Math for Applied Biomedical Engineering Final Exam [Total Points - 100]
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**SHOW ALL WORK FOR FULL CREDIT**

1. [10] a. Find the Fourier series for $f(x)=x$ period from $-\pi$ to $\pi$ (same for all parts).

   b. Integrate the answer from part a. to find a Fourier series for $f(x)=x^2$

   c. Alternately find the Fourier series for $f(x)=x^2$ by finding the appropriate coefficients directly for this function.

   d. Compare your answers from b. and c.

2. [15] Solve the following differential equation using generalized power series or Frobenius series solution method $z\frac{d^2y}{dz^2}+y=0$ by the following steps:

   a. Determine the singular points if any for this equation.

   b. Find and solve the indicial equation

   c. Using the larger root from b. solve for one of the independent solutions for the given D.E.

3. [15] Given the following defining equation for Legendre polynomials

$$P_n(x)=\sum_{m=0}^{M}(-1)^m\frac{(2n-2m)!}{2^n(n-m)!(n-2m)!}x^{n-2m};\quad M=\begin{cases} n/2 & \text{n-even} \\ (n-1)/2 & \text{n- odd}\end{cases}$$

   a. Generate the polynomials for n = 0, 1, 2. Show all work!

   The coefficients for the Fourier-Legendre series, $f(x)=\sum_{n=0}^{\infty}a_nP_n(x)$

   are given by $a_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx$

   b. Calculate the coefficients (for n = 0, 1 and 2) of the Fourier-Legendre series for the function f(x) = x.

   c. Use Rodrigues's formula, $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$ to generate the same Legendre polynomials as in part a. (that is for n = 0, 1, 2). Show all work!

   **Extra credit** [10] Given an even function, f(x) show that the odd coefficients generated by $a_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx$ are zero and alternately for an odd function the even coefficients are zero. Show all work!

4. [10] Find general solutions for the following differential equations. Assume all solutions are functions of x and y, that is u(x,y).

a. $\frac{\partial u}{\partial x} + 4xu = 0$ hint: don't make this harder than it is

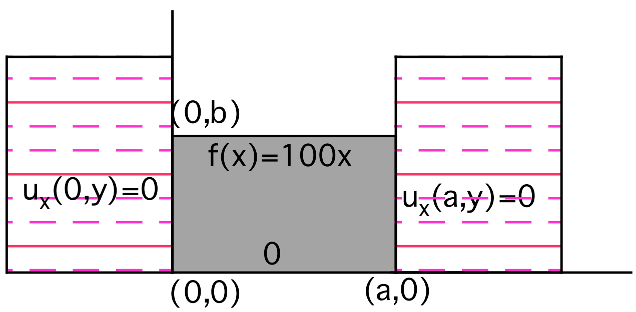
b. $y^2 u_x - x^2 u_y = 0$ hint: the separation variable is simply equal to k

5. [15] Solve the steady state heat equation from scratch, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for the following rectangular plate [dimensions a by b].

Note $u_x \equiv \frac{\partial u}{\partial x}$

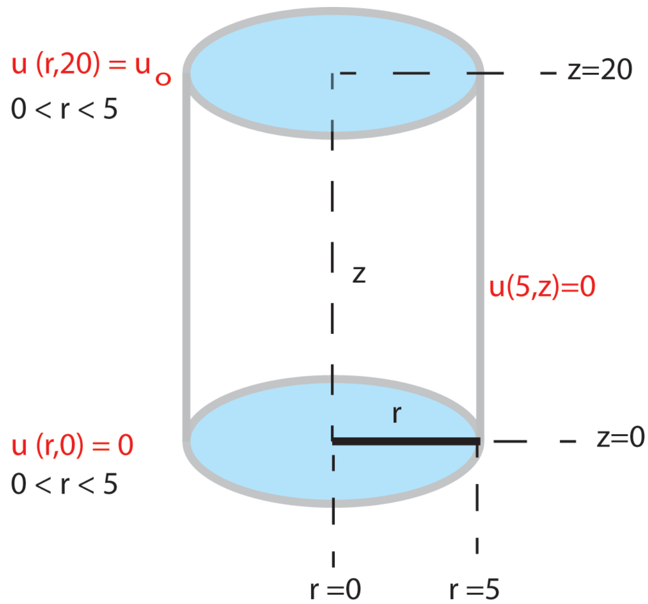
and from the diagram we have

1. On bottom edge (x,0) - we keep the edge at 0 °C
2. On top edge (x,b): we have f(x) = 100x °C
3. and 4. $u_x(0,y) = 0$, $u_x(a,y) = 0$, on left and right respectively, that is perfect insulators - no heat flow!




hint: use k as the separation variable

6. [15] Find the steady state distribution of temperature in the cylinder




given that the sides and bottom are kept at zero temperature and the top remains at a constant of temperature $u_o$ using the following diffusion equation in cylindrical coordinates:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

a. Use the technique of separation of variables wit $u(r,z)=R(r)Z(z)$ to get equations for R and Z. Let the separation constant be " $-k^2$ ".
b. Solve the differential equations in R and Z.
c. Apply the boundary conditions above (in red). Also (as a forth condition) note that the temperature remains bounded at $r = 0$.

hint1: In applying conditions for the function Z(z) at some point you need to rewrite your solution in terms of a "real" hyperbolic function – this substitution should be obvious after looking at the definitions of these functions.

hint2: Finally your superposition solution will look like a series in Bessel functions (see Section 9.5 – Eqs. 9.87 and 9.88) for which you need to solve for the coefficients. At this point you do not need to solve the integral for the coefficients, just set it up!

**Extra credit** [5] Solve the integral equation for the series coefficients from part c. hint: You will need to use the following Bessel function identity

$$\frac{\partial}{\partial r}[rJ_1(r)]=rJ_0(r)$$

7. [10] a. Verify whether the complex function $f(z) = e^{z^2}$ is analytic or not. Show all work!

b. Show that the following function $v = xy$ is harmonic ($v_{xx} + v_{yy} = 0$) and find the corresponding analytic function $f(x,y) = u(x,y) + iv(x,y)$.
hint: use Cauchy-Riemann to help, similar to homework

8. [10] a. Evaluate the following line integral

$$\int_C (x^2 + y)dx + 2xydy$$

on the curve $y = x^2$, from (0,0) to (1,1)

b. Evaluate $\int_C \operatorname{Re} z \, dz$ on line from 1+0i to 0+1i in the complex plane

**Extra credit** [10]

a. Evaluate the following integral $\oint \frac{1}{(z-2)(z-1)^2} dz$ where the closed path is a circle centered at $z = 0$ with radius 4 in the counter-clockwise direction.
hint: residue

b. Evaluate $\oint e^{1/z^n} dz$ around closed path of unit circle centered at $z = 0$ in counter-clockwise direction. hint: Find the Laurent series first