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EN.585.615.81.SP21 Mathematical Methods
Final Exam
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# Question 1

a. $f(x) = x$ is odd on $[-\pi, \pi]$ therefore its Fourier coefficients $a_n$ are $0$ and we need to find its $b_n$ coefficients:

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{4}{2\pi} \int_{0}^{\pi} x \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) dx \end{aligned}$$

Using integration by parts:

$$\begin{aligned} \int_{0}^{\pi} x \sin(nx) dx &= [x(-\frac{\cos(nx)}{n})]_0^{\pi} + \int_{0}^{\pi} 1 \cdot \frac{\cos(nx)}{n} dx \\ &= (-\frac{\pi}{n}) \cos(n\pi) + \frac{1}{n}[\sin(nx)]_0^{\pi} \\ &= \frac{(-1)^{n+1}\pi}{n} \end{aligned}$$

Thus $b_n = \frac{2}{\pi}\frac{(-1)^{n+1}\pi}{n} = \frac{(-1)^{n+1}2}{n}$ and the Fourier series of $x$, on $[-\pi, \pi]$, is:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

b. If we integrate terms by terms the previous expression, the Fourier series of $x$ over $[-\pi, \pi]$, we have:

$$\begin{aligned} \frac{x^2}{2} &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-\frac{\cos(nx)}{n}) + c \;\; c\text{constant of integration} \\ x^2 &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + c \text{ with } 2c \to c \\ &= c + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) \end{aligned}$$

c. $f(x) = x^2$ is an even function, by Fourier Series for even function over symmetric range, we have:

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n x}{2\pi}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \; (1)$$

where

$$a_0 = \frac{4}{2\pi}\int_0^\pi x^2 dx$$
$$= \frac{2}{\pi}[\frac{x^3}{3}]_0^\pi$$
$$= \frac{2}{3}\pi^2$$

$$a_n = \frac{4}{2\pi}\int_0^\pi x^2\cos(\frac{2\pi nx}{2\pi})dx = \frac{2}{\pi}\int_0^\pi x^2\cos(nx)dx$$
$$\int_0^\pi x^2\cos(nx)dx = [x^2\ \frac{\sin(nx)}{n}]_0^\pi - \frac{2}{n}\int_0^\pi x\sin(nx)dx$$
$$= 0 - \frac{2}{n}\frac{(-1)^{n+1}\pi}{n}$$
$$a_n = \frac{2}{\pi}\frac{(-1)^n 2\,\pi}{n^2}$$
$$= (-1)^n\frac{4}{n^2}$$

Substituting for $a_n$ in (1):

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

d. Fourier series of $x^2$ using integration term by term or calculating directly, match, as required, by taking $c = \frac{\pi^2}{3}$ since $x$ is a piecewise smooth function on the specified range.

# Question 2

Consider the differential equation:

$$z\frac{d^2y}{dy^2} + y = 0$$

a. We put the equation in standard form:

$$\frac{d^2y}{dy^2} + \frac{1}{z}y = 0$$

$z\,p(z) = 0$ and $z^2q(z) = z$ therefore $0$ is a regular singular point.

b. Take $y = z^\sigma\sum_{n=0}^{\infty}a_nz^n$ and the usual derivatives in the D.E. gives by substitution

$$z\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_nz^{n+\sigma-2} + \sum_{n=0}^{\infty}a_nz^{n+\sigma} = 0$$
$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_nz^{n+\sigma-1} + \sum_{n=0}^{\infty}a_nz^{n+\sigma} = 0\ (1)$$

Take the term with the lowest power of $z$, which is the first sum with $n = 0$, then since each power of $z$ term must be equal to $0$, we have

$$\sigma(\sigma - 1)a_0 z^{\sigma - 1} = 0$$

Since $a_0 \neq 0$ and $z^{\sigma-1} \neq 0$, therefore $\sigma = 0, 1$.

c. We go back to equation $(1)$ and take $\sigma = 1$ yields

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

Then reindex the second sum to get same power of $z$ in both sums:

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=1}^{\infty} a_{n-1} z^{n} = 0$$

Note, in first term $n = 0$ does not contribute so we can start index at $n = 1$ in the first sum, and combine both sums

$$\sum_{n=1}^{\infty} [n(n+1)a_n + a_{n-1}] z^n = 0$$

Since every power of $z$ term must be $0$ and $z^n \neq 0$, gives:

$$a_n = -\frac{1}{(n+1)n} a_{n-1}$$

Taking $a_0 = 1$, now

$$
\begin{aligned}
n = 1 \; a_1 &= -\frac{1}{2\,1} a_0 = -\frac{1}{2\,1} = \frac{(-1)^1}{2\,1} \\
n = 2 \; a_2 &= -\frac{1}{3\,2} a_1 = \frac{1}{3\,2\,2\,1} = \frac{(-1)^2}{(3\,2\,1)\,(2\,1)} \\
n = 3 \; a_3 &= -\frac{1}{4\,3} a_2 = -\frac{1}{4\,3\,3\,2\,2\,1} = \frac{(-1)^3}{(4\,3\,2\,1)\,(3\,2\,1)} \\
&\vdots \\
a_n &= -\frac{1}{(n+1)n} a_{n-1} = \cdots = \frac{(-1)^n}{((n+1)\,n \cdots 1)\,(n\,(n-1) \cdots 1)} = \frac{(-1)^n}{(n+1)!n!}
\end{aligned}
$$

Therefore one of the independent solution of the ODE is

$$y_1(z) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!n!} z^n$$

# Question 3

a. We have

$$n = 0,\ M = 0,\ P_0(x) = \frac{(-1)^0(2\ 0 - 2\ 0)!}{2^0(0-0)!(0-2\ 0)!}x^{0-2\ 0} = 1$$

$$n = 1,\ M = \frac{1-1}{2} = 0,\ P_1(x) = \frac{(-1)^0(2\ 1 - 2\ 0)!}{2^1(1-0)!(1-2\ 0)!}x^{1-2\ 0} = \frac{1\ 2}{2\ 1!\ 1!}x^1 = x$$

$$n = 2,\ M = \frac{2}{2} = 1,\ P_2(x) = \frac{(-1)^0(2\ 2 - 2\ 0)!}{2^2(2-0)!(2-2\ 0)!}x^{2-2\ 0} + \frac{(-1)^1(2\ 2 - 2\ 1)!}{2^2(2-1)!(2-2\ 1)!}x^{2-2\ 1}$$

$$P_2(x) = \frac{4!}{2^2\ 2!\ 2!}x^2 - \frac{(2\ 2-2)!}{2^2\ 1!\ 0!}x^0$$

$$P_2(x) = \frac{4\ 3\ 2\ 1}{4\ 2\ 2}x^2 - \frac{2!}{4}$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2 - 1)$$

b. From

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx = \frac{2n+1}{2}\int_{-1}^{1} xP_n(x)dx$$

we have

$$n = 0,\ a_0 = \frac{2\ 0+1}{2}\int_{-1}^{1} xP_0(x)dx$$

$$= \frac{1}{2}\int_{-1}^{1} xdx = \frac{1}{2}[\frac{x^2}{2}]_{-1}^{1} = \frac{1}{4}[1^2 - (-1)^2] = 0$$

$$n = 1,\ a_1 = \frac{2\ 1+1}{2}\int_{-1}^{1} xP_1(x)dx$$

$$= \frac{3}{2}\int_{-1}^{1} x^2dx = \frac{3}{2}[\frac{x^3}{3}]_{-1}^{1} = \frac{1}{2}[1^3 - (-1)^3] = \frac{1}{2}\ 2 = 1$$

$$n = 2,\ a_2 = \frac{2\ 2+1}{2}\int_{-1}^{1} xP_2(x)dx$$

$$= \frac{5}{2}\int_{-1}^{1} x[\frac{1}{2}(3x^2-1)]dx = \frac{5}{4}\int_{-1}^{1}(3\ x^3 - x)dx$$

$$= 0 \ \text{ since the powers of } x \text{ in the integrand are odd}$$

Therefore the Fourier-Legendre series of $x$ iss $x = 1 \cdot P_1(x)$ as required.

c. Using Rodrigues's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

we have

$$
\begin{aligned}
n = 0,\ & \frac{d^0}{dx^0}[(x^2-1)^0] = (x^2-1)^0 = 1 \\
& P_0(x) = \frac{1}{2^0\ 0!}\ 1 = 1 \\
n = 1,\ & \frac{d}{dx}(x^2-1) = 2x \\
& P_1(x) = \frac{1}{2^1\ 1!}\ 2x = x \\
n = 2,\ & \frac{d^2}{dx^2}(x^2-1)^2 = \frac{d}{dx}\Big[\frac{d}{dx}(x^2-1)^2\Big] = \frac{d}{dx}\Big[4x(x^2-1)\Big] = \frac{d}{dx}\Big[4x^3-4x\Big] = 12\,x^2-4 \\
& P_2(x) = \frac{1}{2^2\ 2!}\ (12\,x^2-4) = \frac{4}{4\ 2}(3x^2-1) = \frac{1}{2}(3x^2-1)
\end{aligned}
$$

d. Extra Credit

Take $f$ an even function and consider odd coefficients, let $n = 2p+1$. Since n is odd $M = \frac{2p+1-1}{2} = p$, and we have

$$
\begin{aligned}
a_{2p+1} &= \frac{2(2p+1)+1}{2}\int_{-1}^{1} f(x)P_{2p+1}(x)dx \\
&= \frac{4p+3}{2}\int_{-1}^{1} f(x)\sum_{m=0}^{p}(-1)^m\frac{\Big(2(2p+1)-2m\Big)!}{2^{2p+1}(2p+1-m)!(2p+1-2m)!}x^{2p+1-2m}dx \\
&= \frac{4p+3}{2}\sum_{m=0}^{p}(-1)^m\frac{\Big(2(2p+1)-2m\Big)!}{2^{2p+1}(2p+1-m)!(2p+1-2m)!}\int_{-1}^{1} f(x)x^{2p+1-2m}dx \\
&= \frac{4p+3}{2}\sum_{m=0}^{p}(-1)^m\frac{\Big(2(2p+1)-2m\Big)!}{2^{2p+1}(2p+1-m)!(2p+1-2m)!}\int_{-1}^{1} f(x)x^{2(p-m)+1}dx
\end{aligned}
$$

The integral term is $0$ since the integration is over a symmetric range and $f(x)x^{2(p-m)+1}$ is odd. Therefore given an even function $f(x)$ all the odd coefficients for the Fourier-Legendre series are zero. Similarly given an odd function, the even coefficients are zero.

# Question 4

a.

$$
\frac{\partial u}{\partial x} + 4xu = 0
$$

Integration factor is

$$
e^{\int 4x dx} = e^{4\int x dx} = e^{4\frac{x^2}{2}} = e^{2x^2}
$$

Multiply the partial differential equation by the I.F.:

$$e^{2x^2}\frac{\partial u}{\partial x} + 4xe^{2x^2}u = 0$$
$$\frac{\partial}{\partial x}(e^{2x^2}u) = 0$$

Now integrate both sides with respect to x

$$e^{2x^2}u = C \; C\text{:constant}$$
$$u(x) = Ce^{-2x^2}$$

b.

$$y^2u_x - x^2u_y = 0$$

Let $u(x,y) = X(x)Y(y)$, substitution into the D.E. gives

$$y^2X'Y - x^2XY' = 0$$
$$y^2\frac{X'Y}{XY} - x^2\frac{XY'}{XY} = 0$$
$$y^2\frac{X'}{X} - x^2\frac{Y'}{Y} = 0$$
$$\frac{1}{x^2}\frac{X'}{X} = \frac{1}{y^2}\frac{Y'}{Y} = k$$

Integrating $\ln X = \frac{1}{3}kx^3 + \ln(C)$ and $\ln Y = \frac{1}{3}ky^3 + \ln(D)$, so

$$X = Ce^{\frac{1}{3}kx^3}, Y = De^{\frac{1}{3}ky^3}$$

Therefore (with $CD = A$) $u(x,y) = A\, e^{\frac{1}{3}k(x^3+y^3)}$

# Question 5

We have the following problem

$$\Delta u = 0$$
$$u(x,0) = 0$$
$$u(x,b) = 100x$$
$$u_x(0,y) = 0$$
$$u_x(a,y) = 0$$

Assume a solution of the form $u(x,y) = X(x)Y(y)$. Substitute this expression and divide through by $XY$, to get:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$
$$-\frac{X''}{X} = \frac{Y''}{Y}$$

LHS is function of $x$ only and RHS is a function of $y$ only, thus we can write, with $k$ constant

$$-\frac{X''}{X} = \frac{Y''}{Y} = k$$

We see immediately that $X(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$. And $X'(x) = \sqrt{k}\Bigg(B\cos(\sqrt{k}x) - A\sin(\sqrt{k}x)\Bigg)$. Also $u_x(x,y) = X'(x)Y(y)$. Take the boundary condition $u_x(0,y) = X'(0)Y(y) = 0$, since in general $Y(y) \neq 0$ then $X'(0) = 0$. Plug it into $X'(x)$ gives $\sqrt{k}\Bigg(B\,1 - A\,0\Bigg) = 0 \to B = 0$. Now $X(x) = A\cos\Bigg(\sqrt{k}x\Bigg)$ and $X'(x) = -A\sqrt{k}\sin\Bigg(\sqrt{k}x\Bigg)$. Next, with $u_x(a,y) = 0$ gives $X'(a)Y(y) = 0 \to X'(a) = 0$. $X'(a) = 0$ therefore $-A\sqrt{k}\sin\Bigg(\sqrt{k}a\Bigg) = 0 \to \sqrt{k}a = n\pi$. So $k_n = \frac{n^2\pi^2}{a^2}$ and

$$\begin{aligned} X_n(x) &= A_n\cos(\frac{n\pi}{a}x) \\ Y_n(y) &= B_n\cosh(\frac{n\pi}{a}y) + C_n\sinh(\frac{n\pi}{a}y) \end{aligned}$$

The boundary condition $u(x,0) = X(x)Y(0) = 0$ gives the non trivial solution $Y(0) = B_n\,1 + C_n\,0 = 0 \to B_n = 0$. We are left with $u_n(x,y) = A_n\cos(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y)$ (where $A_nC_n \to A_n$). Applying the superposition principle we have

$$u(x,y) = \sum_{n=1}^{\infty} A_n\cos(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y)$$

Finally we apply the initial conditions $u(x,b) = 100x$, therefore

$$u(x,b) = \sum_{n=1}^{\infty} A_n\cos(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}b) = 100x$$

This is a Fourier series and we get

$$\begin{aligned} A_n\sinh(n\pi\frac{a}{b}) &= \frac{2}{a}\int_0^a 100x\cos(\frac{n\pi}{a}x)dx \\ A_n\sinh(n\pi\frac{a}{b}) &= \frac{200}{a}\int_0^a x\cos(\frac{n\pi}{a}x)dx \end{aligned}$$

Now

$$\begin{aligned}
\int_0^a x\cos(\frac{n\pi}{a}x)dx &= \left[x(\frac{a}{n\pi})\sin(\frac{n\pi}{a}x)\right]_0^a - \frac{a}{n\pi}\int_0^a \sin(\frac{n\pi}{a}x)dx \\
&= 0 - \frac{a}{n\pi}\int_0^a \sin(\frac{n\pi}{a}x)dx \\
&= -\frac{a}{n\pi}(-\frac{a}{n\pi})[\cos(\frac{n\pi}{a}x)]_0^a \\
&= \frac{a^2}{n^2\pi^2}(\cos(n\pi)-1) \\
&= \begin{cases} -\frac{2a^2}{n^2\pi^2} & \text{odd n} \\ 0 & \text{even n} \end{cases}
\end{aligned}$$

Substitute it back to get $A_n$

$$\begin{aligned}
A_{n,n\text{ odd}} &= \frac{1}{\sinh(n\pi\frac{a}{b})}\frac{200}{a}(-\frac{2a^2}{n^2\pi^2}) \\
A_{n,n\text{ odd}} &= -\frac{400a}{n^2\pi^2\sinh(n\pi\frac{a}{b})} \\
A_{n,n\text{ odd}} &= -\frac{800a}{n^2\pi^2(e^{n\pi\frac{a}{b}} - e^{-n\pi\frac{a}{b}})} \\
A_{n,n\text{ even}} &= 0
\end{aligned}$$

# Question 6

a. Substituting $u(r,z) = R(r)Z(z)$ into the diffusion equation in cylindrical coordinates gives

$$R''Z + \frac{1}{r}R'Z + RZ'' = 0$$

Diving by $RZ$ gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = 0$$

Separation of variables gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = -k^2$$

or

$$\begin{aligned}
\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} &= -k^2 \\
\frac{Z''}{Z} &= k^2
\end{aligned}$$

b. For $\frac{d^2}{dz^2}Z(z) = k^2 Z(z)$, we immediately see that $Z(z) = c_1e^{kz} + c_2e^{-kz}$ which we can reformulate as $Z(z) = A\sinh(kz) + B\cosh(kz)$.

For $\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=-k^2$, starting with $\frac{d^2R(r)}{dr^2}+\frac{1}{r}\frac{dR(r)}{dr}+k^2R(r)=0$.

$$s=kr, r=\frac{s}{k}, \frac{ds}{dr}=k, R(r)\to R(s)$$
$$\frac{dR}{dr}=\frac{dR}{ds}\frac{ds}{dr}=k\frac{dR}{ds}$$
$$\frac{d^2R}{dr^2}=k\frac{d}{ds}\left(k\frac{dR}{ds}\right)=k^2\frac{d^2R}{ds^2}$$

Substitution into the ODE gives

$$k^2\frac{d^2R(s)}{ds^2}+\frac{1}{\frac{s}{k}}k\frac{dR(s)}{ds}+k^2R(s)=0$$
$$k^2\frac{d^2R(s)}{ds^2}+\frac{1}{s}k^2\frac{dR(s)}{ds}+k^2R(s)=0$$

Multiplying out by $(\frac{s}{k})^2$ gives

$$s^2\frac{d^2R(s)}{ds^2}+s\frac{dR(s)}{ds}+s^2R(s)=0$$
$$s^2\frac{d^2R(s)}{ds^2}+s\frac{dR(s)}{ds}+(s^2-0^2)R(s)=0$$

The last equation being a Bessel equation of order $0$ therefore the solution is of the form

$$R(r)=C_1J_0(kr)+C_2Y_0(kr)$$

Since the temperature remains bounded at $r=0$ thus the term $Y_0(kr)$ has to be discarded, $C_2=0$, and $R(r)=CJ_0(kr), C=C_1$

c. Finally apply boundary conditions. First, $u(r,0)=R(r)Z(0)=0$, since in general $R(r)\neq 0$, thus $Z(0)=0$, which is $A\ 0+B\ 1=0\to B=0$ and $Z(z)=A\sinh(kz)$. Then $u(5,z)=R(5)Z(z)=0\to R(5)=0$, therefore $CJ_0(5k)=0\to J_0(5k)=0$. $5k$ represents the zero crossing for the Bessel function of order 0. We call them $\alpha_m$ and set $5k_m=\alpha_m\to k_m=\frac{\alpha_m}{5}$. Therefore the solutions are $R_m(r)=C_mJ_0(k_mr)=C_mJ_0(\frac{\alpha_m}{5}r)$. Note now that solutions in $z$ are $Z_m(z)=A_m\sinh(k_mz)$. Finally applying the superposition principle, we have

$$u(r,z)=\sum_{m=1}^{\infty}A_m\sinh(k_mz)J_0(\frac{\alpha_m}{5}r)\ \text{ where } A_mC_m\to A_m$$

Applying the last boundary condition $u(r,20)=u_0, 0<r<5$, we get:

$$u(r,20)=\sum_{m=1}^{\infty}A_m\sinh(20k_m)J_0(\frac{\alpha_m}{5}r)=u_0$$

This is a Fourier Bessel series where the coefficients are given by

$$\sinh(20k_m)A_m=\frac{2}{5^2J_1^2(\alpha_m)}\int_0^5 ru_0J_0(\frac{\alpha_m}{5}r)dr=\frac{2u_0}{25J_1(\alpha_m)}\int_0^5 rJ_0(\frac{\alpha_m}{5}r)dr$$

Next, $\sinh(20k_m) = \sinh(20\frac{\alpha_m}{5}) = \sinh(4\alpha_m)$, and we get

$$A_m = \frac{2u_0}{25J_1(\alpha_m)\sinh(4\alpha_m)} \int_0^5 J_0(\frac{\alpha_m}{5}r)rdr$$

Finally using $\frac{\partial}{\partial r}[rJ_1(r)] = rJ_0(r)$

$$\begin{aligned} A_m &= \frac{2u_0}{25J_1(\alpha_m)\sinh(4\alpha_m)} \left[rJ_1(\frac{\alpha_m}{5}r)\right]_0^5 \\ &= \frac{2u_0}{25J_1(\alpha_m)\sinh(4\alpha_m)} \left(5J_1(\frac{\alpha_m}{5}5)\right) \\ &= \frac{2u_0}{5\sinh(4\alpha_m)} \end{aligned}$$

# Question 7

a. Let $z = x + iy$ then $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$ and

$$\begin{aligned} e^{z^2} &= e^{x^2-y^2+2ixy} = e^{x^2-y^2}e^{2ixy} \\ &= e^{x^2-y^2}(\cos 2xy + \sin 2xy) \end{aligned}$$

Now take $u(x,y) = e^{x^2-y^2}\cos 2xy$ and $v(x,y) = e^{x^2-y^2}\sin 2xy$. We have

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x\, e^{x^2-y^2}\cos 2xy + e^{x^2-y^2}(-2y)\sin 2xy \\ \frac{\partial u}{\partial x} &= 2e^{x^2-y^2}(x\cos 2xy - y\sin 2xy) \\ \frac{\partial v}{\partial y} &= -2y\, e^{x^2-y^2}\sin 2xy + e^{x^2-y^2}(2x)\cos 2xy \\ \frac{\partial v}{\partial y} &= 2e^{x^2-y^2}(x\cos 2xy - y\sin 2xy) \\ \Rightarrow \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -2y\, e^{x^2-y^2}\cos 2xy + e^{x^2-y^2}(-2x)\sin 2xy \\ \frac{\partial u}{\partial y} &= -2e^{x^2-y^2}(y\cos 2xy + x\sin 2xy) \\ \frac{\partial v}{\partial x} &= 2x\, e^{x^2-y^2}\sin 2xy + e^{x^2-y^2}(2y)\cos 2xy \\ \frac{\partial v}{\partial x} &= 2e^{x^2-y^2}(y\cos 2xy + x\sin 2xy) \\ \Rightarrow \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

The Cauchy-Riemann conditions are satisfied therefore $e^{z^2}$ is analytic (as required, since both $z^2$ and $e^z$ are entire functions).

b. For $v(x,y) = xy$, we have

$$\begin{aligned} v_x &= y, v_{xx} = 0 \\ v_y &= x, v_{yy} = 0 \\ \rightarrow & v_{xx} + v_{yy} = 0 \end{aligned}$$

Using Cauchy conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, we have $\frac{\partial u}{\partial x} = x$ integration on booth sides gives:

$$\begin{aligned} u &= \int x dx + g(y) \\ u &= \frac{x^2}{2} + g(y) \end{aligned}$$

Next the Cauchy condition $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ yields

$$\begin{aligned} \frac{\partial u}{\partial y} &= g'(y) = -\frac{\partial v}{\partial x} = -y \\ g(y) &= -\frac{y^2}{2} + C \end{aligned}$$

Therefore $u(x,y) = \frac{x^2-y^2}{2} + C$ and $f(x,y) = \frac{x^2-y^2}{2} + ixy$

# Question 8

a. Substitute $y = x^2, dy = 2xdx$ then we get

$$\begin{aligned} \int_C (x^2 + y)dx + 2xydy &= \int_0^1 (x^2 + x^2)dx + 2x\, x^2 2xdx \\ &= \int_0^1 (2x^2 + 4x^4)dx \\ &= 2[\frac{x^3}{3}]_0^1 + 4[\frac{x^5}{5}]_0^1 \\ &= \frac{2}{3} + \frac{4}{5} \\ &= \frac{10+12}{15} = \frac{22}{15} \end{aligned}$$

b. Take $z = x_i y, Re(z) = x$, let parametrize the line from $1$ to $i$ as $z(t) = 1 \cdot (1-t) + i \cdot t, = 1 + (i-1)\,t, dz = (i-1)dt$ for $0 \leq t \leq 1$. We have then

$$\begin{aligned} \int_C Re(z)dz &= \int_0^1 (1-t)(i-1)dt \\ &= (i-1)\int_0^1 (1-t)dt \\ &= (i-1)[t - \frac{t^2}{2}]_0^1 \\ &= (i-1)\frac{1}{2} = \frac{i-1}{2} \end{aligned}$$

# Extra Credit

a. Take $f(z) = \frac{1}{(z-2)(z-1)^2} = \frac{1}{z-2} - \frac{z}{(z-1)^2}$, the poles are at $z = 2$ and $z = 1$. We compute the residues at each pole. At $z = 2$

$$f(z) = \frac{1}{z-2} + \text{something analytic at } 2$$

Therefore the pole is simple and $\text{Res}(f, 2) = 1$ At $z = 1$. Take $z = \xi + 1$ then

$$\begin{aligned}
f(\xi) &= \frac{1}{(\xi - 1)\xi^2} \\
&= -\frac{1}{\xi^2}(1 - \xi)^{-1} \\
&= -\frac{1}{\xi^2}(1 + \xi + \xi^2 + \xi^3 + \cdots) \\
&= -\frac{1}{\xi^2} - \frac{1}{\xi} - \frac{1}{2} - \xi + \cdots
\end{aligned}$$

Therefore the pole is of order 2 and $\text{Res}(f, 1) = -1$, Poles within the circle centered at $z = 0$ with radius 4 are 1 and 2. Applying the residue theorem

$$\oint_C f(z)dz = \oint_C \frac{1}{(z-2)(z-1)^2}dz = 2\pi i(\text{Res}(f, 2) + \text{Res}(f, 1)) = 2\pi i\ (1 - 1) = 0$$

b. We have

$$\begin{aligned}
e^z &= \sum_{k=0}^{\infty} \frac{1}{k!} z^k \\
e^{\frac{1}{z^n}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^{n+k}} \\
&= \frac{1}{z^n} + \frac{1}{z^{n+1}} + \frac{1}{z^{n+2}} + \cdots
\end{aligned}$$

Looking at $\lim_{z\to 0}(z - 0)^\alpha \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^{n+k}}$, there is no largest $\alpha$ such this quantity is finite. Thus 0 is an essential singularity. The Laurent expansion has only negative powers of z starting at n. We see that $\text{Res}(e^{\frac{1}{z^n}}, 0) = 0$. $e^{\frac{1}{z^n}}$ has only the singularity 0 within the unit circle, therefore applying the residue theorem

$$\oint_C e^{\frac{1}{z^n}} dz = 2\pi i\ 0 = 0$$