

Question 1

- a. $f(x) = x$ is odd on $[-\pi, \pi]$ therefore its Fourier coefficients a_n are 0 and we need to find its b_n coefficients:

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \sin\left(\frac{2\pi nx}{2\pi}\right) dx \\ &= \frac{4}{2\pi} \int_0^{\pi} x \sin\left(\frac{2\pi nx}{2\pi}\right) dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx \end{aligned}$$

Using integration by parts:

$$\begin{aligned} \int_0^{\pi} x \sin(nx) dx &= \left[x \left(-\frac{\cos(nx)}{n} \right) \right]_0^{\pi} + \int_0^{\pi} 1 \cdot \frac{\cos(nx)}{n} dx \\ &= \left(-\frac{\pi}{n} \right) \cos(n\pi) + \frac{1}{n} [\sin(nx)]_0^{\pi} \\ &= \frac{(-1)^{n+1} \pi}{n} \end{aligned}$$

Thus $b_n = \frac{2}{\pi} \frac{(-1)^{n+1} \pi}{n} = \frac{2(-1)^{n+1}}{n}$ and the Fourier series of x , on $[-\pi, \pi]$, is:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

- b. If we integrate term by term the previous expression, the Fourier series of x over $[-\pi, \pi]$, we have:

$$\begin{aligned} \frac{x^2}{2} &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(-\frac{\cos(nx)}{n} \right) + c \quad c \text{ constant of integration} \\ x^2 &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + c \quad \text{with } 2c \rightarrow c \\ &= c + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) \end{aligned}$$

- c. $f(x) = x^2$ is an even function, by Fourier Series for even function over symmetric range, we have:

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{2\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad (1)$$

where

$$\begin{aligned} a_0 &= \frac{4}{2\pi} \int_0^\pi x^2 dx \\ &= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^\pi \\ &= \frac{2}{3} \pi^2 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{4}{2\pi} \int_0^\pi x^2 \cos\left(\frac{2\pi nx}{2\pi}\right) dx = \frac{2}{\pi} \int_0^\pi x^2 \cos(nx) dx \\ \int_0^\pi x^2 \cos(nx) dx &= \left[x^2 \frac{\sin(nx)}{n} \right]_0^\pi - \frac{2}{n} \int_0^\pi x \sin(nx) dx \\ &= 0 - \frac{2}{n} \frac{(-1)^{n+1} \pi}{n} \\ a_n &= \frac{2}{\pi} \frac{(-1)^n 2 \pi}{n^2} \\ &= (-1)^n \frac{4}{n^2} \end{aligned}$$

Substituting for a_n in (1):

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

- d. As required, taking $c = \frac{\pi^2}{3}$, since x is a piecewise smooth function on the specified range, the Fourier series of x^2 , using integration term by term or directly calculating, match, .

Question 2

Consider the differential equation:

$$z \frac{d^2 y}{dy^2} + y = 0$$

- a. We put the equation in standard form:

$$\frac{d^2 y}{dy^2} + \frac{1}{z} y = 0$$

$z p(z) = 0$ and $z^2 q(z) = z$ therefore 0 is a regular singular point.

- b. Take $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$ and the usual derivatives in the D.E. gives by substitution

$$\begin{aligned} z \sum_{n=0}^{\infty} (n + \sigma)(n + \sigma - 1) a_n z^{n+\sigma-2} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} &= 0 \\ \sum_{n=0}^{\infty} (n + \sigma)(n + \sigma - 1) a_n z^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} &= 0 \quad (1) \end{aligned}$$

Take the term with the lowest power of z , which is the first sum with $n = 0$, then since each power of z term must be equal to 0, we have

$$\sigma(\sigma - 1)a_0z^{\sigma-1} = 0$$

Since $a_0 \neq 0$ and $z^{\sigma-1} \neq 0$, therefore $\sigma = 0, 1$.

c. We go back to equation (1) and take $\sigma = 1$ yields

$$\sum_{n=0}^{\infty} n(n+1)a_nz^n + \sum_{n=0}^{\infty} a_nz^{n+1} = 0$$

Then reindex the second sum to get same power of z in both sums:

$$\sum_{n=0}^{\infty} n(n+1)a_nz^n + \sum_{n=1}^{\infty} a_{n-1}z^n = 0$$

Note, in first term $n = 0$ does not contribute so we can start index at $n = 1$ in the first sum, and combine both sums

$$\sum_{n=1}^{\infty} [n(n+1)a_n + a_{n-1}]z^n = 0$$

Since every power of z term must be 0 and $z^n \neq 0$, gives:

$$a_n = -\frac{1}{(n+1)n}a_{n-1}$$

Taking $a_0 = 1$

$$\begin{aligned} n=1 \quad a_1 &= -\frac{1}{2 \cdot 1}a_0 = -\frac{1}{2 \cdot 1} = \frac{(-1)^1}{2 \cdot 1} \\ n=2 \quad a_2 &= -\frac{1}{3 \cdot 2}a_1 = \frac{1}{3 \cdot 2 \cdot 2 \cdot 1} = \frac{(-1)^2}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} \\ n=3 \quad a_3 &= -\frac{1}{4 \cdot 3}a_2 = -\frac{1}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = \frac{(-1)^3}{(4 \cdot 3 \cdot 2 \cdot 1)(3 \cdot 2 \cdot 1)} \\ &\vdots \\ a_n &= -\frac{1}{(n+1)n}a_{n-1} = \cdots = \frac{(-1)^n}{((n+1)n \cdots 1)(n(n-1) \cdots 1)} = \frac{(-1)^n}{(n+1)!n!} \end{aligned}$$

Therefore one of the independent solution of the ODE is

$$y_1(z) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!n!} z^n$$

Question 3

a. We have

$$n = 0, M = 0, P_0(x) = \frac{(-1)^0(2 \cdot 0 - 2 \cdot 0)!}{2^0(0 - 0)!(0 - 2 \cdot 0)!} x^{0-2 \cdot 0} = 1$$

$$n = 1, M = \frac{1-1}{2} = 0, P_1(x) = \frac{(-1)^0(2 \cdot 1 - 2 \cdot 0)!}{2^1(1 - 0)!(1 - 2 \cdot 0)!} x^{1-2 \cdot 0} = \frac{1 \cdot 2}{2 \cdot 1! \cdot 1!} x^1 = x$$

$$n = 2, M = \frac{2}{2} = 1, P_2(x) = \frac{(-1)^0(2 \cdot 2 - 2 \cdot 0)!}{2^2(2 - 0)!(2 - 2 \cdot 0)!} x^{2-2 \cdot 0} + \frac{(-1)^1(2 \cdot 2 - 2 \cdot 1)!}{2^2(2 - 1)!(2 - 2 \cdot 1)!} x^{2-2 \cdot 1}$$

$$P_2(x) = \frac{4!}{2^2 \cdot 2! \cdot 2!} x^2 - \frac{(2 \cdot 2 - 2)!}{2^2 \cdot 1! \cdot 0!} x^0$$

$$P_2(x) = \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 2 \cdot 2} x^2 - \frac{2!}{4}$$

$$P_2(x) = \frac{3}{2} x^2 - \frac{1}{2} = \frac{1}{2} (3x^2 - 1)$$

b. From

$$a_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx = \frac{2n+1}{2} \int_{-1}^1 x P_n(x) dx$$

we have

$$\begin{aligned} n = 0, a_0 &= \frac{2 \cdot 0 + 1}{2} \int_{-1}^1 x P_0(x) dx \\ &= \frac{1}{2} \int_{-1}^1 x dx = \frac{1}{2} \left[\frac{x^2}{2} \right]_{-1}^1 = \frac{1}{4} [1^2 - (-1)^2] = 0 \end{aligned}$$

$$\begin{aligned} n = 1, a_1 &= \frac{2 \cdot 1 + 1}{2} \int_{-1}^1 x P_1(x) dx \\ &= \frac{3}{2} \int_{-1}^1 x^2 dx = \frac{3}{2} \left[\frac{x^3}{3} \right]_{-1}^1 = \frac{1}{2} [1^3 - (-1)^3] = \frac{1}{2} \cdot 2 = 1 \end{aligned}$$

$$\begin{aligned} n = 2, a_2 &= \frac{2 \cdot 2 + 1}{2} \int_{-1}^1 x P_2(x) dx \\ &= \frac{5}{2} \int_{-1}^1 x \left[\frac{1}{2} (3x^2 - 1) \right] dx = \frac{5}{4} \int_{-1}^1 (3x^3 - x) dx \\ &= 0 \quad \text{since the powers of } x \text{ in the integrand are odd} \end{aligned}$$

Therefore the Fourier-Legendre series of x is $x = 1 \cdot P_1(x)$, as required.

c. Using Rodrigues's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

we have

$$n = 0, \frac{d^0}{dx^0}[(x^2 - 1)^0] = (x^2 - 1)^0 = 1$$

$$P_0(x) = \frac{1}{2^0 0!} 1 = 1$$

$$n = 1, \frac{d}{dx}(x^2 - 1) = 2x$$

$$P_1(x) = \frac{1}{2^1 1!} 2x = x$$

$$n = 2, \frac{d^2}{dx^2}(x^2 - 1)^2 = \frac{d}{dx} \left[\frac{d}{dx}(x^2 - 1)^2 \right] = \frac{d}{dx} [4x(x^2 - 1)] = \frac{d}{dx} [4x^3 - 4x] = 12x^2 - 4$$

$$P_2(x) = \frac{1}{2^2 2!} (12x^2 - 4) = \frac{4}{4 \cdot 2} (3x^2 - 1) = \frac{1}{2} (3x^2 - 1)$$

d. Extra Credit

Take f an even function and consider odd coefficients, let $n = 2p + 1$. Since n is odd $M = \frac{2p+1-1}{2} = p$, and we have

$$\begin{aligned} a_{2p+1} &= \frac{2(2p+1)+1}{2} \int_{-1}^1 f(x) P_{2p+1}(x) dx \\ &= \frac{4p+3}{2} \int_{-1}^1 f(x) \sum_{m=0}^p (-1)^m \frac{(2(2p+1)-2m)!}{2^{2p+1}(2p+1-m)!(2p+1-2m)!} x^{2p+1-2m} dx \\ &= \frac{4p+3}{2} \sum_{m=0}^p (-1)^m \frac{(2(2p+1)-2m)!}{2^{2p+1}(2p+1-m)!(2p+1-2m)!} \int_{-1}^1 f(x) x^{2p+1-2m} dx \\ &= \frac{4p+3}{2} \sum_{m=0}^p (-1)^m \frac{(2(2p+1)-2m)!}{2^{2p+1}(2p+1-m)!(2p+1-2m)!} \int_{-1}^1 f(x) x^{2(p-m)+1} dx \end{aligned}$$

The integral term is 0 since the integration is over a symmetric range and $f(x)x^{2(p-m)+1}$ is odd. Therefore given an even function $f(x)$ all the odd coefficients for the Fourier-Legendre series are zero. Similarly given an odd function, the even coefficients are zero.

Question 4

a.

$$\frac{\partial u}{\partial x} + 4xu = 0$$

Integration factor is

$$e^{\int 4x dx} = e^{4 \int x dx} = e^{4 \frac{x^2}{2}} = e^{2x^2}$$

Multiply the partial differential equation by the I.F.:

$$e^{2x^2} \frac{\partial u}{\partial x} + 4xe^{2x^2} u = 0$$

$$\frac{\partial}{\partial x}(e^{2x^2} u) = 0$$

Now integrate both sides with respect to x

$$e^{2x^2} u = C \quad C:\text{constant}$$

$$u(x) = Ce^{-2x^2}$$

b.

$$y^2 u_x - x^2 u_y = 0$$

Let $u(x, y) = X(x)Y(y)$, substitution into the D.E. gives

$$y^2 X'Y - x^2 XY' = 0$$

$$y^2 \frac{X'Y}{XY} - x^2 \frac{XY'}{XY} = 0$$

$$y^2 \frac{X'}{X} - x^2 \frac{Y'}{Y} = 0$$

$$\frac{1}{x^2} \frac{X'}{X} = \frac{1}{y^2} \frac{Y'}{Y} = k$$

Integrating $\ln X = \frac{1}{3}kx^3 + \ln(C)$ and $\ln Y = \frac{1}{3}ky^3 + \ln(D)$, so

$$X = Ce^{\frac{1}{3}kx^3}, Y = De^{\frac{1}{3}ky^3}$$

Therefore (with $CD = A$) $u(x, y) = A e^{\frac{1}{3}k(x^3+y^3)}$

Question 5

We have the following problem

$$\Delta u = 0$$

$$u(x, 0) = 0$$

$$u(x, b) = 100x$$

$$u_x(0, y) = 0$$

$$u_x(a, y) = 0$$

Assume a solution of the form $u(x, y) = X(x)Y(y)$. Substitute this expression and divide through by XY , to get:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$-\frac{X''}{X} = \frac{Y''}{Y}$$

LHS is function of x only and RHS is a function of y only, thus we can write, with k constant

$$-\frac{X''}{X} = \frac{Y''}{Y} = k$$

We see immediately that $X(x) = A \cos(\sqrt{k}x) + B \sin(\sqrt{k}x)$. And $X'(x) = \sqrt{k} \left(B \cos(\sqrt{k}x) - A \sin(\sqrt{k}x) \right)$. Also $u_x(x, y) = X'(x)Y(y)$. Take the boundary condition $u_x(0, y) = X'(0)Y(y) = 0$, since in general $Y(y) \neq 0$ then $X'(0) = 0$. Plug it into $X'(x)$ gives $\sqrt{k} \left(B \cdot 1 - A \cdot 0 \right) = 0 \rightarrow B = 0$. Now $X(x) = A \cos(\sqrt{k}x)$ and $X'(x) = -A\sqrt{k} \sin(\sqrt{k}x)$. Next, with $u_x(a, y) = 0$ gives $X'(a)Y(y) = 0 \rightarrow X'(a) = 0$. $X'(a) = 0$ therefore $-A\sqrt{k} \sin(\sqrt{k}a) = 0 \rightarrow \sqrt{k}a = n\pi$. So $k_n = \frac{n^2\pi^2}{a^2}$ and

$$\begin{aligned} X_n(x) &= A_n \cos\left(\frac{n\pi}{a}x\right) \\ Y_n(y) &= B_n \cosh\left(\frac{n\pi}{a}y\right) + C_n \sinh\left(\frac{n\pi}{a}y\right) \end{aligned}$$

The boundary condition $u(x, 0) = X(x)Y(0) = 0$ gives the non trivial solution $Y(0) = B_n \cdot 1 + C_n \cdot 0 = 0 \rightarrow B_n = 0$. We are left with $u_n(x, y) = A_n \cos\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$ (where $A_n C_n \rightarrow A_n$). Applying the superposition principle we have

$$u(x, y) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$$

Finally we apply the initial condition $u(x, b) = 100x$, therefore

$$u(x, b) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}b\right) = 100x$$

This is a Fourier cosine series and we get

$$\begin{aligned} A_n \sinh\left(n\pi \frac{a}{b}\right) &= \frac{2}{a} \int_0^a 100x \cos\left(\frac{n\pi}{a}x\right) dx \\ A_n \sinh\left(n\pi \frac{a}{b}\right) &= \frac{200}{a} \int_0^a x \cos\left(\frac{n\pi}{a}x\right) dx \end{aligned}$$

Now

$$\begin{aligned}
 \int_0^a x \cos\left(\frac{n\pi}{a}x\right)dx &= \left[x\left(\frac{a}{n\pi}\right)\sin\left(\frac{n\pi}{a}x\right)\right]_0^a - \frac{a}{n\pi} \int_0^a \sin\left(\frac{n\pi}{a}x\right)dx \\
 &= 0 - \frac{a}{n\pi} \int_0^a \sin\left(\frac{n\pi}{a}x\right)dx \\
 &= -\frac{a}{n\pi} \left(-\frac{a}{n\pi}\right) \left[\cos\left(\frac{n\pi}{a}x\right)\right]_0^a \\
 &= \frac{a^2}{n^2\pi^2} (\cos(n\pi) - 1) \\
 &= \begin{cases} -\frac{2a^2}{n^2\pi^2} & \text{odd } n \\ 0 & \text{even } n \end{cases}
 \end{aligned}$$

Substitute it back to get A_n

$$\begin{aligned}
 A_{n,n \text{ odd}} &= \frac{1}{\sinh(n\pi \frac{a}{b})} \frac{200}{a} \left(-\frac{2a^2}{n^2\pi^2}\right) \\
 A_{n,n \text{ odd}} &= -\frac{400a}{n^2\pi^2 \sinh(n\pi \frac{a}{b})} \\
 A_{n,n \text{ odd}} &= -\frac{800a}{n^2\pi^2 (e^{n\pi \frac{a}{b}} - e^{-n\pi \frac{a}{b}})} \\
 A_{n,n \text{ even}} &= 0
 \end{aligned}$$

Question 6

- a. Substituting $u(r, z) = R(r)Z(z)$ into the diffusion equation in cylindrical coordinates gives

$$R''Z + \frac{1}{r}R'Z + RZ'' = 0$$

Diving by RZ gives

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{Z''}{Z} = 0$$

Separation of variables gives

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -\frac{Z''}{Z} = -k^2$$

or

$$\begin{aligned}
 \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} &= -k^2 \\
 \frac{Z''}{Z} &= k^2
 \end{aligned}$$

- b. For $\frac{d^2}{dz^2}Z(z) = k^2Z(z)$, we immediately see that $Z(z) = c_1e^{kz} + c_2e^{-kz}$ which we can reformulate as $Z(z) = A \sinh(kz) + B \cosh(kz)$.

For $\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -k^2$, starting with $\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + k^2 R(r) = 0$.

$$\begin{aligned} s = kr, r = \frac{s}{k}, \frac{ds}{dr} &= k, R(r) \rightarrow R(s) \\ \frac{dR}{dr} &= \frac{dR}{ds} \frac{ds}{dr} = k \frac{dR}{ds} \\ \frac{d^2 R}{dr^2} &= k \frac{d}{ds} \left(k \frac{dR}{ds} \right) = k^2 \frac{d^2 R}{ds^2} \end{aligned}$$

Substitution into the ODE gives

$$\begin{aligned} k^2 \frac{d^2 R(s)}{ds^2} + \frac{1}{\frac{s}{k}} k \frac{dR(s)}{ds} + k^2 R(s) &= 0 \\ k^2 \frac{d^2 R(s)}{ds^2} + \frac{k^2}{s} \frac{dR(s)}{ds} + k^2 R(s) &= 0 \end{aligned}$$

Multiplying out by $\frac{s^2}{k^2}$ gives

$$\begin{aligned} s^2 \frac{d^2 R(s)}{ds^2} + s \frac{dR(s)}{ds} + s^2 R(s) &= 0 \\ s^2 \frac{d^2 R(s)}{ds^2} + s \frac{dR(s)}{ds} + (s^2 - 0^2) R(s) &= 0 \end{aligned}$$

The last equation being a Bessel equation of order 0 therefore the solution is of the form

$$R(r) = C_1 J_0(kr) + C_2 Y_0(kr)$$

Since the temperature remains bounded at $r = 0$, the term $Y_0(kr)$ has to be discarded, $C_2 = 0$, and $R(r) = C J_0(kr)$ with $C = C_1$

- c. Finally apply boundary conditions. First, $u(r, 0) = R(r)Z(0) = 0$, since in general $R(r) \neq 0$, thus $Z(0) = 0$, which is $A \cdot 0 + B \cdot 1 = 0 \rightarrow B = 0$ and $Z(z) = A \sinh(kz)$. Then $u(5, z) = R(5)Z(z) = 0 \rightarrow R(5) = 0$, therefore $C J_0(5k) = 0 \rightarrow J_0(5k) = 0$. $5k$ represents the zero crossing for the Bessel function of order 0. We call them α_m and set $5k_m = \alpha_m \rightarrow k_m = \frac{\alpha_m}{5}$. Therefore the solutions are $R_m(r) = C_m J_0(k_m r) = C_m J_0(\frac{\alpha_m}{5} r)$. Note now that solutions in z are $Z_m(z) = A_m \sinh(k_m z)$. Finally applying the superposition principle, we have

$$u(r, z) = \sum_{m=1}^{\infty} A_m \sinh(k_m z) J_0\left(\frac{\alpha_m}{5} r\right) \text{ where } A_m C_m \rightarrow A_m$$

Applying the last boundary condition $u(r, 20) = u_0, 0 < r < 5$, we get:

$$u(r, 20) = \sum_{m=1}^{\infty} A_m \sinh(20k_m) J_0\left(\frac{\alpha_m}{5} r\right) = u_0$$

This is a Fourier Bessel series where the coefficients are given by

$$\sinh(20k_m) A_m = \frac{2}{5^2 J_1^2(\alpha_m)} \int_0^5 r u_0 J_0\left(\frac{\alpha_m}{5} r\right) dr = \frac{2u_0}{25 J_1(\alpha_m)} \int_0^5 r J_0\left(\frac{\alpha_m}{5} r\right) dr$$

Next, $\sinh(20k_m) = \sinh(20\frac{\alpha_m}{5}) = \sinh(4\alpha_m)$, and we get

$$A_m = \frac{2u_0}{25J_1(\alpha_m) \sinh(4\alpha_m)} \int_0^5 J_0\left(\frac{\alpha_m}{5}r\right) r dr$$

Finally using $\frac{\partial}{\partial r}[rJ_1(r)] = rJ_0(r)$ which also gives

$$\begin{aligned} \frac{\partial}{\partial r}[\alpha r J_1(\alpha r)] &= \alpha r J_0(\alpha r) dr \\ \alpha r J_1(\alpha r) &= \int \alpha r J_0(\alpha r) dr \\ r J_1(\alpha r) &= \int r J_0(\alpha r) dr \\ A_m &= \frac{2u_0}{25J_1(\alpha_m) \sinh(4\alpha_m)} \left[r J_1\left(\frac{\alpha_m}{5}r\right) \right]_0^5 \\ &= \frac{2u_0}{25J_1(\alpha_m) \sinh(4\alpha_m)} \left(5J_1\left(\frac{\alpha_m}{5}5\right) \right) \\ &= \frac{2u_0}{5 \sinh(4\alpha_m)} \end{aligned}$$

Question 7

a. Let $z = x + iy$ then $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$ and

$$\begin{aligned} e^{z^2} &= e^{x^2 - y^2 + 2ixy} = e^{x^2 - y^2} e^{2ixy} \\ &= e^{x^2 - y^2} (\cos 2xy + i \sin 2xy) \end{aligned}$$

Now take $u(x, y) = e^{x^2-y^2} \cos 2xy$ and $v(x, y) = e^{x^2-y^2} \sin 2xy$. We have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 2x e^{x^2-y^2} \cos 2xy + e^{x^2-y^2} (-2y) \sin 2xy \\
\frac{\partial u}{\partial x} &= 2e^{x^2-y^2} (x \cos 2xy - y \sin 2xy) \\
\frac{\partial v}{\partial y} &= -2y e^{x^2-y^2} \sin 2xy + e^{x^2-y^2} (2x) \cos 2xy \\
\frac{\partial v}{\partial y} &= 2e^{x^2-y^2} (x \cos 2xy - y \sin 2xy) \\
\Rightarrow \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -2y e^{x^2-y^2} \cos 2xy + e^{x^2-y^2} (-2x) \sin 2xy \\
\frac{\partial u}{\partial y} &= -2e^{x^2-y^2} (y \cos 2xy + x \sin 2xy) \\
\frac{\partial v}{\partial x} &= 2x e^{x^2-y^2} \sin 2xy + e^{x^2-y^2} (2y) \cos 2xy \\
\frac{\partial v}{\partial x} &= 2e^{x^2-y^2} (y \cos 2xy + x \sin 2xy) \\
\Rightarrow \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

The Cauchy-Riemann conditions are satisfied therefore e^{z^2} is analytic (as required, since both z^2 and e^z are entire functions).

b. For $v(x, y) = xy$, we have

$$\begin{aligned}
v_x &= y, v_{xx} = 0 \\
v_y &= x, v_{yy} = 0 \\
\rightarrow v_{xx} + v_{yy} &= 0
\end{aligned}$$

Using Cauchy conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, we have $\frac{\partial u}{\partial x} = x$ integration on both sides, gives:

$$\begin{aligned}
u &= \int x dx + g(y) \\
u &= \frac{x^2}{2} + g(y)
\end{aligned}$$

Next the Cauchy condition $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ yields

$$\begin{aligned}
\frac{\partial u}{\partial y} &= g'(y) = -\frac{\partial v}{\partial x} = -y \\
g(y) &= -\frac{y^2}{2} + C
\end{aligned}$$

Therefore $u(x, y) = \frac{x^2-y^2}{2} + C$ and $f(x, y) = \frac{x^2-y^2}{2} + C + ixy$

Question 8

a. Substitute $y = x^2$, $dy = 2xdx$ then we get

$$\begin{aligned}\int_C (x^2 + y)dx + 2xydy &= \int_0^1 (x^2 + x^2)dx + 2x \cdot x^2 \cdot 2xdx \\ &= \int_0^1 (2x^2 + 4x^4)dx \\ &= 2\left[\frac{x^3}{3}\right]_0^1 + 4\left[\frac{x^5}{5}\right]_0^1 \\ &= \frac{2}{3} + \frac{4}{5} \\ &= \frac{10 + 12}{15} = \frac{22}{15}\end{aligned}$$

b. Take $z = x + iy$, $Re(z) = x$, let parametrize the line from 1 to i as $z(t) = 1 \cdot (1 - t) + i \cdot t$, $= 1 + (i - 1)t$, $dz = (i - 1)dt$ for $0 \leq t \leq 1$. We have then

$$\begin{aligned}\int_C Re(z)dz &= \int_0^1 (1 - t)(i - 1)dt \\ &= (i - 1) \int_0^1 (1 - t)dt \\ &= (i - 1)\left[t - \frac{t^2}{2}\right]_0^1 \\ &= (i - 1)\frac{1}{2} = \frac{i - 1}{2}\end{aligned}$$

Extra Credit

a. Take $f(z) = \frac{1}{(z-2)(z-1)^2} = \frac{1}{z-2} - \frac{z}{(z-1)^2}$, the poles are at $z = 2$ and $z = 1$. We compute the residues at each pole. At $z = 2$

$$f(z) = \frac{1}{z-2} + \text{something analytic at } 2$$

Therefore the pole is simple and $\text{Res}(f, 2) = 1$ At $z = 1$. Take $z = \xi + 1$ then

$$\begin{aligned}f(\xi) &= \frac{1}{(\xi - 1)\xi^2} \\ &= -\frac{1}{\xi^2}(1 - \xi)^{-1} \\ &= -\frac{1}{\xi^2}(1 + \xi + \xi^2 + \xi^3 + \dots) \\ &= -\frac{1}{\xi^2} - \frac{1}{\xi} - \frac{1}{2} - \xi + \dots\end{aligned}$$

Therefore the pole is of order 2 and $\text{Res}(f, 1) = -1$. Poles within the circle centered at $z = 0$ with radius 4, are 1 and 2. Applying the residue theorem

$$\oint_C f(z)dz = \oint_C \frac{1}{(z-2)(z-1)^2} dz = 2\pi i (\text{Res}(f, 2) + \text{Res}(f, 1)) = 2\pi i (1 - 1) = 0$$

b. We have

$$\begin{aligned} e^z &= \sum_{k=0}^{\infty} \frac{1}{k!} z^k \\ e^{\frac{1}{z^n}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^{n+k}} \\ &= \frac{1}{z^n} + \frac{1}{z^{n+1}} + \frac{1}{z^{n+2}} + \dots \end{aligned}$$

Looking at $\lim_{z \rightarrow 0} (z-0)^\alpha \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^{n+k}}$, there is no largest α such this quantity is finite. Thus 0 is an essential singularity. The Laurent expansion has only negative powers of z starting at n . $e^{\frac{1}{z^n}}$ has only the singularity 0 within the unit circle, therefore applying the residue theorem

- $n = 0$, the function is $e^{\frac{1}{z^0}} = e$, take $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$

$$\oint_C e dz = e \oint_C dz = ei \int_0^{2\pi} e^{i\theta} d\theta = ei \left[\frac{e^{i\theta}}{i} \right]_0^{2\pi} = e(e^{i2\pi} - 1) = 0$$

- $n = 1$, $\text{Res}(e^{\frac{1}{z}}, 0) = 1$

$$\oint_C e^{\frac{1}{z}} dz = 2\pi i \cdot 1 = 2\pi i$$

- case $n > 1$ $\text{Res}(e^{\frac{1}{z^n}}, 0) = 0$

$$\oint_C e^{\frac{1}{z^n}} dz = 2\pi i \cdot 0 = 0$$