Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Final Exam
Johns Hopkins University
Student: Yves Greatti

Question 1

a. $f(x) = x$ is odd on $[-\pi, \pi]$ therefore its Fourier coefficients a_n are 0 and we need to find its b_n coefficients:

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{4}{2\pi} \int_{0}^{\pi} x \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) dx \end{aligned}$$

Using integration by parts:

$$\begin{aligned} \int_0^{\pi} x \sin(nx) dx &= [x(-\frac{\cos(nx)}{n})]_0^{\pi} + \int_0^{\pi} 1 \cdot \frac{\cos(nx)}{n} dx \\ &= (-\frac{\pi}{n}) \cos(n\pi) + \frac{1}{n} [\sin(nx)]_0^{\pi} \\ &= \frac{(-1)^{n+1}\pi}{n} \end{aligned}$$

Thus $b_n = \frac{2}{\pi} \frac{(-1)^{n+1}\pi}{n} = \frac{(-1)^{n+1}2}{n}$ and the Fourier series of x, on $[-\pi, \pi]$, is:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

b. If we integrate terms by terms the previous expression, the Fourier series of x over $[-\pi, \pi]$, we have:

$$\begin{aligned} \frac{x^2}{2} &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-\frac{\cos(nx)}{n}) + c \quad c \text{constant of integration} \\ x^2 &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + c \text{ with } 2c \to c \\ &= c + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) \end{aligned}$$

c. $f(x) = x^2$ is an even function, by Fourier Series for even function over symmetric range, we have:

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n x}{2\pi}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \ (1)$$

where

$$
\begin{aligned}
a_0 &= \frac{4}{2\pi}\int_0^{\pi} x^2 dx \\
&= \frac{2}{\pi}[\frac{x^3}{3}]_0^{\pi} \\
&= \frac{2}{3}\pi^2
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{4}{2\pi}\int_0^{\pi} x^2\cos{(\frac{2\pi nx}{2\pi})}dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx \\
\int_0^{\pi} x^2\cos(nx)dx &= [x^2\ \frac{\sin(nx)}{n}]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin(nx)dx \\
&= 0 - \frac{2}{n}\frac{(-1)^{n+1}\pi}{n} \\
a_n &= \frac{2}{\pi}\frac{(-1)^n 2\ \pi}{n^2} \\
&= (-1)^n\frac{4}{n^2}
\end{aligned}
$$

Substituting for a_n in (1):

$$
x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)
$$

d. Fourier series of x^2 using integration terms by terms or calculating directly match, as required, by taking $c = \frac{\pi^2}{3}$ since x is a piecewise smooth function on the specified range.

Question 2

Consider the differential equation:

$$
z\frac{d^2y}{dy^2} + y = 0
$$

a. We put the equation in standard form:

$$
\frac{d^2y}{dy^2} + \frac{1}{z}y = 0
$$

$z\,p(z) = 0$ and $z^2q(z) = z$ therefore 0 is a regular singular point.

b. Take $y = z^{\sigma}\sum_{n=0}^{\infty} a_n z^n$ and the usual derivatives in the D.E. gives by substitution

$$
\begin{aligned}
z\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0 \\
\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0\ (1)
\end{aligned}
$$

Take the term with the lowest power of z, which is the first sum with $n = 0$, then since each power of z term must be equal to 0, we have

$$\sigma(\sigma - 1)a_0 z^{\sigma-1} = 0$$

Since $a_0 \neq 0$ and $z^{\sigma-1} \neq 0$, therefore $\sigma = 0, 1$.

c. We go back to equation (1) and take $\sigma = 1$ yields

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

Then reindex the second sum to get same power of z in both sums:

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=1}^{\infty} a_{n-1} z^n = 0$$

Note, in first term $n = 0$ does not contribute so we can start index at $n = 1$ in the first sum, and combine both sums

$$\sum_{n=1}^{\infty} [n(n+1)a_n + a_{n-1}] z^n = 0$$

Since every power of z term must be 0 and $z^n \neq 0$, gives:

$$a_n = -\frac{1}{(n+1)n} a_{n-1}$$

Taking $a_0 = 1$, now

$$\begin{aligned}
n = 1\ a_1 &= -\frac{1}{2\,1} a_0 = -\frac{1}{2\,1} = \frac{(-1)^1}{2\,1} \\
n = 2\ a_2 &= -\frac{1}{3\,2} a_1 = \frac{1}{3\,2\,2\,1} = \frac{(-1)^2}{(3\,2\,1)\,(2\,1)} \\
n = 3\ a_3 &= -\frac{1}{4\,3} a_2 = -\frac{1}{4\,3\,3\,2\,2\,1} = \frac{(-1)^3}{(4\,3\,2\,1)\,(3\,2\,1)} \\
&\vdots \\
a_n &= -\frac{1}{(n+1)n} a_{n-1} = \cdots = \frac{(-1)^n}{((n+1)\,n \cdots 1)\,(n\,(n-1)\cdots 1)} = \frac{(-1)^n}{(n+1)!n!}
\end{aligned}$$

Therefore one of the independent solution of the ODE is

$$y_1(z) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!n!} z^n$$

Question 3

a. We have

$$
\begin{aligned}
&n=0,\ M=0,\ P_0(x)=\frac{(-1)^0(2\ 0-2\ 0)!}{2^0(0-0)!(0-2\ 0)!}x^{0-2\ 0}=1\\
&n=1,\ M=\frac{1-1}{2}=0,\ P_1(x)=\frac{(-1)^0(2\ 1-2\ 0)!}{2^1(1-0)!(1-2\ 0)!}x^{1-2\ 0}=\frac{1\ 2}{2\ 1!\ 1!}x^1=x\\
&n=2,\ M=\frac{2}{2}=1,\ P_2(x)=\frac{(-1)^0(2\ 2-2\ 0)!}{2^2(2-0)!(2-2\ 0)!}x^{2-2\ 0}+\frac{(-1)^1(2\ 2-2\ 1)!}{2^2(2-1)!(2-2\ 1)!}x^{2-2\ 1}\\
&\qquad P_2(x)=\frac{4!}{2^2\ 2!\ 2!}x^2-\frac{(2\ 2-2)!}{2^2\ 1!\ 0!}x^0\\
&\qquad P_2(x)=\frac{4\ 3\ 2\ 1}{4\ 2\ 2}x^2-\frac{2!}{4}\\
&\qquad P_2(x)=\frac{3}{2}x^2-\frac{1}{2}=\frac{1}{2}(3x^2-1)
\end{aligned}
$$

b. From

$$a_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx=\frac{2n+1}{2}\int_{-1}^{1}xP_n(x)dx$$

we have

$$
\begin{aligned}
n=0,\ a_0&=\frac{2\ 0+1}{2}\int_{-1}^{1}xP_0(x)dx\\
&=\frac{1}{2}\int_{-1}^{1}xdx=\frac{1}{2}[\frac{x^2}{2}]_{-1}^{1}=\frac{1}{4}[1^2-(-1)^2]=0\\
n=1,\ a_1&=\frac{2\ 1+1}{2}\int_{-1}^{1}xP_1(x)dx\\
&=\frac{3}{2}\int_{-1}^{1}x^2dx=\frac{3}{2}[\frac{x^3}{3}]_{-1}^{1}=\frac{1}{2}[1^3-(-1)^3]=\frac{1}{2}\ 2=1\\
n=2,\ a_2&=\frac{2\ 2+1}{2}\int_{-1}^{1}xP_2(x)dx\\
&=\frac{5}{2}\int_{-1}^{1}x[\frac{1}{2}(3x^2-1)]dx=\frac{5}{4}\int_{-1}^{1}(3\ x^3-x)dx\\
&=0\ \text{ since the powers of } x \text{ in the integrand are odd}
\end{aligned}
$$

Therefore the Fourier-Legendre series of x iss $x=1\cdot P_1(x)$ as required.

c. Using Rodrigues's formula

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

we have

$$
\begin{aligned}
n=0,\ & \frac{d^0}{dx^0}[(x^2-1)^0]=(x^2-1)^0=1\\
& P_0(x)=\frac{1}{2^0\ 0!}\ 1=1\\
n=1,\ & \frac{d}{dx}(x^2-1)=2x\\
& P_1(x)=\frac{1}{2^1\ 1!}\ 2x=x\\
n=2,\ & \frac{d^2}{dx^2}(x^2-1)^2=\frac{d}{dx}\Big[\frac{d}{dx}(x^2-1)^2\Big]=\frac{d}{dx}\big[4x(x^2-1)\big]=\frac{d}{dx}\big[4x^3-4x\big]=12\,x^2-4\\
& P_2(x)=\frac{1}{2^2\ 2!}\ (12\,x^2-4)=\frac{4}{4\ 2}(3x^2-1)=\frac{1}{2}(3x^2-1)
\end{aligned}
$$

Question 4

a.

$$\frac{\partial u}{\partial x}+4xu=0$$

Integration factor is

$$e^{\int 4xdx}=e^{4\int xdx}=e^{4\frac{x^2}{2}}=e^{2x^2}$$

Multiply the partial differential equation by the I.F.:

$$
\begin{aligned}
e^{2x^2}\frac{\partial u}{\partial x}+4xe^{2x^2}u&=0\\
\frac{\partial}{\partial x}(e^{2x^2}u)&=0
\end{aligned}
$$

Now integrate both sides with respect to x

$$
\begin{aligned}
e^{2x^2}u&=C\ C\text{:constant}\\
u(x)&=Ce^{-2x^2}
\end{aligned}
$$

b.

$$y^2u_x-x^2u_y=0$$

Let $u(x,y)=X(x)Y(y)$, substitution into the D.E. gives

$$
\begin{aligned}
y^2X'Y-x^2XY'&=0\\
y^2\frac{X'Y}{XY}-x^2\frac{XY'}{XY}&=0\\
y^2\frac{X'}{X}-x^2\frac{Y'}{Y}&=0\\
\frac{1}{x^2}\frac{X'}{X}=\frac{1}{y^2}\frac{Y'}{Y}&=k
\end{aligned}
$$

Integrating $\ln X = \frac{1}{3}kx^3 + \ln(C)$ and $\ln Y = \frac{1}{3}ky^3 + \ln(D)$, so

$$X = Ce^{\frac{1}{3}kx^3}, Y = De^{\frac{1}{3}ky^3}$$

Therefore (with $CD = A$) $u(x, y) = A\ e^{\frac{1}{3}k(x^3+y^3)}$

Question 5

We have the following problem

$$\begin{aligned} \Delta u &= 0 \\ u(x, 0) &= 0 \\ u(x, b) &= 100x \\ u_x(0, y) &= 0 \\ u_x(a, y) &= 0 \end{aligned}$$

Assume a solution of the form $u(x, y) = X(x)Y(y)$. Substitute this expression and divide through by XY, to get:

$$\begin{aligned} \frac{X''}{X} + \frac{Y''}{Y} &= 0 \\ -\frac{X''}{X} &= \frac{Y''}{Y} \end{aligned}$$

LHS is function of x only and RHS is a function of y only, thus we can write, with k constant

$$-\frac{X''}{X} = \frac{Y''}{Y} = k$$

We see immediately that $X(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$. And $X'(x) = \sqrt{k}\Big(B\cos(\sqrt{k}x) - A\sin(\sqrt{k}x)\Big)$. Also $u_x(x, y) = X'(x)Y(y)$. Take the boundary condition $u_x(0, y) = X'(0)Y(y) = 0$, since in general $Y(y) \neq 0$ then $X'(0) = 0$. Plug it into $X'(x)$ gives $\sqrt{k}\Big(B\ 1 - A\ 0\Big) = 0 \to B = 0$. Now $X(x) = A\cos(\sqrt{k}x)$ and $X'(x) = -A\sqrt{k}\sin(\sqrt{k}x)$. Next, with $u_x(a, y) = 0$ gives $X'(a)Y(y) = 0 \to X'(a) = 0$. $X'(a) = 0$ therefore $-A\sqrt{k}\sin(\sqrt{k}a) = 0 \to \sqrt{k}a = n\pi$. Therefore $k_n = \frac{n^2\pi^2}{a^2}$ and

$$X_n(x) = A_n \sin(\frac{n\pi}{a}x)$$

Question 6

a. Substituting $u(r, z) = R(r)Z(z)$ into the diffusion equation in cylindrical coordinates gives

$$R''Z + \frac{1}{r}R'Z + RZ'' = 0$$

Diving by RZ gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = 0$$

Separation of variables gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = -k^2$$

or

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -k^2$$
$$\frac{Z''}{Z} = k^2$$

b. For $\frac{d^2}{dz^2}Z(z) = k^2 Z(z)$, we immediately see that $Z(z) = c_1 e^{kz} + c_2 e^{-kz}$ which we can reformulate as $Z(z) = A\sinh(kz) + B\cosh(kz)$.
For $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -k^2$, starting with $\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + k^2R(r) = 0$.

$$s = kr, r = \frac{s}{k}, \frac{ds}{dr} = k, R(r) \to R(s)$$
$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = k\frac{dR}{ds}$$
$$\frac{d^2R}{dr^2} = k\frac{d}{ds}\left(k\frac{dR}{ds}\right) = k^2\frac{d^2R}{ds^2}$$

Substitution into the ODE gives

$$k^2\frac{d^2R(s)}{ds^2} + \frac{1}{\frac{s}{k}}k\frac{dR(s)}{ds} + k^2R(s) = 0$$
$$k^2\frac{d^2R(s)}{ds^2} + \frac{1}{s}k^2\frac{dR(s)}{ds} + k^2R(s) = 0$$

Multiplying out by $(\frac{s}{k})^2$ gives

$$s^2\frac{d^2R(s)}{ds^2} + s\frac{dR(s)}{ds} + s^2R(s) = 0$$
$$s^2\frac{d^2R(s)}{ds^2} + s\frac{dR(s)}{ds} + (s^2 - 0^2)R(s) = 0$$

The last equation being a Bessel equation of order 0 therefore the solution is of the form

$$R(r) = C_1J_0(kr) + C_2Y_0(kr)$$

Since the temperature remains bounded at $r = 0$ thus the term $Y_0(kr)$ has to be discarded, $C_2 = 0$, and $R(r) = CJ_0(kr), C = C_1$

c. Finally apply boundary conditions. First, $u(r, 0) = R(r)Z(0) = 0$, since in general $R(r) \neq 0$, thus $Z(0) = 0$, which is $A\,0 + B\,1 = 0 \to B = 0$ and $Z(z) = A\sinh(kz)$. Then $u(5, z) = R(5)Z(z) = 0 \to R(5) = 0$, therefore $CJ_0(5k) = 0 \to J_0(5k) = 0$. $5k$ represents the zero crossing for the Bessel function of order 0. We call them α_m and set $5k_m = \alpha_m \to k_m = \frac{\alpha_m}{5}$. Therefore the solutions are $R_m(r) = C_m J_0(k_m r) = C_m J_0(\frac{\alpha_m}{5} r)$. Note now that solutions in z are $Z_m(z) = A_m \sinh(k_m z)$. Finally applying the superposition principle, we have

$$u(r,z) = \sum_{m=1}^{\infty} A_m \sinh(k_m z) J_0(\frac{\alpha_m}{5} r) \text{ where } A_m C_m \to A_m$$

Applying the last boundary condition $u(r, 20) = u_0, 0 < r < 5$, we get:

$$u(r,20) = \sum_{m=1}^{\infty} A_m \sinh(20k_m) J_0(\frac{\alpha_m}{5} r) = u_0$$

This is a Fourier Bessel series where the coefficients are given by

$$\sinh(20k_m)A_m = \frac{2}{5^2 J_1^2(\alpha_m)} \int_0^5 r u_0 J_0(\frac{\alpha_m}{5} r)dr = \frac{2u_0}{25 J_1(\alpha_m)} \int_0^5 r J_0(\frac{\alpha_m}{5} r)dr$$

Next, $\sinh(20k_m) = \sinh(20\frac{\alpha_m}{5}) = \sinh(4\alpha_m)$, and we get

$$A_m = \frac{2u_0}{25 J_1(\alpha_m)\sinh(4\alpha_m)} \int_0^5 J_0(\frac{\alpha_m}{5} r) r dr$$

Finally using $\frac{\partial}{\partial r}[rJ_1(r)] = rJ_0(r)$

$$\begin{aligned} A_m &= \frac{2u_0}{25 J_1(\alpha_m)\sinh(4\alpha_m)} \left[r J_1(\frac{\alpha_m}{5} r) \right]_0^5 \\ &= \frac{2u_0}{25 J_1(\alpha_m)\sinh(4\alpha_m)} \left(5 J_1(\frac{\alpha_m}{5} 5) \right) \\ &= \frac{2u_0}{5\sinh(4\alpha_m)} \end{aligned}$$