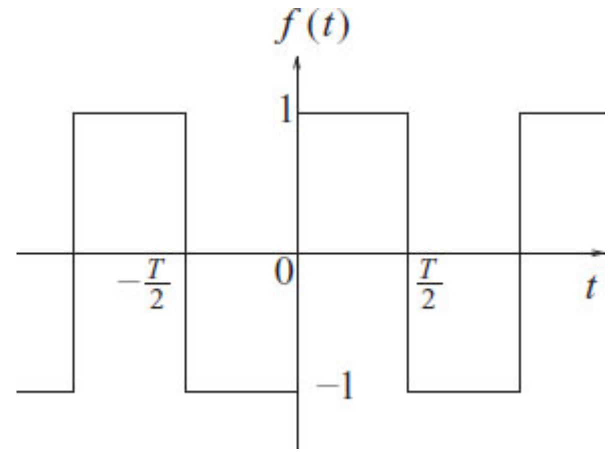
Mathematical Methods for Applied Biomedical Engineering
EN. 585.409

Module 01: Fourier Series I

(B) Symmetry Considerations

Example

Find the Fourier series for this periodic function



$$f(t) = \begin{cases} -1 & \text{for } -\frac{1}{2}T \le t < 0, \\ +1 & \text{for } 0 \le t < \frac{1}{2}T. \end{cases}$$

Take $x_o = -T/2,\ L = T$

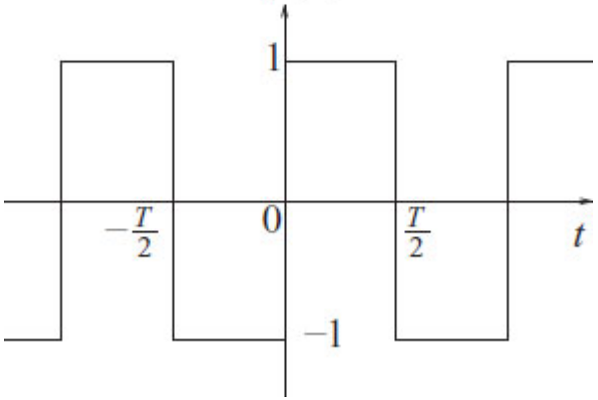
$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{2}{T}\int_{-T/2}^{0} -1\ dt + \frac{2}{T}\int_{0}^{T/2} 1\ dt = \cdots = 0$$

Alternatively if you recognize that the relation for a_0 is in essence that for the average value of the function, that

$$\frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \langle f(t) \rangle$$

Then by inspection of the graph we have $\frac{a_0}{2} = \langle f(t) \rangle = 0 \quad a_0 = 0$

Aside: Even and odd functions
Symmetry about the axis



Lets define even and odd functions

ODD: $f(t) = -f(-t)$ or $-f(t) = f(-t)$

EVEN: $g(t) = g(-t)$

The product of even and odd functions is given by

(ODD)(ODD) = EVEN: $f_1(t)\,f_2(t) = [-f_1(-t)]\,[-f_2(-t)] = f_1(-t)\,f_2(-t)$

(EVEN)(EVEN) = EVEN: $g_1(t)g_2(t) = g_1(-t)\,g_2(-t)$

(ODD)(EVEN) = ODD: $f(t)g(t) = -f(-t)g(-t)$

Fin

Aside

For g(t) an even function (period T) we can calculate the coefficients as follows.

$$a_r = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\cos\left(\frac{2\pi rt}{T}\right)dt = \frac{2}{T}[\int_{-T/2}^{0} g(t)\cos\left(\frac{2\pi rt}{T}\right)dt + \frac{2}{T}\int_{0}^{T/2} g(t)\cos\left(\frac{2\pi rt}{T}\right)dt]$$

Let $t \to -t$ in the first integral from $-T/2$ to 0

$$a_r = \frac{2}{T}[\int_{T/2}^{0} g(-t)\cos\left(\frac{2\pi r(-t)}{T}\right)d(-t) + \frac{2}{T}\int_{0}^{T/2} g(t)\cos\left(\frac{2\pi rt}{T}\right)dt]$$

$$= \frac{2}{T}[-\int_{0}^{T/2} g(t)\cos\left(\frac{2\pi rt}{T}\right)(-dt) + \frac{2}{T}\int_{0}^{T/2} g(t)\cos\left(\frac{2\pi rt}{T}\right)dt]$$

$$= \frac{2}{T}[\int_{0}^{T/2} g(t)\cos\left(\frac{2\pi rt}{T}\right)dt + \frac{2}{T}\int_{0}^{T/2} g(t)\cos\left(\frac{2\pi rt}{T}\right)dt] = \frac{4}{T}\int_{0}^{T/2} g(t)\cos\left(\frac{2\pi rt}{T}\right)dt$$

A similar calculation can be made for b_r however since sine is an odd function we get $b_r = 0$! (do it yourself)

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Therefore for an even function we have

$$a_0 = \frac{4}{L}\int_0^{L/2} f(x)\,dx \qquad a_r = \frac{4}{L}\int_0^{L/2} f(x)\cos\left(\frac{2\pi rx}{L}\right)dx \qquad b_r = 0$$

and for an odd function the coefficients are

$$a_r = 0 \qquad b_r = \frac{4}{L}\int_0^{L/2} f(x)\sin\left(\frac{2\pi rx}{L}\right)dt$$

Therefore for our problem we have

$$a_r = 0 \qquad b_r = \frac{4}{T}\int_0^{T/2}(1)\sin\left(\frac{2\pi rt}{T}\right)dt = \frac{4}{T}\left[\frac{-1}{\left(\frac{2\pi r}{T}\right)}\cos\left(\frac{2\pi rt}{T}\right)\right]\Bigg|_0^{T/2}$$

$$= \frac{-2}{\pi r}\left[\cos\left(\frac{2\pi r(T/2)}{T}\right) - \cos(0)\right] = \frac{-2}{\pi r}[\cos(\pi r) - 1] = \frac{2}{\pi r}[1-(-1)^r]$$

And the Fourier series is $$f(t) = \frac{4}{\pi}\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \cdots\right)$$

Another Example

Find the Fourier series for this periodic function $f(x) = 5 + x$
Whose primary period is from –L/2 to L/2

Note we can separate this into two functions $f_1(x) = 5, f_2(x) = x$
an odd function and 5 an even (and constant).

Aside: The property of linearity for Fourier series

$f(x) = f_1(x) + f_2(x)$
where the Fourier cofficients
for $f_1(x)$ are a_{r1} and b_{r1} and for $f_2(x)$ are a_{r2} and b_{r2}
then the Fourier cofficients for $f(x)$ are
$\mathrm{a}_r = \mathrm{a}_{r1} + a_{r2}$ and $\mathrm{b}_r = b_{r1} + b_{r2}$
and for $f(x) = cf_1(x)$ the Fourier cofficients for are $\mathrm{a}_r = c\mathrm{a}_{r1}$, $b_r = cb_{r1}$

This comes from the linearity property of integrals

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Therefore the coefficients can be calculate separately as

$$f_1(x) = 5 \text{ and } \frac{a_{01}}{2} = \left\langle f(t) \right\rangle = 5 \text{ gives } a_{01} = 10$$

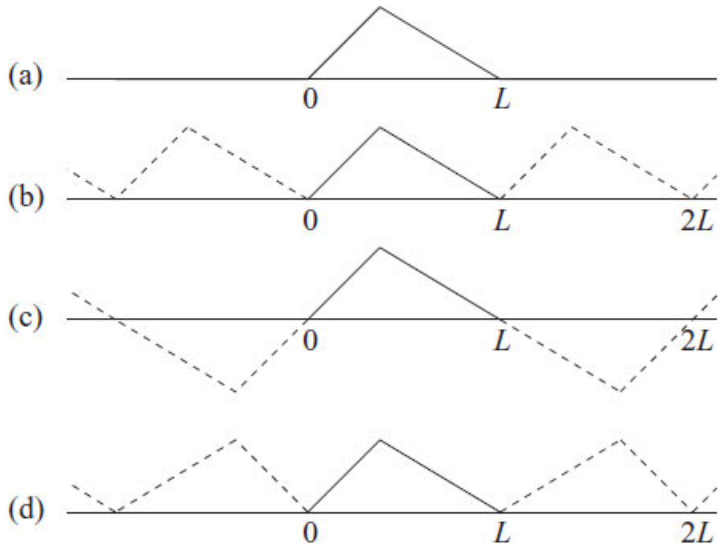
We also have $a_{r1} = 0\ (r \neq 0),\ b_{r1} = 0$

$$f_2(x) = x \text{ and } a_{r2} = 0,\ \mathrm{b}_{r2} = \frac{4}{T}\int_0^{L/2} x \sin\left(\frac{2\pi r x}{L}\right) dx = \cdots = \frac{-2L}{\pi r}\cos(\pi r) = \frac{-2L}{\pi r}(-1)^r$$

And the Fourier series is given by

$$f(x) = 5 + \sum_{r=1}^{\infty} \frac{-2L}{\pi r}(-1)^r \sin\frac{2\pi r x}{L}$$

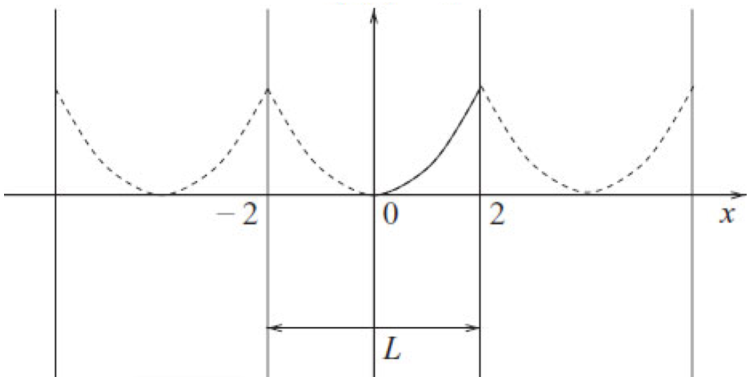
Non-Periodic



If we wish to find the Fourier series of a non-periodic function only within a fixed range then we may *continue* the function outside the range so as to make it periodic.

It must be added that the continuation must not be discontinuous at the end-points of the interval of interest; if it is, the series will not converge to the required value there.

Example – extension of non-periodic function



$$f(x) = x^2 \;\; 0 < x \le 2$$

Taking an even extension (L = 4)

L/2

$$a_0 = \frac{4}{4}\int_0^2 x^2 dx = \frac{8}{3}$$

L=4

$$a_r = \frac{4}{4}\int_0^2 x^2 \cos\left(\frac{2\pi r x}{4}\right) dx = \frac{16}{\pi^2 r^2}\cos(\pi r) \equiv \frac{16}{\pi^2 r^2}(-1)^r$$

By Table; Although it could be done by integration by parts