$$\tilde{y}(s) = e^{-(\gamma+s)t_0}\frac{1}{(s+\gamma)^2+b^2} = e^{-\gamma t_0}e^{-st_0}\frac{1}{(s+\gamma)^2+b^2}$$

$$\rightarrow y(t) = e^{-\gamma t_0}L^{-1}\left\{e^{-st_0}\frac{1}{(s+\gamma)^2+b^2}\right\} \equiv e^{-\gamma t_0}L^{-1}\left\{\tilde{f}(s)\tilde{g}(s)\right\}$$

$$\text{Aside: } L^{-1}\{\tilde{g}(s)\} = L^{-1}\left\{\frac{1}{(s+\gamma)^2+b^2}\right\} = \frac{1}{b}e^{-\gamma t}\sin bt$$

and $L^{-1}\{\tilde{f}(s)\} = L^{-1}\{e^{-st_0}\} = \delta(t-t_0)$ Back:

The integral is delicate to evaluate (using a rigorous method)

First its not in a form we can easily work , i.e. $\delta(t-a)$ therefore make the subst.

$$\tau = u - t_0 \rightarrow u = \tau + t_0, d\tau = du \text{ then } \int_0^t e^{-\gamma\tau}\sin b\tau\ \delta(t-t_0-\tau)d\tau \rightarrow \int_{t_0}^{t+t_0} e^{-\gamma(u-t_0)}\sin b(u-t_0)\ \delta(t-u)du$$

Now we have to evaluate the delta function over a finite inteval whereas its defining integral for a variable (u in this case) would be over all possible values of u, i.e. -∞ to ∞ and we would regularly have $\int_{-\infty}^{\infty} f(u)\delta(t-u)du = f(t)$

Therefore rewrite the integral as (the Heaviside functions restrict the interval!!!)

$$\int_{-\infty}^{\infty} e^{-\gamma(u-t_0)}\sin b(u-t_0)\ [H(u\text{-}t_0) - H(u\text{-}t\text{-}t_0)]\delta(t-u)du =$$

$$\int_{-\infty}^{\infty} e^{-\gamma(u-t_0)}\sin b(u-t_0)\ H(u\text{-}t_0)\delta(t-u)du - \int_{-\infty}^{\infty} e^{-\gamma(u-t_0)}\sin b(u-t_0)\ H(u\text{-}t\text{-}t_0)]\delta(t-u)du$$

$$= e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0) - e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t-t_0) =$$

$$e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0) - e^{-\gamma(t-t_0)}\sin b(t-t_0)H(-t_0) = e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0)$$

Note, the second Heaviside function is evaluated for a negtive value, but the Heaviside function is 0 for negative values, i.e. values less than 0!!! So substiution of the remaining integrated function into the expression for y(t) gives

$$\text{Finally } y(t) = e^{-\gamma t_0}\frac{1}{b}e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0) = \frac{1}{b}e^{-\gamma t}\sin b(t-t_0)H(t-t_0)$$