6.13

System looks like

$A \to B$ with rate $3s^{-1}$

$A \to C$ with rate $1s^{-1}$

$B \to C$ with rate $2s^{-1}$

Therefore

$\frac{dA}{dt} = -3A - A = -4A$ (the rate of change of A is equal to how much of A goes to B at rate $3s^{-1}$, that is -3A

and how much of A goes to C, that is -1A. Notice units the same on both sides of the equation!

$\frac{dB}{dt} = 3A - 2B$ (from first equation , notice -3A change, since this is from $A \to B$, it shows up here as 3A,

Last equation

$\frac{dC}{dt} = A + 2B$

Note $A(0) = x_0$

Therefore three equations to solve. Take Laplace transform and solve 3 equations for 3 unknowns.

$\frac{dA}{dt} = -4A$ $\quad$ $\frac{dB}{dt} = 3A - 2B$ $\quad$ $\frac{dC}{dt} = A + 2B$

$A(0) = x_0,$ $\quad$ $B(0) = 0,$ $\quad$ $C(0) = 0$

Take Laplace transforms

$s\tilde{A}(s) - x_0 = -4\tilde{A}(s)$ $\quad$ $s\tilde{B}(s) = 3\tilde{A}(s) - 2\tilde{B}(s)$ $\quad$ $s\tilde{C}(s) = \tilde{A}(s) + 2\tilde{B}(s)$

Rewrite $(s+4)\tilde{A}(s) = x_0$ $\quad$ $-3\tilde{A}(s) + (s+2)\tilde{B}(s) = 0$ $\quad$ $-\tilde{A}(s) - 2\tilde{B}(s) + s\tilde{C}(s) = 0$

In matrix form

$$\begin{pmatrix} s+4 & 0 & 0 \\ -3 & s+2 & 0 \\ -1 & -2 & s \end{pmatrix} \begin{pmatrix} \tilde{A}(s) \\ \tilde{B}(s) \\ \tilde{C}(s) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} s+4 & 0 & 0 \\ -3 & s+2 & 0 \\ -1 & -2 & s \end{pmatrix}\begin{pmatrix} \tilde{A}(s) \\ \tilde{B}(s) \\ \tilde{C}(s) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix}$$

Solve via Cramers rule

$$\tilde{A}(s) = \frac{\begin{vmatrix} x_0 & 0 & 0 \\ 0 & s+2 & 0 \\ 0 & -2 & s \end{vmatrix}}{\begin{vmatrix} s+4 & 0 & 0 \\ -3 & s+2 & 0 \\ -1 & -2 & s \end{vmatrix}} = \frac{x_0(s+2)s}{(s+4)(s+2)s} = \frac{x_0}{s+4}$$

Similar for $\tilde{B}(s) = \dfrac{3x_0}{(s+4)(s+2)}$, $\tilde{C}(s) = \dfrac{(s+8)x_0}{(s+4)(s+2)s}$

Use partial fractions for $\tilde{B}(s)$ and $\tilde{C}(s)$ to get

$$\tilde{B}(s) = 3x_0\left[-\frac{1}{2}\frac{1}{s+4} + \frac{1}{2}\frac{1}{s+2}\right],\ \tilde{C}(s) = x_0\left[\frac{1}{2}\frac{1}{s+4} - \frac{3}{2}\frac{1}{s+2} + \frac{1}{s}\right]$$

Take inverse transform via Table

$$A(t) = x_0 e^{-4t}$$

$$B(t) = 3x_0\left[-\frac{1}{2}e^{-4t} + \frac{1}{2}e^{-2t}\right]$$

$$C(t) = x_0\left[\frac{1}{2}e^{-4t} - \frac{3}{2}e^{-2t} + 1\right]$$