Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Take Home Project 2
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Question 1

(a) See figure 1

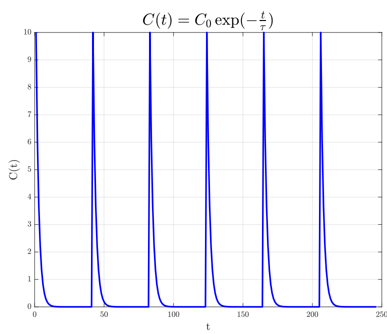


Figure 1

(b) $f(t) = C_0 e^{-\frac{t}{\tau}}$ with period T, so

$$
\begin{aligned}
a_0 &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} dt \\
&= \frac{2C_0}{T}(-\tau)[e^{-\frac{t}{\tau}}]_0^T \\
&= -2C_0 \frac{\tau}{T}[e^{-\frac{T}{\tau}} - 1] \\
&= 2C_0 \frac{\tau}{T}(1 - e^{-\frac{T}{\tau}})
\end{aligned}
$$

If $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_0 \approx 2C_0\frac{\tau}{T}$.

$$\begin{aligned} a_k &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}dt \\ &= \frac{2C_0}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}dt \end{aligned}$$

Using integration by parts with $u = \cos\frac{2k\pi t}{T}, du = -\frac{2k\pi}{T}\sin\frac{2k\pi t}{T}$ and $dv = e^{-\frac{t}{\tau}}, v = (-\tau)e^{-\frac{t}{\tau}}$:

$$\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}dt = (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T - \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}dt$$

Using again integration by parts:

$$\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T}dt = (-\tau)[e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt$$

So

$$\begin{aligned} (1+(\frac{2k\pi\tau}{T}))^2\int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt &= (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau^2}{T}[e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}]_0^T \\ &= (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T + 0 \\ &= \tau(1-e^{-\frac{T}{\tau}}) \\ \int_0^T e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}dt &= \frac{\tau}{1+(\frac{2k\pi\tau}{T})^2}(1-e^{-\frac{T}{\tau}}) \end{aligned}$$

Substituting back into the expression found for a_k yields

$$\begin{aligned} a_k &= 2C_0\frac{\tau}{T}\frac{1}{1+(\frac{2k\pi\tau}{T})^2}(1-e^{-\frac{T}{\tau}}) \\ &= 2C_0\frac{\tau T}{T^2+(2k\pi\tau)^2}(1-e^{-\frac{T}{\tau}}) \end{aligned}$$

With the same assumption $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_k \approx 2C_0\frac{\tau}{T}\frac{1}{1+(\frac{2k\pi\tau}{T})^2}$.

Similarly to compute b_k

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T}\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T}\frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T}\frac{2k\pi\tau}{T}\frac{\tau}{1+(\frac{2k\pi\tau}{T})^2}(1-e^{-\frac{T}{\tau}}) \\
&= 4C_0 k\pi\frac{\tau^2}{T^2+(2k\pi\tau)^2}(1-e^{-\frac{T}{\tau}})
\end{aligned}
$$

Once again, since $e^{-\frac{T}{\tau}} \approx 0$ then $b_k \approx 4C_0(\frac{\tau}{T})^2\frac{1}{1+(\frac{2k\pi\tau}{T})^2}\pi k$

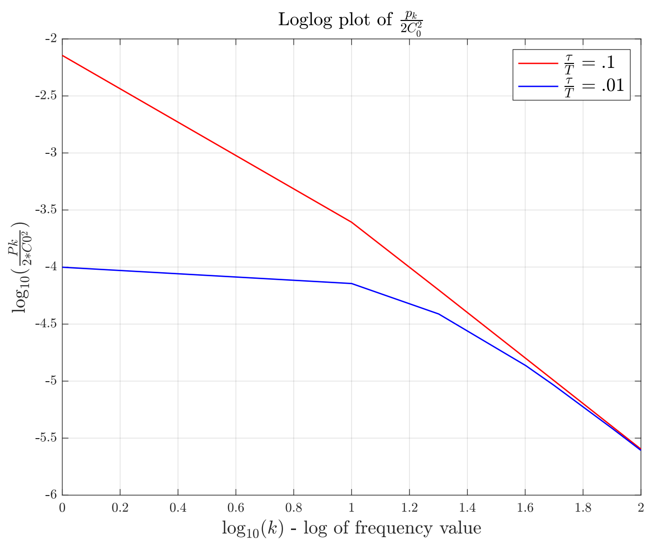
(c) For $k \geq 1$

$$
\begin{aligned}
p_k &= \frac{1}{2}(a_k^2+b_k^2) \\
&= \frac{1}{2}\left[4C_0^2(\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2} + 16C_0^2(\frac{\tau}{T})^4\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\pi^2k^2\right] \\
&= \frac{1}{2}4C_0^2(\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\left[1+4(\frac{\tau}{T})^2\pi^2k^2\right] \\
&= 2C_0^2(\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\left[1+4(\frac{\tau}{T})^2\pi^2k^2\right]
\end{aligned}
$$

(d) We have

$$
\frac{p_k}{2\,C_0^2} = (\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\left[1+4(\frac{\tau}{T})^2\pi^2k^2\right]
$$

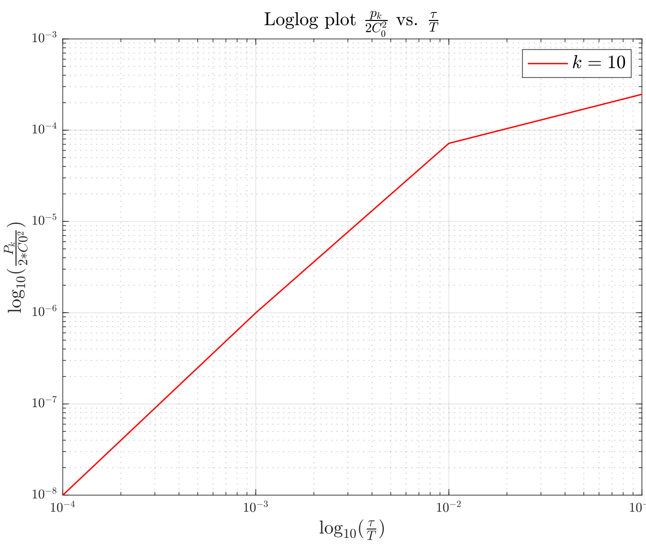
From the plot we can see that the power $\frac{p_k}{2\,C_0^2}$ decreases as the frequency increases. Power is close to 0 starting with a frequency of 10. Looking at the plot, as the pulse τ, becomes narrower, the power decreases linearly. For a greater $\frac{\tau}{T}$, the power starts at a higher value until an inflection point corresponding to frequency of 10 ($1 = \log_{10}(10)$ on the graph). Also for a higher $\frac{\tau}{T}$, the steepest the decrease in power. Eventually the two curves combine in one curve around a frequency of ($2 = \log_{10}(100)$ on the graph).



(e) As the pulses τ narrow or decrease the power decreases: see below loglog plot of power $\frac{p_k}{2\ C_0^2}$ vs. $\frac{\tau}{T}$ for a frequency $k = 10$

(f) We have

$$\begin{aligned} a_k \cos(\frac{k2\pi t}{T}) + b_k \sin(\frac{k2\pi t}{T}) &= \cos(\phi_k)\cos(\frac{k2\pi t}{T}) + \sin(\phi_k)\sin(\frac{k2\pi t}{T}) \\ &= \cos(\frac{k2\pi t}{T} - \phi_k) \end{aligned}$$



where

$$\tan(\phi_k) = \frac{\sin(\phi_k)}{\cos(\phi_k)} = \frac{b_k}{a_k} = 4C_0(\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2}\pi k(2C_0\frac{\tau}{T}\frac{1}{1+(\frac{2k\pi\tau}{T})^2})^{-1}$$
$$= 2\frac{\tau}{T}\pi k$$
$$\phi_k = \arctan(2\frac{\tau}{T}\pi k)$$

For $\frac{\tau}{T} = .1$, $\phi_1 \approx 32.14°$ and $\phi_2 \approx 51.48°$ and for $\frac{\tau}{T} = .01$, $\phi_1 \approx 3.59°$ and $\phi_2 \approx 7.16°$

If we imagine a clock with a hand that turns at constant speed, making a full turn every T seconds, and is pointing straight up at time $t = 0$ when the plasma rises to C_0. The phase ϕ_k is then the angle from the $12 : 00$ position to the current position and indicates how far is the next rise in plasma concentration to C_0. For the same frequency (either $k = 1$ or $k = 2$), with a larger pulse we are closer to the next substance release.

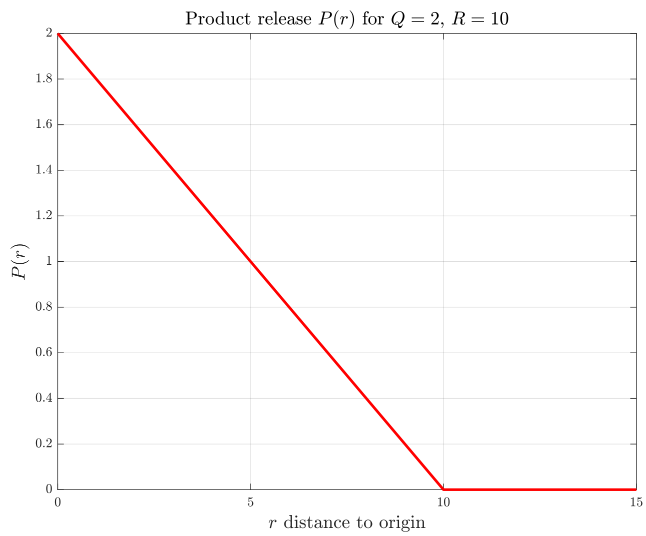
Question 2

(a) One simple way to describe $P(r)$ is to define it as $P(r) = Ar + B$ with the conditions:

$$A \cdot 0 + B = Q$$
$$A \cdot R + B = 0$$

which gives $A = -\frac{Q}{R}$ and $B = Q$. So

$$P(r) = \begin{cases} Q(1 - \frac{r}{R}) & \text{for } 0 \leq r \leq R \\ 0 & \text{for } r > R \end{cases}$$



(b) Since we assume no angular dependence: $\nabla^2 C = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dC}{dr})$, the differential equation is now:

$$\frac{D}{r^2}\frac{d}{dr}(r^2\frac{dC(r)}{dr}) + P(r) = 0$$
$$\frac{d}{dr}(r^2\frac{dC(r)}{dr}) = -\frac{r^2}{D}P(r)$$

(c) Inside the cell $P(r) = Q(1 - \frac{r}{R})$, so we have to solve the differential equation

$$\begin{aligned}\frac{d}{dr}(r^2\frac{dC(r)}{dr}) &= -\frac{r^2}{D}Q(1-\frac{r}{R}) \\ &= \frac{Q}{DR}r^2(r-R) \\ &= \frac{Q}{DR}r^3 - \frac{Q}{D}r^2\end{aligned}$$

Integrating once

$$\begin{aligned}r^2\frac{dC(r)}{dr} &= \frac{Q}{4DR}r^4 - \frac{Q}{3D}r^3 + A \\ \frac{dC(r)}{dr} &= \frac{Q}{4DR}r^2 - \frac{Q}{3D}r + \frac{A}{r^2}\end{aligned}$$

Integrating again

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 - \frac{A}{r} + B \;\; A, B\text{:constants}, C_i\text{:inside cell concentration}$$

Outside the cell $P(r) = 0$ and we want to solve the differential equation

$$\frac{d}{dr}(r^2\frac{dC(r)}{dr}) = 0$$

Which by integration gives

$$\begin{aligned}r^2\frac{dC(r)}{dr} &= C_1 \\ \frac{dC(r)}{dr} &= \frac{C_1}{r^2} \\ C_o(r) &= -\frac{C_1}{r} + C_2 \;\; C_1, C_2\text{:constants}, C_o\text{:outside cell concentration}\end{aligned}$$

(d) Applying the boundary conditions

(i)

$$\lim_{r\to 0} C_i(r) = \lim_{r\to 0} \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 - \frac{A}{r} + B$$

since $\lim_{r\to 0} C_i(r) = \lim_{r\to 0}(-\frac{1}{r} + B) = \infty$ therefore to have finite concentration $C_i(r)$ at $r = 0$ we need $A = 0$

(ii)

$$\lim_{r\to\infty} C_o(r) = \lim_{r\to\infty}\left(-\frac{C_1}{r} + C_2\right) = C_2$$

The concentration goes to zero at infinity implies $C_2 = 0$

(iii) We have now for $C_i(r)$ and $C_o(r)$:

$$\begin{aligned} C_i(r) &= \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 + B \\ C_o(r) &= -\frac{C_1}{r} \end{aligned}$$

$C_i(R) = C_o(R)$ and $\frac{dC_i(r)}{dr} = \frac{dC_o(r)}{dr}|_{r=R}$ yields

$$\begin{aligned} \frac{Q}{12DR}R^3 - \frac{Q}{6D}R^2 + B &= -\frac{C_1}{R} \\ \frac{Q}{4D}R - \frac{Q}{3D}R &= \frac{C_1}{R^2} \end{aligned}$$

Rearranging

$$\begin{aligned} -\frac{Q}{12D}R^2 + B &= -\frac{C_1}{R} \\ -\frac{Q}{12D}R &= \frac{C_1}{R^2} \end{aligned}$$

which gives

$$\begin{aligned} B &= \frac{Q}{6D}R^2 \\ C_1 &= -\frac{Q}{12D}R^3 \end{aligned}$$

substituting back

$$\begin{aligned} C_i(r) &= \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 + \frac{Q}{6D}R^2 \\ &= \frac{Q}{6D}\left[\frac{r^3}{2R} - r^2 + R^2\right] \\ C_o(r) &= \frac{Q}{12D}R^3\frac{1}{r} \end{aligned}$$

(e) Knowing that within the cell since $P(r)$ has maximum value Q at $r = 0$ and then it is zero for $r > R$, we are looking for the value of r for which

$\frac{dC_i(r)}{dr} = 0$:

$$\frac{dC_i(r)}{dr} = \frac{Q}{4DR}r^2 - \frac{Q}{3D}r = \frac{Q}{D}r(\frac{r}{4R} - \frac{1}{3})$$

By the nature of the problem, the maximum concentration is at $r = 0$:

$$C_M = \frac{Q}{6D}R^2$$

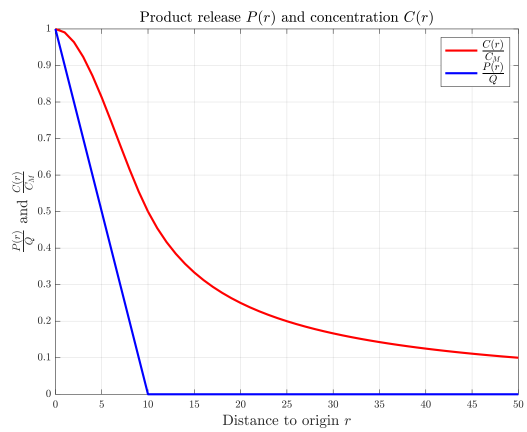
Inside the cell

$$\begin{aligned} C_i(r) &= \frac{Q}{6D}(\frac{1}{2}\frac{r^3}{R} - r^2 + R^2) \\ \frac{C_i(r)}{C_M} &= \frac{6D}{Q}R^{-2}\frac{Q}{6D}(\frac{1}{2}\frac{r^3}{R} - r^2 + R^2) \\ &= \frac{1}{2}(\frac{r}{R})^3 - (\frac{r}{R})^2 + 1 \end{aligned}$$

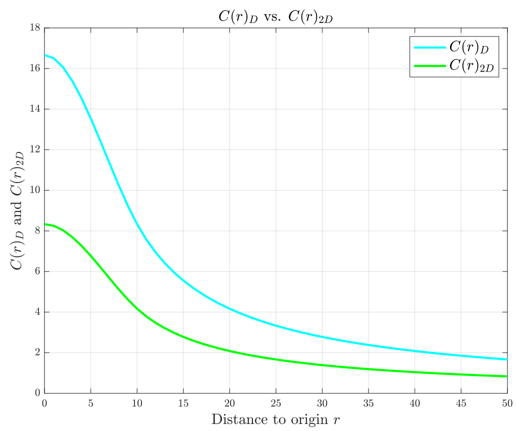
And outside the cell

$$\begin{aligned} C_o(r) &= \frac{Q}{12D}R^3\frac{1}{r} \\ \frac{C_o(r)}{C_M} &= \frac{6D}{Q}R^{-2}\frac{Q}{12D}R^3\frac{1}{r} \\ &= \frac{R}{2r} \end{aligned}$$

When the diffusion constant is doubled, the curve $\{\frac{C_i(r)}{C_M}, \frac{C_0(r)}{C_M}\}$ stays the same since this curve does not depend on the diffusion constant D. See figure below.



However for arbitrary values of Q, R, and D ($Q = 1, R = 10, D = 1$), when the diffusion constant D is doubled, the concentration curve is at a lower level compared to the same concentration curve related to a diffusion constant D as you can expect based on the expressions of $C_i(r)$ and $C_o(r)$ obtained above:


$C(r)_D$ vs. $C(r)_{2D}$
$C(r)_D$
$C(r)_{2D}$
$C(r)_D$ and $C(r)_{2D}$
Distance to origin r
0
2
4
6
8
10
12
14
16
18
0
5
10
15
20
25
30
35
40
45
50