4.22

$$\text{T,Period} = \pi/2$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi nit/(\pi/2)} = \sum_{n=-\infty}^{\infty} c_n e^{4nit}$$

>>>>> Looks good as far as you got except for factor of 2 in exponent.

Can you finish your calculation to get the c_n

$$c_n = \frac{1}{\pi/2}\int_0^{\pi/2}(\sin t)e^{-4nit}dt = \frac{2}{\pi}\left[\frac{e^{-4nit}}{(-4ni)^2+1^2}(-4ni\sin t - \cos t)\right]_0^{\pi/2} = \cdots$$

$$= \frac{2}{\pi}\frac{(-4ni)e^{-2\pi ni}+1}{1-16n^2} = \frac{2}{\pi}\frac{4ni-1}{16n^2-1}$$

where $e^{-2\pi ni} = \cos(-2\pi n) + i\sin(-2\pi ni) = 1 + i0 = 1$

Therefore $c_n = \dfrac{2}{\pi}\dfrac{4ni-1}{16n^2-1}$ (note includes n =0 case, $c_0 = \dfrac{2}{\pi}$)

Note $c^*_{-n} = \dfrac{2}{\pi}\dfrac{4(-n)(-i)-1}{16(-n)^2-1} = c_n$

For the next part use the resultant Fourier series for f(t) and set t = 0

$$f(t) = \sin t = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1}e^{4nit} \rightarrow (\text{at } t = 0)$$

$$\sin 0 = 0 = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1}e^{4ni0} = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1} \rightarrow 0 = \sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1}$$

Seperate sum into three parts

$$\sum_{n=-1}^{-\infty}\frac{4ni-1}{16n^2-1} + \sum_{n=0}^{0}\frac{4ni-1}{16n^2-1} + \sum_{n=1}^{\infty}\frac{4ni-1}{16n^2-1},$$

In first sum let n $\rightarrow$ -n and reindex to $\displaystyle\sum_{n=-1}^{-\infty}\frac{4ni-1}{16n^2-1} \rightarrow \sum_{n=1}^{\infty}\frac{4(-n)i-1}{16(-n)^2-1} = \sum_{n=1}^{\infty}\frac{-4ni-1}{16n^2-1}$

second sum above $\displaystyle\sum_{n=0}^{0}\frac{4ni-1}{16n^2-1} = \sum_{n=0}^{0}\frac{4(0)i-1}{16(0)^2-1} = 1$

Therefore adding all sums

$$0 = \sum_{n=1}^{\infty}\frac{-4ni-1}{16n^2-1} + 1 + \sum_{n=-1}^{\infty}\frac{4ni-1}{16n^2-1} = \sum_{n=1}^{\infty}\frac{-4ni-1}{16n^2-1} + \frac{4ni-1}{16n^2-1} + 1$$

$$\text{or } \sum_{n=1}^{\infty}\frac{-4ni-1+4ni-1}{16n^2-1} + 1 = 0 \rightarrow \sum_{n=1}^{\infty}\frac{-2}{16n^2-1} = -1 \rightarrow \sum_{n=1}^{\infty}\frac{1}{16n^2-1} = \frac{1}{2}$$

For the next part use the resultant Fourier series for f(t) and set $t = \frac{\pi}{2}$

$$f(t) = \sin t = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1}e^{4nit} \to (\text{at } t = \frac{\pi}{2})$$

$$\sin\frac{\pi}{2} = 1 = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1}e^{4ni\frac{\pi}{2}} = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1}e^{2\pi ni} \to 1 = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1}e^{2\pi ni}$$

Seperate sum into three parts

$$\sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1}e^{2\pi ni} = \sum_{n=-1}^{-\infty}\frac{4ni-1}{16n^2-1}e^{2\pi ni} + \sum_{n=0}^{0}\frac{4ni-1}{16n^2-1}e^{2\pi ni} + \sum_{n=1}^{\infty}\frac{4ni-1}{16n^2-1}e^{2\pi ni}$$

In first sum let $n \to -n$ and reindex to $\sum_{n=-1}^{-\infty}\frac{4ni-1}{16n^2-1}e^{2\pi ni} \to \sum_{n=1}^{\infty}\frac{4(-n)i-1}{16(-n)^2-1}e^{-2\pi ni} = \sum_{n=1}^{\infty}\frac{-4ni-1}{16n^2-1}e^{-2\pi ni}$

$e^{-2\pi ni} = \cos(-2\pi n) + i\sin(-2\pi ni) = 1 + i0 = 1$ and first sum is $\sum_{n=1}^{\infty}\frac{-4ni-1}{16n^2-1}$ as before

second sum above $\sum_{n=0}^{0}\frac{4ni-1}{16n^2-1} = \sum_{n=0}^{0}\frac{4(0)i-1}{16(0)^2-1} = 1$ and third sum is same, therefore similar to previous result

$$\sum_{n=1}^{\infty}\frac{-2}{16n^2-1} + 1$$

Substitution gives $1 = \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{4ni-1}{16n^2-1}e^{2\pi ni} = \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\frac{-2}{16n^2-1} + 1\right] = \frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{1}{16n^2-1} + \frac{2}{\pi}$

That is $1 - \frac{2}{\pi} = \frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{1}{16n^2-1}$ or $\sum_{n=1}^{\infty}\frac{1}{16n^2-1} = \frac{-\pi}{4} + \frac{1}{2}$

Compare with previous result $\sum_{n=1}^{\infty}\frac{1}{16n^2-1} = \frac{1}{2}$

CAN ANYONE TELL WHY THESE DO NOT MATCH (hint also see problem HW4.20)

4.25

Use equation on page 182 and split into three parts

$$\frac{1}{L}\int_0^L f(x)g^*(x)dx = \sum_{r=-\infty}^{\infty} c_r\gamma_r^* = \sum_{r=-1}^{-\infty} c_r\gamma_r^* + c_0\gamma_0^* + \sum_{r=1}^{\infty} c_r\gamma_r^*$$

Substitute forms (similar to Eq. 4.13)

$$c_r = \frac{1}{2}(a_r - ib_r); c_{-r} = \frac{1}{2}(a_r + ib_r); c_0 = \frac{1}{2}a_0$$

$$\gamma_r = \frac{1}{2}(\alpha_r - i\beta_r); c_{-r} = \frac{1}{2}(\alpha_r + i\beta_r); \gamma_0 = \frac{1}{2}\alpha_0$$

$$\frac{1}{L}\int_0^L f(x)g^*(x)dx = \sum_{r=-\infty}^{\infty} c_r\gamma_r^* = \sum_{r=-1}^{-\infty} c_r\gamma_r^* + c_0\gamma_0^* + \sum_{r=1}^{\infty} c_r\gamma_r^* =$$

$$\sum_{r=-1}^{-\infty} c_r\gamma_r^* + \sum_{r=1}^{\infty} c_r\gamma_r^* = (\text{let r go to -r in sum}) = \sum_{r=1}^{\infty} c_{-r}\gamma_{-r}^* + \sum_{r=1}^{\infty} c_r\gamma_r^* = \sum_{r=1}^{\infty}(c_{-r}\gamma_{-r}^* + c_r\gamma_r^*)$$

$$\frac{1}{L}\int_0^L f(x)g^*(x)dx = \frac{1}{2}a_0\frac{1}{2}\alpha_0 + \sum_{r=1}^{\infty}(c_{-r}\gamma_{-r}^* + c_r\gamma_r^*) =$$

$$\frac{1}{4}a_0\alpha_0 + \sum_{r=1}^{\infty}\frac{1}{2}(a_r + ib_r)\frac{1}{2}(\alpha_r - i\beta_r) + \frac{1}{2}(a_r - ib_r)\frac{1}{2}(\alpha_r + i\beta_r) = \frac{1}{4}a_0\alpha_0 + \sum_{r=1}^{\infty}\frac{1}{2}(a_r\alpha_r + b_r\beta_r)$$

That is

$$\frac{1}{L}\int_0^L f(x)g^*(x)dx = \frac{1}{4}a_0\alpha_0 + \frac{1}{2}\sum_{r=1}^{\infty}(a_r\alpha_r + b_r\beta_r)$$

For part (a)

$L = 2\pi$ and use equation on pg 173 to compare

$$b_m = \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin mx\,dx$$

For (b)

See equation in problem, on right hand side use

Interval is -1 to 1 and $L=2$

$f(x)=x,\ -1\le x\le 1$

$$g(x)=\begin{cases} -1 & -1\le x<0 \\ 1 & 0<x\le 1 \end{cases}$$

So for our problem $\frac{1}{L}\int_{-L/2}^{L/2} f(x)g(x)dx=\frac{2}{L}\int_0^{L/2} f(x)g(x)dx$

with $L=2$ gives $\int_0^1 x\cdot 1dx=\frac{1}{2}x^2\Big|_0^1=\frac{1}{2}$

$$f(x)=x,\text{odd } -1<x<1 \rightarrow a_r=0; b_r=\frac{2\cdot 2}{2}\int_0^1 x\sin\left(\frac{2\pi rx}{2}\right)dx=\cdots=\frac{-2(-1)^r}{\pi r}$$

$$g(x)=\begin{cases} -1 & -1<x<0 \\ 1 & 0<x<1 \end{cases} \rightarrow \alpha_r=0; \beta_r=\frac{2\cdot 2}{2}\int_0^1 1\sin\left(\frac{2\pi rx}{2}\right)dx=\cdots=\frac{-2[(-1)^r-1]}{\pi r}$$

Now use this all this information to verify Parsevals Th. in our case

$$\frac{1}{2}(\text{from above})=\int_0^1 f(x)g(x)dx=\frac{1}{4}a_0\alpha_0+\frac{1}{2}\sum_{r=1}^{\infty}(a_r\alpha_r+b_r\beta_r)=\frac{1}{2}\sum_{r=1}^{\infty}b_r\beta_r \ (\text{since } a_r\alpha_r=0,\text{above})$$

Therefore $\frac{1}{2}=\frac{1}{2}\sum_{r=1}^{\infty}b_r\beta_r$ and substitution gives

$$\frac{1}{2}=\frac{1}{2}\sum_{r=1}^{\infty}\frac{-2(-1)^r}{\pi r}\frac{-2[(-1)^r-1]}{\pi r}=\frac{2}{\pi^2}\sum_{r=1,odd}^{\infty}\frac{2}{r^2}=\frac{4}{\pi^2}\sum_{r=1,odd}^{\infty}\frac{1}{r^2} \text{ or } \frac{1}{2}=\frac{4}{\pi^2}\sum_{r=1,odd}^{\infty}\frac{1}{r^2}$$

You will need the following result from $\sum_{odd}\frac{1}{r^2}=\frac{\pi^2}{8}$ to finish comparison

$\frac{4}{\pi^2}\sum_{r=1,odd}^{\infty}\frac{1}{r^2}=\frac{4}{\pi^2}\frac{\pi^2}{8}=\frac{1}{2}$ which verifies Parseval's Th.