiguity (weakly) leads to an expansion of the strategy profiles that are equilibria for *some* specification of preferences
- As the Greenberg example illustrates, the new equilibrium outcomes may be particularly interesting
- New possibilities for comparative static results connecting extent of ambiguity aversion and of ambiguity to equilibrium outcomes
- When applying these theories, be as clear as possible about which assumptions, beyond the basic existence of ambiguity and non-neutrality towards it,
    - are being used and
    - are crucial for the main conclusions