9.1

Let's do one of these for l = 1, that is

$$\sum_{m=-1}^{1}\left|Y_1^m(\theta,\phi)\right|^2$$

Aside:

For complex functions a+bi

$$|a+bi|^2 = \left[(a+ib)^*(a+ib)\right]^{1/2}\Big\}^2 = (a+ib)^*(a+ib) = (a-ib)(a+ib)$$

Also $e^{i\phi} = \cos\phi + i\sin\phi$

Therefore for l = 1 (here we are setting the letter 'l' to the number one)

and m ranges -1, 0, 1

Back:

$$\sum_{m=-1}^{1}\left|Y_1^m(\theta,\phi)\right|^2 = Y_1^{-1}(\theta,\phi)^* Y_1^{-1}(\theta,\phi) + Y_1^0(\theta,\phi)^* Y_1^0(\theta,\phi) + Y_1^1(\theta,\phi)^* Y_1^1(\theta,\phi)$$

For

$$Y_1^{-1}(\theta,\phi)^* Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} = \frac{3}{8\pi}\sin^2\theta e^{-i\phi}e^{i\phi}$$

$$Y_1^{-1}(\theta,\phi)^* Y_1^{-1}(\theta,\phi) = \frac{3}{8\pi}\sin^2\theta(\cos\phi - i\sin\phi)(\cos\phi + i\sin\phi)$$

$$Y_1^{-1}(\theta,\phi)^* Y_1^{-1}(\theta,\phi) = \frac{3}{8\pi}\sin^2\theta(\cos^2\phi + \sin^2\phi) = \frac{3}{8\pi}\sin^2\theta$$

Similarly

$$Y_1^1(\theta,\phi)^* Y_1^1(\theta,\phi) = \frac{3}{8\pi}\sin^2\theta$$

and

$$Y_1^0(\theta,\phi)^* Y_1^0(\theta,\phi) = \frac{3}{4\pi}\sin^2\theta$$

Therefore

$$\sum_{m=-1}^{1}\left|Y_1^m(\theta,\phi)\right|^2 = \frac{3}{8\pi}\sin^2\theta + \frac{3}{4\pi}\cos^2\theta + \frac{3}{8\pi}\sin^2\theta$$

$$\sum_{m=-1}^{1}\left|Y_1^m(\theta,\phi)\right|^2 = \frac{3}{4\pi}(\sin^2\theta + \cos^2\theta) = \frac{3}{8\pi} = \frac{(2\cdot 1)+1}{4\pi}$$

Here is a slightly easier way to do the calculation for ℓ=2

Note the real part can be taken out of modulus operaor, since its real!), e.g.

$$\left|Y_2^{-2}(\theta,\phi)\right|^2=\left|\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{in\phi}\right|^2=\left(\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\right)^2\left|e^{in\phi}\right|^2=\cdots$$

Aside:

DO NOT DO THI!S $\left|e^{in\phi}\right|^2\neq\left(e^{in\phi}\right)^2=1$ is not correct notation. Be careful!!

What we have in terms of the definition of the modulus $\left|x+iy\right|=[(x+iy)(x-iy)]^{1/2}=\left(x^2+y^2\right)^{1/2}$

So $\left|e^{in\phi}\right|=\left|\cos n\phi+i\sin n\phi\right|=[(\cos n\phi+i\sin n\phi)(\cos n\phi-i\sin n\phi)]^{1/2}=[\cos^2 n\phi+\sin^2 n\phi]^1=1$

$\left|e^{in\phi}\right|^2=1^2$

Note $\left(e^{in\phi}\right)^2=(\cos n\phi+i\sin n\phi)^2=\cos^2 n\phi-\sin^2 n\phi+2i\cos n\phi\sin n\phi$

Back:

Expand sum , that is terms with m = -2, -1, 0, 1, 2; Note squared term for m = -2,2 and m =-1,1 are the same!

$$\sum_{m=-2}^{2}\left|Y_2^m(\theta,\phi)\right|^2=\left[\left(\sqrt{\frac{15}{32\pi}}\sin^2\theta\right)^2+\left(-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\right)^2+\left(\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)\right)^2\cdots\text{for m = 1 and m = 2}\right]\left|e^{in\phi}\right|^2=$$

(Note factors of 2 come fromm = -2,2 and m =-1,1; m = 0 has only one term)

$$\left[2\frac{15}{32\pi}\sin^4\theta+2\frac{15}{8\pi}\sin^2\theta\cos^2\theta+\frac{5}{16\pi}(3\cos^2\theta-1)^2\right](1)=$$

$$\frac{15}{16\pi}\sin^4\theta+2\frac{2\cdot 15}{2\cdot 8\pi}\sin^2\theta\cos^2\theta+\frac{1}{16\pi}5(3\cos^2\theta-1)^2=$$

(I reversed order of terms, next line - just a choice and factored out common denominator)

$$\frac{1}{16\pi}\{5(3\cos^2\theta-1)^2+4\cdot 15\sin^2\theta\cos^2\theta+15\sin^4\theta\}=$$

Aside: Take $5(3\cos^2\theta-1)^2=5(9\cos^4\theta-6\cos^2\theta+1)=45\cos^4\theta-30\cos^2\theta+5$

Therefore

$$\frac{1}{16\pi}\{45\cos^4\theta-30\cos^2\theta+5+4\cdot 15\sin^2\theta\cos^2\theta+15\sin^4\theta\}=$$

Rearranging $\frac{1}{16\pi}\{15\cos^4\theta+15\sin^4\theta+2\cdot 15\sin^2\theta\cos^2\theta+5-30\cos^2\theta+30\cos^4\theta+30\sin^2\theta\cos^2\theta\}=$

$$\frac{1}{16\pi}\{15(\cos^2\theta+\sin^2\theta)^2+5-30\cos^2\theta+30\cos\theta(\cos^2\theta+\sin^2\theta)\}=\frac{1}{16\pi}\{15+5-30\cos^2\theta+30\cos\theta\}=$$

$$\frac{20}{16\pi}=\frac{5}{4\pi}=\frac{2\cdot 2+1}{4\pi}$$