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 EN.585.615.81.SP21 Mathematical Methods
 Take Home Project 3
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Question 1

The Womersley equation for blood flow is:

$$\rho \frac{\partial w}{\partial t} = \frac{\mu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) + \frac{\partial P}{\partial z}$$

Using $\frac{\partial P}{\partial z} = A e^{int}$ and taking $w(r, t) = u(r) e^{int}$ yields: $\frac{\partial w}{\partial t} = (in) u e^{int}$, $\frac{\partial w}{\partial r} = u'(r) e^{int}$, and $\frac{\partial^2 w}{\partial r^2} = u''(r) e^{int}$, $\frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) = u'(r) e^{int} + r u''(r) e^{int}$. Therefore the Womersley equation becomes:

$$\begin{aligned} \frac{\mu}{r} \left[u'(r) e^{int} + r u''(r) e^{int} \right] + A e^{int} &= \rho (in) u(r) e^{int} \\ \mu \frac{d^2 u(r)}{dr^2} + \frac{\mu}{r} \frac{du(r)}{dr} + A &= (in) \rho u(r) \text{ by dividing through } e^{int} \\ \frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} - \frac{in \rho}{\mu} u &= -\frac{A}{\mu} \text{ by dividing through } \mu \text{ and rearranging} \end{aligned}$$

Finally using $\nu = \frac{\mu}{\rho}$ we have:

$$\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} - \frac{in}{\nu} u = -\frac{A}{\mu}$$

By simple inspection, one particular solution is a constant w.r.t. r , such as $u_p = C$, substituting it into the differential equation gives:

$$-\frac{in \rho}{\mu} u_p = -\frac{A}{\mu}$$

thus $u_p = \frac{A}{in \rho}$. The homogeneous equation is:

$$\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} + \frac{i^3 n}{\nu} u = 0$$

Take $\lambda^2 = \frac{i^3 n}{\nu}$, we now have:

$$\begin{aligned}\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} + \lambda^2 u &= 0 \\ r^2 \frac{d^2 u(r)}{dr^2} + r \frac{du(r)}{dr} + (\lambda r)^2 u &= 0 \quad (1)\end{aligned}$$

Take $x = \lambda r$, then:

$$\begin{aligned}\frac{du(x)}{dr} &= \frac{du(\lambda r)}{dr} = \lambda \frac{du(x)}{dx} \\ \frac{d^2 u(x)}{dr^2} &= \lambda^2 \frac{d^2 u(x)}{dx^2}\end{aligned}$$

Substitute back into (1), we have

$$\begin{aligned}\lambda^2 r^2 \frac{d^2 u(x)}{dx^2} + \lambda r \frac{du(x)}{dx} + (\lambda r)^2 u(x) &= 0 \\ x^2 \frac{d^2 u(x)}{dx^2} + x \frac{du(x)}{dx} + x^2 u &= 0\end{aligned}$$

The last equation is a Bessel's equation of order 0, therefore the solution, u_h , of the homogeneous equation is a solution of a Bessel's equation of order 0:

$$u_h(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$$

And

$$u(r) = u_h(r) + u_p(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r) + \frac{A}{in\rho}$$

Now we apply the boundary conditions to our solution.

$$u'(r) = C_1 J'_0(\lambda r) + C_2 Y'_0(\lambda r)$$

We have

$$\begin{aligned}J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! \Gamma(1+n)} \\ &= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \dots \\ J'_0(x) &= -2 \frac{x}{2^2} + 4 \frac{x^3}{2^2 4^2} - \dots \\ J'_0(0) &= 0 \\ \lim_{r \rightarrow 0} u'(r) &= \lim_{r \rightarrow 0} C_1 J'_0(\lambda r) + C_2 Y'_0(\lambda r) \\ &= 0 + \lim_{r \rightarrow 0} C_2 Y'_0(\lambda r)\end{aligned}$$

Looking at the plot of $Y_0(x)$, we see that in order to have $\frac{\partial w}{\partial r}|_{r=0} = 0$ or $\frac{\partial u}{\partial r}|_{r=0} = 0$, the term in Y_0 must be discarded and we need $C_2 = 0$. Thus

$$u(r) = C_1 J_0(\lambda r) + \frac{A}{i n \rho}$$

Using the second boundary condition $w(R) = u(R) = 0$ we have $C_1 J_0(\lambda R) + \frac{A}{i n \rho} = 0$ or $C_1 = -\frac{A}{i n \rho J_0(\lambda R)}$ Putting everything back

$$\begin{aligned} u(r) &= \frac{A}{\rho i n} \left[1 - \frac{J_0(\lambda r)}{J_0(\lambda R)} \right] \\ &= \frac{A}{\rho i n} \left[1 - \frac{J_0(r \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}})}{J_0(R \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}})} \right] \end{aligned}$$

Take $\alpha = R \sqrt{\frac{\lambda}{\nu}}$ and $y = \frac{r}{R}$ then

$$\begin{aligned} J_0(r \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}}) &= J_0\left(\frac{r}{R} R \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}}\right) = J_0(\alpha y i^{\frac{3}{2}}) \\ J_0(R \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}}) &= J_0(\alpha i^{\frac{3}{2}}) \end{aligned}$$

Lastly

$$w(y, t) = u(r) e^{int} = \frac{A}{\rho i n} \left[1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})} \right] e^{int}$$

Question 2

From

$$Q = 2\pi \int_0^R w(r, t) r dr$$

Make the change of variable $y = \frac{r}{R}$, $dy = \frac{dr}{R}$ and we have

$$Q = 2\pi \int_0^1 w(y, t) R^2 y dy = 2\pi R^2 \int_0^1 w y dy$$

Plugging the expression of w found in the previous question

$$\begin{aligned} Q &= 2\pi R^2 \frac{A}{\rho i n} \int_0^1 \left[1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})} \right] e^{int} y dy \\ &= \frac{2\pi R^2 A}{\rho i n} e^{int} \left[\int_0^1 y dy - \frac{1}{J_0(\alpha i^{\frac{3}{2}})} \int_0^1 y J_0(\alpha y i^{\frac{3}{2}}) dy \right] \end{aligned}$$

$\int_0^1 y dy = [\frac{y^2}{2}]_0^1 = \frac{1}{2}$ and we make the change of variable $s = \alpha i^{\frac{3}{2}} y, ds = \alpha i^{\frac{3}{2}} dy$
so

$$\begin{aligned} \int_0^1 y J_0(\alpha y i^{\frac{3}{2}}) dy &= \int_0^{\alpha i^{\frac{3}{2}}} \frac{s}{\alpha i^{\frac{3}{2}}} J_0(s) \frac{1}{\alpha i^{\frac{3}{2}}} ds \\ &= \frac{1}{\alpha^2 i^3} \int_0^{\alpha i^{\frac{3}{2}}} s J_0(s) ds \\ &= \frac{\alpha i^{\frac{3}{2}}}{\alpha^2 i^3} J_1(\alpha i^{\frac{3}{2}}) \end{aligned}$$

Therefore

$$\begin{aligned} Q &= \frac{2\pi R^2 A}{\rho i n} e^{int} \left[\frac{1}{2} - \frac{\alpha i^{\frac{3}{2}}}{\alpha^2 i^3} \frac{J_1(\alpha i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})} \right] \\ &= \frac{\pi R^2 A}{\rho i n} \left[1 - \frac{2\alpha i^{\frac{3}{2}}}{i^3 \alpha^2} \frac{J_1(\alpha i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})} \right] e^{int} \end{aligned}$$

Question 3

Start with the equation for $w(y, t)$ established in question 1:

$$w(y, t) = \frac{A}{\rho i n} \left[1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})} \right] e^{int}$$

Substituting into the previous equation n with $\alpha = R\sqrt{\frac{n}{\nu}}, n = \nu \frac{\alpha^2}{R^2}$

$$w(y, t) = \frac{A R^2}{i \rho \nu} \left[\frac{J_0(\alpha i^{\frac{3}{2}}) - J_0(\alpha y i^{\frac{3}{2}})}{\alpha^2 J_0(\alpha i^{\frac{3}{2}})} \right] e^{i \frac{\nu t}{R^2} \alpha^2}$$

Let $B = \alpha y i^{\frac{3}{2}}, C = \alpha i^{\frac{3}{2}}$ and $D = i \frac{\nu t}{R^2} \alpha^2$, rewrite the previous equation

$$w(y, t) = \frac{A R^2}{i \rho \nu} \left[\frac{J_0(C) - J_0(B)}{\alpha^2 J_0(C)} \right] e^D$$

When $n \rightarrow 0, \alpha \rightarrow 0$ and we have the indeterminate form for $w(y, t) = \frac{A R^2}{i \rho \nu} \left(\frac{1-n}{0.1} \right)$. $1 = \frac{0}{0}$. Therefore we apply L'Hospital's rule, compute the derivatives of numerator and denominator and taking the limit $\alpha \rightarrow 0$:

$$\begin{aligned} \frac{d}{d\alpha}(J_0(C) - J_0(B))e^D &= \frac{d}{d\alpha}(J_0(C) - J_0(B)) e^D + (J_0(C) - J_0(B)) \frac{d}{d\alpha}e^D \\ \frac{d}{d\alpha}(J_0(C) - J_0(B)) &= -i^{\frac{3}{2}} J_1(C) + i^{\frac{3}{2}} y J_1(B) \\ &= i^{\frac{3}{2}} (y J_1(B) - J_1(C)) e^D \\ \frac{d}{d\alpha}e^D &= \frac{2i\nu t}{R^2} \alpha e^D \end{aligned}$$

So

$$\begin{aligned} \frac{d}{d\alpha}(J_0(C) - J_0(B))e^D &= \left(i^{\frac{3}{2}} (y J_1(B) - J_1(C)) + (J_0(C) - J_0(B)) \frac{2i\nu t}{R^2} \alpha \right) e^D \\ \frac{d}{d\alpha} \alpha^2 J_0(C) &= 2\alpha J_0(C) + \alpha^2 i^{\frac{3}{2}} (-J_1(C)) \\ &= \alpha (2J_0(C) - i^{\frac{3}{2}} \alpha J_1(C)) \end{aligned}$$

And

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{d}{d\alpha}(J_0(C) - J_0(B))e^D &= \left(i^{\frac{3}{2}} (y \cdot 0 - 0) + (0 - 0) \frac{2i\nu t}{R^2} \cdot 0 \right) 1 = 0 \\ \lim_{\alpha \rightarrow 0} \frac{d}{d\alpha} \alpha^2 J_0(C) &= 0 \cdot (2 \cdot 1 - i^{\frac{3}{2}} \cdot 0 \cdot 0) = 0 \end{aligned}$$

We still have the indeterminate form $\frac{0}{0}$, so we apply one more time L'Hospital's

rule

$$\begin{aligned}
\frac{d^2}{d\alpha^2}(J_0(C) - J_0(B))e^D &= \left(i^{\frac{3}{2}}(y \frac{d}{d\alpha} J_1(B) - \frac{d}{d\alpha} J_1(C)) + (\frac{d}{d\alpha} J_0(C) - \frac{d}{d\alpha} J_0(B)) \frac{2i\nu t}{R^2} \alpha \right. \\
&\quad \left. + (J_0(C) - J_0(B)) \frac{2i\nu t}{R^2} \right) e^D + \\
&\quad \left(i^{\frac{3}{2}}(y J_1(B) - J_1(C)) + (J_0(C) - J_0(B)) \frac{2i\nu t}{R^2} \alpha \right) \frac{2i\nu t}{R^2} \alpha e^D \\
&= \left(i^{\frac{3}{2}}(i^{\frac{3}{2}} y^2 \frac{J_0(B) - J_2(B)}{2} - i^{\frac{3}{2}} \frac{J_0(C) - J_2(C)}{2}) + \right. \\
&\quad \left. (i^{\frac{3}{2}} y J_1(B) - i^{\frac{3}{2}} J_1(C)) \frac{2i\nu t}{R^2} \alpha + (J_0(C) - J_0(B)) \frac{2i\nu t}{R^2} \right) e^D + \\
&\quad \left(i^{\frac{3}{2}}(y J_1(B) - J_1(C)) + (J_0(C) - J_0(B)) \frac{2i\nu t}{R^2} \alpha \right) \frac{2i\nu t}{R^2} \alpha e^D \\
&= \left(\frac{i^3 y^2}{2} (J_0(B) - J_2(B)) - \frac{i^3}{2} (J_0(C) - J_2(C)) + \right. \\
&\quad \left. (i^{\frac{3}{2}} y J_1(B) - i^{\frac{3}{2}} J_1(C)) \frac{2i\nu t}{R^2} \alpha + (J_0(C) - J_0(B)) \frac{2i\nu t}{R^2} \right) e^D + \\
&\quad \left(i^{\frac{3}{2}}(y J_1(B) - J_1(C)) + (J_0(C) - J_0(B)) \frac{2i\nu t}{R^2} \alpha \right) \frac{2i\nu t}{R^2} \alpha e^D \\
\frac{d^2}{d\alpha^2} \alpha^2 J_0(C) &= 2J_0(C) - i^{\frac{3}{2}} \alpha J_1(C) + \alpha (2 \frac{d}{d\alpha} J_0(C) \\
&\quad - i^{\frac{3}{2}} J_1(C) - i^{\frac{3}{2}} \alpha \frac{d}{d\alpha} J_1(C)) \\
&= 2J_0(C) - i^{\frac{3}{2}} \alpha J_1(C) + \alpha (2i^{\frac{3}{2}} (-J_1(C)) \\
&\quad - i^{\frac{3}{2}} J_1(C) - i^3 \alpha \frac{J_0(C) - J_2(C)}{2}) \\
&= 2J_0(C) - i^{\frac{3}{2}} J_1(C) - 3i^{\frac{3}{2}} J_1(C) \alpha - \frac{i^3}{2} (J_0(C) - J_2(C)) \alpha
\end{aligned}$$

Next we take the second derivative of the numerator of $w(y, t)$ when $\alpha \rightarrow 0$ using

the expression above:

$$\begin{aligned}
\lim_{\alpha \rightarrow 0} \frac{d^2}{d\alpha^2} (J_0(C) - J_0(B)) e^D &= \left(\frac{i^3 y^2}{2} (1 - 0) - \frac{i^3}{2} (1 - 0) + \right. \\
&\quad \left. (i^{\frac{3}{2}} y \cdot 0 - i^{\frac{3}{2}} \cdot 0) \frac{2i\nu t}{R^2} \cdot 0 + (1 - 1) \frac{2i\nu t}{R^2} \right) \cdot 1 + \\
&\quad \left(i^{\frac{3}{2}} (y \cdot 0 - 0) + (1 - 1) \frac{2i\nu t}{R^2} \cdot 0 \right) \frac{2i\nu t}{R^2} \cdot 0 \cdot 1 \\
&= \frac{i^3}{2} (y^2 - 1) = \frac{i}{2} (1 - y^2)
\end{aligned}$$

Similarly for the limit of $\alpha \rightarrow 0$ of the second derivative of the denominator of $w(y, t)$, we get

$$\begin{aligned}
\lim_{\alpha \rightarrow 0} \frac{d^2}{d\alpha^2} \alpha^2 J_0(C) &= 2 \cdot 1 - i^{\frac{3}{2}} \cdot 0 - 3i^{\frac{3}{2}} \cdot 0 \cdot 0 - \frac{i^3}{2} (1 - 0) \cdot 0 \\
&= 2
\end{aligned}$$

Therefore for a constant input pressure, we have

$$w = \frac{A R^2 i}{i \rho \nu} \frac{1}{4} (1 - y^2) = \frac{A}{4\mu} R^2 (1 - y^2)$$

which is equation 2 in Womersley's paper with $A = \frac{p_1 - p_2}{l}$

Using the expression of the differential equation established in question (1) and with $n = 0$, we want to solve

$$r^2 \frac{d^2 u(r)}{dr^2} + r \frac{du(r)}{dr} = -\frac{A}{\mu} r^2$$

This is an Euler equation or a Legendre ordinary differential equation with $\alpha = 1, \beta = 0$, so we make the change of variable $e^t = r$ or $\ln r = t$. Then $r \frac{du}{dr} = \frac{du}{dt}$ and $r^2 \frac{d^2 u}{dr^2} = \frac{d^2 u}{dt^2} - \frac{du}{dt}$.

which yields for the ODE

$$\begin{aligned}
\frac{d^2 u}{dt^2} - \frac{du}{dt} + \frac{du}{dt} &= -\frac{A}{\mu} e^{2t} \\
\frac{d^2 u}{dt^2} &= -\frac{A}{\mu} e^{2t}
\end{aligned}$$

For the homogeneous equation integrating twice gives $u(t) = C_1 t + C_2$ or $u(r) = C_1 \ln(r) + C_2$. Take for one particular solution of the ODE: $u_p(t) = C_3 e^{2t}$, $u'_p(t) = 2C_3 e^{2t}$, $u''_p(t) = 4C_3 e^{2t}$, substitute into the ODE gives $4C_3 e^{2t} = -\frac{A}{\mu} e^{2t}$ or $C_3 = -\frac{A}{4\mu}$ thus $u_p(t) = -\frac{A}{4\mu} e^{2t}$ or $u_p(r) = -\frac{A}{4\mu} r^2$. Therefore the total solution is

$$u(r) = -\frac{A}{4\mu} r^2 + C_1 \ln(r) + C_2$$

From this, we write $u'(r) = -\frac{A}{2\mu} r + \frac{C_1}{r}$. So to have the boundary condition $\frac{\partial w}{\partial r}|_{r=0} = 0$ or $\frac{\partial u}{\partial r}|_{r=0} = 0$, C_1 has to be zero so $u'(0)$ is finite and equal to 0. The second boundary condition $w(R) = 0$, or $u(R) = 0$, gives $C_2 = \frac{A}{4\mu} R^2$. Finally

$$u(r) = -\frac{A}{4\mu} (r^2 - R^2) = \frac{A}{4\mu} R^2 (1 - (\frac{r}{R})^2) = \frac{A}{4\mu} R^2 (1 - y^2)$$

which is equation (2) in Womersley's paper with $n = 0$, $A = \frac{p_1 - p_2}{l}$ and which is the expression of w we just found in question 3.

Question 4

For Poiseuille's flow

$$w = \frac{p_1 - p_2}{4\mu l} R^2 (1 - y^2)$$

And

$$Q = 2\pi \int_0^R w(r, t) r dr$$

Make the change of variable $y = \frac{r}{R}$, $dy = \frac{dr}{R}$ and we have

$$\begin{aligned} Q &= 2\pi \int_0^1 \frac{p_1 - p_2}{4\mu l} R^2 (1 - y^2) R y R dy \\ &= 2\pi \frac{p_1 - p_2}{4\mu l} R^4 \int_0^1 (1 - y^2) y dy \\ &= 2\pi \frac{p_1 - p_2}{4\mu l} R^4 \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 \\ &= 2\pi \frac{p_1 - p_2}{4\mu l} R^4 \frac{1}{4} \\ &= \frac{p_1 - p_2}{8\mu l} \pi R^4 \end{aligned}$$

= equation 20 in Womersley's paper.