Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Take Home Project 2
Johns Hopkins University
Student: Yves Greatti

# Question 1

(a) Please see attached separate pdf.

(b) $f(t) = C_0 e^{-\frac{t}{\tau}}$ with period $T$, so

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} dt \\ &= \frac{2C_0}{T}(-\tau)[e^{-\frac{t}{\tau}}]_0^T \\ &= -2C_0\frac{\tau}{T}[e^{-\frac{T}{\tau}} - 1] \\ &= 2C_0\frac{\tau}{T}(1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

If $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_0 \approx 2C_0\frac{\tau}{T}$.

$$\begin{aligned} a_k &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \end{aligned}$$

Using integration by parts with $u = \cos\frac{2k\pi t}{T}, du = -\frac{2k\pi}{T}\sin\frac{2k\pi t}{T}$ and $dv = e^{-\frac{t}{\tau}}, v = (-\tau)e^{-\frac{t}{\tau}}$:

$$\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt = (-\tau)[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}]_0^T - \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt$$

Using again integration by parts:

$$\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt = (-\tau)[e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt$$

So

$$\begin{aligned} (1 + (\frac{2k\pi\tau}{T}))^2 \int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt &= (-\tau)[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau^2}{T}[e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T}]_0^T \\ &= (-\tau)[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}]_0^T + 0 \\ &= \tau(1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

$$\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt = \frac{\tau}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}})$$

Substituting back into the expression found for $a_k$ yields

$$\begin{aligned}
a_k &= 2C_0 \frac{\tau}{T} \frac{1}{1 + (\frac{2k\pi\tau}{T})^2} (1 - e^{-\frac{T}{\tau}}) \\
&= 2C_0 \frac{\tau T}{T^2 + (2k\pi\tau)^2} (1 - e^{-\frac{T}{\tau}})
\end{aligned}$$

With the same assumption $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_k \approx 2C_0 \frac{\tau}{T} \frac{1}{1+(\frac{2k\pi\tau}{T})^2}$. Similarly to compute $b_k$

$$\begin{aligned}
b_k &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \int_0^T e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \frac{\tau}{1 + (\frac{2k\pi\tau}{T})^2} (1 - e^{-\frac{T}{\tau}}) \\
&= 4C_0 k\pi \frac{\tau^2}{T^2 + (2k\pi\tau)^2} (1 - e^{-\frac{T}{\tau}})
\end{aligned}$$

Once again, since $e^{-\frac{T}{\tau}} \approx 0$ and $b_k \approx 4C_0 (\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2} \pi k$

(c) For $k \geq 1$

$$\begin{aligned}
p_k &= \frac{1}{2}(a_k^2 + b_k^2) \\
&= \frac{1}{2}\left[4C_0^2 (\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} + 16C_0^2 (\frac{\tau}{T})^4 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \pi^2 k^2\right] \\
&= \frac{1}{2} 4C_0^2 (\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2\right] \\
&= 2C_0^2 (\frac{\tau}{T})^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2\right]
\end{aligned}$$

(d)

(e)

(f) We have

$$\begin{aligned}
a_k \cos(\frac{k2\pi t}{T}) + b_k \sin(\frac{k2\pi t}{T}) &= \cos(\phi_k) \cos(\frac{k2\pi t}{T}) + \sin(\phi_k) \sin(\frac{k2\pi t}{T}) \\
&= \cos(\frac{k2\pi t}{T} - \phi_k)
\end{aligned}$$

where

$$\tan(\phi_k) = \frac{\sin(\phi_k)}{\cos(\phi_k)} = \frac{b_k}{a_k} = 4C_0(\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2}\pi k(2C_0\frac{\tau}{T}\frac{1}{1+(\frac{2k\pi\tau}{T})^2})^{-1}$$
$$= 2\frac{\tau}{T}\pi k$$
$$\phi_k = \arctan(2\frac{\tau}{T}\pi k)$$

For $\frac{\tau}{T} = .1$, $\phi_1 \approx 32.14°$ and $\phi_2 \approx 51.48°$ and for $\frac{\tau}{T} = .01$, $\phi_1 \approx 3.59°$ and $\phi_2 \approx 7.16°$

# Question 2

(a) One simple way to describe $P(r)$ is to define it as $P(r) = Ar + B$ with the conditions:

$$A \cdot 0 + B = Q$$
$$A \cdot R + B = 0$$

which gives $A = -\frac{Q}{R}$ and $B = Q$. So

$$P(r) = \begin{cases} Q(1-\frac{r}{R}) & \text{for } 0 \le r \le R \\ 0 & \text{for } r > R \end{cases}$$

(b) Since we assume no angular dependence: $\nabla^2 C = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dC}{dr})$, and the differential equation is now:

$$\frac{D}{r^2}\frac{d}{dr}(r^2\frac{dC(r)}{dr}) + P(r) = 0 \quad \text{for } 0 \le r \le R$$
$$\frac{d}{dr}(r^2\frac{dC(r)}{dr}) = -\frac{r^2}{D}P(r)$$
$$= -\frac{r^2}{D}Q(1-\frac{r}{R})$$
$$= \frac{Q}{DR}r^2(r-R)$$
$$= \frac{Q}{DR}r^3 - \frac{Q}{D}r^2$$

Integrating once

Diving through by $D$:

$$\frac{d^2C}{dr^2} + \frac{2}{r}\frac{dC}{dr} + \frac{Q}{D}(1-\frac{r}{R}) = 0$$
$$r\frac{d^2C}{dr^2} + 2\frac{dC}{dr} = r\frac{Q}{D}(\frac{r}{R}-1)$$

And otherwise for $r > R$

$$r\frac{d^2C}{dr^2} + 2\frac{dC}{dr} = 0$$

(c) Inside the cell, the homegeneous differential equation in standard form is:

$$\frac{d^2C}{dr^2} + \frac{2}{r}\frac{dC}{dr} - \frac{Q}{RD}r = 0$$

Changing variable notation

$p(r) = \frac{2}{r}$ and $q(r) = -\frac{Q}{RD}$

$r = 0$ is a regular singular point since $rp(r) = 2$ and $r^2q(r) = -r^2\frac{Q}{RD}$ are defined for $r = 0$.

Take $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$, then $y' = \sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1}$, and $y'' = \sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2}$

Therefore