Assigned: March 1, 2011
Due: March 9, 2011

MATH 480: Homework 5
SPRING 2011

Fourier Series:

1. (a) Find the Fourier *sine* series for $f(x) = 1 - x$ defined on the interval $0 \leq x \leq 1$.
   (b) In MATLAB, plot the first 20 terms and the first 200 terms of the sine series in the interval $-3 \leq x \leq 3$.
   (c) To what value does the series converge at $x = 0$?

2. (a) Find the Fourier *cosine* series for $f(x) = 1-x$ defined on the interval $0 \leq x \leq 1$.
   (b) In MATLAB, plot the first 20 terms and the first 200 terms of the cosine series in the interval $-3 \leq x \leq 3$.
   (c) To what value does the series converge at $x = 0$?

3. (a) Find the Fourier series for

$$f(x) = \begin{cases} 0 & \text{if } \ -1 \leq x < 0 \\ 1 - x^2 & \text{if } \ 0 < x \leq 1 \end{cases}$$

   defined on the interval $-1 \leq x \leq 1$.
   (b) In MATLAB, plot the first 20 terms and the first 200 terms of the Fourier series in the interval $-3 \leq x \leq 3$.
   (c) To what value does the series converge at $x = 0$?

4. The Fourier series of the function $f(x) = \cos(ax)$ on the interval $[-\pi, \pi]$, when a is not an integer, is given by

$$\cos(ax) = \frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2 - a^2}\cos(nx)\right] \quad \text{for} \quad -\pi \leq x \leq \pi.$$

   (a) Differentiate both sides of this equation with respect to x, differentiating the series term by term, to find the Fourier series for $\sin(ax)$:

$$\sin(ax) = -\frac{2\sin(a\pi)}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2 - a^2}\sin(nx)\right] \quad \text{for} \quad -\pi < x < \pi.$$

   (b) Explain why this method for computing the Fourier series is valid.
   (c) If you know the Fourier series for $\sin(ax)$ given in (a), why can you not differentiate it term by term with respect to x to derive the Fourier series for $\cos(ax)$.

(d) Now consider the Fourier series of $\sin(ax)$ given in (a) as known. Explain why it can be integrated term by term to get the Fourier expansion of $\cos(ax)$.

(e) Carry out this term by term integration from 0 to x, and use it to show that

$$A_0 = \frac{\sin(a\pi)}{a\pi} = 1 + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 - a^2}.$$

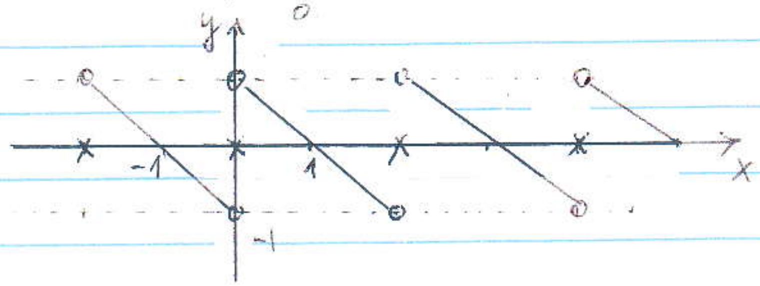
5. Let c_n be the coefficients of the complex Fourier series of $f(x)$. Show that if $f(x)$ is a real-valued function, then $c_{-n} = \bar{c}_n$.

Math 480: HW #5 SOLUTIONS

#1 $f(x) = 1 - x \qquad 0 \le x \le 1 \qquad L = 1$

(a) $f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin n\pi x$

where $B_n = \frac{2}{1} \int_0^1 (1-x) \sin n\pi x \, dx = \frac{2}{n\pi}$



sketch of Fourier sine series

(b) see attached graphs

(c) at $x=0$, Fourier series $\to \frac{1}{2}[f(0^+) + f(0^-)] =$
$= \frac{1}{2}[1 + (-1)] = 0$

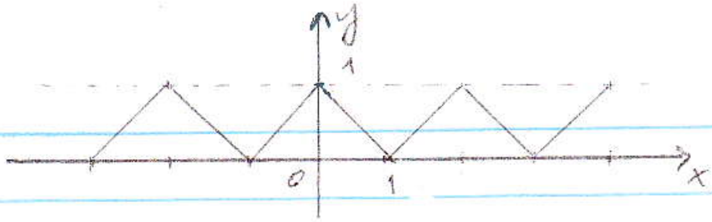
#2 $f(x) = 1 - x \qquad 0 \le x \le 1$

$f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x$

$A_0 = \frac{1}{1} \int_0^1 (1-x)\, dx = \frac{1}{2}$

$A_n = 2 \int_0^1 (1-x) \cos n\pi x \, dx = \frac{2[1-(-1)^n]}{n^2\pi^2}$

(b) see attached graphs



sketch of Fourier cosine series

(c) Fourier series$\Big|_{x=0} = 1$

#3 $f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1-x^2, & 0 < x \le 1 \end{cases}$ $L = 2$

(a) $f(x) \sim A_0 + \sum_{n=1}^{\infty} (A_n \cos n\pi x + B_n \sin n\pi x)$

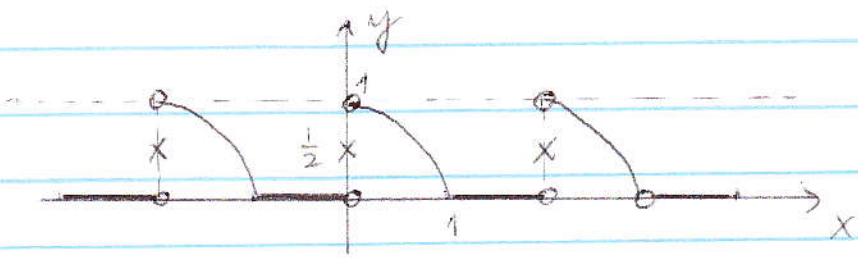
$A_0 = \frac{1}{2}\int_0^1 (1-x^2)\,dx = \frac{1}{3}$

$A_n = \int_0^1 (1-x^2)\cos n\pi x\,dx = \frac{2(-1)^{n+1}}{n^2\pi^2}$

$B_n = \int_0^1 (1-x^2)\sin n\pi x\,dx = \frac{2(-1)^{n+1} + n^2\pi^2 + 2}{n^3\pi^3}$

(b) see attached graphs

(c)



sketch of Fourier series

Fourier series$\Big|_{x=0} = \frac{1}{2}$

Problem # 1 (b)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   HW # 5: pr1b.m
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all; clf;
x=-3:1e-3:3;
NN=20;
FS=0;

for n=1:NN
    Bn=2/(n*pi);
    FS=FS+Bn*sin(n*pi*x);
end

figure(1); clf(1)
subplot(2,1,1), plot(x,FS);
xlabel('x')
ylabel(['First ',num2str(NN), ' terms of Fourier sine series'])

title(['First ',num2str(NN), ...
     ' terms of the Fourier sine series for the function f(x)=1-x defined on 0<=x<=1'])

axis([-3 3 -1.5 1.5])
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

NN=200;
FS=0;

for n=1:NN
    Bn=2/(n*pi);
    FS=FS+Bn*sin(n*pi*x);
end

subplot(2,1,2), plot(x,FS);
 xlabel('x')
 ylabel(['First ' ,num2str(NN), ' terms of Fourier sine series'])

 title(['First ',num2str(NN), ...
     ' terms of the Fourier sine series for the function f(x)=1-x defined on 0<=x<=1'])

axis([-3 3 -1.5 1.5])
print -depsc2 pr1b_graph.eps
```

Problem # 2 (b)

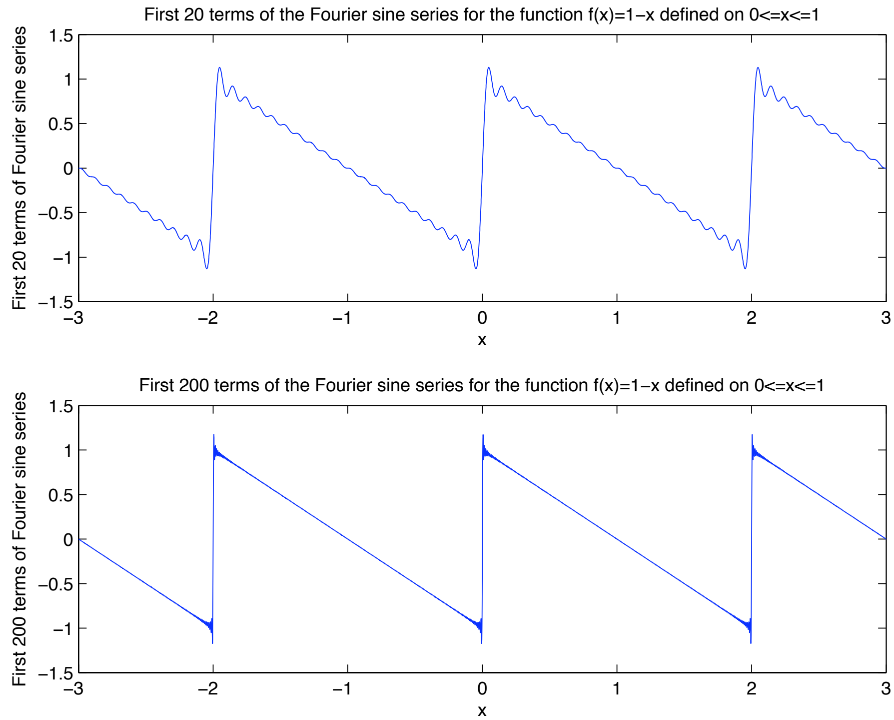


Figure 1: Problem 1 (b): Fourier sine series for $f(x) = 1 - x$. Note Gibbs phenomenon at points of discontinuity of the Fourier sine series: overshoot of about 0.2. This overshoot is about 9% of the function jump: $2 * 0.9 = 0.18$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  HW #5: pr2b.m
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all; clf;
x=-3:1e-3:3;

NN=20;
FS=1/2;

for n=1:NN
    An=2*(1-(-1)^n)/(n^2*pi^2);
    FS=FS+An*cos(n*pi*x);
end

figure(1); clf(1)
subplot(2,1,1), plot(x,FS);
xlabel('x')
ylabel(['First ',num2str(NN), ' terms of Fourier cosine series'])

title(['First ',num2str(NN), ...
```

```
' terms of the Fourier cosine series for the function f(x)=1-x defined on 0<=x<=1'])

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

NN=200;
FS=1/2;

for n=1:NN An=2*(1-(-1)^n)/(n^2*pi^2);
    FS=FS+An*cos(n*pi*x);
end

subplot(2,1,2), plot(x,FS);
xlabel('x')
ylabel(['First ',num2str(NN), ' terms of Fourier cosine series'])

title(['First ',num2str(NN), ...
' terms of the Fourier cosine series for the function f(x)=1-x defined on 0<=x<=1'])

print -depsc2 pr2b_graph.eps
```

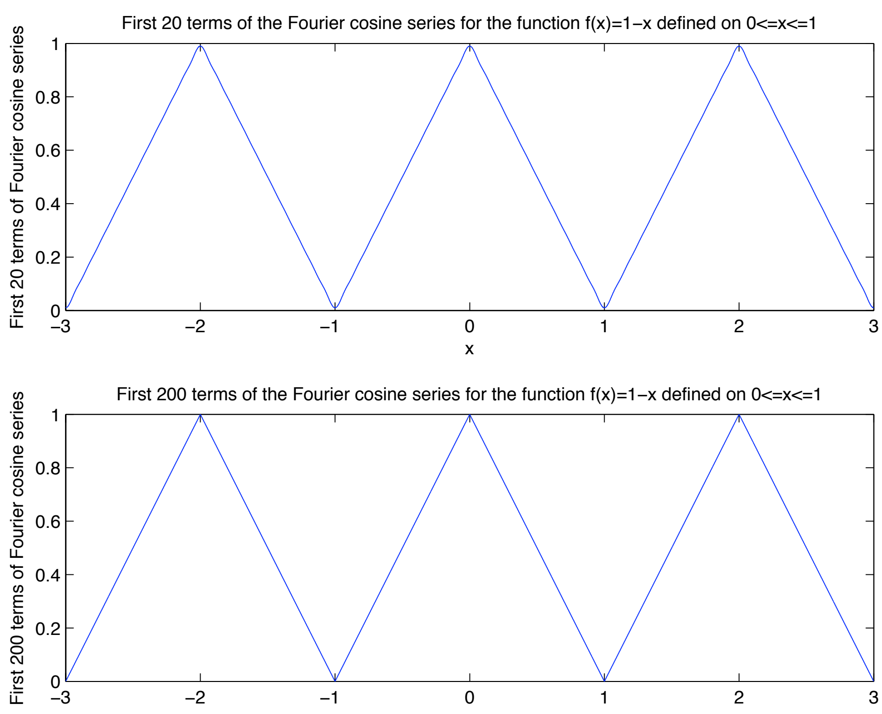


Figure 2: Problem 2 (b): Fourier cosine series for $f(x) = 1 - x$. There is no Gibbs phenomenon since Fourier cosine series for $f(x)$ is continuous

Problem # 3 (b)

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  HW # 5: pr3b.m
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all; clf;
x=-3:1e-3:3;

NN=20;
FS=1/3;

for n=1:NN
  An=2*(-1)^(n+1)/(n^2*pi^2);
  Bn=(2*(-1)^(n+1)+n^2*pi^2+2)/(n^3*pi^3);
  FS=FS+An*cos(n*pi*x)+Bn*sin(n*pi*x);
end

figure(1);clf(1)
subplot(2,1,1), plot(x,FS);
xlabel('x')
ylabel(['First ',num2str(NN), ' terms,of Fourier series'])

title(['First ',num2str(NN), ...
' terms of the Fourier series for the function f(x)=1-x defined on 0<=x<=1'])

axis([-3 3 -0.2 1.2])
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

NN=200;
FS=1/3;

for n=1:NN
  An=2*(-1)^(n+1)/(n^2*pi^2);
  Bn=(2*(-1)^(n+1)+n^2*pi^2+2)/(n^3*pi^3);
  FS=FS+An*cos(n*pi*x)+Bn*sin(n*pi*x);
end

subplot(2,1,2), plot(x,FS);
xlabel('x')
ylabel(['First ',num2str(NN), ' terms of Fourier series'])

title(['First ',num2str(NN), ...
' terms of the Fourier series for the function f(x)=1-x defined on 0<=x<=1'])

axis([-3 3 -0.2 1.2])
```

```
print -depsc2 pr3b_graph.eps
```

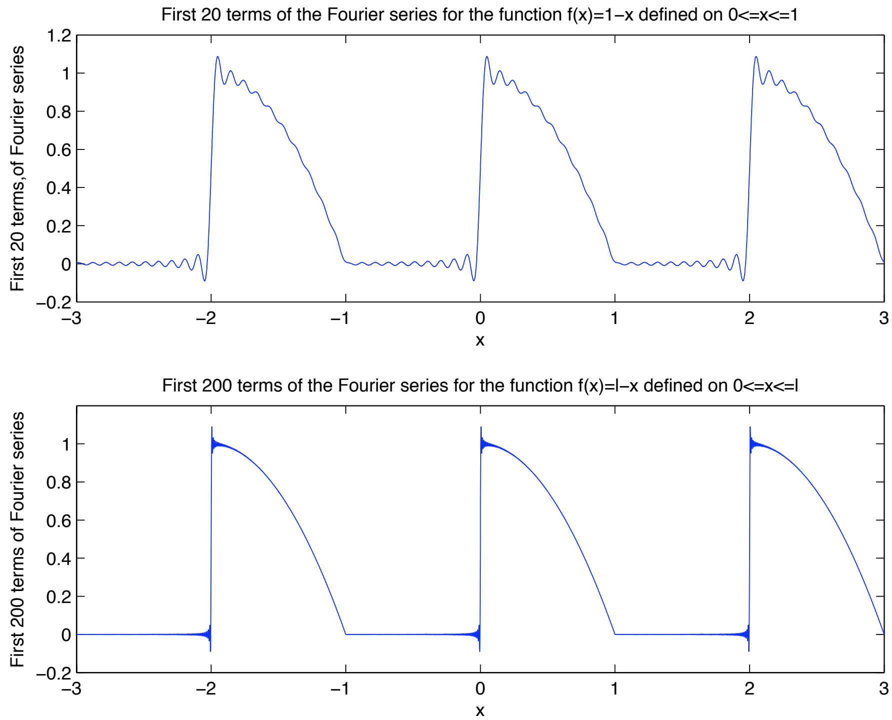


Figure 3: Problem 3 (b): Fourier series for $f(x)$. Note Gibbs phenomenon at points 0, ± 2, $\pm 4, \ldots, \pm 2n, \ldots$, n integer, where Fourier series is discontinuous. The overshoot there is about 0.1 that agrees with the theoretical prediction since function jump is 1, so that 9% of it is $1 * 0.09 = 0.09 \approx 0.1$. At points ± 1, $\pm 3, \ldots, \pm(2n+1), \ldots$, there is no Gibbs phenomenon since Fourier series is continuous there.

#4 $f(x) = \cos(ax) \quad x \in [-\pi, \pi] \quad L = \pi$

$$\cos(ax) = \frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-a^2}\cos(nx)\right], \quad -\pi < x < \pi$$

(a) $\frac{d}{dx}$ of both sides:

$$-a\sin(ax) \sim \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-n)}{n^2-a^2}\sin nx$$

$$\therefore \sin(ax) = -\frac{2\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \cdot n}{n^2-a^2}\sin nx, \quad -\pi < x < \pi$$

(b) term-by-term differentiation is valid since $f(x) = \cos(ax)$ is continuous on $-\pi \le x \le \pi$ and $f(L) = f(-L) = \cos aL$

(c) $g(x) = \sin(ax)$
Fourier series for $g(x)$ cannot be differentiated term-by-term since $g(-L) \neq g(L)$ even though $\sin(ax)$ is continuous on $[-\pi, \pi]$. Moreover, $\sin(ax)$ has Fourier sine series since $\sin(ax)$ is an odd function. To differentiate Fourier sine series, we need $g(0) = g(L) = 0$ but
$g(0) = \sin 0 = 0$ ✓ and $g(L) = g(\pi) = \sin a\pi \neq 0$
since a is not an integer

(d) Fourier series can always be integrated term-by-term

(e) $\sin(ax) = -\frac{2\sin(a\pi)}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2-a^2}\sin nx\right], \quad -\pi<x<\pi$

$$\int_0^x \sin(a\xi)d\xi = -\frac{2\sin(a\pi)}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2-a^2}\int_0^x \sin n\xi\, d\xi\right]$$

$$\int_0^x \sin(a\xi)\,d\xi = -\frac{1}{a}\cos(a\xi)\Big|_{\xi=0}^{\xi=x} = -\frac{1}{a}(\cos ax - 1)$$

$$\int_0^x \sin(n\xi)\,d\xi = -\frac{1}{n}\cdot\cos(n\xi)\Big|_{\xi=0}^{\xi=x} = -\frac{1}{n}(\cos nx - 1)$$

$$\therefore\ -\frac{1}{a}(\cos ax - 1) = -\frac{2\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2-a^2}\left(-\frac{1}{n}\right)(\cos nx - 1)$$

$$\cos ax = 1 + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-a^2}(\cos nx - 1) =$$

$$= 1 + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-a^2}\cos nx + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2}$$

$$= A_0 + \sum_{n=1}^{\infty} A_n\cos nx$$

where

$$A_0 = 1 + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2}$$

$$A_n = \frac{2a\sin(a\pi)}{\pi}\,\frac{(-1)^{n+1}}{n^2-a^2}$$

Compare with the original Fourier series for $\cos(ax)$

$\therefore$

$$A_0 = \frac{2a\sin(a\pi)}{\pi}\cdot\frac{1}{2a^2} = 1 + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2}$$

i.e. $$A_0 = \frac{\sin(a\pi)}{a\pi} = 1 + \frac{2a\sin(a\pi)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-a^2}$$

#5

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}$$

where
$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{in\pi x/L}\,dx$$

Let $f(x)$ be real-valued. $\Rightarrow \overline{f(x)} = f(x)$

$$\overline{c_n} = \frac{1}{2L}\int_{-L}^{L} \overline{f(x) e^{in\pi x/L}}\,dx = \frac{1}{2L}\int_{-L}^{L} \overline{f(x)} \cdot \overline{e^{in\pi x/L}}\,dx =$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\,dx = c_{-n}$$