1 Fourier Series

Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos(\frac{2\pi rx}{L}) + b_r \sin(\frac{2\pi rx}{L}) \right]$$

with

$$a_0 = \frac{2}{L} \int_{x_0}^{x_0+L} f(x)dx$$
$$a_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi rx}{L}\right) dx$$
$$b_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi rx}{L}\right) dx$$

For an even function, we have:

$$a_0 = \frac{4}{T} \int_0^{\frac{L}{2}} f(x)dx \quad a_r = \frac{4}{T} \int_0^{\frac{L}{2}} f(x) \cos\left(\frac{2\pi rx}{L}\right) dx \quad b_r = 0$$

For an odd function, we have:

$$a_0 = 0 \quad b_r = \frac{4}{T} \int_0^{\frac{L}{2}} f(x) \sin\left(\frac{2\pi rx}{L}\right) dx$$

Complex Fourier series

$$f(x) = \sum_{r=-\infty}^{r=\infty} c_r e^{\frac{i2\pi rx}{L}}$$

where

$$c_r = \frac{1}{L} \int_{x_0}^{x_0+L} f(x) e^{-\frac{i2\pi rx}{L}} dx$$
$$c_r = \frac{1}{2}(a_r - ib_r)$$

Parseval Identity

$$\frac{1}{L} \int_{x_0}^{x_0+L} |f(x)|^2 dx = \sum_{r=-\infty}^{r=\infty} |c_r|^2$$

2 Fourier transforms

Fourier transform of $f(t)$:

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt}\, dt$$

And its inverse defined by:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w) e^{iwt}\, dw$$

3 The Dirac δ-Function

$\delta(t) = 0$ for $t = 0$. Provided the range of integration includes the point $t = a$:

$$\int f(t)\delta(t-a)\,dt = f(a)$$

otherwise the integral equals 0. This leads to:

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(t) f(t) dt &= f(0) \\
\int_{-a}^{b} \delta(t)\,dt &= 1 \text{ for all } a, b > 0 \\
\int \delta(t-a)\,dt &= 1 \text{ if range of integration includes a} \\
\delta(t) &= \delta(-t) \\
\delta(bt) &= \frac{1}{|b|}\delta(t) \\
t\delta(t) &= 0 \\
\delta(h(t)) &= \sum_i \frac{\delta(t-t_i)}{|h'(t_i)|} \text{ where the } t_i \text{ are the zeros of } h(t)
\end{aligned}$$

The derivatives $\delta^n(t)$ are defined by:

$$\int_{-\infty}^{\infty} f(t)\delta^n(t)\,dt = (-1)^n f^n(0)$$

Integral representation:

$$\delta(t-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iw(t-u)}\,dw$$

Dirac function's relation to the sinc function:

$$\delta(t) = \lim_{\Omega\to\infty} \frac{\sin(\Omega t)}{\pi t}$$

4 The Heaviside function

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$H'(t) = \delta(t)$.

5 Properties of Fourier transforms

- $L\{t^n\} = \frac{n!}{s^{n+1}}$
- $F\{e^{\alpha t} f(t)\} = \tilde{f}(w + \alpha t)$
- $F\{f(t + a)\} = e^{iwa} \tilde{f}(w)$
- $F\{f(at)\} = \frac{1}{a} \tilde{f}(\frac{w}{a})$
- $\mathcal{F}\big[f^n(t)\big] = (i)^n w^n \tilde{f}(w)$
- $\mathcal{F}\big[\int^t f(s)ds\big] = \frac{1}{iw} f(\tilde{w}) + 2\pi c\delta(w)$

The Fourier sine transform: $\tilde{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(wt) dt$

6 Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk$$

7 Laplace transform

By definition:

$$\bar{f}(s) = \int_0^\infty f(t) e^{-st} \, dt$$

And

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}$$

Laplace transform properties:

- $L\{e^{at} f(t)\} = \bar{f}(s - a)$ or $L^{-1}\{\bar{f}(s - a)\} = e^{at} f(t)$
- $L\{f(t - b)H(t - b)\} = e^{-bs} \bar{f}(s)$ or $(t - b)H(t - b) = L^{-1}\{e^{-bs} \bar{f}(s)\}$
- $L\{f'(t)\} = -f(0) + s\tilde{f}(s)$
- $L\{f''(t)\} = -sf(0) - f'(0) + s^2 \tilde{f}(s)$
- $L\{\int_0^t f(x)dx\} = \frac{1}{s} \tilde{f}(s)$

8 Convolution

$$h(z) = \int_{-\infty}^{\infty} f(x) g(z - x) dx$$

Fourier transform of a convolution: $\tilde{h}(k) = \sqrt{2\pi} \tilde{f}(k) \tilde{g}(k)$
Laplace transform of a convolution: $\mathcal{L}\big[\int_0^t f(u) g(t - u) du\big] = \bar{f}(s) \bar{g}(s)$

9 Legendre Linear differential equations

$$a_n(\alpha x+\beta)^n\frac{d^n y}{dx^n}+\cdots+a_1(\alpha x+\beta)\frac{d\,y}{dx}+a_0y=f(x)$$

Change of variable: $\alpha x+\beta=e^t$ or $x=(e^t-\beta)/\alpha$ or $t=\ln(\alpha x+\beta)$. Relations:

- $(\alpha x+\beta)\frac{dy}{dx}=\alpha\frac{dy}{dt}$
- $(\alpha x+\beta)^2\frac{d^2y}{dx^2}=\alpha^2\frac{d}{dt}\left(\frac{d}{dt}-1\right)y$

10 Variation of parameters

Used to find a complementary solution to the homogeneous equation: $y_p(x)=k_1(x)y_1(x)+k_2(x)y_2(x)+\cdots+k_n(x)y_n(x)$

$$\begin{aligned}k_1'(x)y_1(x)+k_2'(x)y_2(x)+\cdots+k_n'(x)y_n(x)&=0\\ k_1'(x)y_1'(x)+k_2'(x)y_2'(x)+\cdots+k_n'(x)y_n'(x)&=0\\ &\vdots\\ k_1'(x)y_1^{n-1}(x)+k_2'(x)y_2^{n-1}(x)+\cdots+k_n'(x)y_n^{n-1}(x)&=\frac{f(x)}{a_n(x)}\end{aligned}$$

11 Green function

Looking for a solution of a differential equation with some boundary conditions on some interval (a,b), e.g. $y(a)=y(b)=0$ of the form: $y(x)=\int_a^b G(x,z)f(z)dz$.

$$\begin{aligned}&\mathcal{L}y(x)=\mathcal{L}\int_a^b G(x,z)f(z)dz=\int_a^b\mathcal{L}G(x,z)f(z)dz=f(x)=\int_a^b\delta(x-z)f(z)dz\\ &\mathcal{L}G(x,z)=\delta(x-z)\end{aligned}$$

Green's function properties for an nth-order linear ODE

- Obeys the boundary conditions on $y(x)$: $G(a,z)=G(b,z)=0$
- The derivatives of $G(x,z)$ w.r.t. x up to order $n-2$ are continuous a $x=z$
- But the $n-1$th-order derivatives has a discontinuity of $\frac{1}{a_n}$ at this point.

12 Wronskian

Consider the linear ODE in standard form: $y''+p(x)y'+q(x)y=0$. Two solutions $y_1(x)$ and $y_2(x)$ for the solution $y(x)=c_1y_1(x)+c_2y_2(x)$ of the differential homogeneous equation are independant if the Wronskian is non zero:

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0$$
$$W = Ce^{-\int^x p(u)du}$$

13 Singular point

$p(z)$ and $q(z)$ diverge at $z = z_0$, z_0 is a regular singular point if $(z-z_0)p(z)$ and $(z-z_0)^2q(z)$ are both analytic. For $\pm\infty$, make the change of variable $z = \frac{1}{w}$ and investigate if $p(z)$ and $q(z)$ are defined at $w = 0$.

14 Series solution of ODE at ordinary point

The second order ODE solution includes two arbitrary constants and the series solution method generates two linearly independent solutions. Solution can be represented as: $y(z) = \sum_{n=0}^{\infty} a_n z^n$

$$y'(z) = \sum_{n=0}^{\infty} na_n z^{n-1} = \sum_{n=1}^{\infty} na_n z^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n$$
$$y''(z) = \sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n$$

15 Series solution of ODE at regular singular point

We look for a solution of the form: $y(z) = z^{\sigma}\sum_{n=0}^{\infty} a_n z^n$ where the a_n and σ are constants.

$$y(z) = \sum_{n=0}^{\infty} a_n z^{\sigma+n}$$
$$y'(z) = \sum_{n=0}^{\infty}(\sigma+n)a_n z^{\sigma+n-1}$$
$$y(z) = \sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_n z^{\sigma+n-2}$$

Find the roots of the indicial equation $\sigma_1 \geq \sigma_2$:

- the indicial equation has two different roots where their difference is not an integer, find recurrence relation
  - calculation of $y(z, \sigma_2)$
  - second solution of the form: $y_2(z) = z^{\sigma_2}\sum_{n=0}^{\infty} a_n z^n$
- double root: derivative or Wronskian method:

- calculation of $\left[\frac{\partial y(z,\sigma)}{\partial \sigma}\right]_{\sigma=\sigma_1} \Rightarrow y_2(z) = y_1(z)\ln z + z^{\sigma_1}\sum_{n=0}^{\infty} b_n z^n$
- or Wronskian method and $y_2(z) = y_1(z)\int^z \frac{g(u)}{y_1^2(u)}du$ where $g(u) = \exp\left\{-\int^u p(v)dv\right\}$

- two roots where their difference is an integer:
  - $\left[\frac{\partial[(\sigma-\sigma_2)y(z,\sigma)]}{\partial \sigma}\right]_{\sigma=\sigma_2} \Rightarrow y_2(z) = cy_1(z)\ln z + z^{\sigma_2}\sum_{n=0}^{\infty} b_n z^n$
  - or Wronskian method and $y_2(z) = y_1(z)\int^z \frac{g(u)}{y_1^2(u)}du$ where $g(u) = \exp\left\{-\int^u p(v)dv\right\}$

16 Series solution of an ODE at ordinary poit with arbitrary coefficient in its no-differential term

Polynomial solutions of ODEs

$$y'' - 2zy' + \lambda y = 0$$

17 Strum-Liouville differential equations

The standard form for a Strum-Liouville differential equation:

$$p(x)\frac{d^2y}{dx^2} + r(x)\frac{dy}{dx} + q(x)y + \lambda\rho(x)y = 0 \; r(x) = \frac{dp(x)}{dx}$$

In case $r(x) \neq \frac{dp(x)}{dx}$, compute $F(x) = e^{\int^x \frac{r(u)-p'(u)}{p(u)}}du$ and multiple the differential equation by it.

18 Legendree Functions

Legendre's defining differential equation:

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

Solution of Legendre's differential equation:

$$y(x) = c_1 P_l(x) + c_2 Q_l(x)$$

Rodrigues' formula:

$$p_l(x) = \frac{1}{l!2^l}\frac{d^l}{dx^l}(x^2-1)^l$$

19 Some results

$$
\begin{aligned}
L\{tH(t)\} &= e^{-s\cdot 0}\frac{1}{s^2} \\
L\{(t-a)H(t-a)\} &= e^{-as}\frac{1}{s^2} \\
\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n \\
\frac{z}{(1-z)^2} &= \sum_{n=0}^{\infty} nz^n \\
\sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\
\cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}
\end{aligned}
$$

From

$$\sum_{l=0}^{\infty} a_l[b-l(l+1)]P_l(x) = f(x)$$

$$
\begin{aligned}
\sum_{l=0}^{\infty} a_l[b-l(l+1)]P_k(x)P_l(x) &= P_k(x)f(x)dx \\
\int_{-1}^{1}\sum_{l=0}^{\infty} a_l[b-l(l+1)]P_k(x)P_l(x) &= \int_{-1}^{1} P_k(x)f(x)dx \text{ integrating each side} \\
&\text{swap integral and infinite sum} \\
\sum_{l=0}^{\infty} a_l[b-l(l+1)]\int_{-1}^{1} P_k(x)P_l(x) &= \int_{-1}^{1} P_k(x)f(x)dx
\end{aligned}
$$

For each term in l, we apply orthogonalization and normalization property (see proof in example just before generating function chapter, for me page 346). Each term for which $l \neq k$ the term is zero, which leaves the only term for which $l = k$:

$$
\begin{aligned}
a_k[b-l(l+1)]\int_{-1}^{1} P_k(x)P_k(x) &= \int_{-1}^{1} P_k(x)f(x)dx \\
a_k[b-l(l+1)]\frac{2}{2k+1} &= \int_{-1}^{1} P_k(x)f(x)dx \\
a_k &= \frac{2k+1}{2(b-l(l+1))}\int_{-1}^{1} P_k(x)f(x)dx
\end{aligned}
$$