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Green's function approach to solving an inhomogeneous differential equations is a way of solving for the particular solution for an arbitrary generic function given the solution to the response of the differential equation to a delta (or impulse) function.

Development of the Green's function approach

Start with a general nth order linear inhomogeneous differential differential equation

$$a_n(x)\frac{d^n y}{dx^n}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x)$$

Next identify the differential operator part of this differential equation $\mathcal{L}=a_n(x)\frac{d^n}{dx^n}+\cdots+a_1(x)\frac{d}{dx}+a_0(x)$

Therefore $\mathcal{L}y=f(x)$

Next we presume that the differential equation will have a solution with imposed boundary conditions on some interval (a,b), e.g. take $y(a)=y(b)=0$
and the solution takes the following form

$$y(x)=\int_a^b G(x,z)f(z)dz$$

Where $G(x,z)$ is our **Green's function**.

Now comes the important part!

Substitute this solution into our differential equation (with differential operator) and interchange the order of differentiation and integration (assumed well behaved) and note that the function f(x), as shown previously can be represented by a delta function Integral representation (essentially part of our initial definition of a delta function!)

$$\boxed{\mathcal{L}y(x)=\mathcal{L}\int_a^b G(x,z)f(z)dz=\int_a^b \boxed{\mathcal{L}G(x,z)}f(z)dz=f(x)=\int_a^b \boxed{\delta(x-z)}f(z)dz}$$

Therefore the Green's function represents the response of the differential equation to a delta input

$$\mathcal{L}G(x,z)=\delta(x-z)$$

We can again looking at the solution of our differential equation in terms of the Green's function. Except now we see that this represents the superposition of the Green's function weighted by the values of f(z). Therefore since the Green's function represented the response to a delta input for our differential equation the integrated Green's function represents the entire response to the input function f(x). Thus once we have the Green's function. We can solve our differential equation for any function f(x) provided that we can do the integral (more than likely by numerical integration except in "simple" but often important cases).

$$y(x)=\int_a^b G(x,z)f(z)dz$$

Of course the next problem is how to construct this Green's function.

First look again at out integral representation of the solution (above). Applying our Initial conditions gives , e.g.

$$y(a)=\int_a^b G(a,z)f(z)dz=0$$

First condition

Since in general $f(z)\neq 0$ then $G(a,z)=0$

Similarly we get $G(b,z)=0$ so we have $\boxed{G(a,z)=G(b,z)=0}$

Secondly we look at the behavior of

$$\mathcal{L}G(x,z) = [a_n(x)\frac{d^n}{dx^n} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)]G(x,z) = \delta(x-z)$$

near x =z.

We do this by looking at the integrated expression near x = z (dividing by $a_n(x)$ first and note since G is a function of x and z we change the derivative operator to a partial derivative), that is

$$\lim_{\varepsilon\to 0}\int_{z-\varepsilon}^{z+\varepsilon}[\frac{\partial^n}{\partial x^n} + \cdots + \frac{a_1(x)}{a_n(x)}\frac{\partial}{\partial x} + \frac{a_0(x)}{a_n(x)}]G(x,z)dx = \lim_{\varepsilon\to 0}\int_{z-\varepsilon}^{z+\varepsilon}\frac{1}{a_n(x)}\delta(x-z)dx$$

The expression on the right is easily seen to equal $1/a_n(z)$

Term with jump

Integration of the left hand side gives

$$\lim_{\varepsilon\to 0}\int_{z-\varepsilon}^{z+\varepsilon}[\frac{\partial^n}{\partial x^n} + \cdots + \frac{a_1(x)}{a_n(x)}\frac{\partial}{\partial x} + \frac{a_0(x)}{a_n(x)}]G(x,z)dx = \lim_{\varepsilon\to 0}[\boxed{\frac{\partial^{n-1}G(x,z)}{\partial x^{n-1}}} + \cdots \text{lower order}]\Big|_{z-\varepsilon}^{z+\varepsilon}$$

This consist of a number of differences of the expression evaluated at $z+\varepsilon$ and $z-\varepsilon$

KEY: The term with the n-1 derivative gives rise to a nonzero difference (a jump) and all lower order derivatives do not have a discontinuity and therefore have no jumps associated with them as $\varepsilon \to 0$ and all these differences are 0. The discontinuity is of course equal to the term on the right hand side (calculated above) $1/a_n(z)$

Finally

$$\lim_{\varepsilon\to 0}\left[\frac{\partial^{n-1}G(x,z)}{\partial x^{n-1}}+\cdots\text{lower order}\right]\Bigg|_{z-\varepsilon}^{z+\varepsilon}=\lim_{\varepsilon\to 0}\left[\frac{\partial^{n-1}G(x,z)}{\partial x^{n-1}}+0\right]\Bigg|_{z-\varepsilon}^{z+\varepsilon}=$$

$$\boxed{\lim_{\varepsilon\to 0}\left[\frac{\partial^{n-1}G(z+\varepsilon,z)}{\partial x^{n-1}}-\frac{\partial^{n-1}G(z-\varepsilon,z)}{\partial x^{n-1}}\right]=\frac{1}{a_n(z)}}$$

Second condition

An example of solving a differential using a Green's function approach

Let's go back to the differential equation we solved using variation of parameters.

$$\frac{d^2y}{dx^2}+y=f(x)=\csc x,\ \ y(0)=y(\pi/2)=0$$

We start the Green's function approach almost the same as for the variation of parameters approach, except we look for functions, e.g. A(z), near x = z (instead of the functions, e.g. $k_1(x)$), that is

$$G(x,z)=\begin{cases} A(z)\sin x+B(z)\cos x & x<z \\ C(z)\sin x+D(z)\cos x & x>z \end{cases}$$

Apply the first condition for ours Green's function $G(0,z)=0$ and $G(\pi/2,z)=0$

For $x=0<z$ we have $G(0,z)=A(z)\sin 0+B(z)\cos 0=0\ \ x=0<z$

That is $B(z)=0\ \ x=0<z$

For $x=\pi/2>z$ we have $G(\pi/2,z)=C(z)\sin\pi/2+D(z)\cos\pi/2=0\ \ x=\pi/2>z$

That is $C(z)=0\ \ x=\pi/2>z$

Therefore $G(x,z)=\begin{cases} A(z)\sin x & x<z \\ D(z)\cos x & x>z \end{cases}$

And at x = z $A(z)\sin z=D(z)\cos z$ or $-A(z)\sin z+D(z)\cos z=0$

Next we look at our second condition $\lim_{\varepsilon\to 0}\left[\frac{\partial^{n-1}G(z+\varepsilon,z)}{\partial x^{n-1}}-\frac{\partial^{n-1}G(z-\varepsilon,z)}{\partial x^{n-1}}\right]=\frac{1}{a_n(z)}$

For this problem n=2, a_2=1 therefore $\lim_{\varepsilon\to 0}\left[\frac{\partial G(z+\varepsilon,z)}{\partial x}-\frac{\partial G(z-\varepsilon,z)}{\partial x}\right]=\frac{1}{1}$

Using our previously derived expression for G(x,z) $\rightarrow$ $\frac{\partial G(x,z)}{\partial x}=\begin{cases} A(z)\cos x & x<z \\ D(z)(-\sin x) & x>z \end{cases}$

Substitution into the expression above gives

$$\lim_{\varepsilon\to 0}[-D(z)(\sin(z+\varepsilon)-A(z)\cos(z-\varepsilon)]=-D(z)\sin z-A(z)\cos z=1$$

Therefore from our two conditions on G(x,z) we have the following two equations

$$-A(z)\sin z+D(z)\cos z=0$$
$$-D(z)\sin z-A(z)\cos z=1$$

These are easily solved for A(z) and D(z)

$$A(z)=-\cos z$$
$$D(z)=-\sin z$$

Therefore $\boxed{G(x,z)=\begin{cases} -\cos z\sin x & x<z \\ -\sin z\cos x & x>z \end{cases}}$

Our Green's function for this differential equation

At this point we are ready to generate our answer using the Green's function using the following integration

$$y(x)=\int_0^{\pi/2} G(x,z)f(z)dz$$

The **key** to doing this integral with our given Green's function is to split the integral into two parts, that is

z<x

z>x

$$y(x)=\int_0^{x} G(x,z)f(z)dz+\int_x^{\pi/2} G(x,z)f(z)dz$$

Substitution for G(x,z) into the expression gives

$$y(x)=\int_0^{x} -\sin z\cos x\csc z dz+\int_x^{\pi/2} -\cos z\sin x\csc z dz=$$

$$\cos x\int_0^{x} -\sin z\frac{1}{\sin z}dz+\sin x\int_x^{\pi/2} -\cos z\frac{1}{\sin z}dz=-\cos x\int_0^{x} 1dz-\sin x\int_x^{\pi/2} \cot z dz$$

Evaluating this integral gives

$$\boxed{y(x)}=-\cos x[z\big|_0^x]-\sin x[\ln(\sin z)\big|_x^{\pi/2}]=-\cos x[x]-[\ln(\sin\pi/2)-\ln(\sin x)]$$

$$=-\cos x[x]-\sin x[\ln(1)-\ln(\sin x)]=-\cos x[x]-\sin x[0-\ln(\sin x)]=\boxed{\sin x\ln(\sin x)-x\cos x}$$

The same answer we arrived at by variation of parameter, however as mentioned this method is more general in that the Green's function provides us a very direct way to solve this form of a differential equation for **any** f(x).