Johns Hopkins
Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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This module presents methods to solve Legendre linear equations differential equations with variable coefficients having a specific format

What does a Legendre linear differential equation look like?

$$a_n(\alpha x+\beta)^n \frac{d^n y}{dx^n}+\cdots+a_1(\alpha x+\beta)\frac{dy}{dx}+a_0 y=f(x)$$

Where α, β, a_n are constants

A general way of solving this type of differential equations is to make a change of variable in order to It's complexity. In particular, in this case, to reduce this to a differential equation with constant coefficients.

Let's take a detailed look at how to change variables in this differential equation since it is useful in many other situations!

The key is knowing what variable expression to convert to. In many cases it is obvious, however while it may not be obvious in this situation this equation has been well studied so we know how to construct it.

The new variable is , the functional relationship between the original variable x and t is $\alpha x+\beta=e^t$ or $x=(e^t-\beta)/\alpha$ and $t=\ln(\alpha x+\beta)$

Therefore we can convert the following original differential operators with respect to x to that of the variable t by using the chain rule for differentiation.

$$(\alpha x+\beta)\frac{dy}{dx}=e^{t}\frac{dy}{dx}=e^{t}\frac{dt}{dx}\frac{dy}{dt}=e^{t}\frac{d[\ln(\alpha x+\beta)]}{dx}\frac{dy}{dt}=e^{t}\alpha\frac{1}{\alpha x+\beta}\frac{dy}{dt}=e^{t}\alpha\frac{1}{e^{t}}\frac{dy}{dt}=\alpha\frac{dy}{dt}$$

Looking at the second term in the string of equalities above we also have $e^{t}\frac{dy}{dx}=\alpha\frac{dy}{dt}$ or $\frac{dy}{dx}=\alpha e^{-t}\frac{dy}{dt}$

Using the above results we can construct the second derivative term

$$(\alpha x+\beta)^{2}\frac{d^{2}y}{dx^{2}}=(e^{t})^{2}\frac{d}{dx}\left(\frac{dy}{dx}\right)=e^{2t}\alpha e^{-t}\frac{d}{dt}\left[\alpha e^{-t}\frac{dy}{dt}\right]=\alpha^{2}e^{t}\frac{d}{dt}\left[e^{-t}\frac{dy}{dt}\right]$$

$$=\alpha^{2}e^{t}\left[-e^{-t}\frac{dy}{dt}+e^{-t}\frac{d^{2}y}{dt^{2}}\right]=\alpha^{2}e^{t}e^{-t}\left[\frac{d^{2}y}{dt^{2}}-\frac{dy}{dt}\right]=\alpha^{2}\frac{d}{dt}\left[\frac{dy}{dt}-y\right]=\alpha^{2}\frac{d}{dt}\left[\frac{d}{dt}-1\right]y$$

Try to develop an expression for the third order derivative term yourself (compare with the general result below).

The extension to multiple or n-derivative (note most of the time we will only deal with at most second order derivatives) is below

$$\boxed{(\alpha x+\beta)^{n}\frac{d^{n}y}{dx^{n}}=\alpha^{n}\frac{d}{dt}\left[\frac{d}{dt}-1\right]\cdots\left[\frac{d}{dt}-n+1\right]y}$$

An example of solving a Legendre linear differential equation using variable substitution

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$$

Note that this is simply a homogeneous ODE with no initial conditions given so that Our solution will simply consist of a linear combination of two linearly independent solution with arbitrary coefficients. We identify this equations as a Legendre ODE with $a_2=1,\ a_1=1,\ a_0=-4$, however since $\beta=0$ and $\alpha=1$ call this an Euler ODE a special form of a Legendre's ODE.

We start by changing the variables as previously described, thus we have

$$1x+0=e^t \text{ or simply } \boxed{x=e^t} \text{ or also equivalently } t=\ln(x)$$

$$(1x+0)\frac{dy}{dx}=(1)\frac{dy}{dt} \text{ or simply } \boxed{x\frac{dy}{dx}=\frac{dy}{dt}}$$

$$(1x+0)^2\frac{d^2y}{dx^2}=(1)^2\frac{d}{dt}\left[\frac{d}{dt}-1\right]y \text{ or simply } \boxed{x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}}$$ Corrected

Substitution into the differential equation gives

$$\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+\frac{dy}{dt}-4y=0$$ This immediately simplifies to $$\frac{d^2y}{dt^2}-4y=0$$

The resulting differential equation, with y now a function of (t) is easily solved by elementary techniques as follows. $\frac{d^2y}{dt^2} - 4y = 0$

Take as the solution $y(t) = Ae^{\lambda t}$ and $\frac{d^2y}{dt^2} = \lambda^2 Ae^{\lambda t}$

Substitution into the differential equation gives this characteristic equation (after canceling out the terms in common)

$$\lambda^2 Ae^{\lambda t} - 4Ae^{\lambda t} = 0 \text{ or } \lambda^2 - 4 = 0 \text{ so } \lambda = -2,2$$

And the solution consist of a linear combination of two generated independent solutions, that is

$$y(t) = c_1 e^{2t} + c_2 e^{-2t}$$

However this solution is in terms of t so we need to substitute for t as a function of x.

That is $\boxed{y(x)} = c_1 e^{2(\ln x)} + c_2 e^{-2(\ln x)} = c_1[e^{(\ln x)}]^2 + c_2[e^{(\ln x)}]^{-2} = \boxed{c_1 x^2 + c_2 x^{-2}}$

Aside: Another method, not involving variable substitution is to assume the form

$$y(x) = Ax^m \text{ and } \frac{d^2y}{dx^2} = m(m-1)Ax^{m-2}$$

Substitution in to the original differential equations as a function of x gives a similar characteristic equation in m, that is $m^2 - 4 = 0$ so $m = -2,2$ and leads immediately to

$$\boxed{y(x) = c_1 x^2 + c_2 x^{-2}}$$