6.24a

Solve $\frac{d^2y}{dx^2} - y = x^n$ where homogeneous $\frac{d^2y}{dx^2} - y = 0$ solution $\rightarrow y_h(x) = c_1e^x + c_2e^{-x}$

By variation of parameters take $y_p(x) = k_1e^x + k_2e^{-x}$

Use variation of parameters to get (similar to example pg. 252)

$k_1^{'}e^x + k_2^{'}e^{-x} = 0$ and $k_1^{'}e^x + k_2^{'}(-e^{-x}) = x^n$

Solve for $k_1^{'}(x) = \frac{1}{2}e^{-x}x^n$ and $k_2^{'}(x) = -\frac{1}{2}e^x x^n$

From here it get a little more complicated! One quick way!

Look it up in "good" extensive table of integration formulas:

$$\int k_1^{'}(x)dx = \int \frac{1}{2}e^{-x}x^n dx = \frac{1}{2}\int e^{-x}x^n dx \rightarrow k_1(x) = \frac{1}{2}e^{-x}\sum_{r=0}^{n}\frac{(-1)^r n! x^{n-r}}{(n-r)!(-1)^{r+1}} =$$

$\frac{(-1)^r}{(-1)^{r+1}} = -1$ and n! not a function of index r therefore $k_1(x) = \frac{1}{2}e^{-x}(-1)n!\sum_{r=0}^{n}\frac{x^{n-r}}{(n-r)!}$

OR: A student asked about derivation of the integral $\int e^{-x}x^n dx$ Aside:

So this can been done by multiple applications of integration by parts. Lets set this up.

Note we need to reduce powers in x^n so let

$u = x^n \rightarrow du = nx^{n-1}dx$ and $dv = e^{-x}dx \rightarrow v = (-1)e^{-x}$

Then $\int e^{-x}x^n dx = x^n(-1)e^{-x} - \int (-1)e^{-x}nx^{n-1}\,dx = x^n(-1)e^{-x} + n\int e^{-x}x^{n-1}\,dx$

Do it again

$u = x^{n-1} \rightarrow du = (n-1)x^{n-2}dx$ and $dv = e^{-x}dx \rightarrow v = (-1)e^{-x}$

Then $x^n(-1)e^{-x} + n\int e^{-x}x^{n-1}\,dx = x^n(-1)e^{-x} + n[x^{n-1}(-1)e^{-x} - \int (-1)e^{-x}(n-1)x^{n-2}\,dx] =$

$(-1)x^n e^{-x} + (-1)nx^{n-1}e^{-x} + n(n-1)\int e^{-x}x^{n-2}\,dx$

One more time (to see pattern)

$u = x^{n-2} \rightarrow du = (n-2)x^{n-3}dx$ and $dv = e^{-x}dx \rightarrow v = (-1)e^{-x}$

Then $(-1)x^n e^{-x} + (-1)nx^{n-1}e^{-x} + n(n-1)\int e^{-x}x^{n-2}\,dx =$

$(-1)x^n e^{-x} + (-1)nx^{n-1}e^{-x} + n(n-1)[x^{n-2}(-1)e^{-x} - \int (-1)e^{-x}(n-2)x^{n-3}\,dx =$

$(-1)x^n e^{-x} + (-1)nx^{n-1}e^{-x} + (-1)n(n-1)x^{n-2}e^{-x} + n(n-1)(n-2)\int e^{-x}x^{n-3}\,dx$

Now note that $\frac{n!}{(n-r)!} = n\cdots(n-r+1)$ So continuing this process we get $\int e^{-x}x^n dx =$

$(-1)\frac{n!}{(n-0)!}x^n e^{-x} + (-1)\frac{n!}{(n-1)!}x^{n-1}e^{-x} + (-1)\frac{n!}{(n-2)!}x^{n-2}e^{-x} + \cdots =$

$(-1)e^{-x}\sum_{r=0}^{n}\frac{n!}{(n-r)!}x^{n-r} = (-1)e^{-x}n!\sum_{r=0}^{n}\frac{x^{n-r}}{(n-r)!}$ Back:

Substitution into $k_1(x) = \frac{1}{2}\int e^{-x}x^n dx \rightarrow \frac{1}{2}(-1)e^{-x}n!\sum_{r=0}^{n}\frac{x^{n-r}}{(n-r)!} \;\; = -\frac{1}{2}e^{-x}n!\sum_{r=0}^{n}\frac{x^{n-r}}{(n-r)!}$

Finally let $m = n-r \rightarrow k_1(x) = -\frac{1}{2}e^{-x}n!\sum_{m=0}^{n}\frac{x^m}{m!}$

Similar for $k_2(x) = -\frac{1}{2}e^{x}n!(-1)^n\sum_{m=0}^{n}\frac{(-1)^m x^m}{m!}$

Substitution(above) into $y_p(x) = \; k_1e^x + k_2e^{-x} = \left[-\frac{1}{2}e^{-x}n!\sum_{m=0}^{n}\frac{x^m}{m!}\right]e^x + \left[-\frac{1}{2}e^{x}n!(-1)^n\sum_{m=0}^{n}\frac{(-1)^m x^m}{m!}\right]e^{-x} =$

Finally $y_p(x) = -\frac{n!}{2}\sum_{m=0}^{n}\frac{x^m}{m!} + -\frac{n!}{2}\sum_{m=0}^{n}\frac{(-1)^{m+n}x^m}{m!} = -\frac{n!}{2}\sum_{m=0}^{n}\frac{[1+(-1)^{m+n}]x^m}{m!}$

Solution $y(x) = y_h(x) + y_p(x) = \; c_1e^x + c_2e^{-x} - \frac{n!}{2}\sum_{m=0}^{n}\frac{[1+(-1)^{m+n}]x^m}{m!}$

(b) This one is easier – fill in the steps yourself

Again solve homogenous and particular (variation of parameters)

This one is straightforward!

$y_h(x) = c_1e^x + c_2xe^x$

Then using variation of parameters methods with $y_p(x) = k_1e^x + k_2xe^x$

we get $k_1 = -\frac{2x^3}{3}$ and $k_2 = x^2$

Therefore

$$y(x) = c_1e^x + c_2xe^x - \frac{2x^3}{3}e^x + x^2xe^x = c_1e^x + c_2xe^x + \frac{x^3}{3}e^x$$