Professor Rio
EN.585.615.81.SP21 Mathematical Methods
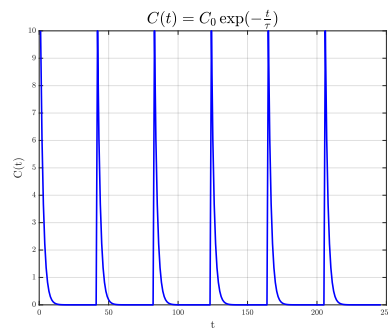Take Home Project 2
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Question 1

(a) See figure 1

$$C(t) = C_0 \exp\left(-\frac{t}{\tau}\right)$$

Figure 1

(b) $f(t) = C_0 e^{-\frac{t}{\tau}}$ with period T, so

$$
\begin{aligned}
a_0 &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} dt \\
&= \frac{2C_0}{T}(-\tau)[e^{-\frac{t}{\tau}}]_0^T \\
&= -2C_0 \frac{\tau}{T}[e^{-\frac{T}{\tau}} - 1] \\
&= 2C_0 \frac{\tau}{T}(1 - e^{-\frac{T}{\tau}})
\end{aligned}
$$

If $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_0 \approx 2C_0 \frac{\tau}{T}$.

$$
\begin{aligned}
a_k &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt
\end{aligned}
$$

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Using integration by parts with $u = \cos\frac{2k\pi t}{T}$, $du = -\frac{2k\pi}{T}\sin\frac{2k\pi t}{T}$ and $dv = e^{-\frac{t}{\tau}}$, $v = (-\tau)e^{-\frac{t}{\tau}}$:

$$\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt = (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T - \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}dt$$

Using again integration by parts:

$$\int_0^T e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}dt = (-\tau)[e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt$$

So

$$(1+(\frac{2k\pi\tau}{T}))^2\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt = (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau^2}{T}[e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}]_0^T$$

$$= (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T + 0$$

$$= \tau(1 - e^{-\frac{T}{\tau}})$$

$$\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt = \frac{\tau}{1+(\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}})$$

Substituting back into the expression found for a_k yields

$$a_k = 2C_0\frac{\tau}{T}\frac{1}{1+(\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}})$$

$$= 2C_0\frac{\tau T}{T^2+(2k\pi\tau)^2}(1 - e^{-\frac{T}{\tau}})$$

With the same assumption $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_k \approx 2C_0\frac{\tau}{T}\frac{1}{1+(\frac{2k\pi\tau}{T})^2}$. Similarly to compute b_k

$$b_k = \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}dt$$

$$= \frac{2C_0}{T}\int_0^T e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}dt$$

$$= \frac{2C_0}{T}\frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt$$

$$= \frac{2C_0}{T}\frac{2k\pi\tau}{T}\frac{\tau}{1+(\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}})$$
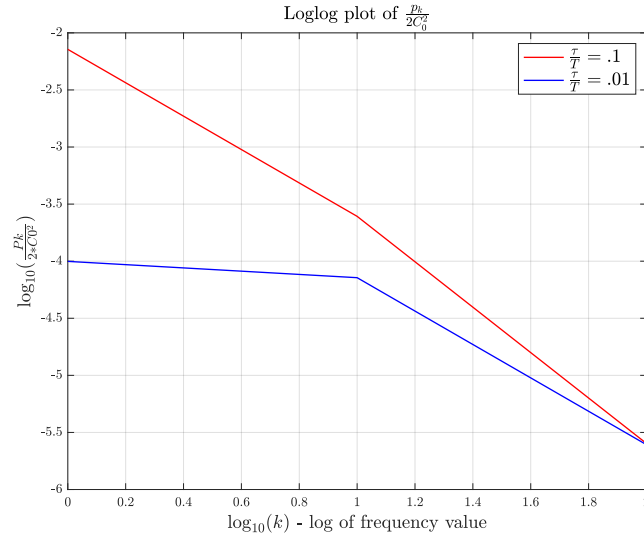
$$= 4C_0k\pi\frac{\tau^2}{T^2+(2k\pi\tau)^2}(1 - e^{-\frac{T}{\tau}})$$

Once again, since $e^{-\frac{T}{\tau}} \approx 0$ then $b_k \approx 4C_0(\frac{\tau}{T})^2\frac{1}{1+(\frac{2k\pi\tau}{T})^2}\pi k$

(c) For $k \geq 1$

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$$p_k = \frac{1}{2}(a_k^2 + b_k^2)$$

$$= \frac{1}{2}\left[4C_0^2(\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2} + 16C_0^2(\frac{\tau}{T})^4\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\pi^2k^2\right]$$

$$= \frac{1}{2}4C_0^2(\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\left[1+4(\frac{\tau}{T})^2\pi^2k^2\right]$$

$$= 2C_0^2(\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\left[1+4(\frac{\tau}{T})^2\pi^2k^2\right]$$
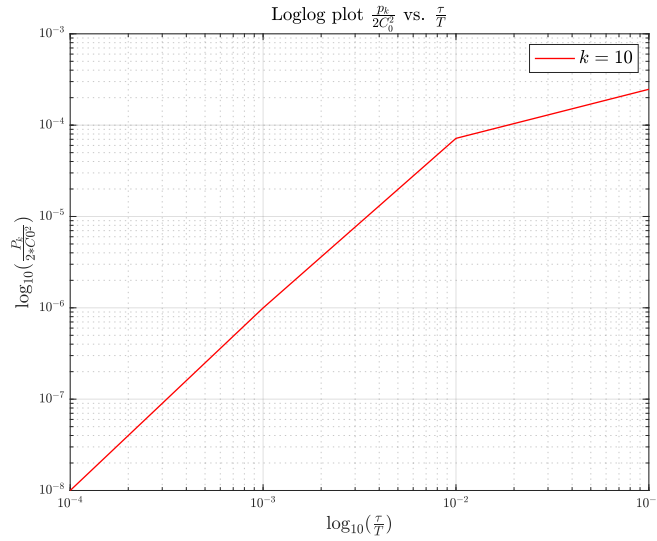
(d) We have

$$\frac{p_k}{2\,C_0^2} = (\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\left[1+4(\frac{\tau}{T})^2\pi^2k^2\right]$$

From the plot we can see that the power $\frac{p_k}{2\,C_0^2}$ decreases as the frequency increases. Power is close to 0 starting with a frequency of 10.

(e) Looking at the plot in part (d), as the pulse τ, becomes narrower, the power decreases linearly. For a greater $\frac{\tau}{T}$, the power starts at a higher value until an inflection point corresponding to frequency of 10 ($1 = \log_{10}(10)$ on the graph). Also for a higher $\frac{\tau}{T}$, the steepest the decrease in power. Eventually the two curves combine in one curve around a frequency of ($2 = \log_{10}(100)$ on the graph). The same observations are obtained with a loglog plot of power $\frac{p_k}{2\,C_0^2}$ vs. $\frac{\tau}{T}$ for a frequency $k = 10$.

Loglog plot $\frac{p_k}{2C_0^2}$ vs. $\frac{\tau}{T}$

(f) We have

$$a_k \cos(\frac{k2\pi t}{T}) + b_k \sin(\frac{k2\pi t}{T}) = \cos(\phi_k)\cos(\frac{k2\pi t}{T}) + \sin(\phi_k)\sin(\frac{k2\pi t}{T})$$
$$= \cos(\frac{k2\pi t}{T} - \phi_k)$$

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where

$$\tan(\phi_k) = \frac{\sin(\phi_k)}{\cos(\phi_k)} = \frac{b_k}{a_k} = 4C_0(\frac{\tau}{T})^2 \frac{1}{1 + (\frac{2k\pi\tau}{T})^2} \pi k (2C_0 \frac{\tau}{T} \frac{1}{1 + (\frac{2k\pi\tau}{T})^2})^{-1}$$

$$= 2\frac{\tau}{T}\pi k$$

$$\phi_k = \arctan(2\frac{\tau}{T}\pi k)$$

For $\frac{\tau}{T} = .1$, $\phi_1 \approx 32.14°$ and $\phi_2 \approx 51.48°$ and for $\frac{\tau}{T} = .01$, $\phi_1 \approx 3.59°$ and $\phi_2 \approx 7.16°$

If we imagine a clock with a hand that turns at constant speed, making a full turn every T seconds, and is pointing straight up at time $t = 0$ when the plasma rises to C_0. The phase ϕ_k is then the angle from the $12 : 00$ position to the current position and indicates how far is the next rise in plasma concentration to C_0. For the same frequency (either $k = 1$ or $k = 2$), with a larger pulse we are closer to the next substance release.
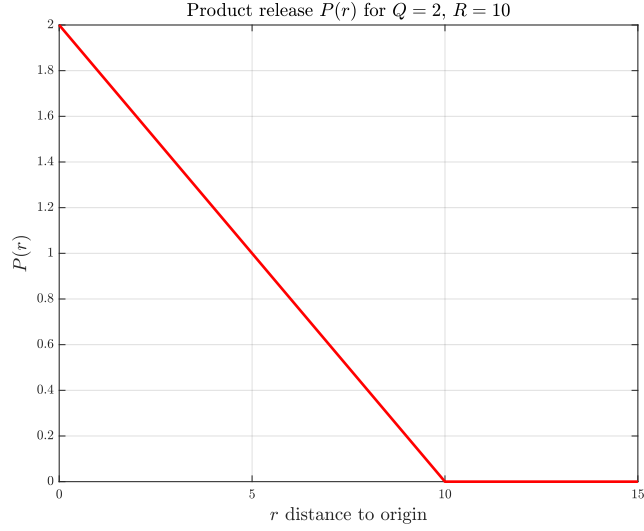
Question 2

(a) One simple way to describe $P(r)$ is to define it as $P(r) = Ar + B$ with the conditions:

$$A \cdot 0 + B = Q$$
$$A \cdot R + B = 0$$

which gives $A = -\frac{Q}{R}$ and $B = Q$. So

$$P(r) = \begin{cases} Q(1 - \frac{r}{R}) & \text{for } 0 \leq r \leq R \\ 0 & \text{for } r > R \end{cases}$$

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Product release $P(r)$ for $Q = 2$, $R = 10$

(b) Since we assume no angular dependence: $\nabla^2 C = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dC}{dr})$, the differential equation is now:

$$\frac{D}{r^2}\frac{d}{dr}(r^2\frac{dC(r)}{dr}) + P(r) = 0$$
$$\frac{d}{dr}(r^2\frac{dC(r)}{dr}) = -\frac{r^2}{D}P(r)$$

(c) Inside the cell $P(r) = Q(1 - \frac{r}{R})$, so we have to solve the differential equation

$$\frac{d}{dr}(r^2\frac{dC(r)}{dr}) = -\frac{r^2}{D}Q(1 - \frac{r}{R})$$
$$= \frac{Q}{DR}r^2(r - R)$$
$$= \frac{Q}{DR}r^3 - \frac{Q}{D}r^2$$

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Integrating once

$$r^2 \frac{dC(r)}{dr} = \frac{Q}{4DR}r^4 - \frac{Q}{3D}r^3 + A$$

$$\frac{dC(r)}{dr} = \frac{Q}{4DR}r^2 - \frac{Q}{3D}r + \frac{A}{r^2}$$

Integrating again

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 - \frac{A}{r} + B \quad A, B\text{:constants}, C_i\text{:inside cell concentration}$$

Outside the cell $P(r) = 0$ and we want to solve the differential equation

$$\frac{d}{dr}(r^2 \frac{dC(r)}{dr}) = 0$$

Which by integration gives

$$r^2 \frac{dC(r)}{dr} = C_1$$

$$\frac{dC(r)}{dr} = \frac{C_1}{r^2}$$

$$C_o(r) = -\frac{C_1}{r} + C_2 \quad C_1, C_2\text{:constants}, C_o\text{:outside cell concentration}$$

(d) Applying the boundary conditions

(i)

$$\lim_{r \to 0} C_i(r) = \lim_{r \to 0} \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 - \frac{A}{r} + B$$

since $\lim_{r \to 0} C_i(r) = \lim_{r \to 0}(-\frac{1}{r} + B) = \infty$ therefore to have finite concentration $C_i(r)$ at $r = 0$ we need $A = 0$

(ii)

$$\lim_{r \to \infty} C_o(r) = \lim_{r \to \infty} \left(-\frac{C_1}{r} + C_2 \right) = C_2$$

The concentration goes to zero at infinity implies $C_2 = 0$

(iii) We have now for $C_i(r)$ and $C_o(r)$:

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 + B$$

$$C_o(r) = -\frac{C_1}{r}$$

$C_i(R) = C_o(R)$ and $\frac{dC_i(r)}{dr} = \frac{dC_o(r)}{dr}|_{r=R}$ yields

$$\frac{Q}{12DR}R^3 - \frac{Q}{6D}R^2 + B = -\frac{C_1}{R}$$

$$\frac{Q}{4D}R - \frac{Q}{3D}R = \frac{C_1}{R^2}$$

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Rearranging

$$-\frac{Q}{12D}R^2 + B = -\frac{C_1}{R}$$

$$-\frac{Q}{12D}R = \frac{C_1}{R^2}$$

which gives

$$B = \frac{Q}{6D}R^2$$

$$C_1 = -\frac{Q}{12D}R^3$$

substituting back

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 + \frac{Q}{6D}R^2$$

$$= \frac{Q}{6D}\left[\frac{r^3}{2R} - r^2 + R^2\right]$$

$$C_o(r) = \frac{Q}{12D}R^3\frac{1}{r}$$

(e) Knowing that within the cell since $P(r)$ has maximum value Q at $r = 0$ and then it is zero for $r > R$, we are looking for the value of r for which $\frac{dC_i(r)}{dr} = 0$:

$$\frac{dC_i(r)}{dr} = \frac{Q}{4DR}r^2 - \frac{Q}{3D}r = \frac{Q}{D}r\left(\frac{r}{4R} - \frac{1}{3}\right)$$

Also note that

$$\frac{d^2C_i(r)}{dr^2} = -\frac{Q}{3D} < 0$$

So finding an r for which $\frac{dC_i(r)}{dr} = 0$ and $\frac{d^2C_i(r)}{dr^2} < 0$ then this r is a local maximum.

Discarding the solution $r = 0$ we are left that concentration maximum is for $r = \frac{4}{3}R$ and it is

$$C_M = \frac{Q}{12DR}\left(\frac{4}{3}\right)^3 R^3 - \frac{Q}{6D}\left(\frac{4}{3}\right)^2 R^2 + \frac{Q}{6D}R^2$$

$$= \frac{Q}{6D}R^2\left[\frac{4^3}{2\,3^3} - \frac{4^2}{3^2} + 1\right]$$

$$= \frac{11}{162}\frac{Q}{D}R^2$$

Inside the cell

$$C_i(r) = \frac{Q}{6D}\left(\frac{1}{2}\frac{r^3}{R} - r^2 + R^2\right)$$

$$\frac{C_i(r)}{C_M} = \frac{Q}{6D}\frac{162}{11}\frac{D}{Q}R^{-2}\left(\frac{1}{2}\frac{r^3}{R} - r^2 + R^2\right)$$

$$= \frac{162}{6\,11}\left[\frac{1}{2}\left(\frac{r}{R}\right)^3 - \left(\frac{r}{R}\right)^2 + 1\right]$$

$$= \frac{27}{11}\left[\frac{1}{2}\left(\frac{r}{R}\right)^3 - \left(\frac{r}{R}\right)^2 + 1\right]$$
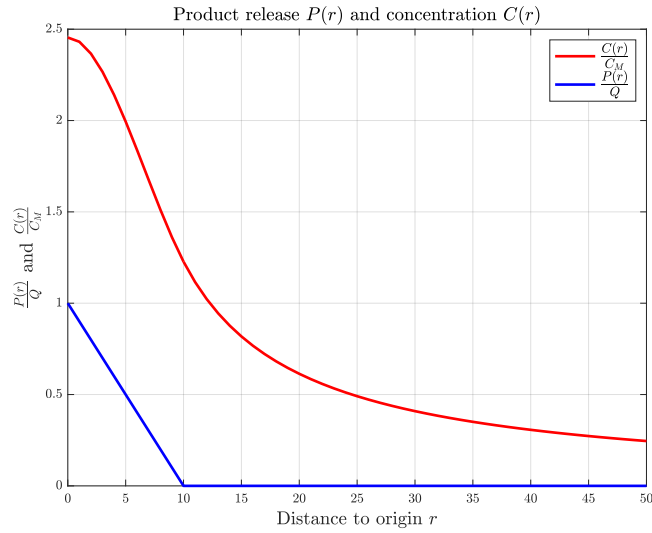
And outside the cell

$$C_o(r) = \frac{Q}{12D} R^3 \frac{1}{r}$$

$$\frac{C_o(r)}{C_M} = \frac{Q}{12D} \frac{162}{11} \frac{D}{Q} R^{-2} R^3 \frac{1}{r}$$

$$= \frac{27}{22} \frac{R}{r}$$

When the diffusion constant is doubled, the curve $\frac{C_i(r)}{C_M}$ stays the same since $\frac{C_i(r)}{C_M}$ does not depend on the diffusion constant D.

However for arbitrary values of Q, R, and D ($Q = 2, R = 10, D = 1$), when the diffusion constant D is doubled, the concentration curve is at a lower level compared to the same concentration curve related to a diffusion constant D as you can expect based on the expressions of $C_i(r)$ and $C_o(r)$ obtained above:

$C(r)_D$ vs. $C(r)_{2D}$