7.4

For $\frac{d^2f}{dz^2}+2(z-\alpha)\frac{df}{dz}+4f=0$ make the obvious substitution $x=z-\alpha$

This leads to $dx=dz$ and $\frac{d}{dz}=\frac{d}{dx},\ \frac{d^2}{dz^2}=\frac{d^2}{dx^2}$ and gives

$\frac{d^2f}{dx^2}+2x\frac{df}{dx}+4f=0$, with f= f(x) Take $f(x)=\sum_{n=0}^{\infty}a_n x^n$

and taking derivatives with respect to x and substitute into this DE gives

$$\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}+2x\sum_{n=0}^{\infty}na_n x^{n-1}+4\sum_{n=0}^{\infty}a_n x^n=0$$

Combine terms two and three together gives

$$\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}+\sum_{n=0}^{\infty}[2n+4]a_n x^n=0$$

and reindexing the first sum gives

$$\sum_{n=-2}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(2n+4)a_n x^n=0$$

Note in the first sum for n = -2, -1 that the terms are zero so can start this sum at n = 0!, therefore

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+(2n+4)a_n]x^n=0$$

From the sum above we have

$$(n+2)(n+1)a_{n+2}+(2n+4)a_n]=0$$

or

$$a_{n+2}=\frac{-(2n+4)}{(n+2)(n+1)}a_n=\frac{-2(n+2)}{(n+2)(n+1)}a_n=\frac{-2}{n+1}a_n$$

Take $a_0=1, a_1=0$ and generate even coefficients

n=0 then $a_2=\frac{-2}{1}a_0=\frac{-2}{1}(1)=\frac{-2}{1}$

n=2 then $a_4=\frac{-2}{3}a_2$

n=4 then $a_6=\frac{-2}{5}a_4$

n=6 then $a_8=\frac{-2}{7}a_6$

To get solution in the book we take

$$a_2=\frac{-2}{1}\left(\frac{2}{2}\right)=\frac{-4}{1\cdot 2}$$

$$a_4=\frac{-2}{3}a_2=\frac{-2}{3}\left(\frac{-4}{1\cdot 2}\right)=\frac{-2}{3}\left(\frac{-4}{1\cdot 2}\right)\left(\frac{4}{4}\right)=\frac{2(4)^2}{4!}$$

$$a_6=\frac{-2}{5}\frac{2(4)^2}{4!}\left(\frac{6}{6}\right)=\frac{-2}{5}\frac{2(4)^2}{4!}\left(\frac{2\cdot 3}{6}\right)=\frac{-2\cdot 3\cdot 2\cdot 2(4)^2}{6!}=\frac{-2\cdot 3(4)^3}{6!}$$

$$a_8=\frac{-2}{7}\left(\frac{-2\cdot 3(4)^3}{6!}\right)\left(\frac{8}{8}\right)=\cdots=\frac{2\cdot 3\cdot 4(4)^4}{8!}=\frac{4!(4)^4}{8!}$$

In general

$$a_{2n}=\frac{(-1)^n n!(4)^n}{(2n)!}\text{ therefore } f_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n n!(4)^n}{(2n)!}x^{2n}$$

or $f_1(z,\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n n!(4)^n}{(2n)!}(z-\alpha)^{2n}$ (note my index is n instead of m like in book - doesn't matter)

$a_{n+2} = \dfrac{-2}{n+1} a_n$

Take $a_0 = 0, a_0 = 1$ and generate odd coefficients

n=1 then $a_3 = \dfrac{-2}{2} a_1 = \dfrac{-2}{2}(1) = -1$

n=3 then $a_3 = \dfrac{-2}{4} a_1 = \dfrac{-2}{4}(-1) = \dfrac{1}{2}$

n=5 then $a_5 = \dfrac{-2}{6} a_3 = \dfrac{-1}{3}\left(\dfrac{1}{2}\right) = \dfrac{1}{3 \cdot 2}$

Start

with $n = 0$ then

In general $a_{2n+1} = \dfrac{(-1)^n}{n!}$

therefore $f_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!} x^{2n+1}$

or

$$f_2(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x^2)^n$$

Compare this to Taylor expansion for

$$e^x = \sum_{n=0}^{\infty} \frac{1^n}{n!} x^n$$

So $e^{-x^2} = \sum_{n=0}^{\infty} \dfrac{1}{n!}(-x^2)^n = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!}(x^2)^n$

Therefore

$$f_2(x) = xe^{-x^2}$$

The total solution is

$$f(x) = f_2(x) + f_1(x) = Axe^{-x^2} + B\sum_{n=0}^{\infty} \frac{(-1)^n n!(4)^n}{(2n)!} x^{2n}$$

and in terms of z and α, that is $x = z - \alpha$

$$f(z,\alpha) = A(z-\alpha)e^{-(z-\alpha)^2} + B\sum_{n=0}^{\infty} \frac{(-1)^n n!(4)^n}{(2n)!}(z-\alpha)^{2n}$$