Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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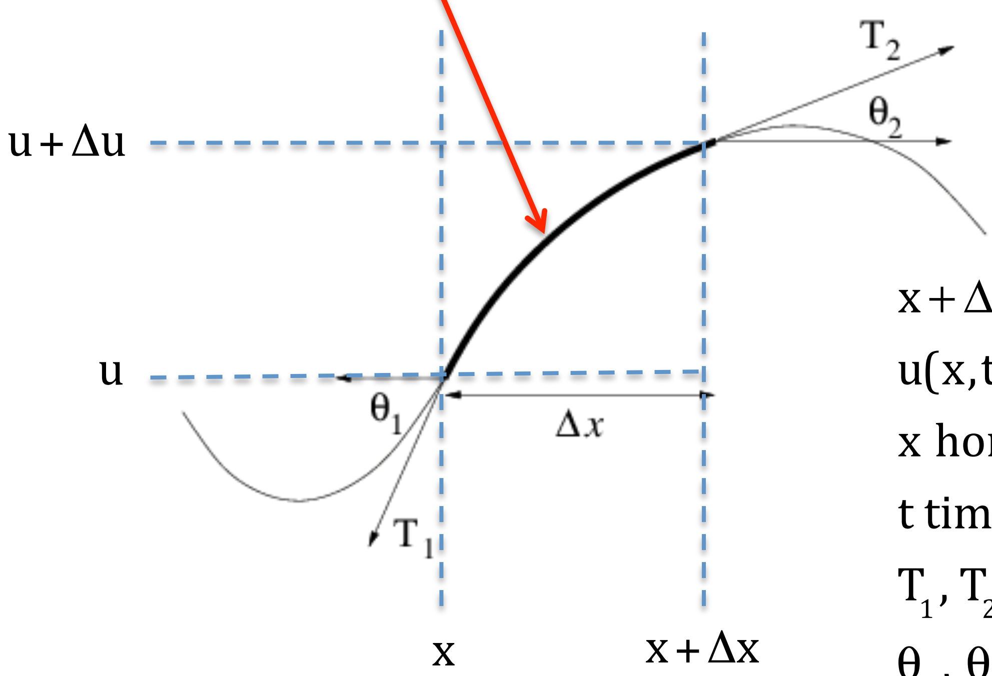
JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

Derivation of the one dimensional wave equation

Start with a small segment of string with small angular deflection (in the diagram below this is exaggerated). Newton's equation of motion is applied, where the force(s) along the string are referred to as tension.

$$F = ma = \sum \text{sum of Forces where} \sum \text{sum of Forces} = T_1 + T_2$$



$x + \Delta x$

$u(x,t)$ vertical displacement

x horizontal position

t time

T_1, T_2, T tension or force along string

θ_1, θ_2 tangent angles

$$F = ma = \sum \text{sum of Forces}$$

$$\sum \text{sum of Forces} = T_1 + T_2$$

$$T_2 = T\cos\theta_2 i + T\sin\theta_2 j$$

$$T_1 = -T\cos\theta_1 i - T\sin\theta_1 j$$

Therefore

$$\sum \text{sum of Forces} = [T\cos\theta_2 - T\cos\theta_1]i + [T\sin\theta_2 - T\sin\theta_1]j$$

No motion in horizontal (or j direction) $\rightarrow T\cos\theta_2 - T\cos\theta_1 = 0$

For small angles $\tan\theta = \dfrac{\sin\theta}{\cos\theta} \cong \dfrac{\sin\theta}{1} = \sin\theta$

Therefore $\sum \text{sum of Forces} = [T\sin\theta_2 - T\sin\theta_1]j$

Furthermore looking at the basic definition for the sine function at θ_2 and θ_1 we can write

$$\sin\theta_2 \cong \tan\theta_2 = \left.\frac{\Delta u}{\Delta x}\right|_{x+\Delta x} \text{ and } \sin\theta_1 \cong \tan\theta_1 = \left.\frac{\Delta u}{\Delta x}\right|_{x}$$

Also drop the directional vector j since only motion is vertical!

Substitution into our equation of motion gives $ma \approx T\left[\left.\frac{\Delta u}{\Delta x}\right|_{x+\Delta x} - \left.\frac{\Delta u}{\Delta x}\right|_{x}\right]$

The mass, m can be represented in terms of a linear density ρ therefore $m = \rho\Delta x$ and the acceleration, $a = \frac{\partial^2 u}{\partial t^2}$

Substitution gives $\frac{\partial^2 u}{\partial t^2} \approx \frac{T}{\rho}\frac{\left[\left.\frac{\Delta u}{\Delta x}\right|_{x+\Delta x} - \left.\frac{\Delta u}{\Delta x}\right|_{x}\right]}{\Delta x}$

Then taking the limit $\Delta x \to 0$ and letting $c^2 = \frac{T}{\rho}$ gives

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial x^2} \text{ or } \boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$$

Wave equation!

Finally let's look at the units of our equation as a simple check.

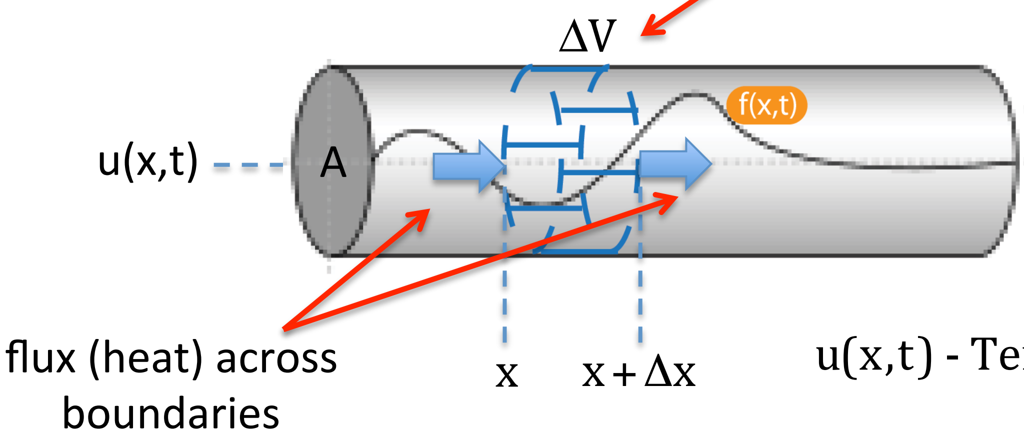
$$c^2 = \frac{T\{\text{units of force}\}}{\rho\{\text{units of linear density}\}} \text{ that is } \frac{\{kg - m/s^2\}}{\{kg/m\}} = \frac{m^2}{s^2} \to c\left\{\frac{m}{s}\right\}, \text{ velocity}$$

Therefore in terms of units $$\frac{\partial^2 u}{\partial x^2}\left\{\frac{m}{m^2}\right\} = \frac{1}{c^2\{m^2/s^2\}}\frac{\partial^2 u}{\partial t^2}\left\{\frac{m}{s^2}\right\} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\left\{\frac{s^2}{m^2}\right\}\left\{\frac{m}{s^2}\right\} \text{or} \left\{\frac{m}{m^2}\right\}$$

Derivation of the one dimensional diffusion equation

Identify a segment of material where we have isolated a small disk of length Δx and cross section A. We will use conservation of heat in this disk (volume $\Delta V = A\Delta x$) to derive the one-dimensional partial differential equation for diffusion (in this case heat diffusion).

Net change in heat in ΔV = Net change in heat flux across boundaries + Total heat generated in ΔV



u(x,t) - Temperature $\{{}^{\circ}\mathrm{C}\}$

f(x,t) - linear heat source $\left\{\dfrac{\mathrm{cal}}{\mathrm{cm}^3\mathrm{s}}\right\}$

$$\text{Total heat inside } \Delta V \; \{\text{cal}\} = \int_{x}^{x+\Delta x} \sigma\rho A \, u(s,t)\, ds$$

σ thermal capacity $\left\{\dfrac{\text{cal}}{\text{g}\,^{o}\text{C}}\right\}$, ρ density $\left\{\dfrac{\text{g}}{\text{cm}^3}\right\}$, A area $\left\{\text{cm}^2\right\}$

$u(s,t)\ \left\{^{o}\text{C}\right\}$, $ds\ \left\{\text{cm}\right\}$

$$\text{Net } \boxed{\text{change}} \text{ in total heat} \left\{\frac{\text{cal}}{\text{s}}\right\} = \boxed{\frac{\partial}{\partial t}}\left[\int_{x}^{x+\Delta x} \sigma\rho A \, u(s,t)\, ds\right]$$

$$\text{Net change in heat across boundaries} \left\{\frac{\text{cal}}{\text{s}}\right\} = \text{OUT} - \text{IN} = kA\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right]$$

k is thermal conductivity $\left\{\dfrac{\text{cal}}{^{o}\text{C cm s}}\right\}$ and $\dfrac{\partial u}{\partial x}\left\{\dfrac{^{o}\text{C}}{\text{cm}}\right\}$ $A\{\text{cm}^2\}$

$$\text{Total heat generated inside } \Delta V \left\{\frac{\text{cal}}{\text{s}}\right\} = A\int_{x}^{x+\Delta x} f(s,t)\, ds$$

f(s,t) is linear heat source $\left\{\dfrac{\text{cal}}{\text{cm}^3\ \text{s}}\right\}$ and $A\{\text{cm}^2\}$ $ds\{\text{cm}\}$

Combining these quantities in our conservation of heat equation gives

$$\frac{\partial}{\partial t}\left[\int_{x}^{x+\Delta x} \sigma\rho A\, u(s,t)\, ds\right] = kA\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] + A\int_{x}^{x+\Delta x} f(s,t)\, ds$$

Xi

Application of the mean value theorem $\int_a^b f(x)\,dx = f(\xi)(b-a)$

to the term on the left hand side and last term on the right hand side gives

$$\frac{\partial}{\partial t}\left[\int_{x}^{x+\Delta x} \sigma\rho A\, u(s,t)\, ds\right] = \frac{\partial}{\partial t}\left[\sigma\rho A\, u(\xi,t)(x+\Delta x - x)\right] = \sigma\rho A\, \frac{\partial}{\partial t} u(\xi,t)\Delta x$$

and

$$A\int_{x}^{x+\Delta x} f(s,t)\, ds = A\, f(\xi,t)\Delta x$$

Substitution gives

$$\sigma\rho A\ \frac{\partial}{\partial t}u(\xi,t)\Delta x = kA\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] + A\ f(\xi,t)\Delta x$$

$$\frac{\partial}{\partial t}u(\xi,t) = \frac{1}{\sigma\rho A\Delta x}kA\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right] + \frac{1}{\sigma\rho A\Delta x}A\ f(\xi,t)\Delta x$$

$$\frac{\partial}{\partial t}u(\xi,t) = \frac{k}{\sigma\rho}\frac{\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right]}{\Delta x} + \frac{1}{\sigma\rho}\ f(\xi,t)$$

Finally take the limit as $\Delta x \rightarrow 0$ gives $\lim\limits_{\Delta x \rightarrow 0}\frac{\left[\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right]}{\Delta x} = \frac{\partial^2 u}{\partial x^2}$

Note as $\Delta x \rightarrow 0$ that since $\xi \in \Delta x$ the mean values

$u(\xi,t) \rightarrow u(x,t),\ f(\xi,t) \rightarrow f(x,t)$

$$\frac{\partial u}{\partial t} = \frac{k}{\sigma\rho}\frac{\partial^2 u}{\partial x^2} + \frac{1}{\sigma\rho}\ f(x,t)$$

or $\boxed{k\frac{\partial^2 u}{\partial x^2} + f(x,t) = \sigma\rho\frac{\partial u}{\partial t}}$ $\boxed{\left\{\frac{\text{cal}}{\text{s}}\right\}}$

Diffusion equation!