Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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JOHNS HOPKINS
WHITING SCHOOL
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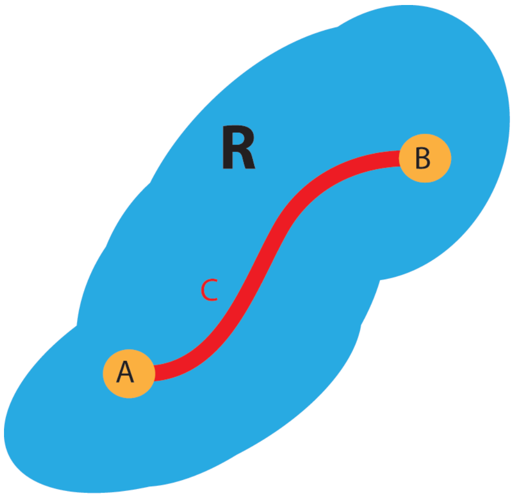
Complex Integrals

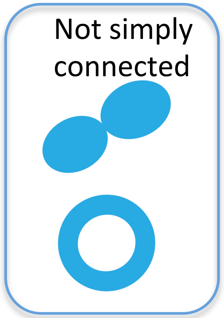
R is a simply connected region



Integration along path from A to B

$$x = x(t)$$
$$y = y(t)$$
$$\text{where } \alpha \le t \le \beta \text{ and}$$
$$\alpha \Rightarrow \text{pt } A(x,y)$$
$$\beta \Rightarrow \text{pt } B(x,y)$$

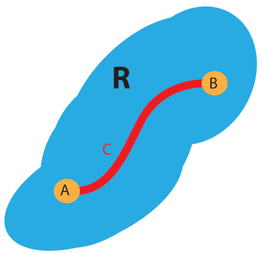
Aside:



KEY: If a complex function is single valued and continuous in some region R its value in general will depend on the path taken! On the other hand some paths may be different and the value of the integral does not depend on the path only the beginning and end points.

We will start at by looking at the general technique of performing a path integral in the complex plane of a function of a complex variable. The technique depends on parameterization of the function to restrict it to the path and is very similar to that studied in a multivariate calculus class however we will discover some powerful theorems in the complex domain.

General representation of a path integral in complex plane



$$\int_C f(z)dz = \int_{\alpha(x,y)}^{\beta(x,y)} [u(x,y)+iv(x,y)](dx+idy) =$$

$$\int_{\alpha(x,y)}^{\beta(x,y)} u(x,y)dx - v(x,y)dy + i[u(x,y)dy + u(x,y)dx] =$$

Letting for example $u(x,y) \rightarrow u(x(t),y(t)) \equiv u(t)$ and $dx = \frac{dx}{dt}dt$

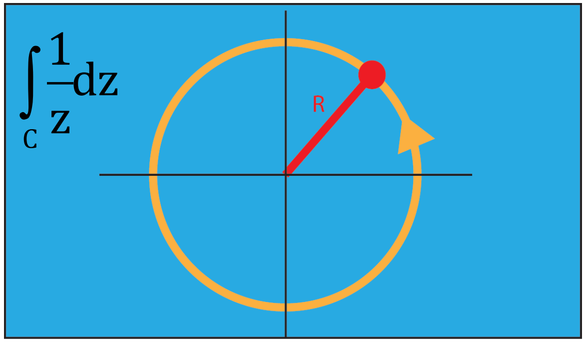
and separating integral

$$\int_{\alpha(t)}^{\beta(t)} u(t)\frac{dx}{dt}dt - v(t)\frac{dy}{dt}dt + i\int_{\alpha(t)}^{\beta(t)} u(t)\frac{dy}{dt}dt + v(t)\frac{dx}{dt}dt =$$

$$\boxed{\int_{\alpha(t)}^{\beta(t)} \left(u(t)\frac{dx}{dt} - v(t)\frac{dy}{dt} \right) dt + i\int_{\alpha(t)}^{\beta(t)} \left(u(t)\frac{dy}{dt} + v(t)\frac{dx}{dt} \right) dt}$$

Remember $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$ where $\alpha \le t \le \beta$ and $\begin{matrix} \alpha \Rightarrow \text{pt } A(x,y) \\ \beta \Rightarrow \text{pt } B(x,y) \end{matrix}$

Examples of path integrals in complex plane



Integrate $f(z)=\dfrac{1}{z}$

Path C: Counter clockwise circle of radius R that starts and ends on real axis

Parameterize path in terms of t:

$$z(t)=x+iy=R(\cos t+i\sin t)\ \alpha=0\le t\le 2\pi=\beta$$

taking $x=R\cos t$ and $y=R\sin t$

Write f(z) in "standard form", that is

$$f(z)=\frac{1}{z}=\frac{1}{x+iy}=\frac{1}{x+iy}\frac{x-iy}{x-iy}=\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}=f(x,y)=u(x,y)+iv(x,y)$$

KEY: Restrict x and y in f(z) = f(x,y) to path by substitution of our x and y from path above.

$$u(x,y)=\frac{x}{x^2+y^2}\rightarrow u(t)=\frac{R\cos t}{(R\cos t)^2+(R\sin t)^2}=\frac{R\cos t}{R^2(\cos^2 t+\sin^2 t)}=\frac{R\cos t}{R^2}=\frac{\cos t}{R}$$

$$v(x,y)=\frac{-y}{x^2+y^2}\rightarrow u(t)=\frac{-R\sin t}{(R\cos t)^2+(R\sin t)^2}=\frac{-R\sin t}{R^2(\cos^2 t+\sin^2 t)}=\frac{-R\sin t}{R^2}=\frac{-\sin t}{R}$$

Next convert the derivatives to a function of t

$$x = R\cos t \rightarrow \frac{dx}{dt} = -R\sin t \text{ and } y = R\sin t \rightarrow \frac{dy}{dt} = R\cos t$$

Then to evaluate our complex integral we use our previously constructed integration as a function of the parameter t to restrict the integral to the path C.

$$\int_C f(z)dz = \int_{\alpha(t)}^{\beta(t)} \left(u(t)\frac{dx}{dt} - v(t)\frac{dy}{dt} \right) dt + i\int_{\alpha(t)}^{\beta(t)} \left(u(t)\frac{dy}{dt} + v(t)\frac{dx}{dt} \right) dt$$

Key: This restricts our integration to a path mapped out by the parameter t. As t varies from 0 to 2pi it maps out our circle.

$$\int_0^{2\pi} \left[\frac{\cos t}{R}(-R\sin t) - \frac{-\sin t}{R}(R\cos t) \right] dt + i\int_0^{2\pi} \left[\frac{\cos t}{R}(R\cos t) + \frac{-\sin t}{R}(-R\sin t) \right] dt =$$

FIXED

$$\int_0^{2\pi} \left[-\cos t \sin t + \sin t \cos t) \right] dt + i\int_0^{2\pi} \left[\cos^2 t + \sin^2 t \right] dt = \int_0^{2\pi} 0\,dt + i\int_0^{2\pi} dt = 2\pi i$$

Alternately, we can convert the complex integral from z to t in place, that is

$$z = x + iy = R\cos t + iR\sin t = R(\cos t + i\sin t)$$

Then taking a derivative $\frac{dz}{dt} = R(-\sin t + i\cos t) \rightarrow dz = R(-\sin t + i\cos t)dt$

Substitution into our orginal integral

$$\int_C \frac{dz}{z} \rightarrow \int_0^{2\pi} \frac{R(-\sin t + i\cos t)dt}{R(\cos t + i\sin t)} = \int_0^{2\pi} \frac{(-\sin t + i\cos t)}{(\cos t + i\sin t)}dt$$

Multiple and divide by i gives $\int_0^{2\pi} \frac{i}{i}\frac{(-\sin t + i\cos t)}{(\cos t + i\sin t)}dt = i\int_0^{2\pi} \frac{(-\sin t + i\cos t)}{i(\cos t + i\sin t)}dt =$

$$i\int_0^{2\pi} \frac{(-\sin t + i\cos t)}{i\cos t - \sin t}dt = i\int_0^{2\pi} dt = 2\pi i$$

The same answer as before!

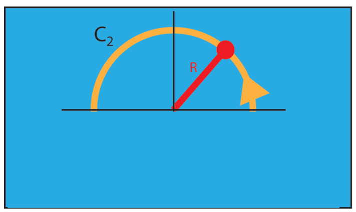
More Examples of path integrals in complex plane

Let's continue our example for the same function but choose some different paths.

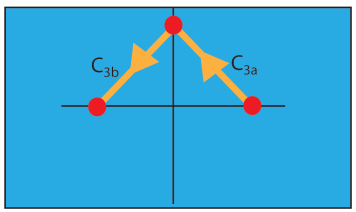
Path 2: A semi-circle on half-plane

Path 3: Made up of
Path 3a: From (x,y) = (R,0) to (0,R)
Path 3b: From (x,y) = (0,R) to (-R,0)

Path 2



Paths 3a and 3b



For path 2 just perform the integral as before but limit the path to the half circle. Picking up at the point when we previously evaluated the integral as a function of t we have

As we did previuosly let $s = \cos t \; ds = -\sin t dt$

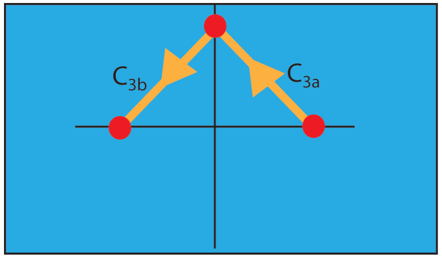
Except in this case take the endpoint for the variable t = π, therefore

$$2\int_{s=\cos 0=1}^{s=\cos\pi=-1} s\,ds + i\int_{0}^{\pi} dt = 2\left.\frac{s^2}{2}\right|_{1}^{-1} + \pi i = (-1)^2 - (1)^2 + \pi i = \pi i$$

Half the value we previously calculated!

For path 3 we need to do additional work. First let's set up the parameter t for each path 3a and 3b.

Paths 3a and 3b



For path 3a: (x,y) goes from $(R,0)\rightarrow(0,R)$

Therefore on this path of a straight line

we represent $z = x+iy$ in terms of the parameter t

where $x=R(1-t)$, $y=Rt$ and $0\le t\le 1$, that is

$z_a = R(1-t)+iRt$ (subscript a to denote path 3a)

Then $dz_a = -Rdt+iRdt=(-R+iR)dt$

On path 3b: (x,y) goes from $(0,R)\rightarrow(-R,0)$

Similarly we find

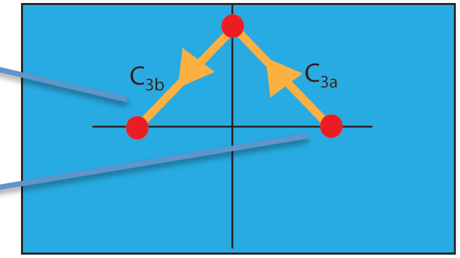
$z_b = -Rt+iR(1-t)$ where $0\le t\le 1$

Then $dz_b = -Rdt+iRdt=(-R-iR)dt$

Then the integral for path $C_3 = C_{3a}+C_{3b}$ can be represented as

$$\int_{C_3}\frac{1}{z}dz=\int_{C_{3a}}\frac{1}{z}dz+\int_{C_{3b}}\frac{1}{z}dz$$

Substitution for each part of the path 3a and 3b gives



$$\int_{C_3} \frac{1}{z} dz \to \int_0^1 \frac{1}{R(1-t)+iRt}(-R+iR)dt + \int_0^1 \frac{1}{-Rt+iR(1-t)}(-R-iR)dt =$$

Next cancel out all the constants R and put integrand in "standard" form

$$\int_0^1 \frac{-R+iR}{R(1-t)+iRt}dt + \int_0^1 \frac{-R-iR}{-Rt+iR(1-t)}dt = \int_0^1 \frac{-1+i}{(1-t)+it}dt + \int_0^1 \frac{-1-i}{-t+i(1-t)}dt =$$

$$\int_0^1 \frac{-1+i}{(1-t)+it}\left[\frac{(1-t)-it}{(1-t)-it}\right]dt + \int_0^1 \frac{-1-i}{-t+i(1-t)}\left[\frac{-t-i(1-t)}{-t-i(1-t)}\right]dt =$$

$$\int_0^1 \frac{2t-1+i}{1-2t+2t^2}dt + \int_0^1 \frac{2t-1+i}{1-2t+2t^2}dt = 2\int_0^1 \frac{2t-1+i}{1-2t+2t^2}dt =$$

$$2\int_0^1 \frac{2t-1}{1-2t+2t^2}dt + 2i\int_0^1 \frac{1}{1-2t+2t^2}dt$$

Next we need to evaluate the integrals.

Now use integration by substitution for the first integral

with $u = 1-2t+2t^2, \ \dfrac{du}{dt} = -2+4t \ \rightarrow \dfrac{du}{2} = (2t-1)dt$

and complete the square in denominator of second integral

$$1-2t+2t^2 = 2\left[t^2 - t + \frac{1}{2}\right] = 2\left[t^2 - t + \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 + \frac{1}{2}\right] = \cdots = 2\left[\left(t-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]$$

Substitution into the integrals gives

$$2\int_0^1 \frac{2t-1}{1-2t+2t^2}dt + 2i\int_0^1 \frac{1}{1-2t+2t^2}dt \rightarrow 2\int_{u=1-2\cdot 0+2\cdot 0^2=1}^{u=1-2\cdot 1+2\cdot 1^2=1} \frac{du/2}{u} + 2i\int_0^1 \frac{1}{2\left[\left(t-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]}dt =$$

$$0 + i\int_0^1 \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}dt = i\frac{1}{\left(\frac{1}{2}\right)}\tan^{-1}\left[\frac{t-\frac{1}{2}}{\frac{1}{2}}\right]\Bigg|_0^1 = 2i[\tan^{-1}1 - \tan^{-1}(-1)] = 2i\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi i$$

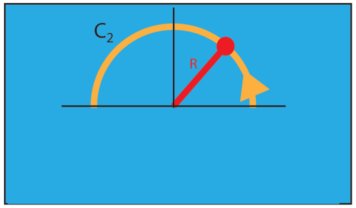
Use integral table lookup or by trigonometric substitution

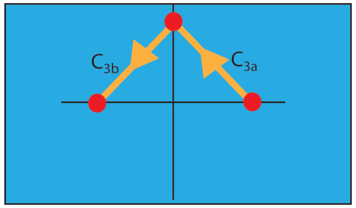
Conclusions from our examples

For the path integrals of the function $f(z)=\frac{1}{z}$

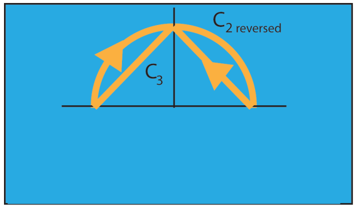
Key: Singularity at z =0

1. Paths C_2 and C_3 which have the same end points but whose specific paths differ give the same value πi for the integral.

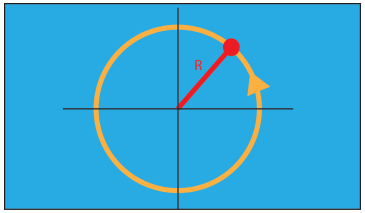




2. If we take a closed path C_3 + reverse path for C_2 (from –R,0 to R,0) then the values on these paths cancel out to give 0. That is the closed path integration gives a zero value.



3. If however we take a closed path C around ending at the same place as the start, that is (R,0) the value of the path integral is $2\pi i$.



Key: If the closed path includes a singular point of the function being integrated the answer differs!