4.26

$$E_m = \int_{-\pi}^{\pi}[f(x) - \sum_{n=1}^{m} b_n \sin nx]^2 dx$$

is the residual difference (or error) of the function over its primary interval -π to π

We minimize this quantity with respet to a particular b_n in this case b_p

we take a derivative with respect to b_p (a particular p in sum of n = 1 to m <<< IMPORTANT)

The partial derivative is moved inside integral (assume integral converges) and applied!

$$\frac{\partial E_m}{\partial b_p} = \int_{-\pi}^{\pi} 2[f(x) - \sum_{n=1}^{m} b_n \sin nx][-\sin px]dx = 0$$

$$-2\int_{-\pi}^{\pi} f(x)\sin px\,dx + 2\int_{-\pi}^{\pi}[\sum_{n=1}^{m} b_n \sin nx]\sin px\,dx = 0$$

The second integral has only a contribution when n = p!!!!

The first integral $-2\int_{-\pi}^{\pi} f(x)\sin px\,dx$ and the second integral reduces to $2b_p\int_{-\pi}^{\pi}\sin^2 px\,dx = 2b_n\pi$

$$-2\int_{-\pi}^{\pi} f(x)\sin px dx + 2b_p\pi = 0 \rightarrow \; b_p = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin px\,dx$$

A quick approximate way to calculate the error is as follows and allows us to gauge the convergence as we add terms

Use the following form for error, b_r above and m = 3 and do integral

$$b_r = \frac{4}{\pi r^3}[(-1)^r - 1] = \begin{cases} 0 & r \text{ even} \\ \dfrac{-8}{\pi r^3} & r \text{ odd} \end{cases}$$ {try it yourself - answer on last page}

$$E_m = \int_{-\pi}^{\pi}[\sum_{r=1}^{\infty} b_r \sin rx - \sum_{r=1}^{m} b_r \sin rx]^2 dx = \int_{-\pi}^{\pi}[\sum_{r=m+1}^{\infty} b_r \sin rx]^2 dx$$

is calculated as entire sum minus the first three term!! And can be further simplified

$$\equiv \sum_{r=m+1}^{\infty} b_r^2 \int_{-\pi}^{\pi} \sin^2 rxdx$$ <<< INCLUDED EXPLANATION FOR THIS STEP BELOW

where I dropped cross terms, e.g. $\sin 2x \sin 3x$ since the integral for these type of terms is zero because of the orthogonality condition leaving only terms with $\sin rx \sin rx \equiv \sin^2 rx$ as part of sum!!

For the second part we have

Note since no contribution from even values of r lower limit starts not at 3+1 but r = 5!!

$$E_3 = \sum_{\substack{r=3+1 \\ \text{odd only}}}^{\infty} b_r^2 \int_{-\pi}^{\pi} \sin^2 rx dx \equiv \sum_{r=5,\text{ odd only}}^{\infty} b_r^2 \int_{-\pi}^{\pi} \sin^2 rx dx$$

Using $\int_{-\pi}^{\pi} \sin^2 rx dx = \pi$, for any r and substitute for b_r

$$E_3 = \sum_{r=5,\text{ odd only}}^{\infty} \left(\frac{-8}{\pi r^3} \right)^2 \pi = \frac{8^2}{\pi} \sum_{r=5,\text{ odd only}}^{\infty} \frac{1}{r^6} \approx \frac{8^2}{\pi} \sum_{r=5,\text{ odd only}}^{R} \frac{1}{r^6}$$

Taking upper bound for sum as R=5 we get error as .0013

Taking upper bound for sum as R=7 we get error as .0015

Taking upper bound for sum as R=9 we get error as .0015

$$E_3 = \frac{64}{\pi} \sum_{r=5}^{\infty} \frac{1}{r^6} \cong .0015$$

which is a very good approximation since sum over $\frac{1}{r^6}$ converges very quickly.

appendix – calculation of coefficients:

$$b_r = \frac{4}{\pi r^3}[(-1)^r - 1] = \begin{cases} 0 & \text{r even} \\ \dfrac{-8}{\pi r^3} & \text{r odd} \end{cases}$$

Here is the calculation for b_r FIRST $f(x) = \begin{cases} -x(\pi - x) & -\pi \le x < 0 \\ x(x-\pi) & 0 \le x < \pi \end{cases}$ is ODD!!!!

$$b_r(\text{for odd}) = 2\frac{2}{L}\int_0^{\pi} x(x-\pi)\sin\frac{2\pi r}{L}x dx;\ L{=}2\pi \rightarrow b_r = \frac{2}{\pi}\int_0^{\pi} x(x-\pi)\sin rx dx \quad \text{Using table}$$

$$b_r = \frac{2}{\pi}\int_0^{\pi} x(x-\pi)\sin rx dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin rx dx - 2\int_0^{\pi} x \sin rx dx =$$

$$\frac{2}{\pi}\left\{\left[\frac{2}{r^2}\sin r\pi - \left(\frac{r^2\pi^2 - 2}{r^3}\right)\cos r\pi\right] - \left[\frac{2}{r^2}\sin r0 - \left(\frac{0^2\pi^2 - 2}{r^3}\right)\cos r0\right]\right\} -$$

$$2\left\{\left[\frac{1}{r^2}\sin r\pi - \frac{\pi}{r}\cos r\pi\right] - \left[\frac{1}{r^2}\sin r0 - \frac{0}{r}\cos r0\right]\right\} =$$

$$\frac{2}{\pi}\left\{\left[\frac{2}{r^2}0 - \left(\frac{r^2\pi^2 - 2}{r^3}\right)\cos r\pi\right] - \left[\frac{2}{r^2}0 - \left(\frac{0^2\pi^2 - 2}{r^3}\right)\cos r0\right]\right\} - 2\left\{\left[\frac{1}{r^2}0 - \frac{\pi}{r}\cos r\pi\right] - \left[\frac{1}{r^2}0 - 0\right]\right\} =$$

$$\frac{2}{\pi}\left\{\left[-\left(\frac{r^2\pi^2 - 2}{r^3}\right)\cos r\pi\right] - \left[-\left(\frac{0^2\pi^2 - 2}{r^3}\right)\cos r0\right]\right\} - 2\left\{\left[-\frac{\pi}{r}\cos r\pi\right] - 0\right\} =$$

$$\frac{2}{\pi}\left[-\left(\frac{r^2\pi^2 - 2}{r^3}\right)\cos r\pi - \frac{2}{r^3}1\right] + \frac{2\pi}{r}\cos r\pi = -\frac{2\pi}{r}\cos r\pi + \frac{4}{\pi r^3}\cos r\pi - \frac{4}{\pi r^3}1 + \frac{2\pi}{r}\cos r\pi =$$

$$\frac{4}{\pi r^3}\cos r\pi - \frac{4}{\pi r^3} = \frac{4}{\pi r^3}\left(\cos r\pi - 1\right) \equiv \frac{4}{\pi r^3}[(-1)^r - 1]$$