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Mathematical Methods for Applied Biomedical Engineering
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JOHNS HOPKINS
WHITING SCHOOL
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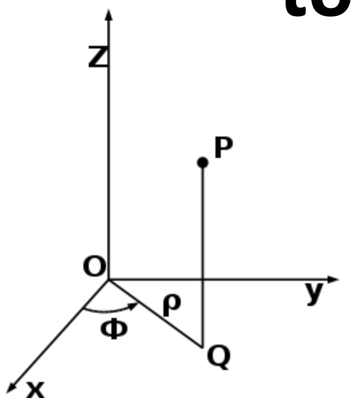
Separation of variables in polar (cylindrical) coordinates – application to Laplacian equation



$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

Laplacian operator in cylindrical coordinates

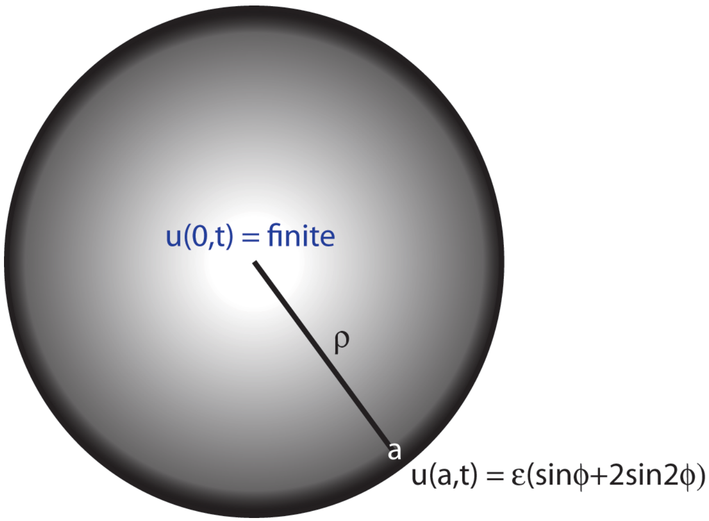
$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

https://en.wikibooks.org/wiki/Calculus/Vectors

The Laplacian equation in plane (no z dependence) is

$$\nabla^2 u(\rho,\phi) = \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]u(\rho,\phi) = 0$$

Application of Laplacian equation in cylindrical coordinates (no z dependence)



Assume separation of variable solution

$$u(\rho,\phi)=P(\rho)\Phi(\phi)$$

Substitution gives

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]P(\rho)\Phi(\phi)=0$$

$$\text{or } \Phi(\phi)\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)P(\rho)+\frac{1}{\rho^2}P(\rho)\frac{d^2}{d\phi^2}\Phi(\phi)=0$$

Dividing by $P(\rho)\Phi(\phi)$ and multiple by ρ^2

$$\frac{\rho}{P(\rho)}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)P(\rho)+\frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi)=0$$

Let the separation constant be n^2 therefore we have

$$\frac{\rho}{P(\rho)}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)P(\rho)=n^2 \text{ and } \frac{1}{\Phi(\phi)}\frac{d^2}{d\phi^2}\Phi(\phi)=-n^2$$

This gives us two equations

For the equation in ρ expand the derivative and multiple by $P(\rho\ )$

$$\frac{\rho}{P(\rho)}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)P(\rho)=\frac{\rho}{P(\rho)}\left(\frac{dP(\rho)}{d\rho}+\rho\frac{d^2P(\rho)}{d\rho^2}\right)=n^2 \text{ or}$$

$$\rho^2\frac{d^2P(\rho)}{d\rho^2}+\rho\frac{dP(\rho)}{d\rho}-n^2P(\rho)=0$$

and in ϕ we have

$$\frac{d^2}{d\phi^2}\Phi(\phi)+n^2\Phi(\phi)=0$$

The solution looks different when $n=0$, so lets look at that case first.

For $n=0$ we have $\rho^2\frac{d^2P(\rho)}{d\rho^2}+\rho\frac{dP(\rho)}{d\rho}=0$

$$\rho^2 \frac{d^2P(\rho)}{d\rho^2} + \rho \frac{dP(\rho)}{d\rho} = 0$$ is an Euler ODE (previosly covered!)

Making the substitution $\rho = e^x$ gives $\dfrac{d^2P(x)}{dx^2} = 0$

The solution is easily seen (integration twice) to be $P(x) = Cx + D$

Upon substitution we have $\boxed{P_0(\rho) = C_0 \ln\rho + D_0}$ and the subscript represents n =0

For $\dfrac{d^2}{d\phi^2}\Phi(\phi) = 0$ (n = 0 again)

Its solution is $\Phi(\phi) = A\phi + B$

Now ϕ is a periodic variable around boundary (value should be the same whether

$\phi = 0$ or $2\pi \rightarrow A = 0$

Therefore $\Phi_0(\phi) = B_0$ (the subscript represents n =0 index case again)

Construct $\boxed{u_0(\rho,\phi) = P_0(\rho)\Phi_0(\phi) = (C_0 \ln\rho + D_0)B_0 \equiv C_0 \ln\rho + D_0}$ (absorb B_0 into C_0, D_0)

$$\text{For } n \neq 0 \ \rho^2 \frac{d^2P(\rho)}{d\rho^2} + \rho \frac{dP(\rho)}{d\rho} - n^2 P(\rho) = 0$$

Again this has an Euler form. letting $\rho = e^x \rightarrow x = \ln\rho$ and $P(\rho) \rightarrow P(x)$

$$\frac{d^2P(x)}{dx^2} - n^2 P(x) = 0$$

The solution (note index n) is $P_n(x) = Ce^{nx} + De^{-nx} \rightarrow (x = \ln\rho) \rightarrow \boxed{P_n(\rho) = C_n \rho^n + D_n \rho^{-n}}$

For the angular dependent equation we have

$$\frac{1}{\Phi(\phi)} \frac{d^2}{d\phi^2} \Phi(\phi) = -n^2 \ \text{ or } \ \frac{d^2}{d\phi^2} \Phi(\phi) + n^2 \Phi(\phi) = 0$$

We have seen this before (where ϕ is circular variable) its solution is

$$\boxed{\Phi_n(\phi) = A_n \cos n\phi + B_n \sin n\phi}$$

Application of the superposition property (and including the n = 0 term) gives

$$\boxed{\begin{array}{l} u(\rho,\phi) = u_0(\rho,\phi) + \sum_{n=1}^{\infty} u_n(\rho,\phi) \\ \text{where } u_0(\rho,\phi) = C\ln\rho + D \text{ and } u_n(\rho,\phi) = \Phi_n(\phi) P_n(\rho) = (A_n \cos n\phi + B_n \sin n\phi)(C_n \rho^n + D_n \rho^{-n}) \end{array}}$$

We have the solution $u(\rho,\phi) = C_0 \ln\rho + D_0 \; + \sum_{n=1}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)(C_n\rho^n + D_n\rho^{-n})$

Applying the boundary condition $u(0,\phi)$ finite $\rightarrow C_0$ since $\ln 0$ undefined

KEY

Also for $D_n\rho^{-nx} = \dfrac{D_n}{\rho^n} \rightarrow (\rho = 0) \rightarrow \text{undef}$, therefore $D_n = 0$

See image in previous slide for these boundary conditions

We are left with $u(\rho,\phi) = D_0 \; + \sum_{n=1}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)C_n\rho^n$

Next apply $u(a,\phi) = \varepsilon(\sin\phi + 2\sin 2\phi) = D_0 \; + \sum_{n=1}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)C_n a^n$

Equating left and right hand sides gives the following relations

$D_0 = 0$ leaving $u(\rho,\phi) = \sum_{n=1}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)(C_n\rho^n + D_n\rho^{-n})$

$u(a,\phi) = \sum_{n=1}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)(C_n a^n + D_n a^{-n}) = \varepsilon(\sin\phi + 2\sin 2\phi)$

Further inspection (expanding and equating left to right side terms) gives

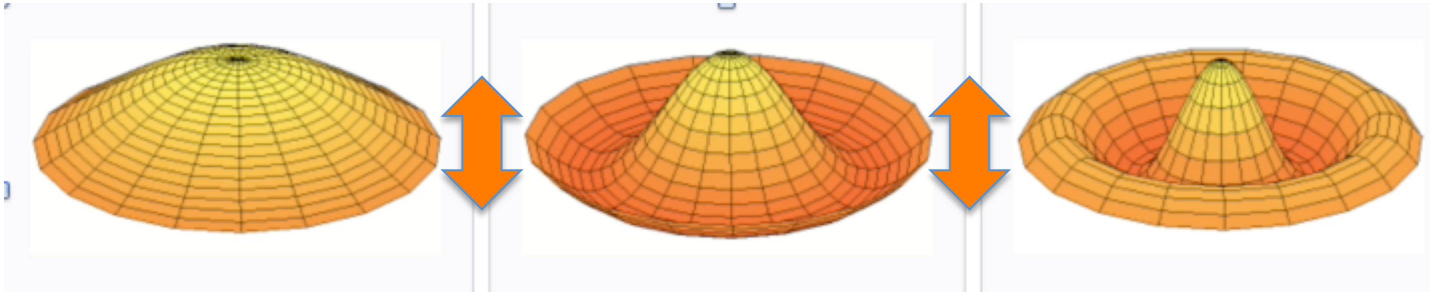
$A_1C_1a^1 = 0,\; B_1C_1a^1 = \varepsilon,\; A_2C_2a^2 = 0,\; B_2C_2a^2 = 2\varepsilon$

KEY

All other combination of constants and constants with index $n > 2$ are 0!!

Therefore $A_1 = 0,\; B_1C_1 = \dfrac{\varepsilon}{a},\; A_2 = 0,\; B_2C_2 = \dfrac{2\varepsilon}{a^2}$ and finally $u(\rho,\phi) = \dfrac{\varepsilon}{a}\rho\sin\phi + \dfrac{2\varepsilon}{a^2}\rho^2\sin 2\phi$

Vibrating circular membrane – circular symmetry



https://en.wikipedia.org/wiki/Vibrations_of_a_circular_membrane

$$\nabla^2 u(\rho,\phi,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(\rho,\phi,t)$$

Expanding we get

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]u(\rho,\phi,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(\rho,\phi,t)$$

With circular symmetry $u(\rho,\phi,t) \rightarrow u(\rho,t)$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)u(\rho,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(\rho,t)$$

$u(\rho,t)=f(\rho)$, integrable function, $u(0,t)$ finite and $u(a,t)=0$

Assume seperation of variable solution $u(\rho,t) = P(\rho)T(t)$

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Substitution gives

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)\right]P(\rho)T(t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}P(\rho)T(t)$$

Dividing by $P(\rho)\Phi(\phi)$

$$\frac{1}{P(\rho)}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)\right]P(\rho) = \frac{1}{c^2}\frac{1}{T(t)}\frac{d^2}{dt^2}T(t)$$

Let the separation constant be $-k^2$ therefore we have

$$\frac{1}{P(\rho)}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)\right]P(\rho) = -k^2 \text{ or expanding } \left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right]P(\rho) = -k^2P(\rho)$$

and $\dfrac{1}{c^2}\dfrac{1}{T(t)}\dfrac{d^2}{dt^2}T(t) = -k^2$ or $\dfrac{d^2}{dt^2}T(t) = -k^2c^2T(t)$

Starting with $\frac{d^2P(\rho)}{d\rho^2}+\frac{1}{\rho}\frac{dP(\rho)}{d\rho}+k^2P(\rho)\ =0$

To put this into standard form we apply the following transformation (k constant)

$$s=k\rho\rightarrow\rho=\frac{s}{k}\ \text{and}\ \frac{ds}{d\rho}=k,\ \ P(\rho)\rightarrow P(s)$$

Also $\frac{dP}{d\rho}=\frac{dP}{ds}\frac{ds}{d\rho}=\frac{dP}{ds}k\equiv k\frac{dP}{ds},\ \frac{d^2P}{d\rho^2}=\ k\frac{d}{ds}\left(k\frac{dP}{ds}\right)=k^2\frac{d^2P}{ds^2}$

Substitution into our ODE gives

$$k^2\frac{d^2P(s)}{ds^2}+\frac{1}{\left(\frac{s}{k}\right)}k\frac{dP(s)}{ds}+k^2P(s)\ =0\rightarrow\ k^2\frac{d^2P(s)}{ds^2}+\frac{1}{s}k^2\frac{dP(s)}{ds}+k^2P(s)\ =0$$

Multiplying by out $\frac{s^2}{k^2}$ gives $s^2\frac{d^2P(s)}{ds^2}+s\frac{dP(s)}{ds}+s^2P(s)\ =0$

or $\boxed{s^2\frac{d^2P(s)}{ds^2}+s\frac{dP(s)}{ds}+(s^2-0^2)P(s)\ =0}$

KEY: Important, this has the form of a Bessel equation for order 0

$$P(s)=J_0(s)\equiv J_0(k\rho)$$

Finally apply boundary condition

$u(a,t) = P(a)T(t)=0 \rightarrow P(a)=0$

Therefore $P(a)=J_0(ka) = 0$

ka represents the zero crossing for Bessel function of order 0,

We will call them α_m (indexed by m since there are an infinite number of them)

and set $k_m a = \alpha_m \rightarrow k_m = \dfrac{\alpha_m}{a}$

Therefore the solution are $P(\rho) = J_0(k_m\rho) = J_0(\dfrac{\alpha_m}{a}\rho)$

Note when $\rho = a$ $P(a) = J_0(\dfrac{\alpha_m}{a}a) = J_0(\alpha_m) = 0$

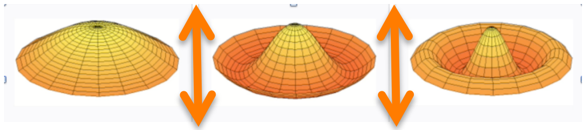
Now $\dfrac{d^2}{dt^2}T(t) = -k_m^2c^2T(t) \rightarrow \dfrac{d^2}{dt^2}T(t) + k_m^2c^2T(t)=0$

Let $\lambda_m^2 = k_m^2c^2$ or $\lambda_m = k_m c$ then

standard solution is $T_m(t) = A_m \cos\lambda_m t + B_m \sin\lambda_m t$

Finally (as usual) applying the superposition principle

$$\boxed{u(\rho,t)=\sum_{m=1}^{\infty}(A_m\cos\lambda_m t+B_m\sin\lambda_m t)J_0(\frac{\alpha_m}{a}\rho)}$$ **General Solution**



As often for the velocity, $\frac{\partial}{\partial t}u(\rho,t)\Big|_{t=0}=0\rightarrow B_m=0$

And applying the initial condition $u(\rho,0)=f(\rho)$ gives

$$u(\rho,0)=\sum_{m=1}^{\infty}A_m\cos(\lambda_m 0)J_0(\frac{\alpha_m}{a}\rho)=\sum_{m=1}^{\infty}A_m\cos(\lambda_m 0)J_0(\frac{\alpha_m}{a}\rho)=\sum_{m=1}^{\infty}A_m J_0(\frac{\alpha_m}{a}\rho)=f(\rho)$$

KEY This is a Fourier Bessel series where the coefficients are given by

$$A_m=\frac{2}{a^2J_1(\alpha_m)}\int_0^a\rho f(\rho)J_0(\frac{\alpha_m}{a}\rho)d\rho$$