Dynamic Decision Making Under Uncertainty:

Tools

Lars Peter Hansen (University of Chicago)

IMSI Tutorial on Decision Making and Uncertainty

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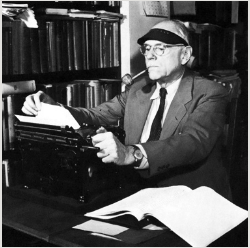
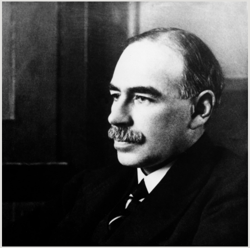
Keynes and Knight



"Uncertainty must be taken in a sense radically distinct from the familiar notion of Risk, from which it has never been properly separated.... and there are far-reaching and crucial differences in the bearings of the phenomena depending on which of the two is really present and operating."
Knight (1921)



"We have, as a rule, only the vaguest idea of any but the most direct consequences of our acts... Our knowledge of the future is fluctuating, vague, and uncertain."
Keynes (1937)

Decision theory

- families of structured models - confront unknown parameters and parameters that change over time
  framework: Gilboa-Schmeidler, Chen-Epstein, Epstein-Schneider, Klibanoff-Marinacci-Mukerji, Hansen-Miao
- families of unstructured models - confront potential model misspecification
  framework: robust control adapted and modified as in Hansen-Sargent

Formal approach

- two-player zero-sum differential game
  - stochastic differential equations for state evolution
  - one player is a "fictitious planner" engaged in maximizing social well-being
  - another player investigates the adverse consequences of uncertainty about probabilistic inputs through minimization
- use "relative entropy" to limit or bound the probabilistic uncertainty
- use numerical PDE methods along with some extra twists for computations

Alternative probabilities

Apply Girsanov theory:

- Use positive martingales M with unit expectations to represent alternative probabilities relative to some baseline:

$$
\begin{aligned}
M_t^H &= \exp\left(\int_0^t H_u \cdot dW_u - \frac{1}{2}\int_0^t |H_u|^2 du\right) \\
dM_t^H &= M_t^H H_t \cdot dW_t
\end{aligned}
$$

- Implied change in probability measure

  $M^H \quad \Rightarrow \quad$ drift distortion $\quad H_t dt \quad$ to a Brownian increment $\quad dW_t$

  for all $t \geq 0$.

- M^S denotes an alternative structured probability model where $H = S$.
- M^U denotes alternative unstructured probability models where $H = U$.

Long run macroeconomic risk model

- Initial model

$$dY_t = (.01)\left(\widehat{\alpha}_y + \widehat{\beta}_y Z_t\right)dt + (.01)\sigma_y \cdot dW_t$$

$$dZ_t = \widehat{\alpha}_z dt - \widehat{\beta}_z Z_t dt + \sigma_z \cdot dW_t$$

- W is a multivariate Brownian motion
- Y is log capital, consumption, or output
- Z generates “long-run risk” or growth rate uncertainty

The coefficient $\widehat{\beta}_y$ captures the exposure to growth rate uncertainty and $\widehat{\beta}_z$ captures persistence in the macro growth rate.

Family of Restricted Models

- parameters: $\alpha_y, \beta_y, \alpha_z, \beta_z$
- evolution:

$$
\begin{aligned}
dY_t &= .01\left(\alpha_y + \beta_y Z_t\right) dt + .01\sigma_y \cdot dW_t^S \\
dZ_t &= \alpha_z dt - \beta_z Z_t dt + \sigma_z \cdot dW_t^S
\end{aligned}
$$

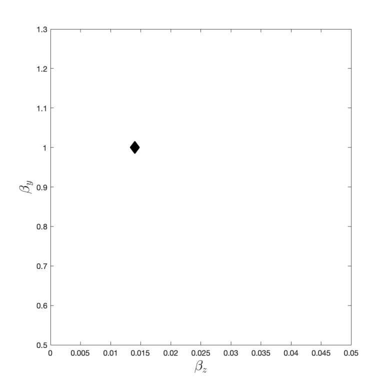
- Construct drift distortion for the Brownian motion $dW_t = S_t dt + dW_t^S$ where $S_t = \eta(Z_t) \equiv \eta_0 + \eta_1 Z_t$ and where

$$
\sigma = \begin{bmatrix} {\sigma_y}' \\ {\sigma_z}' \end{bmatrix},
$$

and

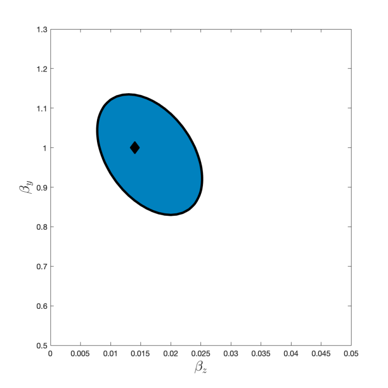
$$
\sigma\eta_0 = \begin{bmatrix} \alpha_y - \widehat{\alpha}_y \\ \alpha_z - \widehat{\alpha}_z \end{bmatrix} \quad \sigma\eta_1 = \begin{bmatrix} \beta_y - \widehat{\beta}_y \\ \widehat{\beta}_z - \beta_z \end{bmatrix}
$$

Growth rate ambiguity



Ambiguity sets: parameter values constrained by relative entropy where β_y quantifies exposure to the macro growth-rate process and β_z quantifies the persistence of that process. The single point is the baseline.

Growth rate ambiguity



Ambiguity sets: parameter values constrained by relative entropy where β_y quantifies exposure to the macro growth-rate process and β_z quantifies the persistence of that process. The single point is the baseline and the region is implied by an ambiguity set. (Construction described later.)

Relative entropy

Recall:

$$dM_t^H = M_t^H H_t \cdot dW_t$$

- Local evolution

$$d\left(M_t^H \log M_t^H\right) = \frac{1}{2} M_t^H |H_t|^2 dt + \left(M_t^H + M_t^H \log M_t^H\right) H_t \cdot dW_t$$

- Discounted entropy relative to the baseline:

$$\begin{aligned}\Delta\left(M^H; 1 \mid \mathcal{F}_0\right) &= \delta \int_0^\infty \exp(-\delta t) E\left(M_t^H \log M_t^H \middle| \mathcal{F}_0\right) dt \\ &= \frac{1}{2} \int_0^\infty \exp(-\delta t) E\left(M_t^H \mid H_t \mid^2 \middle| \mathcal{F}_0\right) dt\end{aligned}$$

Entropy relative to a structured model

Let $H = S$ be a structured model and $H = U$ an unstructured model

Discounted entropy of U relative to S

$$\begin{aligned}\Delta\left(M^U; M^S \mid \mathcal{F}_0\right) &= \delta \int_0^\infty \exp(-\delta t) E\left[M_t^U\left(\log M_t^U - \log M_t^S\right) \Big| \mathcal{F}_0\right) dt \\ &= \frac{1}{2}\int_0^\infty \exp(-\delta t) E\left(M_t^U \mid U_t - S_t \mid^2 \Big| \mathcal{F}_0\right) dt\end{aligned}$$

Confronting misspecification

Use entropy of unstructured relative to alternative structured probability models

$$\Delta\left(M^U; M^S \mid \mathcal{F}_0\right) = \frac{1}{2}\int_0^\infty \exp(-\delta t) E\left(M_t^U \mid U_t - S_t \mid^2 \Big| \mathcal{F}_0\right) dt.$$

multiplied by a penalty parameter to "limit" the consequences of misspecification.

Observations:

- Add a penalty $\theta\frac{|u-s|^2}{2}$ to the HJB equation of a fictitious social planner.
- To accommodate misspecification , we minimize over u given s and to accommodate model ambiguity, we minimize over s subject to a constraint or penalty

The resulting M^{U^*} process is a martingale contribution to valuation.

Two decision theory approaches

- Max-min utility
- Smooth ambiguity

Continuous-time recursive versions of the first is given in Chen-Epstein (2002) and versions of the second in Hansen-Miao (2018)

Recursive Preferences

Let $\{V_t : t \geq 0\}$ be a continuation value process.
Preferences restrict $(\upsilon_t, \varsigma_t)$ where

$$dV_t = \upsilon_t dt + \varsigma_t \cdot dW_t.$$

Discounted expected utility under the baseline model:

$$V_t = \mathbb{E}\left[\int_0^\infty \exp(-\tau\delta)\psi_{t+\tau} d\tau \mid \mathfrak{F}_t\right]$$

Then

$$0 = \psi_t - \delta V_t + \upsilon_t.$$

No restriction on ς_t. Risk neutral in units of utility.

Recursive preferences: distortion

Preferences restrict $(\upsilon_t, \varsigma_t)$ where

$$dV_t = \upsilon_t dt + \varsigma_t \cdot dW_t$$

Discounted expected utility under an alternative model:

$$V_t = \mathbb{E}\left[\int_0^\infty \exp(-\tau\delta)\left(\frac{M^H_{t+\tau}}{M^H_t}\right)\psi_{t+\tau} d\tau \mid \mathfrak{F}_t\right]$$

$$0 = \psi_t - \delta V_t + H_t \cdot \varsigma_t + \upsilon_t$$

since under the H_t is a drift distortion in dW_t under the change in probability measure.

ς_t now matters.

Recursive preferences: robustness I

Preferences restrict $(\upsilon_t, \varsigma_t)$ where

$$dV_t = \upsilon_t dt + \varsigma_t \cdot dW_t$$

Discounted expected utility under an alternative model:

$$\begin{aligned} V_t = \min_U \mathbb{E} &\left[\int_0^\infty \exp(-\tau\delta) \left(\frac{M_{t+\tau}^U}{M_t^U} \right) \psi_{t+\tau} d\tau \mid \mathfrak{F}_t \right] \\ &+ \xi_u \mathbb{E} \left[\int_0^\infty \exp(-\tau\delta) \left(\frac{M_{t+\tau}^U}{M_t^U} \right) \left(\log M_{t+\tau}^U - \log M_t^U \right) d\tau \mid \mathfrak{F}_t \right] \end{aligned}$$

Restriction

$$0 = \min_{U_t} \psi_t - \delta V_t + U_t \cdot \varsigma_t + \frac{\xi_u}{2} |U_t|^2 + \upsilon_t.$$

Recursive preferences: robustness II

Preferences restrict $(\upsilon_t, \varsigma_t)$ where

$$dV_t = \upsilon_t dt + \varsigma_t \cdot dW_t$$

Minimization:

$$0 = \min_{U_t} \psi_t - \delta V_t + U_t \cdot \varsigma_t + \frac{\xi_u}{2} |U_t|^2 + \upsilon_t.$$

Solution:

$$\begin{aligned} U_t &= -\frac{1}{\xi_u} \varsigma_t \\ 0 &= \psi_t - \delta V_t - \frac{1}{2\xi_u} |\varsigma_t|^2 + \upsilon_t \end{aligned}$$

Including structured models: I

Preferences restrict $(\upsilon_t, \varsigma_t)$ where

$$dV_t = \upsilon_t dt + \varsigma_t \cdot dW_t$$

Recursive max-min utility with misspecification concerns:
Minimization:

$$\begin{aligned}
0 &= \min_{S_t \in \Xi_t} \min_{U_t} \psi_t - \delta V_t + U_t \cdot \varsigma_t + \frac{\xi_u}{2} |U_t - S_t|^2 + \upsilon_t \\
&= \min_{S_t \in \Xi_t} \min_{U_t - S_t} \psi_t - \delta V_t + S_t \cdot \varsigma_t + (U_t - S_t) \cdot \varsigma_t + \frac{\xi_u}{2} |U_t - S_t|^2 + \upsilon_t \\
&= \min_{S_t \in \Xi_t} \psi_t - \delta V_t + S_t \cdot \varsigma_t - \frac{1}{2\xi_u} |\varsigma_t|^2 + \upsilon_t
\end{aligned}$$

where Ξ_t is a convex, compact random set.

Including structured models: II

Preferences restrict $(\upsilon_t, \varsigma_t)$ where

$$dV_t = \upsilon_t dt + \varsigma_t \cdot dW_t$$

Write $S_t(\theta)$ where θ is an unknown parameter. Let π_t be a date t posterior over θ where $\int S_t(\theta)\pi_t(d\theta) = 0$.

Minimization:

$$\begin{aligned} 0 = \min_{G_t(\theta), \int G_t(\theta)\pi_t(d\theta)=1} & \psi_t - \delta V_t + \left[\int_\Theta S_t(\theta) G_t(\theta)\pi_t(d\theta)\right] \cdot \varsigma_t \\ & + \xi_s \int_\Theta \log G_t(\theta) G_t(\theta)\pi_t(d\theta) - \frac{1}{2\xi_u}|\varsigma_t|^2 + \upsilon_t \end{aligned}$$

Including structured models: II

$$\min_{G_t(\theta), \int G_t(\theta)\pi_t(d\theta)=1} \left[\int_\Theta S_t(\theta) G_t(\theta)\pi_t(d\theta)\right] \cdot \varsigma_t + \xi_s \int_\Theta \log G_t(\theta) G_t(\theta)\pi_t(d\theta)$$

Minimizer

$$G_t(\theta) \propto \exp\left[-\frac{1}{\xi_s} S_t(\theta) \cdot \varsigma_t\right]$$

Solution

$$-\xi_s \log\left(\int \exp\left[-\frac{1}{\xi_s} S_t(\theta) \cdot \varsigma_t\right] \pi_t(d\theta)\right)$$

Equivalent to smooth ambiguity adjustment.

From preferences to HJB's

Write $V_t = f(X_t)$ where $\{X_t : t \geq 0\}$ is a Markov diffusion:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

Since $dV_t = \upsilon_t dt + \varsigma_t \cdot dW_t$

- ▷ $$\upsilon_t = \frac{\partial f}{\partial x}(X_t) \cdot \mu(X_t) + \frac{1}{2}\text{trace}\left[\sigma(X_t)' \frac{\partial^2 f}{\partial x \partial x'}(X_t)\sigma(X_t)\right]$$
- ▷ $$\varsigma_t = \sigma(X_t)' \frac{\partial f}{\partial x}(X_t)$$

With unknown parameters, replace $\mu(X_t)$ with $\int_\Theta \mu(X_t \mid \theta)\pi_t(d\theta)$.