Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Final Exam
Johns Hopkins University
Student: Yves Greatti

Question 1

a. $f(x) = x$ is odd on $[-\pi, \pi]$ therefore its Fourier coefficients a_n are 0 and we need to find its b_n coefficients:

$$b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(\frac{2\pi n x}{2\pi}) dx$$

$$= \frac{4}{2\pi} \int_{0}^{\pi} x \sin(\frac{2\pi n x}{2\pi}) dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) dx$$

Using integration by parts:

$$\int_{0}^{\pi} x \sin(nx) dx = [x(-\frac{\cos(nx)}{n})]_0^\pi + \int_{0}^{\pi} 1 \cdot \frac{\cos(nx)}{n} dx$$

$$= (-\frac{\pi}{n}) \cos(n\pi) + \frac{1}{n}[\sin(nx)]_0^\pi$$

$$= \frac{(-1)^{n+1}\pi}{n}$$

Thus $b_n = \frac{2}{\pi} \frac{(-1)^{n+1}\pi}{n} = \frac{2(-1)^{n+1}}{n}$ and the Fourier series of x, on $[-\pi, \pi]$, is:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

b. If we integrate terms by terms the previous expression, the Fourier series of x over $[-\pi, \pi]$, we have:

$$\frac{x^2}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-\frac{\cos(nx)}{n}) + c \quad c \text{ constant of integration}$$

$$x^2 = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + c \text{ with } 2c \to c$$

$$= c + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

c. $f(x) = x^2$ is an even function, by Fourier Series for even function over symmetric range, we have:

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{2\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \ (1)$$

1

where

$$a_0 = \frac{4}{2\pi} \int_0^\pi x^2 dx$$

$$= \frac{2}{\pi}[\frac{x^3}{3}]_0^\pi$$

$$= \frac{2}{3}\pi^2$$

$$a_n = \frac{4}{2\pi} \int_0^\pi x^2 \cos\left(\frac{2\pi n x}{2\pi}\right)dx = \frac{2}{\pi} \int_0^\pi x^2 \cos(nx)dx$$

$$\int_0^\pi x^2 \cos(nx)dx = [x^2 \frac{\sin(nx)}{n}]_0^\pi - \frac{2}{n} \int_0^\pi x \sin(nx)dx$$

$$= 0 - \frac{2}{n}\frac{(-1)^{n+1}\pi}{n}$$

$$a_n = \frac{2}{\pi}\frac{(-1)^n 2\,\pi}{n^2}$$

$$= (-1)^n \frac{4}{n^2}$$

Substituting for a_n in (1):

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^\infty \frac{(-1)^n}{n^2} \cos(nx)$$

d. As required, taking $c = \frac{\pi^2}{3}$, since x is a piecewise smooth function on the specified range, the Fourier series of x^2, using integration term by term or directly calculating, match, .

Question 2

Consider the differential equation:

$$z\frac{d^2y}{dy^2} + y = 0$$

a. We put the equation in standard form:

$$\frac{d^2y}{dy^2} + \frac{1}{z}y = 0$$

$z\,p(z) = 0$ and $z^2 q(z) = z$ therefore 0 is a regular singular point.

b. Take $y = z^\sigma \sum_{n=0}^\infty a_n z^n$ and the usual derivatives in the D.E. gives by substitution

$$z\sum_{n=0}^\infty (n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} + \sum_{n=0}^\infty a_n z^{n+\sigma} = 0$$

$$\sum_{n=0}^\infty (n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-1} + \sum_{n=0}^\infty a_n z^{n+\sigma} = 0 \;(1)$$

2

Take the term with the lowest power of z, which is the first sum with $n = 0$, then since each power of z term must be equal to 0, we have

$$\sigma(\sigma - 1)a_0 z^{\sigma-1} = 0$$

Since $a_0 \neq 0$ and $z^{\sigma-1} \neq 0$, therefore $\sigma = 0, 1$.

c. We go back to equation (1) and take $\sigma = 1$ yields

$$\sum_{n=0}^{\infty} n(n + 1)a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

Then reindex the second sum to get same power of z in both sums:

$$\sum_{n=0}^{\infty} n(n + 1)a_n z^n + \sum_{n=1}^{\infty} a_{n-1} z^n = 0$$

Note, in first term $n = 0$ does not contribute so we can start index at $n = 1$ in the first sum, and combine both sums

$$\sum_{n=1}^{\infty} [n(n + 1)a_n + a_{n-1}]z^n = 0$$

Since every power of z term must be 0 and $z^n \neq 0$, gives:

$$a_n = -\frac{1}{(n + 1)n}a_{n-1}$$

Taking $a_0 = 1$

$$n = 1 \; a_1 = -\frac{1}{2\,1}a_0 = -\frac{1}{2\,1} = \frac{(-1)^1}{2\,1}$$

$$n = 2 \; a_2 = -\frac{1}{3\,2}a_1 = \frac{1}{3\,2\,2\,1} = \frac{(-1)^2}{(3\,2\,1)\,(2\,1)}$$

$$n = 3 \; a_3 = -\frac{1}{4\,3}a_2 = -\frac{1}{4\,3\,3\,2\,2\,1} = \frac{(-1)^3}{(4\,3\,2\,1)\,(3\,2\,1)}$$

$$\vdots$$

$$a_n = -\frac{1}{(n + 1)n}a_{n-1} = \cdots = \frac{(-1)^n}{((n + 1)\, n \cdots 1)\,(n\,(n - 1) \cdots 1)} = \frac{(-1)^n}{(n + 1)!n!}$$

Therefore one of the independent solution of the ODE is

$$y_1(z) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + 1)!n!}z^n$$

3

Question 3

a. We have

$$n = 0, \ M = 0, \ P_0(x) = \frac{(-1)^0(2\ 0 - 2\ 0)!}{2^0(0-0)!(0-2\ 0)!}x^{0-2\ 0} = 1$$

$$n = 1, \ M = \frac{1-1}{2} = 0, \ P_1(x) = \frac{(-1)^0(2\ 1 - 2\ 0)!}{2^1(1-0)!(1-2\ 0)!}x^{1-2\ 0} = \frac{1\ 2}{2\ 1!\ 1!}x^1 = x$$

$$n = 2, \ M = \frac{2}{2} = 1, \ P_2(x) = \frac{(-1)^0(2\ 2 - 2\ 0)!}{2^2(2-0)!(2-2\ 0)!}x^{2-2\ 0} + \frac{(-1)^1(2\ 2 - 2\ 1)!}{2^2(2-1)!(2-2\ 1)!}x^{2-2\ 1}$$

$$P_2(x) = \frac{4!}{2^2\ 2!\ 2!}x^2 - \frac{(2\ 2 - 2)!}{2^2\ 1!\ 0!}x^0$$

$$P_2(x) = \frac{4\ 3\ 2\ 1}{4\ 2\ 2}x^2 - \frac{2!}{4}$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2 - 1)$$

b. From

$$a_n = \frac{2n+1}{2}\int_{-1}^1 f(x)P_n(x)dx = \frac{2n+1}{2}\int_{-1}^1 xP_n(x)dx$$

we have

$$n = 0, \ a_0 = \frac{2\ 0 + 1}{2}\int_{-1}^1 xP_0(x)dx$$

$$= \frac{1}{2}\int_{-1}^1 xdx = \frac{1}{2}[\frac{x^2}{2}]_{-1}^1 = \frac{1}{4}[1^2 - (-1)^2] = 0$$

$$n = 1, \ a_1 = \frac{2\ 1 + 1}{2}\int_{-1}^1 xP_1(x)dx$$

$$= \frac{3}{2}\int_{-1}^1 x^2dx = \frac{3}{2}[\frac{x^3}{3}]_{-1}^1 = \frac{1}{2}[1^3 - (-1)^3] = \frac{1}{2}\ 2 = 1$$

$$n = 2, \ a_2 = \frac{2\ 2 + 1}{2}\int_{-1}^1 xP_2(x)dx$$

$$= \frac{5}{2}\int_{-1}^1 x[\frac{1}{2}(3x^2 - 1)]dx = \frac{5}{4}\int_{-1}^1 (3\ x^3 - x)dx$$

$$= 0 \ \text{ since the powers of } x \text{ in the integrand are odd}$$

Therefore the Fourier-Legendre series of x is $x = 1 \cdot P_1(x)$, as required.

c. Using Rodrigues's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2 - 1)^n]$$

we have

$$n = 0, \quad \frac{d^0}{dx^0}[(x^2 - 1)^0] = (x^2 - 1)^0 = 1$$

$$P_0(x) = \frac{1}{2^0 \, 0!} \, 1 = 1$$

$$n = 1, \quad \frac{d}{dx}(x^2 - 1) = 2x$$

$$P_1(x) = \frac{1}{2^1 \, 1!} \, 2x = x$$

$$n = 2, \quad \frac{d^2}{dx^2}(x^2 - 1)^2 = \frac{d}{dx}\Big[\frac{d}{dx}(x^2 - 1)^2\Big] = \frac{d}{dx}\big[4x(x^2 - 1)\big] = \frac{d}{dx}\big[4x^3 - 4x\big] = 12\,x^2 - 4$$

$$P_2(x) = \frac{1}{2^2 \, 2!} \, (12\,x^2 - 4) = \frac{4}{4\,2}(3x^2 - 1) = \frac{1}{2}(3x^2 - 1)$$

d. Extra Credit

Take f an even function and consider odd coefficients, let $n = 2p + 1$. Since n is odd $M = \frac{2p+1-1}{2} = p$, and we have

$$a_{2p+1} = \frac{2(2p+1)+1}{2} \int_{-1}^{1} f(x)P_{2p+1}(x)dx$$

$$= \frac{4p+3}{2} \int_{-1}^{1} f(x) \sum_{m=0}^{p}(-1)^m \frac{\Big(2(2p+1) - 2m\Big)!}{2^{2p+1}(2p+1-m)!(2p+1-2m)!}x^{2p+1-2m}dx$$

$$= \frac{4p+3}{2} \sum_{m=0}^{p}(-1)^m \frac{\Big(2(2p+1) - 2m\Big)!}{2^{2p+1}(2p+1-m)!(2p+1-2m)!} \int_{-1}^{1} f(x)x^{2p+1-2m}dx$$

$$= \frac{4p+3}{2} \sum_{m=0}^{p}(-1)^m \frac{\Big(2(2p+1) - 2m\Big)!}{2^{2p+1}(2p+1-m)!(2p+1-2m)!} \int_{-1}^{1} f(x)x^{2(p-m)+1}dx$$

The integral term is 0 since the integration is over a symmetric range and $f(x)x^{2(p-m)+1}$ is odd. Therefore given an even function $f(x)$ all the odd coefficients for the Fourier-Legendre series are zero. Similarly given an odd function, the even coefficients are zero.

Question 4

a.

$$\frac{\partial u}{\partial x} + 4xu = 0$$

Integration factor is

$$e^{\int 4x dx} = e^{4 \int x dx} = e^{4\frac{x^2}{2}} = e^{2x^2}$$

Multiply the partial differential equation by the I.F.:

$$e^{2x^2}\frac{\partial u}{\partial x} + 4xe^{2x^2}u = 0$$

$$\frac{\partial}{\partial x}(e^{2x^2}u) = 0$$

Now integrate both sides with respect to x

$$e^{2x^2}u = C \quad C\text{:constant}$$

$$u(x) = Ce^{-2x^2}$$

b.

$$y^2 u_x - x^2 u_y = 0$$

Let $u(x,y) = X(x)Y(y)$, substitution into the D.E. gives

$$y^2 X'Y - x^2 XY' = 0$$

$$y^2\frac{X'Y}{XY} - x^2\frac{XY'}{XY} = 0$$

$$y^2\frac{X'}{X} - x^2\frac{Y'}{Y} = 0$$

$$\frac{1}{x^2}\frac{X'}{X} = \frac{1}{y^2}\frac{Y'}{Y} = k$$

Integrating $\ln X = \frac{1}{3}kx^3 + \ln(C)$ and $\ln Y = \frac{1}{3}ky^3 + \ln(D)$, so

$$X = Ce^{\frac{1}{3}kx^3}, Y = De^{\frac{1}{3}ky^3}$$

Therefore (with $CD = A$) $u(x,y) = A\, e^{\frac{1}{3}k(x^3+y^3)}$

Question 5

We have the following problem

$$\Delta u = 0$$
$$u(x,0) = 0$$
$$u(x,b) = 100x$$
$$u_x(0,y) = 0$$
$$u_x(a,y) = 0$$

Assume a solution of the form $u(x,y) = X(x)Y(y)$. Substitute this expression and divide through by XY, to get:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$-\frac{X''}{X} = \frac{Y''}{Y}$$

LHS is function of x only and RHS is a function of y only, thus we can write, with k constant

$$-\frac{X''}{X} = \frac{Y''}{Y} = k$$

We see immediately that $X(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$. And $X'(x) = \sqrt{k}\left(B\cos(\sqrt{k}x) - A\sin(\sqrt{k}x)\right)$. Also $u_x(x,y) = X'(x)Y(y)$. Take the boundary condition $u_x(0,y) = X'(0)Y(y) = 0$, since in general $Y(y) \neq 0$ then $X'(0) = 0$. Plug it into $X'(x)$ gives $\sqrt{k}\left(B\,1 - A\,0\right) = 0 \to B = 0$.

Now $X(x) = A\cos\left(\sqrt{k}x\right)$ and $X'(x) = -A\sqrt{k}\sin\left(\sqrt{k}x\right)$. Next, with $u_x(a,y) = 0$ gives $X'(a)Y(y) = 0 \to X'(a) = 0$. $X'(a) = 0$ therefore $-A\sqrt{k}\sin\left(\sqrt{k}a\right) = 0 \to \sqrt{k}a = n\pi$. So $k_n = \frac{n^2\pi^2}{a^2}$ and

$$X_n(x) = A_n \cos(\frac{n\pi}{a}x)$$
$$Y_n(y) = B_n \cosh(\frac{n\pi}{a}y) + C_n \sinh(\frac{n\pi}{a}y)$$

The boundary condition $u(x,0) = X(x)Y(0) = 0$ gives the non trivial solution $Y(0) = B_n\,1 + C_n\,0 = 0 \to B_n = 0$. We are left with $u_n(x,y) = A_n \cos(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y)$ (where $A_nC_n \to A_n$). Applying the superposition principle we have

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y)$$

Finally we apply the initial condition $u(x,b) = 100x$, therefore

$$u(x,b) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}b) = 100x$$

This is a Fourier cosine series and we get

$$A_n \sinh(n\pi\frac{a}{b}) = \frac{2}{a}\int_0^a 100x \cos(\frac{n\pi}{a}x)dx$$
$$A_n \sinh(n\pi\frac{a}{b}) = \frac{200}{a}\int_0^a x \cos(\frac{n\pi}{a}x)dx$$

7

Now

$$\int_0^a x\cos(\frac{n\pi}{a}x)dx = \left[x(\frac{a}{n\pi})\sin(\frac{n\pi}{a}x)\right]_0^a - \frac{a}{n\pi}\int_0^a \sin(\frac{n\pi}{a}x)dx$$

$$= 0 - \frac{a}{n\pi}\int_0^a \sin(\frac{n\pi}{a}x)dx$$

$$= -\frac{a}{n\pi}(-\frac{a}{n\pi})[\cos(\frac{n\pi}{a}x)]_0^a$$

$$= \frac{a^2}{n^2\pi^2}(\cos(n\pi) - 1)$$

$$= \begin{cases} -\frac{2a^2}{n^2\pi^2} & \text{odd n} \\ 0 & \text{even n} \end{cases}$$

Substitute it back to get A_n

$$A_{n,n\text{ odd}} = \frac{1}{\sinh(n\pi\frac{a}{b})}\frac{200}{a}(-\frac{2a^2}{n^2\pi^2})$$

$$A_{n,n\text{ odd}} = -\frac{400a}{n^2\pi^2\sinh(n\pi\frac{a}{b})}$$

$$A_{n,n\text{ odd}} = -\frac{800a}{n^2\pi^2(e^{n\pi\frac{a}{b}} - e^{-n\pi\frac{a}{b}})}$$

$$A_{n,n\text{ even}} = 0$$

Question 6

a. Substituting $u(r,z) = R(r)Z(z)$ into the diffusion equation in cylindrical coordinates gives

$$R''Z + \frac{1}{r}R'Z + RZ'' = 0$$

Diving by RZ gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = 0$$

Separation of variables gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = -k^2$$

or

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -k^2$$
$$\frac{Z''}{Z} = k^2$$

b. For $\frac{d^2}{dz^2}Z(z) = k^2 Z(z)$, we immediately see that $Z(z) = c_1 e^{kz} + c_2 e^{-kz}$ which we can reformulate as $Z(z) = A\sinh(kz) + B\cosh(kz)$.

8

For $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -k^2$, starting with $\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + k^2R(r) = 0$.

$$s = kr, r = \frac{s}{k}, \frac{ds}{dr} = k, R(r) \to R(s)$$

$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = k\frac{dR}{ds}$$

$$\frac{d^2R}{dr^2} = k\frac{d}{ds}\left(k\frac{dR}{ds}\right) = k^2\frac{d^2R}{ds^2}$$

Substitution into the ODE gives

$$k^2\frac{d^2R(s)}{ds^2} + \frac{1}{\frac{s}{k}}k\frac{dR(s)}{ds} + k^2R(s) = 0$$

$$k^2\frac{d^2R(s)}{ds^2} + \frac{k^2}{s}\frac{dR(s)}{ds} + k^2R(s) = 0$$

Multiplying out by $\frac{s^2}{k^2}$ gives

$$s^2\frac{d^2R(s)}{ds^2} + s\frac{dR(s)}{ds} + s^2R(s) = 0$$

$$s^2\frac{d^2R(s)}{ds^2} + s\frac{dR(s)}{ds} + (s^2 - 0^2)R(s) = 0$$

The last equation being a Bessel equation of order 0 therefore the solution is of the form

$$R(r) = C_1 J_0(kr) + C_2 Y_0(kr)$$

Since the temperature remains bounded at $r = 0$, the term $Y_0(kr)$ has to be discarded, $C_2 = 0$, and $R(r) = C J_0(kr)$ with $C = C_1$

c. Finally apply boundary conditions. First, $u(r, 0) = R(r)Z(0) = 0$, since in general $R(r) \neq 0$, thus $Z(0) = 0$, which is $A\,0 + B\,1 = 0 \to B = 0$ and $Z(z) = A\sinh(kz)$. Then $u(5, z) = R(5)Z(z) = 0 \to R(5) = 0$, therefore $C J_0(5k) = 0 \to J_0(5k) = 0$. $5k$ represents the zero crossing for the Bessel function of order 0. We call them α_m and set $5k_m = \alpha_m \to k_m = \frac{\alpha_m}{5}$. Therefore the solutions are $R_m(r) = C_m J_0(k_m r) = C_m J_0(\frac{\alpha_m}{5}r)$. Note now that solutions in z are $Z_m(z) = A_m\sinh(k_m z)$. Finally applying the superposition principle, we have

$$u(r, z) = \sum_{m=1}^{\infty} A_m \sinh(k_m z) J_0(\frac{\alpha_m}{5}r) \quad \text{where } A_m C_m \to A_m$$

Applying the last boundary condition $u(r, 20) = u_0, 0 < r < 5$, we get:

$$u(r, 20) = \sum_{m=1}^{\infty} A_m \sinh(20 k_m) J_0(\frac{\alpha_m}{5}r) = u_0$$

This is a Fourier Bessel series where the coefficients are given by

$$\sinh(20 k_m) A_m = \frac{2}{5^2 J_1^2(\alpha_m)} \int_0^5 r u_0 J_0(\frac{\alpha_m}{5}r)dr = \frac{2u_0}{25 J_1^2(\alpha_m)} \int_0^5 r J_0(\frac{\alpha_m}{5}r)dr$$

9

Next, $\sinh(20k_m) = \sinh(20\frac{\alpha_m}{5}) = \sinh(4\alpha_m)$, and we get

$$A_m = \frac{2u_0}{25 J_1^2(\alpha_m)\sinh(4\alpha_m)} \int_0^5 J_0(\frac{\alpha_m}{5}r)r\,dr$$

Finally using $\frac{\partial}{\partial r}[r J_1(r)] = r J_0(r)$ which also gives

$$\frac{\partial}{\partial r}[\alpha r J_1(\alpha r)] = \alpha r J_0(\alpha r)dr$$

$$\alpha r J_1(\alpha r) = \int \alpha r J_0(\alpha r)dr$$

$$r J_1(\alpha r) = \int r J_0(\alpha r)dr$$

$$A_m = \frac{2u_0}{25 J_1^2(\alpha_m)\sinh(4\alpha_m)}\left[r J_1(\frac{\alpha_m}{5}r)\right]_0^5$$

$$= \frac{2u_0}{25 J_1^2(\alpha_m)\sinh(4\alpha_m)}\left(5J_1(\frac{\alpha_m}{5}5)\right)$$

$$= \frac{2u_0}{5\sinh(4\alpha_m)J_1(\alpha_m)}$$

Question 7

a. Let $z = x + iy$ then $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$ and

$$e^{z^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2}e^{2ixy}$$
$$= e^{x^2-y^2}(\cos 2xy + \sin 2xy)$$

10

Now take $u(x, y) = e^{x^2-y^2} \cos 2xy$ and $v(x, y) = e^{x^2-y^2} \sin 2xy$. We have

$$\frac{\partial u}{\partial x} = 2x\, e^{x^2-y^2} \cos 2xy + e^{x^2-y^2}(-2y) \sin 2xy$$

$$\frac{\partial u}{\partial x} = 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy)$$

$$\frac{\partial v}{\partial y} = -2y\, e^{x^2-y^2} \sin 2xy + e^{x^2-y^2}(2x) \cos 2xy$$

$$\frac{\partial v}{\partial y} = 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y\, e^{x^2-y^2} \cos 2xy + e^{x^2-y^2}(-2x) \sin 2xy$$

$$\frac{\partial u}{\partial y} = -2e^{x^2-y^2}(y \cos 2xy + x \sin 2xy)$$

$$\frac{\partial v}{\partial x} = 2x\, e^{x^2-y^2} \sin 2xy + e^{x^2-y^2}(2y) \cos 2xy$$

$$\frac{\partial v}{\partial x} = 2e^{x^2-y^2}(y \cos 2xy + x \sin 2xy)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Cauchy-Riemann conditions are satisfied therefore e^{z^2} is analytic (as required, since both z^2 and e^z are entire functions).

b. For $v(x, y) = xy$, we have

$$v_x = y, v_{xx} = 0$$
$$v_y = x, v_{yy} = 0$$
$$\rightarrow v_{xx} + v_{yy} = 0$$

Using Cauchy conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, we have $\frac{\partial u}{\partial x} = x$ integration on both sides, gives:

$$u = \int x dx + g(y)$$

$$u = \frac{x^2}{2} + g(y)$$

Next the Cauchy condition $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ yields

$$\frac{\partial u}{\partial y} = g'(y) = -\frac{\partial v}{\partial x} = -y$$

$$g(y) = -\frac{y^2}{2} + C$$

Therefore $u(x, y) = \frac{x^2-y^2}{2} + C$ and $f(x, y) = \frac{x^2-y^2}{2} + C + ixy$

11

Question 8

a. Substitute $y = x^2, dy = 2xdx$ then we get

$$\int_C (x^2 + y)dx + 2xydy = \int_0^1 (x^2 + x^2)dx + 2x\ x^2 2xdx$$

$$= \int_0^1 (2x^2 + 4x^4)dx$$

$$= 2[\frac{x^3}{3}]_0^1 + 4[\frac{x^5}{5}]_0^1$$

$$= \frac{2}{3} + \frac{4}{5}$$

$$= \frac{10 + 12}{15} = \frac{22}{15}$$

b. Take $z = x + iy$, $Re(z) = x$, let parametrize the line from 1 to i as $z(t) = 1 \cdot (1 - t) + i \cdot t, = 1 + (i - 1)\,t, dz = (i - 1)dt$ for $0 \le t \le 1$. We have then

$$\int_C Re(z)dz = \int_0^1 (1 - t)(i - 1)dt$$

$$= (i - 1)\int_0^1 (1 - t)dt$$

$$= (i - 1)[t - \frac{t^2}{2}]_0^1$$

$$= (i - 1)\frac{1}{2} = \frac{i - 1}{2}$$

Extra Credit

a. Take $f(z) = \frac{1}{(z-2)(z-1)^2} = \frac{1}{z-2} - \frac{z}{(z-1)^2}$, the poles are at $z = 2$ and $z = 1$. We compute the residues at each pole. At $z = 2$

$$f(z) = \frac{1}{z - 2} + \text{something analytic at } 2$$

Therefore the pole is simple and $\text{Res}(f, 2) = 1$ At $z = 1$. Take $z = \xi + 1$ then

$$f(\xi) = \frac{1}{(\xi - 1)\xi^2}$$

$$= -\frac{1}{\xi^2}(1 - \xi)^{-1}$$

$$= -\frac{1}{\xi^2}(1 + \xi + \xi^2 + \xi^3 + \cdots)$$

$$= -\frac{1}{\xi^2} - \frac{1}{\xi} - \frac{1}{2} - \xi + \cdots$$

12

Therefore the pole is of order 2 and $\text{Res}(f, 1) = -1$. Poles within the circle centered at $z = 0$ with radius 4, are 1 and 2. Applying the residue theorem

$$\oint_C f(z)dz = \oint_C \frac{1}{(z-2)(z-1)^2}dz = 2\pi i(\text{Res}(f, 2) + \text{Res}(f, 1)) = 2\pi i\,(1 - 1) = 0$$

b. We have

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

$$e^{\frac{1}{z^n}} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^{n+k}}$$

$$= \frac{1}{z^n} + \frac{1}{z^{n+1}} + \frac{1}{z^{n+2}} + \cdots$$

Looking at $\lim_{z \to 0}(z - 0)^{\alpha} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^{n+k}}$, there is no largest α such this quantity is finite. Thus 0 is an essential singularity. The Laurent expansion has only negative powers of z starting at n. $e^{\frac{1}{z^n}}$ has only the singularity 0 within the unit circle, therefore applying the residue theorem

- $n = 0$, the function is $e^{\frac{1}{z^0}} = e$, take $z = e^{i\theta}, dz = ie^{i\theta}d\theta$

$$\oint_C e\,dz = e\oint_C dz = ei\int_0^{2\pi} e^{i\theta}d\theta = ei[\frac{e^{i\theta}}{i}]_0^{2\pi} = e(e^{i2\pi} - 1) = 0$$

- $n = 1$, $\text{Res}(e^{\frac{1}{z}}, 0) = 1$

$$\oint_C e^{\frac{1}{z}}dz = 2\pi i\,1 = 2\pi i$$

- case $n > 1$ $\text{Res}(e^{\frac{1}{z^n}}, 0) = 0$

$$\oint_C e^{\frac{1}{z^n}}dz = 2\pi i\,0 = 0$$