Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Final Exam
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Question 1

a. $f(x) = x$ is odd on $[-\pi, \pi]$ therefore its Fourier coefficients a_n are 0 and we need to find its b_n coefficients:

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{4}{2\pi} \int_{0}^{\pi} x \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) dx \end{aligned}$$

Using integration by parts:

$$\begin{aligned} \int_{0}^{\pi} x \sin(nx) dx &= [x(-\frac{\cos(nx)}{n})]_0^{\pi} + \int_{0}^{\pi} 1 \cdot \frac{\cos(nx)}{n} dx \\ &= (-\frac{\pi}{n}) \cos(n\pi) + \frac{1}{n}[\sin(nx)]_0^{\pi} \\ &= \frac{(-1)^{n+1}\pi}{n} \end{aligned}$$

Thus $b_n = \frac{2}{\pi}\frac{(-1)^{n+1}\pi}{n} = \frac{(-1)^{n+1}2}{n}$ and the Fourier series of x, on $[-\pi, \pi]$, is:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

b. If we integrate terms by terms the previous expression, the Fourier series of x over $[-\pi, \pi]$, we have:

$$\begin{aligned} \frac{x^2}{2} &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-\frac{\cos(nx)}{n}) + c \quad c \text{constant of integration} \\ x^2 &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + c \text{ with } 2c \to c \\ &= c + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) \end{aligned}$$

c. $f(x) = x^2$ is an even function, by Fourier Series for even function over symmetric range, we have:

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n x}{2\pi}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \ (1)$$

where

$$
\begin{aligned}
a_0 &= \frac{4}{2\pi}\int_0^\pi x^2 dx \\
&= \frac{2}{\pi}[\frac{x^3}{3}]_0^\pi \\
&= \frac{2}{3}\pi^2
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{4}{2\pi}\int_0^\pi x^2 \cos{(\frac{2\pi n x}{2\pi})} dx = \frac{2}{\pi}\int_0^\pi x^2\cos(nx)dx \\
\int_0^\pi x^2\cos(nx)dx &= [x^2\ \frac{\sin(nx)}{n}]_0^\pi - \frac{2}{n}\int_0^\pi x\sin(nx)dx \\
&= 0 - \frac{2}{n}\frac{(-1)^{n+1}\pi}{n} \\
a_n &= \frac{2}{\pi}\frac{(-1)^n 2\ \pi}{n^2} \\
&= (-1)^n\frac{4}{n^2}
\end{aligned}
$$

Substituting for a_n in (1):

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

d. Fourier series of x^2 using integration terms by terms or calculating directly match, as required, by taking $c = \frac{\pi^2}{3}$ since x is a piecewise smooth function on the specified range.

Question 2

Consider the differential equation:

$$z\frac{d^2y}{dy^2} + y = 0$$

a. We put the equation in standard form:

$$\frac{d^2y}{dy^2} + \frac{1}{z}y = 0$$

$z\,p(z) = 0$ and $z^2 q(z) = z$ therefore 0 is a regular singular point.

b. Take $y = z^\sigma \sum_{n=0}^{\infty} a_n z^n$ and the usual derivatives in the D.E. gives by substitution

$$
\begin{aligned}
z\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0 \\
\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0\ (1)
\end{aligned}
$$

Take the term with the lowest power of z, which is the first sum with $n = 0$, then since each power of z term must be equal to 0, we have

$$\sigma(\sigma - 1)a_0 z^{\sigma-1} = 0$$

Since $a_0 \neq 0$ and $z^{\sigma-1} \neq 0$, therefore $\sigma = 0, 1$.

c. We go back to equation (1) and take $\sigma = 1$ yields

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

Then reindex the second sum to get same power of z in both sums:

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=1}^{\infty} a_{n-1} z^n = 0$$

Note, in first term $n = 0$ does not contribute so we can start index at $n = 1$ in the first sum, and combine both sums

$$\sum_{n=1}^{\infty} [n(n+1)a_n + a_{n-1}]z^n = 0$$

Since every power of z term must be 0 and $z^n \neq 0$, gives:

$$a_n = -\frac{1}{(n+1)n} a_{n-1}$$

Taking $a_0 = 1$, now

$$\begin{aligned}
n = 1\ a_1 &= -\frac{1}{2\ 1} a_0 = -\frac{1}{2\ 1} = \frac{(-1)^1}{2\ 1} \\
n = 2\ a_2 &= -\frac{1}{3\ 2} a_1 = \frac{1}{3\ 2\ 2\ 1} = \frac{(-1)^2}{(3\ 2\ 1)\ (2\ 1)} \\
n = 3\ a_3 &= -\frac{1}{4\ 3} a_2 = -\frac{1}{4\ 3\ 3\ 2\ 2\ 1} = \frac{(-1)^3}{(4\ 3\ 2\ 1)\ (3\ 2\ 1)} \\
&\vdots \\
a_n &= -\frac{1}{(n+1)n} a_{n-1} = \cdots = \frac{(-1)^n}{((n+1)\ n \cdots 1)\ (n\ (n-1) \cdots 1)} = \frac{(-1)^n}{(n+1)!n!}
\end{aligned}$$

Therefore one of the independent solution of the ODE is

$$y_1(z) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!n!} z^n$$

Question 3

a. We have

$$n=0,\ M=0,\ P_0(x)=\frac{(-1)^0(2\ 0-2\ 0)!}{2^0(0-0)!(0-2\ 0)!}x^{0-2\ 0}=1$$
$$n=1,\ M=\frac{1-1}{2}=0,\ P_1(x)=\frac{(-1)^0(2\ 1-2\ 0)!}{2^1(1-0)!(1-2\ 0)!}x^{1-2\ 0}=\frac{1\ 2}{2\ 1!\ 1!}x^1=x$$
$$n=2,\ M=\frac{2}{2}=1,\ P_2(x)=\frac{(-1)^0(2\ 2-2\ 0)!}{2^2(2-0)!(2-2\ 0)!}x^{2-2\ 0}+\frac{(-1)^1(2\ 2-2\ 1)!}{2^2(2-1)!(2-2\ 1)!}x^{2-2\ 1}$$
$$P_2(x)=\frac{4!}{2^2\ 2!\ 2!}x^2-\frac{(2\ 2-2)!}{2^2\ 1!\ 0!}x^0$$
$$P_2(x)=\frac{4\ 3\ 2\ 1}{4\ 2\ 2}x^2-\frac{2!}{4}$$
$$P_2(x)=\frac{3}{2}x^2-\frac{1}{2}=\frac{1}{2}(3x^2-1)$$

b. From

$$a_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx=\frac{2n+1}{2}\int_{-1}^{1}xP_n(x)dx$$

we have

$$n=0,\ a_0=\frac{2\ 0+1}{2}\int_{-1}^{1}xP_0(x)dx$$
$$=\frac{1}{2}\int_{-1}^{1}xdx=\frac{1}{2}[\frac{x^2}{2}]_{-1}^{1}=\frac{1}{4}[1^2-(-1)^2]=0$$
$$n=1,\ a_1=\frac{2\ 1+1}{2}\int_{-1}^{1}xP_1(x)dx$$
$$=\frac{3}{2}\int_{-1}^{1}x^2dx=\frac{3}{2}[\frac{x^3}{3}]_{-1}^{1}=\frac{1}{2}[1^3-(-1)^3]=\frac{1}{2}\ 2=1$$
$$n=2,\ a_2=\frac{2\ 2+1}{2}\int_{-1}^{1}xP_2(x)dx$$
$$=\frac{5}{2}\int_{-1}^{1}x[\frac{1}{2}(3x^2-1)]dx=\frac{5}{4}\int_{-1}^{1}(3\ x^3-x)dx$$
$$=0\ \text{ since the powers of } x \text{ in the integrand are odd}$$

Therefore the Fourier-Legendre series of x iss $x=1\cdot P_1(x)$ as required.

c. Using Rodrigues's formula

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

we have

$$
\begin{aligned}
n=0,\ &\frac{d^0}{dx^0}[(x^2-1)^0]=(x^2-1)^0=1\\
&P_0(x)=\frac{1}{2^0\,0!}\,1=1\\
n=1,\ &\frac{d}{dx}(x^2-1)=2x\\
&P_1(x)=\frac{1}{2^1\,1!}\,2x=x\\
n=2,\ &\frac{d^2}{dx^2}(x^2-1)^2=\frac{d}{dx}\Big[\frac{d}{dx}(x^2-1)^2\Big]=\frac{d}{dx}\big[4x(x^2-1)\big]=\frac{d}{dx}\big[4x^3-4x\big]=12\,x^2-4\\
&P_2(x)=\frac{1}{2^2\,2!}\,(12\,x^2-4)=\frac{4}{4\,2}(3x^2-1)=\frac{1}{2}(3x^2-1)
\end{aligned}
$$

Question 4

a.

$$\frac{\partial u}{\partial x}+4xu=0$$

Integration factor is

$$e^{\int 4x dx}=e^{4\int x dx}=e^{4\frac{x^2}{2}}=e^{2x^2}$$

Multiply the partial differential equation by the I.F.:

$$
\begin{aligned}
e^{2x^2}\frac{\partial u}{\partial x}+4xe^{2x^2}u=0\\
\frac{\partial}{\partial x}(e^{2x^2}u)=0
\end{aligned}
$$

Now integrate both sides with respect to x

$$
\begin{aligned}
e^{2x^2}u&=C\ C\text{:constant}\\
u(x)&=Ce^{-2x^2}
\end{aligned}
$$

b.

$$y^2u_x-x^2u_y=0$$

Let $u(x,y)=X(x)Y(y)$, substitution into the D.E. gives

$$
\begin{aligned}
y^2X'Y-x^2XY'=0\\
y^2\frac{X'Y}{XY}-x^2\frac{XY'}{XY}=0\\
y^2\frac{X'}{X}-x^2\frac{Y'}{Y}=0\\
\frac{1}{x^2}\frac{X'}{X}=\frac{1}{y^2}\frac{Y'}{Y}=k
\end{aligned}
$$

Integrating $\ln X = \frac{1}{3}kx^3 + \ln(C)$ and $\ln Y = \frac{1}{3}ky^3 + \ln(D)$, so

$$X = Ce^{\frac{1}{3}kx^3}, Y = De^{\frac{1}{3}ky^3}$$

Therefore (with $CD = A$) $u(x,y) = A\ e^{\frac{1}{3}k(x^3+y^3)}$

Question 5

We have the following problem

$$\begin{aligned}\Delta u &= 0\\ u(x,0) &= 0\\ u(x,b) &= 100x\\ u_x(0,y) &= 0\\ u_x(a,y) &= 0\end{aligned}$$

Assume a solution of the form $u(x,y) = X(x)Y(y)$. Substitute this expression and divide through by XY, to get:

$$\begin{aligned}\frac{X''}{X} + \frac{Y''}{Y} &= 0\\ -\frac{X''}{X} &= \frac{Y''}{Y}\end{aligned}$$

LHS is function of x only and RHS is a function of y only, thus we can write, with k constant

$$-\frac{X''}{X} = \frac{Y''}{Y} = k$$

We see immediately that $X(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$. And $X'(x) = \sqrt{k}\Bigg(B\cos(\sqrt{k}x) - A\sin(\sqrt{k}x)\Bigg)$. Also $u_x(x,y) = X'(x)Y(y)$. Take the boundary condition $u_x(0,y) = X'(0)Y(y) = 0$, since in general $Y(y) \neq 0$ then $X'(0) = 0$. Plug it into $X'(x)$ gives $\sqrt{k}\Bigg(B\ 1 - A\ 0\Bigg) = 0 \to B = 0$. Now $X(x) = A\cos\Bigg(\sqrt{k}x\Bigg)$ and $X'(x) = -A\sqrt{k}\sin\ \Bigg(\sqrt{k}x\Bigg)$. Next, with $u_x(a,y) = 0$ gives $X'(a)Y(y) = 0 \to X'(a) = 0$. $X'(a) = 0$ therefore $-A\sqrt{k}\sin\Bigg(\sqrt{k}a\Bigg) = 0 \to \sqrt{k}a = n\pi$. So $k_n = \frac{n^2\pi^2}{a^2}$ and

$$\begin{aligned}X_n(x) &= A_n\cos(\frac{n\pi}{a}x)\\ Y_n(y) &= B_n\cosh(\frac{n\pi}{a}y) + C_n\sinh(\frac{n\pi}{a}y)\end{aligned}$$

The boundary condition $u(x,0) = X(x)Y(0) = 0$ gives the non trivial solution $Y(0) = B_n\,1 + C_n\,0 = 0 \to B_n = 0$. We are left with $u_n(x,y) = A_n\cos(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y)$ (where $A_nC_n \to A_n$). Applying the superposition principle we have

$$u(x,y) = \sum_{n=1}^{\infty} A_n\cos(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y)$$

Finally we apply the initial conditions $u(x,b) = 100x$, therefore

$$u(x,b) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}b) = 100x$$

This is a Fourier series and we get

$$\begin{aligned} A_n \sinh(n\pi\frac{a}{b}) &= \frac{2}{a}\int_0^a 100x\cos(\frac{n\pi}{a}x)dx \\ A_n \sinh(n\pi\frac{a}{b}) &= \frac{200}{a}\int_0^a x\cos(\frac{n\pi}{a}x)dx \end{aligned}$$

Now

$$\begin{aligned} \int_0^a x\cos(\frac{n\pi}{a}x)dx &= \left[x(\frac{a}{n\pi})\sin(\frac{n\pi}{a}x)\right]_0^a - \frac{a}{n\pi}\int_0^a \sin(\frac{n\pi}{a}x)dx \\ &= 0 - \frac{a}{n\pi}\int_0^a \sin(\frac{n\pi}{a}x)dx \\ &= -\frac{a}{n\pi}(-\frac{a}{n\pi})[\cos(\frac{n\pi}{a}x)]_0^a \\ &= \frac{a^2}{n^2\pi^2}(\cos(n\pi) - 1) \\ &= \begin{cases} -\frac{2a^2}{n^2\pi^2} & \text{odd n} \\ 0 & \text{even n} \end{cases} \end{aligned}$$

Substitute it back to get A_n

$$\begin{aligned} A_{n,n \text{ odd}} &= \frac{1}{\sinh(n\pi\frac{a}{b})}\frac{200}{a}(-\frac{2a^2}{n^2\pi^2}) \\ A_{n,n \text{ odd}} &= -\frac{400a}{n^2\pi^2\sinh(n\pi\frac{a}{b})} \\ A_{n,n \text{ odd}} &= -\frac{800a}{n^2\pi^2(e^{n\pi\frac{a}{b}} - e^{-n\pi\frac{a}{b}})} \\ A_{n,n \text{ even}} &= 0 \end{aligned}$$

Question 6

a. Substituting $u(r,z) = R(r)Z(z)$ into the diffusion equation in cylindrical coordinates gives

$$R''Z + \frac{1}{r}R'Z + RZ'' = 0$$

Diving by RZ gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = 0$$

Separation of variables gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = -k^2$$

or

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=-k^2$$
$$\frac{Z''}{Z}=k^2$$

b. For $\frac{d^2}{dz^2}Z(z)=k^2Z(z)$, we immediately see that $Z(z)=c_1e^{kz}+c_2e^{-kz}$ which we can reformulate as $Z(z)=A\sinh(kz)+B\cosh(kz)$.
For $\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=-k^2$, starting with $\frac{d^2R(r)}{dr^2}+\frac{1}{r}\frac{dR(r)}{dr}+k^2R(r)=0$.

$$s=kr, r=\frac{s}{k}, \frac{ds}{dr}=k, R(r)\to R(s)$$
$$\frac{dR}{dr}=\frac{dR}{ds}\frac{ds}{dr}=k\frac{dR}{ds}$$
$$\frac{d^2R}{dr^2}=k\frac{d}{ds}\left(k\frac{dR}{ds}\right)=k^2\frac{d^2R}{ds^2}$$

Substitution into the ODE gives

$$k^2\frac{d^2R(s)}{ds^2}+\frac{1}{\frac{s}{k}}k\frac{dR(s)}{ds}+k^2R(s)=0$$
$$k^2\frac{d^2R(s)}{ds^2}+\frac{1}{s}k^2\frac{dR(s)}{ds}+k^2R(s)=0$$

Multiplying out by $(\frac{s}{k})^2$ gives

$$s^2\frac{d^2R(s)}{ds^2}+s\frac{dR(s)}{ds}+s^2R(s)=0$$
$$s^2\frac{d^2R(s)}{ds^2}+s\frac{dR(s)}{ds}+(s^2-0^2)R(s)=0$$

The last equation being a Bessel equation of order 0 therefore the solution is of the form

$$R(r)=C_1J_0(kr)+C_2Y_0(kr)$$

Since the temperature remains bounded at $r=0$ thus the term $Y_0(kr)$ has to be discarded, $C_2=0$, and $R(r)=CJ_0(kr), C=C_1$

c. Finally apply boundary conditions. First, $u(r,0)=R(r)Z(0)=0$, since in general $R(r)\neq 0$, thus $Z(0)=0$, which is $A\,0+B\,1=0\to B=0$ and $Z(z)=A\sinh(kz)$. Then $u(5,z)=R(5)Z(z)=0\to R(5)=0$, therefore $CJ_0(5k)=0\to J_0(5k)=0$. $5k$ represents the zero crossing for the Bessel function of order 0. We call them α_m and set $5k_m=\alpha_m\to k_m=\frac{\alpha_m}{5}$. Therefore the solutions are $R_m(r)=C_mJ_0(k_mr)=C_mJ_0(\frac{\alpha_m}{5}r)$. Note now that solutions in z are $Z_m(z)=A_m\sinh(k_mz)$. Finally applying the superposition principle, we have

$$u(r,z)=\sum_{m=1}^{\infty}A_m\sinh(k_mz)J_0(\frac{\alpha_m}{5}r)\ \text{ where } A_mC_m\to A_m$$

Applying the last boundary condition $u(r, 20) = u_0, 0 < r < 5$, we get:

$$u(r, 20) = \sum_{m=1}^{\infty} A_m \sinh(20k_m) J_0(\frac{\alpha_m}{5} r) = u_0$$

This is a Fourier Bessel series where the coefficients are given by

$$\sinh(20k_m) A_m = \frac{2}{5^2 J_1^2(\alpha_m)} \int_0^5 r u_0 J_0(\frac{\alpha_m}{5} r) dr = \frac{2u_0}{25 J_1(\alpha_m)} \int_0^5 r J_0(\frac{\alpha_m}{5} r) dr$$

Next, $\sinh(20k_m) = \sinh(20\frac{\alpha_m}{5}) = \sinh(4\alpha_m)$, and we get

$$A_m = \frac{2u_0}{25 J_1(\alpha_m) \sinh(4\alpha_m)} \int_0^5 J_0(\frac{\alpha_m}{5} r) r dr$$

Finally using $\frac{\partial}{\partial r}[r J_1(r)] = r J_0(r)$

$$\begin{aligned} A_m &= \frac{2u_0}{25 J_1(\alpha_m) \sinh(4\alpha_m)} \left[r J_1(\frac{\alpha_m}{5} r) \right]_0^5 \\ &= \frac{2u_0}{25 J_1(\alpha_m) \sinh(4\alpha_m)} \left(5 J_1(\frac{\alpha_m}{5} 5) \right) \\ &= \frac{2u_0}{5 \sinh(4\alpha_m)} \end{aligned}$$