Chapter 1

Problem 1.5

Using the properties of determinants, solve with a minimum of calculations the following equation for x.

(a)
$$D = \begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

1) For $x = a$, first and last column are the same by a factor of a; hence $D = 0$

2) For $x = b$, second and third column are identical then $D = 0$

3) And for $x = c$, 3^{RD} and last row are identical and again $D = 0$

(b)
$$\begin{vmatrix} x+2 & x+4 & x-3 \\ x+3 & x & x+5 \\ x-2 & x-1 & x+1 \end{vmatrix} = 0$$

Let D this determinant $x = 1$ is a solution since $D = \begin{vmatrix} 1 & 3 & -4 \\ 2 & -1 & 4 \\ -3 & -2 & 0 \end{vmatrix}$

Adding row 1 to row 2 and replacing row 2 by this combination of these two rows gives us:

$D = \begin{vmatrix} 1 & 3 & -4 \\ 3 & 2 & 0 \\ -3 & -2 & 0 \end{vmatrix} = 0$ since row 3 and row 3 are identical by a factor of (-1)

Problem 1.16

Solve the following simultaneous equations for x_1, x_2 and x_3, using matrix method:

$$(S) \begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 3x_1 + 4x_2 + 5x_3 = 2 \\ x_1 + 3x_2 + 4x_3 = 3 \end{cases}$$

We can rewrite this system of 3 equations as:

$Ax = b$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$; $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$|A| = 1\begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & 5 \\ 1 & 4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= 1 - 14 + 5 = 6 - 14 = -8$$

$|A| \neq 0$, the system has a unique solution!

$$\underline{x} = A^{-1}\,\underline{b}$$

Using the method of cofactors:

$$C = \begin{bmatrix} 16-15 & -(12-5) & 9-4 \\ -(8-3) & 4-1 & -(3-2) \\ 10-4 & -(5-3) & 6-4 \end{bmatrix} = \begin{bmatrix} 1 & -7 & 5 \\ -5 & 3 & -1 \\ 6 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{(-8)}\, C^T = 1/8 \begin{bmatrix} -1 & 5 & -6 \\ 7 & -3 & 2 \\ -5 & 1 & -12 \end{bmatrix}$$

And

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1/8 \begin{bmatrix} -9 \\ 7 \\ 3 \end{bmatrix}$$

Problem 1.22

Find the eigenvalues and a set of eigenvectors of the matrix $A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 4 & -2 \\ -1 & -2 & 2 \end{pmatrix}$

Verify that its eigenvectors are mutually orthogonal.

The characteristic polynomial is:

$$|A - \lambda I| = 0 \quad \text{for } \lambda \in \mathbb{C}$$

$$|A-\lambda I| = (1-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ -2 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 3 & -1 \\ 4-\lambda & -2 \end{vmatrix}$$

$$= -\lambda^3 + 7\lambda^2 - 6 = 0$$

We notice that $\lambda = 1$ is a root of the cubic equation so we are looking for an expression of the polynomial characteristic $P(\lambda) = (\lambda - 1)(a\lambda^2 + b\lambda + c)$ $a, b, c \in \mathbb{C}$

$$(\lambda-1)(a\lambda^2 + b\lambda + c) = -\lambda^3 + 7\lambda^2 - 6$$

After equaling terms on LHS and RHS, we find

$$a = -1, \quad b = c = 6$$

Thus $P(\lambda) = (\lambda - 1)\ (-\lambda^2 + 6\lambda + 6)$

The roots of the polynomial of degree 2 is: $r_{1,2} = 3 \pm \sqrt{15}$

$\Rightarrow P(\lambda) = (\lambda - 1)\ (\lambda - r_1)\ (\lambda - r_2)$

<u>First eigenvector</u>

$Ax = x \Rightarrow \begin{cases} x_1 = x_2 \\ 3x_2 = x_3 \end{cases} \quad \underline{V_1} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1 \end{pmatrix}$

<u>Second eigenvector</u>

$Ax = r_1 x$ which gives after few algebra

$$\underline{V_2} = \begin{pmatrix} -\sqrt{15} - 5 \\ \sqrt{15} + 2 \\ 1 \end{pmatrix}$$

<u>Third eigenvector</u>

$Ax = r_2 x$ leads to the third eigenvector $V_3 = \begin{pmatrix} +\sqrt{15} - 5 \\ -\sqrt{15} + 2 \\ 1 \end{pmatrix}$

- $\underline{V_1} \cdot \underline{V_2} = \frac{1}{3}\left(-\sqrt{15} - 5 + \sqrt{15} + 2\right) + 1 = -1 + 1 = 0$

Similarly we show that $\underline{V_1} \cdot \underline{V_3} = \underline{V_2} \cdot \underline{V_3} = 0$

Problem 1.36

Find an orthogonal transformation that takes the quadratic form $Q \equiv -x_1^2 - 2x_2^2 - x_3^2 - 8x_2x_3 + 6x_1x_3 + 8x_1x_2$

onto the form $\mu_1 y_1^2 + \mu_2 y_2^2 + 4y_3^2$

and determine μ_1 and μ_2.

---

let $Q = x^T A x = (x_1\ x_2\ x_3)\begin{pmatrix} -1 & 4 & 3 \\ 4 & -2 & 4 \\ 3 & 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

By finding the eigenvalues of A, we can express Q as

$$Q = x^T A x = (x')^T Q^T A Q x' = (x')^T \Lambda x'$$

where $x' = Q^{-1} x = Q^T x$

and $A = Q \Lambda Q^T$

which gives as $Q = \mu_1 y_1^2 + \mu_2 \mu_2^2 + \mu_3 y_3^3$

where μ_1, y_2, μ_3 eigenvalues of A $= \begin{bmatrix} \mu_1 & 0 & 6 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}$

$\mu_1 = -6,\ \mu_2 = -4,\ \mu_3 = 6$

and $Q = 6y_1^2 + 6y_2^2 - 4y_3^2$

Problem 2-6

Find the areas of the given surfaces using parametric coordinates

(a) Using the parametrization $x = u\cos\phi$, $y = u\sin\phi$,
$z = u \cot \Omega$

find the sloping surface area of a right circular cone of semi-angle Ω whose base has radius a.
Verify that it is equal $= \frac{1}{2} \times$ perimeter of the base $\times$ slope height

(b) Using same parametrization in (a) for x and y, and an appropriate choice for z, find the surface area between the planes $z = 0$ and $z = 2$ of the paraboloid of revolution $z = a(x^2 + y^2)$

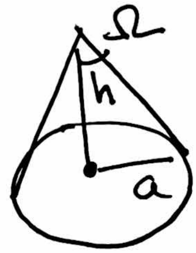
(a)

Using the parametrization

$$x = u\cos\phi,\quad y = u\sin\phi,\quad z = u\,\mathrm{cotg}\,\Omega$$

we can set $\gamma(u,\phi,\Omega) = (u\cos\phi,\ u\sin\phi,\ u\,\mathrm{cotg}\,\Omega)$

where $\tan\Omega = \dfrac{a}{h}$ h being the height of the cone



$u \in [0, h\tan\Omega)$

$\phi \in [0, 2\pi)$

$$\frac{\partial\gamma}{\partial u} = (\cos\phi,\ \sin\phi,\ \mathrm{cotg}\,\Omega)$$

$$\frac{\partial\gamma}{\partial\varphi} = (-u\sin\phi,\ u\cos\phi,\ 0)$$

$$\left|\frac{\partial\gamma}{\partial u}\times\frac{\partial\gamma}{\partial\varphi}\right| = \left|\left(\frac{-u\cos\varphi}{\tan\Omega},\ \frac{-u\sin\varphi}{\tan\Omega},\ u\right)\right|$$

$$= \frac{u\sec\Omega}{\tan\Omega}$$

$$A = \iint_R \left|\frac{\partial\gamma}{\partial u}\times\frac{\partial\gamma}{\partial\varphi}\right| du\,d\varphi$$

$$= \int_0^{2\pi}\int_0^{\tan\Omega} \frac{u\sec\Omega}{\tan\Omega}\,du\,d\varphi = 2\pi\,\frac{\sec\Omega}{\tan\Omega}\left[\frac{u^2}{2}\right]_0^{h\tan\Omega}$$

$$A = 2\pi \cdot \frac{\sec\Omega}{\tan\Omega} \cdot \frac{h^2\tan^2\Omega}{2} = \pi h^2 \sec\Omega \tan\Omega$$

$$A = \pi h^2 \frac{a}{h} \cdot \frac{\sqrt{h^2+a^2}}{h} = \pi a\sqrt{h^2+a^2}$$

$$= \frac{1}{2} \text{ Perimeter} \times \sqrt{h^2+a^2}$$

Problem 2.6.b

(b) The surface area of the paraboloid $z = f(x,y) = a(x^2+y^2)$ between $z=0$ and $z=Z$ is defined by:

$$S = \iint_D \sqrt{(f'_x)^2 + (f'_y)^2 + 1}\; dx\,dy$$

$$= \iint_D 2a\sqrt{x^2+y^2+1}\; dx\,dy.$$

$$x^2+y^2 = u^2\cos^2\phi + u^2\sin^2\phi = u^2$$

u varies for $z=0$ to $Z = a u^2 \Rightarrow u^2 = \frac{Z}{a}$ we use $u = \sqrt{\frac{Z}{a}}$

$$S = \int_0^{2\pi} \int_0^{\sqrt{\frac{Z}{a}}} (2a)\sqrt{u^2+1}\; u\,du\,d\theta$$

$$= 4\pi a \int_0^{\sqrt{Z/a}} u\sqrt{1+u^2}\,du$$

Problem 2-6.b

$$S = 4\pi a \left[\frac{(1+u^2)^{3/2}}{3} \right]_{\cancel{-\sqrt{z/a}}\; 0}^{+\sqrt{z/a}}$$

$$S = \frac{4}{3}\pi a \left[\left(1 + \frac{z}{a}\right)^{3/2} - 1 \right]$$

Problem 2.11

Evaluate the Laplacian of the function

$$\psi(x, y, z) = \frac{z x^2}{x^2+y^2+z^2}$$

(a) Directly in cartesian coordinates

(b) After changing to spherical polar coordinates.

Verify, as they must, the two methods give same result.

Let $V = x^2+y^2+z^2$

$$\frac{\partial \psi}{\partial x} = \frac{2xz\,(y^2+z^2)}{V^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2z\,(y^2+z^2)\,(-3x^2+y^2+z^2)}{V^3}$$

$$\frac{\partial \psi}{\partial y} = -\frac{2x^2yz}{V^2}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \left(-\frac{2x^2z}{V^3}\right)(x^2-3y^2+z^2)$$

$$\frac{\partial \psi}{\partial z} = \frac{x^2(x^2+y^2-z^2)}{V^2} \qquad \frac{\partial^2 \psi}{\partial z^2} = -\frac{2x^2z\,(3x^2+3y^2-z^2)}{(x^2+y^2+z^2)^3}$$

Problem 3.2

A vector field Q is defined as

$$Q = [3x^2(y+z) + y^3 + z^3]\, i + [3y^2(x+z) + z^3 + x^3]\, j$$
$$+ [3z^2(x+y) + x^3 + y^3]\, k$$

Show that Q is a conservative field, construct its potential function and hence evaluate the integral $I = \int Q \cdot dr$ along any line connecting the point A at $(1,-1,1)$ to B at $(2,1,2)$.

$$\frac{\partial Q_z}{\partial y} = 3z^2 + 3y^2 \qquad \frac{\partial Q_y}{\partial z} = 3y^2 + 3z^2$$

$$\Rightarrow \quad \frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z} = 0$$

$$\frac{\partial Q_x}{\partial z} = 3x^2 + 3z^2 \qquad \frac{\partial Q_z}{\partial x} = 3z^2 + 3x^2$$

$$\Rightarrow \quad \frac{\partial Q_x}{\partial z} - \frac{\partial Q_z}{\partial x} = 0$$

And $\dfrac{\partial Q_y}{\partial x} = 3y^2 + 3x^2 \qquad \dfrac{\partial Q_x}{\partial y} = 3x^2 + 3y^2$

$$\Rightarrow \quad \frac{\partial Q_y}{\partial x} - \frac{\partial Q_x}{\partial y} = 0$$

Problem 3.2

Taking the curl of Q we have then $\nabla \times Q = 0$ So Q is a conservative vector field.

Since Q is conservative, we can write $Q = \nabla \phi$

and by inspection we ~~must~~ have:

$$\frac{\partial \phi}{\partial x} = 3x^2(y+z) + y^3 + z^3 \cancel{+ f(y,z)}$$

$$\Rightarrow \phi(x,y,z) = x^3(y+z) + y^3 + z^3 + f(y,z)$$

And $$\frac{\partial \phi}{\partial y} = x^3 + 3y^2 + \frac{\partial f}{\partial y} = 3y^2(x+z) + z^3 + x^3$$

$$\Rightarrow \frac{\partial f}{\partial y} = 3y^2(x+z-1) + z^3$$

$$\Rightarrow f(y,z) = y^3(x+z-1) + z^3 y + g(z)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left(x^3(y+z) + y^3 + z^3 + y^3(x+z-1) + z^3 y + g(z) \right]$$

$$= x^3 + 3z^2 + y^3 + 3z^2 y + \frac{dg(z)}{dz}$$

$$= 3z^2(x+y) + x^3 + y^3$$

Problem 3.2

$$\Rightarrow \quad \frac{dg(z)}{dz} = 3xz^2 - 3z^2 \Rightarrow g(z) = z^3(x-1) + C$$

And finally

$$\phi(x,y,z) = x^3(y+z) + y^3 + z^3 + y^3(x+z-1) + z^3 y + z^3(x-1) + C$$

C: constant

let $A = (1,-1,1)$ $B = (2,1,2)$

$$\int_A^B Q \cdot dr = \phi(B) - \phi(A)$$

$\phi(A) = C$

$\phi(B) = 52 + C \quad \Rightarrow \quad I = \int_A^B Q \cdot dr = 52$

Problem 3.7

Evaluate the line integral

$$I = \oint_C [y(4x^2 + y^2)dx + x(2x^2 + 3y^2)dy$$

around the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let $P(x,y) = y(4x^2+y^2)$ $Q(x,y) = x(2x^2+3y^2)$

We want to evaluate the line integral I around the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, around the simple closed curve C

by Green's theorem

$$I = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\frac{\partial Q}{\partial x} = 6x^2 + 3y^2 \quad \frac{\partial P}{\partial y} = 4x^2 + 3y^2$$

therefore $I = \iint_D 2x^2\,dA = 2\int_0^1 \int_0^{2\pi} (a r \cos\theta)^2\, ab r\, dr\, d\theta$

$$= 2a^3b \int_0^1 r^3\,dr \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= 2a^3b \times \left[\frac{r^4}{4}\right]_0^1 \times \pi$$

$$= \frac{a^3 b \pi}{2}$$

328 A vector force field F is defined in cartesian coordinates by

$$F = F_0 \left[\left(\frac{y^3}{3a^3} + \frac{y}{a} e^{xy/a^2} + 1 \right) i + \left(\frac{xy^2}{a^3} + \frac{x+y}{a} e^{xy/a^2} \right) j + \frac{z}{a} e^{xy/a^2} k \right]$$

Use Stokes' theorem to calculate $\oint_L F \cdot dr$ where L is the perimeter of the rectangle ABCD given by $A = (0, a, 0)$ $B = (a, a, 0)$, $C = (a, 3a, 0)$ and $D = (0, 3a, 0)$

Using stokes' theorem

$$\oint_L F \cdot dr = \iint_S (\nabla \times F) \, dS$$

$$\nabla \times F = F_0 \begin{bmatrix} \frac{xz}{a^3} \quad e^{xy/a^2} \\ -\frac{yz}{a^3} \quad e^{xy/a^2} \\ y^2 \left(\frac{1}{3} + \frac{e^{xy/a^2}}{a^3} \right) \end{bmatrix}$$

$$\Rightarrow \oint_L F \cdot dr = \int_0^a \int_a^{3a} \langle \nabla \times F_x, \nabla \times F_y, \nabla \times F_z \rangle \cdot \langle 0, 0, 1 \rangle \, dx \, dy$$

$$= \int_0^a \int_a^{3a} F_0 \cdot y^2 \left(1/3 + \frac{e^{xy/a^2}}{a^3} \right) dx \, dy$$

Yves GREATTI

(R1) Integrate $I=\int_0^2\int_x^{2x}(x+y)^2\,dx\,dy$

$$I=\int_0^2\int_x^{2x}(x^2+y^2+2xy)\,dx\,dy$$

then $\int_0^2\int_x^{2x} x^2\,dx\,dy=\int_0^2 x^3\,dx=\frac{16}{4}=4$

$$\int_0^2\int_x^{2x} y^2\,dx\,dy=\int_0^2 \frac{1}{3}\left[y^3\right]_x^{2x}\,dx$$

$$=\frac{7}{3}\int_0^2 x^3\,dx=\frac{7}{12}\left[x^4\right]_0^2$$

$$=\frac{28}{3}$$

$$2\int_0^2\int_x^{2x} xy\,dx\,dy=\frac{3\times 16}{4}=12$$

$$\Rightarrow I=4+12+\frac{28}{3}=\frac{76}{3}$$

Yves GREATTI

(R.3) Solve $y^{(4)} + 2y'' + y = 0$

The characteristic equation is: $r^4 + 2r^2 + 1 = 0$

By the change of variable $\sigma = r^2$, the equation becomes:

$(\sigma + 1)^2 = 0$ the roots are $\pm i$ and

$r^4 + 2r^2 + 1 = (r - i)^2 (r + i)^2$

$\pm i$ are a pair of complex roots with multiplicity 2

The general solution is:

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$$

(R4) Solve $4y'' - 4y' - 3y = 0$

with $y(-2) = e$ $y'(-2) = -e/2$

The characteristic equation is: $4r^2 - 4r - 3 = 0$

This equation has $r_1 = 3/2$ and $r_2 = -1/2$ for roots

and since they have multiplicity one, the general

solution is: $y(t) = c_1 e^{-1/2 t} + c_2 e^{3/2 t}$

(R4) Yves Grbatti

To determine the constants c_1, c_2 we use the initial conditions which must be satisfied by the general solution:

$$y(-2) = c_1 e^{-1} + c_2 e^{-3} = e$$

$$y'(-2) = -c_1/2\, e^{-1} + (3c_2)/2\, e^{-3} = -e/2$$

This system of two equations in two unknown c_1, c_2 has for unique solution: $c_1 = e^2$, $c_2 = 0$

Therefore the solution is $y(t) = e^{2-t/2}$

(R5) Solve the following system

$$\frac{dx}{dt} = -k_1 x + k_2 y, \quad x(0) = 2 \qquad (1)$$

$$\frac{dy}{dt} = k_1 x - k_2 y, \quad y(0) = 0 \qquad (2)$$

When $k_1 = k_2$ then adding (1) and (2) we have: $\frac{dx(t)}{dt} = -\frac{dy(t)}{dt}$

which implies that $x(t) = -y(t) + c$

with $x(0) = 2 = -y(0) + c$
$= c$

$\Rightarrow x(t) = -y(t) + 2$

(R4)

Yücel Greatti

20/21

Using the operator $D \equiv d/dt$ we can rewrite this system as:

$$\begin{cases} (D+k_1)\,x - k_2\,y = 0 & (1) & x(0)=2 \\ k_1\,x - (D+k_2)\,y = 0 & (2) & y(0)=0 \end{cases}$$

Multiplying the first equation by $(D+k_2)$ and the second equation by $(-k_2)$ we obtain by adding them together:

$$(D+k_1)(D+k_2)\,x - k_1k_2\,x = 0$$

$$\Rightarrow \quad x''(t) + (k_1+k_2)\,x'(t) = 0.$$

The characteristic equation is: $r^2 + (k_1+k_2)\,r = 0$

$$\Rightarrow r\,(r + (k_1+k_2)) = 0$$

The roots of this equation are $r_0 = 0$ and $r_1 = -(k_1+k_2)$

The general solution of the system for $x(t)$ is of the form:

$$x(t) = c_1 + c_2\,e^{-(k_1+k_2)t}$$

Let $K = k_1+k_2$ and assume $k_1 \neq -k_2$.

$$x(t) = c_1 + c_2\,e^{-Kt}$$

substituting $x(t)$ back into equation (1) gives:

$$k_2\,y(t) = (D+k_1)\,x = (D+k_1)\,(c_1 + c_2e^{-Kt})$$

(Ru)

YG
21/4

$$k_2 y(t) = f(c_2 K) \; e^{-Kt} + k_1 (c_1 + c_2 e^{-Kt})$$

Assuming $k_2 \neq 0$ $y(t) = \frac{k_1 c_1}{k_2} + \frac{f_2}{k_2}(k_1 - K)\, e^{-Kt}$

$$y(t) = \frac{k_1 c_1}{k_2} - c_2 \, e^{-Kt}$$

$x(t)$ and $y(t)$ have to satisfy the initial conditions:

$$\left.\begin{matrix} x(0) = 2 \\ y(0) = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} c_1 + c_2 = 2 \\ \frac{k_1}{k_2} c_1 - c_2 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} c_1 = 2\frac{k_2}{k_1+k_2} \\ c_2 = \frac{2k_1}{k_1+k_2} \end{matrix}$$

$$\begin{cases} x(t) = \frac{2k_2}{K} + \frac{2k_1}{K} e^{-Kt} = \frac{2}{K}(k_2 + k_1 e^{-Kt}) \\ y(t) = \frac{2k_1}{K}(1 - e^{-Kt}) \end{cases}$$

If $k_2 = 0$ we have again $\frac{dx(t)}{dt} = -\frac{dy(t)}{dt}$

for which we found the solutions $x(t)$ and $y(t)$.