4.25 (a)

Start with

$$\frac{1}{L}\int_0^L f(x)g^*(x)dx = \frac{1}{4}a_0\alpha_0 + \frac{1}{2}\sum_{r=1}^{\infty}(a_r\alpha_r + b_r\beta_r)$$

where $a_0 \alpha_0 a_r \alpha_r a_r b_r \beta_r$ are the usual Fourier coefficents for f(x) and g(x)

Take $g(x) = \sin mx \equiv g^*(x)$

Aside: *, denotes the complex conjugate. Note the complex conjugate of real valued function is the function itself.

Now for $\sin mx$ since it is an odd function α_0, α_r are all 0! We are left with

$$\frac{1}{L}\int_0^L f(x)\sin mx\,dx = \frac{1}{2}\sum_{r=1}^{\infty} b_r\beta_r$$

Next the critical part is to calculate the β_rs (Fourier coefficient for f(x)=sinmx, L = 2π, Eq. 4.7 in bool

$$\beta_r = \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin rx\,dx = \frac{1}{2\pi}\int_0^{2\pi}\sin mx \sin rx\,dx$$

These functions are orthogonal therefore only a value when r = m (see Eq 4.1 - 4.3) therefore

$$\beta_{r=m} \equiv \beta_m = \frac{1}{2\pi}\int_0^{2\pi}\sin mx \sin rx\,dx = \frac{1}{2\pi}2\pi = 1 \text{ and note when } r \neq m, \beta_{r\neq m} = 0$$

Finally only left with one term in sum using these result, that is

$$\frac{1}{L}\int_0^L f(x)\sin mx\,dx = \frac{1}{2}\sum_{r=1}^{\infty} b_r\beta_r \rightarrow \frac{1}{2}b_m\beta_m \rightarrow \frac{1}{2}b_m(1) = \frac{1}{2}b_m$$

OR

$$b_m = \frac{2}{L}\int_0^L f(x)\sin mx\,dx =$$