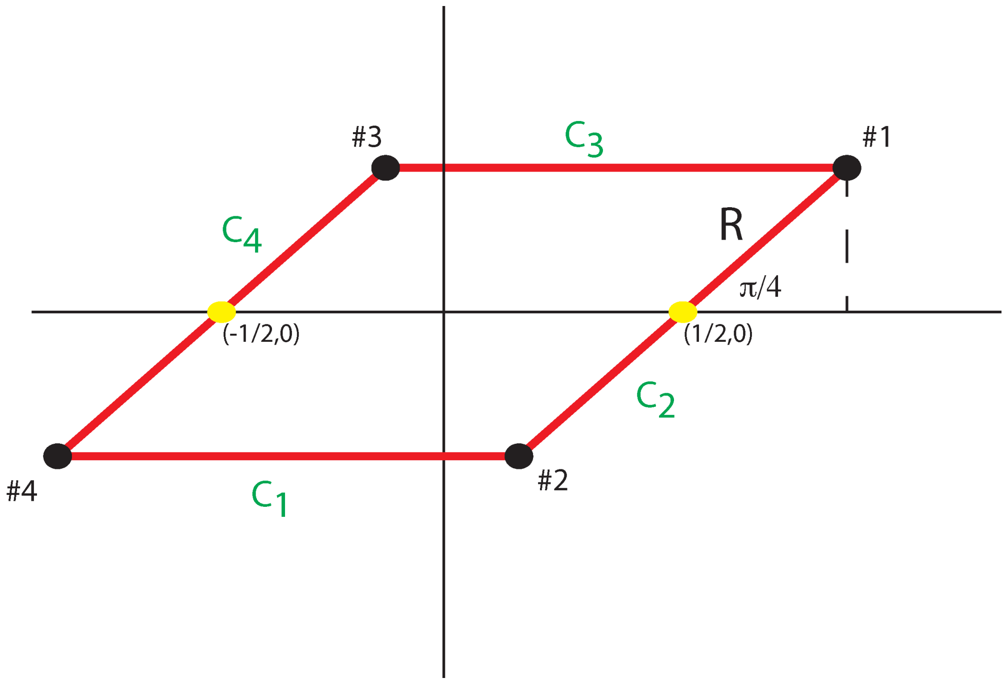
14.13



(a) Four corners at

#1: $\frac{1}{2}+R(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

#2: $\frac{1}{2}-R(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

#3: $-\frac{1}{2}+R(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

#4: $-\frac{1}{2}-R(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

Evalute integral $\int_C e^{i\pi z^2}\csc\pi z dz=\int_C e^{i\pi z^2}\frac{1}{\sin\pi z}dz=\int_C \frac{e^{i\pi z^2}}{\sin\pi z}dz$

where $C = C_1+C_2+C_3+C_4$

and $g(z)= e^{i\pi z^2}, h(z)=\sin\pi z$

Poles at $\sin\pi z = 0 \rightarrow z = z_0 = 0$ simple pole

using Eq. 14.56 residue, $\mathbf{R}(0) = \left.\dfrac{g(z_0)}{h'(z_0)}\right|_{z_0=0} = \left.\dfrac{e^{i\pi 0^2}}{\pi\cos\pi 0}\right|_{z_0=0} = \dfrac{1}{\pi}$

Therefore by residue theorem $\displaystyle\int_C \frac{e^{i\pi z^2}}{\sin\pi z}dz = 2\pi i\mathbf{R}(0) = 2\pi i\frac{1}{\pi} = 2i$

Next evaluate the integral around contour $C = C_1 + C_2 + C_3 + C_4$

as $R \rightarrow \infty$

First evaluate integral on

$C_1 :\ z\ =\ t - Re^{i\pi/4},\ -\dfrac{1}{2} \le t \le \dfrac{1}{2}$

$$\int_{C_1} \frac{e^{i\pi z^2}}{\sin\pi z}dz = \lim_{R\to\infty}\int_{-1/2}^{1/2} \frac{e^{i\pi(t-Re^{i\pi/4})^2}}{\sin\pi(t-Re^{i\pi/4})}dt = \lim_{R\to\infty}\int_{-1/2}^{1/2} \frac{e^{i\pi(t^2-2tRe^{i\pi/4}+R^2e^{i\pi/2})}}{\sin\pi(t-Re^{i\pi/4})}dt$$

$$= \int_{-1/2}^{1/2} \lim_{R\to\infty}\frac{e^{i\pi(t^2-2tRe^{i\pi/4}+R^2e^{i\pi/2})}}{\sin\pi(t-Re^{i\pi/4})}dt$$

Look at $\displaystyle\lim_{R\to\infty}\frac{e^{i\pi(t^2-2tRe^{i\pi/4}+R^2e^{i\pi/2})}}{\sin\pi(t-Re^{i\pi/4})} \equiv \lim_{R\to\infty}\frac{e^{i\pi(R^2e^{i\pi/2})}}{\sin\pi(-Re^{i\pi/4})}$

Note $e^{i\pi/2} = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = 0 + 1i = i$

Therefore $\displaystyle\lim_{R\to\infty}\frac{e^{i\pi(R^2e^{i\pi/2})}}{\sin\pi(-Re^{i\pi/4})} = \lim_{R\to\infty}\frac{e^{i\pi(R^2 i)}}{-\sin\pi(Re^{i\pi/4})} = \lim_{R\to\infty}\frac{e^{-\pi R^2}}{-\sin\pi(Re^{i\pi/4})} = 0$

and

$$\int_{-1/2}^{1/2} \lim_{R\to\infty}\frac{e^{i\pi(t^2-2tRe^{i\pi/4}+R^2e^{i\pi/2})}}{\sin\pi(t-Re^{i\pi/4})}dt = \int_{-1/2}^{1/2} 0dt = 0$$

Similar for contour C_3

Next evaluate integral on C_3, C_4 as $R \to \infty$

$C_3: \; z = \frac{1}{2} + te^{i\pi/4}$, $-R \le t \le R$, $dz = e^{i\pi/4}\,dt$

$C_4: \; z = -\frac{1}{2} + te^{i\pi/4}$, $-R \le t \le R$, $dz = e^{i\pi/4}\,dt$

$$\int_{C_2+C_4} \frac{e^{i\pi z^2}}{\sin\pi z}dz = \lim_{R\to\infty}\left[\int_{-R}^{R} \frac{e^{i\pi(\frac{1}{2}+te^{i\pi/4})^2}}{\sin\pi(\frac{1}{2}+te^{i\pi/4})}e^{i\pi/4}dt + \int_{R}^{-R} \frac{e^{i\pi(-\frac{1}{2}+te^{i\pi/4})^2}}{\sin\pi(-\frac{1}{2}+te^{i\pi/4})}e^{i\pi/4}dt\right] =$$

$$\lim_{R\to\infty}\left[\int_{-R}^{R} \frac{e^{i\pi(\frac{1}{2}+te^{i\pi/4})^2}}{\sin\pi(\frac{1}{2}+te^{i\pi/4})}e^{i\pi/4}dt - \int_{-R}^{R} \frac{e^{i\pi(-\frac{1}{2}+te^{i\pi/4})^2}}{\sin\pi(-\frac{1}{2}+te^{i\pi/4})}e^{i\pi/4}dt\right] =$$

$$\lim_{R\to\infty}\int_{-R}^{R}\left[\frac{e^{i\pi(\frac{1}{2}+te^{i\pi/4})^2}}{\sin\pi(\frac{1}{2}+te^{i\pi/4})} - \frac{e^{i\pi(-\frac{1}{2}+te^{i\pi/4})^2}}{\sin\pi(-\frac{1}{2}+te^{i\pi/4})}\right]e^{i\pi/4}dt$$

Note $\sin\pi(\frac{1}{2}+te^{i\pi/4}) = \sin(\frac{\pi}{2}+\pi te^{i\pi/4}) = \sin\frac{\pi}{2}\cos\pi te^{i\pi/4} + \cos\frac{\pi}{2}\sin\pi te^{i\pi/4} =$

$1\cos\pi te^{i\pi/4} + 0\sin\pi te^{i\pi/4} = \cos\pi te^{i\pi/4}$

Similarly $\sin\pi(-\frac{1}{2}+te^{i\pi/4}) = -\cos\pi te^{i\pi/4}$

Next

Note $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

Therefore $e^{i\pi(\frac{1}{2}+te^{i\pi/4})^2} = e^{i\pi(\frac{1}{4}+te^{i\pi/4}+t^2e^{i\pi/2})} = e^{i\pi(\frac{1}{4}+te^{i\pi/4}+t^2 i)} = e^{\frac{i\pi}{4}+i\pi te^{i\pi/4}-\pi t^2}$

Similarly $e^{i\pi(-\frac{1}{2}+te^{i\pi/4})^2} = e^{\frac{i\pi}{4}-i\pi te^{i\pi/4}-\pi t^2}$

Substitution of all previous constructs gives

$$\lim_{R\to\infty}\left[\int_{-R}^{R}\frac{e^{i\pi(\frac{1}{2}+te^{i\pi/4})^2}}{\sin\pi(\frac{1}{2}+te^{i\pi/4})}e^{i\pi/4}dt-\int_{R}^{-R}\frac{e^{i\pi(-\frac{1}{2}+te^{i\pi/4})^2}}{\sin\pi(-\frac{1}{2}+te^{i\pi/4})}e^{i\pi/4}dt\right]=\lim_{R\to\infty}\left[\int_{-R}^{R}\frac{e^{\frac{i\pi}{4}+i\pi te^{i\pi/4}-\pi t^2}}{\cos\pi te^{i\pi/4}}e^{i\pi/4}dt-\int_{R}^{-R}\frac{e^{\frac{i\pi}{4}-i\pi te^{i\pi/4}-\pi t^2}}{-\cos\pi te^{i\pi/4}}e^{i\pi/4}dt\right]=$$

$$\lim_{R\to\infty}\left[\int_{-R}^{R}\frac{e^{\frac{i\pi}{4}}e^{\pi te^{i\pi/4}}e^{-\pi t^2}}{\cos\pi te^{i\pi/4}}e^{i\pi/4}dt+\int_{R}^{-R}\frac{e^{\frac{i\pi}{4}}e^{-\pi te^{i\pi/4}}e^{-\pi t^2}}{\cos\pi te^{i\pi/4}}e^{i\pi/4}dt\right]=\lim_{R\to\infty}\left[\int_{-R}^{R}\frac{e^{\frac{i\pi}{4}}e^{\pi te^{i\pi/4}}e^{-\pi t^2}}{\cos\pi te^{i\pi/4}}+\frac{e^{\frac{i\pi}{4}}e^{-\pi te^{i\pi/4}}e^{-\pi t^2}}{\cos\pi te^{i\pi/4}}e^{i\pi/4}dt\right]$$

$$=\lim_{R\to\infty}\int_{-R}^{R}\left[\frac{e^{\pi te^{i\pi/4}}}{\cos\pi te^{i\pi/4}}+\frac{e^{-\pi te^{i\pi/4}}}{\cos\pi te^{i\pi/4}}\right]e^{\frac{i\pi}{4}}e^{-\pi t^2}e^{i\pi/4}dt=\lim_{R\to\infty}\int_{-R}^{R}\left[\frac{e^{\pi te^{i\pi/4}}+e^{-\pi te^{i\pi/4}}}{\cos\pi te^{i\pi/4}}\right]e^{-\pi t^2}e^{2i\pi/4}dt$$

$e^{2i\pi/4}=e^{i\pi/2}=i$ (as before) and $e^{\pi te^{i\pi/4}}+e^{-\pi te^{i\pi/4}}=2\cos(\pi te^{i\pi/4})$ therefore

$$\lim_{R\to\infty}\int_{-R}^{R}\left[\frac{2\cos(\pi te^{i\pi/4})}{\cos\pi te^{i\pi/4}}\right]e^{-\pi t^2}idt=\int_{-\infty}^{\infty}2e^{-\pi t^2}idt=2i\int_{-\infty}^{\infty}e^{-\pi t^2}dt$$

Finally since

$$\int_C\frac{e^{i\pi z^2}}{\sin\pi z}dz=2\pi i\mathbf{R}(0)=2\pi i\frac{1}{\pi}=2i$$

and

$$\int_C\frac{e^{i\pi z^2}}{\sin\pi z}dz=\int_{C_2+C_4}\frac{e^{i\pi z^2}}{\sin\pi z}dz=2i\int_{-\infty}^{\infty}e^{-\pi t^2}dt$$

We have

$$2i\int_{-\infty}^{\infty}e^{-\pi t^2}dt=2i \text{ or } \int_{-\infty}^{\infty}e^{-\pi t^2}dt=1$$