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Strum-Liouville differential equations

The standard form for a Strum-Liouville differential equation is

$$p(x)\frac{d^2y}{dx^2}+r(x)\frac{dy}{dx}+q(x)y+\lambda\rho(x)y=0,\quad r(x)=\frac{dp(x)}{dx}$$

Taking $\mathcal{L}=-\left[p(x)\frac{d^2}{dx^2}+r(x)\frac{d}{dx}+q(x)\right]$ as a differential operator

We can then write $\mathcal{L}y(x)=\lambda\rho(x)y(x)$

Another more compact way of writing this operator, since r(x) is just the derivative of p(x), is

$$\mathcal{L}=-\left[p(x)\frac{d^2}{dx^2}+\frac{dp(x)}{dx}\frac{d}{dx}+q(x)\right]=-\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]$$

KEY: Any second order differential equation can be put in Strum-Liouiviille form By the following process.

If the second order differential does not have the required form we can create a related equation in which this relationship exist. Construct the following function F(x) – similar to completing a total differential operator

$$F(x) = e^{\int^x \frac{r(u) - p'(u)}{p(u)} du}$$

and multiple our differential equation by it. So suppose we have an equation In which r(x) is not the derivative of p(x), that is

$$\tilde{p}(x)\frac{d^2y}{dx^2} + \tilde{r}(x)\frac{dy}{dx} + \tilde{q}(x)y + \lambda\tilde{\rho}(x)y = 0, \;\; \tilde{r}(x) \neq \frac{d\tilde{p}(x)}{dx}$$

Then the new equation will conform to the Strum-Louiville form, that is

$$F(x)\tilde{p}(x)\frac{d^2y}{dx^2} + F(x)\tilde{r}(x)\frac{dy}{dx} + F(x)\tilde{q}(x)y + \lambda F(x)\tilde{\rho}(x)y =$$

$$p(x)\frac{d^2y}{dx^2} + r(x)\frac{dy}{dx} + q(x)y + \lambda\rho(x)y = 0$$

where $p(x) = F(x)\tilde{p}(x), q(x) = F(x)\tilde{q}(x), \; \rho(x)=F(x)\tilde{\rho}(x)$

and especially $r(x)=F(x)\tilde{r}(x)=\dfrac{dF(x)\tilde{p}(x)}{dx} = \dfrac{dp(x)}{dx}$

The process is easily performed provided the integral is tractable!

Show that the Strum-Liouville operator is Hermitian

We need to show $\boxed{\left\langle \mathcal{L} y_i^* \middle| y_j \right\rangle = \left\langle y_i^* \middle| \mathcal{L} y_j \right\rangle}$ where $\mathcal{L} = -\left[\frac{d}{dx}\left(p(x) \frac{d}{dx} \right) + q(x) \right]$

Start the calculation with

$$\left\langle y_i^* \middle| \mathcal{L} y_j \right\rangle = \int_a^b y_i^* \left[\mathcal{L} y_j \right] dx = \int_a^b y_i^* \left[-\frac{d}{dx}\left(p(x) \frac{dy_j}{dx} \right) - q(x) y_j \right] dx$$

KEY: Is integration by parts, therefore

$$\left\langle y_i^* \middle| \mathcal{L} y_j \right\rangle = \int_a^b y_i^* \left[\mathcal{L} y_j \right] dx = \int_a^b y_i^* \left[-\frac{d}{dx}\left(p(x) \frac{dy_j}{dx} \right) - q(x) y_j \right] dx =$$

$$\int_a^b y_i^* \left[-\frac{d}{dx}\left(p(x) \frac{dy_j}{dx} \right) \right] dx - \int_a^b y_i^* q(x) y_j \, dx$$

In the first integral above let $u = y_i^*$, $du = \frac{dy_i^*}{dx} dx$

and $dv = -\frac{d}{dx}\left(p(x) \frac{dy_j}{dx} \right) dx$, $v = -p(x) \frac{dy_j}{dx}$

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We need to show $\boxed{\left\langle \mathcal{L} y_i^* \middle| y_j \right\rangle = \left\langle y_i^* \middle| \mathcal{L} y_j \right\rangle}$ where $\mathcal{L} = -\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right]$

Start the calculation with

$$\boxed{\left\langle y_i^* \middle| \mathcal{L} y_j \right\rangle} = \int_a^b y_i^* \left[\mathcal{L} y_j\right] dx = \int_a^b y_i^* \left[-\frac{d}{dx}\left(p(x)\frac{dy_j}{dx}\right) - q(x) y_j\right] dx$$

KEY: Is integration by parts, therefore

$$\boxed{\int_a^b y_i^* \left[-\frac{d}{dx}\left(p(x)\frac{dy_j}{dx}\right)\right] dx} - \int_a^b y_i^* q(x) y_j \, dx$$

In the first integral above let

$$u = y_i^*,\ du = \frac{dy_i^*}{dx} dx \text{ and } dv = -\frac{d}{dx}\left(p(x)\frac{dy_j}{dx}\right),\ v = -p(x)\frac{dy_j}{dx}$$

$$\boxed{\left[y_i^*\left(-p(x)\frac{dy_j}{dx}\right)\right]_a^b - \int_a^b \left[-p(x)\frac{dy_j}{dx}\right]\frac{dy_i^*}{dx} dx} - \int_a^b y_i^* q(x) y_j \, dx =$$

$$-p(x) y_i^* \frac{dy_j}{dx}\Bigg|_a^b + \int_a^b \left[p(x)\frac{dy_i^*}{dx}\right]\frac{dy_j}{dx} dx - \int_a^b y_i^* q(x) y_j \, dx$$

$$-p(x)y_i^*\frac{dy_j}{dx}\bigg|_a^b+\int_a^b\left[p(x)\frac{dy_i^*}{dx}\right]\frac{dy_j}{dx}dx-\int_a^b y_i^*q(x)y_j\,dx=$$

In the first integral let

$$u=p(x)\frac{dy_i^*}{dx},\ du=\left[\frac{dp(x)}{dx}\frac{dy_i^*}{dx}+p(x)\frac{d^2y_i^*}{dx^2}\right]dx \text{ and } dv=\frac{dy_j}{dx}dx,\ v=y_j$$

$$-p(x)y_i^*\frac{dy_j}{dx}\bigg|_a^b+p(x)\frac{dy_i^*}{dx}y_j\bigg|_a^b-\int_a^b y_j\left[\frac{dp(x)}{dx}\frac{dy_i^*}{dx}+p(x)\frac{d^2y_i^*}{dx^2}\right]dx-\int_a^b y_i^*q(x)y_j\,dx=$$

$$-p(x)y_i^*\frac{dy_j}{dx}\bigg|_a^b+p(x)\frac{dy_i^*}{dx}y_j\bigg|_a^b-\int_a^b\left[p(x)\frac{d^2y_i^*}{dx^2}+\frac{dp(x)}{dx}\frac{dy_i^*}{dx}+y_i^*q(x)y_j\right]dx=$$

Applying the boundary conditions, for instance $y(a)=y(b)=0$: $y(a)=y'(b)=0$; etc. gives

$$0+\int_a^b-\left[\frac{d}{dx}\left(p(x)\frac{dy_i^*}{dx}\right)+y_i^*q(x)y_j\right]y_j\,dx=\boxed{\left\langle \mathcal{L}y_i^*\middle|y_j\right\rangle}$$

Some Strum-Liouville systems we have or will be looking at in more detail:

Equation	p(x)	q(x)	λ	$\rho(x)$
Harmonic (Fourier)	1	0	ω^2	1
Legendre	$1-x^2$	0	$\ell(\ell+1)$	1
Bessel$(x \rightarrow \frac{x}{\alpha})$	x^2	$\frac{-\nu^2}{x}$	α^2	x