Professor Rio EN.585.615.81.SP21 Mathematical Methods Take Home Project 2 Johns Hopkins University

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Question 1

(a) See figure 1

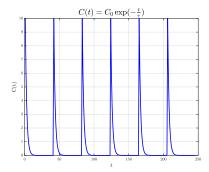


Figure 1

(b) $f(t) = C_0 e^{-\frac{t}{\tau}}$ with period T, so

$$a_0 = \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} dt$$

$$= \frac{2C_0}{T} (-\tau) [e^{-\frac{t}{\tau}}]_0^T$$

$$= -2C_0 \frac{\tau}{T} [e^{-\frac{T}{\tau}} - 1]$$

$$= 2C_0 \frac{\tau}{T} (1 - e^{-\frac{T}{\tau}})$$

If $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_0 \approx 2C_0 \frac{\tau}{T}$.

$$a_k = \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt$$
$$= \frac{2C_0}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt$$

Using integration by parts with $u = \cos \frac{2k\pi t}{T}$, $du = -\frac{2k\pi}{T}\sin \frac{2k\pi t}{T}$ and $dv = e^{-\frac{t}{\tau}}$, $v = (-\tau)e^{-\frac{t}{\tau}}$:

$$\int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt = (-\tau) \left[e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} \right]_0^T - \frac{2k\pi \tau}{T} \int_0^T e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt$$

Using again integration by parts:

$$\int_0^T e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt = (-\tau) \left[e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} \right]_0^T + \frac{2k\pi \tau}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt$$

So

$$(1 + (\frac{2k\pi\tau}{T}))^2 \int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt = (-\tau) \left[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} \right]_0^T + \frac{2k\pi\tau^2}{T} \left[e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} \right]_0^T$$

$$= (-\tau) \left[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} \right]_0^T + 0$$

$$= \tau (1 - e^{-\frac{T}{\tau}})$$

$$\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt = \frac{\tau}{1 + (\frac{2k\pi\tau}{T})^2} (1 - e^{-\frac{T}{\tau}})$$

Substituting back into the expression found for a_k yields

$$a_k = 2C_0 \frac{\tau}{T} \frac{1}{1 + (\frac{2k\pi\tau}{T})^2} (1 - e^{-\frac{T}{\tau}})$$
$$= 2C_0 \frac{\tau T}{T^2 + (2k\pi\tau)^2} (1 - e^{-\frac{T}{\tau}})$$

With the same assumption $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_k \approx 2C_0 \frac{\tau}{T} \frac{1}{1+(\frac{2k\pi\tau}{T})^2}$. Similarly to compute b_k

$$b_{k} = \frac{2}{T} \int_{0}^{T} C_{0} e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt$$

$$= \frac{2C_{0}}{T} \int_{0}^{T} e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt$$

$$= \frac{2C_{0}}{T} \frac{2k\pi \tau}{T} \int_{0}^{T} e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt$$

$$= \frac{2C_{0}}{T} \frac{2k\pi \tau}{T} \frac{\tau}{1 + (\frac{2k\pi \tau}{T})^{2}} (1 - e^{-\frac{T}{\tau}})$$

$$= 4C_{0}k\pi \frac{\tau^{2}}{T^{2} + (2k\pi \tau)^{2}} (1 - e^{-\frac{T}{\tau}})$$

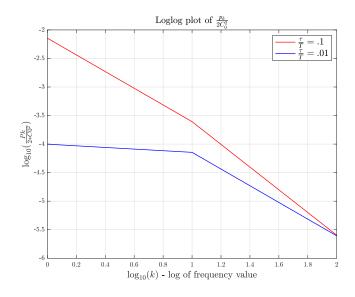
Once again, since $e^{-\frac{T}{\tau}} \approx 0$ then $b_k \approx 4C_0(\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2} \pi k$

(c) For $k \ge 1$

$$\begin{aligned} p_k &= \frac{1}{2} (a_k^2 + b_k^2) \\ &= \frac{1}{2} \left[4C_0^2 (\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} + 16C_0^2 (\frac{\tau}{T})^4 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \pi^2 k^2 \right] \\ &= \frac{1}{2} 4C_0^2 (\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2 \right] \\ &= 2C_0^2 (\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2 \right] \end{aligned}$$

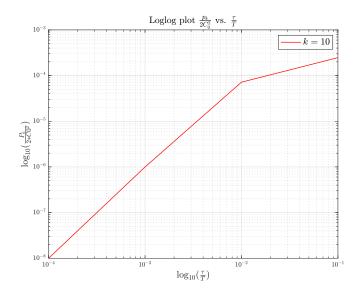
(d) We have

$$\frac{p_k}{2C_0^2} = \left(\frac{\tau}{T}\right)^2 \frac{1}{\left(1 + \left(\frac{2k\pi\tau}{T}\right)^2\right)^2} \left[1 + 4\left(\frac{\tau}{T}\right)^2 \pi^2 k^2\right]$$



From the plot we can see that the power $\frac{p_k}{2C_0^2}$ decreases as the frequency increases. Power is close to 0 starting with a frequency of 10.

(e) Looking at the plot in part (d), as the pulse τ , becomes narrower, the power decreases linearly. For a greater $\frac{\tau}{T}$, the power starts at a higher value until an inflection point corresponding to frequency of 10 ($1 = \log_{10}(10)$ on the graph). Also for a higher $\frac{\tau}{T}$, the steepest the decrease in power. Eventually the two curves combine in one curve around a frequency of ($2 = \log_{10}(100)$ on the graph). The same observations are obtained with a loglog plot of power $\frac{p_k}{2C_0^2}$ vs. $\frac{\tau}{T}$ for a frequency k = 10.



(f) We have

$$a_k \cos(\frac{k2\pi t}{T}) + b_k \sin(\frac{k2\pi t}{T}) = \cos(\phi_k) \cos(\frac{k2\pi t}{T}) + \sin(\phi_k) \sin(\frac{k2\pi t}{T})$$
$$= \cos(\frac{k2\pi t}{T} - \phi_k)$$

where

$$\tan(\phi_k) = \frac{\sin(\phi_k)}{\cos(\phi_k)} = \frac{b_k}{a_k} = 4C_0(\frac{\tau}{T})^2 \frac{1}{1 + (\frac{2k\pi\tau}{T})^2} \pi k (2C_0 \frac{\tau}{T} \frac{1}{1 + (\frac{2k\pi\tau}{T})^2})^{-1}$$
$$= 2\frac{\tau}{T} \pi k$$
$$\phi_k = \arctan(2\frac{\tau}{T} \pi k)$$

For $\frac{\tau}{T}=.1$, $\phi_1\approx 32.14^\circ$ and $\phi_2\approx 51.48^\circ$ and for $\frac{\tau}{T}=.01$, $\phi_1\approx 3.59^\circ$ and $\phi_2\approx 7.16^\circ$

If we imagine a clock with a hand that turns at constant speed, making a full turn every T seconds, and is pointing straight up at time t=0 when the plasma rises to C_0 . The phase ϕ_k is then the angle from the 12:00 position to the current position and indicates how far is the next rise in plasma concentration to C_0 . For the same frequency (either k=1 or k=2), with a larger pulse we are closer to the next substance release.

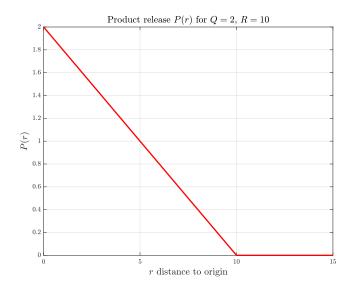
Question 2

(a) One simple way to describe P(r) is to define it as P(r) = Ar + B with the conditions:

$$A \cdot 0 + B = Q$$
$$A \cdot R + B = 0$$

which gives $A = -\frac{Q}{R}$ and B = Q. So

$$P(r) = \begin{cases} Q(1 - \frac{r}{R}) & \text{for } 0 \le r \le R \\ 0 & \text{for } r > R \end{cases}$$



(b) Since we assume no angular dependence: $\nabla^2 C = \frac{1}{r^2} \frac{d}{dr} (r^2 \frac{dC}{dr})$, the differential equation is now:

$$\frac{D}{r^2}\frac{d}{dr}(r^2\frac{dC(r)}{dr}) + P(r) = 0$$
$$\frac{d}{dr}(r^2\frac{dC(r)}{dr}) = -\frac{r^2}{D}P(r)$$

(c) Inside the cell $P(r)=Q(1-\frac{r}{R})$, so we have to solve the differential equation

$$\frac{d}{dr}(r^2 \frac{dC(r)}{dr}) = -\frac{r^2}{D}Q(1 - \frac{r}{R})$$
$$= \frac{Q}{DR}r^2(r - R)$$
$$= \frac{Q}{DR}r^3 - \frac{Q}{D}r^2$$

Integrating once

$$r^{2} \frac{dC(r)}{dr} = \frac{Q}{4DR} r^{4} - \frac{Q}{3D} r^{3} + A$$
$$\frac{dC(r)}{dr} = \frac{Q}{4DR} r^{2} - \frac{Q}{3D} r + \frac{A}{r^{2}}$$

Integrating again

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 - \frac{A}{r} + B$$
 A, B:constants, C_i :inside cell concentration

Outside the cell P(r) = 0 and we want to solve the differential equation

$$\frac{d}{dr}(r^2\frac{dC(r)}{dr}) = 0$$

Which by integration gives

$$r^2\frac{dC(r)}{dr}=C_1$$

$$\frac{dC(r)}{dr}=\frac{C_1}{r^2}$$

$$C_o(r)=-\frac{C_1}{r}+C_2 \ \ C_1,C_2\text{:constants},C_o\text{:outside cell concentration}$$

(d) Applying the boundary conditions

(i) $\lim_{r \to 0} C_i(r) = \lim_{r \to 0} \frac{Q}{12DR} r^3 - \frac{Q}{6D} r^2 - \frac{A}{r} + B$

since $\lim_{r\to 0} C_i(r) = \lim_{r\to 0} (-\frac{1}{r} + B) = \infty$ therefore to have finite concentration $C_i(r)$ at r=0 we need A=0

(ii) $\lim_{r \to \infty} C_o(r) = \lim_{r \to \infty} \left(-\frac{C_1}{r} + C_2 \right) = C_2$

The concentration goes to zero at infinity implies $C_2 = 0$

(iii) We have now for $C_i(r)$ and $C_o(r)$:

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 + B$$
$$C_o(r) = -\frac{C_1}{r}$$

$$C_i(R) = C_o(R)$$
 and $\frac{dC_i(r)}{dr} = \frac{dC_o(r)}{dr}|_{r=R}$ yields

$$\frac{Q}{12DR}R^{3} - \frac{Q}{6D}R^{2} + B = -\frac{C_{1}}{R}$$
$$\frac{Q}{4D}R - \frac{Q}{3D}R = \frac{C_{1}}{R^{2}}$$

Rearranging

$$-\frac{Q}{12D}R^{2} + B = -\frac{C_{1}}{R}$$
$$-\frac{Q}{12D}R = \frac{C_{1}}{R^{2}}$$

which gives

$$B = \frac{Q}{6D}R^2$$

$$C_1 = -\frac{Q}{12D}R^3$$

substituting back

$$C_{i}(r) = \frac{Q}{12DR}r^{3} - \frac{Q}{6D}r^{2} + \frac{Q}{6D}R^{2}$$
$$= \frac{Q}{6D} \left[\frac{r^{3}}{2R} - r^{2} + R^{2} \right]$$
$$C_{o}(r) = \frac{Q}{12D}R^{3} \frac{1}{r}$$

(e) We are looking for the value of r for which $\frac{dC_i(r)}{dr} = 0$:

$$\frac{dC_{i}(r)}{dr} = \frac{Q}{4DR}r^{2} - \frac{Q}{3D}r = \frac{Q}{D}r(\frac{r}{4R} - \frac{1}{3})$$

Discarding the solution r=0 we are left that concentration maximum is for $r=\frac{4}{3}R$ and it is

$$C_M = \frac{Q}{12DR} (\frac{4}{3})^3 R^3 - \frac{Q}{6D} (\frac{4}{3})^2 R^2 + \frac{Q}{6D} R^2$$
$$= \frac{Q}{6D} R^2 \left[\frac{4^3}{2 \cdot 3^3} - \frac{4^2}{3^2} + 1 \right]$$
$$= \frac{11}{162} \frac{Q}{D} R^2$$

Inside the cell

$$C_{i}(r) = \frac{Q}{6D} \left(\frac{1}{2} \frac{r^{3}}{R} - r^{2} + R^{2}\right)$$

$$\frac{C_{i}(r)}{C_{M}} = \frac{Q}{6D} \frac{162}{11} \frac{D}{Q} R^{-2} \left(\frac{1}{2} \frac{r^{3}}{R} - r^{2} + R^{2}\right)$$

$$= \frac{162}{611} \left[\frac{1}{2} \left(\frac{r}{R}\right)^{3} - \left(\frac{r}{R}\right)^{2} + 1\right]$$

$$= \frac{27}{11} \left[\frac{1}{2} \left(\frac{r}{R}\right)^{3} - \left(\frac{r}{R}\right)^{2} + 1\right]$$

And outside the cell

$$C_o(r) = \frac{Q}{12D}R^3 \frac{1}{r}$$

$$\frac{C_o(r)}{C_M} = \frac{Q}{12D} \frac{162}{11} \frac{D}{Q} R^{-2} R^3 \frac{1}{r}$$

$$= \frac{27}{22} \frac{R}{r}$$

When the diffusion constant is doubled, the curve $\frac{C_i(r)}{C_M}$ stays the same.

