Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
EN. 585.409

JOHNS HOPKINS
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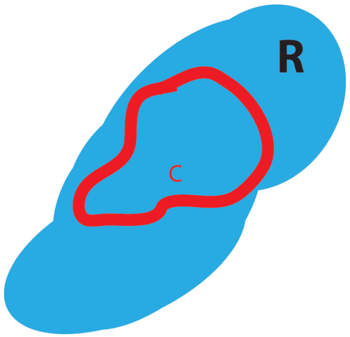
of ENGINEERING

Cauchy's theorem

Given f(z) is analytic and differentiable within a region **R** that is simply connected then for a closed path we have the following result.

$$\oint_C f(z)dz = 0$$



Let's derive this!
But first let's look at a theorem from multivariate calculus we will need to use.

Key: Green's theorem $$\iint_R \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) dxdy = \oint_C p\,dx - q\,dy$$

where the closed path C is encloses the simply connected region R and p(x,y) and q(x,y) and their derivatives are single valued inside and on the boundary.

Start with $\oint_C f(z)dz$ where in this case the closed path C encloses a simple connected region **R**.

Take $f(z)=u(x,y)+iv(x,y)$ and $z = x+iy$, $dz=dx+idy$

Substitution gives

$$\oint_C f(z)dz=\oint_C u(x,y)+iv(x,y)(dx+i\,dy)=\oint_C u(x,y)dx-v(x,y)dy+i\oint_C v(x,y)dx+u(x,y)dy$$

Next take the first integral and rewrite it and apply Green's theorem

$$\oint_C u\,dx-v\,dy=-\oint_C v\,dy-u\,dx=\boxed{-\iint_R\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dxdy}$$

Then take the second integral and rewrite it and apply Green's theorem

$$i\oint_C v\,dx+u\,dy=i\oint_C u\,dy-(-v)dx=i\iint_R\left(\frac{\partial u}{\partial x}+\frac{\partial(-v)}{\partial y}\right)dxdy=\boxed{i\iint_R\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dxdy}$$

Key: Next since this is an analytic function it satisfies the Cauchy-Riemann conditions.

Cauchy – Riemann gives us $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Substitute our Green's theorem results and using these relationships in our integral gives

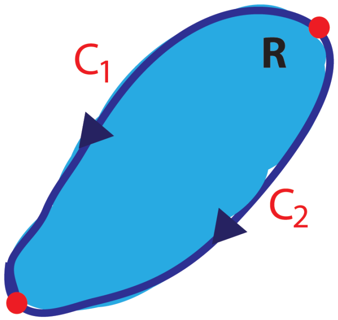
$$\boxed{\oint_C f(z)dz} = \oint_C u(x,y)dx - v(x,y)dy + i\oint_C v(x,y)dx + u(x,y)dy =$$

$$-\iint_R \left(-\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} \right) dxdy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y} \right) dxdy = -\iint_R (0) dxdy + i\iint_R (0) dxdy = \boxed{0}$$

Cauchy's Theorem

A couple of quick but important results applying Cauchy's theorem

Path independent for analytic function in simple region



$C = C_1 - C_2$

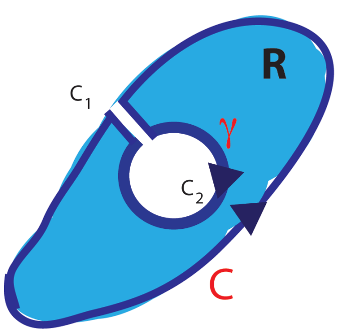
Using Cauchy's theorem we have

$$\oint_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

Since $C = C_1 - C_2$ (Paths in diagram in the same direction)

Therefore immediately $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

Two closed paths embedded within each other for analytic function in simple region are equivalent



Using Cauchy's theorem we have

$$\oint_{\text{Total path}} f(z)dz = \int_C f(z)dz - \int_\gamma f(z)dz$$

where total path outside and inside

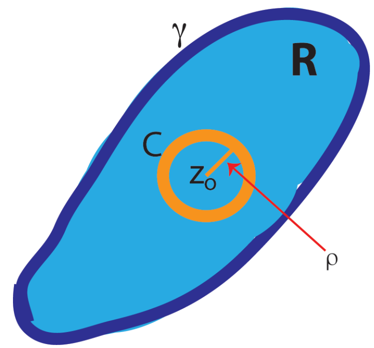
Now shrink take $\lim_{\text{gap}\to 0} \oint_{\text{Total path}} f(z)dz = 0$

Again immediately $\int_C f(z)dz = \int_\gamma f(z)dz = 0$

Cauchy's Integral theorem

Given f(z) is analytic except at z = z_o within a region **R** that is simply connected then for a closed path γ enclosing z_o we have the following result

$$f(z_o) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_o}dz \quad \text{or} \quad \oint_\gamma \frac{f(z)}{z - z_o}dz = 2\pi i f(z_o)$$



The proof of this involves a result from our previous derivation of Cauchy's theorem. $\oint_\gamma g(z)dz \equiv \oint_C g(z)dz$

Then we can write for our integral $$\oint_\gamma \frac{f(z)}{z - z_o}dz \equiv \oint_C \frac{f(z)}{z - z_o}dz$$

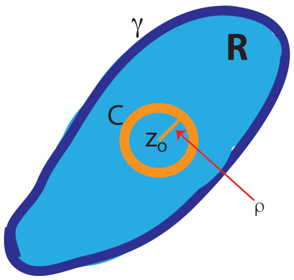
But in particular we will let the close path C be represented as in the diagram above. That is a circle of radius ρ centered at z_o, $z = z_o + \rho e^{i\theta}$ where θ goes from 0 to 2π.

$$\oint_\gamma \frac{f(z)}{z-z_o}dz \equiv \oint_C \frac{f(z)}{z-z_o}dz$$

Now taking our path as a function in terms of the parameterization

$z = z_o + \rho e^{i\theta}$ and noting that $\frac{dz}{d\theta} = i\rho e^{i\theta} \rightarrow dz = i\rho e^{i\theta} d\theta$ then

$$\int_0^{2\pi} \frac{f(z_o+\rho e^{i\theta})}{z_o+\rho e^{i\theta}-z_o} i\rho e^{i\theta}\, d\theta = i\int_0^{2\pi} \frac{f(z_o+\rho e^{i\theta})}{\rho e^{i\theta}} \rho e^{i\theta}\, d\theta = i\int_0^{2\pi} f(z_o+\rho e^{i\theta})d\theta$$



Finally the key is to let the radius of the path go to 0, that is

$$\oint_\gamma \frac{f(z)}{z-z_o}dz = \lim_{\rho\to 0} i\int_0^{2\pi} f(z_o+\rho e^{i\theta})d\theta = i\int_0^{2\pi} f(z_o)d\theta = if(z_o)\int_0^{2\pi} d\theta = 2\pi i f(z_o)$$

Extension to Cauchy's Integral Theorem to the derivative of a function

The proof of this is fairly easy and starts with Cauchy's integral theorem.

$$f(z_o) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_o} dz$$

Take the nth partial derivative with respect to z_o, that is

$$\frac{\partial^n}{\partial z_o^n} f(z_o) = \frac{\partial^n}{\partial z_o^n} \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_o} dz \;= \frac{1}{2\pi i}\oint_\gamma f(z) \frac{\partial^n}{\partial z_o^n} \frac{1}{z - z_o} dz$$

Now focusing on our derivative in the integrand

$$\frac{\partial^1}{\partial z_o^1} \frac{1}{z - z_o} = \frac{1!}{(z - z_o)^2} \qquad \frac{\partial^2}{\partial z_o^2} \frac{1}{z - z_o} = \frac{2!}{(z - z_o)^3} \qquad \frac{\partial^3}{\partial z_o^3} \frac{1}{z - z_o} = \frac{3!}{(z - z_o)^4}$$

In general $\frac{\partial^n}{\partial z_o^n} \frac{1}{z - z_o} = \frac{n!}{(z - z_o)^{n+1}}$ and substitution in our integral gives

$$\frac{\partial^n}{\partial z_o^n} f(z_o) \equiv f^{(n)}(z_o) = \frac{1}{2\pi i}\oint_\gamma f(z) \frac{n!}{(z - z_o)^{n+1}} dz = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_o)^{n+1}} dz$$

$$\boxed{f^{(n)}(z_o) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_o)^{n+1}} dz}$$