Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Take Home Project 2
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Question 1

(a) Please see attached separate pdf.

(b) $f(t) = C_0 e^{-\frac{t}{\tau}}$ with period T, so

$$\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} dt \\
&= \frac{2C_0}{T}(-\tau)[e^{-\frac{t}{\tau}}]_0^T \\
&= -2C_0\frac{\tau}{T}[e^{-\frac{T}{\tau}} - 1] \\
&= 2C_0\frac{\tau}{T}(1 - e^{-\frac{T}{\tau}})
\end{aligned}$$

If $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_0 \approx 2C_0\frac{\tau}{T}$.

$$\begin{aligned}
a_k &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt
\end{aligned}$$

Using integration by parts with $u = \cos\frac{2k\pi t}{T}, du = -\frac{2k\pi}{T}\sin\frac{2k\pi t}{T}$ and $dv = e^{-\frac{t}{\tau}}, v = (-\tau)e^{-\frac{t}{\tau}}$:

$$\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt = (-\tau)[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}]_0^T - \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt$$

Using again integration by parts:

$$\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt = (-\tau)[e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt$$

So

$$\begin{aligned}
(1 + (\frac{2k\pi\tau}{T}))^2 \int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt &= (-\tau)[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau^2}{T}[e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T}]_0^T \\
&= (-\tau)[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}]_0^T + 0 \\
&= \tau(1 - e^{-\frac{T}{\tau}}) \\
\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt &= \frac{\tau}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}})
\end{aligned}$$

Substituting back into the expression found for a_k yields

$$\begin{aligned}
a_k &= 2C_0 \frac{\tau}{T} \frac{1}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}}) \\
&= 2C_0 \frac{\tau T}{T^2 + (2k\pi\tau)^2}(1 - e^{-\frac{T}{\tau}})
\end{aligned}$$

With the same assumption $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_k \approx 2C_0 \frac{\tau}{T} \frac{1}{1+(\frac{2k\pi\tau}{T})^2}$. Similarly to compute b_k

$$\begin{aligned}
b_k &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \int_0^T e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \frac{\tau}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}}) \\
&= 4C_0 k\pi \frac{\tau^2}{T^2 + (2k\pi\tau)^2}(1 - e^{-\frac{T}{\tau}})
\end{aligned}$$

Once again, since $e^{-\frac{T}{\tau}} \approx 0$ and $b_k \approx 4C_0(\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2}\pi k$

(c) For $k \geq 1$

$$\begin{aligned}
p_k &= \frac{1}{2}(a_k^2 + b_k^2) \\
&= \frac{1}{2}\left[4C_0^2(\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} + 16C_0^2(\frac{\tau}{T})^4 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2}\pi^2 k^2\right] \\
&= \frac{1}{2} 4C_0^2(\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2}\left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2\right] \\
&= 2C_0^2(\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2}\left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2\right]
\end{aligned}$$