4.16

Take even extension for $f(t)=\cosh(t-1)$, Period $=2$

$$a_r=\frac{2}{2}\int_{-1}^{1}\cosh(t-1)t\cos\frac{2\pi rt}{2}dt$$

Since even $a_r=\frac{2\cdot 2}{2}\int_{0}^{1}\cosh(t-1)t\cos\frac{2\pi rt}{2}dt,\ b_r=0$

Use $2\cosh(t-1)=e^{t-1}+e^{-(t-1)}=e^{-1}e^{t}+e^{1}e^{-t}$

$$a_r=\int_0^1 2\cosh(t-1)t\cos\pi rt\,dt=\int_0^1[e^{-1}e^{t}+e^{1}e^{-t}]\cos\pi rt\,dt=$$

$$\int_0^1[e^{-1}e^{t}+e^{1}e^{-t}]\cos\pi rt\,dt=e^{-1}\int_0^1 e^{t}\cos\pi rt\,dt+e\int_0^1 e^{-t}\cos\pi rt\,dt=$$

$\cdots$(use table for integrals) $=\frac{-e^{-1}}{1+(\pi r)^2}+\frac{e^{1}}{1+(\pi r)^2}=\frac{2\sinh 1}{1+(\pi r)^2}$; Note for r=0 $a_0=\sinh 1$

Therefore $\cosh(t-1)=\sinh 1+2\sinh 1\sum_{r=1}^{\infty}\frac{\cos\pi rt}{1+(\pi r)^2}$

or $\sum_{r=1}^{\infty}\frac{\cos\pi rt}{1+(\pi r)^2}=\frac{\cosh(t-1)-\sinh 1}{2\sinh 1}$

Next use t = 0 then cos(0) = 1

we get $\sum_{r=1}^{\infty}\frac{\cos\pi rt}{1+(\pi r)^2}\to\sum_{r=1}^{\infty}\frac{1}{1+(\pi r)^2}$

and $\frac{\cosh(t-1)-\sinh 1}{2\sinh 1}\to\frac{\cosh(-1)-\sinh 1}{2\sinh 1}=\frac{1}{e^2-1}$

using identities $\cosh(-1)=\frac{e^{-1}+e^{1}}{2}$ $\sinh(1)=\frac{e^{1}-e^{-1}}{2}$

Therefore $\sum_{r=1}^{\infty}\frac{1}{1+(\pi r)^2}=\frac{1}{e^2-1}$

Now use t = 1we get $\sum_{r=1}^{\infty}\frac{(-1)^r}{1+(\pi r)^2}=\frac{1}{e^{1}-e^{-1}}-\frac{1}{2}=\frac{e}{e^2-1}-\frac{1}{2}$

Add togther $\sum_{r=1}^{\infty}\frac{1}{1+(\pi r)^2}+\sum_{r=1}^{\infty}\frac{(-1)^r}{1+(\pi r)^2}=\frac{1}{e^2-1}+\frac{e}{e^2-1}-\frac{1}{2}$

or $\sum_{r=1}^{\infty}\frac{1+(-1)^r}{1+(\pi r)^2}=\frac{3-e}{2(e-1)}\to\sum_{r=1,\text{even}}^{\infty}\frac{2}{1+(\pi r)^2}=\frac{3-e}{2(e-1)}$ or $\sum_{r=1,\text{even}}^{\infty}\frac{1}{1+(\pi r)^2}=\frac{3-e}{4(e-1)}$

Finally using two results above

$$\sum_{r=1,\text{odd}}^{\infty}\frac{1}{1+(\pi r)^2}=\sum_{r=1}^{\infty}\frac{1}{1+(\pi r)^2}-\sum_{r=1,\text{even}}^{\infty}\frac{1}{1+(\pi r)^2}=\frac{1}{e^2-1}-\frac{3-e}{4(e-1)}=\frac{e-1}{4(e+1)}$$

4.20

Even

$$a_r = \frac{2\cdot 2}{2\pi}\int_0^\pi \sin\theta\cos r\theta\, d\theta$$

For r = 1 doing integral gives $a_1 = \frac{2\cdot 2}{2\pi}\int_0^\pi \sin\theta\cos\theta\, d\theta = 0$

For $r \neq 1$ we have $a_r = \frac{2\cdot 2}{2\pi}\int_0^\pi \sin\theta\cos r\theta\, d\theta =$

$$a_r = \frac{2}{\pi}\left[\frac{-\cos(1-r)\theta}{2(1-r)} - \frac{\cos(1+r)\theta}{2(1+r)}\right]_0^\pi =$$

$$\frac{1}{\pi}\left\{\left[\frac{-\cos(1-r)\pi}{(1-r)} - \frac{\cos(1+r)\pi}{(1+r)}\right] - \left[\frac{-\cos 0}{(1-r)} - \frac{\cos 0}{(1+r)}\right]\right\} =$$

$$\frac{-1}{\pi}\left\{\left[\frac{\cos(1-r)\pi}{(1-r)} + \frac{\cos(1+r)\pi}{(1+r)}\right] - \left[\frac{\cos 0}{(1-r)} + \frac{\cos 0}{(1+r)}\right]\right\} =$$

$$\frac{-1}{\pi}\left\{\left[\frac{(-1)^{r-1}}{r-1} + \frac{(-1)^{r-1}}{r-1}\right] + \left[\frac{1}{r+1} - \frac{1}{r+1}\right]\right\} = \cdots = \frac{-4}{\pi(r^2-1)}, r-\text{even};\ = 0\ r-\text{odd}$$

$$f(\theta) = |\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{\text{even}}\frac{1}{r^2-1}\cos r\theta \to (r = 2m) \to |\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}\cos 2m\theta$$

Use $\theta = 0$ then $\sin 0 = 0$, $\cos 0 = 1$

$$|\sin 0| = 0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}\cos 2m\cdot 0 \to 0\frac{2}{\pi} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}$$

or $\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1} = \frac{1}{2}$

Next use $\theta = \pi/2$

leads to $\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)^2-1} = \frac{1}{2} - \frac{\pi}{4}$

Add two sums together and re-index m = 2n leads to result

$$\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}=\frac{1}{2}$$

$$f(\theta)=\left|\sin\theta\right|=\frac{2}{\pi}-\frac{4}{\pi}\sum_{\text{even}}\frac{1}{r^2-1}\cos r\theta\to(r=2m)\to\left|\sin\theta\right|=\frac{2}{\pi}-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}\cos 2m\theta$$

Use $\theta=0$ then $\sin 0=0$, $\cos 0=1$

$$\left|\sin 0\right|=0=\frac{2}{\pi}-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}\cos 2m\cdot 0\to 0\frac{2}{\pi}-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}$$

or $\displaystyle\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}=\sum_{m=0}^{\infty}\frac{1}{4m^2-1}=\frac{1}{2}$ (sum 1)

Next use $\theta=\frac{\pi}{2}$

$$\left|\sin\frac{\pi}{2}\right|=1=\frac{2}{\pi}-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}\cos 2m\cdot\frac{\pi}{2}\to 1=\frac{2}{\pi}-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos m\pi}{(2m)^2-1}=\frac{2}{\pi}-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)^2-1}$$

Therefore

$$-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)^2-1}=1-\frac{2}{\pi}\to\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)^2-1}=-\frac{\pi}{4}+\frac{1}{2}\ \text{(sum 2)}$$

Add sums 1 and 2 together

$$\sum_{m=0}^{\infty}\frac{1}{(2m)^2-1}+\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)^2-1}=\sum_{m=0}^{\infty}\frac{1+(-1)^m}{(2m)^2-1}=\frac{1}{2}+\left(-\frac{\pi}{4}+\frac{1}{2}\right)=1-\frac{\pi}{4}$$

$$1+(-1)^m=\begin{cases}0 & m\text{ odd}\\ 2 & m\text{ even}\end{cases}$$

Therefore

$$\sum_{m=0}^{\infty}\frac{1+(-1)^m}{(2m)^2-1}=\sum_{m=0,\,\text{even}}^{\infty}\frac{2}{4m^2-1}=1-\frac{\pi}{4}$$

Let m = 2n then sum is reindexed $\displaystyle\sum_{n=0}^{\infty}\frac{2}{4(2n)^2-1}=1-\frac{\pi}{4}\to\sum_{n=0}^{\infty}\frac{1}{16n^2-1}=\frac{1}{2}-\frac{\pi}{8}$