4.8 Convergence of series

Provided you get the coefficients as I have for 4.8 - presented below, then the following algebra show why this converges as r^{-3}

$$b_r = \begin{cases} \frac{1}{\pi}[\frac{(-1)^{r-1}-1}{(r-1)^2} - \frac{(-1)^{r+1}-1}{(r+1)^2}] & r-\text{even} \\ 0 & r-\text{odd} \end{cases}$$

Take a look at this term and rearrange

$$\frac{(-1)^{r-1}-1}{(r-1)^2} - \frac{(-1)^{r+1}-1}{(r+1)^2} = \frac{(-1)^r(-1)^{-1}-1}{(r-1)^2} - \frac{(-1)^r(-1)^1-1}{(r+1)^2}$$

$$= \frac{-(-1)^r-1}{(r-1)^2} - \frac{-(-1)^{r1}-1}{(r+1)^2} = [-(-1)^r-1]\left[\frac{1}{(r-1)^2} - \frac{1}{(r+1)^2}\right]$$

$$= [-(-1)^r-1]\left[\frac{(r+1)^2-(r-1)^2}{(r-1)^2(r+1)^2}\right] = [-(-1)^r-1]\left[\frac{4r}{(r-1)^2(r+1)^2}\right] = 4[-(-1)^r-1]\left[\frac{r}{r^4+\cdots}\right]$$

Therefore taking the largest power of r in the denom and that in the numerator we have $\frac{r}{r^4}$ or $\frac{1}{r^3}$

So we say this function is $O(r^{-3})$, that is big-O and in sum would converge as $\frac{1}{r^3}$ with respect to only the even terms

4,10 Derivative problem

From 4.9

$$a_0 = \frac{2}{2}\int_{-1}^{1} e^x\,dx = e^x\Big|_{-1}^{1} = e^x - e^{-x} = (\text{use def. for sinh}) = 2\sinh(1) \text{ and } \frac{a_0}{2} = \sinh(1)$$

$$a_r = \frac{2\sinh(1)(-1)^r}{1+\pi^2 r^2}, \text{ note also includes r = 0 case}$$

$$b_r = \frac{2\pi r\sinh(1)(-1)^{r+1}}{1+\pi^2 r^2}$$

Note period from -1 to 1, T = 2

$$f(x) = e^x = \sinh(1) + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}\cos\pi rx + \frac{\pi r(-1)^{r+1}}{1+\pi^2 r^2}\sin\pi rx\right]$$

$$= \sinh(1) + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}\cos\pi rx - \frac{\pi r(-1)^r}{1+\pi^2 r^2}\sin\pi rx\right] =$$

NOW TRY TO DIFFERENTIATE

$$\frac{d}{dx}\left\{\sinh(1) + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}\cos\pi rx - \frac{\pi r(-1)^r}{1+\pi^2 r^2}\sin\pi rx\right]\right\}$$

$$= \sinh(1) + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}\frac{d}{dx}\cos\pi rx - \frac{\pi r(-1)^r}{1+\pi^2 r^2}\frac{d}{dx}\sin\pi rx\right]$$

$$= \sinh(1) + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}(-\pi r\sin\pi rx) - \frac{\pi r(-1)^r\pi r}{1+\pi^2 r^2}(\pi r\cos\pi rx)\right]$$

$$= \sinh(1) + 2\sinh(1)\sum_{r=1}^{\infty}\left[\frac{(-1)^r}{1+\pi^2 r^2}(-\pi r\sin\pi rx) - \frac{(-1)^r\pi^2 r^2}{1+\pi^2 r^2}\cos\pi rx\right]$$

LOOK AT SECOND TERM $\frac{\pi^2 r^2}{1+\pi^2 r^2}(-1)^r\cos\pi rx$

Then as $r \to \infty$ the term $\frac{\pi^2 r^2}{1+\pi^2 r^2} \to \frac{\pi^2 r^2}{\pi^2 r^2} = 1$ does not go to zero

THEREFORE THE SECOND SUM TERM

$\sum_{r=1}^{\infty}\frac{(-1)^r\pi^2 r^2}{1+\pi^2 r^2}\cos\pi rx$ DOES NOT CONVERGE (AS IT MUST)

AND THE DERIVATIVE DOES NOT EXIST!!!