# 1 Fourier Series

Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[ a_r \cos(\frac{2\pi rx}{L}) + b_r \sin(\frac{2\pi rx}{L}) \right]$$

with

$$a_0 = \frac{2}{L} \int_{x_0}^{x_0+L} f(x)dx$$
$$a_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi rx}{L}\right) dx$$
$$b_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi rx}{L}\right) dx$$

For an even function, we have:

$$a_0 = \frac{4}{T} \int_0^{\frac{L}{2}} f(x)dx \quad a_r = \frac{4}{T} \int_0^{\frac{L}{2}} f(x) \cos\left(\frac{2\pi rx}{L}\right) dx \quad b_r = 0$$

For an odd function, we have:

$$a_0 = 0 \quad b_r = \frac{4}{T} \int_0^{\frac{L}{2}} f(x) \sin\left(\frac{2\pi rx}{L}\right) dx$$

Complex Fourier series

$$f(x) = \sum_{r=-\infty}^{r=\infty} c_r e^{\frac{i2\pi rx}{L}}$$

where

$$c_r = \frac{1}{L} \int_{x_0}^{x_0+L} f(x) e^{-\frac{i2\pi rx}{L}} dx$$
$$c_r = \frac{1}{2}(a_r - ib_r)$$

Parseval Identity

$$\frac{1}{L} \int_{x_0}^{x_0+L} |f(x)|^2 dx = \sum_{r=-\infty}^{r=\infty} |c_r|^2$$

# 2 Fourier transforms

Fourier transform of $f(t)$:

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt}\, dt$$

And its inverse defined by:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w) e^{iwt}\, dw$$

# 3 The Dirac $\delta$-Function

$\delta(t) = 0$ for $t = 0$. Provided the range of integration includes the point $t = a$:

$$\int f(t)\delta(t-a)\,dt = f(a)$$

otherwise the integral equals $0$. This leads to:

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(t) f(t) dt &= f(0) \\
\int_{-a}^{b} \delta(t)\,dt &= 1 \text{ for all } a, b > 0 \\
\int \delta(t-a)\,dt &= 1 \text{ if range of integration includes a} \\
\delta(t) &= \delta(-t) \\
\delta(bt) &= \frac{1}{|b|}\delta(t) \\
t\delta(t) &= 0 \\
\delta(h(t)) &= \sum_i \frac{\delta(t-t_i)}{|h'(t_i)|} \text{ where the } t_i \text{ are the zeros of } h(t)
\end{aligned}$$

The derivatives $\delta^n(t)$ are defined by:

$$\int_{-\infty}^{\infty} f(t)\delta^n(t)\,dt = (-1)^n f^n(0)$$

Integral representation:

$$\delta(t-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iw(t-u)}\,dw$$

Dirac function's relation to the sinc function:

$$\delta(t) = \lim_{\Omega\to\infty} \frac{\sin(\Omega t)}{\pi t}$$

# 4 The Heaviside function

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$H'(t) = \delta(t)$.

## 5 Properties of Fourier transforms

- $L\{t^n\} = \frac{n!}{s^{n+1}}$
- $F\{e^{\alpha t} f(t)\} = \tilde{f}(w + \alpha t)$
- $F\{f(t+a)\} = e^{iwa}\tilde{f}(w)$
- $F\{f(at)\} = \frac{1}{a}\tilde{f}(\frac{w}{a})$
- $\mathcal{F}\big[f^n(t)\big] = (i)^n w^n \tilde{f}(w)$
- $\mathcal{F}\big[\int^t f(s)ds\big] = \frac{1}{iw} f(\tilde{w}) + 2\pi c\delta(w)$

The Fourier sine transform: $\tilde{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(wt)dt$

## 6 Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk$$

## 7 Laplace transform

By definition:

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st} \, dt$$

And

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}$$

Laplace transform properties:

- $L\{e^{at} f(t)\} = \bar{f}(s-a)$ or $L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t)$
- $L\{f(t-b)H(t-b)\} = e^{-bs}\bar{f}(s)$ or $(t-b)H(t-b) = L^{-1}\{e^{-bs}\bar{f}(s)\}$
- $L\{f'(t)\} = -f(0) + s\tilde{f}(s)$
- $L\{f''(t)\} = -sf(0) - f'(0) + s^2\tilde{f}(s)$
- $L\{\int_0^t f(x)dx\} = \frac{1}{s}\tilde{f}(s)$