Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Take Home Project 2
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Question 1

(a) Please see attached separate pdf.

(b) $f(t) = C_0 e^{-\frac{t}{\tau}}$ with period T, so

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} dt \\ &= \frac{2C_0}{T}(-\tau)[e^{-\frac{t}{\tau}}]_0^T \\ &= -2C_0\frac{\tau}{T}[e^{-\frac{T}{\tau}} - 1] \\ &= 2C_0\frac{\tau}{T}(1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

If $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_0 \approx 2C_0\frac{\tau}{T}$.

$$\begin{aligned} a_k &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \end{aligned}$$

Using integration by parts with $u = \cos\frac{2k\pi t}{T}, du = -\frac{2k\pi}{T}\sin\frac{2k\pi t}{T}$ and $dv = e^{-\frac{t}{\tau}}, v = (-\tau)e^{-\frac{t}{\tau}}$:

$$\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt = (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T - \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}dt$$

Using again integration by parts:

$$\int_0^T e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}dt = (-\tau)[e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt$$

So

$$\begin{aligned} (1 + (\frac{2k\pi\tau}{T}))^2\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt &= (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau^2}{T}[e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}]_0^T \\ &= (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T + 0 \\ &= \tau(1 - e^{-\frac{T}{\tau}}) \\ \int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt &= \frac{\tau}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

Substituting back into the expression found for a_k yields

$$\begin{aligned} a_k &= 2C_0 \frac{\tau}{T} \frac{1}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}}) \\ &= 2C_0 \frac{\tau T}{T^2 + (2k\pi\tau)^2}(1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

With the same assumption $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_k \approx 2C_0 \frac{\tau}{T} \frac{1}{1+(\frac{2k\pi\tau}{T})^2}$. Similarly to compute b_k

$$\begin{aligned} b_k &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T} \int_0^T e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \frac{\tau}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}}) \\ &= 4C_0 k\pi \frac{\tau^2}{T^2 + (2k\pi\tau)^2}(1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

Once again, since $e^{-\frac{T}{\tau}} \approx 0$ and $b_k \approx 4C_0(\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2} \pi k$

(c) For $k \geq 1$

$$\begin{aligned} p_k &= \frac{1}{2}(a_k^2 + b_k^2) \\ &= \frac{1}{2}\left[4C_0^2(\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} + 16C_0^2(\frac{\tau}{T})^4 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2}\pi^2 k^2\right] \\ &= \frac{1}{2} 4C_0^2(\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2}\left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2\right] \\ &= 2C_0^2(\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2}\left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2\right] \end{aligned}$$

(d)

(e)

(f) We have

$$\begin{aligned} a_k \cos(\frac{k2\pi t}{T}) + b_k \sin(\frac{k2\pi t}{T}) &= \cos(\phi_k)\cos(\frac{k2\pi t}{T}) + \sin(\phi_k)\sin(\frac{k2\pi t}{T}) \\ &= \cos(\frac{k2\pi t}{T} - \phi_k) \end{aligned}$$

where

$$\tan(\phi_k) = \frac{\sin(\phi_k)}{\cos(\phi_k)} = \frac{b_k}{a_k} = 4C_0(\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2}\pi k (2C_0 \frac{\tau}{T}\frac{1}{1+(\frac{2k\pi\tau}{T})^2})^{-1}$$
$$= 2\frac{\tau}{T}\pi k$$
$$\phi_k = \arctan(2\frac{\tau}{T}\pi k)$$

For $\frac{\tau}{T} = .1$, $\phi_1 \approx 32.14°$ and $\phi_2 \approx 51.48°$ and for $\frac{\tau}{T} = .01$, $\phi_1 \approx 3.59°$ and $\phi_2 \approx 7.16°$

Question 2

(a) One simple way to describe $P(r)$ is to define it as $P(r) = Ar + B$ with the conditions:

$$A \cdot 0 + B = Q$$
$$A \cdot R + B = 0$$

which gives $A = -\frac{Q}{R}$ and $B = Q$. So

$$P(r) = \begin{cases} Q(1-\frac{r}{R}) & \text{for } 0 \le r \le R \\ 0 & \text{for } r > R \end{cases}$$

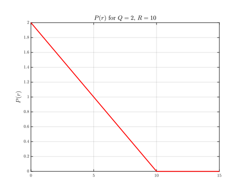

Figure 1: $P(r)$

(b) Since we assume no angular dependence: $\nabla^2 C = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dC}{dr})$, the differential equation is now:

$$\frac{D}{r^2}\frac{d}{dr}(r^2\frac{dC(r)}{dr}) + P(r) = 0$$
$$\frac{d}{dr}(r^2\frac{dC(r)}{dr}) = -\frac{r^2}{D}P(r)$$

(c) Inside the cell $P(r) = Q(1 - \frac{r}{R})$, so we have to solve the differential equation

$$\begin{aligned}
\frac{d}{dr}(r^2 \frac{dC(r)}{dr}) &= -\frac{r^2}{D}Q(1 - \frac{r}{R}) \\
&= \frac{Q}{DR}r^2(r - R) \\
&= \frac{Q}{DR}r^3 - \frac{Q}{D}r^2
\end{aligned}$$

Integrating once

$$\begin{aligned}
r^2\frac{dC(r)}{dr} &= \frac{Q}{4DR}r^4 - \frac{Q}{3D}r^3 + A \\
\frac{dC(r)}{dr} &= \frac{Q}{4DR}r^2 - \frac{Q}{3D}r + \frac{A}{r^2}
\end{aligned}$$

Integrating again

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 - \frac{A}{r} + B \quad A, B\text{:constants}, C_i\text{:inside cell concentration}$$

Outside the cell $P(r) = 0$ and the we want to solve the differential equation

$$\frac{d}{dr}(r^2\frac{dC(r)}{dr}) = 0$$

Which by integration gives

$$\begin{aligned}
r^2\frac{dC(r)}{dr} &= C_1 \\
\frac{dC(r)}{dr} &= \frac{C_1}{r^2} \\
C_o(r) &= -\frac{C_1}{r} + C_2 \quad C_1, C_2\text{:constants}, C_o\text{:outside cell concentration}
\end{aligned}$$

(d) Applying the boundary conditions

(i)

$$\lim_{r\to 0} C_i(r) = \lim_{r\to 0} \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 - \frac{A}{r} + B$$

since $\lim_{r\to 0} C_i(r) = \frac{1}{r} = \infty$ therefore to have finite concentration $C_i(r)$ at $r = 0$ we need $A = 0$

(ii)

$$\lim_{r\to\infty} C_o(r) = \lim_{r\to\infty}\left(-\frac{C_1}{r} + C_2\right) = C_2$$

The concentration goes to zero at infinity implies $C_2 = 0$

(iii) We have now for $C_i(r)$ and $C_o(r)$:

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 + B$$
$$C_o(r) = -\frac{C_1}{r}$$

$C_i(R) = C_o(R)$ and $\frac{dC_i(r)}{dr} = \frac{dC_o(r)}{dr}|_{r=R}$ yields

$$\frac{Q}{12DR}R^3 - \frac{Q}{6D}R^2 + B = -\frac{C_1}{R}$$
$$\frac{Q}{4D}R - \frac{Q}{3D}R = \frac{C_1}{R^2}$$

Rearranging

$$-\frac{Q}{12D}R^2 + B = -\frac{C_1}{R}$$
$$-\frac{Q}{12D}R = \frac{C_1}{R^2}$$

which gives

$$B = \frac{Q}{6D}R^2$$
$$C_1 = -\frac{Q}{12D}R^3$$

substituting back

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 + \frac{Q}{6D}R^2$$
$$C_o(r) = \frac{Q}{12D}R^3\frac{1}{r}$$

(e) The concentration maximum happens within the cell since $P(r)$ has maximum value Q at $r = 0$ and then it is zero for $r > R$. We are looking for the value of r for which $\frac{dC_i(r)}{dr} = 0$:

$$\frac{dC_i(r)}{dr} = \frac{Q}{4DR}r^2 - \frac{Q}{3D}r = \frac{Q}{D}r(\frac{r}{4R} - \frac{1}{3})$$

Discarding the solution $r = 0$ we are left that concentration maximum is for $r = \frac{4}{3}R$ and it is

$$C_M = \frac{Q}{12DR}(\frac{4}{3})^3R^3 - \frac{Q}{6D}(\frac{4}{3})^2R^2 + \frac{Q}{6D}R^2$$
$$= \frac{Q}{6D}R^2\left[\frac{4^3}{2\ 3^3} - \frac{4^2}{3^2} + 1\right]$$
$$= \frac{11}{162}\frac{Q}{D}R^2$$

Inside the cell

$$C_i(r) = \frac{Q}{6D}(\frac{1}{2}\frac{r^3}{R} - r^2 + R^2)$$
$$\frac{C_i(r)}{C_M} = \frac{Q}{6D}\frac{162}{11}\frac{D}{Q}R^{-2}(\frac{1}{2}\frac{r^3}{R} - r^2 + R^2)$$
$$= \frac{162}{6\ \ 11}\left[\frac{1}{2}(\frac{r}{R})^3 - (\frac{r}{R})^2 + 1\right]$$