Interactive Assignment 13 9 Pages

Problems

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Problem 4.12

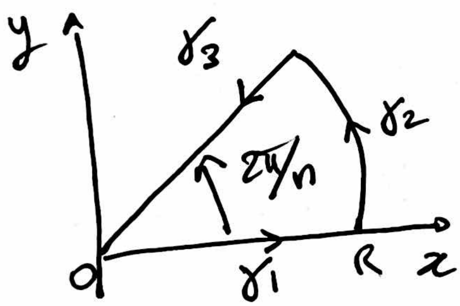
Consider the function $f(z) = \frac{1}{1+z^n}$

Poles of $f(z)$ are points: $1+z^n=0 \rightarrow z^n=-1=e^{i(\pi+2k\pi)}$

$$\rightarrow z = e^{i\pi/n} e^{i\frac{2k\pi}{n}}$$

$$n=0,1,2,\dots,n-1$$

Consider the close contour given by the following curves:



1) $\gamma_1(z)=x \quad 0\le x\le R$

2) $\gamma_2(z)=Re^{i\theta} \quad 0\le\theta\le 2\pi/n$

3) $\gamma_3(z)=xe^{i2\pi/n} \quad 0\le x\le R$ from R to 0

The only pole inside the closed contour is $z=e^{i\pi/n}$

$$\text{Res}\left(e^{i\pi/n}\right) = \lim_{z\to e^{i\pi/n}} \frac{1}{(1+z^n)'} = \lim_{z\to e^{i\pi/n}} \frac{1}{n z^{n-1}}$$

$$= \frac{1}{n e^{i\pi/n (n-1)}} = \frac{1}{n e^{i\pi} e^{-i\pi/n}}$$

$$= -\frac{1}{n e^{-i\pi/n}}$$

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Then by the residue theorem, we get:

$$\int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \int_{\gamma_3} f\,dz = 2\pi i \,\mathrm{Res}_{e^{i\pi/n}}(f)$$

$$= 2\pi i \left(-\frac{1}{n e^{-i\pi/n}}\right)$$

$$= -\frac{2\pi i}{n e^{-i\pi/n}}$$

<u>on γ_1</u> $z = x$ and $\int_{\gamma_1} f(z)\,dz = \int_0^R \frac{dx}{1+x^n}$

$dz = dx$

<u>on γ_3</u> $z = x e^{i2\pi/n}$ and $dz = e^{i2\pi/n}\,dx$

$$\int_{\gamma_3} f(z)\,dz = \int_R^0 \frac{e^{i2\pi/n}}{1+(xe^{i2\pi/n})^n}\,dx = (-e^{i2\pi/n}) \int_0^R \frac{dx}{1+x^n e^{i2\pi}}$$

$$= -e^{i2\pi/n} \int_0^R \frac{dx}{1+x^n}$$

<u>on γ_2</u> $z = Re^{i\theta}$ $dz = iRe^{i\theta}\,d\theta$

$$\int_{\gamma_2} f(z)\,dz = \int_0^{2\pi/n} \frac{1}{1+(Re^{i\theta})^n}\, iRe^{i\theta}\,d\theta$$

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$$\int_{\gamma_2} f(z)\,dz = i\int_0^{2\pi/n} \frac{R\,e^{i\theta}\,d\theta}{1+R^n e^{in\theta}}$$

$$\int_0^{2\pi/n} \frac{Re^{i\theta}d\theta}{1+R^n e^{in\theta}} \leq \left|\int_0^{2\pi/n} \frac{Re^{i\theta}d\theta}{1+R^n e^{in\theta}}\right| \leq \int_0^{2\pi/n} \frac{R\,|e^{i\theta}|}{|1+R^n e^{in\theta}|}\,d\theta$$

$|R^n e^{in\theta}+1| \geq \big|\,|R^n e^{in\theta}|-1\big| = R^n-1$ for R large enough

So $$\left|\int_0^{2\pi/n} \frac{Re^{i\theta}d\theta}{1+R^n e^{in\theta}}\right| \leq \int_0^{2\pi/n} \frac{R}{R^n-1}\,d\theta = \frac{R}{R^n-1}\cdot\frac{2\pi}{n}$$

Thus $$\lim_{R\to\infty}\left|\int_0^{2\pi/n} \frac{Re^{i\theta}d\theta}{1+R^n e^{in\theta}}\right| = 0$$

And as $R\to\infty$ $\int_{\gamma_2} f(z)\,dz \to 0$

Therefore taking the limit $R\to\infty$ of the equation obtained by the residue theorem gives

$$\int_{\gamma_1+\gamma_3} f(z)\,dt = -\frac{2\pi i}{n e^{-i\pi/n}}$$

or $$\left(1-e^{i2\pi/n}\right)\int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n e^{-i\pi/n}}$$

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$$\int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{(1-e^{i2\pi/n})\; n e^{-i\pi/n}}$$

$$= -\frac{i\pi}{n}\,\frac{2}{(e^{-i\pi/n}-e^{i\pi/n})}$$

$$\sin\frac{\pi}{n} = \frac{e^{i\pi/n}-e^{-i\pi/n}}{2i}$$ therefore $$\int_0^\infty \frac{dx}{1+x^n} = -\frac{i\pi}{n}\times\frac{1}{(-i\sin\pi/n)}$$

$$= \frac{\pi}{n}\,\frac{1}{\sin\pi/n}$$

$$= \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

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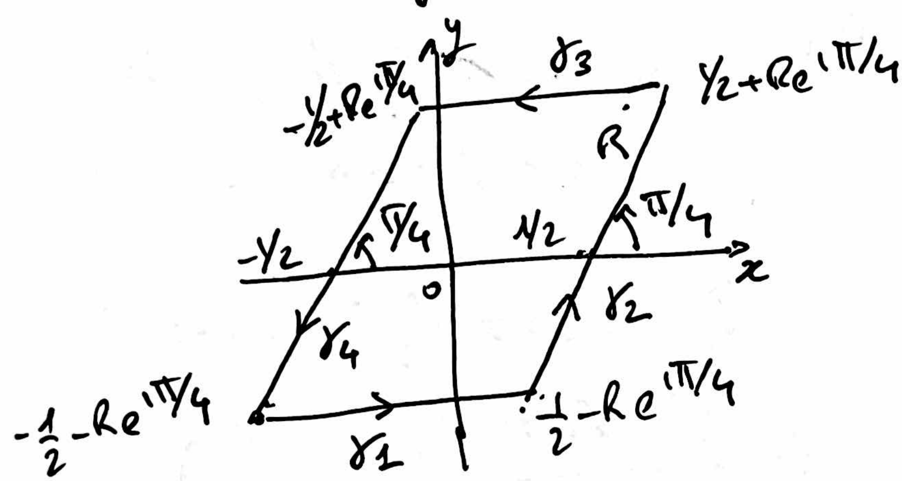
(a) Prove that the integral of $\exp(i\pi z^2)\,\mathrm{cosec}\,\pi z$ around the parallelogram with corner $\pm 1/2 \pm R\exp(i\pi/4)$ has the value $2i$

The parallelogram is defined as:



$f(z) = \dfrac{e^{i\pi z^2}}{\sin \pi z}$ poles are defined such that $\sin \pi z = 0$

$\rightarrow \pi z = n\pi$

$z = n$, n integer

Within the parallelogram, 0 is the only integer on the real axis

From the residue theorem: $\oint_C f(z)\,dz = 2\pi i \operatorname{Res}_0(f)$

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$z=0$ is a simple pole thus $\operatorname{Res}_0(f) = \left.\frac{g(z)}{h'(z)}\right|_{z=0} = \frac{e^{i\pi \cdot 0^2}}{\pi\cos(\pi \cdot 0)} = 1/\pi$

Therefore $\oint_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} f(z)\,dz = 2\pi i \; 1/\pi = 2i$

(b) Show that the parts of the contour parallel to the real axis do not contribute when $R \to \infty$

<u>On γ_1</u> $\quad z = x - Re^{i\pi/4}$ for $x \in [-1/2, 1/2]$ $\quad dz = dx$

$$\int_{\gamma_1} f\,dz = \int_{-1/2}^{1/2} \frac{e^{i\pi(x-Re^{i\pi/4})^2}}{\sin[\pi(x-Re^{i\pi/4})]}\,dx$$

$$\text{so } \left|\int_{\gamma_1} f(z)\,dz\right| \leq \int_{-1/2}^{1/2} \left|\frac{e^{i\pi(x-Re^{i\pi/4})^2}}{\sin[\pi(x-Re^{i\pi/4})]}\right| dx$$

$$\begin{aligned}
e^{i\pi(x-Re^{i\pi/4})^2} &= e^{i\pi(x^2+R^2e^{i\pi/2}-2xRe^{i\pi/4})} \\
&= e^{i\pi(x^2+iR^2-2xRe^{i\pi/4})} \\
&= e^{i\pi x^2}\, e^{-\pi R^2}\, e^{i\pi(-2xR\frac{1}{\sqrt{2}}(1+i))} \\
&= e^{i\pi x^2}\, e^{-\pi R^2}\, e^{-i\pi\sqrt{2}xR(1+i)} \\
&= e^{i\pi x^2}\, e^{-i\pi\sqrt{2}xR}\, e^{\pi(\sqrt{2}xR-R^2)}
\end{aligned}$$

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So $\left|e^{i\pi(x-Re^{i\pi/4})^2}\right| = \left|e^{i\pi x^2} e^{-\pi\sqrt{2}xR} e^{i\pi(\sqrt{2}xR-R^2)}\right|$

$$= e^{\pi R^2\left(\frac{\sqrt{2}x}{R}-1\right)}$$

Thus $\lim_{R\to\infty}\left|\frac{e^{i\pi(x-Re^{i\pi/4})^2}}{\sin[i\pi(x-Re^{i\pi/4})]}\right| = \lim_{R\to\infty}\frac{e^{\pi R^2\left(\frac{\sqrt{2}x}{R}-1\right)}}{\left|\sin[\pi(x-Re^{i\pi/4})]\right|}$

$$=0$$

Therefore $\lim_{R\to\infty}\int_{\gamma_1} f(z)\,dz \le \lim_{R\to\infty}\left|\int_{\gamma_1} f\,dz\right|$

$$=0$$

Similarly $\lim_{R\to\infty}\int_{\gamma_3} f(z)\,dz = 0$

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On γ_4 $$\int_{\gamma_4} f\,dz = \int_{R}^{-R} \frac{e^{i\pi(z'-1/2)^2}}{\sin[\pi(z'-1/2)]} e^{i\pi/4} dr$$

$$= -\int_{-R}^{R} \frac{e^{i\pi(z'-1/2)^2}}{\sin[\pi(z'-1/2)]} e^{i\pi/4} dr$$

$$e^{i\pi(z'-1/2)^2} = e^{i\pi(z'^2 - z' + 1/4)} = e^{i\pi/4} e^{-\pi r^2} e^{-i\pi z'}$$

$$\sin \pi(z'-1/2) = \sin(\pi z' - \pi/2) = -\cos(\pi z')$$

$$\int_{\gamma_4} f(z)\,dz = -\int_{-R}^{R} \frac{e^{i\pi/4} e^{-\pi r^2} e^{-i\pi z'}}{-\cos(\pi z')} e^{i\pi/4} dr$$

$$= e^{i\pi/2} \int_{-R}^{R} \frac{e^{-\pi r^2} e^{-i\pi z'}}{\cos(\pi z')} dr$$

$$= i \int_{-R}^{R} \frac{e^{-\pi r^2} e^{-i\pi z'}}{\cos(\pi z')} dr$$

$$\int_{\gamma_2} f\,dz + \int_{\gamma_4} f\,dz = i \int_{-R}^{R} \frac{e^{-\pi r^2}}{\cos(\pi z')} \left(e^{i\pi z'} + e^{-i\pi z'}\right) dr$$

$$= i \int_{-R}^{R} \frac{e^{-\pi r^2}}{\cos(\pi z')} 2\cos(\pi z')\, dr$$

$$= 2i \int_{-R}^{R} e^{-\pi r^2} dr$$

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Using the residue theorem and result from part (a):

$$\int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \int_{\gamma_3} f\,dz + \int_{\gamma_4} f\,dz = 2i$$

$$\int_{\gamma_1} f\,dz + \int_{\gamma_3} f\,dz + \int_{\gamma_2+\gamma_4} f\,dz = 2i$$

Taking the limit $R \to \infty$ on both sides yields:

$$0 + 0 + 2i \int_{-\infty}^{\infty} e^{-\pi r^2}\,dr = 2i$$

therefore $\int_{-\infty}^{\infty} e^{-\pi r^2}\,dr = 1$