9.3

Start with Generating function

$$\frac{1}{(1-2xh+h^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)h^n$$

Integrate both sides

$$\int_0^1 \frac{1}{(1-2xh+h^2)^{1/2}}dx = \int_0^1 \sum_{n=0}^{\infty} P_n(x)h^n dx$$

To integrate LHS let $u = 1-2xh+h^2$ and $du = -2hdx$

$$\int \frac{1}{(1-2xh+h^2)^{1/2}}dx = \int \frac{1}{(u)^{1/2}} \frac{du}{-2h} = \frac{1}{-2h}\int u^{-1/2}du = -\frac{u^{1/2}}{h}$$

Therefore

$$\int_0^1 \frac{1}{(1-2xh+h^2)^{1/2}}dx = -\frac{(1-2xh+h^2)^{1/2}}{h}\Bigg|_0^1$$

$$\int_0^1 \frac{1}{(1-2xh+h^2)^{1/2}}dx = -\frac{(1-2h+h^2)^{1/2}}{h} - -\frac{(1+h^2)^{1/2}}{h}$$

$$\int_0^1 \frac{1}{(1-2xh+h^2)^{1/2}}dx = \frac{(1+h^2)^{1/2} - [(1-h)^2]^{1/2}}{h} = \frac{(1+h^2)^{1/2} - (1-h)}{h}$$

This is the LHS term now – we need to do one more manipulation on it below to get this into form with powers of h. We need this because we want to match powers of h on the RHS. The trick is to use the binomial expansion!

Need to deal with

$$\frac{(1+h^2)^{1/2}-(1-h)}{h}$$

Use binomial Theorem

$$(1+x)^n = 1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$$

Therefore

$$(1+h^2)^{1/2} = 1+\frac{(\frac{1}{2})}{1!}h^2+\frac{(\frac{1}{2})((\frac{1}{2})-1)}{2!}h^4+\frac{(\frac{1}{2})((\frac{1}{2})-1)((\frac{1}{2})-2)}{3!}h^6+\cdots$$

and

$$\frac{(1+h^2)^{1/2}-(1-h)}{h} = \frac{1}{h}\left\{[1+\frac{(\frac{1}{2})}{1!}h^2+\frac{(\frac{1}{2})((\frac{1}{2})-1)}{2!}h^4+\frac{(\frac{1}{2})((\frac{1}{2})-1)((\frac{1}{2})-2)}{3!}h^6+\cdots]-(1-h)\right\}$$

$$\frac{(1+h^2)^{1/2}-(1-h)}{h} = 1+\frac{(\frac{1}{2})}{1!}h+\frac{(\frac{1}{2})((\frac{1}{2})-1)}{2!}h^3+\frac{(\frac{1}{2})((\frac{1}{2})-1)((\frac{1}{2})-2)}{3!}h^5+\cdots$$

From RHS (at the beginning after integration) above we have

$$\int_0^1 \sum_{n=0}^{\infty} P_n(x)h^n dx = \sum_{n=0}^{\infty}\left[\int_0^1 P_n(x dx)\right]h^n$$

Match powers of h from LHS (our previous binomial expansion) to RHS (directly above)

For n = 0 $\int_0^1 P_0(x)dx = 1$

For n even $\int_0^1 P_{2n}(x)dx = 0,\ n = 1,2,\cdots$ since no even powers on LHS

For n odd $\int_0^1 P_{2n-1}(x)dx) = \dfrac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\cdots(\frac{1}{2}-n+1)}{n!}$, n=1,2,$\cdots$

Multiply numerator and denominator by 2^n (taking care of all the $\frac{1}{2}$ factors, also do sign change!)

$$\int_0^1 P_{2n-1}(x)dx) = \frac{2^n}{2^n}\frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\cdots(\frac{1}{2}-n+1)}{n!} = \frac{(1)(1-2)(1-4)(1-6)\cdots(1-2n+2)}{2^n n!}$$

Reduce some terms

$$\int_0^1 P_{2n-1}(x)dx = \frac{(1)(-1)(2-1)(-1)(4-1)(-1)(6-1)\cdots(-1)(2n-3)}{2^n n!} = \frac{(-1)^n(1)(3)(5)\cdots(2n-3)}{2^n n!}$$

Note product missing terms in numerator to complete factorial

$$(2)(4)(6)\ldots 2(n-1) = 2(1)2(2)2(3)\cdots 2(n-1) = 2^{n-1}(n-1)!$$

Therefore multiply numerator and denominator by $2^{n-1}(n-1)!$ by gives

$$\int_0^1 P_{2n-1}(x)dx) = \frac{(-1)^{n-1}(2n-2)!}{2^n n! 2^{n-1}(n-1)!} = \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!}$$

Finally let $2n-1 = 2m+1$ to match book, that is n = m+1 and substitute

$$\int_0^1 P_{2m+1}(x)dx = \frac{(-1)^{(m+1-1)}(2(m+1)-2)!}{2^{2(m+1)-1}(m+1)!(m+1-1)!} = \frac{(-1)^m(2m)!}{2^{2m+1}(m+1)!(m)!}$$

Same as the book with this m equal to n in the book!!!!