Math for Applied Biomedical Engineering Midterm Exam [Points - 100]
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All work must be done independently- Points in brackets

1. [15] a. Graph the following function $f(x) = x,\ 0 < x < 2$, whose primary period is from -2 to 2, L = 4 assuming an **even** extension.
   b. Assume this even function is periodic and find its Fourier series - show all steps.
   c. Present the coefficients solved for in part b. and given the following form for Parseval's identity for Fourier series

$$\frac{1}{L}\int_{x_0}^{x_0+L}\left[f(x)\right]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

   derive the following summation formula (show all steps) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$

2. [10] a. Graph the function $f(t) = \begin{cases} A & 0 < t < \tau \\ 0 & \text{otherwise} \end{cases}$
   b. Represent this function in terms of the Heaviside functions.
   c. Find the Fourier transform (not the series) for the function represented in b.
   d. Letting $A = \frac{1}{\tau}$ in the answer from part c. find the Fourier transform of $\lim_{\tau\to 0} f(t)$.
   e. Compare your answer to the Fourier transform representation for $\delta(t)$ in the book, Eq. 5.27.

3. [15] a. Use the basic integral definition to find the Laplace transform for $g(t) = \sin 5t$
   b. Find the Laplace transform of $g(t) = t\sin 5t$ using just the Laplace transform table and properties from the book. Do not use the basic integral definition.

   c. Compute the convolution of the functions $f(t) = t$ and $g(t) = e^{-t}$ using the integral definition.

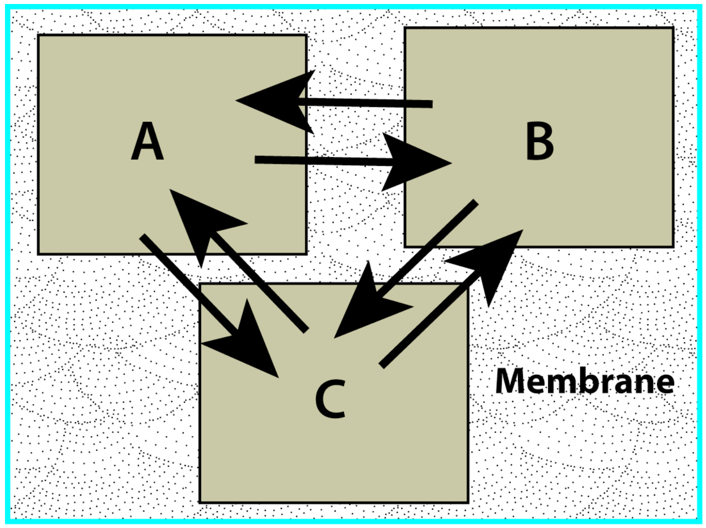
   Extra credit [3]: Use the Laplace transform to calculate the convolution of the functions in c.

4. [10] Solve the following differential equation by Laplace transform:

$$y'' + 4y' - 5y = \delta(t\text{-}1);\ y(0) = 0,\ y'(0) = 3$$

   [Note: the Laplace transform of $\delta(t\text{-}1)$ is e^{-s}]
   by the following steps:
   a. Write the differential equation after the transform has been applied as a function of s.
   b. Solve for $\tilde{y}(s)$
   c. Solve for y(t) using part b.

5. [20] A simple model for a membrane system is represented by the following diagram.



The transport in both directions is at the same rate for the salt concentrate in the reservoirs A, B and C and represented by the arrows to the left and right.

That is $A \to B$, $A \leftarrow B$, $C \to B$, $C \leftarrow B$, $A \to C$, $A \leftarrow C$ all rates are 10min^{-1} and the reservoirs A, B and C are the same size (containing either water or salt water). You can take the movement of salt across the membranes in units of g/min.
Reservoirs B and C contain NO salt dissolved in fresh water at time $t = 0$.
Reservoir A contains 20g of salt dissolved in fresh water at time $t = 0$.

a. Write down the differential equations and initial conditions that describe this system.
b. Take the Laplace transform of the equations a. and present their representation in s space
c. Solve the equations from part b. in s space - show all steps.
d. Finish the problem by finding the solutions for A(t), B(t) and C(t) - show all steps.

Extra credit [2] By the way what are the values of A(t), B(t) and C(t) as $t \to \infty$**?**

6. [10] Another special function generated by a Sturm-Liouville differential equation is that for the Laguerre polynomials. They can be generated by the following formulation:

$$L_0(x) = 1 \text{ and } L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \quad n = 1, 2, \cdots$$

Show by using the above formulation that: (Note, show all steps)

$$L_1(x) = 1 - x,\ L_2(x) = 1 - 2x + \frac{x^2}{2} \text{ and } L_3(x) = 1 - 3x + \frac{3x^2}{2} - \frac{x^3}{6}$$

Extra credit [10] Show that the Laguerre polynomials are orthogonal on the positive axis $0 \le x < \infty$ with respect to the weight function e^{-x}. It suffices to show that the following integral $\int e^{-x} x^k L_n(x) = 0$ since the highest power of L_n is x^n and taking $k < n$.

7. [20] For the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x$, $y(x)\big|_{x=e}=y(e)=0,\ \ y'(x)\big|_{x=e}=2$

a. Make the usual substitution (show all work), $x = e^t$ to restructure this equation as a differential equation with constant coefficients with respect to the variable t – present this equation in terms of the variable t.

b. For this equation in part a., written now as a function of t, solve for the homogeneous solution (need not show steps – the easy way!).

c. Solve the same homogenous equation that was solved in part b. (function of t), however this time use the method of series solution – show all steps. [Hint: To find the two independent solutions for this homogenous equation first take $a_0 = 1;\ a_1 = 1$ then take $a_0 = 1;\ a_1 = -1$]

Extra credit [5] Show that the answers in parts b. and c. are equivalent (provided you make certain assumptions and use additional derivation).

d. Using the inhomogeneous differential equation from part a. in terms of the variable t and using the homogeneous solution from part b. solve for the particular solution in terms of the variable t using variation of parameters.

e. Write the total solution y(t) to the differential equation in the variable t derived in part a. using the solutions from parts b. and d. Then write it in terms of the variable x.

f. Apply the initial conditions to the total solution in part e. and finish solving for the coefficients associated with the homogeneous part of the solution.