8.2

$L(y;\lambda) = (py')' + qy + \lambda\rho y = 0$ with $y(x)$ and $y(a) = y(b) = 0$

Also

$L(z;\lambda) = F(x)$ where z is related to x, that is $z(x)$

and having the same boundary conditions as $y(x)$, that is $z(a) = z(b) = 0$

Then prove this relationship

$$\int_a^b y(x)F(x)dx = 0$$

Start integral and substitute (Note y and z are both functions of x)

$$\int_a^b y(x)F(x)dx = \int_a^b y(x)L(z;\lambda)dx = \int_a^b y[(pz')' + qz + \lambda\rho z]dx = \int_a^b y(pz')'dx + \int_a^b (yqz + y\lambda\rho z)dx$$

Take the first integral and use integration by parts $\int u\,dv = uv - \int v\,du$

let $u = y$ therefore $\frac{du}{dx} = \frac{dy}{dx}$ or $du = \frac{dy}{dx}dx \equiv y'dx$

and $dv = (pz')'$ therefore $v = pz'$

$$\int_a^b y(pz')'dx = ypz'\Big|_a^b - \int_a^b pz'y'dx$$

Note the first term is zero since for $y(x)$ and bounds a and b both $y(b)$ and $y(a)$ are 0!

We are left with

$$-\int_a^b pz'y'dx + \int_a^b (yqz + y\lambda\rho z)dx$$

Again use integration by parts on the first integral

with $u = py'$ and $du = (py')'dx$; $dv = z'dx$ and $v=z$

$$-\int_a^b pz'y'dx = -\left[zpy'\Big|_a^b - \int_a^b (py')'zdx\right] = \int_a^b (py')'zdx$$

Similar to before the first term is zero since for $z(x)$ and bounds a and b both $z(b)$ and $z(a)$ are 0!

Therefore

$$-\int_a^b pz'y'dx + \int_a^b (yqz + y\lambda\rho z)dx = \int_a^b (py')'zdx + \int_a^b (yqz + y\lambda\rho z)dx$$

$$= \int_a^b (py')'z + (yqz + y\lambda\rho z)dx = \int_a^b [(py')' + yq + y\lambda\rho]zdx = \int_a^b L(y;\lambda)zdx$$

Now since $L(y;\lambda) = 0$ we get the result needed

$$\int_a^b y(x)F(x)dx = \int_a^b L(y;\lambda)zdx = 0$$

(b) Take

$p(x)=1;\ q(x)=0;\ \rho(x)=1$
$a=-1;\ b=1$ and correspondingly $y(-1)=y(1)=0$
$z(x)=1-x^2$

First look at $L(y;\lambda)$ to get $y(x)$
$L(y;\lambda)=y''+\lambda y=0$ It has the usual solution $y(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$
Plug in boundary conditions
$y(-1)=A\cos\sqrt{\lambda}(-1)+B\sin\sqrt{\lambda}(-1)\equiv A\cos\sqrt{\lambda}(1)-B\sin\sqrt{\lambda}(1)=0$
$y(1)=A\cos\sqrt{\lambda}(1)+B\sin\sqrt{\lambda}(1)=0$
Sibtracting y(1) from y(-1) together gives
$-2B\sin\sqrt{\lambda}(1)=0$
Therefore take B = 0
Leaves $y(x)=A\cos\sqrt{\lambda}x$

Next evaluate $L(z;\lambda)=z''+\lambda z=(1-x^2)''+\lambda((1-x^2)=-2+\lambda((1-x^2)=F(x)$

Finally evaluate the integral from part (a) $\int_a^b y(x)F(x)dx$ for this particular case, that is
$\int_{-1}^{1} A\cos\sqrt{\lambda}x[-2+\lambda((1-x^2)]dx$
Doing the integration above (using Table of integral formulas) gives 0 if done corrctly.

Aside: EXTRA hint to simplify integrated results is
y(1) = 0 $\rightarrow y(1)=A\cos\sqrt{\lambda}1=A\cos\sqrt{\lambda}=0$ BUT A $\neq 0$ otherwise no solution therefore
$\cos\sqrt{\lambda}\rightarrow\sqrt{\lambda}=(2n+1)\pi/2$ and that also means $\sin\sqrt{\lambda}=\sin(2n+1)\pi/2=(-1)^n$
Back:

Do it!
This shows verification for the formula from part (a) for this case.