Integral transforms

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1 Fourier transforms

Fourier transform of $f(t)$:

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} \, dt$$

And its inverse defined by:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w) e^{iwt} \, dw$$

2 The Dirac δ-Function

$\delta(t) = 0$ for $t = 0$. Provided the range of integration includes the point $t = a$:

$$\int f(t)\delta(t-a) \, dt = f(a)$$

otherwise the integral equals 0. This leads to:

$$\begin{aligned}
\int_{-a}^{b} \delta(t) \, dt &= 1 \text{ for all } a, b > 0 \\
\int \delta(t-a) \, dt &= 1 \text{ if range of integration includes a} \\
\delta(t) &= \delta(-t) \\
\delta(bt) &= \frac{1}{|b|}\delta(t) \\
t\delta(t) &= 0 \\
\delta(h(t)) &= \sum_i \frac{\delta(t-t_i)}{|h'(t_i)|}
\end{aligned}$$

where the t_i are the zeros of $h(t)$. The derivatives $\delta^n(t)$ are defined by: $\int_{-\infty}^{\infty} f(t)\delta^n(t) \, dt = (-1)^n f^n(0)$. The heaviside function $H(t)$, which is defined as $H(t) = 1$ for $t > 0$ and $H(t) = 0$ for $t < 0$ has the property $H'(t) = \delta(t)$.

Integral representation:

$$\delta(t-u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iw(t-u)} \, dw$$

3 Properties of Fourier transforms

- $\mathcal{F}\big[f^n(t)\big] = (i)^n w^n \tilde{f}(w)$
- $\mathcal{F}\big[\int^t f(s)ds\big] = \frac{1}{iw} f(\tilde{w}) + 2\pi c\delta(w)$
- $\mathcal{F}\big[f(at)\big] \frac{1}{a}\tilde{f}(\frac{w}{a})$

4 Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk$$

5 Laplace transform

By definition:

$$\bar{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-st} \, dt$$

And

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}$$