Interactive Assignment 6

Problems

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Chapter 7 - Problem 7.7

Use the derivative method to obtain, as a second (independent) solution of Bessel's equation for the case when $\nu = 0$, the following expression:

$$J_0(z)\ln z - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sum_{r=1}^{n} \frac{1}{r}\right) \left(\frac{z}{2}\right)^{2n}$$

given that the first solution is $J_0(z)$, as specified by 9.76

The recurrence relation (9.75) leads to coefficient a_n:

$$a_n = \frac{(-1)}{n(n+2\sigma)} a_{n-2}$$

For $n=2$ $a_2 = \frac{(-1)}{2(2+2\sigma)} a_0$ $\quad n=4$ $a_4 = \frac{-1}{4(4+2\sigma)} a_2 = \frac{(-1)^2}{4\cdot 2(4+2\sigma)(2+2\sigma)} a_0$

$n=6$ $a_6 = \frac{-1}{6(6+2\sigma)} a_4 = \dots = \frac{(-1)^3}{6\cdot 4\cdot 2\,(6+2\sigma)(4+2\sigma)(2+2\sigma)} a_0$

In general $a_{2n} = \frac{(-1)^n}{2^n n!\,(2+2\sigma)(4+2\sigma)(6+2\sigma)\cdots(2n+2\sigma)} a_0$

Thus $y_1(x,\sigma) = x^\sigma \sum_{n=0}^{\infty} a_{2n} x^{2n} = x^\sigma a_0 x^0 + x^\sigma \sum_{n=1}^{\infty} a_{2n} x^{2n}$

Using the derivative method, the second solution is:

$$y_2(x) = \left[\frac{\partial}{\partial\sigma} y_1(x,\sigma)\right]_{\sigma=0} \quad \text{for double indicial roots}$$

$$= \left[\frac{\partial}{\partial\sigma}\left(x^\sigma a_0 x^0 + x^\sigma \sum_{n=1}^{\infty} a_{2n} x^{2n}\right)\right]_{\sigma=0}$$

$$= \left[\frac{\partial}{\partial\sigma}\left(x^\sigma a_0 + x^\sigma \sum_{n=1}^{\infty} a_{2n} x^{2n}\right)\right]_{\sigma=0}$$

$$= \left[\frac{\partial}{\partial\sigma} x^\sigma a_0 + \left(\frac{\partial}{\partial\sigma} x^\sigma\right)\sum_{n=1}^{\infty} a_{2n} x^{2n} + x^\sigma \sum \frac{\partial}{\partial\sigma} a_{2n} x^{2n}\right]$$

Note that $\frac{\partial}{\partial\sigma} x^\sigma = \frac{\partial}{\partial\sigma} e^{\sigma \ln x} = \ln x \; e^{\sigma \ln x} = (\ln x)\, x^\sigma$

And $\frac{\partial}{\partial\sigma} a_{2n} = \frac{\partial}{\partial\sigma}\left[\frac{(-1)^n}{2^n n!\,(2+2\sigma)(4+2\sigma)\cdots(2n+2\sigma)} a_0\right]$

Take the case for example $n=2$

$$\frac{\partial}{\partial\sigma} \frac{1}{(4+2\sigma)(2+2\sigma)} = \frac{(-1)(2)}{(2+2\sigma)(4+2\sigma)}\left[\frac{1}{2+2\sigma} + \frac{1}{4+2\sigma}\right]$$

Therefore $\frac{\partial}{\partial\sigma} a_{2n} = \frac{(-1)\,(-1)^n (2)}{2^n n!\,(2+2\sigma)(4+2\sigma)\cdots(2n+2\sigma)}\left[\frac{1}{2+2\sigma} + \frac{1}{4+2\sigma} \cdots + \frac{1}{2n+2\sigma}\right]$

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Substitution back into $y_2(x)$ gives:

$$y_2(x) = \Big[(\ln x)\, x^{\sigma} + (\ln x)\, x^{\sigma} \sum_{n=1}^{\infty} a_{2n}\, x^{2n}$$

$$+ x^{\sigma} \sum_{n=1}^{\infty} \frac{(-1)\,(-1)^n\,(2)}{2^n n!\,(2+2\sigma)(4+2\sigma)\cdots(2n+2\sigma)}\, a_0 \left[\frac{1}{2+2\sigma} + \frac{1}{4+2\sigma} + \cdots + \frac{1}{2n+2\sigma}\right] x^{2n}\Big]_{\sigma=0}$$

$$= \Big[(\ln x)\, x^{\sigma}\left(1 + \sum_{n=1}^{\infty} a_{2n}\, x^{2n}\right)$$

$$+ x^{\sigma} \sum_{n=1}^{\infty} \frac{(-1)\,(-1)^n\,(2)}{2^n n!\,(2+2\sigma)(4+2\sigma)\cdots(2n+2\sigma)}\, a_0 \left[\frac{1}{2+2\sigma} + \frac{1}{4+2\sigma} + \cdots + \frac{1}{2n+2\sigma}\right] x^{2n}\Big]_{\sigma=0}$$

Note that with $s=0$

$$y_1(x,0) = x^0 \sum_{n=0}^{\infty} a_{2n}\, x^{2n} = a_0 + \sum_{n=1}^{\infty} a_{2n}\, x^{2n}$$

And from the expression of a_{2n}: $a_{2n}(\sigma) = \dfrac{-1}{n(n+2s)}$

$$\text{and } a_0 = \frac{1}{2^{\sigma}\,\Gamma(\sigma+1)}$$

$$\Rightarrow a_0(0) = 1$$

Thus $1 + \sum_{n=1}^{\infty} a_{2n}(0)\, x^{2n} = y_1(x)$.

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Coming back to $y_2(x)$:

$$y_2(x) = (\ln x)\, x^0 y_1(x) + \left[x^s \sum_{n=1}^{\infty} \frac{a_0}{2^n n! (2+2s)(4+2s)\cdots(2n+2s)} \left[\frac{1}{2+2s} + \frac{1}{4+2s} + \cdots + \frac{1}{2n+2s} \right] x^{2n} \right]_{s=0}$$

$x^0 = 1$ and $y_1(x) = J_0(x)$ gives by substitution with $s=0$

$$y_2(x) = (\ln x) J_0(x) + x^0 \sum_{n=1}^{\infty} \frac{(-1)(-1)^n (2)}{2^n n! (2)(4)\cdots(2n)} \left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} \right] x^{2n}$$

$$= (\ln x) J_0(x) - \sum_{n=1}^{\infty} \frac{(-1)^n (2)}{2^n n! (2)(4)\cdots(2n)} \left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} \right] x^{2n}$$

The term in bracket could be expressed as:

$$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} = \frac{1}{2}\left[1 + \frac{1}{2} + \cdots + \frac{1}{n}\right] = 1/2 \sum_{r=1}^{n} 1/r$$

leading to:

$$y_2(x) = (\ln x) J_0(x) - \sum_{n=1}^{\infty} \frac{(-1)^n (2)}{2^n n! (2.1)\cdots(2.n)} \frac{1}{2} \cdot \left(\sum_{r=1}^{n} 1/r \right) x^{2n}$$

$$= (\ln x) J_0(x) - \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n} (n!)^2} \left(\sum_{r=1}^{n} 1/r \right) x^{2n} =$$

$$= (\ln x) J_0(x) - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sum_{r=1}^{n} 1/r \right) \left(\frac{x}{2} \right)^{2n}$$

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Consider a series solution of the equation:

$$z y'' - 2y' + yz = 0 \qquad (*)$$

about its regular singular point.

(a) Show that its indicial equation has roots that differ by an integer but that the two roots nevertheless generate linearly independent solutions

$$y_1(z) = 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n\, z^{2n+1}}{(2n+1)!}$$

$$y_2(z) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n-1)\, z^{2n}}{2n!}$$

Putting $(*)$ in its standard form:

$$y'' - \frac{2}{z} y' + y = 0$$

$z\,p(z) = -2$ and $z^2 q(z) = z^2$ are finite at the singular point $z=0$, it is a regular singular points and we expect to find a solution of the form: $y = z^{\sigma} \sum_{n=0}^{\infty} a_n z^n$

Substituting it in the previous equation, gives:

$$\sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-1)\, a_n z^{n+\sigma-2} - \frac{2}{z} \sum_{n=0}^{\infty} (n+\sigma)\, a_n z^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0$$

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Dividing through by $z^{\sigma-2}$, gives:

$$\sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-1)\, a_n z^n - 2\sum_{n=0}^{\infty}(n+\sigma)\, a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+2} = 0$$

Collecting coefficients of same power of z together and simplifying yields:

$$\sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-1-2)\, a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-3)\, a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+2} = 0$$

Reindexing the second sum: $\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-3)a_n z^n + \sum_{n=2}^{\infty} a_{n-2} z^n = 0$

$$\sigma(\sigma-3)\,a_0 + (\sigma+1)(\sigma+1-3)\,a_1 z + \sum_{n=2}^{\infty}\left[(n+\sigma)(n+\sigma-3)a_n + a_{n-2}\right]z^n = 0$$

If we set $z=0$ the all terms in the sum and in coefficient a_1 disappear, and we obtain the indicial equation:

$$\sigma(\sigma-3)=0$$

which has roots $\sigma=0$ and $\sigma=3$ and the roots differ by an integer 3.

If we are looking for a solution for $z \neq 0$ then, we obtain the recurrence relation:

$$(n+\sigma)(n+\sigma-3)\,a_n + a_{n-2} = 0$$

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We choose the largest root $s = 3$, the recurrence relation is:

$$(n+3)\, n\, a_n = -a_{n-2}$$

$$a_n = -\frac{a_{n-2}}{n(n+3)}$$

Setting $a_1 = 0$ and considering only even coefficients since odd coefficients vanish, one solution is:

$$y_1(z) = z^3 \sum_{n=0}^{\infty} a_n z^n \quad \text{for } n \text{ even}$$

$$= z^3 \left(a_0 - \frac{a_0}{2\times 5} z^2 + \frac{a_0}{2\times 4\times 5\times 7} z^4 + \cdots + \frac{(-1)^n}{2\times 4\times\, 2n(2n+3)} z^{2n} + \cdots\right.$$

$$= a_0 z^3 - \frac{a_0}{2\times 5} z^5 + \frac{a_0}{2\times 4\times 5\times 7} z^7 + \cdots + \frac{(-1)^n}{2\times 4\times \cdots \times 2n(2n+3} z^{2n+3} + \cdots$$

$$= \frac{3\times 2}{3!} a_0 z^3 - \frac{a_0 \times 3\times 4}{5!} z^5 + \frac{a_0 \times 3\times 6}{7!} z^7 + \cdots + \cdots$$

by reindexing with $n' = n+1$

$$= \frac{3\times 2}{3!} a_0 z^3 - \frac{a_0 \times 3 \times 2\times 2}{5!} \cdot z^{2\times 2+1} + \cdots + \frac{(-1)^{n+1+1}\, 3\times 2(n+1)}{2\times 4\times \cdots \times 2n\,(2(n+1)+1)!} z^{2(n+1)+1}$$

$$= 3a_0 \quad \frac{2}{3!} - \frac{3\times 2\times 2}{5!} z^{2\times 2+1} + \cdots + \frac{(-1)^{n'+1}\, 3\times 2\, n'}{(2n'+1)!} z^{2n'+1}$$

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Therefore

$$y_1(z) = 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2n\, z^{2n+1}}{(2n+1)!}$$

To obtain the second solution, we set $s=0$ in the recurrence relation, which gives:

$$n(n-3)\, a_n = -a_{n-2}$$

$$a_n = -\frac{a_{n-2}}{n(n-3)}$$

setting $a_1 = 0$ makes all the odd coefficients to vanish, we are left with the second solution $y_2(z) = z^0 \sum_{n=0}^{\infty} a_n z^n$ for n even

a_0 non zero, we have:

$$a_2 = -\frac{a_0}{2(-1)} = \frac{a_0}{2} = \frac{(-1)^2 (2\times1-1)}{(2\times1)!}$$

$$a_4 = -\frac{a_2}{4\times1} = -\frac{a_0}{4\times2} = \frac{(-1)^{2+1}\times 3}{4!}\, a_0$$

$$\vdots$$

$$a_{2n} = \frac{(-1)^{n+1}(2n-1)}{2n!}$$

therefore

$$y_2(z) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{2n!}\, z^{2n}$$

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(b) Show that $y_1(z)$ is equal to $3a_0(\sin z - z\cos z)$ by expanding the sinusoidal function. Then, using the Wronskian method, find an expression for $y_2(z)$ in terms of sinusoids.

We have $y_1(z) = 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n\, z^{2n+1}}{(2n+1)!}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n\, z^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n 2n\, z^{2n+1}}{(2n+1)!} = -\sum_{n=0} (-1)^n \left[\frac{1}{2n!} - \frac{1}{2n+1!}\right] z^{2n+1}$$

$$= -\left[z \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} z^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \right]$$

$$= -\left[z\cos z - \sin z \right] = \sin z - z\cos z$$

Thus: $y_1(z) = 3a_0(\sin z - z\cos z)$

The initial equation in standard form is:

$$y'' - \frac{2}{z} y' + y = 0 \quad \text{and } p(z) = -2/z$$

Using the Wronskian method

$$y_2(z) = y_1(z) \int^z \frac{e^{-\int^u p(v)\,dv}}{[y_1[u]]^2} du$$

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First : $e^{-\int^u p(v)\,dv} = e^{-\int^u (-2/v)\,dv} = e^{2\int^u \frac{dv}{v}}$

$$= e^{2\ln u} = e^{\ln u^2} = u^2$$

Substituting this onto the expression of $y_2(z)$ gives:

$$y_2(z) = y_1(z)\int^z \frac{u^2}{[y_1(u)]^2}\,du = 3a_0(\sin z - z\cos z)\int^z \frac{u^2}{[3a_0(\sin u - u\cos u)]^2}\,du$$

$$= \frac{3a_0}{9a_0^2}(\sin z - z\cos z)\int^z \frac{u^2}{(\sin u - u\cos u)^2}\,du$$

$$\int^z \frac{u^2}{(\sin u - u\cos u)^2}\,du = \int^z \frac{u}{\sin u}\;\frac{u\sin u}{(\sin u - u\cos u)^2}\,du$$

Note that $\frac{d}{du}\sin u - u\cos u = \cos u - \cos u + u\sin u = u\sin u$

Integration by parts gives:

$$\int^z \frac{u^2}{(\sin u - u\cos u)^2}\,du = \left[\frac{u}{\sin u}\left(-\frac{1}{\sin u - u\cos u}\right)\right]^z$$

$$+\int \frac{1}{\sin u - u\cos u}\;\frac{\sin u - u\cos u}{(\sin u)^2}\,du$$

$$= -\frac{z}{\sin z(\sin z - z\cos z)} + \int \frac{du}{(\sin u)^2}$$

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Therefore

$$\int^{z} \frac{a^2}{(\sin u - u\cos u)^2}\, du = -\frac{z}{\sin z\,(\sin z - z\cos z)} - \frac{\cos z}{\sin z}$$

$$= -\frac{\cos z \sin z - z\cos z^2 + z}{\sin z\,(\sin z - z\cos z)}$$

$$= -\frac{\cos z \sin z - z\cos^2 z + z\cos^2 z + z\sin^2 z}{\sin z\,(\sin z - z\cos z)}$$

$$= -\frac{\cos z \sin z + z\sin^2 z}{\sin z\,(\sin z - z\cos z)} = -\frac{\cos z + z\sin z}{\sin z - z\cos z}$$

$$= \frac{\cos z + z\sin z}{z\cos z - \sin z}$$

And $y_2(z) = \dfrac{1}{3a_0}(\sin z - z\cos z)\dfrac{\cos z + z\sin z}{z\cos z - \sin z}$

$= A.(\cos z + z\sin z)$ A = constant $= -1/3a_0$

c) Confirm that the two solutions are linearly independent by showing that their Wronskian is equal to $-z^2$,

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The Wronskian is $W = y_1 y_2' - y_2 y_1'$

$$y_1(z) = 3a_0(\sin z - z\cos z)$$

$$y_1'(z) = 3a_0 \cancel{\sin}\, z\sin z$$

$$y_2(z) = -\frac{1}{3a_0}(\cos z + z\sin z) \quad \text{then} \quad y_2'(z) = -\frac{1}{3a_0} z\cos z$$

Therefore
$$W = 3a_0\left(-\frac{1}{3a_0}\right)(\sin z - z\cos z)\, z\cos z$$
$$- \left(-\frac{1}{3a_0}\right)(\cos z + z\sin z)\, 3a_0\, z\sin z$$
$$= -\cancel{3}\, z\cos z(\sin z - z\cos z) + z\sin z(z\sin z + \cos z)$$
$$= z^2$$

For $z \neq 0$, $W \neq 0$ and the solutions $y_1(z)$ and $y_2(z)$ are linearly independent.

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Chapter 7 - Problem 7.12

Prove that the Laguerre equation,

$$z \frac{d^2y}{dz^2} + (1-z)\frac{dy}{dz} + \lambda y = 0$$

has polynomial solutions $L_N(z)$ if λ is a non-negative integer N, and determine the recurrence relationship for the polynomial coefficients. Hence show that an expression for $L_N(z)$, normalized in such a way that $L_N(0) = N!$, is

$$L_N(z) = \sum_{n=0}^{N} \frac{(-1)^n (N!)^2}{(N-n)!\,(n!)^2} z^n$$

Evaluate $L_3(z)$ explicitly.

Dividing through by z to put the equation in normal form, we obtain

$$y'' + \frac{1-z}{z} y' + \lambda \frac{y}{z} = 0 \qquad (1)$$

With the standard form, we identify $p(z) = 1-z/z$ and $q(z) = \lambda/z$

$z=0$ is a singular point, but since $zp(z) = 1-z$ and $z^2 q(z) = \lambda z$ are finite for $z=0$, it is a regular singular point of the equation above.

We substitute the Frobenius series $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$ into equation (1), and we obtain:

$$\sum_{n=0}^{\infty} (n+\sigma)(n+\sigma-1) a_n z^{n+\sigma-2} + (1-z) \sum_{n=0}^{\infty} (n+\sigma) a_n z^{n+\sigma-2} + \lambda \sum_{n=0}^{\infty} a_n z^{n+\sigma-1} = 0$$

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Dividing through by $z^{\sigma-2}$ gives:

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^n+(1-z)\sum_{n=0}^{\infty}(n+\sigma)a_n z^n+\lambda\sum_{n=0}^{\infty}a_n z^{n+1}=0$$

collating coefficients in same power of z:

$$\sum_{n=0}^{\infty}\left[(n+\sigma)(n+\sigma-1)+(n+\sigma)\right]a_n z^n$$

$$+\sum_{n=0}^{\infty}\left[-(n+\sigma)+\lambda\right]a_n z^{n+1}=0$$

Reindexing the second summation in z^n:

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1+1)a_n z^n+\sum_{n=1}^{\infty}\left[-(n-1+\sigma)+\lambda\right]a_{n-1}z^n=0$$

Simplifying the first summation yields:

$$\sum_{n=0}^{\infty}(n+\sigma)^2 a_n z^n+\sum_{n=1}^{\infty}(-(n-1+\sigma)+\lambda)a_{n-1}z^n=0$$

or $$\sigma^2 a_0 z^0+\sum_{n=1}^{\infty}\left[(n+\sigma)^2 a_n+(-(n-1+\sigma)+\lambda)a_{n-1}\right]z^n=0$$

If we set $z=0$ then all the terms in the sum vanish, and we obtain the indicial equation: $\sigma^2 a_0=0$

If we discard the trivial solution $z=0$ then $a_0\neq 0$ since the recurrence relation shows that coefficients are dependent on a_0. This implies that $\sigma=0$ is a double root.

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Demanding that the coefficients of z^n vanish separately, we obtain the recurrence relation:

$$a_n = \frac{(n-1)+\sigma-\lambda}{(n+\sigma)^2}\, a_{n-1} \quad \text{for } n \geq 1$$

substitute $\sigma=0$ gives: $a_n = \dfrac{n-1-\lambda}{n^2}\, a_{n-1}$; $n \geq 1$

For a given $\lambda = N$, $a_{N+1} = \dfrac{N+1-1-N}{(N+1)^2}\, a_N = 0$, and all subsequent terms $n \geq N+1$ are zeros. Therefore the solution is a polynomial $f_N(z) = \sum_{n=0}^{N} a_n z^n$

Expanding each coefficient gives:

$$a_1 = \frac{1-1-N}{1^2}\, a_0 = \frac{(-1)^1 N}{1!^2}\, a_0$$

$$a_2 = \frac{1-N}{2^2}\, a_1 = \frac{(-1)^2 N(N-1)}{(2!)^2}\, a_0$$

$$\vdots$$

$$a_n = \frac{(-1)^n N(N-1)\dots(N-(n-1))}{(n!)^2}\, a_0$$

$$a_n = \frac{(-1)^n N(N-1)\dots(N-n+1)(N-n)\dots 1}{(N-n)!\,(n!)^2}\, a_0$$

$$a_n = \frac{(-1)^n N!}{(N-n)!\,(n!)^2}\, a_0$$

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We want $L_N(z)$ normalized such that $L_N(0) = N!$

$$L_N(0) = a_0 z^0 = a_0 \Rightarrow a_0 = N!$$

Plugging back in the expression of the coefficients a_n:

$$a_n = \frac{(-1)^n N!}{(N-n)!(n!)^2} \cdot N! = \frac{(-1)^n (N!)^2}{(N-n)!(n!)^2}$$

And therefore $L_N(z) = \displaystyle\sum_{n=0}^{N} \frac{(-1)^n (N!)^2}{(N-n)!(n!)^2} z^n$

Thus

$$L_3(z) = 3! + \frac{(-1)^1 (3!)^2}{(3-1)!(1!)^2} z + \frac{(-1)^2 (3!)^2}{(3-2)!(2!)^2} z^2 + \frac{(-1)^3 (3!)^2}{(3-3)!(3!)^2} z^3$$

$$= 3+2\times 1 + \frac{(-1)(3\times 2\times 1)^2}{2!\ 1!} z + \frac{(3\times 2\times 1)^2}{1!\ (2\times 1)^2} z^2 + \frac{(-1)(3\times 2\times 1)^2}{0! \times (3!)^2} z^3$$

$$= 6 - 18z + 9z^2 - z^3$$