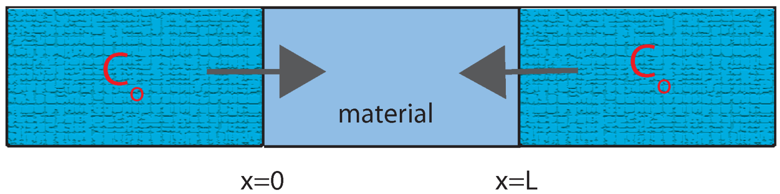
11.16 Essentially a 1-d diffusion problem



$$K\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ \ u(0,t)=u(L,t)=C_0$$

and at the start u(x,t) for 0<x<L is zero,

except at x = 0 and x = L, that is $u(x,0)=0,\ 0<x<L$

however as $t\to\infty$ material becomes saturated for all x

and $\lim\limits_{t\to\infty}u(x,t)=C_0$ (for all x)

KEY: Take as solution u(x,t)= steady state + transient = $C_0+v(x,t)$

Substitution into PDE gives

$$K\frac{\partial^2(C_0+v(x,t))}{\partial x^2}=\frac{\partial(C_0+v(x,t))}{\partial t}$$

or

$$K\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}\ \text{ since } C_0 \text{ is constant!}$$

For boundary conditions $u(0,t)=C_0+v(0,t)=C_0\Rightarrow v(0,t)=0$

Similarly $u(L,t)=C_0+v(L,t)=C_0\Rightarrow v(L,t)=0$

Also $u(x,0)=C_0+v(x,0)=0\ \Rightarrow\ v(x,0)=-C_0$ for 0<x<L

So for v(x,t) system we have

$$K\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}$$

with $v(0,t)=0,\ v(L,t)=0$ and $v(x,0)=-C_0$

Take $v(x,t) = X(x)T(t)$

$K\frac{\partial^2 XT}{\partial x^2} = \frac{\partial XT}{\partial t}$ or $KX''T = X\dot{T}$

Dividing by XT gives $K\frac{X''}{X} = \frac{\dot{T}}{T} = -\lambda^2$ using $-\lambda^2$ as

seperation constant since boundary conditions at 0 and L go to zero!!

The usual gives $K\frac{X''}{X} = -\lambda^2 \Rightarrow KX'' - \lambda^2 X = 0 \Rightarrow$

$X(x) = A_1 e^{-i\lambda/\sqrt{K}x} + B_1 e^{i\lambda/\sqrt{K}x} \equiv X(x) = A\cos\frac{\lambda}{\sqrt{K}}x + B\sin\frac{\lambda}{\sqrt{K}}x$

Apply boundary conditions with $v(0,t) = 0,\ v(L,t) = 0$ gives

$X(0) = A\cos\frac{\lambda}{\sqrt{K}}0 + B\sin\frac{\lambda}{\sqrt{K}}0 = 0 \Rightarrow A \cdot 1 + B \cdot 0 = 0 \Rightarrow A = 0$

Then $X(L) = B\sin\frac{\lambda}{\sqrt{K}}L = 0 \Rightarrow \frac{\lambda}{\sqrt{K}}L = n\pi \Rightarrow \lambda_n = n\pi\frac{\sqrt{K}}{L}$

Therefore $X_n(x) = B_n \sin\frac{\lambda_n}{\sqrt{K}}x = B_n \sin\frac{\left[n\pi\frac{\sqrt{K}}{L}\right]}{\sqrt{K}}x = B_n \sin\frac{n\pi}{L}x$

For T equation $\frac{\dot{T}}{T} = -\lambda^2 = -n^2\pi^2\frac{K}{L^2} \Rightarrow \dot{T} = -n^2\pi^2\frac{K}{L^2}T$

Usual solution is $T(t) = e^{-n^2\pi^2\frac{K}{L^2}t}$

So $v_n(x,t) = B_n \sin\frac{n\pi}{L}x \cdot e^{-n^2\pi^2\frac{K}{L^2}t}$ and $v(x,t) = \sum_{n=1}^{\infty} v_n(x,t)$

Therefore $v(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x \cdot e^{-n^2\pi^2\frac{K}{L^2}t}$

Next apply initial condition $v(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x \cdot e^{-n^2\pi^2\frac{K}{L^2}0} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x = -C_0$

which gives a Fourier series where $B_n = \int_0^L -C_0 \sin\frac{n\pi}{L}x\,dx \Rightarrow B_n = \frac{2C_0}{n\pi}(\cos n\pi - 1) = \begin{cases} 0 & n \text{ even} \\ \frac{-4C_0}{n\pi} & n \text{ odd} \end{cases}$

Finally $u(x,t) = C_0 + v(x,t) = C_0 - \frac{4C_0}{\pi}\sum_{n=1, n\ \text{odd}}^{\infty} \frac{1}{n}\sin\frac{n\pi}{L}x \cdot e^{-n^2\pi^2\frac{K}{L^2}t}$

Note at t = 0 $u(x,0) = 0$ and as $t \to \infty$ $u(x,t) = C_0$