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Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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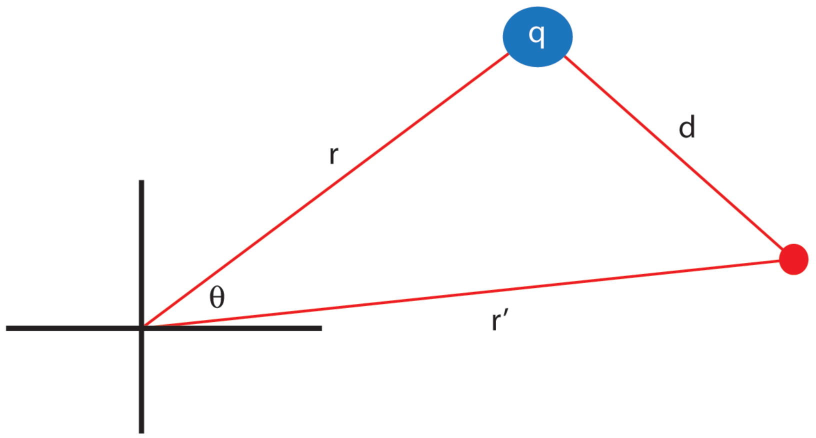
Generating function for Legendre polynomials of the first kind

Suppose we want to find the potential of a point charge q at the red point in the diagram. Given the distances from the origin of the two positions and the angle between them. Since the potential is proportional to 1/d we need to calculate this quantity.
This also leads us to what is also called a **generating function** for Legendre polynomials (next slide).



For a more general mathematical treatment see the paper “Generating functions of Legendre polynomials: A tribute to Fred Brafman”, by James Wan and Wadim Zudilin, Journal of Approximation Theory 164 (2012) 488–503.

Taking the distance as a function of θ we have $d^2 = r^2 + r'^2 - 2rr'\cos\theta$

and the inverse distance $\dfrac{1}{d} = \dfrac{1}{(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}} = \dfrac{1}{r\left[1^2 + \left(\dfrac{r'}{r}\right)^2 - 2\left(\dfrac{r'}{r}\right)\cos\theta\right]^{1/2}} =$

Letting $h = \dfrac{r'}{r}$, $x = \cos\theta$ we have $\dfrac{1}{r(1+h^2-2hx)^{1/2}}$

Next let $y = 2hx - h^2$ and use a Taylor expansion near 0 for the root

that is, $(1-y)^{-1/2} = \dfrac{1}{(1-y)^{1/2}} = 1 + \dfrac{1}{2}y + \dfrac{3}{8}y^2 + \dfrac{5}{16}y^3 + \cdots$

Now let's just FOCUS on this expansion of $(1-y)^{-1/2}$, that is

$$\frac{1}{(1+h^2-2hx)^{1/2}} = \frac{1}{(1-(2hx-h^2))^{1/2}} = 1 + \frac{1}{2}(2hx-h^2) + \frac{3}{8}(2hx-h^2)^2 + \cdots =$$

$$1 + \left(hx - \frac{h^2}{2}\right) + \left(\frac{3}{2}h^2x^2 - \frac{3}{2}h^3x + \frac{3}{8}h^4\right) + \cdots = 1 + hx + h^2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \cdots = P_0(x) + P_1(x)h + P_2(x)h^2 + \cdots$$

Therefore $\boxed{G(x,h) = \dfrac{1}{(1+h^2-2hx)^{1/2}} = P_0(x) + P_1(x)h + P_2(x)h^2 + \cdots = \sum_{n=0}^{\infty} P_n(x)h^n}$

Alternatively we can show that the generating function representation in fact presents us with the Legendre polynomials that solve Legendre's differential equation! (Next slide->)

1. Start with $G(x,h)=\dfrac{1}{(1+h^2-2hx)^{1/2}}\equiv(1+h^2-2hx)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)h^n$

First a derivative with respect to x of G(x,h) is $\dfrac{\partial}{\partial x}G(x,h)=\dfrac{\partial}{\partial x}(1+h^2-2hx)^{-1/2}=\dfrac{\partial}{\partial x}\sum_{n=0}^{\infty}P_n(x)h^n$

This gives $-\dfrac{1}{2}(1+h^2-2hx)^{-3/2}(-2h)=(1+h^2-2hx)^{-3/2}h=\sum_{n=0}^{\infty}P_n{}'(x)h^n$

2. Next take a derivative with respect to h of G(x,h), that is $\dfrac{\partial}{\partial h}(1+h^2-2hx)^{-1/2}=\dfrac{\partial}{\partial h}\sum_{n=0}^{\infty}P_n(x)h^n$

This gives $(1+h^2-2hx)^{-3/2}(x-h)=\sum_{n=0}^{\infty}P_n(x)nh^{n-1}$

3. Next multiply G(x,h) by h, this gives $\boxed{(1+h^2-2hx)^{-1/2}h=h\sum_{n=0}^{\infty}P_n(x)h^n}$

and then multiply $\dfrac{\partial G}{\partial x}$ (derived in 1.) by $(1+h^2-2hx)$

This gives $(1+h^2-2hx)^{-3/2}h(1+h^2-2hx)=\boxed{(1+h^2-2hx)^{-1/2}h=(1+h^2-2hx)\sum_{n=0}^{\infty}P_n{}'(x)h^n}$

Equating these two expressions gives $h\sum_{n=0}^{\infty}P_n(x)h^n=(1+h^2-2hx)\sum_{n=0}^{\infty}P_n{}'(x)h^n$

or $\sum_{n=0}^{\infty}P_n(x)h^{n+1}=\sum_{n=0}^{\infty}P_n{}'(x)h^n+\sum_{n=0}^{\infty}P_n{}'(x)h^{n+2}-2x\sum_{n=0}^{\infty}P_n{}'(x)h^{n+1}$

4. Reindex $\sum_{n=0}^{\infty} P_n{}'(x)h^n$, letting $n \to n+1$ gives $\sum_{n+1=1}^{\infty} P_{n+1}{}'(x)h^{n+1} \equiv \sum_{n=-1}^{\infty} P_{n+1}{}'(x)h^{n+1}$

Note for n = -1 $P_{-1+1}{}'(x) = P_0{}'(x) = 0$ (remember $P_0(x) = 1$)

Therefore $\sum_{n=0}^{\infty} P_n{}'(x)h^n = \sum_{n=-1}^{\infty} P_{n+1}{}'(x)h^{n+1} \equiv \sum_{n=0}^{\infty} P_{n+1}{}'(x)h^{n+1}$

For $\sum_{n=0}^{\infty} P_n{}'(x)h^{n+2}$ let $n \to n-1$ gives $\sum_{n-1=0}^{\infty} P_{n-1}{}'(x)h^{n+1} = \sum_{n=1}^{\infty} P_{n-1}{}'(x)h^{n+1}$

Substitution these results into the results from step 3., that is

$$\sum_{n=0}^{\infty} P_n(x)h^{n+1} = \boxed{\sum_{n=0}^{\infty} P_n{}'(x)h^n} + \boxed{\sum_{n=0}^{\infty} P_n{}'(x)h^{n+2}} - 2x\sum_{n=0}^{\infty} P_n{}'(x)h^{n+1}$$

gives

$$\sum_{n=0}^{\infty} P_n(x)h^{n+1} = \sum_{n=0}^{\infty} P_{n+1}{}'(x)h^{n+1} + \sum_{n=1}^{\infty} P_{n-1}{}'(x)h^{n+1} - 2x\sum_{n=0}^{\infty} P_n{}'(x)h^{n+1}$$

So for $n \geq 1$ we can match powers of h and we have $P_n(x) = P_{n+1}{}'(x) + P_{n-1}{}'(x) - 2xP_n{}'(x)$

5. Multiplying $\frac{\partial G}{\partial x}$ (step 2.)by (x-h) gives $(1+h^2-2hx)^{-3/2}h(x-h)=(x-h)\sum_{n=0}^{\infty}P_n{}'(x)h^n$

and $\frac{\partial G}{\partial h}$ (step 3.) by h gives $(1+h^2-2hx)^{-3/2}(x-h)h=\sum_{n=0}^{\infty}P_n(x)nh^{n-1}h=\sum_{n=0}^{\infty}P_n(x)nh^n$

Equation these expressions gives

$$(x-h)\sum_{n=0}^{\infty}P_n{}'(x)h^n=\sum_{n=0}^{\infty}P_n(x)nh^n$$

or $x\sum_{n=0}^{\infty}P_n{}'(x)h^n-\sum_{n=0}^{\infty}P_n{}'(x)h^{n+1}=\sum_{n=0}^{\infty}P_n(x)nh^n$

6. Again note $P_n{}'(x)=0$ in term 1.

Also reindexing term 2 gives $\sum_{n=0}^{\infty}P_n{}'(x)h^{n+1}\to\sum_{n=1}^{\infty}P_{n-1}{}'(x)h^n$

Finally on the RHS $\sum_{n=0}^{\infty}P_n(x)nh^n\equiv\sum_{n=1}^{\infty}P_n(x)nh^n$ since the n = 0 term does not contribute

Substitution into the expression in 5. gives

$$\sum_{n=1}^{\infty}xP_n{}'(x)h^n-\sum_{n=1}^{\infty}P_{n-1}{}'(x)h^n=\sum_{n=1}^{\infty}P_n(x)nh^n$$

For $n\geq 1$ matching powers of h gives $xP_n{}'(x)-P_{n-1}{}'(x)=nP_n(x)$

Given the two derived identities

$$1.\ P_n(x) = P_{n+1}'(x) + P_{n-1}'(x) - 2xP_n'(x)$$

$$2.\ xP_n'(x) - P_{n-1}'(x) = nP_n(x)$$

Rearranging 2. gives $P_{n-1}'(x) = xP_n'(x) - nP_n(x)$

and substitution in 1. gives

$$P_n(x) = P_{n+1}'(x) + [xP_n'(x) - nP_n(x)] - 2xP_n'(x)$$

or

$$(1+n)P_n(x) = P_{n+1}'(x) - xP_n'(x)$$

Let $n \rightarrow n\text{-}1$ gives expression

$$3.\ \ nP_{n-1}(x) = P_n'(x) - xP_{n-1}'(x)$$

Next take 2. again and multiply by x giving $x^2P_n'(x) - xP_{n-1}'(x) = nxP_n(x)$

and add to 3.

$$nP_{n-1}(x) + [x^2P_n'(x) - xP_{n-1}'(x)] = P_n'(x) - xP_{n-1}'(x) + [nxP_n(x)]$$

Simplifying and collecting similar terms gives

$$(1\text{-}x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$$

Taking a derivative of this expression gives

$$-2xP_n'(x)+(1-x^2)P_n''(x)=nP'_{n-1}(x)-[nP_n(x)+nxP_n'(x)]$$

Using 2. again multiply by n and rearranging gives

$$nP_{n-1}'(x)=nxP_n'(x)-n^2P_n(x)$$

Substitution of this into the expression above gives

$$-2xP_n'(x)+(1-x^2)P_n''(x)=[nxP_n'(x)-n^2P_n(x)]-[nP_n(x)+nxP_n'(x)]$$

Simplification gives

$$(1-x^2)P_n''(x)-2xP_n'(x)=-n^2P_n(x)-nP_n(x)=-n(n+1)P_n(x)$$

Finally we have Legendre's differential equation

$$\boxed{(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0}$$

Note we also derived a couple of identities along the way!