# Johns Hopkins Engineering for Professionals

**Mathematical Methods for Applied Biomedical Engineering**
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JOHNS HOPKINS
WHITING SCHOOL
*of* ENGINEERING

# One more simple function of a complex variable

Let's start out by looking at $e^{iy} = \cos y + i\sin y$

Then $\left(e^{iy}\right)^n = e^{iny} = \cos ny + i\sin ny$

This can also be proven directly by induction, that is prove the base case

$$(\cos y + i\sin y)^1 = \cos(1\cdot y) + i\sin(1\cdot y)$$

Assume its true for n = k $(\cos y + i\sin y)^k = \cos ky + i\sin ky$

And prove $\boxed{(\cos y + i\sin y)^{k+1} = \cos(k+1)y + i\sin(k+1)y}$

Taking the polar form for a complex variable $z = re^{i\theta} \equiv re^{i(\theta+2k\pi)}$

**KEY: The exponential function Is a circular function!**

Now for w and z complex we have $\ln z = w \leftrightarrow z = e^w$

**KEY**

$$\ln z = \ln\left[re^{i(\theta+2k\pi)}\right] = \ln r + \ln\left[e^{i(\theta+2k\pi)}\right] = \ln r + i(\theta + 2k\pi)$$

The primary value $\ln z = \ln r + i\theta,\ r = |z| \in R,\ -\pi<\theta<\pi$

# Roots of a complex variable

Take $t^z = e^{z\ln t}$ Now take $z = \frac{1}{n}$ then we have $t^{\frac{1}{n}} = e^{\frac{1}{n}\ln t}$

With $t = x + iy = re^{i(\theta+2k\pi)}$ on the right habd side we have

$$t^{\frac{1}{n}} = e^{\frac{1}{n}\ln t} = e^{\frac{1}{n}\ln[re^{i(\theta+2k\pi)}]} = e^{\frac{1}{n}[\ln r+\ln e^{i(\theta+2k\pi)}]} = e^{[\frac{1}{n}\ln r+\frac{1}{n}i(\theta+2k\pi)]} =$$

$$e^{\frac{1}{n}\ln r}e^{i(\theta+2k\pi)/n} = \left(e^{\ln r}\right)^{\frac{1}{n}} e^{i(\theta+2k\pi)/n} = r^{\frac{1}{n}}e^{i(\theta+2k\pi)/n}$$

Note k = 0,…,n-1 since when k = n same angular value for $2\pi$ as 0

Therefore $\boxed{t^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i(\theta+2k\pi)/n}}$ where $r = |t| = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Let's fine the square roots of 1, of course for $1 \in \text{Real}$ we know they are $\pm 1$

But what about when $1 \in \text{Complex}$ ,that is when t = 1= 1+i0

With $t = 1 + i0 \in \text{Complex}$ and n = 2 we have $1^{\frac{1}{2}} \equiv (1+i0)^{\frac{1}{2}} = r^{\frac{1}{2}}e^{i(\theta+2k\pi)/2} = r^{\frac{1}{2}}e^{i\phi}$

$r = |1+i0| = \sqrt{1^2+0^2} = 1 \in \text{Real} \rightarrow 1^{\frac{1}{2}} = 1 \in \text{Real}$ Also $\theta = \tan^{-1}\left(\frac{0}{1}\right) = \tan^{-1}0 = 0$

Then for $k = 0,\ 1$ (n-1 with n = 2) Also taking $\phi = (\theta + 2k\pi)/2$ with $\theta = 0 \rightarrow \begin{cases} k=0 & \phi = 0 \\ k=1 & \phi = \pi \end{cases}$

Therefore $1^{\frac{1}{2}} \equiv (1+i0)^{\frac{1}{2}} = \begin{cases} k=0 & 1^{\frac{1}{2}}e^{i0} = 1e^0 = 1 \\ k=1 & 1^{\frac{1}{2}}e^{i\pi} = 1(-1) = -1 \end{cases}$

**The same as for the Real case!**
**Note for n = 2 we get 2 roots**

Next let's look at the n = 3 case or cubic root of 1, of course for $1 \in \mathrm{Real}$ the cube root Is just 1. **Where are the other roots?**

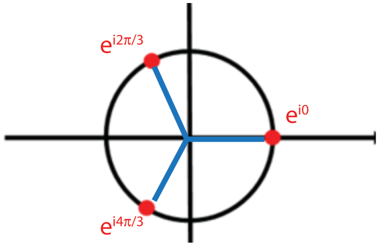
Let's instead look at the complex domain with $t = 1 + i0$ again

but taking n = 3 this time. We have $1^{\frac{1}{3}} = (1+i0)^{\frac{1}{3}} = r^{\frac{1}{3}} e^{i(\theta+2k\pi)/3} = r^{\frac{1}{3}} e^{i\phi}$

$$r = |1+i0| = \sqrt{1^2+0^2} = 1 \in \mathrm{Real} \to 1^{\frac{1}{3}} = 1 \text{ and } \theta = \tan^{-1}\left(\frac{0}{1}\right) = \tan^{-1} 0 = 0$$

and $k = 0,\ 1,\ 2$ (n-1 with n = 3) Also taking $\phi = (\theta + 2k\pi)/3$ with $\theta = 0 \to \begin{cases} k=0 & \phi = 0 \\ k=1 & \phi = 0 + 2\pi/3 \\ k=2 & \phi = 0 + 4\pi/3 \end{cases}$

$$\text{Finally } 1^{\frac{1}{3}} = \begin{cases} k=0 & \phi = 0 & 1^{\frac{1}{3}} e^{i0} = 1 + i0 \\ k=1 & \phi = 0 + 2\pi/3 & 1^{\frac{1}{3}} e^{i2\pi/3} = -1/2 + i\sqrt{3}/2 \\ k=2 & \phi = 0 + 4\pi/3 & 1^{\frac{1}{3}} e^{i4\pi/3} = -1/2 - i\sqrt{3}/2 \end{cases}$$




**Unlike the Real case we get all 3 roots!**

# Singularities and zeros of complex functions

A singular point of a function of a complex variable is any point in an Argand diagram for which the function is not analytic.

If f(z) has a singular point at $z = z_0$ but is analytic in some neighborhood including $z_0$ we call this an isolated singularity. We can write

KEY

$$f(z)=\frac{g(z)}{(z-z_0)^n},\ n=1,2,\cdots \text{ where } g(x)\neq 0 \text{ is analytic}$$

We say that f(x) has a pole of order n at $z = z_0$. Equivalently we can write

KEY

$$\lim_{z\to z_0}[(z-z_0)^n f(x)]=a,\ a\neq 0 \text{ is finite and} \in \text{Complex}$$

If a = 0 then $z=z_0$ is a pole order less than n

If a = $\infty$ then $z=z_0$ is a pole order greater than n

If no value of n can be found that satisfies the above relation then $z=z_0$ is an essential singularity

Example: Find the singularities for $f(x)=\dfrac{1}{1-z}-\dfrac{1}{1+z}$

Write in "standard" form

$$f(x)=\frac{1}{1-z}-\frac{1}{1+z}=\frac{1}{1-z}\left(\frac{1+z}{1+z}\right)-\frac{1}{1+z}\left(\frac{1-z}{1-z}\right)=\frac{1+z-(1-z)}{(1+z)(1-z)}=\boxed{\frac{2z}{(1+z)(1-z)}}$$

Therefore pole of order n = 1 at $z_0=-1,1$

Example: Find the singularities for

$$f(x)=\tanh z=\frac{\sinh z}{\cosh z}=\text{(using the definitions)}=\frac{\dfrac{e^z-e^{-z}}{2}}{\dfrac{e^z+e^{-z}}{2}}=\frac{e^z-e^{-z}}{e^z+e^{-z}}$$

Therefore we have singularity when $e^z+e^{-z}=0$ or $e^z=-e^{-z}=-1e^{-z}$

KEY Now -1 (complex) $=1e^{i(2n+1)\pi}=e^{i(2n+1)\pi}$ and substition gives $e^z=[e^{i(2n+1)\pi}]e^{-z}=e^{-z+i(2n+1)\pi}$

Equating the exponents $z=-z+i(2n+1)\pi$ or $2z=i(2n+1)\pi\to z_0=(n+\frac{1}{2})\pi i$

Next using are limit definition for poles we have

$$\lim_{z\to z_0}[(z-z_0)f(z)]=\lim_{z\to(n+\frac{1}{2})\pi i}[(z-(n+\frac{1}{2})\pi i)\frac{\sinh z}{\cosh z}]$$

$$\lim_{z\to(n+\frac{1}{2})\pi i}\frac{\left[z-(n+\frac{1}{2})\pi i\right]\sinh z}{\cosh z}$$ Note as $z\to(n+\frac{1}{2})\pi i$ we have $z-(n+\frac{1}{2})\pi i\to 0$

and as $z\to(n+\frac{1}{2})\pi i$ we have $\cosh\left[(n+\frac{1}{2})\pi i\right]=\frac{e^{(n+\frac{1}{2})\pi i}+e^{-(n+\frac{1}{2})\pi i}}{2}=$

Let's just look at the case n = 0 (you can generalize to any integer $n\geq 0$)

$$\frac{e^{\frac{\pi}{2}i}+e^{-\frac{\pi}{2}i}}{2}=\frac{\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]+\left[\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right]}{2}=\frac{2\cos\frac{\pi}{2}}{2}=\frac{2\cdot 0}{2}=0$$

**KEY** Therefore $\lim_{z\to(n+\frac{1}{2})\pi i}\left[\frac{(z-(n+\frac{1}{2})\pi i)\sinh z}{\cosh z}\right]$ is of the form $\frac{0}{0}$ and we can use L'Hospitals rule

$$\lim_{z\to(n+\frac{1}{2})\pi i}\left[\frac{(z-(n+\frac{1}{2})\pi i)\sinh z}{\cosh z}\right]=\lim_{z\to(n+\frac{1}{2})\pi i}\left[\frac{(1)\sinh z+(z-(n+\frac{1}{2})\pi i)\cosh z}{\sinh z}\right]=$$

$$\frac{(1)\sinh(n+\frac{1}{2})\pi i+\left[(n+\frac{1}{2})\pi i-(n+\frac{1}{2})\pi i)\right]\cosh z}{\sinh(n+\frac{1}{2})\pi i}=\frac{\sinh(n+\frac{1}{2})\pi i}{\sinh(n+\frac{1}{2})\pi i}=1$$