4.10

From 4.9

$a_0 = \sinh(1)$

$a_r = \dfrac{2\sinh(1)(-1)^r}{1+\pi^2 r^2}$, note also includes r = 0 case

$b_r = \dfrac{2\pi r\sinh(1)(-1)^{r+1}}{1+\pi^2 r^2}$

Note period from -1 to 1, T = 2

$$f(x) = e^x = \sinh(1) + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}\cos\pi rx + \frac{\pi r(-1)^{r+1}}{1+\pi^2 r^2}\sin\pi rx\right]$$

$$= \sinh(1) + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}\cos\pi rx - \frac{\pi r(-1)^r}{1+\pi^2 r^2}\sin\pi rx\right] =$$

NOW INTEGRATE!!!

(EXPANDED) Integrate with respect to x

Integrating RHS $-$ note r and sinh(1) are constants with respect to x

$$\text{RHS: } \int \sinh(1)dx + 2\sinh(1)\int\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}\cos\pi rx - \frac{\pi r(-1)^r}{1+\pi^2 r^2}\sin\pi rx\right]dx + c =$$

$$\int \sinh(1)dx + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}\int\cos\pi rx\,dx - \frac{\pi r(-1)^r}{1+\pi^2 r^2}\int\sin\pi rx\,dx\right] + c =$$

$$\sinh(1)x + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}\frac{\sin\pi rx}{\pi r} - \frac{\pi r(-1)^r}{1+\pi^2 r^2}\frac{-\cos\pi rx}{\pi r}\right] + c =$$

Aside: we need Fourier series for x here to subst where we see sinh(1)x

Take f(x) = x odd function interval -1 to 1 then find $b_r = \dfrac{-2(-1)^r}{\pi r}$

Therefore we have $x = \dfrac{-2}{\pi}\sum_{r=1}^{\infty}\dfrac{(-1)^r}{r}\sin\pi rx$

Back:

RHS – cont.

Subst. for x

$$\sinh(1)\left[\frac{-2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\sin\pi rx\right]+2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2r^2}\frac{\sin\pi rx}{\pi r}-\frac{\pi r(-1)^r}{1+\pi^2r^2}\frac{(-\cos\pi rx)}{\pi r}\right]+c=$$

$$\sinh(1)\left[\frac{-2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\sin\pi rx\right]+2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2r^2}\frac{\sin\pi rx}{\pi r}+\right]+2\sinh(1)\sum_{r=1}^{\infty}\frac{(-1)^r}{1+\pi^2r^2}\cos\pi rx+c=$$

$$2\sinh(1)\left[\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{\pi r}\sin\pi rx+\frac{(-1)^r}{1+\pi^2r^2}\frac{\sin\pi rx}{\pi r}\right]+2\sinh(1)\sum_{r=1}^{\infty}\frac{(-1)^r}{1+\pi^2r^2}\cos\pi rx+c=$$

$$2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^{r+1}}{\pi r}+\frac{(-1)^r}{(\pi r)1+\pi^2r^2}\right]\sin\pi rx+2\sinh(1)\sum_{r=1}^{\infty}\frac{(-1)^r}{1+\pi^2r^2}\cos\pi rx+c=$$

Aside: simplify bracket $\left[\frac{(-1)^{r+1}}{\pi r}+\frac{(-1)^r}{(\pi r)1+\pi^2r^2}\right]=\cdots=\frac{-\pi r(-1)^r}{1+\pi^2r^2}$ Back: then subst.

$$2\sinh(1)\sum_{r=1}^{\infty}\frac{-\pi r(-1)^r}{1+\pi^2r^2}\sin\pi rx+2\sinh(1)\sum_{r=1}^{\infty}\frac{(-1)^r}{1+\pi^2r^2}\cos\pi rx+c=$$

$$2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2r^2}\cos\pi rx\frac{-\pi r(-1)^r}{1+\pi^2r^2}\sin\pi rx\right]+c$$

Compare with original result for $e^x=\sinh(1)+2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2r^2}\cos\pi rx\frac{-\pi r(-1)^r}{1+\pi^2r^2}\sin\pi rx\right]$

Therefore $c=\sinh(1)$

Of course the we know from calculus that the integral of e^x is e^x up to
a constant as we hve shown here!!

4.14

Even

$$a_0 = \frac{2\cdot 2}{2\pi}\int_0^{\pi} x\,dx = \pi$$

$$a_r = \frac{2\cdot 2}{2\pi}\int_0^{\pi} x\cos rx\,dx = \frac{2}{\pi r^2}[(-1)^r - 1]$$

$$y(x) = |x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{1}{r^2}[(-1)^r - 1]\cos rx = (\text{no even terms})$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{r=1,\text{odd}}^{\infty}\frac{-2}{r^2}\cos rx = \frac{\pi}{2} + \frac{2}{\pi}\sum_{p=0}^{\infty}\frac{-2}{(2p+1)^2}\cos(2p+1)x =$$

Therefore $|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{\cos(2p+1)x}{(2p+1)^2}$

Integrate left hand $\int_0^x |\tilde{x}|d\tilde{x} = \begin{cases} \int_0^x -\tilde{x}d\tilde{x} = \frac{-x^2}{2} & x<0 \\ \int_0^x \tilde{x}d\tilde{x} = \frac{x^2}{2} & x>0 \end{cases}$

Integrate right hand

$$\int^x \frac{\pi}{2} - \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{\cos(2p+1)\tilde{x}}{(2p+1)^2}d\tilde{x} = \frac{\pi}{2}x - \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{\sin(2p+1)x}{(2p+1)^3} + c$$

Therefore

$$\int_0^x |\tilde{x}|d\tilde{x} = \left\{\begin{array}{cc} \frac{-x^2}{2} & x<0 \\ \frac{x^2}{2} & x>0 \end{array}\right\} = \frac{\pi}{2}x - \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{\sin(2p+1)x}{(2p+1)^3}$$

Pick $x = \frac{\pi}{2}$ then - Note $\sin(2p+1)\left(\frac{\pi}{2}\right) = (-1)^p$

$$\frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{\pi}{2}\left(\frac{\pi}{2}\right) - \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{\sin(2p+1)\left(\frac{\pi}{2}\right)}{(2p+1)^3}$$

$$\rightarrow \frac{\pi^2}{8} = \frac{\pi^2}{4} - \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)^3} \rightarrow \frac{\pi^2}{8}\left(\frac{\pi}{4}\right) = \sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)^3}$$

or

$$\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} + \cdots = \frac{\pi^3}{32}$$