Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
EN. 585.409

Johns Hopkins
Whiting School of Engineering

Taylor and Laurent series

A Taylor power series expression for a functions of a complex variables can derived starting with Cauchy's integral formula.

$$\text{Starting with } f(z)=\frac{1}{2\pi i}\oint_{\gamma}\frac{f(\xi)}{\xi-z}d\xi=\frac{1}{2\pi i}\oint_{\gamma}\frac{1}{\xi-z}f(\xi)d\xi$$

$$\text{Take } \frac{1}{\xi-z}=\frac{1}{(\xi-z_o)+(z_o-z)}=\frac{1}{\xi-z_o}\frac{1}{\dfrac{[(\xi-z_o)+(z_o-z)]}{\xi-z_o}}=$$

$$\frac{1}{\xi-z_o}\frac{1}{\left[\dfrac{(\xi-z_o)}{(\xi-z_o)}+\dfrac{(z_o-z)}{(\xi-z_o)}\right]}=\frac{1}{\xi-z_o}\frac{1}{1-r} \text{ where } r=\frac{-(z_o-z)}{(\xi-z_o)}=\frac{(z-z_o)}{(\xi-z_o)}$$

$$\text{Therefore } \frac{1}{\xi-z}=\frac{1}{\xi-z_o}(1-r)^{-1}$$

Now we will use a formula for geometric progression

$$\sum_{n=0}^{\infty}a_1 r^n=\frac{a_1}{1-r} \text{ with } a_1=1 \text{ we have } (1-r)^{-1}=\sum_{n=0}^{\infty}r^n$$

$$\text{Therefore } \frac{1}{\xi-z}=\frac{1}{\xi-z_o}\sum_{n=0}^{\infty}r^n=\frac{1}{\xi-z_o}\sum_{n=0}^{\infty}\left(\frac{z-z_o}{\xi-z_o}\right)^n$$

Substitution of our geometric expression gives

$$f(z)=\frac{1}{2\pi i}\oint_{\gamma}\frac{1}{\xi-z}f(\xi)d\xi=\frac{1}{2\pi i}\oint_{\gamma}\frac{1}{\xi-z_o}\sum_{n=0}^{\infty}\left(\frac{z-z_o}{\xi-z_o}\right)^n f(\xi)d\xi=\frac{1}{2\pi i}\oint_{\gamma}\sum_{n=0}^{\infty}\frac{(z-z_o)^n}{(\xi-z_o)^{n+1}}f(\xi)d\xi=$$

$$\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_o)^n\oint_{\gamma}\frac{f(\xi)}{(\xi-z_o)^{n+1}}d\xi$$

Finally using Cauchy's integral theorem for derivatives $\oint_{\gamma}\frac{f(\xi)}{(\xi-z_o)^{n+1}}d\xi=\frac{2\pi i}{n!}f^{(n)}(z_o)$

to substitute for our integral we have

$$\boxed{f(z)}=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_o)^n\frac{2\pi i}{n!}f^{(n)}(z_o)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_o)}{n!}(z-z_o)^n=\boxed{\sum_{n=0}^{\infty}a_n(z-z_o)^n \text{ and } a_n=\left.\frac{f^{(n)}(z)}{n!}\right|_{z=z_o}}$$

Therefore we have an expression for the Taylor series of a function of a complex variable!

What happens if our function f(z) is not analytic and has a singularity at z = z_o?

Start with $\boxed{f(z)=\dfrac{g(z)}{(z-z_o)^p}}$ or $g(z)=f(z)(z-z_o)^p$ where g(z) is analytic,

that is $g(z)=\sum_{n=0}^{\infty} b_n(z-z_o)^n$

Therefore $f(z)=\dfrac{1}{(z-z_o)^p}\sum_{n=0}^{\infty} b_n(z-z_o)^n=\sum_{n=0}^{\infty} b_n(z-z_o)^{n-p}$

Reindex $n \rightarrow n+p$ gives

$$f(z)=\sum_{n=0}^{\infty} b_{n+p}(z-z_o)^{n+p-p}=\sum_{n=-p}^{\infty} b_{n+p}(z-z_o)^n$$

We can make the following associations $a_n=b_{n+p}$ and $a_{-p}=b_0$

Therefore this defines an extension of a Taylor series for f(z) as

$$\boxed{f(z)=\frac{a_{-p}}{(z-z_o)^p}+\cdots+\frac{a_{-1}}{(z-z_o)^1}+\sum_{n=0}^{\infty} a_n(z-z_o)^n}$$ where $a_{-p}\neq 0$

Now using Cauchy's integral theorem for derivatives again we have

$$b_n = \frac{g^{(n)}(z_o)}{n!} = \frac{1}{2\pi i}\oint_\gamma \frac{g(z)}{(z-z_o)^{n+1}}dz$$

Therefore $a_n = b_{n+p} = \frac{1}{2\pi i}\oint_\gamma \frac{g(z)}{(z-z_o)^{n+p+1}}dz = \frac{1}{2\pi i}\oint_\gamma \frac{g(z)}{(z-z_o)^p}\frac{1}{(z-z_o)^{n+1}}dz =$

Finally $\boxed{a_n = \frac{1}{2\pi i}\oint_\gamma f(z)\frac{1}{(z-z_o)^{n+1}}dz}$, n positive or negative

For n >= 0 terms in Laurent series are analytic

n < 0 remaining terms consist of inverse powers of $(z\text{-}z_o)$ called **principle points**

Then the Laurent series is $\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_o)^n}$ where we define the **residue** to be the value of the coefficient a_{-1}

Key: The residue will play an important roll soon!

Key: If f(z) is not analytic at $z = z_o$ there are two possible cases:

(i) Possible to find integer p such that $a_{-p} \neq 0$ but for $\forall_{k>0}\ a_{-p-k} = 0$ there is min p

(ii) It is not possible to find a lowest value -p

Example for finding Laurent series

Take $f(z)=\dfrac{1}{z(z-2)^3}\equiv\dfrac{1}{(z-0)^1(z-2)^3}$ This function has Singularities (poles) at $z_o=0$ of order 1, $z_o=2$ of order 3

First look at expansion of of f(z) as function of z , that is singularity at $z_o= 0$

$$f(z)=\frac{1}{z(z-2)^3}=\frac{1}{z\left[2\left(\frac{z}{2}-1\right)\right]^3}=\frac{1}{8z\left(\frac{z}{2}-1\right)^3}=\frac{-1}{8z}\frac{1}{\left(1-\frac{z}{2}\right)^3}=$$

This form is needed to take expansion as powers of z (geometric series), that is

$$\left(1-\frac{z}{2}\right)^{-3}=1+\frac{3}{1!}\left(\frac{z}{2}\right)^1+\frac{3\cdot(3+1)}{2!}\left(\frac{z}{2}\right)^2+\frac{3\cdot(3+1)\cdot(3+2)}{3!}\left(\frac{z}{2}\right)^3+\cdots=$$

$$1+\frac{3}{1!}\left(\frac{1}{2}\right)^1 z+\frac{3\cdot 4}{2}\left(\frac{1}{2}\right)^2 z^2+\frac{3\cdot 4\cdot 5}{3!}\left(\frac{1}{2}\right)^3 z^3+\cdots=1+\frac{3}{2}z+\frac{3}{2}z^2+\frac{5}{4}z^3+\cdots$$

Finally

$$f(z)=\frac{-1}{8z}\frac{1}{\left(1-\frac{z}{2}\right)^3}=\frac{-1}{8z}\left(1+\frac{3}{2}z+\frac{3}{2}z^2+\frac{5}{4}z^3+\cdots\right)=\boxed{-\frac{1}{8z}}\boxed{-\frac{3}{16}-\frac{3}{16}z-\frac{5}{32}z^2+\cdots}$$

Therefore a_{-1}=-1/8 lowest index value and also residue for pole of order n=1

Next let's look for pole at $z_o = 2$

Let $z=\xi+2 \rightarrow \xi=z-2$ Therefore $f(z)=\dfrac{1}{z(z-2)^3} \rightarrow f(\xi)=\dfrac{1}{(2+\xi)(\xi)^3}$

Then $f(\xi)=\dfrac{1}{(2+\xi)(\xi)^3}=\dfrac{1}{\xi^3 2\left[1+\dfrac{\xi}{2}\right]}=\dfrac{1}{\xi^3 2}\left(1+\dfrac{\xi}{2}\right)^{-1}$

Similar to the previous series expansion used but with $1+r$, $r=\dfrac{\xi}{2}$ and $n=-1$

Explicitly $\boxed{\left(1+r\right)^{-n}=1-\dfrac{n}{1!}r^1+\dfrac{n\cdot(n+1)}{2!}r^2-\dfrac{n\cdot(n+1)\cdot(n+2)}{3!}r^3+\cdots}$ and in our case

$$\left(1+\frac{\xi}{2}\right)^{-1}=1-\frac{1}{1!}\left(\frac{\xi}{2}\right)^1+\frac{1\cdot(1+1)}{2!}\left(\frac{\xi}{2}\right)^2-\frac{1\cdot(1+1)\cdot(1+2)}{3!}\left(\frac{\xi}{2}\right)^3+\cdots=1-\frac{1}{2}\xi+\frac{1}{4}\xi^2-\frac{1}{8}\xi^3+\cdots$$

Therefore $f(\xi)=\dfrac{1}{\xi^3 2}\left(1+\dfrac{\xi}{2}\right)^{-1}=\dfrac{1}{\xi^3 2}\left[1-\dfrac{1}{2}\xi+\dfrac{1}{4}\xi^2-\dfrac{1}{8}\xi^3+\cdots\right]=\dfrac{1}{2\xi^3}-\dfrac{1}{4\xi^2}+\dfrac{1}{8\xi}-\dfrac{1}{16}+\cdots$

Finally with $\xi=z-2$ we have $\boxed{f(z)=\dfrac{1}{2(z-2)^3}-\dfrac{1}{4(z-2)^2}+\dfrac{1}{8(z-2)}-\dfrac{1}{16}+\cdots}$

Therefore $a_{-1}= 1/8$ is the residue and lowest index value $a_{-3}= 1/2$ with order of pole 3.

A final quick unproven result for residues

If a function $f(z) = \dfrac{g(z)}{h(z)}$ has a simple pole at z_o then the residue is $\boxed{\mathrm{Res}(z_o) = \dfrac{g(z_o)}{h'(z_o)}}$