1 Legendre functions

The Legendre polynomials are defined over the interval [-1,1] with a weighting $\rho(x) = 1$. The original D.E. defining Legendre polynomials:

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

$$\int_{-1}^{1} P_l(x)P_k(x) = \begin{cases} 0 & \text{if } l \neq k \\ \frac{2}{2l+1} & \text{if } l = k \end{cases}$$

Some identities:

1. $P_n(x) = P'_{n+1}(x) + P'_{n-1}(x) - 2xP'_n(x)$
2. $xP_n(x) - P'_{n-1}(x) = nP_n(x)$
3. $(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0$

Generating function:

$$G(x,h) = \frac{1}{(1+h^2-2h\,x)^{\frac{1}{2}}} = P_0(x) + P_1(x)h + P_2(x)h^2 + \cdots = \sum_{n=0}^{\infty} P_n(x)h^n$$

2 Bessel Functions

Bessel's differential equation:

$$x^2y'' + xy' = (x^2 - \nu^2)y = 0$$

Bessel functions of the first kind

1. case 1 for $\pm\nu$ non integer (and $\nu \neq m/2$, m integer)

$$J_\nu = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{2n+\nu} \quad J_{-\nu} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-\nu+n+1)} \left(\frac{x}{2}\right)^{2n-\nu}$$

2. case 2 for integer values, that is $\sigma = \pm\nu = \pm m$

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m} n!\Gamma(n+m+1)} x^{2n+m}$$

In particular for $m = 0$, $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!\Gamma(n+1)} x^{2n}$, also $J_m(x) = (-1)^m J_m(x)$.

3. case 3 $\sigma = \nu = \frac{m}{2}, m = 1, 3, \cdots$, for $\nu = \frac{1}{2}$, $j_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi\,x}} \sin x,\ j_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi\,x}} \cos x$

Some identities:

- $\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x)$
- $\int x^\nu J_{\nu-1}(x)d\nu = x^\nu J_\nu(x)$
- $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$

3 Gamma Function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx, n > 0, \text{ real}$$

4 Partial Differential Equations

- Wave equation $\frac{\partial u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t^2}$
- Diffusion equation $k \frac{\partial u}{\partial x^2} + f(x,t) = \sigma \rho \frac{\partial u}{\partial t}$ or $k \nabla u(x,y,t) = \frac{\partial u(x,y,t)}{\partial t}$ If not time dependence $\frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 0$
- Laplace equation in two dimensions $\frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 0$