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Put the ODE in standard form:

$$y'' - \frac{3z}{1-z^2} y' + \frac{\lambda}{1-z^2} y = 0$$

thus $p(z) = -\frac{3z}{1-z^2}$ and $q(z) = \frac{\lambda}{1-z^2}$

$p(0) = 0$ and $q(0) = \lambda$ then $z=0$ is an ordinary point, we can then represent the solutions in terms of a power series: $y(z) = \sum_{n=0}^{\infty} a_n z^n$

$$y'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

$$y''(z) = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2}$$

Plug back these into the initial ODE gives:

$$(1-z^2) \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - 3z \sum_{n=0}^{\infty} n a_n z^{n-1} + \lambda \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n z^n - 3 \sum_{n=0}^{\infty} n a_n z^n + \lambda \sum_{n=0}^{\infty} a_n z^n = 0$$

Reindexing the first term:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n - \sum_{n=0}^{\infty} n(n-1) a_n z^n - 3 \sum_{n=0}^{\infty} n a_n z^n + \lambda \sum_{n=0}^{\infty} a_n z^n = 0$$

Chapter 7 - Problem 7.1

Collecting the coefficients for same power of z:

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n^2+2n-\lambda) a_n \right] z^n = 0$$

Discarding the degenerate solution for $z^n = 0$ yields the recurrent relation:

$$a_{n+2} = \frac{n^2+2n-\lambda}{(n+1)(n+2)} a_n$$

Note that for $\lambda = n(n+2)$ all the a_m terms for $m > n$ are 0!
The solution in this case is the polynomial of degree n: $y(z) = \sum_{k=0}^{n} a_k z^k$

For $n=2$, $\lambda = 8$, take $a_1 = 0$ and $a_2 = \frac{(-8)}{2 \times 1} a_0 = -4 a_0$

therefore for $n=2$, we get $y(z) = U_2(z) = a_0 - 4 a_0 z^2$
$= a_0 (1 - 4z^2)$

For $n=3$, $\lambda = 3(3+2) = 15$, take $a_0 = 0$ and $a_1 \neq 0$ to generate a power series with odd terms

$$a_{1+2} = a_3 = \frac{3-15}{(1+2)(1+1)} = -\frac{12}{6} a_1 = -2a_1$$

Therefore for $n=3$ $y(z) = U_3(z) = a_1 (z - 2z^3)$

Chapter 7 - Problem 7.4

By the change of variable $x = z - a$ and using the chain rule:

$$\frac{df}{dz} = \frac{df}{dx}\frac{dx}{dz} = \frac{df}{dx}\cdot 1 = \frac{df}{dx}$$

$$\frac{d^2f}{dz^2} = \frac{dx}{dz}\frac{d}{dx}\left(\frac{df}{dz}\right) = 1\cdot\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2}$$

Therefore the ODE becomes:

$$\frac{d^2f}{dx^2} + 2x\frac{df}{dx} + 4f = 0$$

The solution can be represented at any point x as:

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ since there is no singular point.

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \text{ and } f''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

Substitution of these derivatives into the ODE gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + (2n+4)a_n\right]x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + 2(n+2)a_n\right]x^n = 0$$

Chapter 7 - Problem 7.4

Avoiding the degenerate solution $x=0$, for each n

$$(n+2)(n+1)a_{n+2} + 2(n+2)a_n = 0$$

for all n :
$$a_{n+2} = -\frac{2}{n+1}a_n$$

. For even n: $n=2p$
$$a_{2p} = -\frac{2}{2p-1}a_{2p-2} = \frac{(-2)}{2p-1}\frac{(-2)}{(2p-3)}a_{2p-4}$$

$$= \frac{(-2)(-2)(-2)(-2)}{(2p-1)(2p-3)(2p-5)(2p-7)}a_{2p-8}$$

$$= \dots = \frac{(-2)^{p+1}}{(2p-1)(2p-3)\cdots 3}a_2$$

$$= \frac{(-2)^{p+2}}{(2p-1)(2p-3)\cdots 3}a_0$$

$$= \frac{(-2)^p\, 2p\,(2p-2)(2p-4)\cdots 2}{(2p)!}a_0$$

$$= \frac{(-2)^p 2^p\, p!}{(2p)!}a_0 = \frac{(-4)^p\, p!}{(2p)!}a_0$$

Shifting the indices:

$$a_n = \frac{-2}{n-1}a_{n-2}$$

And for odd n:
$$a_{2p-1} = \frac{-2}{2p-1-1}a_{2p-3} = \frac{-2}{2(p-1)}a_{2p-3}$$

Chapter 7 - Problem 7.4

We have

$$a_{2p+1} = \frac{(-2)}{2p} a_{2p-1} = \frac{(-1)}{p} a_{2p-1} = \frac{(-1)(-1)}{p(p-1)} a_{2p-3}$$

$$= \dots = \frac{(-1)\dots(-1)}{p(p-1)(p-2)\dots 2 \cdot 1} a_1 = \frac{(-1)^p}{p!} a_1$$

We want to generate two independent solutions: So for one case set $a_1 = 0$ then $a_{2p+1} = 0$ and $a_{2p} = \frac{(-4)^p p!}{2p!}$

This independent solution is of the form:

$$f(x) = \sum_{p=0}^{\infty} a_{2p} x^{2p} = \sum_{p=0}^{\infty} \frac{(-4)^p p!}{2p!} a_0 \cdot x^{2p}$$

$$= a_0 \sum_{p=0}^{\infty} \frac{(-4)^p p!}{2p!} x^{2p}$$

For the second case set $a_0 = 0$ then $a_{2p} = 0$ and $a_{2p+1} = \frac{(-1)^p}{p!} a_1$.

which yields:

$$f(x) = \sum_{p=0}^{\infty} a_{2p+1} x^{2p+1} = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} a_1 \, x^{2p+1}$$

$$= a_1 x \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} x^{2p}$$

$$= a_1 x \sum_{p=0}^{\infty} \frac{(-x^2)^p}{p!} = a_1 x \, e^{-x^2}$$

Chapter 7 - Problem 7.4

Substitute back in the two proposed independent solutions $x = z - a$, the general solution of

$$\frac{d^2 f}{dz^2} + 2(z-a)\frac{df}{dz} + 4f = 0$$

is therefore

$$f(z,\alpha) = A(z-a)e^{-(z-a)^2} + B\sum_{m=0}^{\infty}\frac{(-4)^m\, m!}{2m!}(z-a)^{2m}$$