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JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

We will look at two main topics in this presentation

- Wronskian and independent solutions of an ODE
- Classification of ordinary and singular points of ODEs

Wronskian and independent solutions of an ODE

We start by looking at second order linear differential homogeneous equation in standard form

$$y'' + p(x)y' + q(x)y = 0$$

It has been mentioned that the solution to this second order differential equation consist of two linearly independent solutions, that is

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

So if we have two solutions how do we verify that they are in fact independent?

For the complementary solution above we set y(x) =0, essentially forming a linear combination of the two solutions and setting it equal to zero.

Then the solutions $y_1(x)$ and $y_2(x)$ are linear independent if the only possible way this can happen is if both constants are equal to zero otherwise these two solutions are linearly dependent. Note since we have two constants to Investigate we need another condition to construct. This condition is chosen by letting the derivative $y_c'(x)$ also be zero. Thus we have

$$c_1 y_1(x) + c_2 y_2(x) = 0$$
$$c_1 y_1'(x) + c_2 y_2'(x) = 0$$

This is easily generalize to an nth order differential equation. Then $y_c(x)=\sum_{i=1}^{n} c_i y_i(x)$

$$
\begin{aligned}
c_1 y_1(x)+\cdots+c_n y_n(x)&=0\\
c_1 y_1^{(1)}(x)+ c_n y_n^{(1)}(x)&=0\\
&\vdots\\
c_1 y_1^{(n-1)}(x)+ c_n y_n^{(n-1)}(x)&=0
\end{aligned}
$$

In matrix form this is

$$
\begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}=Y(x)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}=Yc=\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},\ c=\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}
$$

Let $\boxed{Y=\begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}}$, $c=\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ and $0=\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Gives us the following equation $Yc=0$ and the matrix c is zero, that is the constants c_1 to c_n are all zero only if the determinate of Y is zero.

This quantity is called the Wronskian and denoted W(x), that is det(Y).

Going back to the simpler case of a second order differential equation we have for the Wronskian

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

Taking the derivative of this of the Wronskian (using the product rule) and
The following relationships for the two solutions from the original ODE

$$y_1'' = -p(x)y_1' - q(x)y_1$$
$$y_2'' = -p(x)y_2' - q(x)y_2$$

We get $W'(x) = -p(x)W(x)$ or $\dfrac{W'(x)}{W(x)} = -p(x)$

This of course is the form of an exponential decay differential equation,
Integration with respect to x easily gives

$$W(x) = Ce^{-\int^{x} p(u)du}$$

Note that if p(x)=0 in our original differential equation then W(x) = constant.

Classification of ordinary and singular points in ODEs

Again we will focus on second order ODEs, that is $y''+p(z)y'+q(z)y=0$

Let's look at the functions $p(z)$, $q(z)$at some particular point $z=z_0$.

These functions are said to be analytic provided they can be represented as a sum having the following form

$$p(z)=\sum_{n=0}^{\infty} p_n (z-z_0)^n,\ q(z)=\sum_{n=0}^{\infty} q_n (z-z_0)^n$$

Where the p_n, q_n are constants. Then z_0 is said to be an **ordinary point**.

If however p(z) or q(z) cannot be expressed in this form, that is they diverge at $z=z_0$
Then the point is called a singular point. Singular points can be of two major types when Looking for solutions od ODE, regular or essential singularity.

Definition: singular points of ODE

If $(z-z_0)p(z)$, $(z-z_0)^2 q(z)$ are both analytic then we call z_0 a regular singular point.

Otherwise $z=z_0$ is an essential singular point.

Examples of ordinary and singular points of ODE

We start by looking at an example of an important second order ODE.

$$(1-z^2)y''-2zy'+\ell(\ell+1)y=0$$ **Legendre ODE**

Put this in standard form and identify p(z) and q(z)

$$y''-\frac{2z}{(1-z^2)}y'+\frac{\ell(\ell+1)}{(1-z^2)}y=0, \text{ then } p(z)=-\frac{2z}{(1-z^2)}, \quad q(z)=\frac{\ell(\ell+1)}{(1-z^2)}$$

Next let's pick a number of points $z=z_0=0,\ \pm 1,\ \pm\infty$ to investigate and identify whether they are singular.

For $z_0=0=z$ $\quad p(0)=0,\ q(0)=\ell(\ell+1)$ $\quad$ Ordinary point at z =0

$z_0=\pm 1=z$ $\quad p(z),\ q(z)$ undefined $\quad$ Singular points

So for $z=\pm 1$ we need to identify what type of singular point

First construct $(z-z_0)p(z),\ (z-z_0)^2 q(z)$ with z_0=1 and then evaluate the resultant function at z =1

$$(z-1)p(z)=(z-1)\left[-\frac{2z}{1-z^2}\right]=\frac{2z}{1+z} \text{ and at } z=1 \quad \frac{2(1)}{1+1}=1$$

$$(z-1)^2 q(z)=(z-1)^2\left[\frac{\ell(\ell+1)}{(1-z^2)}\right]=\frac{-(z-1)\ell(\ell+1)}{(1+z)} \text{ and at } z=1 \quad \frac{-(0)\ell(\ell+1)}{1+1}=0$$

Both are well defined at z = 1 so this is a regular singular point.
A similar result is obtained for z = -1, therefore it is also a regular singular point.

A more complex case to look at is whether the Legendre equation is well behaved at

$$z = \pm\infty$$

Start with the original ODE not in standard form. $(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0$

While it is difficult to tell what it's behavior is at Infinity directly we can make a change of variable which makes it much easier to investigate what type of points these are.

Therefore let $z = 1/w$ (or $w = 1/z$) and look at w = 0 (essentially z at infinities)

We change the derivatives with respect to z to the variable w as follows

$$\frac{dy}{dz} = \frac{dy}{dw}\frac{dw}{dz} = \frac{dy}{dw}\frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2}\frac{dy}{dw} = -w^2\frac{dy}{dw}$$

$$\frac{d^2y}{dz^2} = -w^2\frac{d}{dw}\left(-w^2\frac{dy}{dw}\right) = w^2\frac{d}{dw}\left(w^2\frac{dy}{dw}\right) = w^2\left(2w\frac{dy}{dw} + w^2\frac{d^2y}{dw^2}\right)$$

$$= w^3\left(w\frac{d^2y}{dw^2} + 2\frac{dy}{dw}\right)$$

Next substitution into our differential equation

Start with the original ODE not in standard form. $(1-z^2)y''-2zy'+\ell(\ell+1)y=0$

While it is difficult to tell what it's behavior is at Infinity directly we can make a change of variable which makes it much easier to investigate what type of points these are.

Therefore let $z=1/w$ (or $w=1/z$) and look at w = 0 (essentially z at infinities)

We change the derivatives with respect to z to the variable w as follows

$$(1-\frac{1}{w^2})w^3\left(w\frac{d^2y}{dw^2}+2\frac{dy}{dw}\right)-2\left(\frac{1}{w}\right)\left(-w^2\frac{dy}{dw}\right)+\ell(\ell+1)y=0$$

Simplification gives

$$(w^3-w)\left(w\frac{d^2y}{dw^2}+2\frac{dy}{dw}\right)+2w\frac{dy}{dw}+\ell(\ell+1)y=(w^3-w)\left(w\frac{d^2y}{dw^2}\right)+w^3 2\frac{dy}{dw}+\ell(\ell+1)y0$$

In standard form we get

$$\frac{d^2y}{dw^2}+\frac{w^3 2}{w^2(w^2-1)}\frac{dy}{dw}+\ell(\ell+1)y=\frac{d^2y}{dw^2}+\frac{2w}{w^2-1}\frac{dy}{dw}+\frac{\ell(\ell+1)}{w^2(w^2-1)}y=0$$

That gives
$$p(w)=\frac{2w}{w^2-1}$$
$$q(w)=\frac{\ell(\ell+1)}{w^2(w^2-1)}$$
and at w = 0
$$p(w)=0$$
$$q(w)\text{ is undefined}$$

So w is a singular point. Finally identify what type using

$$(w-0)p(w)=w\left(\frac{2w}{w^2-1}\right)=\frac{2w^2}{w^2-1}\text{and at w = 0}\quad\frac{2(0)^2}{0^2-1}=0$$
$$(w-0)^2q(w)=w^2\left[\frac{\ell(\ell+1)}{w^2(w^2-1)}\right]=\frac{\ell(\ell+1)}{w^2-1}\text{and at w = 0}\quad\frac{\ell(\ell+1)}{0^2-1}=-\ell(\ell+1)$$

We that both expressions analytic and thus the point w – 0.

The differential equations at z plus/minus infinity behaves like a regular singular point.