7.12

$zy''+(1-z)y'+\lambda y=0$

Put into standard form

$y''+\dfrac{1-z}{z}y'+\dfrac{\lambda}{z}y=0$

$zp(z)=1-z,\ z^2q(z)=z\lambda$

Therefore $z=0$ regular singular point

Take $y=z^{\sigma}\sum_{n=0}^{\infty}a_nz^n$ and the usual derivatives in the D.E. gives

Substitution

$$z\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_nz^{n+\sigma-2}+(1-z)\sum_{n=0}^{\infty}(n+\sigma)a_nz^{n+\sigma-1}+\lambda\sum_{n=0}^{\infty}a_nz^{n+\sigma}=0$$

Simplify

$$\sum_{n=0}^{\infty}[(n+\sigma)(n+\sigma-1)+(n+\sigma)]a_nz^{n+\sigma-1}+\sum_{n=0}^{\infty}[-(n+\sigma)+\lambda]a_nz^{n+\sigma}=0$$

Take term with lowest power of z, that is first sum with $n=0$, then since each power of z term must be equal to zero we have

$[(\sigma)(\sigma-1)+(\sigma)]a_0z^{\sigma-1}=0$

Now $a_0\neq 0$ and $z^{\sigma-1}\neq 0$ therefore $(\sigma)(\sigma-1)+(\sigma)=\sigma^2=0$ and $\sigma=0,0$

Next go back to sums above and take $\sigma=0$, that is

$$\sum_{n=0}^{\infty}[(n)(n-1)+(n)]a_nz^{n-1}+\sum_{n=0}^{\infty}[-n+\lambda]a_nz^{n}=\sum_{n=0}^{\infty}n^2a_nz^{n-1}+\sum_{n=0}^{\infty}[-n+\lambda]a_nz^{n}=0$$

Then reindex $n \rightarrow n-1$ second sum to get same power of z in both sums

$$\sum_{n=0}^{\infty}n^2a_nz^{n-1}+\sum_{n=1}^{\infty}[-(n-1)+\lambda]a_{n-1}z^{n-1}=0$$

Note in first term $n=0$ does not contribute so therefore we
can start index at n = 1 in sum one and therefore we can combine
both sums

$$\sum_{n=1}^{\infty}\{n^2 a_n + [-(n-1)+\lambda]a_{n-1}\}z^{n-1} = 0$$

Therefore taking $n^2 a_n + [-(n-1)+\lambda]a_{n-1} = 0$ since every power of z term
must be 0 and $z^{n-1} \neq 0$. This gives

$$a_n = \frac{[n-1-\lambda]}{n^2}a_{n-1}$$

IMPORTANT: Now take λ=N so $a_n = \dfrac{[n-1-N]}{n^2}a_{n-1}$ and when $N = n-1$ or n = N+1
then $a_n = 0$ and all terms with index greater are also 0 so only take terms up to N!!!
Now take

$$n=1 \quad a_1 = \frac{[1-1-N]}{1^2}a_{1-1} = \frac{-N}{1^2}a_0$$

$$n=2 \quad a_2 = \frac{[2-1-N]}{2^2}a_{2-1} = \frac{-(1-N)N}{2^2 1^2}a_0$$

$$n=3 \quad a_2 = \frac{[3-1-N]}{3^2}a_{3-1} = \cdots = \frac{-(2-N)(1-N)N}{3^2 2^2 1^2}a_0 = \frac{(-1)^3(N-2)(N-1)N}{3^2 2^2 1^2}a_0$$

Note $(N-2)(N-1)N = \dfrac{N!}{(N-3)!}$ and substitution infers for general a_n

$$a_n = \frac{(-1)^n N!}{(N-n)!(n!)^2}a_0$$

Therefore taking the sum for y(z) only up to N gives

$$y_N(z) = \sum_{n=0}^{N}\frac{(-1)^n N!}{(N-n)!(n!)^2}a_0 z^n$$

Next lets find normalization such that $L_N(0) = N!$

Take N = 0

$$y_0(z) = a_0 \sum_{n=0}^{0} \frac{(-1)^n 0!}{(0-n)!(n!)^2} z^n = a_0 \frac{(-1)^0 0!}{(0-0)!(0!)^2} z^0 = a_0 \text{ so take } a_0 = 0!$$

$$\text{and } y_0(z) = 0! \sum_{n=0}^{0} \frac{(-1)^n 0!}{(0-n)!(n!)^2} z^n = \sum_{n=0}^{0} \frac{(-1)^n (0!)^2}{(0-n)!(n!)^2} z^n = L_0(z), \text{note } L_0(z) = 0!$$

Lets try N = 1

$$y_1(z) = a_0 \sum_{n=0}^{1} \frac{(-1)^n 1!}{(1-n)!(n!)^2} z^n = a_0 \left[\frac{(-1)^0 1!}{(1-0)!(0!)^2} z^0 + \frac{(-1)^1 1!}{(1-1)!(1!)^2} z^1 \right] = a_0(1-z)$$

Therefore $y_1(0) = a_0(1-0) = a_0$ so taking $a_0 = 1!$ satisfies normalization for N=1 and consistent with previous a_0 in N=0

$$y_1(z) = 1! \sum_{n=0}^{1} \frac{(-1)^n 1!}{(1-n)!(n!)^2} z^n = \sum_{n=0}^{1} \frac{(-1)^n (1!)^2}{(1-n)!(n!)^2} z^n = L_1(z) \text{ and note } L_1(0) = 1!$$

Lets try N = 2

$$y_2(z) = a_0 \sum_{n=0}^{2} \frac{(-1)^n 2!}{(2-n)!(n!)^2} z^n = a_0 \left[\frac{(-1)^0 2!}{(2-0)!(0!)^2} z^0 + \frac{(-1)^1 2!}{(2-1)!(1!)^2} z^1 + \frac{(-1)^2 2!}{(2-2)!(2!)^2} z^2 \right] = a_0(1 - 2z + \frac{1}{2} z^2)$$

Therefore $y_2(0) = a_0(1-0+0) = a_0$ so taking $a_0 = 2!$ is consistent and

$$y_2(z) = 2! \sum_{n=0}^{2} \frac{(-1)^n 2!}{(2-n)!(n!)^2} z^n = \sum_{n=0}^{2} \frac{(-1)^n (2!)^2}{(2-n)!(n!)^2} z^n = L_2(z) \text{and note } L_2(0) = 2!$$

Generalizing for all N gives

$$L_N(z) = \sum_{n=0}^{N} \frac{(-1)^n (N!)^2}{(N-n)!(n!)^2} z^n$$

I will leave the writing out of all terms of $L_3(z)$ to you - don't forget to expand sum when you do this.