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Legendre Functions

Legendre's defining differential equation is

$$(1-x^2)y''+-2xy'+\ell(\ell+1)y=0$$

As previously noted +/-1 and infinity are regular singular points and at x = 0 we have an ordinary point. For Legendre functions we will be looking for solutions near x = 0 where the interval of interest is -1 to 1.

As usual $y(x)=\sum_{n=0}^{\infty}a_n x^n,\quad y'(x)=\sum_{n=0}^{\infty}na_n x^{n-1},\quad y''(x)=\sum_{n=0}^{\infty}(n-1)na_n x^{n-2}$

Substitution off our proposed series solutions and it's derivatives into our original ODE gives

$$(1-x^2)y''+-2xy'+\ell(\ell+1)y=(1-x^2)\sum_{n=0}^{\infty}(n-1)na_n x^{n-2}-2x\sum_{n=0}^{\infty}na_n x^{n-1}+\ell(\ell+1)\sum_{n=0}^{\infty}a_n x^n=$$

$$\left[\sum_{n=0}^{\infty}(n-1)na_n x^{n-2}-\sum_{n=0}^{\infty}(n-1)na_n x^n\right]-\sum_{n=0}^{\infty}2na_n x^n+\ell(\ell+1)\sum_{n=0}^{\infty}a_n x^n=$$

$$\sum_{n=0}^{\infty}(n-1)na_n x^{n-2}+\left[-\sum_{n=0}^{\infty}(n-1)n-2n+\ell(\ell+1)a_n x^n\right]=$$

$$\sum_{n=0}^{\infty}(n-1)na_n x^{n-2}+\sum_{n=0}^{\infty}[-(n-1)n+2n+\ell(\ell+1)]a_n x^n=0$$

Re-index the first sum. Let $n \to n+2$

$$\sum_{n=-2}^{\infty}(n+1)(n+2)a_{n+2}x^n+\sum_{n=0}^{\infty}[-(n-1)n+2n+\ell(\ell+1)]a_n x^n \equiv$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n+\sum_{n=0}^{\infty}[-(n-1)n+2n+\ell(\ell+1)]a_n x^n =$$

$$\sum_{n=0}^{\infty}\{(n+1)(n+2)a_{n+2}+[-(n-1)n+2n+\ell(\ell+1)]a_n\}x^n=0$$

As usual since in general $x^n \neq 0$ $\quad (n+1)(n+2)a_{n+2}+[-(n-1)n+2n+\ell(\ell+1)]a_n=0$

$$a_{n+2}=\frac{[n(n+1)-\ell(\ell+1)]}{(n+1)(n+2)}a_n$$

Since this is an ordinary point we are guaranteed two linearly independent solutions.
So take the two cases

$$a_0=1,\ a_1=0 \text{ and } a_0=0,\ a_1=1$$

First take $a_0 = 1,\ a_1 = 0$

For n = 0 $\quad a_2 = \dfrac{[0(0+1)-\ell(\ell+1)]}{(0+1)(0+2)}(1) = \dfrac{-\ell(\ell+1)}{2!}$

For n = 2 $\quad a_4 = \dfrac{[2(2+1)-\ell(\ell+1)]}{(2+1)(2+2)}\left[\dfrac{-\ell(\ell+1)}{2!}\right] = \dfrac{[2\cdot 3-\ell(\ell+1)][-\ell(\ell+1)]}{4!} =$

$$= \frac{-(\ell+1)[6\ell-\ell^3-\ell^2)]}{4!} = \frac{(\ell+1)\ell[\ell^2+\ell-6]}{4!} = \frac{\ell(\ell+1)(\ell+3)(\ell-2)}{4!} = \frac{(\ell-2)\ell(\ell+1)(\ell+3)}{4!}$$

Therefore $\quad y_1(x) = \sum_{n=0,n-\text{even}}^{\infty} a_n x^n = a_0x^0 + a_2x^2 + a_2x^2 + \cdots =$

$$a_0 + \frac{-\ell(\ell+1)}{2!}x^2 + \frac{(\ell-2)\ell(\ell+1)(\ell+3)}{4!}x^2 + \cdots$$

Similar for the second solution, where we have

$$y_2(x) = \sum_{n=0,n-\text{odd}}^{\infty} a_n x^n = a_1x^1 + a_3x^3 + a_5x^5 + \cdots =$$

$$x - \frac{(\ell-1)(\ell+2)}{3!}x^3 + \frac{(\ell-3)(\ell-1)(\ell+2)(\ell+4)}{5!}x^5 + \cdots$$

Let's take another look at our recursion relationship for the coefficients

$$a_{n+2} = \frac{[n(n+1) - \ell(\ell+1)]}{(n+1)(n+2)} a_n$$

Note this leads to finite length polynomial solutions when $\ell = n$

Take $\ell = n$, even then $y_1(x)$ terminates, however $y_2(x)$ does not.
The opposite occurs when we take $\ell = n$ odd. Let's look at $y_1(x)$

Therefore

For n = 0 $\quad y_1(x) = 1 + \frac{-0(0+1)}{0!} x^2 = 1 + \frac{-0}{1} x^2 = 1$

For n = 2 $\quad y_1(x) = 1 + \frac{-2(2+1)}{2!} x^2 + \frac{(2-2)2(2+1)(2+3)}{4!} x^4 = 1 - 3x^2$

$\vdots$

Above we have generated un-normalized Legendre polynomials for n = 0, 2.
The final requirement is to normalize the polynomials such that, $P_n(1)=1$

For n = 0, since $y_1(1) = 1 \equiv P_0(1)$

For n = 2 since $y_1(1) = 1 - 3 \cdot 1^2 = -2 \neq P_2(1)$

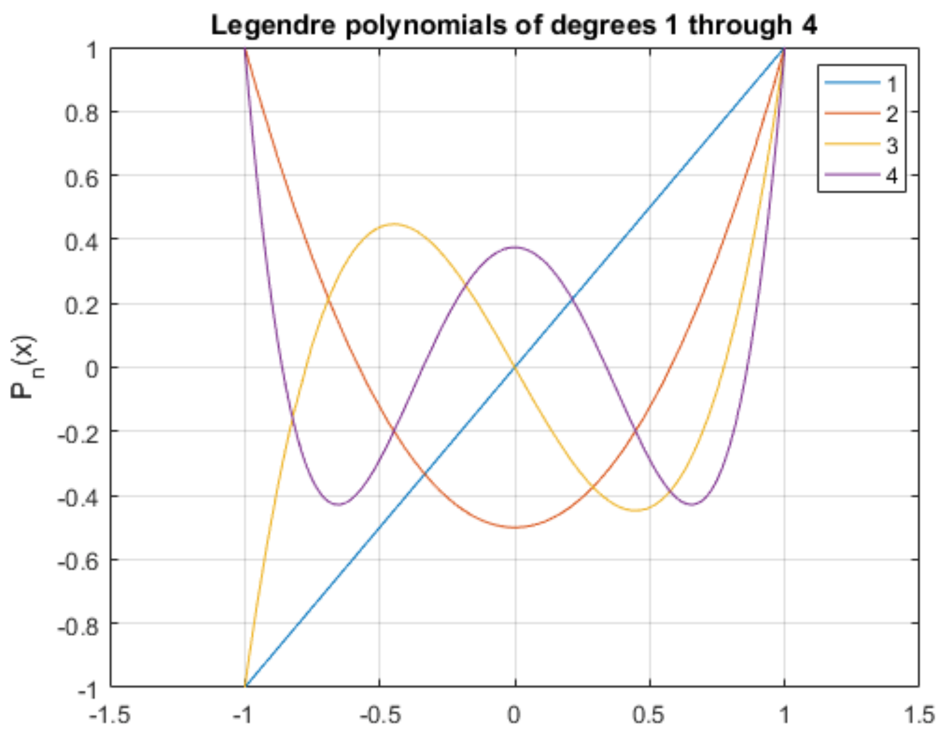
Therefore setting $ay_1(1) = a(1 - 3 \cdot 1^2) = a(-2) = 1 = P_2(1)$

we get $a = -\frac{1}{2}$ and $P_2(x) = ay_1(x; n = \ell = 2) = -\frac{1}{2}(1 - 3x^2) = \frac{1}{2}(3x^2 - 1)$

Legendre polynomials of the first kind

$$P_1(x) = x \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$



Weisstein, Eric W. "Legendre Polynomial." From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/LegendrePolynomial.html

Legendre polynomials of the second kind

Legendre polynomials of the second kind are composed from the non-terminating series solutions $y_2(x)$.

The normalization factor for these polynomials are slightly different for even or odd, that is

$$\begin{aligned} Q_\ell(x) &= \alpha_\ell\, y_2(x)\Big|_\ell \quad \ell \text{ even} \\ Q_\ell(x) &= \beta_\ell\, y_1(x)\Big|_\ell \quad \ell \text{ odd} \end{aligned}$$

Let's look at a specific case.

We define $Q_0(x) = \alpha_\ell\, y_2(x)\Big|_{\ell=0}$ as the zero order Legendre polynomial of the second kind.

With $\alpha_\ell\Big|_{\ell=0} = \dfrac{(-1)^{\ell/2} 2^\ell}{\ell!}\left[\left(\dfrac{\ell}{2}\right)!\right]^2\Bigg|_{\ell=0} = \dfrac{(-1)^{0/2} 2^0}{0!}\left[\left(\dfrac{0}{2}\right)!\right]^2 = \dfrac{(1)(1)}{(1)}\left[1\right]^2 = 1$

and

$$y_2(x)\Big|_{\ell=0} = x - \frac{(\ell-1)(\ell+2)}{3!}x^3 + \frac{(\ell-3)(\ell-1)(\ell+2)(\ell+4)}{5!}x^5 + \cdots\Bigg|_{\ell=0} =$$

$$x - \frac{(0-1)(0+2)}{3!}x^3 + \frac{(0-3)(0-1)(0+2)(0+4)}{5!}x^5 + \cdots =$$

$$x + \frac{2}{3!}x^3 + \frac{3\cdot 8}{5!}x^5 + \cdots$$

Therefore $Q_0(x) = 1\left[x + \dfrac{2}{3!}x^3 + \dfrac{3\cdot 8}{5!}x^5 + \cdots\right] = x + \dfrac{2}{3!}x^3 + \dfrac{3\cdot 8}{5!}x^5 + \cdots$

By the way because both p(x) from the standard form of Legendre's D.E. and the Legendre polynomial $P_0(x)=1$ of the first kind is simple it is possible to get a closed form solution for $Q_0(x)$. Without presenting the details of the calculations we have

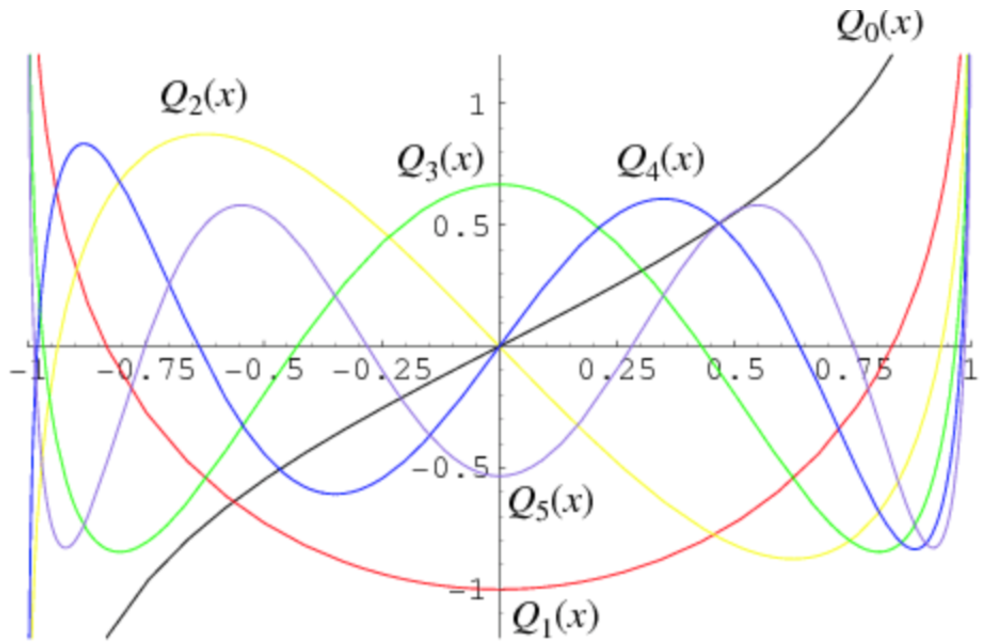
$$Q_0(x)=P_0(x)\int^x \frac{e^{-\int^u p(v)dv}}{[P_0(u)]^2}du \rightarrow \left\{ \begin{array}{c} p(v)=\dfrac{-2v}{1-v^2} \\ P_0(u)=1 \end{array} \right\} \rightarrow \int^x e^{\int^u \frac{2v}{1-v^2}dv} du=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Therefore

$$\boxed{Q_0(x)=x+\frac{2}{3!}x^3+\frac{3\cdot 8}{5!}x^5+\cdots=x+\frac{1}{3}x^3+\frac{1}{5}x^5+\cdots=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$$

By the way you can check this by finding a Taylor series expansion for The closed form of Q_0.

Legendre polynomials of the second kind



Weisstein, Eric W. "Legendre Function of the Second Kind." From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/LegendreFunctionoftheSecondKindl.html

Solutions of Legendre's differential equation

The general solution to Legendre's differential equation is a superposition of the two independent solutions, that is

$$\boxed{y(x) = c_1 P_\ell(x) + c_2 Q_\ell(x)}$$

Rodrigues' formula for Legendre polynomials of the first kind

To derive Rodrigues' formula to construct Legendre polynomials we start out with two key formulas:

- $u(x) = (x^2 - 1)^{\ell}$
- $\dfrac{d^n}{dx^n} uv = \left[\dfrac{d^n u}{dx^n}\right] v + \left[n \dfrac{d^{n-1} u}{dx^{n-1}}\right] \dfrac{dv}{dx} + \left[\dfrac{n(n-1)}{2!} \dfrac{d^{n-2} u}{dx^{n-2}}\right] \dfrac{d^2 v}{dx^2} + \cdots + \dfrac{du}{dx}\left[n \dfrac{d^{n-1} v}{dx^{n-1}}\right] + u\left[\dfrac{d^n v}{dx^n}\right]$

or

$$\frac{d^n}{dx^n} uv = \sum_{r=0}^{n} \binom{n}{r} u^{(n-r)} v^{(r)}$$

Start by taking the first derivative of u(x), that is

$$u'(x) = \ell(x^2 - 1)^{\ell-1}(2x) \rightarrow u'(x) - \ell(x^2 - 1)^{\ell-1}(2x) = 0$$

Multiply by (x^2-1) gives

$$(x^2 - 1)u'(x) - 2x\ell(x^2 - 1)^{\ell} = (x^2 - 1)u'(x) - 2x\ell u(x) = 0$$

Next take $\boxed{\dfrac{d^{\ell+1}}{dx^{\ell+1}}}\left\{(x^2-1)u'(x)-2\ell xu(x)\right\}=0$

$= 0$ since for $n>2$ $\dfrac{d^n}{dx^n}(x^2-1)=0$

For the first term

$$\frac{d^{\ell+1}}{dx^{\ell+1}}(x^2-1)u'(x)=$$

$$\boxed{\left[\frac{d^{\ell+1}(x^2-1)}{dx^{\ell+1}}\right]u'+\left[n\frac{d^{\ell}(x^2-1)}{dx^{\ell}}\right]\frac{du'}{dx}+\cdots}+\frac{d^2(x^2-1)}{dx^2}\left[\frac{(\ell+1)(\ell)}{2!}\frac{d^{\ell+1-2}u'}{dx^{\ell+1-2}}\right]+$$

$$\frac{d(x^2-1)}{dx}\left[\frac{(\ell+1)}{1!}\frac{d^{\ell+1-1}u'}{dx^{\ell}}\right]+(x^2-1)\left[\frac{d^{\ell+1}u'}{dx^{\ell+1}}\right]=$$

$$0+2\left[\frac{(\ell+1)(\ell)}{2!}\frac{d^{\ell-1}u'}{dx^{\ell-1}}\right]+2x\left[\frac{(\ell+1)}{1!}\frac{d^{\ell}u'}{dx^{\ell}}\right]+(x^2-1)\left[\frac{d^{\ell+1}u'}{dx^{\ell+1}}\right]=$$

$$\boxed{(\ell+1)(\ell)\frac{d^{\ell}u}{dx^{\ell}}+2x(\ell+1)\frac{d^{\ell+1}u}{dx^{\ell+1}}+(x^2-1)\frac{d^{\ell+2}u}{dx^{\ell+2}}}$$

Similarly for the Second term

$$\frac{d^{\ell+1}}{dx^{\ell+1}}\left\{(x^2-1)u'(x)-2\ell xu(x)\right\}=0$$

$$\frac{d^{\ell+1}}{dx^{\ell+1}}2\ell xu(x)=2\ell\frac{d^{\ell+1}}{dx^{\ell+1}}xu(x)=2\ell\left[0+(\ell+1)\frac{d^{\ell+1-1}u}{dx^{\ell+1-1}}+x\frac{d^{\ell+1}u}{dx^{\ell+1}}\right]=$$

$$\boxed{2\ell(\ell+1)\frac{d^{\ell}u}{dx^{\ell}}+2\ell x\frac{d^{\ell+1}u}{dx^{\ell+1}}}$$

Substitution gives

$$\frac{d^{\ell+1}}{dx^{\ell+1}}\left\{(x^2-1)u'(x)-2\ell xu(x)\right\}=$$

$$(\ell+1)(\ell)\frac{d^{\ell}u}{dx^{\ell}}+2x(\ell+1)\frac{d^{\ell+1}u}{dx^{\ell+1}}+(x^2-1)\frac{d^{\ell+2}u}{dx^{\ell+2}}-\left(2\ell(\ell+1)\frac{d^{\ell}u}{dx^{\ell}}+2\ell x\frac{d^{\ell+1}u}{dx^{\ell+1}}\right)=$$

$$(x^2-1)\frac{d^{\ell+2}u}{dx^{\ell+2}}+2x(\ell+1)\frac{d^{\ell+1}u}{dx^{\ell+1}}-2\ell x\frac{d^{\ell+1}u}{dx^{\ell+1}}-2\ell(\ell+1)\frac{d^{\ell}u}{dx^{\ell}}=$$

$$(x^2-1)\frac{d^{\ell+2}u}{dx^{\ell+2}}+2x\frac{d^{\ell+1}u}{dx^{\ell+1}}-2\ell(\ell+1)\frac{d^{\ell}u}{dx^{\ell}}=$$

$$\boxed{(1-x^2)\frac{d^{\ell+2}u}{dx^{\ell+2}}-2x\frac{d^{\ell+1}u}{dx^{\ell+1}}+2\ell(\ell+1)\frac{d^{\ell}u}{dx^{\ell}}=0}$$

That is $\frac{d^{\ell}u}{dx^{\ell}}\equiv u^{(\ell)}=c_{\ell}P_{\ell}(x)$ is a solution to Legendre's differential equation .

To find the constant c_{ℓ} taking derivatives of u(x) gives

$$\frac{d^{\ell}}{dx^{\ell}}u(x)=\frac{d^{\ell-1}}{dx^{\ell-1}}\ell(x^2-1)^{\ell-1}(2x)=\frac{d^{\ell-2}}{dx^2}[\ell(\ell+1)(x^2-1)^{\ell-2}(2x)^2+\ell(x^2-1)^{\ell-1}(2)]=$$

$$\cdots=\ell!(x^2-1)^0(2x)^{\ell}+\text{other terms }(x^2-1)^n,\ n>0$$

Note when x = 1 we retain only the leading term since it is the only term that is non zero! $\left.(x^2-1)^0\right|_{x=1}=1$

Finally $\left.\frac{d^\ell}{dx^\ell}u(x)\right|_{x=1} = \ell!(2\cdot 1)^\ell = c_\ell P_\ell(1) = c_\ell \cdot 1 \rightarrow c_\ell = \ell! 2^\ell$

$$\boxed{P_\ell(x) = \frac{1}{c_\ell}\frac{d^\ell}{dx^\ell}u(x) = \frac{1}{\ell! 2^\ell}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell}$$

Rodrigues' formula!