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JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

Series solution of an ODE at an ordinary point with an arbitrary coefficient in its non-differential term or polynomial solutions of ODEs

Here is an ODE having no singular points, however there is a n arbitrary constant for the coefficient of its non-differential term

$$y''-2zy'+\lambda y=0$$

For a simple series solution of this differential equation we will have two linearly independent solutions. Let's quickly find them.

Take a simple series solution (and its derivatives)

$$y(z)=\sum_{n=0}^{\infty}a_n z^n,\ \ y'(z)=\sum_{n=0}^{\infty}na_n z^{n-1},\ \ y''(z)=\sum_{n=0}^{\infty}n(n-1)a_n z^{n-2}$$

Substitution off our proposed series solutions and it's derivatives into our original ODE gives

$$y''-2zy'+\lambda y=\sum_{n=0}^{\infty}n(n-1)a_n z^{n-2}-2z\sum_{n=0}^{\infty}na_n z^{n-1}+\lambda\sum_{n=0}^{\infty}a_n z^n=$$

$$\sum_{n=0}^{\infty}n(n-1)a_n z^{n-2}-\sum_{n=0}^{\infty}2na_n z^n+\sum_{n=0}^{\infty}\lambda a_n z^n=\sum_{n=0}^{\infty}n(n-1)a_n z^{n-2}+\sum_{n=0}^{\infty}(-2n+\lambda)a_n z^n=0$$

Re-index second sum (let $n \to n-2$) to match powers of z. Also note change lower Index initial value in first sum, since n = 0, 1 terms do not contribute to this sum. Finally multiple both sums by z^2 to make order of z greater than or equal to 0.

$$z^2\left\{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=2}^{\infty}(-2(n-2)+\lambda)a_{n-2}z^{n-2}\right\} =$$

$$\sum_{n=2}^{\infty}\left\{n(n-1)a_n + (-2(n-2)+\lambda)a_{n-2}\right\}z^n = 0$$

Since as usual z or its powers cannot be zero we take the coefficients as zero, that is

$$n(n-1)a_n + (-2(n-2)+\lambda)a_{n-2} = 0 \to \boxed{a_n = \frac{2(n-2)-\lambda}{n(n-1)}a_{n-2}, \quad n \geq 2}$$

Therefore we have two arbitrary constants, a_0 and a_1. One choice for these constants is a_0=1 and a_1=0.

Then we have

$$\text{For } n=2 \quad a_2 = \frac{2(2-2)-\lambda}{2(2-1)}a_0 = \frac{-\lambda}{2!}(1) = \frac{-\lambda}{2!}$$

$$\text{For } n=4 \quad a_4 = \frac{2(4-2)-\lambda}{4(4-1)}a_2 = \frac{4-\lambda}{4(3)}\left(\frac{-\lambda}{2!}\right) = \frac{-\lambda(4-\lambda)}{4!}$$

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^n = 1 + \frac{(-\lambda)}{2!} z^2 + \frac{-\lambda(4-\lambda)}{4!} z^4 + \cdots$$

Another choice for these constants is $a_0=0$ and $a_1=1$.

Then we have

$$\text{For } n=3 \quad a_3 = \frac{2(3-2)-\lambda}{3(3-1)} a_1 = \frac{2-\lambda}{3\cdot 2}(1) = \frac{2-\lambda}{3!}$$

$$\text{For } n=5 \quad a_5 = \frac{2(5-2)-\lambda}{5(5-1)} a_3 = \frac{6-\lambda}{5(4)}\left(\frac{2-\lambda}{3!}\right) = \frac{(6-\lambda)(2-\lambda)}{5!}$$

And we have $$y_2(z) = \sum_{n=0}^{\infty} a_n z^n = z + \frac{(2-\lambda)}{3!} z^3 + \frac{(6-\lambda)(2-\lambda)}{5!} z^5 + \cdots$$

However for integer values of λ one of these sums may terminate. For example take

$$\lambda = 4,\ a_4 = \frac{-\lambda(4-\lambda)}{4!} \rightarrow a_4 = \frac{-4(4-4)}{4!} = 0$$

and all subsequent a_n (n even) are zero since they are defined recursively.
Therefore

$$y_1(z:\lambda=4) = 1 + \frac{-4}{2!} z^2 = 1 - 2z^2$$

Note y_2 does not terminate as $4-\lambda$ is not a factor in any of the coefficients a_n (n odd).