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Bessel functions

Bessel's differential equation is

$$x^2y''+xy'+(x^2-\upsilon^2)y=0$$

In standard form

$$y''+\frac{1}{x}y'+(1-\frac{\upsilon^2}{x^2})y=0$$

$$p(x)=\frac{1}{x},\ q(x)=1-\frac{\upsilon^2}{x^2}$$

We are interested at the solution near or at x = 0.

We see that x= 0 is a singular point and identify it as a regular singular point since

$$(x-0)p(x)\Big|_{x=x_0}=x\left[\frac{1}{x}\right]\Bigg|_{x=0}=1\Big|_{x=0}=1$$

$$(x-0)^2q(x)\Big|_{z=z_0}=(x)^2\left[1-\frac{\upsilon^2}{x^2}\right]\Bigg|_{z=0}=(x^2-\upsilon^2)\Big|_{x=0}=-\upsilon^2$$

Our proposed series solutions for a regular singular points is

$$y(x)=\sum_{n=0}^{\infty}a_nx^{\sigma+n},\quad y'(x)=\sum_{n=0}^{\infty}(\sigma+n)a_nx^{\sigma+n-1},\quad y''(x)=\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_nx^{\sigma+n-2}$$

Substitution of our proposed series solutions and it’s derivatives into our original ODE gives

$$x^2y''+xy'+(x^2-\upsilon^2)y=x^2\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_n x^{\sigma+n-2}+x\sum_{n=0}^{\infty}(\sigma+n)a_n x^{\sigma+n-1}+(x^2-\upsilon^2)\sum_{n=0}^{\infty}a_n x^{\sigma+n}=$$

$$\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_n x^{\sigma+n}+\sum_{n=0}^{\infty}(\sigma+n)a_n x^{\sigma+n}+(x^2-\upsilon^2)\sum_{n=0}^{\infty}a_n x^{\sigma+n}=$$

$$\sum_{n=0}^{\infty}[(\sigma+n-1)(\sigma+n)+(\sigma+n)-\upsilon^2]a_n x^{\sigma+n}+\sum_{n=0}^{\infty}a_n x^{\sigma+n+2}=0$$

Take the lowest power of x to find the indicial equation. That happens with n = 0 in the first sum.

$$(\sigma+0-1)(\sigma+0)+(\sigma+0)-\upsilon^2]a_0=0$$

$$a_0\neq 0$$

$$(\sigma-1)\sigma+\sigma-\upsilon^2=0$$

$$\sigma^2=\upsilon^2$$

$$\boxed{\sigma=\pm\upsilon}$$

Since the indical equation corresponds to the n = 0 term in the fist sum we need not include it since this cofficent has been forced to be zero by are choice of σ! That is $\sigma = \pm\upsilon$.

Also in the second sum reindex by letting $n \to n-2$. Therefore we have

$$\sum_{n=1}^{\infty}[(\pm\upsilon+n-1)(\pm\upsilon+n)+(\pm\upsilon+n)-\upsilon^2]a_n x^{\pm\upsilon+n}+\sum_{n=2}^{\infty}a_{n-2}x^{\pm\upsilon+n}=$$

$$\sum_{n=1}^{\infty}[(\pm\upsilon+n)^2-\upsilon^2]a_n x^{\pm\upsilon+n}+\sum_{n=2}^{\infty}a_{n-2}x^{\pm\upsilon+n}=0$$

Now let's look at n = 1 in this same first sum. Since the coefficients for all powers of x are zero as they must be (since in general $x^{\sigma+n} \neq 0$) for these sums to be equal to zero on the RHS! We get $[(\pm\upsilon+1)^2-\upsilon^2]a_1 = \boxed{(\pm2\upsilon+1)a_1 = 0}$

For $n \geq 2$ we are then left with

$$\sum_{n=2}^{\infty}[(\pm\upsilon+n)^2-\upsilon^2]a_n x^{\pm\upsilon+n}+\sum_{n=2}^{\infty}a_{n-2}x^{\pm\upsilon+n}=\sum_{n=2}^{\infty}\{[(\pm\upsilon+n)^2-\upsilon^2]a_n+a_{n-2}\}x^{\pm\upsilon+n}=0$$

Therefore for $n \geq 2$ we have $\boxed{[(\pm\upsilon+n)^2-\upsilon^2]a_n+a_{n-2}=n(n\pm2\upsilon)a_n+a_{n-2}=0}$

Therefore we have $(\pm 2\upsilon+1)a_1 = 0$ and $[(\pm\upsilon+n)^2-\upsilon^2]a_n + a_{n-2} = 0$

Case 1: For $\pm\nu$ non integer (and $\nu \neq m/2$, m an integer) we have the following

First take $a_0 = \dfrac{1}{2^{\pm\nu}\Gamma(1\pm\nu)}$ as is customary

Gamma function
- we will study this in a coming lecture

Then $(\pm 2\upsilon+1)a_1 = 0 \rightarrow a_1 = 0$ since $\pm 2\upsilon + 1 \neq 0$

and $(n \geq 2)$ we have $a_n = \dfrac{-a_{n-2}}{(\pm\upsilon+n)^2-\upsilon^2} = \dfrac{-a_{n-2}}{n(n\pm 2\upsilon)}$

Starting with n = 2 we have $a_2 = \dfrac{-a_{2-2}}{(\upsilon+2)^2-\upsilon^2} = \dfrac{-a_0}{4(\upsilon+1)}$

For n = 3 $a_2 = \dfrac{-a_{3-2}}{(\upsilon+3)^2-\upsilon^2} = \dfrac{-a_1}{(\upsilon+3)^2-\upsilon^2} = \dfrac{-0}{(\upsilon+3)^2-\upsilon^2} = 0$

For n = 4 $a_4 = \dfrac{-a_{4-2}}{(\upsilon+4)^2-\upsilon^2} = \dfrac{-a_2}{(\upsilon+4)^2-\upsilon^2} = \dfrac{-1}{2^3(\upsilon+2)}\dfrac{-a_0}{4(\upsilon+1)} = \dfrac{a_0}{2^{2\cdot 2}2(\upsilon+1)(\upsilon+2)}$

$\vdots$

Note since since $a_1 = 0$ the recursive relation gives $\boxed{a_m = 0,\ m = 1,3,5,\cdots}$

Based on these results (without a detailed demonstration) the general pattern is

$$\boxed{a_n = \frac{(-1)^n a_0}{2^{2n} n![(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)]} = \frac{(-1)^n}{2^{2n}2^{\pm\nu} n!\Gamma(\upsilon+n+1)}}$$

$$a_n = \frac{(-1)^n a_0}{2^{2n} n![(\upsilon+1)(\upsilon+2)\cdots(\upsilon+n)]} = \frac{(-1)^n}{2^{2n}2^{\pm\nu} n!\Gamma(\upsilon+n+1)}$$

Therefore our solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^{2n\pm\nu} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}2^{\pm\nu} n!\Gamma(\upsilon+n+1)} x^{2n\pm\nu}$$

or

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\upsilon+n+1)} \left(\frac{x}{2}\right)^{2n+\nu} \quad \text{and} \quad J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-\upsilon+n+1)} \left(\frac{x}{2}\right)^{2n-\nu}$$

These are called Bessel functions of the first kind!

Case 2: For Integer values, that is $\sigma = \pm \upsilon = \pm m$

For $\upsilon = m$, an integer take $a_0 = \dfrac{1}{2^m m!}$

Again $a_1 = 0$ so we have

and $a_n = \dfrac{(-1)^n}{2^{2n+m} n!(m+n)!} \equiv \dfrac{(-1)^n}{2^{2n+m} n!\Gamma(n+m+1)}$

Therefore our solution for is $\upsilon = n$

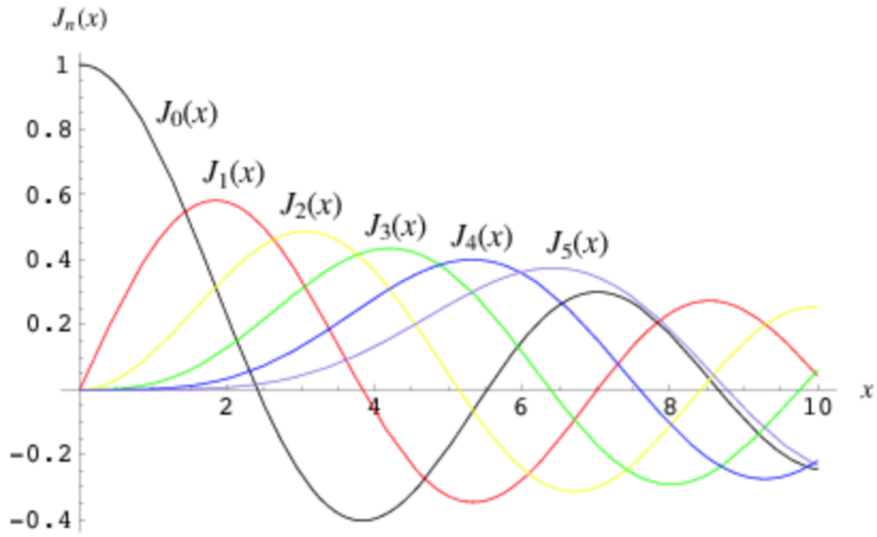
$$y(x) = \sum_{n=0}^{\infty} a_n x^{2n+m} = \sum_{m=0}^{\infty} \frac{(-1)^n}{2^{2n+m} n!(m+n)!} x^{2n+m} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m} n!\Gamma(n+m+1)} x^{2n+m}$$

or similar to our previous result for non integers.

$$\boxed{J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+m} n!\Gamma(n+m+1)} x^{2n+m}}$$

In particular for m = 0 we have $\boxed{J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!\Gamma(n+1)} x^{2n}}$

Graph of some Bessel functions for the first kind for integer values m = 0 to 5



http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

Finally as we have previously discussed series solution that differ by an integral value for $\sigma = \pm\upsilon = \pm m$ are not independent. In fact

$$J_{-m}(x) = (-1)^m J_m(x)$$

Therefore we need to identify a second independent solution! We define (without proof) Bessel functions of the second kind, independent from the first kind for integer as well as non integer values as

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

Finally there is a last case, Case 3 for values $\sigma = \upsilon = \frac{m}{2}$, $m = 1, 3, \cdots$

For $\upsilon = \frac{1}{2}$ (that is $m=1$) we have $J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1/2+n+1)} \left(\frac{x}{2}\right)^{2n+1/2} =$

$$\frac{(-1)^0}{0!\Gamma(1/2+0+1)} \left(\frac{x}{2}\right)^{2(0)+1/2} + \frac{(-1)^1}{1!\Gamma(1/2+1+1)} \left(\frac{x}{2}\right)^{2(1)+1/2} + \cdots =$$

$$\frac{1}{\Gamma(3/2)} \left(\frac{x}{2}\right)^{1/2} + \frac{-1}{\Gamma(5/2)} \left(\frac{x}{2}\right)^{5/2} + \cdots =$$

Using the results on Gamma functions $\Gamma(1/2) = \sqrt{\pi}$

Soon to be derived!!

and using the Gamma function identity $\Gamma(x+1) = x\Gamma(x)$

We have $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

$$J_{1/2}(x) = \frac{1}{\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^{1/2} + \frac{-1}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^{5/2} + \cdots = \frac{1}{\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^{1/2} + \frac{-1}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^{5/2} + \cdots$$

$$= \frac{1}{\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^{1/2} \left[1 - \frac{x^2}{3!} + \cdots\right] = \frac{1}{\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^{1/2} \left[\frac{\sin x}{x}\right] = \sqrt{\frac{2}{\pi x}} \sin x,$$ Similarly $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Using a Taylor series expansion

Example derivation of a Bessel function identity

Let's derive a Bessel identity using the series representation for Bessel functions.

$$\frac{d}{dx}[x^{\nu}\ J_{\nu}(x)]=\frac{d}{dx}\left[x^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!\Gamma(\upsilon+n+1)}\left(\frac{x}{2}\right)^{2n+\nu}\right]=$$

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!\Gamma(\upsilon+n+1)}\frac{x^{2n+2\nu}}{2^{2n+\nu}}\right]=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!\Gamma(\upsilon+n+1)}\frac{(2n+2\nu)x^{2n+2\nu-1}}{2^{2n+\nu}}=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!\Gamma(\upsilon+n+1)}\frac{2(n+\nu)x^{2n+2\nu-1}}{2^{2n+\nu}}=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\frac{(n+\nu)}{\Gamma(\upsilon+n+1)}\frac{x^{2n+2\nu-1}}{2^{2n+\nu-1}}=$$

Using $\Gamma(\upsilon+n+1)=(n+\nu)\Gamma(n+\nu)$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\frac{(n+\nu)}{(n+\nu)\Gamma(n+\nu)}\frac{x^{2n+2\nu-1}}{2^{2n+\nu-1}}=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\frac{1}{\Gamma(n+\nu)}\frac{x^{\nu}x^{2n+\nu-1}}{2^{2n+\nu-1}}=$$

$$x^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\frac{1}{\Gamma(n+\nu)}\frac{x^{2n+\nu-1}}{2^{2n+\nu-1}}=x^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\frac{1}{\Gamma(\nu-1+n+1)}\left(\frac{x}{2}\right)^{2n+\nu-1}=x^{\nu}J_{\nu-1}(x)$$

That is

$$\boxed{\frac{d}{dx}[x^{\nu}\ J_{\nu}(x)]=x^{\nu}J_{\nu-1}(x)}$$

Example using Bessel function identities

Here is an example of calculating a coefficient for a Bessel-Fourier series

$$a_s = \frac{2}{R^2 J_1^2(\alpha_s)} \int_0^R (1 - r^2) J_0(\alpha_s r)\, r dr$$

where α_s are the zero crossing for the Bessel functions

and we will take R = 1.

Start by make the substitution $x = \alpha_s r$ or $r = \dfrac{x}{\alpha_s}$, $dr = \dfrac{dx}{\alpha_s}$

$$a_s = \frac{2}{J_1^2(\alpha_s)} \int_0^{\alpha_s} \left[1 - \left(\frac{x}{\alpha_s} \right)^2 \right] J_0(x)\ \frac{x}{\alpha_s} \frac{dx}{\alpha_s} =$$

$$\frac{2}{J_1^2(\alpha_s)} \int_0^{\alpha_s} \left[1 - \left(\frac{x}{\alpha_s} \right)^2 \right] J_0(x)\ \frac{x}{\alpha_s} \frac{dx}{\alpha_s} = \frac{2}{J_1^2(\alpha_s)} \int_0^{\alpha_s} \left[\frac{x}{\alpha_s^2} - \frac{x^3}{\alpha_s^4} \right] J_0(x)\ dx =$$

That is

$$a_s = \frac{2}{J_1^2(\alpha_s)}\left\{\frac{1}{\alpha_s^2}\int_0^{\alpha_s} xJ_0(x)\,dx - \frac{1}{\alpha_s^4}\int_0^{\alpha_s} x^3J_0(x)\,dx\right\}$$

We have the following indentities for Bessel functions

$$\frac{d}{dx}x^{\nu}J_{\nu}(x) = x^{\nu}J_{\nu-1}(x) \rightarrow \int x^{\nu}J_{\nu-1}(x)dv = x^{\nu}J_{\nu}(x)$$

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_{\nu}(x)$$

Apply the first identity to the first integral $(\nu = 1)$ gives

$$\int_0^{\alpha_s} xJ_0(x)\,dx = x^1J_1(x)\Big|_0^{\alpha_s} = \alpha_s J_1(\alpha_s)$$

For the second integral use the second identity with $\nu = 1$

that is $J_{1-1}(x) + J_{1+1}(x) = \frac{2\cdot 1}{x}J_1(x)$ or $J_0(x) = \frac{2}{x}J_1(x) - J_2(x)$

and $\int_0^{\alpha_s} x^3J_0(x)\,dx = \int_0^{\alpha_s} x^3\left[\frac{2}{x}J_1(x) - J_2(x)\right]dx = \int_0^{\alpha_s} 2x^2J_1(x) - x^3J_2(x)dx =$

Then use identity 1 again with the first integral ($\nu = 2$) and second integral ($\nu = 3$). That is

$$\int_0^{\alpha_s} 2x^2 J_1(x) - x^3 J_2(x)\,dx = 2x^2 J_2(x) - x^3 J_3(x)\Big|_0^{\alpha_s} = 2\alpha_s{}^2 J_2(\alpha_s) - \alpha_s{}^3 J_3(\alpha_s)$$

Putting all this together gives us

$$a_s = \frac{2}{J_1^2(\alpha_s)}\left\{\frac{1}{\alpha_s^2}\alpha_s J_1(\alpha_s) - \frac{1}{\alpha_s^4}[2\alpha_s{}^2 J_2(\alpha_s) - \alpha_s{}^3 J_3(\alpha_s)]\right\} =$$

$$\frac{2}{J_1^2(\alpha_s)}\left\{\frac{1}{\alpha_s}J_1(\alpha_s) - \frac{2}{\alpha_s^2}J_2(\alpha_s) + \frac{1}{\alpha_s}J_3(\alpha_s)\right\} = \frac{2}{J_1^2(\alpha_s)}\left\{\frac{1}{\alpha_s}[J_1(\alpha_s) + J_3(\alpha_s)] - \frac{2}{\alpha_s^2}J_2(\alpha_s)\right\}$$

Finally using identity 2 with ($\nu = 2$), that is $J_1(\alpha_s) + J_3(\alpha_s) = \frac{4}{\alpha_s}J_2(\alpha_s)$

$$\boxed{a_s} = \frac{2}{J_1^2(\alpha_s)}\left\{\frac{1}{\alpha_s}[\frac{4}{\alpha_s}J_2(\alpha_s)] - \frac{2}{\alpha_s^2}J_2(\alpha_s)\right\} = \frac{2}{J_1^2(\alpha_s)}\left\{\frac{4}{\alpha_s^2}J_2(\alpha_s) - \frac{2}{\alpha_s^2}J_2(\alpha_s)\right\} = \boxed{\frac{4J_2(\alpha_s)}{\alpha_s^2 J_1^2(\alpha_s)}}$$