Chapter 4 16 pages

Problems

Yves GREATTI

Chapter 4 - Problem 4.16

By finding a cosine Fourier series of period 2 for the function $f(t)$ that takes the form $f(t) = \cosh(t-1)$ in the range $0 \le t \le 1$, prove that $\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2} = \frac{1}{e^2-1}$

Deduce values for the sums $\Sigma (n^2\pi^2+1)^{-1}$ over odd n and even n separately.

By changing to a new variable $t' = t-1$, we express the function $f(t)$ as $f(t) = \cosh t'$ over the range $-1 \le t' \le 0$.

Extending on the interval $[-1, 1]$, we can compute its Fourier expansion as a cosine series.

$$a_0 = \frac{2}{2} \cdot \int_{-1}^{1} \cosh(t)\, dt = \sinh(1)$$

$$a_n = \frac{2}{2} \cdot \int_{-1}^{1} \cosh(t) \cos\left(\frac{2n\pi t}{2}\right) dt$$

$$= \int_{-1}^{1} \cosh(t) \cos(n\pi t)\, dt = \frac{2\sinh(1)(-1)^n}{1+(n\pi)^2}$$

Therefore

$$f(t) = \sinh(1) + 2 \sum_{n=1}^{\infty} \frac{\sinh(1)(-1)^n}{1+(n\pi)^2} \cos(n\pi t)$$

And $\cosh(t-1) = \sinh(1) + 2\sinh(1) \sum_{n=1}^{\infty} \frac{\cos(n\pi t)}{1+(n\pi)^2}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n\pi t)}{1+(n\pi)^2} = \frac{\cosh(t-1) - \sinh(1)}{2\sinh(1)}$$

For $t=0$, we have then: $\sum_{n=1}^{\infty} \frac{1}{1+(n\pi)^2} = \frac{\cosh(-1) - \sinh(1)}{2\sinh(1)}$

$$= \frac{e^{-1}}{e^1 - e^{-1}} = \frac{1}{e^2-1} \qquad (1)$$

For $t=1$, we have: $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2} = \frac{\cosh(0) - \sinh(1)}{2\sinh(1)} = \frac{1}{e^1 - e^{-1}} - \frac{1}{2}$

since $\cos(n\pi) = (-1)^n$

or $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2} = \frac{e}{e^2-1} - 1/2$

which we can write as: $-\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{1+(n\pi)^2} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{1}{1+(n\pi)^2} = \frac{e}{e^2-1} - 1/2 \quad (2)$

Adding (1) and (2): $2 \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{1}{1+(n\pi)^2} = \frac{1}{e^2-1} + \frac{e}{e^2-1} - \frac{1}{2} = \frac{e+1}{e^2-1} - \frac{1}{2}$

$$= \frac{1}{e-1} - \frac{1}{2}$$

$$\Leftrightarrow \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{1}{1+(n\pi)^2} = \frac{3-e}{4(e-1)}$$

Reporting into (1): $\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{1+(n\pi)^2} = \frac{1}{e^2-1} - \frac{3-e}{4(e-1)} = \frac{e-1}{4(e+1)}$

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Chapter 4. Problem 4.20

Show that the Fourier series for $|\sin\theta|$ in the range $-\pi \le \theta \le \pi$ is given by:

$$|\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2m\theta}{4m^2-1}$$

By setting $\theta=0$ and $\theta=\pi/2$, deduce values for

$$\sum_{m=1}^{\infty}\frac{1}{4m^2-1} \quad \text{and} \quad \sum_{m=1}^{\infty}\frac{1}{16m^2-1}$$

Let $f(x)=|\sin x|$, $f(x)$ is even, the Fourier coefficients b_n in its Fourier series expansion are zero. Thus we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin(x)\,dx$$
$$= \frac{2}{\pi}\left[-\cos(x)\right]_0^{\pi} = 4/\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin(x)\cos\left(\frac{2n\pi x}{\pi}\right)dx$$
$$= \frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(2nx)\,dx$$

$$\int_0^{\pi}\sin(x)\cos(2nx)\,dx = \frac{1}{2}\int_0^{\pi}\left(\sin(2n+1)x + \sin(1-2n)x\right)dx$$

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$$\int_0^\pi \sin(x)\cos(2nx)\,dx = 1/2 \left[-\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(2n-1)x}{2n-1} \right]_0^\pi$$

$$= \frac{1}{2} \times \frac{4}{1-4n^2} = \frac{2}{1-4n^2}$$

$$\Rightarrow a_n = \frac{2}{\pi} \times \frac{2}{1-4n^2} = \frac{4}{\pi(1-4n^2)}$$

$$\Rightarrow |\sin x| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos(2nx)$$

for $x \in [-\pi, \pi]$

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2-1}$$

Set $\theta = 0 \Rightarrow$ $$0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{4m^2-1}$$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{1}{4m^2-1} = 1/2$$

For $\theta = \pi/2$ we have that

$$1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos m\pi}{4m^2-1}$$

$$\Rightarrow \sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2-1} = \frac{2-\pi}{4}$$

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$$\sum_{m=1}^{\infty} \frac{1}{4m^2-1} = 1/2 \Rightarrow \sum_{\substack{m=1\\ m\,odd}}^{\infty} \frac{1}{4m^2-1} + \sum_{\substack{m=2\\ m\,even}}^{\infty} \frac{1}{4m^2-1} = 1/2$$

$$\Rightarrow \sum_{\substack{m=1\\ m\,odd}}^{\infty} \frac{1}{4m^2-1} = \frac{1}{2} - \sum_{\substack{m=2\\ m\,even}}^{\infty} \frac{1}{4m^2-1} \quad (1)$$

And
$$\sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2-1} = \sum_{\substack{m=2\\ m\,even}}^{\infty} \frac{1}{4m^2-1} - \sum_{\substack{m=1\\ m\,odd}}^{\infty} \frac{1}{4m^2-1} = \frac{2-\pi}{4} \quad (2)$$

Putting both expressions (1) and (2) together gives:

$$\sum_{\substack{m=2\\ even}}^{\infty} \frac{1}{4m^2-1} - \left(\frac{1}{2} - \sum_{\substack{m=2\\ m\,even}}^{\infty} \frac{1}{4m^2-1}\right) = \frac{2-\pi}{4}$$

$$\Rightarrow 2 \times \sum_{\substack{m=2\\ m\,even}}^{\infty} \frac{1}{4m^2-1} = \frac{2-\pi}{4} + \frac{1}{2} = \frac{4-\pi}{4}$$

$$\sum_{\substack{m=2\\ m=even}}^{\infty} \frac{1}{4m^2-1} = \frac{4-\pi}{8}$$

which we can rewrite with $m=2p$ for $p=1$ to ∞:

$$\sum_{p=1}^{\infty} \frac{1}{16p^2-1} = \frac{4-\pi}{8}$$

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Chapter 4 - Problem 4.22

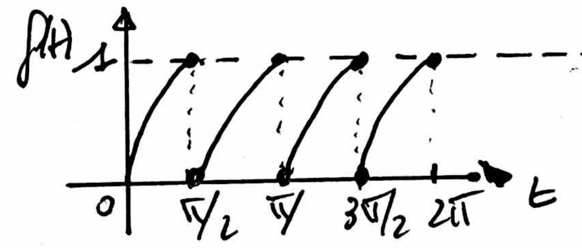
The repeating output from an electronic oscillator takes the form of a sine wave $f(t) = \sin t$ for $0 \le t \le \pi/2$; it then drops instantaneously to zero and starts again. The output is to be represented by a complex Fourier series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{4ni}$$

Sketch the function and find an expression for c_n. Verify that $c_{-n} = c_n^*$. Demonstrate that setting $t=0$ and $t=\pi/2$ produces differing values for the sum $\sum_{n=1}^{\infty} \frac{1}{16n^2-1}$

Determine the correct value and check it using the result of problem 4.20

(*) see note



$$c_0 = \frac{1}{\pi/2}\int_0^{\pi/2} \sin(t)\, dt = \frac{2}{\pi} \cdot [-\cos t]_0^{\pi/2} = \frac{2}{\pi}$$

*: assuming t is time
And measurements start at $t=0$

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Chapter 4 - Problem 4.22

For $n \neq 0$

$$c_n = \frac{1}{\pi/2} \int_0^{\pi/2} \sin(t)\, e^{-\frac{2\pi i n t}{\pi/2}}\, dt$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \sin(t)\, e^{-4int}\, dt$$

$$\int_0^{\pi/2} \sin(x) e^{-4inx} dx = \frac{1}{(-4in)} \left[\sin(x) e^{-4inx}\right]_0^{\pi/2}$$

$$- \int_0^{\pi/2} \cos(x) \left(-\frac{1}{4in} e^{-4inx}\right) dx$$

$$= \left(-\frac{1}{4in}\right) \cdot 1 + \frac{1}{4in} \int_0^{\pi/2} \cos(x)\, e^{-4inx} dx$$

$$\int_0^{\pi} \cos(x) e^{-4inx} dx = \left(-\frac{1}{4in}\right) \left[\cos(x)\, e^{-4inx}\right]_0^{\pi/2}$$

$$- \frac{1}{4in} \int_0^{\pi/2} \sin(x)\, e^{-4inx} dx$$

let $I = \int_0^{\pi/2} \sin(x)\, e^{-4inx} dx$

$$\Rightarrow \quad I = -\frac{1}{4in} + \frac{1}{4in} \left(\frac{1}{4in} - \frac{I}{4in}\right)$$

$$\Rightarrow \quad I = \frac{4in-1}{16n^2-1} \quad \Rightarrow \quad c_n = \frac{2}{\pi} \times \frac{4in-1}{16n^2-1}$$

And $\quad c_{-n} = \frac{2}{\pi} \frac{-4in-1}{16n^2-1} = c_n^*$

The Fourier series is thus:

$$f(t) = \sin t = \frac{2}{\pi} \sum_{n=-\infty}^{+\infty} \frac{4in-1}{16n^2-1} e^{4int}$$

$$= \frac{2}{\pi} + \frac{2}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{4in-1}{16n^2-1} e^{4int}.$$

Considering terms in pair: n and $-n$;

$$(4in-1)\, e^{4int} - (4in+1)\, e^{-4int}$$

$$= 4in\left(e^{4int} - e^{-4int}\right) - \left(e^{4int} + e^{-4int}\right)$$

$$= -8 \sin 4nt - 2\cos 4nt$$

$$\Rightarrow f(t) = \sin(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{16n^2-1} (\cos 4nt + 4\sin 4nt)$$

Setting $t=0$ $\quad 0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{16n^2-1} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{16n^2-1} = 1/2$

For $t = \pi/2$ $\quad 1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{16n^2-1} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{16n^2-1} = \frac{2-\pi}{4}$

We obtain two different values because $f(t)$ is discontinuous at the ends of the interval. And from section 4.4, the value of the function at the discontinuity is half-way between the upper and lower values. $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{16n^2-1} = \frac{1}{2}\left(\frac{1}{2} + \frac{2-\pi}{4}\right) = \frac{4-\pi}{8}$

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chapter 4 - Problem 4.25

Show that Parseval's theorem for two real functions whose Fourier expansions have cosine and sine coefficients a_n, b_n and α_n, β_n takes the form

$$\frac{1}{L}\int_0^L f(x)\,g(x)\,dx = \frac{1}{4}a_0\alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n)$$

(a) Demonstrate that for $g(x) = \sin mx$ or $\cos mx$ this reduces to the definition of the Fourier coefficients.

(b) Explicitly verify the above result for the case in which $f(x) = x$ and $g(x)$ is the square-wave function, both in the interval $-1 \le x \le 1$.

Given $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi n x}{L} + b_n \sin\frac{2\pi n x}{L}$$

$$g(x) = \frac{\alpha_0}{2} + \sum_{m=1}^{\infty} \alpha_m \cos\frac{2\pi m x}{L} + \beta_m \sin\frac{2\pi m x}{L}$$

$$f(x)\,g(x) = \frac{a_0\alpha_0}{4} + \frac{a_0}{2}\sum_{m=1}^{\infty} \alpha_m \cos\frac{2\pi m x}{L} + \beta_m \sin\frac{2\pi m x}{L}$$

$$+ \frac{\alpha_0}{2}\sum_{n=1}^{\infty} a_n \cos\frac{2\pi n x}{L} + b_n \sin\frac{2\pi n x}{L}$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Big[a_n\alpha_m \cos\frac{2\pi n x}{L}\cos\frac{2\pi m x}{L} + a_n\beta_m \cos\frac{2\pi n x}{L}\sin\frac{2\pi m x}{L}$$

$$+ b_m \alpha_m \sin\frac{2\pi n x}{L}\cos\frac{2\pi m x}{L} + b_n \beta_m \sin\frac{2\pi n x}{L}\sin\frac{2\pi m x}{L}]$$

Integrating:

$$\int_0^L f(x)g(x)\,dx = \frac{a_0\alpha_0}{4}\int_0^L dx + \frac{a_0}{2}\int_0^L [\sum_{m=1}^{\infty} \alpha_m \cos\frac{2\pi m x}{L} + \beta_m \sin\frac{2\pi m x}{L}]\,dx$$

$$+ \frac{\alpha_0}{2}\int_0^L [\sum_{n=1}^{\infty} a_n \cos\frac{2\pi n x}{L} + b_n \sin\frac{2\pi m x}{L}]\,dx$$

$$+ \int_0^L \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [a_n \alpha_n \cos\frac{2\pi n x}{L}\cos\frac{2\pi m x}{L} + a_n \beta_m \cos\frac{2\pi n x}{L}\sin\frac{2\pi m x}{L}$$

$$+ b_m \alpha_m \sin\frac{2\pi n x}{L}\cos\frac{2\pi m x}{L} + b_n \beta_m \sin\frac{2\pi n x}{L}\sin\frac{2\pi m x}{L})$$

Assuming convergence of the Fourier series, we can switch Σ and $\int$

On an interval of length L:

$$\int_0^L \cos\frac{2\pi m x}{L}\,dx = \int_0^L \sin\frac{2\pi n x}{L}\,dx = \int_0^L \cos\frac{2\pi m x}{L}\,dx = \int_0^L \sin\frac{2\pi m x}{L}\,dx = 0$$

And the cross-terms in cosine and sine also disappears leaving the terms in $a_n \alpha_n$ and $b_n \beta_n$. Therefore

$$\frac{1}{2}\int_0^L f(x)\,g(x)\,dx = \frac{a_0\alpha_0}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n \alpha_n + b_n \beta_n)$$

where we have used $\int_0^L \cos\frac{2\pi n x}{L}\sin\frac{2\pi m x}{L} = \frac{L}{2}\delta_{mn}$

$$\int_0^L \sin\frac{2\pi n x}{L}\cos\frac{2\pi m x}{L} = \frac{L}{2}\delta_{mn}$$

δ: delta Kronecker $= 1$ if $m=0$, 0 otherwise

Chapter 4. Problem 4.25

(a)

Applying the Parseval's theorem for $g(x) = \cos mx$

gives us $\frac{1}{L}\int_0^L f(x)\cos mx\,dx = \frac{1}{4}a_0 \cdot 0 + \frac{1}{2}a_m \cdot 1$

$$\Rightarrow a_m = \frac{2}{L}\int_0^L f(x)\cos mx\,dx$$

since the Fourier series of $g(x) = \cos mx = 1 \cdot \cos mx$

similarly of $g(x) = \sin mx$

$$\frac{1}{L}\int_0^L f(x)\sin mx\,dx = \frac{1}{4}a_0 \cdot 0 + \frac{1}{2}b_m \cdot 1$$

$$\Rightarrow b_m = \frac{2}{L}\int_0^L f(x)\sin(mx)\,dx$$

(b) let $L = 2$

$$\frac{1}{2}\int_{-1}^1 f(x)g(x)\,dx = \frac{1}{2}\int_{-1}^1 x\,g(x)\,dx$$

where $g(x)\begin{cases} -1 & -1 \leq x < 0 \\ 1 & 0 \leq x \leq 1 \end{cases}$

$$\int_{-1}^1 x\,g(x)\,dx = \int_{-1}^0 (-x)\,dx + \int_0^1 x\,dx = 2\int_0^1 x\,dx = \frac{1}{2}$$

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Chapter 4. Problem 4. 25

We now compute the Fourier series of $f(x)=x$ in $[-1,1]$.
$f(x)$ is an odd function, its Fourier coefficients $a_n=0$

$$b_n = \frac{2}{2}\int_{-1}^{1} x \sin n\pi x \, dx = \frac{2\cdot(-1)^{n+1}}{\pi n}$$

$g(x)$ is also an odd function and

$$\beta_n = 2\int_0^1 \sin\frac{2\pi n x}{2}\,dx = 2\int_0^1 \sin \pi n x \, dx$$

$$= \left(-\frac{2}{\pi n}\right)\left[\cos \pi n x\right]_0^1 = \frac{-2}{\pi n}\left((-1)^n - 1\right)$$

$$\frac{1}{2}\sum_{n=1}^{\infty} b_n \beta_n = 1/2 \sum_{n=1}^{\infty}\left(\frac{-2}{\pi n}\right)\left((-1)^n-1\right) \times 2\,\frac{(-1)^{n+1}}{\pi n}$$

$$= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)\cdot\left((-1)^n-1\right)(-1)^{n+1}}{n^2}$$

$$= \frac{4}{\pi}\sum_{\substack{n=1\\ n\ \text{odd}}}^{\infty}\frac{1}{n^2} = \frac{4}{\pi}\,\frac{\pi^2}{8} = \frac{1}{2}$$

where we use the result $\displaystyle\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$

(problem 4.19)

we just verified $\displaystyle\frac{1}{2}\int_{-1}^{1} f(x)\,g(x)\,dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n\beta_n$

An odd function $f(x)$ of period 2π is to be approximated by a Fourier sine series having only m terms. The error in this approximation is measured by the square deviation

$$E_m = \int_{-\pi}^{\pi} \left[f(x) - \sum_{n=1}^{m} b_n \sin nx \right]^2 dx$$

By differentiating E_m with respect to the coefficients b_n find the values of b_n that minimizes E_m.
Sketch the graph of the function $f(x)$ where

$$f(x) = \begin{cases} -x(\pi + x) & \text{for } -\pi \leq x < 0 \\ x(x-\pi) & \text{for } 0 \leq x < \pi \end{cases}$$

If $f(x)$ is to be approximated by the first three terms of a Fourier sine series, what values should the coefficients have so as to minimize E_3? What is the resulting value of E_3?

$$\frac{\partial E_m}{\partial b_l} = \int_{-\pi}^{\pi} \left[f(x) - \sum_{n=1}^{m} b_n \sin nx \right] (-2 \sin lx)\, dx$$

$$= -2 \int_{-\pi}^{\pi} \left[f(x) \sin lx - \sum_{n=1}^{m} b_n \sin nx \sin lx \right] dx$$

All the sine terms disappear but the term $n = l$ and is equal to: $\frac{2\pi}{2} = \pi$

$$\Rightarrow \frac{\partial E_m}{\partial b_l} = -2\left[\int_{-\pi}^{\pi} f(x) \sin lx \, dx - b_l \pi \right]$$

b_l minimizes E_m if $b_l = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin lx \, dx$

Graph of the function $f(x)$: see attached pdf (page 16)

The function $f(x)$ is odd of period 2π.

Thus the coefficients which minimize E_3 are:

$$b_l = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin lx \, dx$$

$$b_l = \frac{2}{\pi} \times \int_0^{\pi} x(x-\pi) \sin lx \, dx$$

$$\int_0^{\pi} x(x-\pi) \sin lx \, dx = \left(\frac{-1}{l}\right) \left[x(x-\pi) \cos lx \right]_0^{\pi} + \frac{1}{l} \int_0^{\pi} (2x-\pi) \cos lx \, dx$$

And $\int_0^{\pi} 2x \cos lx \, dx = \frac{2}{l^2} \left((-1)^l - 1 \right)$

$$\Rightarrow \int_0^{\pi} x(x-\pi) \sin lx \, dx = \frac{2}{l^3} \left((-1)^l - 1 \right)$$

$$\Rightarrow b_l = \frac{4}{\pi l^3} \left[(-1)^l - 1 \right]$$

$$\Rightarrow b_1 = -8/\pi, \quad b_2 = 0 \text{ and } b_3 = -\frac{8}{27\pi}$$

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Chapter 4 - Problem 4.26

We have:

$$E_m = \int_{-\pi}^{\pi} \left[f(x) - \sum_{n=1}^{m} b_n \sin nx\right]^2 dx$$

$$= \int_{-\pi}^{\pi} [f(x)]^2 dx - 2 \sum_{n=1}^{m} \int_{-\pi}^{\pi} f(x)\, b_n \sin nx\, dx$$

$$+ \int_{-\pi}^{\pi} \left[\sum_{n=1}^{m} b_n \sin nx\right]^2 dx$$

$$= \int_{-\pi}^{\pi} f(x)^2 dx - 2\pi \sum_{n=1}^{m} b_n^2 + \pi \sum_{n=1}^{m} b_n^2$$

$$= \int_{-\pi}^{\pi} f(x)^2 dx - \pi \sum_{n=1}^{m} b_n^2$$

Hence

$$E_3 = 2 \times \int_{0}^{\pi} x^2 (x-\pi)^2\, dx - \pi \left(b_1^2 + b_3^2\right)$$

$$= \frac{\pi^5}{15} - \pi \left(\frac{8^2}{\pi^2} + \frac{8^2}{27^2 \pi^2}\right) \simeq 0.0015$$

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Chapter 4 - Problem 4.26

Graph of the function f(x)