Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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WHITING SCHOOL
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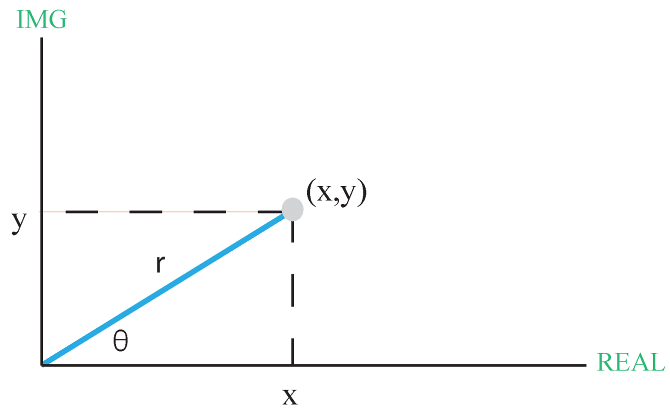
Introduction to Complex Variables



Argand diagram

Notice: We can also have a polar representation for a complex variables and we will investigate this later.

Let z be a complex variable equal to $x+iy$ where the real part of z can be denoted as $\mathrm{Re}(z)=x$ and the imaginary part as $\mathrm{Im}(z)=y$.

As usual $i^2=-1 \rightarrow i=\sqrt{-1}$

Functions of a complex variable, f(z) can also be represented in terms of a real and imaginary part. The standard representation of such a function is denoted

$$f(z)=f(x,y)=u(x,y)+iv(x,y)$$

where u is the real part and v is the imaginary part, that is

A couple of examples of differentiation of a function of a complex variable

Let's differentiate the function of a complex variable $f(z) = z^2$

Remember z is a complex variable equal to $x + iy$ where the real part of z can be denoted as Re(z) = x and the imaginary part as Im(z) = y.

Substitution for z into our function and expansion gives us the following standard form

$$f(z) = z^2 = (x + iy)^2 = \boxed{x^2 - y^2 + i2xy = f(x,y)}$$

Differentiation of this function is defined as

$$\boxed{f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + \Delta x, y + \Delta y) - f(x,y)}{\Delta x + i\Delta y}}$$

We will calculate the derivative of this function a few different ways!

First we will use $f'(x,y)=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{f(x+\Delta x,y+\Delta y)-f(x,y)}{\Delta x+i\Delta y}$

Substitution for $f(x,y)=x^2-y^2+i2xy$ gives

$$f'(x,y)=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{[(x+\Delta x)^2-(y+\Delta y)^2+i2(x+\Delta x)(y+\Delta y)]-[x^2-y^2+i2xy]}{\Delta x+i\Delta y}$$

Take the limit as $\Delta x\to 0$ first then $\Delta y\to 0$ therefore

$$f'(x,y)=\lim_{\Delta y\to 0}\frac{[(x)^2-(y+\Delta y)^2+i2(x)(y+\Delta y)]-[x^2-y^2+i2xy]}{i\Delta y}=$$

$$\lim_{\Delta y\to 0}\frac{x^2-(y+\Delta y)^2+i2(x)(y+\Delta y)]-[x^2-y^2+i2xy]}{i\Delta y}=$$

$$\lim_{\Delta y\to 0}\frac{x^2-(y+\Delta y)^2+i2xy+i2x\Delta y-x^2+y^2-i2xy}{i\Delta y}=$$

$$\lim_{\Delta y\to 0}\frac{-(y+\Delta y)^2+i2x\Delta y+y^2}{i\Delta y}=\lim_{\Delta y\to 0}\frac{-y^2-2y\Delta y-\Delta y^2+i2x\Delta y+y^2}{i\Delta y}=$$

$$\lim_{\Delta y\to 0}\frac{-2y\Delta y-\Delta y^2+i2x\Delta y}{i\Delta y}=\lim_{\Delta y\to 0}\frac{\Delta y(-2y-\Delta y+i2x)}{i\Delta y}=$$

$$\lim_{\Delta y\to 0}\frac{(-2y-\Delta y+i2x)}{i}=\frac{-2y+i2x}{i}\left(\frac{i}{i}\right)=\frac{-2yi+i^2 2x}{i^2}=\frac{-2yi-2x}{-1}=2(x+iy)\equiv 2z$$

First we will use $f'(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x + i\Delta y}$

Substitution for $f(x,y) = x^2 - y^2 + i2xy$ gives

Starting with the same construct for the derivatives

$$f'(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[(x+\Delta x)^2 - (y+\Delta y)^2 + i2(x+\Delta x)(y+\Delta y)] - [x^2 - y^2 + i2xy]}{\Delta x + i\Delta y}$$

But in this case take the limit as $\Delta y \to 0$ first and then $\Delta x \to 0$ (Try it yourself)

It gives the same answer

Finally let's just use the derivative as defined by the variable z, where $f(z) = z^2$ that is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + \Delta z^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z} =$$

$$\lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z)}{\Delta z} = \lim_{\Delta z \to 0} 2z + \Delta z = 2z$$

Again the same answer! – So what can go wrong?

Let's use $f'(x,y)=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(x+\Delta x,y+\Delta y)-f(x,y)}{\Delta x+i\Delta y}$

But for this example let $f(x,y)=2y+ix$ Then substituting as before

$$f'(x,y)=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{[2(y+\Delta y)+i(x+\Delta x)]-[2y+ix]}{\Delta x+i\Delta y}$$

Take the limit as $\Delta x\to 0$ first then $\Delta y\to 0$ therefore

$$f'(x,y)=\lim_{\Delta y\to 0}\frac{[2(y+\Delta y)+i(x)]-[2y+ix]}{\Delta x+i\Delta y}=\lim_{\Delta y\to 0}\frac{2y+2\Delta y+ix-2y-ix}{i\Delta y}=$$

$$\lim_{\Delta y\to 0}\frac{2\Delta y}{i\Delta y}=\frac{2}{i}=-2i$$

Alternatively take the limit as $\Delta y\to 0$ first and then $\Delta x\to 0$

$$f'(x,y)=\lim_{\Delta x\to 0}\frac{[2(y)+i(x+\Delta x)]-[2y+ix]}{\Delta x}=\lim_{\Delta x\to 0}\frac{2y+ix+i\Delta x-2y-ix}{\Delta x}=$$

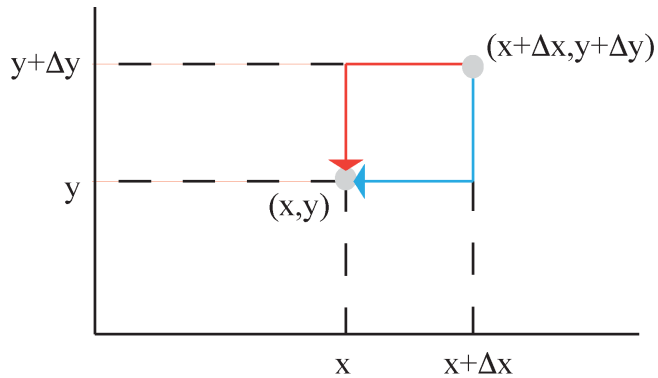
$$\lim_{\Delta x\to 0}\frac{i\Delta x}{\Delta x}=i$$

These are completely different answers!

KEY: For some functions of a complex variable the path by which $\Delta z\to 0$ that is equivalently $\Delta x\to 0$ and $\Delta y\to 0$ matters! If you remember this is also true in multivariate calculus. The question then is for which functions of a complex variable can we define a "standard" single valued derivative.

Cauchy-Riemann conditions

For certain functions the derivative of a complex function is path independent. These functions are called analytic and the necessary relation they satisfy is called the Cauchy-Riemann condition. Let's derive it



KEY: Start with the definition of the derivative in terms of the real and imaginary variables x,y and assume the function has only one derivative regardless of the path taken to construct it!.

$$f'(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x + i\Delta y}$$

Case 1: Take the blue path and let $\Delta y \to 0$ then $\Delta x \to 0$

Case 2: Take the red path and let $\Delta x \to 0$ then $\Delta y \to 0$

Take the form for the function of a complex variable in terms of its real and imaginary form.

$$f(z)=f(x,y)=u(x,y)+iv(x,y)$$

Then for case 1 substituting for f(z) gives

$$f'(x,y)=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(x+\Delta x,y+\Delta y)-f(x,y)}{\Delta x+i\Delta y}=$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{[u(x+\Delta x,y+\Delta y)+iv(x+\Delta x,y+\Delta y)]-[u(x,y)+iv(x,y)]}{\Delta x+i\Delta y}=$$

Let $\Delta y\to 0$ then

$$\lim_{\Delta x\to 0}\frac{[u(x+\Delta x,y)+iv(x+\Delta x,y)]-[u(x,y)+iv(x,y)]}{\Delta x}$$

Rearrange and collecting terms gives

$$\lim_{\Delta x\to 0}\frac{[u(x+\Delta x,y)-u(x,y)]+i[v(x+\Delta x,y)-v(x,y)]}{\Delta x}=$$

$$\lim_{\Delta x\to 0}\frac{[u(x+\Delta x,y)-u(x,y)]}{\Delta x}+\lim_{\Delta x\to 0}\frac{i[v(x+\Delta x,y)-v(x,y)]}{\Delta x}$$

Therefore $f'(x,y)=\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}$

For case 2 we have

$$f'(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\Delta x + i\Delta y} =$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y} =$$

Let $\Delta x \to 0$ then

$$\lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) + iv(x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{i\Delta y}$$

Rearrange and collecting terms gives

$$\lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y - u(x,y)] + i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y} =$$

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y - u(x,y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y} =$$

$$\lim_{\Delta y \to 0} \frac{-i[u(x, y+\Delta y - u(x,y)]}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

Therefore $f'(x,y) = -i\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y}$

From case 1 we have $$f'(x,y)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

From case 2 we have $$f'(x,y)=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Finally setting the derivatives equal to each other and correspondingly the real and imaginary parts equal to each other gives

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ \text{and}\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}}$$

The Cauchy-Riemann conditions

Let's take a look at a couple of simple examples

Example 1: Using the previous function discussed $f(x,y)=(x+iy)^2=x^2-y^2+i2xy$

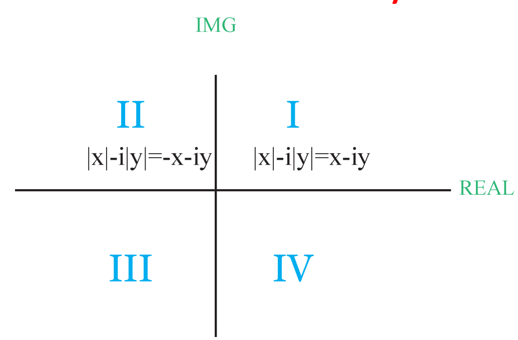
Then $u(x,y)=x^2-y^2$ and $v(x,y)=2xy$

and

$$\frac{\partial u}{\partial x}=2x,\ \frac{\partial v}{\partial y}=2x \text{ and } \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ Also } \frac{\partial v}{\partial x}=2y \text{ and } \frac{\partial u}{\partial y}=-2y \text{ and } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

So yes it does satisfy the Cauchy-Riemann conditions and is an analytic function.

Example 2: Take $f(x,y)=|x|-i|y|$



Key: The value of the absolute values depends on the quadrant we are in!

In quadrant I $\frac{\partial u}{\partial x}=1,\ \frac{\partial v}{\partial y}=-1 \text{ and } \frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$ It does not satisfy Cauchy-Riemann

However in quadrant II

$$\frac{\partial u}{\partial x}=-1,\ \frac{\partial v}{\partial y}=-1 \text{ and } \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ Also } \frac{\partial v}{\partial x}=0,\ \frac{\partial u}{\partial y}=0 \text{ and } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

It does satisfy Cauchy-Riemann