Dynamically Consistent Updating, including under Ambiguity

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Concerned with how one should update preferences upon receiving new information.

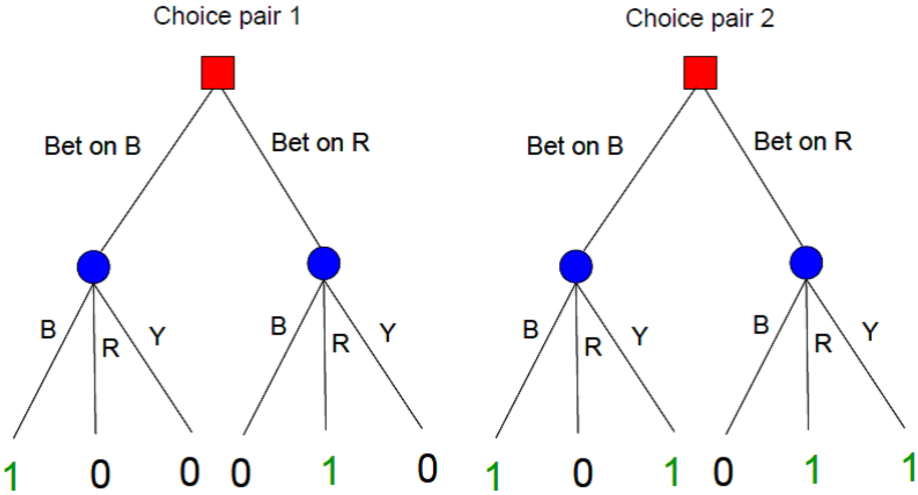
Approach will be to investigate update rules delivering the following properties:

- Dynamic Consistency: Ex-ante optimal information contingent choices are respected by updated preferences
- Closure: If limit consideration ex-ante to a certain model, then updated preferences should remain within that model
- Update beliefs, not tastes (when separated)

- Hanany and Klibanoff ('07) pursues these issues in the context of a specific model of preferences: Maxmin expected utility (MEU, Gilboa and Schmeidler '89)
- Hanany and Klibanoff ('09) provides a characterization of dynamically consistent update rules that applies to a significantly broader set of preferences.
- Also applies this general result to some specific models of recent interest in the ambiguity literature and the regret literature, while delivering the first rules for updating these models in a consistent way.

3-Color Ellsberg Example

30 (B)lack, 60 (R)ed or (Y)ellow



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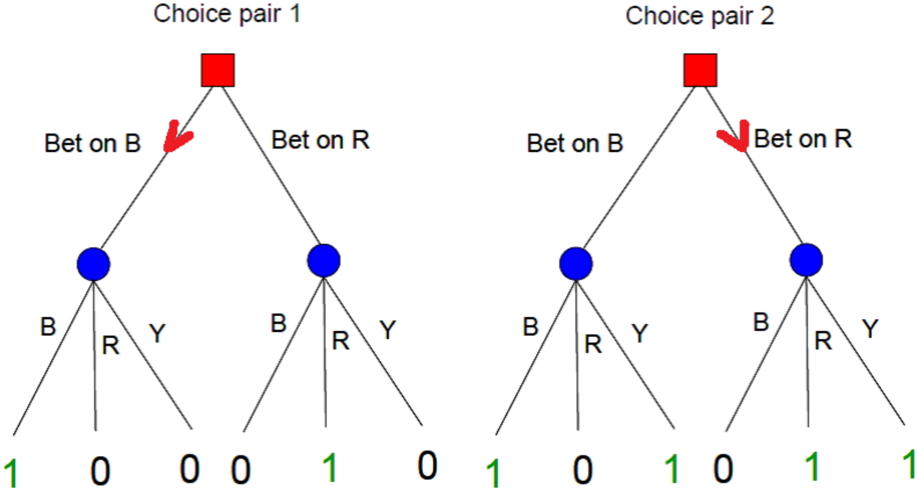
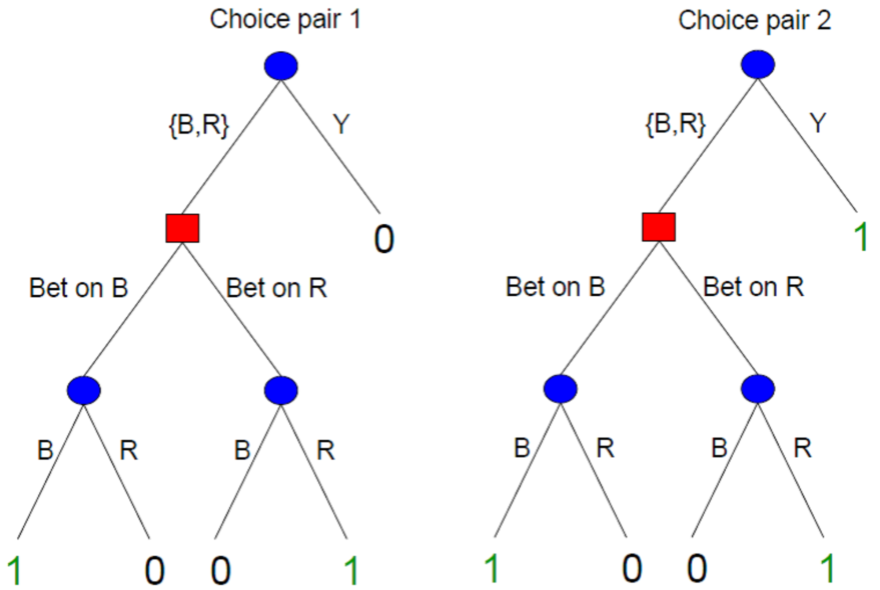


A Dynamic Variation



Literature

- Non-expected utility updating, consistency is achieved by having later choices influenced by earlier ones (Machina '89, McClennen '90, Machina and Schmeidler '92, Epstein and Le Breton '93, Eichberger and Grant '97, Segal '97, Wakker '97, Grant, Kajii, Polak '00)
- Dynamically Consistent updating (Hanany & Klibanoff '07, '09)
- Recursive models:
  - Dynamic Variational preferences (Maccheroni, Marinacci, Rustichini '06), includes Recursive MEU (Epstein & Schneider '03, Wang '03, Hayashi '05, '09)
  - Recursive Smooth Ambiguity preferences (Klibanoff, Marinacci, Mukerji '09)
- Sophistication / Consistent Planning (á la Strotz '55) under ambiguity (Siniscalchi '11)

Literature

- Any theory implying backward induction forces choices to be the same across the two pairs (this applies to both ex-ante and realized choices)
  - Recursive models: Epstein & Schneider '03, Wang '03, Hayashi '05, Maccheroni, Marinacci & Rustichini '06, Klibanoff, Marinacci & Mukerji '09, Gul & Pesendorfer '21
  - Sophistication / Consistent Planning: Siniscalchi '11
- Also under Naïveté – dynamic inconsistency in that if ex-ante choices are Ellsberg, realized choices are not:
  - Full (or generalized) Bayesian updating (Jaffray '92, '94, Fagin & Halpern '89, Walley '91, Sarin & Wakker '98, Pires '02, Wang '03, Epstein & Schneider '03)
  - Maximum Likelihood updating and, more broadly, all "classical" update rules (Gilboa & Schmeidler '03)
  - Dempster-Shafer updating of capacities (Dempster '68, Shafer '76, Gilboa & Schmeidler '03, Wang '03, Nishimura & Ozaki '03)

Recursive Models

- restrict dynamic consistency to a fixed event tree (or information filtration)
- recursively construct preferences on that tree (which guarantees dynamic consistency—on that tree only)

Main advantage: can apply toolbox of recursive optimization methods (solving signal by signal)
Main disadvantage: recursion restricts the scope of ambiguity-averse behavior

- restricted ex-ante preferences, e.g. MEU with rectangular priors, rules out resolution of ambiguity on one dimension (e.g. signals) impacting ambiguity on other dimensions (e.g. states)
- forecloses a wide range of ambiguity hedging behavior: partially hedging ambiguity over states by offsetting positions across different signals

Sophistication / Consistent Planning

- drop dynamic consistency
- assume sophistication to pin down behavior

Main advantage: allows for full scope of ambiguity-averse behavior
Main disadvantage: more complex to solve than recursion

- Introduces additional phenomena such as a preference for commitment
  - a sophisticated individual will in general be willing to pay less for information than a recursive (or dynamically consistent) one
  - may even be willing to pay a cost not to observe information: not because information per se is harmful, but because the intrinsic value of information may be less than the value of commitment

Preferences

- Anscombe-Aumann framework, objects of choice are acts, $f : S \to X$, functions from states to lotteries
- Consider non-degenerate, continuous, monotonic preference relations that, when restricted to constant acts, obey the vNM expected utility axioms.
- Preference representation

$$V(u \circ \cdot)$$

is quasi-concave if and only if preferences satisfy Schmeidler's Uncertainty (Ambiguity) Aversion axiom:

$$f \succsim h \text{ implies } \alpha f + (1-\alpha) h \succsim h \text{ for } \alpha \in [0, 1] .$$

Preferences (cont.)

- Early models satisfying ambiguity aversion include the now-classic:
  - Maxmin Expected Utility (Gilboa and Schmeidler '89)
  - concave Choquet Expected Utility (Schmeidler '89); concave Cumulative Prospect Theory over gains (Tversky and Kahneman, '92)
- More recent advances have led to several more flexible models of ambiguity aversion, including:
  - concave Smooth Ambiguity preferences (Klibanoff, Marinacci, Mukerji '05; Nau '06; Seo '09)
  - Variational preferences (Maccheroni, Marinacci, Rustichini '06)
  - Ambiguity Averse Preferences (Cerreja, Maccheroni, Marinacci, Montrucchio '11)
- Many regret models satisfy ambiguity aversion given a feasible set, e.g. minimax regret with multiple priors (Hayashi '08, Stoye '08).

Update Rules

- Quadruple:
  - A preference representation, (V, u)
  - A convex, compact feasible set of acts, B
  - An (interior) act, g, optimal in B
  - A non-null event, $E \subseteq S$
- An update rule U is a function defined on a set of such quadruples that maps each quadruple to a preference representation $(V_{E,g,B}, u_{E,g,B})$ that makes E^c a null event.
- Restrict attention to rules for which updated preferences are ambiguity averse ($V_{E,g,B}$ quasiconcave) and risk preferences are unchanged, $u_{E,g,B} = u$.
- $\succsim_{E,g,B}$ denotes the updated preference, possibly depends on g, B.
- Indifference between utility acts $u \circ f$ that are identical on E (consequentialism).

Example of inconsistency with full Bayesian updating

- Dynamic Ellsberg example with Max-min EU multiple priors model (MEU) of Gilboa & Schmeidler '89.

$$V(u \circ f) = \min_{q \in C} \int (u \circ f) dq$$

- $u(z) = z$ for $z \in \mathbb{R}$ and $C = \left\{ \left(\frac{1}{3}, \alpha, \frac{2}{3} - \alpha \right) \mid \alpha \in \left[\frac{3}{12}, \frac{5}{12} \right] \right\}$. Ex-ante optimum is "Bet on B" in Pair 1 ($(1, 0, 0)$), and "Bet on R" for Pair 2 ($(0, 1, 1)$).
- Full Bayesian updating conditional on $E = \{\mathrm{B}, \mathrm{R}\}$ gives $C_E = \left\{ (\alpha, 1 - \alpha, 0) \mid \alpha \in \left[\frac{4}{9}, \frac{4}{7} \right] \right\}$ in each problem.
- Since $\frac{4}{9} > \frac{3}{7}$, "Bet on B" is preferred in each choice problem, reversing the ex-ante choice in Pair 2.

Dynamic Consistency

- Optimal contingent plans should be respected when a planned for contingency arises:
  an ex-ante optimal act cannot be feasibly improved conditional on any planned-for event.
- Why?
  - Often put forward as a rationality requirement – successful planning ahead
  - Violations may lead to choosing strictly dominated outcomes (Machina '89, McClennen '90, Segal '97, Seidenfeld '04)
  - Violations may lead to negative value of information
  - Easier to make welfare statements in dynamic models
  - Psychologically: Rationalization in the sense of supporting past choices/plans.
- How to formalize?

Dynamic Consistency of an Update Rule

Formally (Hanany & Klibanoff 2007):

Axiom

DC (Dynamic Consistency): For each quadruple in the domain of the update rule, if $f \in B$ with $f = g$ on E^c, then

$$g \succsim_{E,g,B} f.$$

- Recall $g \succsim f$ for all $f \in B$.
- Preferences for acts other than g need not be preserved
- Comparison with feasible acts identical on E^c

A General Characterization

Definition: For a quasiconcave V, the measures supporting the conditional indifference curve at h on E are:

$$T_{E,h}(V) = \left\{ \begin{array}{c} q \in \Delta(E) \mid \int (u \circ f) dq > \int (u \circ h) dq \\ \text{for all } f \in \mathcal{A} \text{ such that } V(u \circ f) > V(u \circ h) \end{array} \right\}.$$

Normalized Greenberg–Pierskalla ('73) differential in quasiconvex analysis.
Definition: The measures supporting the conditional optimality of h are:

$$Q^{E,h,B} = \left\{ \begin{array}{c} q \in \Delta \mid q(E) > 0 \text{ and } \int (u \circ h) dq \geq \int (u \circ f) dq \\ \text{for all } f \in B \text{ with } f = h \text{ on } E^c \end{array} \right\},$$

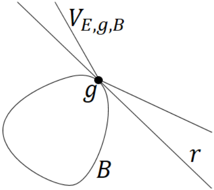
$Q_E^{E,h,B}$ denotes the set of Bayesian conditionals on E of $q \in Q^{E,h,B}$.

Theorem

An update rule satisfies DC if and only if $T_{E,g}(V_{E,g,B}) \cap Q_E^{E,g,B} \neq \emptyset$ for each quadruple in the domain of the update rule.

A General Characterization (proof)

- Let $r \in T_{E,g}\left(V_{E,g,B}\right) \cap Q_E^{E,g,B}$.
  For any $f \in B$ with $f = g$ on E^c, $\int (u \circ g) dr \geq \int (u \circ f) dr$, therefore $V_{E,g,B}\left(u \circ g\right) \geq V_{E,g,B}\left(u \circ f\right)$, and DC holds.



- Assume DC. By a separating hyperplane theorem, there exists r such that

$$\int (u \circ f') dr > \int (u \circ g) dr \geq \int (u \circ f) dr$$

  for all $f \in B$ with $f = g$ on E^c and f' with $V\left(u \circ f'\right) > V\left(u \circ g\right)$.
  Therefore $r \in T_{E,g}\left(V_{E,g,B}\right) \cap Q_E^{E,g,B} \neq \emptyset$.

A General Characterization (cont.)

- This gives us a test for dynamic consistency of an update rule for any quasiconcave model:
  - Calculate $T_{E,g}\left(V_{E,g,B}\right)$
  - Calculate $Q_E^{E,g,B}$
  - See if they intersect.
- If $V_{E,g,B}$ is concave, $T_{E,g}\left(V_{E,g,B}\right)$ can be replaced by a set of measures derived from the superdifferential of $V_{E,g,B}$ at $u \circ g$. If differentiable then look at gradient.
- Similarly, smoothness of the relevant slice of B at $u \circ g$ simplifies calculation of $Q_E^{E,g,B}$ as a singleton.

Dynamically consistent updating for MEU preferences

- MEU preferences:

$$V(u \circ f) = \min_{q \in C} \int (u \circ f) dq$$

- The rules apply Bayes' rule to subsets of C, the subset updated will in general vary with the choice problem:
  - Past choices and/or the feasible sets from which they were selected matter for updating.
- Two Steps:
  1. Propose 4 axioms for MEU update rules and show they are equivalent to requiring rules to work by applying Bayes' rule to some measures in C.
  2. Describe the rules satisfying dynamic consistency plus the above axioms.

Step 1: An Intermediate Result

- Axiom CL (Closure under MEU): $\succsim_{E,g,B}$ is an MEU preference.
- Axiom UT (Unchanged Tastes): For all constant acts (i.e., lotteries) x and y,

$$x \succsim y \Leftrightarrow x \succsim_{E,g,B} y.$$

- Axiom NC (Null Complement): For all acts f and h,

$$f \sim_{E,g,B} f_E h.$$

Let $\mathcal{U}^{\mathrm{MEU}}$ denote the set of all update rules delivering MEU preferences having updated sets of measures, $C_{E,g,B} \subseteq \Delta(E)$.

Proposition

$\mathcal{U}^{MEU}$ is the set of all update rules satisfying CL, UT and NC.

No connection required between $C_{E,g,B}$ and C.

Step 1: A Fourth Axiom

- Axiom IIW (Information Improves the Worst-case): For all acts f and constant acts x,

$$\text{if } f_E x \sim x \text{ then } f \succsim_{E,g,B} x.$$

  - Equivalently:

$$\text{if } \min_{q \in C} \int (u \circ f)\, dq_E = u(x) \text{ then } \min_{q \in C_{E,g,B}} \int (u \circ f)\, dq \geq u(x)$$

  - Motivation: information (weakly) reduces ambiguity concerns.
  - Weakening of an axiom Pires '02 used to characterize full Bayesian updating for MEU.
- Definition: $\mathcal{U}^{Bayes} = \{U \in \mathcal{U}^{\text{MEU}} \mid C_{E,g,B} \subseteq \{q_E \mid q \in C\}\}$.
  - Set of update rules that work by applying Bayes' rule to some measures in C.

Proposition

$\mathcal{U}^{Bayes}$ is the set of all update rules satisfying CL, NC, UT and IIW.

Step 2: Dynamically Consistent Update Rules

Lemma

$T_{E,h}(V) = \arg\min_{q \in C} \int (u \circ h) dq.$

Definition:
$\mathcal{U}^{DC} = \left\{ U \in \mathcal{U}^{Bayes} \mid Q_E^{E,g,B} \cap \arg\min_{q \in C_{E,g,B}} \int (u \circ g) dq \neq \emptyset \right\}.$

Proposition

$\mathcal{U}^{DC}$ is the set of all update rules satisfying CL, NC, UT, IIW and DC.

- g is conditionally optimal within $\{f \in B \text{ with } f = g \text{ on } E^c\}$ when that set and the conditional indifference curve containing g have a tangent in common on E at g.
- Corollary: $\mathcal{U}^{DC} =$
$$\left\{ \begin{array}{c} U \in \mathcal{U}^{Bayes} \mid \text{for some } r \in Q^{E,g,B}, \\ r_E \in C_{E,g,B} \subseteq \{q_E \mid q \in C \text{ and } \int (u \circ g) dq_E \geq \int (u \circ g) dr_E\}. \end{array} \right\}.$$

Back to the Dynamic Ellsberg Example

- Recall the example and again consider MEU preferences with u the identity and $C = \left\{ \left(\frac{1}{3}, \alpha, \frac{2}{3} - \alpha \right) \mid \alpha \in \left[\frac{3}{12}, \frac{5}{12} \right] \right\}$. This delivers the Ellsberg choices ex-ante. Let $E = \{\mathrm{B}, \mathrm{R}\}$.
- $\mathcal{U}^{DC}$ tells us the updates consistent with DC and thus with carrying out the Ellsberg choices.
- For choice pair 1, $B = co\left\{(1,0,0),(0,1,0)\right\}$ and $Q^{E,(1,0,0),B} = \left\{ q \in C \mid q_E(\mathrm{B}) \geq \frac{1}{2} \right\}$.
- Applying our characterization of $\mathcal{U}^{DC}$, $C_{E,(1,0,0),B}$ must be a closed, convex subset of $\left\{ q \in \Delta(E) \mid \frac{1}{2} \leq q(\mathrm{B}) \leq \frac{4}{7} \right\}$, the updates of measures in C putting more weight on black than red.
- Similarly, for choice pair 2, $B' = co\left\{(1,0,1),(0,1,1)\right\}$ and $Q^{E,(0,1,1),B'} =$ $\left\{ q \in C \mid q_E(\mathrm{B}) \leq \frac{1}{2} \right\}$ and $C_{E,(0,1,1),B'}$ must be a closed, convex subset of $\left\{ q \in \Delta(E) \mid \frac{4}{9} \leq q(\mathrm{B}) \leq \frac{1}{2} \right\}$, the updates of measures in C putting less weight on black than red.

Existence and Ambiguity Maximization

- Things worked fine for the example, but how do we know $\mathcal{U}^{DC}$ is always non-empty? How large a subset of measures in C may be updated while maintaining consistency?
- Definition: U^{DCmax} is the update rule in $\mathcal{U}^{Bayes}$ such that

$$C_{E,g,B} = \{q_E \mid q \in C \text{ and } \int (u \circ g) dq_E \geq \min_{p \in Q_E^{E,g,B}} \int (u \circ g) dp\}.$$

Proposition

U^{DCmax} exists and is the unique ambiguity maximizing update rule in $\mathcal{U}^{DC}$ (i.e., satisfying CL, NC, UT, IIW and DC).

Can/Should Dynamic Consistency be Strengthened?

- DC is a weak requirement compared to many formalizations of dynamic consistency in the literature. Can it be strengthened while still admitting update rules for MEU? If so, how can this be done and what are the implications for updating?
- We begin by introducing three candidate strengthenings of DC and show that each leads to the impossibility of updating MEU preferences.
- Following this we consider feasible strengthenings of DC that may be attractive and set out their implications for updating.

3 Candidate Strengthenings of Dynamic Consistency

- Axiom DC (Dynamic Consistency): For all acts $f \in B$ with $f = g$ on E^c,

$$g \succsim_{E,g,B} f.$$

- Axiom DC1: For all acts $f, h \in B$ with $f = h = g$ on E^c and $f \succsim h$,

$$f \succsim_{E,g,B} h.$$

- Axiom DC2: For all acts f with $f = g$ on E^c and $g \succsim f$,

$$g \succsim_{E,g,B} f.$$

- Axiom DC3: For all acts $f \in B$ with $f = g$ on E^c,

$$g \succsim_{E,g,B} f$$

and if additionally $g \succ f$ then $g \succ_{E,g,B} f$.

Non-existence of Update Rules

Proposition

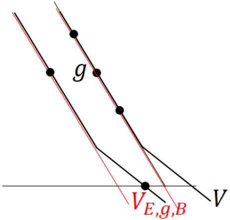
No update rule satisfying axioms CL, NC, and UT satisfies either DC1, DC2 or DC3.

- Proof by Example: Consider again the same state space, information structure and preferences as in the dynamic Ellsberg example. So, MEU preferences with u the identity and $C = \left\{ \left(\frac{1}{3}, \alpha, \frac{2}{3} - \alpha \right) \mid \alpha \in \left[\frac{1}{4}, \frac{5}{12} \right] \right\}$. The conditioning event is $E = \{\mathrm{B}, \mathrm{R}\}$.
  - By our earlier result, axioms CL, NC, and UT imply that $\succsim_{E,g,B}$ can be represented using the same u and using a set of measures contained in $\Delta(E)$. Thus, fixing g and B, all acts that give a weakly higher payoff in state 1 than in state 2 should be evaluated using the same measure. Denote this measure by $(\beta, 1-\beta, 0)$.
  - We assume all feasible acts will give a payoff of 1 if the state is Yellow, as in choice pair 2, however the attainable payoffs on Black and Red will be varied to show non-existence in the three cases.

- Axiom DC1: For all acts $f, h \in B$ with $f = h = g$ on E^c and $f \succsim h$,

$$f \succsim_{E,g,B} h.$$

  - Let $B = co\{$all acts of the form $(a, b, 1)$ such that $4a + 3b \leq 29$ and $a, b \geq 0\}$, includes $(\frac{17}{4}, 4, 1), (\frac{23}{4}, 2, 1), (\frac{27}{4}, 0, 1), (4, 3, 1)$.
  - According to $\succsim$, $g = (5, 3, 1)$ is optimal within B.

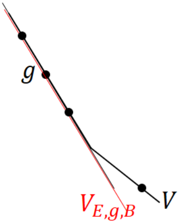


    - Suppose $\beta < \frac{4}{7}$, then $(\frac{17}{4}, 4, 1) \succ_{E,g,B} g$ in violation of DC.
    - Suppose $\beta > \frac{4}{7}$, then $(\frac{23}{4}, 2, 1) \succ_{E,g,B} g$ in violation of DC.
    - Therefore $\beta = \frac{4}{7}$, but then $(\frac{27}{4}, 0, 1) \succ_{E,g,B} (4, 3, 1)$ contradicting the unconditional preference $(4, 3, 1) \sim (\frac{27}{4}, 0, 1)$ and thus violating DC1.

- Axiom DC2: For all acts f with $f = g$ on E^c and $g \succsim f$,

$$g \succsim_{E,g,B} f.$$

  - Let $g = (5,3,1)$ and let $B = \{g\}$.
    $g \sim (\frac{17}{4},4,1) \sim (\frac{23}{4},2,1) \sim (\frac{31}{4},0,1)$.

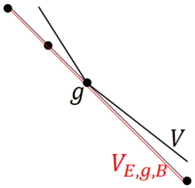


    - Suppose $\beta < \frac{4}{7}$, then $(\frac{17}{4},4,1) \succ_{E,g,B} g$ in violation of DC2.
    - Suppose $\beta > \frac{4}{7}$, then $(\frac{23}{4},2,1) \succ_{E,g,B} g$ in violation of DC2.
    - Therefore $\beta = \frac{4}{7}$, but then $(\frac{31}{4},0,1) \succ_{E,g,B} g$ violating DC2.

- Axiom DC3: For all acts $f \in B$ with $f = g$ on E^c, $g \succsim_{E,g,B} f$, and if additionally $g \succ f$ then

$$g \succ_{E,g,B} f.$$

  - Let $B = co\{(1,2,1), (3,0,1)\}$, includes $(\frac{3}{2}, \frac{3}{2}, 1)$.
  - According to $\succsim$, $g = (2,1,1)$ is strictly optimal within B.



    - Suppose $\beta < \frac{1}{2}$, then $(\frac{3}{2}, \frac{3}{2}, 1) \succ_{E,g,B} g$ in violation of DC.
    - Suppose $\beta > \frac{1}{2}$, then $(3,0,1) \succ_{E,g,B} g$ in violation of DC.
    - Therefore $\beta = \frac{1}{2}$, but then $(\frac{3}{2}, \frac{3}{2}, 1) \sim_{E,g,B} g$ violating DC3.

- All other previous formalizations of consistency that allow dependence on the choice problem (e.g., McClennen '90, Machina '89, Machina & Schmeidler '92, Gul & Lantto '90, Segal '97) also generate impossibility results in our setting.

Strengthenings of Dynamic Consistency that work

- Axiom PFI (Preservation of Feasible optimal Indifference):
  For all acts $f \in B$ with $f = g$ on E^c and $g \sim f$,

$$g \sim_{E,g,B} f.$$

- Motivation: Robustness to indifference being resolved conditionally differently than it was unconditionally, in the sense that all feasible acts agreeing with g on E^c that were unconditionally indifferent to g should remain optimal choices conditional on E. In some sense no real commitment to the choice of g over such acts need have been made yet.

- What are the implications of adding this to DC for updating?

- Definition: $\mathcal{U}^{DC\cap PFI} =$

$$\left\{ \begin{array}{c} U \in \mathcal{U}^{Bayes} \mid \text{for some } r \in Q^{E,g,B},\ r_E \in C_{E,g,B} \subseteq \{q_E \mid q \in C \\ \text{and } \int (u \circ f) dq_E \geq \int (u \circ g) dr_E \\ \text{for all } f \in B \text{ with } f = g \text{ on } E^c \text{ and } f \sim g\}. \end{array} \right\}$$

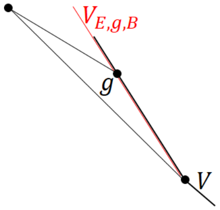
Proposition

$\mathcal{U}^{DC\cap PFI}$ is the set of all update rules satisfying CL, NC, UT, IIW, DC and PFI.

- Also show existence and characterize unique ambiguity maximizing rule in $\mathcal{U}^{DC\cap PFI}$.
- A further strengthening would be replacing $f = g$ on E^c with $u \circ f = u \circ g$ on E^c. Analysis wouldn't change much – all the characterization results and proofs would go through with only the same replacement.

Example where PFI matters

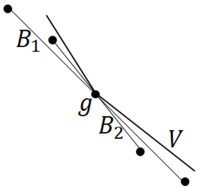
- Consider again the same state space, information structure and preferences as in the dynamic Ellsberg example. So, MEU preferences with u the identity and $C = \left\{ \left(\frac{1}{3}, \alpha, \frac{2}{3} - \alpha \right) \mid \alpha \in \left[\frac{3}{12}, \frac{5}{12} \right] \right\}$. The conditioning event is $E = \{\mathrm{B}, \mathrm{R}\}$.
- Let $g = (\frac{4}{7}, \frac{4}{7}, 0)$ and $B = co \left\{ (\frac{4}{7}, \frac{4}{7}, 0), (1, 0, 0), (0, 1, 0) \right\}$.



Any convex set of conditionals of measures in C will satisfy DC in this example.
However, since $(\frac{4}{7}, \frac{4}{7}, 0) \sim (1, 0, 0)$ only $C_{E,g,B} = \{(\frac{4}{7}, \frac{3}{7}, 0)\}$ will also satisfy PFI.

Additional Directions

- Results concerning the extent to which updating must depend on the choice problem and when or in what ways they need not.
  - Most importantly, it is necessary to allow dependence on the feasible set B, as opposed to simply g, a novel feature.
  - Proof: States, information and preferences as in dynamic Ellsberg, $B_1 = co\{(1,2,1), (3,0,1)\}$, $B_2 = co\{\left(\frac{14}{9}, \frac{14}{9}, 1\right), (\frac{22}{9}, \frac{4}{9}, 1)\}$. $g = (2,1,1)$ strictly optimal in each.



$\beta < \frac{5}{9}$ implies $(\frac{14}{9}, \frac{14}{9}, 1) \succ_{E,g,B_2} g$ and $\beta > \frac{1}{2}$ implies $(3,0,1) \succ_{E,g,B_1} g$, violating DC.
  - This necessity stems from the kinked nature of MEU indifference curves – when g is not located at a preference kink, dependence on B is not required to attain dynamic consistency.

Additional Directions (cont.)

- Our ambiguity maximizing rules are shown to extend the recursive MEU model (Epstein & Schneider '03), in that for all events that can be updated on in recursive MEU, the two models agree.

MEU summary/conclusion

- Theory of ambiguity-sensitive preferences that can maintain, in dynamic extensions, typical behavior (a la Ellsberg) under ambiguity.
- Propose and characterize update rules for MEU preferences.
- The dynamic consistency axioms we use are weak, yet effectively capture the fundamental idea that unconditionally optimal contingent plans remain optimal.

Smooth Ambiguity model

Definition: A smooth ambiguity preference (KMM 2005) over acts has the following representation:

$$V(u \circ f) = \mathbb{E}_\mu \phi \left(\mathbb{E}_\pi u \circ f \right) = \int_{\pi \in \Delta} \phi \left(\int (u \circ f) d\pi \right) d\mu,$$

where ϕ is an increasing real-valued function and captures ambiguity attitude (ϕ concave $=$ ambiguity aversion) and μ is a subjective probability over probability measures π on the state space. Ambiguity is captured through disagreement in the support of μ about the probability of an event.

Smooth Ambiguity model (cont.)

Model of decision-making under ambiguity that simultaneously allows:

- separation of ambiguity attitude from perception of ambiguity
- flexibility in ambiguity attitude
- subjective and flexible perception of which events are ambiguous
- the tractability of smooth preferences
- expected utility as a special case for any given ambiguity attitude.

DC updating of SA preferences

Assume ϕ differentiable. Let $\mathcal{U}^{\mathrm{SA}}$ denote the set of all update rules delivering SA preferences that:

- Leave ϕ unchanged
- Do not depend on the feasible set B given (ϕ, u, μ, E, g).

Theorem

$U \in \mathcal{U}^{SA}$ is dynamically consistent iff

$$\frac{\mathbb{E}_{\mu_{E,g}}[\phi'(\mathbb{E}_{\pi_E}(u \circ g))\pi_E(s)]}{\mathbb{E}_{\mu_{E,g}}[\phi'(\mathbb{E}_{\pi_E}(u \circ g))]} = \frac{\mathbb{E}_{\mu}[\phi'(\mathbb{E}_{\pi}(u \circ g))\pi(s)]}{\mathbb{E}_{\mu}[\phi'(\mathbb{E}_{\pi}(u \circ g))\pi(E)]}$$

for all $s \in E$.

Proof: $T_{E,g}\left(V_{E,g,B}\right) = \{$normalized gradient of $\mathbb{E}_{\mu_{E,g}}\phi\left(\mathbb{E}_{\pi_E}(u \circ g)\right)\}$, $Q_E^{E,g,B} = \{$normalized gradient of $\mathbb{E}_{\mu}\phi\left(\mathbb{E}_{\pi}(u \circ g)\right)$ conditional on $E\}$ for some B. The general theorem implies that the two must be equal.

- Bayesian updating of normalized gradient (local measure) at $u \circ g$.

Application: Subjective Expected Utility

Let's take a step back – to expected utility (SEU).
Subjective Expected utility corresponds to the special case where ϕ is affine, so ϕ' is constant. The only beliefs that matter in this case are the reduced measures

$$p = \mathbb{E}_{\mu}\pi \text{ and } p_{E,g} = \mathbb{E}_{\mu_{E,g}}\pi_E$$

Corollary

U is dynamically consistent iff $p_{E,g}$ is the Bayesian update of p.

Smooth Ambiguity model

- Consider again ambiguity averse Smooth Ambiguity (SA) preferences.
- Will Bayesian updating work here?
- No! Easy to show in the dynamic Ellsberg example. Also follows from our Theorem.
- Since Bayesian updating is dynamically inconsistent, to find a rule that satisfies consistency for SA preferences as Bayes' rule does for EU preferences we need to look elsewhere.
- We will look at reweightings of Bayes' rule.

Bayes' Rule

Beliefs conditional on event E:

$$\forall \pi \in \Delta, s \in E, \qquad \hat{\mu}^*(\pi, s) = \mu(\pi)\,\pi(s)$$
$$\mu_E(\pi) = \hat{\mu}^*_E(\pi, E) = \frac{\mu(\pi)\pi(E)}{\mathbb{E}_{\mu}\hat{\pi}(E)}$$
$$\pi_E(s) = \frac{\hat{\mu}^*_E(\pi,s)}{\hat{\mu}^*_E(\pi,E)} = \frac{\pi(s)}{\pi(E)}$$

Conditional preference:

$$f \succsim_E h \qquad \Longleftrightarrow \qquad \mathbb{E}_{\mu_E}[\phi(\mathbb{E}_{\pi_E}(u \circ f))] \geq \mathbb{E}_{\mu_E}[\phi(\mathbb{E}_{\pi_E}(u \circ h))]$$

Reweighted Bayes' Rules

Beliefs conditional on event E:

$$\forall \pi \in \Delta, s \in E, \qquad \hat{\mu}^*(\pi, s) = \mu(\pi)\,\pi(s)\alpha(\pi, \phi, u, g, E)$$
$$\mu_{E,g}(\pi) = \hat{\mu}^*_{E,g}(\pi, E) = \frac{\mu(\pi)\pi(E)\alpha(\pi,\phi,u,g,E)}{\mathbb{E}_\mu \hat{\pi}(E)\alpha(\pi,\phi,u,g,E)}$$
$$\pi_E(s) = \frac{\hat{\mu}^*_{E,g}(\pi,s)}{\hat{\mu}^*_{E,g}(\pi,E)} = \frac{\pi(s)}{\pi(E)}$$

Conditional preference:

$$f \succsim_{E,g} h \qquad \Longleftrightarrow \qquad \mathbb{E}_{\mu_{E,g}}[\phi(\mathbb{E}_{\pi_E}(u \circ f))] \geq \mathbb{E}_{\mu_{E,g}}[\phi(\mathbb{E}_{\pi_E}(u \circ h))]$$

Proposition

A reweighted Bayes' rule is dynamically consistent iff $\mu_{E,g}$ is the measure generated by setting

$$\alpha(\pi, \phi, u, g, E) = \frac{\phi'(\mathbb{E}_{\pi}(u \circ g))}{\phi'(\mathbb{E}_{\pi_E}(u \circ g))}$$

if $\pi(E) > 0$, and 0 otherwise.

Proof: Satisfies DC because for each π, $\mathbb{E}_{\mu_{E,g}}[\phi'(\mathbb{E}_{\pi_E}(u \circ g))\pi_E(s)] \propto \mathbb{E}_{\mu}[\pi(E)\frac{\phi'(\mathbb{E}_{\pi}(u \circ g))}{\phi'(\mathbb{E}_{\pi_E}(u \circ g))}\phi'(\mathbb{E}_{\pi_E}(u \circ g))\pi_E(s)] = \mathbb{E}_{\mu}[\phi'(\mathbb{E}_{\pi}(u \circ g))\pi(s)]$.

DC implies this proportionality for each π whenever the reweighted Bayes' rule involves two distinct π_E.

Properties of the Smooth Rule

- This rule generalizes Bayesian updating of beliefs.
- Exactly coincides with Bayesian updating when:
  - preferences are ambiguity neutral, or
  - the ex-ante optimum, g, is constant in utilities.

In both cases the reweighting factor vanishes.

- Departs from Bayesian updating in a manner that depends on:
  - The ex-ante optimum, g
  - the ambiguity attitude of the decision-maker.

Properties of the Smooth Rule (cont.)

Compared to Bayes' Rule, the smooth rule overweights measures that result in higher conditional valuations of g relative to unconditional valuations of g.

$$\mu_{E,g}(\pi) \propto \mu(\pi)\,\pi(E)\,\frac{\phi'\left(\mathbb{E}_{\pi}(u \circ g)\right)}{\phi'\left(\mathbb{E}_{\pi_E}(u \circ g)\right)}$$

Properties of the Smooth Rule (cont.)

The Smooth rule is commutative. Commutativity means that the order of information received does not affect updating.
Specifically, for any non-null events E and F having non-null intersection:

$$\left(\mu_{E,g}\right)_{F,g} = \mu_{E\cap F,g} = \left(\mu_{F,g}\right)_{E,g}$$

Properties of the Smooth Rule (cont.)

- Similarly to MEU preferences, no update rule in $\mathcal{U}^{\mathsf{SA}}$ satisfies the strengthening of DC to either DC1, DC2.
- Nevertheless, the Smooth rule does satisfy a strict version of dynamic consistency when there is strict ambiguity aversion (ϕ is strictly concave):

Axiom Strict DC: Add to DC that if g is ex-ante strictly preferred (indifferent) to some $f \in B$ with $f = g$ on E^c, then g remains strictly preferred (indifferent) to f after learning E.

Uncertainty Averse preferences

Definition: An ambiguity averse preference (CMMM 2011) over acts has the following representation:

$$\inf_{p \in \Delta} G^{\star}\left(\int (u \circ f)\, dp, p\right)$$

where $G^{\star} : u(X) \times \Delta \to (-\infty, \infty]$ is an uncertainty aversion index given by

$$G^{\star}(t, p) = \sup_{f \in \mathcal{A}} \left\{ u\left(x^{f}\right) : \int (u \circ f)\, dp \leq t \right\},$$

and $x^{f} \in X$ satisfies $x^{f} \sim f$.

DC updating of Uncertainty Averse preferences

Lemma

$T_{E,h}(V) = \arg\min_{p\in\Delta} G^{\star}\left(\int (u \circ h)\, dp, p\right).$

Let $\mathcal{U}^{\mathrm{UA}}$ denote the set of all update rules delivering variational preferences.

Proposition

$U \in \mathcal{U}^{UA}$ is dynamically consistent iff

$$\arg\min_{p\in\Delta} G^{\star}_{E,g,B}\left(\int (u \circ g)\, dp, p\right) \cap Q_E^{E,g,B} \neq \emptyset.$$

Variational preferences

Definition: A variational preference (MMR 2006) over acts has the following concave representation:

$$\min_{p\in\Delta}\left(\int (u\circ f)\,dp + c\,(p)\right)$$

where $u : X \rightarrow \mathbb{R}$ is a vN-M utility function and $c : \Delta \rightarrow [0, \infty]$ is grounded, convex and lower semicontinuous.

- Special cases include:
  - MEU (Gilboa and Schmeidler 1989): $c\,(p) = 0$ if $p \in C$ and $+\infty$ otherwise.
  - Robust Control Preferences (Hansen and Sargent 2001): $c\,(p) = \theta\mathbb{E}_p(\ln \frac{dp}{dq})$ if $p \ll q$ and $+\infty$ otherwise.

DC updating of Variational preferences

Lemma

$T_{E,h}(V) = \arg\min_{p\in\Delta}\left(\int (u\circ h)\, dp + c(p)\right).$

Let $\mathcal{U}^{\mathrm{VR}}$ denote the set of all update rules delivering variational preferences.

Proposition

$U \in \mathcal{U}^{VR}$ is dynamically consistent iff

$$\arg\min_{p\in\Delta}\left(\int (u\circ g)\, dp + c_{E,g,B}(p)\right) \cap Q_E^{E,g,B} \neq \emptyset.$$

Regret-based models

Representation $V(\hat{u}_B(f))$ with V non-constant, continuous, weakly increasing and quasiconcave, where

$$\hat{u}_B(f)(s) = u \circ f(s) - \max_{h \in B} u \circ h(s), \forall s$$

is the "regret-adjusted" utility profile.
Example: minimax regret with multiple priors:

$$V(\hat{u}_B(f)) = \min_{p \in C} \int \hat{u}_B(f) dp.$$

Proposition

U is dynamically consistent iff

$$\hat{Q}_E^{E,g,B} \cap \arg\min_{q \in C_{E,g,B}} \int \hat{u}_B(g) dq \neq \emptyset,$$

where $\hat{Q}_E^{E,g,B}$ differs from $Q_E^{E,g,B}$ only in replacing $u \circ f$ and $u \circ g$ with $\hat{u}_B(f)$ and $\hat{u}_B(g)$ respectively.

Conclusions

- General result characterizing dynamic consistency for convex (ambiguity/uncertainty averse) preferences.
- Characterization of DC updating for:
  - MEU preferences
  - ambiguity averse Smooth Ambiguity preferences, and identify a natural generalization of Bayes' rule
  - Variational preferences (also have further results on specific rules), show Bayesian updating is dynamically consistent for Multiplier preferences
  - various regret models such as minimax regret with multiple priors.