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Mathematical Methods for Applied Biomedical Engineering
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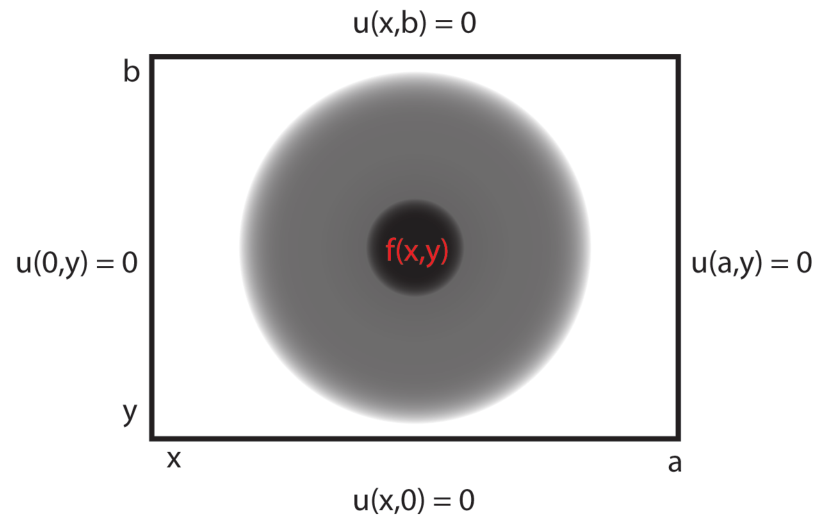
Example of wave equation of rectangular membrane

Let's look at a rectangular membrane. As our next example we will derive a time dependent solution of the wave equation in two spatial dimensions.

The diffusion equation in two spatial dimensions is

$$\nabla^2 u(x,y,t) \equiv \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] u(x,y,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u(x,y,t)$$



There are four spatial derivatives and two temporal derivatives. Each of these require either a boundary condition or initial condition. For this problem we are setting up the following conditions:

Initial position

$$\text{Initial conditions: } u(x,y,0) = f(x,y), \quad \left.\frac{\partial}{\partial t} u(x,y,t)\right|_{t=0} = g(x,y) = 0$$

Initial velocity – Which we are setting to 0 as a simplification

$$\text{Boundary conditions: } u(0,y,t) = u(a,y,t) = 0, \; u(x,0,t) = u(x,b,t) = 0$$

Take $u(x,y,t)=X(x)Y(y)T(t)$ and substitute

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]X(x)Y(y)T(t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}X(x)Y(y)T(t)$$

$$\frac{\partial^2}{\partial x^2}X(x)Y(y)T(t)+\frac{\partial^2}{\partial y^2}X(x)Y(y)T(t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}X(x)Y(y)T(t)$$

Can be written as $Y(y)T(t)\frac{d^2}{dx^2}X(x)+X(x)T(t)\frac{d^2}{dy^2}Y(y)=X(x)Y(y)\frac{1}{c^2}\frac{d^2}{dt^2}T(t)$

and dividing by $X(x)Y(y)T(t)$ gives

$$\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)+\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=\frac{1}{c^2}\frac{1}{T(t)}\frac{d^2}{dt^2}T(t)$$

Our seperation constants are such that

$$\boxed{\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)=-l^2,\ \frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=-m^2 \text{ and } \frac{1}{c^2}\frac{1}{T(t)}\frac{d^2}{dt^2}T(t)=-\mu^2}$$

such that $l^2+m^2=\mu^2$ and of course each constant, eg. $-m^2$ is strictly negative!

As previously stated the solutions for X and Y are

$$X(x)=A\cos(lx)+N\sin(lx),\quad Y(y)=C\cos(my)+D\sin(my)$$

First lets apply the boundary conditions for X and Y.

As before $u(0,y,t)=0 \rightarrow X(0)=0,\ u(a,y,t)=0 \rightarrow X(a)=0$

$$u(x,0,t)=0 \rightarrow Y(0)=0,\ u(x,b,t)=0 \rightarrow Y(b)=0$$

Applying these boundary conditions gives conditions for X and Y similar to previous results, that is

$$la=n_1\pi \rightarrow l=\frac{n_1\pi}{a},\ mb=n_2\pi \rightarrow m=\frac{n_2\pi}{b}$$

And we have (leaving off the constants for now)

$$X_{n_1}(x)=\sin\left(\frac{n_1\pi}{a}x\right),\quad Y_{n_2}(y)=\sin\left(\frac{n_2\pi}{b}y\right)$$

Therefore $\boxed{X_{n_1}(x)Y_{n_2}(y)=\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)}$

For the temporal equation we have the following solution

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{d^2}{dt^2}T(t) = -\mu^2 \text{ or } \frac{d^2}{dt^2}T(t) + \mu^2 c^2 T(t) = 0$$

Now $l^2 + m^2 = \mu^2 \rightarrow \left(\frac{n_1\pi}{a}\right)^2 + \left(\frac{n_2\pi}{b}\right)^2 = \pi^2\left[\left(\frac{n_1}{a}\right)^2 + \left(\frac{n_2}{b}\right)^2\right] = \mu^2$

Unit analysis allows us to set $\mu^2 c^2 = \lambda_{n_1n_2}^2$ or $\lambda_{n_1n_2} = c\pi\left[\left(\frac{n_1}{a}\right)^2 + \left(\frac{n_2}{b}\right)^2\right]$

Substitution into the temporal equation

$$\frac{d^2}{dt^2}T(t) + \lambda_{n_1n_2}^2 T(t) = 0$$

Similar to previous solutions of this simple second order equation we have

$$T_{n_1n_2}(t) = B_{n_1n_2}\cos\lambda_{n_1n_2}t + B_{n_1n_2}^{*}\sin\lambda_{n_1n_2}t$$

Therefore we can write $u_{n_1n_2}(x,y,t) = X_{n_1}(x)Y_{n_2}(y)T_{n_1n_2}(t)$

That is $u_{n_1n_2}(x,y,t) = \sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)(B_{n_1n_2}\cos\lambda_{n_1n_2}t + B_{n_1n_2}^{*}\sin\lambda_{n_1n_2}t)$

We next apply the superposition principle, that is

$$u(x,y,t)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}u_{n_1n_2}(x,y,t)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)(B_{n_1n_2}\cos\lambda_{n_1n_2}t+B^*_{n_1n_2}\sin\lambda_{n_1n_2}t)$$

Finally we apply the initial conditions

First $\boxed{\left.\frac{\partial}{\partial t}u(x,y,t)\right|_{t=0}=g(x,y)=0}$, therefore

$$\left.\frac{\partial}{\partial t}u(x,y,t)\right|_{t=0}=\frac{\partial}{\partial t}\left[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)(B_{n_1n_2}\cos\lambda_{n_1n_2}t+B^*_{n_1n_2}\sin\lambda_{n_1n_2}t)\right]_{t=0}=0$$

or $$\left[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)(B_{n_1n_2}\frac{\partial}{\partial t}\cos\lambda_{n_1n_2}t+B^*_{n_1n_2}\frac{\partial}{\partial t}\sin\lambda_{n_1n_2}t)\right]_{t=0}=0$$

Evaluation of the derivative and substitution of t = 0 gives

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)(B_{n_1n_2}(-\lambda_{n_1n_2})\sin\lambda_{n_1n_2}0+B^*_{n_1n_2}(\lambda_{n_1n_2})\cos\lambda_{n_1n_2}0)=$$

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)(0+B^*_{n_1n_2}(\lambda_{n_1n_2})(1))=B^*_{n_1n_2}\lambda_{n_1n_2}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)=0$$

Since neither $\sin\left(\frac{n_1\pi}{a}x\right)$, $\sin\left(\frac{n_2\pi}{b}y\right)$ or $\lambda_{n_1n_2}$ can be 0 that leaves $\boxed{B^*_{n_1n_2}=0}$, as expected!

We are left with

$$u(x,y,t)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}B_{n_1n_2}\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)\cos\lambda_{n_1n_2}t$$

Applying the final initial conditions $\boxed{u(x,y,0)=f(x,y)}$ gives

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}B_{n_1n_2}\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)\cos(\lambda_{n_1n_2}0)=f(x,y) \text{ or}$$

KEY: A 2-D Fourier series

$$\boxed{\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}B_{n_1n_2}\sin\left(\frac{n_1\pi}{a}x\right)\sin\left(\frac{n_2\pi}{b}y\right)=f(x,y)}$$

KEY: We need to solve for the 2-D Fourier coefficients, in this case $B_{n_1n_2}$

KEY

First we take $\boxed{K_{n_2}(y) = \sum_{n_2=1}^{\infty} B_{n_1 n_2} \sin\left(\frac{n_2 \pi}{b} y\right)}$

Substitution into our original Fourier series gives

$$f(x,y) = \sum_{n_1=1}^{\infty} K_{n_2}(y) \sin\left(\frac{n_1 \pi}{a} x\right)$$

Solving for $K_{n_2}(y)$ – its now the coefficient associated with 1-D Fourier series gives

$$K_{n_2}(y) = \frac{2}{a}\int_0^a f(x,y) \sin\left(\frac{n_1 \pi}{a} x\right) dx$$

But from our first equation above for $K_{n_2}(y) = \sum_{n_2=1}^{\infty} B_{n_1 n_2} \sin\left(\frac{n_2 \pi}{b} y\right)$

we have $B_{n_1 n_2} = \frac{2}{b}\int_0^b K_{n_2}(y) \sin\left(\frac{n_2 \pi}{a} y\right) dy$

Substitution for $K_{n_2}(y)$ finally gives us

$$\boxed{B_{n_1 n_2}} = \frac{2}{b}\int_0^b \frac{2}{a}\int_0^a f(x,y) \sin\left(\frac{n_1 \pi}{a} x\right) dx \sin\left(\frac{n_2 \pi}{a} y\right) dy = \boxed{\frac{4}{ab}\int_0^b \int_0^a f(x,y) \sin\left(\frac{n_1 \pi}{a} x\right) \sin\left(\frac{n_2 \pi}{a} y\right) dxdy}$$