Johns Hopkins
Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
EN. 585.409

JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

The Dirac δ-function

The δ-function is different from most functions encountered in the physical sciences but we will see that a rigorous mathematical definition exists

The formal properties of the δ-function may be summarized as follows

$$\delta(t) = 0 \quad \text{for } t \neq 0,$$

$$\boxed{\int f(t)\delta(t-a)\,dt = f(a),}$$

A particular form of the above is $\int_{-\infty}^{\infty} \delta(t-0) f(t)\,dt = \int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$

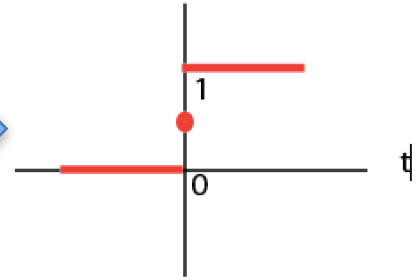
$$\int_{-a}^{b} \delta(t)\,dt = 1 \quad \text{for all } a, b > 0$$

$$\delta(t) = \delta(-t),$$

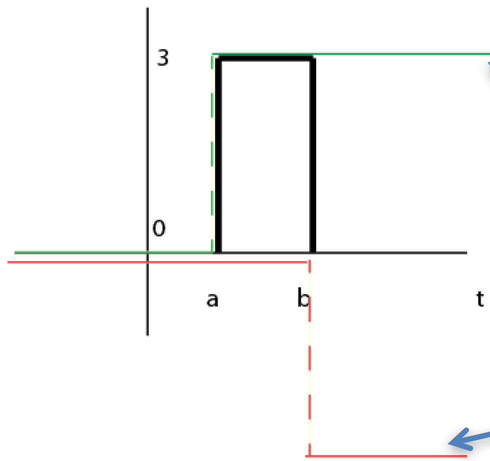
The Heaviside function

Define

$$H(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$



Example



$$f(t)=3[\boxed{H(t-a)}-\boxed{H(t-b)}]$$

Relation between the Heaviside function and the Dirac δ-function

Start with the following integral $\int_{-\infty}^{\infty} f(t)H'(t)dt$

It can be evaluated by using integration by parts

Aside: Integration by parts $\int u\,dv = uv - \int v\,du$

In our derivation let
$$u = f(t),\ du = f'(t)dt$$
$$dv = H'(t)dt,\ v = H(t)$$

Therefore $\int_{-\infty}^{\infty} f(t)H'(t)dt = f(t)H(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(t)f'(t)dt$

$$\int_{-\infty}^{\infty} f(t)H'(t)dt = f(t)H(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(t)f'(t)dt = \lim_{t\to\infty} f(t)(1) - \lim_{t\to-\infty} f(t)(0) - \left[\int_{0}^{\infty} f'(t)(1)dt\right]$$

$$= \lim_{t\to\infty} f(t) - \left[\int_{0}^{\infty} f'(t)dt\right] = \lim_{t\to\infty} f(t) - \left[\lim_{t\to\infty} f(t) - f(0)\right] = f(0) \equiv \int_{-\infty}^{\infty} f(t)\delta(t-0)dt = \int_{-\infty}^{\infty} f(t)\delta(t)dt$$

$$\boxed{H'(t) = \delta(t)}$$

Derivation of a few Dirac function identities

The following derivation is given in the book for the function $f(t) = bt$

b>0

$$\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt = \int_{-\infty}^{\infty} f\left(\frac{t'}{b}\right)\delta(t')\frac{dt'}{b} = \frac{1}{b}f(0) = \frac{1}{b}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt,$$

b = -c <0

$$\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt = \int_{\infty}^{-\infty} f\left(\frac{t'}{-c}\right)\delta(t')\left(\frac{dt'}{-c}\right) = \int_{-\infty}^{\infty} \frac{1}{c} f\left(\frac{t'}{-c}\right)\delta(t')\,dt'$$

$$= \frac{1}{c}f(0) = \frac{1}{|b|}f(0) = \frac{1}{|b|}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt,$$

Note we made use of the following identity $f(0) = \int_{-\infty}^{\infty} \delta(t) f(t)\,dt$

On the other hand for h(t) a general function with i zero crossings we have the following derivation (not given in the text)

Start with

Only need to evaluate near where h(t)=0, That is at t_i

$$\int_{-\infty}^{\infty} \delta[h(t)]f(t)dt = \boxed{\sum_i \int_{t_i - r_i}^{t_i + r_i}} \delta[h(t)]f(t)dt$$

Let h(t) be a function with n number of roots, t_i. Near these roots we take intervals of radius r_i where the function h(t) is differentiable and has an inverse inside interval. That is $h[h_i^{-1}(t)] = t$ Therefore let $u = h(t),\ du = h'(t)dt$ and $t = h_i^{-1}(u)$

$$\int_{-\infty}^{\infty} \boxed{\delta[h(t)]} f(t)dt = \sum_i \int_{t_i - r_i}^{t_i + r_i} \delta[h(t)] \frac{h'(t)}{h'(t)} f(t)dt = \sum_i \int_{t_i - r_i}^{t_i + r_i} \delta[h(t)] \frac{f(t)}{h'(t)} h'(t)dt$$

$$= \sum_i \int_{h(t_i - r_i)}^{h(t_i + r_i)} \delta[u] \frac{f(h_i^{-1}(u))}{h'(h_i^{-1}(u))} du = \sum_i \frac{f(h_i^{-1}(0))}{h'(h_i^{-1}(0))} = \sum_i \frac{f(t_i)}{|h'(t_i)|} = \sum_i \frac{\int_{-\infty}^{\infty} \delta(t - t_i)f(t)dt}{|h'(t_i)|}$$

$$= \int_{-\infty}^{\infty} \boxed{\sum_i \frac{\delta(t - t_i)}{|h'(t_i)|}} f(t)dt \qquad \text{where we have used } f(t_i) = \int_{-\infty}^{\infty} \delta(t - t_i)f(t)dt$$

Therefore $\delta[h(t)] = \sum_i \frac{\delta(t - t_i)}{|h'(t_i)|}$

One last derivation involving the derivative of of the delta function

$$\int_{-\infty}^{\infty} \delta'(t)f(t)dt$$

Using integration by parts

$$\int_{-\infty}^{\infty} \delta'(t)f(t)dt = \delta(t)f(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t)f'(t)dt$$

We require that f(t) be well behaved at limits, that is $\lim_{t \to \pm\infty} f(t) = 0$

Therefore

$$\int_{-\infty}^{\infty} \delta'(t)f(t)dt = -\int_{-\infty}^{\infty} \delta(t)f'(t)dt = -f'(0)$$

Relation of Dirac function to the Fourier integral

Taking the Fourier and inverse Fourier transform of f(t) we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \int_{-\infty}^{\infty} du\; f(u)\, e^{-i\omega u}$$
$$= \int_{-\infty}^{\infty} du\;\, f(u) \boxed{\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega\right\}}.$$

We also have from the defining equation of the delta function

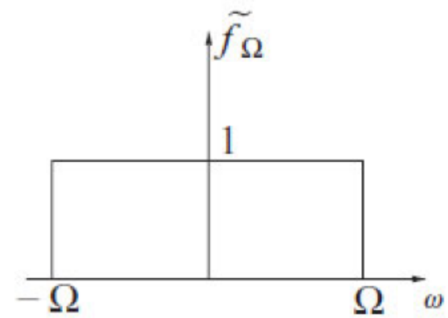
$$f(t) = \int_{-\infty}^{\infty} \delta(t-u) f(u) du$$

Therefore

$$\delta(t-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega.$$

Dirac function's relation to the sinc function

Lastly let's investigate the Fourier transform of the following function in the frequency domain

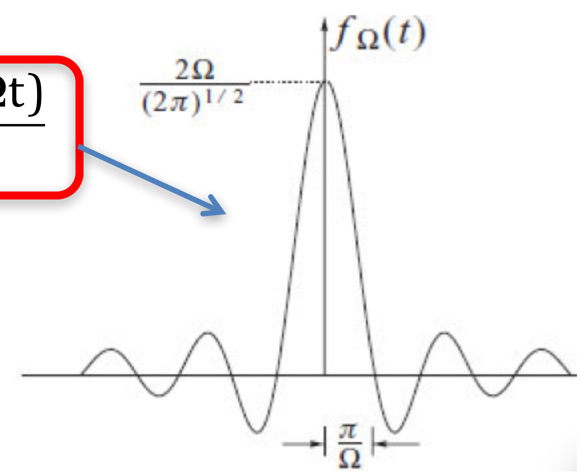


Take the inverse Fourier transform of the rectangular function in the frequency domain

$$f(t) = F^{-1}\{\tilde{f}(\omega)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 1[H(\omega+\Omega) - H(\omega-\Omega)]e^{i\omega t}\,d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\Omega}^{\Omega} 1e^{i\omega t}\,d\omega$$

After integration use the identity $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$f_\Omega(t) = \frac{1}{\sqrt{2\pi}}\int_{-\Omega}^{\Omega} 1e^{i\omega t}\,d\omega = \frac{1}{\sqrt{2\pi}}\frac{1}{it}2i\sin(\Omega t) = \boxed{\frac{2\Omega}{\sqrt{2\pi}}\frac{\sin(\Omega t)}{\Omega t}}$$



Let $\Omega \to \infty$ but from our previous derivation

$$\delta(t-0) = \delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-0)t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,d\omega$$

Gives $$\boxed{\delta(t) = \lim_{\Omega\to\infty}\frac{\sin(\Omega t)}{\pi t}}$$