CHAPTER 8

Blood flow in compliant vessels

The arterial system is a complex network of viscoelastic vessels, with complex geometries, nonlinear elastic properties. Additionally the blood rheology is complex and the blood has generally non-Newtonian properties. Pressure and flow waves are set up by the heart and interacts with the arterial system. Thus, the arterial system will continuously modify the pressure and flow waves from the heart.

In this section the 1D governing equations for pressure and flow waves will be derived on a basis where most of the complicating factors mentioned above will be discarded.

The first section 8.1, we consider Poiseuille flow in a compliant vessel, i.e. only a simplified version of the momentum equation is included, whereas convective terms, and thus wave propagation phenomena, are not accounted for. However, an important feature for flow in compliant vessels, namely chocked flow, is illustrated with this simple model.

The following sections 8.2 and 8.3 are devoted to derivation of 1D governing equations for wave propagation in compliant vessels. In section 8.2 a flat velocity profile is assumed, whereas a velocity profile is accounted for in 8.3.

8.1 POISEUILLE FLOW IN A COMPLIANT VESSEL

Simplified solution for flow in compliant vessel. In general Navier-Stokes equations for blood, Navier equations for vessel wall, must be solved simultaneously.

Simplification; Poiseuille flow for the fluid, $p(A)$ relation for vessel wall.

Poiseuille flow relates pressure gradient to volume flow:

$$\frac{dp}{dx} = -\frac{8\mu}{\pi a^4}\, Q = -\frac{8\pi\mu}{A^2}\, Q \tag{8.1}$$

Compliance $\qquad C = \dfrac{\partial A}{\partial p}$

$$\frac{dp}{dx} = \frac{\partial p}{\partial A}\frac{dA}{dx} = \frac{1}{C}\frac{dA}{dx} = -\frac{8\pi\mu}{A^2}\, Q \tag{8.2}$$

$$A^2\,\frac{dA}{dx} = \frac{1}{3}\frac{d}{dx}\left(A^3\right) = -8\pi\mu C Q \tag{8.3}$$

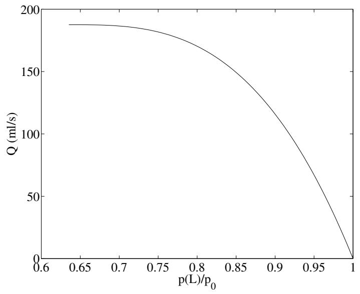


Figure 8.1: Flow versus normalized outlet pressure for Poiseuille flow in compliant pipe.

Integration

$$A(x)^3 = A(0)^3 - 24\pi\mu C Q x \tag{8.4}$$

Constitutive model: $A(p) = A_0 + C\,(p - p_0)$

Pressure and flow for stationary flow in compliant vessel

$$Q(x) = \frac{A(0)^3 - A(x)^3}{24\pi\mu C x}, \quad p(x) = p_0 + \frac{A(x) - A(0)}{C} \tag{8.5}$$

8.2 Infinitesimal derivation of the 1D governing equations for a compliant vessel

In this section the 1D governing equations for mass and momentum will be derived in a somewhat simpler way than in section 8.3. In the derivation we will first consider mass and momentum for a control volume. However, the length of the control volume will later be reduced to zero. Thus, higher order terms may be neglected for several terms.

Some of the first scientific papers on this issue include [2, 15, 37]

8.2.1 Conservation of mass

Blood may normally be considered incompressible (ρ = constant) and thus conservation of mass per time unit reduces to:

$$\dot{V} = Q_i - Q_o \tag{8.6}$$

where $\dot{V}$ denotes rate of change in volume, whereas Q_i and Q_o is volume flow rate in and out of the volume, respectively (see Fig. 8.2.1).

By adopting the convention in (equation (8.42)) of section 8.3 the volumetric influx into the volume Q_i (Fig. 8.2.1) may be expressed by:

$$Q_i = \int_{A_0} v_3 \,\mathrm{dA} = vA \tag{8.7}$$

For the outflux the spatial changes in the velocity in the

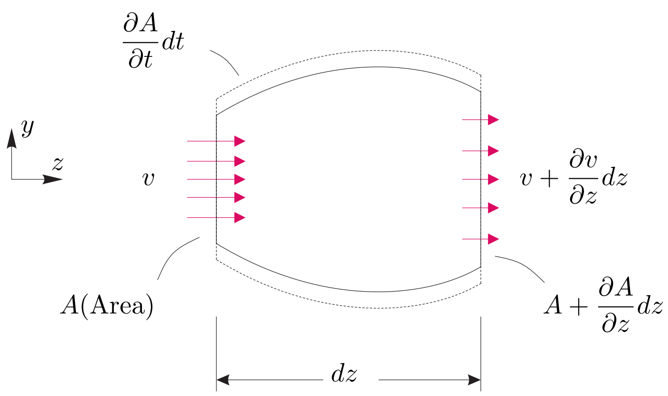


Figure 8.2: A deformable control volume with fixed endpoints.

axial direction must be accounted
for:

$$Q_o = \int_A \left(v_3 + \frac{\partial v_3}{\partial z}\, dz \right)\ \mathrm{dA} \approx vA + \frac{\partial (vA)}{\partial z}\, \mathrm{dz}, \quad A_0 = A + \frac{\partial A}{\partial z}\, dz \tag{8.8}$$

The deformation of the volume is assumed to be homogeneous in the axial direction and consequently the rate of change in volume may be expressed by:

$$\dot{V} \approx \frac{\partial A}{\partial t}\, dz \tag{8.9}$$

Note, that the assumption of homogeneous deformation has no significant bearings as the length dz control volume collapse to zero in the final expression.

Finally, the equation for conservation of mass is obtained by combination of (8.6), (8.7), (8.8) and (8.9) and subsequent division by dz :

$$\frac{\partial A}{\partial t} + \frac{\partial Av}{\partial z} = 0 \tag{8.10}$$

The equation for mass conservation in equation (8.10) may alternatively be derived by using the Reynolds transport theorem for deformable control volumes equation (2.28). As for the former derivation in this section, we assume a control volume which deforms with the compliant vessels, but tether the endpoints (see Fig. 8.2.1). The mass equation is obtained by letting the generic intensive property b be the mass density ρ:

$$\dot{m} = \frac{dm}{dt} = \frac{d}{dt} \int\limits_{V_c(t)} \rho\, dV + \int\limits_{A_c(t)} \rho\, (\mathbf{v} - \mathbf{v}_c) \cdot \mathbf{n}\, dA = 0 \tag{8.11}$$

Note, that the control volume velocity $\mathbf{v}_c = 0$ at the fixed endpoints z_1 and z_2 of $V_c(t)$ the control volume, whereas $\mathbf{v}_c = \mathbf{v}$ at the compliant vessel wall. Thus, the flux terms in equation (8.11) will only give contributions at the endpoints, as the flux at the the vessel wall be zero. Further, the time derivative of the volume integral in equation (8.11) may be put inside the integral sign if the axial coordinate z is iterated first:

$$\dot{m} = \int\limits_{z_1}^{z_2} \frac{\partial (\rho A)}{\partial t}\, dz + (\rho v A)_2 - (\rho v A)_1 = 0 \tag{8.12}$$

where $v = \bar{v}_z$, i.e. the cross-sectional averaged z-component of $\mathbf{v}$, $A = A(z)$ is the cross-sectional area at any location z. The subscripts of the flux terms refers to location z_1 and z_2, respectively. By assuming a constant density ρ of the fluid in the compliant vessel, it may be eliminated from equation (8.12) to yield:

$$\dot{m} = \int\limits_{z_1}^{z_2} \frac{\partial A}{\partial t} + \frac{\partial v A}{\partial z}\, dz = 0 \tag{8.13}$$

As equation (8.13) must be valid for any choice of $V_c(t)$, i.e. z_1 and z_2, the integrand must vanish, and thus the differential form of the mass conservation equation given in equation (8.10) is obtained. As $vA = Q$ an equivalent equation for mass conservation is:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0 \tag{8.14}$$

8.2.2 The momentum equation

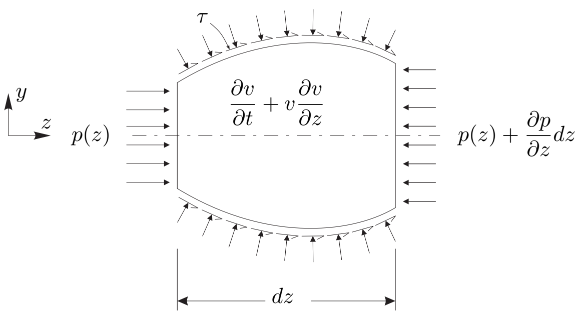


Figure 8.3: Outline of control volume with pressure and velocity

The derivation of the momentum equation is based on Euler's first axiom which states that the rate of linear momentum is balanced balanced by the net force. In this section we will express the various contributions to the net force and subsequently present an expression for the rate of linear momentum, which will lead us to a momentum equation.

The first force contribution we will consider is that of pressure. As can be seen from Fig. 8.2.2, pressure will contribute on all surfaces of our control volume. The pressure force acting on the left hand side will be:

$$F_{p_i} = pA \tag{8.15}$$

On the right hand side the expression for the pressure contribution is somewhat more complicated:

$$F_{p_o} - \left(p + \frac{\partial p}{\partial z}\,dz\right) \tag{8.16}$$

The pressure acting on the surface A will also have an axial contribution which may be shown to be:

$$F_{p_A} = p\frac{\partial A}{\partial z}\,dz \tag{8.17}$$

Wall friction may be accounted for by introducing the wall shear stress τ, which will be a function of the cross-wise velocity profile (see equation equation (8.69)). In order to derive the momentum equation it will suffice to present the viscous force contribution as:

$$F_\tau = \tau\pi D\,dz \tag{8.18}$$

Finally, the net force is found by summation of Eqs. (8.15), (8.16), (8.17), and (8.18):

$$F_{\text{net}} = -A\frac{\partial p}{\partial z}\,dz + \tau\pi D\,dz \tag{8.19}$$

Further, we need an expression for the rate of change of linear momentum $\mathbf{P}$ (see equation (2.5)) in the z-direction, and employ the Reynolds transport theorem for deformable control volumes, in the same manner as for the derivation of the mass conservation equation.

$$\dot{P}_z = \frac{d}{dt}\int\limits_{V_c(t)} \rho v_z\,dV + \int\limits_{A_c(t)} \rho v_z\,(\mathbf{v} - \mathbf{v}_c)\cdot\mathbf{n})\,dA = 0 \tag{8.20}$$

which, by arguing in the same manner as for the mass conservation derivation, may be simplified to:

$$\dot{P}_z = \int_{z_1}^{z_2} \frac{\partial \rho \overline{v_z} A}{\partial t}\, dz + (\rho \overline{v_z^2} A)_2 - (\rho \overline{v_z^2} A)_1 \tag{8.21}$$

By assuming a flat velocity profile we have $\overline{v_z^2} = \bar{v}_z^2$ and by further simplification of the notation by $v = \bar{v}_z$ we get:

Finally, a momentum equation may be formed by assembling the rate of change of linear momentum in equation (8.21) and the net force in equation (8.19):

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\pi D}{\rho A}\,\tau \tag{8.22}$$

Note, that conceptually we have a problem with equation (8.22) as the left hand side is derived for invicid flow, whereas the right hand side has a viscous friction term. However, in this section equation (8.22) will serve as an approximation to the more elaborated expression derived in equation (8.69). The friction term depends on the local, time-dependent velocity profile and must be estimated in some appropriate way.

Together, mass conservation equation equation (8.10) and balance of linear momentum equation (8.22) form a system of partial differential equations:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\pi D}{\rho A}\,\tau \tag{8.23}$$

which are the governing equations for wave propagation in blood vessels.

It can be shown, by using the mass conservation equation, that an alternative formulation of the governing equations, with volume flow Q rather that mean velocity v as the primary variable, satisfy:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0 \tag{8.24a}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial z}\left(\frac{Q^2}{A}\right) = -\frac{A}{\rho}\frac{\partial p}{\partial z} + \frac{\pi\, D}{\rho}\,\tau \tag{8.24b}$$

Regardless of the chosen formulation, the governing equations constitute two differential equations, with three primary variables (either p, v, and A or p, Q, and A). Consequently a constitutive equation, i.e. a relation between pressure and area, is needed to close the system of equations. An example of a simple linear constitutive model is:

$$A(p) = A_0 + C\,(p - p_0), \quad C = \frac{\partial A}{\partial p} \tag{8.25}$$

where the subindex of zero refers to a given state of reference. From equation (8.25) and the chain rule for derivation we get:

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial p}\frac{\partial p}{\partial t} = C\frac{\partial p}{\partial t} \tag{8.26}$$

8.3 Integral derivation of the 1D governing equations for a compliant vessel

8.3.1 1D transport equation

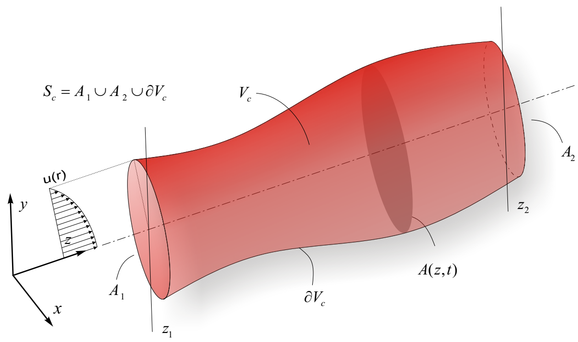


Figure 8.4: A compliant vessel with axial coordinate z and surface area $A(z,t)$

A fluid with constant mass density ρ is flowing through a compliant vessel of volume V_c. The volume is bounded by two *fixed* spatial planes in the xy-plane, denoted A_1 and A_2, with corresponding coordinates z_1 and z_2. Vector components with respect to a fixed spatial coordinate system are denoted with subscripts 1, 2, 3, corresponding to respective spatial directions x, y, z, e.g. v_1, v_2, v_3 are the components of the velocity vector $\mathbf{v}$. The lateral surface (luminary boundary) A of the vessel is allowed to move. However, note that it does not necessarily have to be a material surface with respect to the fluid, as fluid may be allowed to pass through the vessel wall [15]. Thus, the vessel volume V_c, which we will consider as our control volume, has a surface $S_c = A_1 \cup A_2 \cup A_3$.

The Reynolds transport theorem for a moving control volume (equation (2.28) generic density β takes the form:

$$\dot{B} = \frac{d}{dt} \int\limits_{V(t)} \beta \, dV = \frac{d}{dt} \int\limits_{V_c(t)} \beta \, dV + \int\limits_{S_c(t)} \beta \, (\mathbf{v} - \mathbf{v}_c) \cdot \mathbf{n} \, dA \tag{8.27}$$

where $\mathbf{v}_c$ denotes the velocity vector of the moving control volume $V_c(t)$. In order to derive a 1D transport equation for a generic specific property, we will assume a particular controle volume V_c, which is fixed in the axial direction (z-direction in figure 8.4), but follows the walls of the compliant vessel in the directions orthogonal to the chosen spatial direction (i.e. x- and y-directions in figure 8.4). By adopting the conventions of equation (2.21) a the more compact represenation of the RTT is obtained:

$$\frac{dB}{dt} = \frac{dB_c}{dt} + \int_{S_c(t)} \beta(\mathbf{v} - \mathbf{v}_c) \cdot \mathbf{n} \, dA \tag{8.28}$$

The surface integrals may be simplified and split as $S_c = A_1 \cup A_2 \cup A_3$ and $\mathbf{v}_c = 0$ for all $\mathbf{v}_c \in \{A_1, A_2\}$. By focusing on the first term of equation (8.28), we get from *Leibniz's rule* for 3D integrals (see equation (A.44)) or equation (2.27) for dB_c/dt:

$$\frac{dB_c}{dt} = \int_{V_c(t)} \frac{\partial \beta}{\partial t} \, dV + \int_{A_3} \beta \, \mathbf{v}_c \cdot \mathbf{n} \, dA \tag{8.29}$$

To simplify the integral in equation (8.29), we define an iterated them first in the direction which we want to express the spatial dependency and then in the orthogonal directions over which we will average:

$$\int_{V_c} (\cdot)\, dV = \int_{z_1}^{z_2} \left\{ \int_{A(x,t)} (\cdot), dA \right\} dz \quad \text{and} \quad \int_{A_3} (\cdot)\, dA = \int_{z_1}^{z_2} \left\{ \oint_{C(x,t)} (\cdot)\, dl \right\} dz \tag{8.30}$$

where C is the closed curve bounding A_3 orthogonal to the streamwise z-direction, and dl is the corresponding differential line element. Now, both integrals of equation (8.29) may be integrated along the z-axis:

$$\frac{dB_c}{dt} = \int_{z_1}^{z_2} \left(\int_{A_c} \frac{\partial \beta}{\partial t}\, dA + \oint_C \beta\, \mathbf{v}_c \cdot \mathbf{n}\, dl \right) dz \tag{8.31}$$

Further, the cross-sectional mean value $\bar{\beta}$ is defined conventionally as:

$$\bar{\beta} = \frac{1}{A} \int_A \beta\, dA \tag{8.32}$$

and for convenience we define B_s:

$$B_s \equiv A\bar{\beta} = \int_{A(t)} \beta dA \tag{8.33}$$

Based on the definitions in equation (8.33) we get from Leibniz's rule for 2D integrals (see equation (A.40)):

$$\frac{dB_s}{dt} = \int_{A(t)} \frac{\partial \beta}{\partial t}\, dA + \oint_C \mathbf{v}_c \cdot \mathbf{n}\, dl = \frac{d}{dt}\left(A\bar{\beta}\right) = \frac{\partial A\bar{\beta}}{\partial t} \tag{8.34}$$

The latter identity follows as this integral is valid for a fixed z only. This result in equation (8.34) may be substituted into equation (8.31) to yield:

$$\frac{dB_c}{dt} = \int_{z_1}^{z_2} \frac{\partial A\bar{\beta}}{\partial t}\, dz \tag{8.35}$$

To proceed further, the surface integral of equation (8.28) may be split in the following manner:

$$\int_{S_c(t)} \beta\,(\mathbf{v}-\mathbf{v}_c)\cdot\mathbf{n}\, dA = \int_{A_1} \beta\,(\mathbf{v}-\mathbf{v}_c)\cdot\mathbf{n}\, dA + \int_{A_2} \beta\,(\mathbf{v}-\mathbf{v}_c)\cdot\mathbf{n}\, dA + \int_{A_3(t)} \beta\,(\mathbf{v}-\mathbf{v}_c)\cdot\mathbf{n}\, dA \tag{8.36}$$

At the inlet where $z = z_1$, we have $\mathbf{n}_1 = [0, 0, -1]$, whereas at the outlet where $z = z_2$, the normal vector points in positive z-direction $\mathbf{n}_2 = [0, 0, 1]$. Further, $\mathbf{v}_c = 0$ at both z_1 and z_2. Thus, the first and second surface integrals of equation (8.36) may be simplified to:

$$\int_{A_1} \beta\,(\mathbf{v}-\mathbf{v}_c)\cdot\mathbf{n}\, dA + \int_{A_2} \beta\,(\mathbf{v}-\mathbf{v}_c)\cdot\mathbf{n}\, dA = \int_{A_2} \beta\, v_3\, dA - \int_{A_1} \beta\, v_3\, dA = \int_{z_1}^{z_2} \frac{\partial}{\partial z}\left(A(\overline{\beta v_3})\right) dz \tag{8.37}$$

Furthermore, a leakage may be allowed for by introducing the normal component of the relative velocity $v_n = (\mathbf{v} - \mathbf{v}_c) \cdot \mathbf{n}$, and thus the last integral in equation (8.36) may be represented:

$$\int\limits_{A_3(t)} \beta\,(\mathbf{v} - \mathbf{v}_c) \cdot \mathbf{n}\, dA = \int_{z_1}^{z_2} \oint_{C(t)} \beta v_n \, dl\, dz \tag{8.38}$$

Now, by substitution of equations (8.38) and (8.37) into equation (8.36), which again may be substituted into equation (8.28) together with equation (8.35), we obtain the:

1D transport equation for a generic density

$$\frac{dB}{dt} = \int_{z_1}^{z_2} \frac{\partial}{\partial t}(A\bar{\beta}) + \frac{\partial}{\partial z}\Big(A(\overline{\beta v_3})\Big) + \oint_C \beta v_n \, dl\, dz \tag{8.39}$$

Equation (8.39) represent a 1D transport equation for a generic specific property in a compliant vessel, which we will use in the derivation of the mass and momentum equations below.

8.3.2 Mass conservation

The differential equation for mass conservation is obtained from equation (8.39), simply by setting $\beta = 1$. Obviously the right hand side of equation (8.39), vanishes for a constant β and we get:

$$\frac{\partial A}{\partial t} + \frac{\partial A\overline{v_3}}{\partial z} + \oint_C v_n \, dl = 0 \tag{8.40}$$

Further, we define volumetric outflow per unit length and time as:

$$\psi = \oint_C v_n \, dl \tag{8.41}$$

As the equation is 1D we drop the subscripts and the bar for averaged vector components in the streamwise direction:

$$v = \bar{v}_3 = \frac{1}{A}\int\limits_A v_3 \, dA \tag{8.42}$$

and thus equation (8.40) becomes:

$$\frac{\partial A}{\partial t} + \frac{\partial Av}{\partial z} + \psi = 0 \tag{8.43}$$

However, in most application the volumetric source term is neglected and the mass conservation equation takes the form:

$$\frac{\partial A}{\partial t} + \frac{\partial Av}{\partial z} = \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0 \tag{8.44}$$

where the flow rate $Q = Av$ has been introduced as the flow variable in the second expression in equation (8.44).

8.3.3 Momentum equation

By letting the specific property be $\beta = v_3$ (i.e. linear momentum per unit mass in the streamwise direction) in (8.39) we get:

$$\frac{\partial}{\partial t}(Av) + \frac{\partial}{\partial z}\left(A\overline{v_3^2}\right) + \oint_C v_3 v_n \, dl = \int_A \dot{v_3} \, dA \tag{8.45}$$

By repeated use of the chain rule, introduction of the mass equation equation (8.43), and the mathematical identity $\partial v^2/\partial z = 2v\partial v/\partial z$ equation (8.45) may be reformulated:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} + \frac{1}{A}\frac{\partial}{\partial z}\left(A(\overline{v_3^2} - v^2)\right) = \frac{1}{A}\int_A \dot{v_3} \, dA + \frac{v}{A}\psi - \frac{1}{A}\oint_C v_3 v_n \, dl \tag{8.46}$$

For 1D flows it is natural to define a material derivative:

$$\dot{v} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} \tag{8.47}$$

Then, by substitution of equation (8.47) and (8.41) into equation (8.46), we get:

$$\dot{v} + \frac{1}{A}\frac{\partial}{\partial z}\left(A(\overline{v_3^2} - v^2)\right) = \frac{1}{A}\int_A \dot{v_3} \, dA + \frac{1}{A}\oint_C (v - v_3) v_n \, dl \tag{8.48}$$

Notice that both v and v_3 intentionally, appear in equation (8.48). The aim in the following is to derive a momentum equation formulated by means of cross-sectionally averaged quantities only.

To evaluate $\dot{v_3}$ on the right hand side of equation (8.48) we step back to Cauchy's equations for balance of linear momentum:

$$\dot{\mathbf{v}} = \frac{1}{\rho}\nabla \cdot \mathbf{T} + \mathbf{b} \tag{8.49}$$

where $\mathbf{T}$ is the stress tensor and $\mathbf{b}$ is the body force vector. For a Newtonian fluid the constitutive equation is given by:

$$\mathbf{T} = -p\,\mathbf{I} + 2\mu\,\mathbf{D} \tag{8.50}$$

where $\mathbf{I}$ is the identity tensor, μ is the dynamic viscosity and $\mathbf{D}$ is the rate of deformation tensor. Substitution of equation (8.50) into equation (8.49) yields the Navier-Stokes equations:

$$\dot{\mathbf{v}} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{b} \tag{8.51}$$

Here the kinematic viscosity is denoted by $\nu = \mu/\rho$. On component form equation (8.51) reads[1]:

$$\dot{v_i} = -\frac{1}{\rho}p_{,i} + \nu v_{i,kk} + b_i \tag{8.52}$$

[1] Here we use the standard Einstein convention of summation of repeated indexes and comma for partial differentiation.

Motivated by the arguments of Hughes and Lubliner in [15], we define appropriate velocity scales:

$$V = \max(v_3), \qquad U = \max(v_2, v_3) \tag{8.53}$$

and assume that:

$$\epsilon = \frac{U}{V} \ll 1 \tag{8.54}$$

i.e. that mean transverse velocities are small compared to mean axial velocities. Further, as spatial scale we define a mean "radius" to be $R = (A_0/\pi)^{\frac{1}{2}}$, where A_0 is the mean cross sectional vessel area over a cycle. Based on these scales we define the following nondimensional primed numbers:

$$\begin{aligned}
x_3 &= \frac{R}{\epsilon}\, x_3', & x_\alpha &= R\, x_\alpha' \\
v_3 &= V\, v_3', & v_\alpha &= U\, v_\alpha' \\
t &= \frac{R}{U}\, t', & p &= \rho V^2\, p' \\
b_3 &= \frac{UV}{R}\, b_3', & b_\alpha &= \frac{U^2}{R}\, b_\alpha'
\end{aligned} \tag{8.55}$$

where α takes the values $1, 2$. By introduction of the scales in equation (8.55) into equation (8.52), and letting $\epsilon \to 0$, and the resulting dimensional equations become:

$$\dot{v}_3 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial}{\partial x_\alpha}\left(\nu \frac{\partial v_3}{\partial x_\alpha}\right) + b_3 \tag{8.56}$$

$$\frac{\partial p}{\partial x_\alpha} = 0 \tag{8.57}$$

Note that through this scaling, the streamwise derivative has disappeared ($\partial^2 v_3/\partial x^2 \propto \epsilon$) in the streamwise momentum equation equation (8.56) (as α only takes the values $1, 2$). Further, equation (8.57) means that the pressure is approximately constant (i.e. $p(z,t) = \bar{p}(z,t)$) in the crosswise directions.

Now we may integrate equation (8.56) over the vessel cross section, and use the Gauss theorem to obtain:

$$\int_A \dot{v}_3\, dA = -\frac{A}{\rho}\frac{\partial p}{\partial z} + \oint_C \nu \frac{\partial v_3}{\partial x_\alpha}\, n_\alpha\, dl + Ab \tag{8.58}$$

where $\mathbf{n} = [n_1, n_2]$ is the outward unit vector[2] but on C in the xy-plane and $b = 1/A \int b_3\, dA$.

A 1D equation for balance of linear momentum for pulsatile flow in a compliant vessel is then obtained by substitution of the expression for $\int_A \dot{v}_3\, dA$ in equation (8.58) equation (8.48):

$$\dot{v} + \frac{1}{A}\frac{\partial}{\partial z}\left(A(\overline{v_3^2} - v^2)\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b + \frac{1}{A}\oint_C \nu \frac{\partial v_3}{\partial x_\alpha}\, n_\alpha + (v - v_3)v_n\, dl \tag{8.59}$$

The equation still contains both v and v_3 and in order to proceed further, assumptions have to be made for the velocity profile of v_3.

[2]We could have used three dimensions for both the normal vector and the velocity gradient, here as well, but we use α-notation to stress that the streamwise derivative has been discarded.

Example 8.1 Momentum equations for invicid flow

A simple zeroth order approximation is to abandon the no-slip boundary condition by assuming $v_3 = v$ which:

$$\dot{v} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b \tag{8.60}$$

which is nothing but the inviscid Navier-Stokes equations in 1D.

Example 8.2 Momentum equations for polynomial velocity profiles

In order to account for viscous losses and thus provide estimates of local wall shear stresses, a crosswise velocity profile must be introduced for v_3 in some way. Hughes and Lubliner simply assumes

$$v_3 = \phi v \tag{8.61}$$

the profile function must satisfy the conditions:

$$\phi|_C = 0 \tag{8.62}$$

$$\overline{v_3^2} - v^2 = \delta v^2 \tag{8.63}$$

where a nonlinear correction factor has been introduced to simplify the expressions in Eq. (8.59). The correction factor is given by:

$$\delta = \frac{1}{A}\int\limits_A (\phi^2 - 1)\, dA \tag{8.64}$$

For axisymmetric case one might postulate a polynomial velocity profile on the form:

$$\phi = C_1(1 - (\frac{r}{R})^n) = C_1(1 - \tilde{r}^n) \tag{8.65}$$

where R is the radius of the vessel, whereas $\tilde{r} = r/R$ is a nondimensional radius, and C_1 is an arbitrary constant to be determined below. For $n = 2$ equation (8.65) correspond to a parabolic profile, while the profile becomes blunter for higher values of n and approaches a flat profile as $n \to \infty$. For this polynomial velocity profile the correction factor integral equation (8.64) has an analytical solution:

$$\begin{aligned}
\delta &= \frac{1}{\pi R^2}\int_0^R (\phi^2 - 1)\, 2\pi r\, dr = 2\int_0^1 (\phi^2 - 1)\, \tilde{r}\, d\tilde{r} \\
&= 2C_1^2 \int_o^1 (1 - C_1^{-2})\tilde{r} - 2\tilde{r}^{n+1} + \tilde{r}^{2n+1}\, d\tilde{r} \\
&= 2C_1^2 \left[\frac{1 - C_1^{-2}}{2}\tilde{r}^2 - \frac{2}{n+2}\tilde{r}^{n+2} + \frac{1}{2(n+1)}\tilde{r}^{2(n+1)}\right]_0^1 \\
&= \frac{C_1^2 n^2 - (n+2)(n+1)}{(n+2)(n+1)}
\end{aligned} \tag{8.66}$$

Now, by inspection of equation (8.66) we see that if C_1 is chosen to be:

$$C_1 = \frac{n+2}{n} \Rightarrow \phi = \frac{n+2}{n}(1 - \tilde{r}^n) \tag{8.67}$$

the expression for the correction factor in equation (8.66) reduces to:

$$\delta = \frac{1}{n+1} \tag{8.68}$$

Thus, $\delta = 1/3$ for Poiseuille flow (n=2) while $\delta \to 0$ as the profile becomes blunter (i.e. $n \to \infty$). Substitution of equation (8.63) into (8.59) and assumption of zero outflow $w_n = 0$ yields:

$$\dot{v} + \frac{\delta}{A}\frac{\partial}{\partial z}\left(Av^2\right)) = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b + \frac{v}{A}\oint_C \nu \frac{\partial \phi}{\partial x_\alpha} n_\alpha \, dl \tag{8.69}$$

This is the simplest form of momentum balance accounting for viscous forces based on a velocity profile function. Note that equation (8.69) is valid as long as equation (8.61) and (8.63) are satisfied, i.e. no assumption of a polynomial, axisymmetric, velocity profile is mandatory.

By multiplying equation (8.69) with A we get:

$$A\frac{\partial v}{\partial t} + Av\frac{\partial v}{\partial z} + \delta\frac{\partial}{\partial z}\left(Av^2\right) = -\frac{A}{\rho}\frac{\partial p}{\partial z} + Ab + v\oint_C \nu \frac{\partial \phi}{\partial x_\alpha} n_\alpha \, dl \tag{8.70}$$

which by using the chain rule may be re-written as:

$$\frac{\partial Q}{\partial t} - v\frac{\partial A}{\partial t} - v\frac{\partial Q}{\partial z} + \frac{\partial}{\partial z}\left(\frac{Q^2}{A}\right) + \delta\frac{\partial}{\partial z}\left(Av^2\right) = -\frac{A}{\rho}\frac{\partial p}{\partial z} + Ab + v\oint_C \nu \frac{\partial \phi}{\partial x_\alpha} n_\alpha \, dl \tag{8.71}$$

The second and third term vanish due to conservation of mass equation (8.44), which has the same mathematical representation regardless of whether the velocity profile is accounted for or not. Thus, a conservative formulation of equation (8.69) is:

$$\frac{\partial Q}{\partial t} + (1+\delta)\frac{\partial}{\partial z}\left(\frac{Q^2}{A}\right) = -\frac{A}{\rho}\frac{\partial p}{\partial z} + Ab + v\oint_C \nu \frac{\partial \phi}{\partial x_\alpha} n_\alpha \, dl \tag{8.72}$$

8.4 THE WAVE NATURE OF THE PRESSURE AND FLOW EQUATIONS

By introducing equation (8.26) into the linearized and inviscid form of equation (8.24a) and (8.24b), the derivative of the cross-sectional area is eliminated we get:

$$C\frac{\partial p}{\partial t} + \frac{\partial Q}{\partial z} = 0 \tag{8.73}$$

$$\frac{\partial Q}{\partial t} = -\frac{A}{\rho}\frac{\partial p}{\partial z} \tag{8.74}$$

By cross-derivation and subtraction of equation (8.73) and (8.74) the following differential equations are obtained:

$$\frac{\partial^2 p}{\partial t^2} - c_0^2\frac{\partial^2 p}{\partial z^2} = 0 \tag{8.75}$$

$$\frac{\partial^2 Q}{\partial t^2} - c_0^2\frac{\partial^2 Q}{\partial z^2} = 0 \tag{8.76}$$

where we have introduced the *pulse wave velocity* for inviscid flows:

$$c_0^2 = \frac{\partial p}{\partial A}\frac{A}{\rho} = \frac{1}{C}\frac{A}{\rho} \tag{8.77}$$

Thus, equation (8.75) and (8.76) have both the form of a classical wave equation and one may show that they together have the general solutions:

$$p = p_0\, f(z - c\,t) + p_0^*\, g(z + c\,t) \tag{8.78}$$
$$Q = Q_0\, f(z - c\,t) + Q_0^*\, g(z + c\,t) \tag{8.79}$$

where f and g represents waves traveling with wave speed c forward and backward, respectively.

8.4.1 The Moens-Korteweg formula for pulse wave velocity

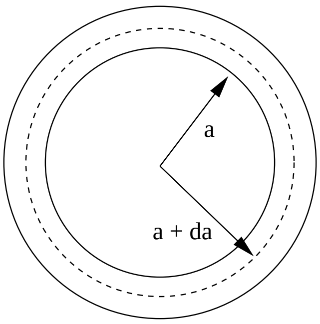


Figure 8.5: Thin walled cylinder with inner radius a and a small change in the radius of $a + da$.

To obtain the Moens-Korteweg formula for pulse wave velocity the circumferential stress and strain are assumed to satisfy Hooke's law:

$$\Delta\sigma_\theta = \eta\,\varepsilon_\theta = \eta\,\frac{da}{a} \tag{8.80}$$

as the circumferential strain ε_θ is given by:

$$\varepsilon_\theta = \frac{2\pi(a + da) - 2\pi a}{2\pi a} = \frac{da}{a} \tag{8.81}$$

Under the assumption of small deformations, the circumferential stresses before and after deformation ($\sigma_{\theta 1}$ and $\sigma_{\theta 2}$) are given by:

$$\sigma_{\theta 1} = \frac{pa}{h}, \qquad \sigma_{\theta 2} \approx \frac{(p + dp)a}{h} \tag{8.82}$$

which yields:

$$\Delta\sigma_\theta = \sigma_{\theta 2} - \sigma_{\theta 1} \approx \frac{dp\;a}{h} \tag{8.83}$$

Then by combination of equation (8.80) and (8.83):

$$\eta\frac{da}{a} = \eta\frac{dA}{2\,A} = \frac{dp\;a}{h} \tag{8.84}$$

which yields the expression:

$$\frac{dp}{dA} = \frac{\eta h}{2Aa} = \frac{1}{C} \tag{8.85}$$

This is an expression for the inverse compliance which may be substituted into the general expression for the pulse wave velocity in equation (8.77), to give the Moens-Korteweg formula for pulse wave velocity:

$$c_0^2 = \frac{1}{C}\frac{A}{\rho} = \frac{\eta h}{2\rho a} \tag{8.86}$$

8.5 CHARACTERISTIC IMPEDANCE

By introducing equation (8.78) and (8.79) into equation (8.74) one obtains:

$$\begin{aligned} -Q_0\, c\, f' + Q_0^*\, c\, g' &= -\frac{A}{\rho}\,(p_o f' + p_0^* g') \\ f'\left(p_0 \frac{A}{\rho} - Q_0 c\right) + g'\left(p_0^* \frac{A}{\rho} + Q_0^* c\right) &= 0 \end{aligned} \tag{8.87}$$

As equation (8.87) must hold for arbitrarily chosen f and g, an expression for the *characteristic impedance* Z_c is obtained:

$$Z_c \equiv \frac{p_0}{Q_0} = \frac{\rho\, c}{A} = -\frac{p_0^*}{Q_0^*} \tag{8.88}$$

From the expression above characteristic impedance is seen to be the ratio of the pulsatile pressure and flow components in the case of a unidirectional wave, i.e. in absence of reflections. The Z_c can also be shown to express the ratio of local inertance (ρ/A) to compliance capacity (C) as:

$$c = \sqrt{\frac{A}{\rho}\frac{1}{C}}, \quad C = \frac{\partial A}{\partial p} \tag{8.89}$$

which by substitution into equation (8.88) yields:

$$Z_c = \sqrt{\frac{\rho}{A}\frac{1}{C}} \tag{8.90}$$

Thus, the characteristic impedance is a quantity that relates to both geometry and the elastic properties of the vessel.

8.6 INPUT IMPEDANCE

The input impedance Z_{in} is defined in a very similar manner as the characteristic impedance, namely as the ratio of the pulsatile components of pressure and flow:

$$Z_{in}(\omega) = \frac{P(\omega)}{Q(\omega)} \tag{8.91}$$

where uppercase denotes the Fourier-component of the corresponding lowercase primary variable for a given angular frequency ω. However, the input impedance is *not* restricted to unidirectional waves, i.e. reflected wave components are included. Thus, Z_{in} is a global quantity that characterize the properties distal (downstream) to the measuring point. The cumulative effect of all distal contributions is incorporated in the input impedance. In the aorta Z_{in} represents the afterload on the heart.

In Fig. 8.6 the input impedance of a young healthy subject is depicted. For high frequencies the phase angles are close to zero, as high frequency components are more damped and reflections tend to cancel out. The negative phase angle for the low-frequency components correspond to that flow components lead pressure, i.e. the aorta first sees flow and the pressure.

8.7 WAVE REFLECTIONS

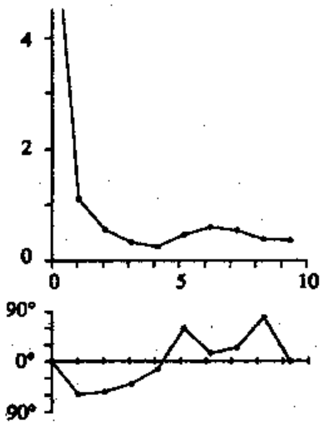


Figure 8.6: The input impedance of a young healthy subject (Adapted from O'Rourke 1992).

So far we have discussed propagation of pressure and flow waves in an infinitely long, straight, cylindrical, elastic vessel filled with an incompressible inviscid fluid. However, for real blood vessels we have:

- short, curved, tapered, bifurcating vessels
- the vessel walls exhibit nonlinear viscoelastic properties
- the blood is viscous

It turns out that the effect of nonlinear viscoelasticity is not so severe, as the Womersley number often is sufficiently large. However, the "infinitely long" assumption must be removed. When pressure and flow reach an end they must conform to the end conditions. Consequently, the waves will be modified and reflections will occur. The concept of wave reflection is easily understood for a single pulse which travels to the end of the vessel and is reflected. However, for a train of pulses or continuous oscillations, the reflected waves will interfere with the original pulse. In this case, the only evidence of reflections is spatial variations in amplitude of the waves. Reflections occur wherever there is a change in the characteristic impedance (mismatch in impedance).

The reflection factor

Let p_f denote the oscillatory pressure associated with the forward propagating wave and p_b associated with the reflected, backward propagating wave. Further, let the flow wave be split into forward and backward components in the same manner. These components superimpose to form the actual values:

$$p = p_f + p_b \tag{8.92}$$

The reflection factor Γ is then defined as:

$$\Gamma \equiv \frac{p_b}{p_f} = -\frac{Q_b}{Q_f} \tag{8.93}$$

The reflection factor may also be expressed in terms of the input impedance $Z_{in} = p/Q$ and the characteristic impedance $Z_c = p_f/Q_f = -p_b/Q_b$:

$$Q = \frac{p}{Z_{in}} = \frac{p_f}{Z_c} - \frac{p_b}{Z_c} \tag{8.94}$$

$$\frac{Z_{in}}{Z_c} = \frac{p_f + p_b}{p_f - p_b} = \frac{1+\Gamma}{1-\Gamma} \tag{8.95}$$

And thus:

$$\Gamma = \frac{Z_{in} - Z_c}{Z_{in} + Z_c} \tag{8.96}$$

The quarter wavelength formula

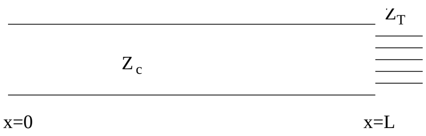


Figure 8.7: An elastic vessel terminated with Z_T

To illustrate the effect of reflections on the input impedance, let us consider a frictionless vessel with a total reflective impedance at the end, i.e. $\Gamma = 1$ (see Fig. 8.7). Let the forward waves at the inlet of the vessel take the form:

$$p_f = p_0 \, e^{j\omega t}, \quad Q_f = \frac{p_0}{Z_c} \, e^{j\omega t} \tag{8.97}$$

These waves travel the length L with pulse wave velocity c, are totally reflected, and must then travel the same distance back. Thus, the expressions for the reflected waves are:

$$p_b = p_0 \, e^{j\omega(t-2L/c)}, \quad Q_b = -\frac{p_0}{Z_c} \, e^{j\omega(t-2L/c)} \tag{8.98}$$

And the input impedance at the inlet of the vessel is:

$$Z_{in} = \frac{p_f + p_b}{Q_f + Q_b} = Z_c \, \frac{e^{j\omega t} + e^{j\omega(t-2L/c)}}{e^{j\omega t} - e^{j\omega(t-2L/c)}} \tag{8.99}$$

From equation (8.99) we see that $Z_{in} = 0$ whenever:

$$\frac{2\omega L}{c} = \pi \tag{8.100}$$

As $\omega = 2\pi f$ and $\lambda = c/f$ *the quarter wave length formula* is obtained:

$$L = \frac{\lambda}{4} \tag{8.101}$$

Thus, from the first minimum of Z_{in} an indication of "the effective length" to the major reflection site of the arterial system may be obtained.

For example if the first minimum of $|Z_{in}|$ is found at 3.8 Hz, an estimate of $L \approx 0.33$ m may be obtained by assuming a typical pulse wave velocity of $c \approx 5$ m/s.

8.8 General equations with reflection and friction

For the linearized, frictionless case obtained previously in equation (8.78):

$$p = p_0 \, f(z - ct) + p_0^* \, g(z - ct) = p_f + p_b$$

and for a frictionless vessel the forward component was conveniently expressed as:

$$p_f = p_0 \, f(z - ct) = P_0 \, e^{j\omega(t - x/c)}$$

Viscous friction may conveniently be incorporated by the introduction of a complex *propagation coefficient*:

$$\gamma = \frac{j\omega}{\hat{c}} = a + jb \tag{8.102}$$

where a is the *attenuation constant*, b the *phase constant*, and $\hat{c}$ a complex pulse wave velocity. An expression a may be obtained from the equation for the shear stress equation (8.18), however this is discarded here for brevity. The phase constant b is related to the pulse wave velocity by: $c = \omega/b$. Having adopted this convention, the forward propagating waves incorporating viscous friction may be expressed:

$$p_f = p_0 \, e^{j\omega t} \, e^{-\gamma z}, \quad Q_f = Q_0 \, e^{j\omega t} \, e^{-\gamma z}$$

And further, by assuming that the waves propagates in a vessel depicted in Fig. 8.7, one may obtain expressions for the backward propagating components also:

$$\begin{aligned} p_f(L) &= p_f(0) e^{-\gamma L}, \qquad p_f(0) = p_0 e^{j\omega t} \\ p_b(L) &= \Gamma \, p_f(L) = \Gamma \, p_f(0) \, e^{-\gamma L} \\ p_b(0) &= p_b(L) \, e^{-\gamma L} = \Gamma \, p_f(0) \, e^{-2\gamma L} \end{aligned}$$

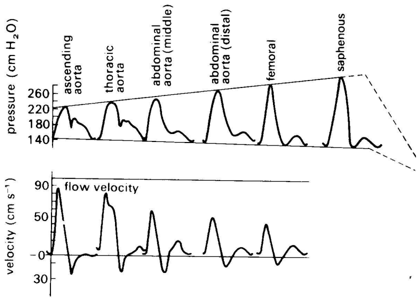


Figure 8.8: Spatial variation in pressure and velocity in the arterial tree (McDonald, 1974)

The forward and backward components at the inlet of the vessel may subsequently be superimposed to form the resulting pressure:

$$p(0) = p_f(0) + p_b(0) = p_f(0) \, (1 + \Gamma e^{-2\gamma L}) \tag{8.103}$$

This solution incorporates both viscous friction and reflections.

By taking into account that the reflected flow wave opposes the forward wave, an expression for the total flow at the inlet may be obtained in the same manner:

$$\begin{aligned} Q_f(L) &= Q_f(0) e^{-\gamma L}, \qquad Q_f(0) = Q_0 e^{j\omega t} \\ Q_b(L) &= -\Gamma \, Q_f(L) = -\Gamma \, Q_f(0) \, e^{-\gamma L} \\ Q_b(0) &= Q_b(L) \, e^{-\gamma L} = -\Gamma \, Q_f(0) \, e^{-2\gamma L} \end{aligned}$$

which yields:

$$Q(0) = Q_f(0) + Q_b(0) = Q_f(0) \, (1 - \Gamma e^{-2\gamma L}) \tag{8.104}$$

Further, from equation (8.103) and (8.104) an expression for the input impedance may also be found:

$$Z_{in} = Z_c \, \frac{1 + \Gamma \, e^{-2\gamma L}}{1 - \Gamma \, e^{-2\gamma L}}, \quad Z_c = \frac{p_f(0)}{Q_f(0)} \tag{8.105}$$

or alternatively:

$$Z_{in} = Z_c \, \frac{Z_T + Z_c \, \tanh(\gamma L)}{Z_c + Z_T \, \tanh(\gamma L)} \tag{8.106}$$

where:

$$\Gamma = \frac{Z_T - Z_c}{Z_T + Z_c}, \qquad Z_T = \frac{p(L)}{Q(L)}$$

From equation (8.103) and (8.104) it is clearly seen that a positive reflection factor Γ will cause an *amplification in pressure*, whereas the flow will be reduced. Thus, the presence of reflections will cause the pressure and flow waves to have different forms. Such reflections are believed to explain the streamwise increase in pressure amplitude in the arterial tree (Fig. 8.8).

8.9 WAVE PROPAGATION IN BLOOD VESSELS

Use [41, chap 6] and own stuff.

8.10 WAVE SEPARATION

Several methods have been suggested to separate measured pressure and flow into forward and backward traveling components. This has been motivated by the notion that waves traveling from the heart toward the periphery contain information related to the heart, while the reflected waves contain information related to the periphery. The simplest method, which is still the method of choice for most applications, was suggested by Westerhof et al. [47]. This method is based on the linearized and inviscid form of the governing equations.

$$p = p_f + p_b, \quad Q = Q_f + Q_b = \frac{p_f}{Z_c} - \frac{p_b}{Z_c} \tag{8.107}$$

which by simple algebraic elimination yield:

$$p_f = \frac{p + Z_c Q}{2}, \qquad p_b = \frac{p - Z_c Q}{2} \tag{8.108a}$$

$$Q_f = \frac{Z_c Q + p}{2Z_c}, \qquad Q_b = \frac{Z_c Q - p}{2Z_c} \tag{8.108b}$$

In a normal healthy subject the reflections in the aorta come in the diastole after the aortic valve has closed. Thus, they are believed of enhance coronary perfusion. However, for elderly subjects with "stiffer" vessels the reflections may arrive in the systole. Thus, the reflections will increase the aortic systolic pressure and thereby increase the heart load. Such an condition may lead to hypertrophy.

8.11 WAVE TRAVEL AND REFLECTION

8.12 NETWORKS 1D COMPLIANT VESSELS

The propagation of pressure and flow waves in the arterial system and how they influence stenotic regions, aneurysms and other vascular diseases, has been the subject of many studies [8, 9, 16, 28, 32, 35, 43, 44].

Recently there have also been a renewed interest for such 1D network models, as they both may provide valuable physiological insight in patient specific modalities (morphometric rendering of the arterial network) in addition to that they may provide better boundary conditions for 3D FSI models [3, 5, 6, 33, 34, 42].

Network models have also been used for various segments of the arterial tree. To provide better understanding of the genesis of ischemia that develops in the gastrointestinal system, a model for the mesenteric arterial system have been developed to simulated realistic blood flow during normal conditions [22].

One of the early papers on this topic was on the blood flow in the human arm [4]. It used FEM to discretize the equations and a modified Windkessel model for the distal boundary conditions.

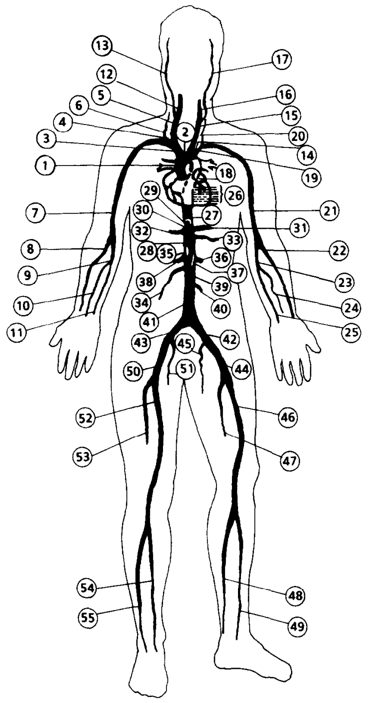


Figure 8.9: A model of the human arterial system (from Stergiopulos et al.).

A hybrid on-dimensional/Womersley model of pulsatile blood flow in the entire coronary arterial tree have also been developed recently[16].

Blood flow in the circle of Willis (CoW) has been modeled using the 1D equations of pressure and flow wave propagation in compliant vessels [1]. The model starts at the left ventricle and includes the largest arteries that supply the CoW. Based on published physiological data, it is able to capture the main features of pulse wave propagation along the aorta, at the brachiocephalic bifurcation and throughout the cerebral arteries.

8.12.1 Numerical solution of the 1D equations for compliant vessels and boundary conditions

When the governing 1D equations for compliant vessels (see e.g. equation (8.44) and (8.72)) are to be solved numerically, it is beneficial to recast them in the following generic form:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{M}\frac{\partial \mathbf{u}}{\partial x} = 0 \tag{8.109}$$

In particular, the formulation equation (8.109) and what follows this section is important for the implementation of the boundary conditions. For the internal domain, any kind of numerical discretization may be employed.

By choosing the state vector as $\mathbf{u} = [p, Q]^T$, the equations may be cast into the canonical given in equation (8.109) with:

$$\mathbf{M} = \begin{bmatrix} 0 & \dfrac{1}{C} \\ \dfrac{A}{\rho} - (1+\delta)\, v^2 C & 2(1+\delta)\, v \end{bmatrix} \tag{8.110}$$

The off diagonal elements in $\mathbf{M}$ represent a coupling in time and space between the components in the state vector, in this case the pressure p and the flow rate Q.

In the following we will transform the equation system represented by equation (8.109), to a form where the rows of the equation system are independent. In order to do this we introduce the the diagonal eigenvalue-matrix of $\mathbf{M}$:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{8.111}$$

the relation between $\mathbf{M}$ and it's eigenvalues and corresponding eigenvectors may be written:

$$\mathbf{M}\,\mathbf{R} = \mathbf{R}\,\mathbf{\Lambda} \tag{8.112}$$

The right eigenmatrix $\mathbf{R}$ is composed of the right eigenvectors as *columns*. From basic linear algebra one may prove that the left eigenmatrix is $\mathbf{L}$ related to the right eigenmatrix by $\mathbf{L} = \mathbf{R}^{-1}$. The i-th *row* of $\mathbf{L}$, denoted $\mathbf{l}_i$, is the left eigenvector of $\mathbf{M}$ corresponding to the i-th eigenvalue λ_i. From equation (8.112) we see that $\mathbf{M} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{L}$ and thus by premultiplication of $\mathbf{L}$ in equation (8.109), we obtain:

$$\mathbf{L}\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\Lambda}\mathbf{L}\frac{\partial \mathbf{u}}{\partial x} = 0 \tag{8.113}$$

Further, we may introduce a change of variables such that

$$\frac{\partial \boldsymbol{\omega}}{\partial \mathbf{u}} = \mathbf{L} \tag{8.114}$$

where $\boldsymbol{\omega} = [\omega_1, \omega_2]^T$ is the vector of characteristic variables, also commonly denoted Riemann invariants, which in general satisfy:

$$\boldsymbol{\omega} = \int_{\mathbf{u}_0}^{\mathbf{u}} \frac{\partial \boldsymbol{\omega}}{\partial \mathbf{u}}\, d\mathbf{u} = \int_{\mathbf{u}_0}^{\mathbf{u}} \mathbf{L}\; d\mathbf{u} \approx \mathbf{L}(\mathbf{u}_\epsilon)\,\Delta\mathbf{u}, \qquad \text{and} \quad \Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0 \tag{8.115}$$

The latter approximation follows from the mean value theorem A.4.3 and is valid for some $\mathbf{u}_0 \leq \mathbf{u}_\epsilon \leq \mathbf{u}$. In particular, the approximation is good if the change in the state variable vector from $\mathbf{u}_0$ to $\mathbf{u}$ is small. This will typically be the case for an explicit numerical scheme where $\mathbf{u}_0$ refers to the state vector at the previous timestep and the timestep is small due to the CFL-limitation.

With this change of variables and use of the chain rule equation (8.113) transforms to a decoupled system of canonical wave equations:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\Lambda}\frac{\partial \boldsymbol{\omega}}{\partial x} = 0 \tag{8.116}$$

which on component form reads:

$$\frac{\partial \omega_1}{\partial t} + \lambda_1\frac{\partial \omega_1}{\partial x} = 0 \tag{8.117a}$$

$$\frac{\partial \omega_2}{\partial t} + \lambda_2\frac{\partial \omega_2}{\partial x} = 0 \tag{8.117b}$$

For real eigenvalues λ_1 and λ_2, the Riemann invariants ω_1 and ω_2 are scalars which will propagate as waves, with wavespeeds λ_1 and λ_2, respectively. That is they have solutions:

$$\omega_1 = \hat{\omega}_1\, f(x - \lambda_1 t) \tag{8.118}$$

$$\omega_2 = \hat{\omega}_2\, g(x - \lambda_2 t) \tag{8.119}$$

Note that when λ_1 and λ_2 have different signs, the waves will travel in opposite directions. For our particular case in equation (8.110) on may find that the eigenvalues are:

$$\begin{aligned} \lambda_{1,2} &= (1+\delta)v \pm \sqrt{(1+\delta)^2 v^2 + c^2 - (1+\delta)v^2} \\ &= \delta' v \pm c^* \end{aligned} \tag{8.120}$$

where we have introduced for simplicity:

$$\delta' = (1+\delta), \qquad c^* = c\sqrt{1 + \delta'(\delta' - 1)\mathcal{M}^2} \qquad \text{and} \qquad \mathcal{M} = v/c \tag{8.121}$$

Here, c denote the invisicid pulse wave velocity as defined in equation (8.77). The δ' is simply a modified correction factor, whereas c^* is the modified wave speed due to the velocity field, and $\mathcal{M}$ is a Mach number. Note that a flat velocity profile corresponds to $\delta = 0$, and consequently $\delta' = 1$ and $c^* = c$. Normally, $v \ll c^*$ and therefore we see from equation (8.120) that $\lambda_1 > 0$ and $\lambda_2 < 0$, and therefore ω_1 will travel to the right (or in the positive coordinate direction) whereas ω_2 will travel to the left (or in the negative coordinate direction) according to equation (8.117).

The corresponding left eigenmatrix may be taken as:

$$\mathbf{L} = \begin{bmatrix} 1 & -1/\lambda_2 C \\ 1 & -1/\lambda_1 C \end{bmatrix} = \begin{bmatrix} 1 & Z_c^b \\ 1 & -Z_c^f \end{bmatrix} \tag{8.122}$$

where we have introduced the modified characteristic impedances for the forward and backward traveling waves as Z_c^f and Z_c^b, respectively. From equations (8.115) and (8.122) we get:

$$\boldsymbol{\omega} = [\omega_1, \omega_2]^T = \mathbf{L}(\mathbf{u}_\epsilon)\,\Delta\mathbf{u} = [\Delta p + Z_c^b \Delta Q, \quad \Delta p - Z_c^f \Delta Q]^T \tag{8.123}$$

Note, that the $Z_c^f = Z_c^f(p, Q)$ and $Z_c^b = Z_c^b(p, Q)$, and are to be evalued in the given range $\mathbf{u}_0 \leq \mathbf{u}_\epsilon \leq \mathbf{u}$. Normally, the value at the previous timestep may be used.

The Riemann invariants $\boldsymbol{\omega}$ are conventionally found from the internal field and from the boundary conditions. Subsequently the updated values of the primary variables are found at the boundary by:

$$\Delta\mathbf{u} = \mathbf{L}(\mathbf{u}_\epsilon)^{-1}\boldsymbol{\omega} = \mathbf{R}(\mathbf{u}_\epsilon)\boldsymbol{\omega} \tag{8.124}$$

where:

$$\mathbf{R} = \frac{1}{Z_c^f + Z_c^b} \begin{bmatrix} Z_c^f & Z_c^b \\ 1 & -1 \end{bmatrix} \tag{8.125}$$

The increments in the primary variables may also be expressed by means of the Riemann invariants by rearrangements of the expressions in equation (8.123):

$$\Delta Q = \frac{\omega_1 - \omega_2}{Z_c^f + Z_c^b}, \qquad \Delta p = \frac{Z_c^f \omega_1 + Z_c^b \omega_2}{Z_c^f + Z_c^b} \tag{8.126}$$

Note that for waves superimposed on a quiescent fluid fluid, i.e. for $v = 0$ in equation (8.120), then $\lambda_1 = -\lambda_2$ and $Z_c^f = Z_c^b = Z_c$, and consequently equation (8.126) simplifies to:

$$\Delta Q = \frac{\omega_1 - \omega_2}{2 Z_c}, \qquad \Delta p = \frac{\omega_1 + \omega_2}{2} \tag{8.127}$$

now we may define:

$$\Delta Q_f = \frac{\omega_1}{Z_c^f + Z_c^b}, \qquad \Delta Q_b = \frac{-\omega_2}{Z_c^f + Z_c^b} \tag{8.128}$$

$$\Delta p_f = \frac{Z_c^f\,\omega_1}{Z_c^f + Z_c^b}, \qquad \Delta p_b = \frac{Z_c^b\,\omega_2}{Z_c^f + Z_c^b} \tag{8.129}$$

which by substitution into equation (8.126) yields:

$$\Delta Q = \Delta Q_f + \Delta Q_b \tag{8.130}$$

$$\Delta p = \Delta p_f + \Delta p_b \tag{8.131}$$

Inlet boundary conditions. A prescribed flow at the inlet boundary may be imposed in such a way that the wave approaching this boundary from the interior field is also handled in a way which respects the physics or the differential equations for the problem. Clearly, as flow is imposed we know the value of Q_{in}, the pressure, however, has to be computed and will be a function both of the flow at the inlet and of waves coming from the interior. The mathematical consequence of this outset is that two conditions need to be fulfilled, namely that the flow Q_{in} may be taken as a time varying function $Q_{in} = Q(t)$, whereas the outgoing wave will be accounted for by imposing $\mathbf{l_2} \cdot \mathbf{u_{in}} = \omega_2$. Note we assume here that the inlet is taken at the left boundary, i.e. where the x-coordinate is at minimum.

Now, to compute the resulting pressure from the imposed flow Q_{in} and the (potential) wave ω_2 from the interior, we construct an equation system similar to equation (8.115). As we need to respect the physics from the interior we keep the second row of $\mathbf{L}$, corresponding to the wave from the interior ω_2, but modify the first row to ensure that the flow is imposed:

$$\mathbf{L}_{in}\mathbf{u}_{in} = \boldsymbol{\omega}_{in} \tag{8.132}$$

where:

$$\mathbf{L}_{in} = \begin{bmatrix} 0 & 1 \\ 1 & -Z_c^f \end{bmatrix} \quad \text{and} \quad \boldsymbol{\omega}_{in} = \begin{bmatrix} Q(t) \\ \omega_2 \end{bmatrix} \tag{8.133}$$

By inspecting equations (8.132) and (8.133), we see that the above mentioned conditions are satisfied. The inverse of $\mathbf{L}_{in}$ may be computed as:

$$\mathbf{L}_{in}^{-1} = \begin{bmatrix} Z_c^f & 1 \\ 1 & 0 \end{bmatrix} \tag{8.134}$$

and the resulting pressure p_{in} (and flow) from the imposed flow may be computed as:

Imposed flow

$$\mathbf{u}_{in} = \begin{bmatrix} p_{in} \\ Q_{in} \end{bmatrix} = \mathbf{L}_{in}^{-1} \cdot \boldsymbol{\omega}_{in} = \begin{bmatrix} Z_c^f Q(t) + \omega_2 \\ Q(t) \end{bmatrix} \tag{8.135}$$

and we clearly see from equation (8.135), that the pressure has contributions from both the imposed flow and the wave from the interior ω_2. Observe also that with no reflected wave present (i.e. $\omega_2 = 0$, pressure and flow are related with the characteristic impedane Z_c^f, as expected.

A similar strategy may be taken for an imposed time varying pressure $p_i = p(t)$:

$$\mathbf{L}_{in} = \begin{bmatrix} 1 & 0 \\ 1 & -Z_c^f \end{bmatrix} \quad \text{and} \quad \mathbf{L}_{in}^{-1} = \begin{bmatrix} 1 & 0 \\ -1/Z_c^f & 1/Z_c^f \end{bmatrix} \quad \text{and} \quad \boldsymbol{\omega}_{in} = \begin{bmatrix} p(t) \\ \omega_2 \end{bmatrix} \tag{8.136}$$

which yields the following equations for the inlet:

Imposed pressure

$$\mathbf{u}_{in} = \begin{bmatrix} p_{in} \\ Q_{in} \end{bmatrix} = \begin{bmatrix} p(t) \\ \dfrac{\omega_2 - p(t)}{Z_c^f} \end{bmatrix} \tag{8.137}$$

We observe from equation (8.137) that the inlet flow Q_{in} has contributions both from the imposed pressure $p(t)$ and from the interior wave ω_2. In the absence of a reflected wave ω_2, pressure and flow are related with the characteristic impdeance Z_c^f as expected for unidirectional waves.

Impose forward flow/pressure. Impose $\Delta Q_f = Q_0$. From equation (8.128) ω_1 may be computed as:

$$\omega_1 = (Z_c^f + Z_c^b)\Delta Q_f = (Z_c^f + Z_c^b)\Delta Q_0 \tag{8.138}$$

Extrapolation from interior field:

$$\omega_2 = w_2(t^n, \lambda_2 \Delta t) \tag{8.139}$$

and compute values at the inlet by:

$$\boldsymbol{\Delta}\mathbf{u}_{in} = \mathbf{R}\boldsymbol{\omega} \tag{8.140}$$

Outlet boundary conditions. In the numerical scheme for the governing equations (8.44) and (8.72), a values for ω_1 are ω_2 are needed in order to prescribe the pressure/flow at the outlet in a way which repsects both the wave ω_1 coming from the inside of the vessel and the wave entering the vessel ω_2. This amounts to solving equation (8.124) at the boundary.

For the outlet (i.e. for $x = L$) the outgoing Riemann invariant may be extrapolated from the interior field from the previous timestep. Whenever source terms are negligible in equation (8.117a), the outgoing Riemann invariant at the outlet is constant along the characteristic curve $dx/dt = \lambda_1$, and thus we can approximate:

$$\omega_1^{n+1} = \omega_1(t_n, L - \lambda_1^n \Delta t) \tag{8.141}$$

The question now is how one may obtain a value for ω_2, which represents the incoming wave due to the two element Windkessel. Terminal vessels are often modeled with lumped models Windkessel model, which is represented by the differential equation (7.14b). On incremental form the two element Windkessel mode has the corresponding differential equation:

$$\Delta Q = C \frac{\partial \Delta p}{\partial t} + \frac{\Delta p}{R} \tag{8.142}$$

Now, by substiution of equation (8.127), which expresses the primary variables p, and Q by means of the Riemann invariants, into the differential equation (8.142) which models the physics of the exterior, a differential equation for ω_2 is obtained:

$$Z_c^b C \frac{d\omega_2}{dt} + \omega_2 \left(1 + \frac{Z_c^b}{R}\right) = Z_c^f C \frac{d\omega_1}{dt} + \omega_1 \left(1 - \frac{Z_c^f}{R}\right) \tag{8.143}$$

As the right hand side of equation (8.143) is known from previous values of ω_1 and from the extrapolated current value given by equation (8.141), an updated value for ω_2 may readily be obtained by an appropriate discretization of equation (8.143), e.g. backward Euler.

8.12.2 Lumped heart model: varying elastance model

The varying elastance model, is a phenemenological model of the left ventricular function, originally proposed by Suga et al. in [38, 39]

$$E(t) = \frac{p_v(t)}{V(t) - V_0} \tag{8.144}$$

Modified version

$$E(t) = \frac{p_v(t)}{(V(t) - V_0)\,(1 + KQ_v(t))} \tag{8.145}$$

with K as a resistance in the left ventricle. Normally, K is rather small, and therefore a first approximation is to assume it to be zero as in equation (8.144).

As $V(t)$ is the instantaneous left ventricular volume, the flow rate $Q_v(t)$ ejected by the left ventricle is given by $Q_v(t) = -dV/dt$. From the varying elastance model in equation (8.144) we get:

$$Q_v(t) = -\frac{dV}{dt} = \frac{dE}{dt}\left(\frac{1}{E^2}\right) p_v - \frac{1}{E}\frac{dp_v}{dt} \tag{8.146}$$

from the representation we see that $1/E$ plays the role of a compliance and:

$$\frac{d}{dt}\left(\frac{1}{E}\right) \tag{8.147}$$

the role of a resistance [10].

The varying elastance model may be used as an inlet condition, subject to the assumption that $Q = Q_v$ and $p = p_v$ at the inlet. As we now are at the inlet, we may extraplate the outgoing Riemann invariant ω_2 in the same manner as in equation (8.141):

$$\omega_2^{n+1} = \omega_2(t_n, \lambda_2^n \Delta t) \tag{8.148}$$

Further, the differential equation for ω_1 may be obtained by substitution of equation (8.126) into the differential equation (8.146) for the varying elastance model:

$$\frac{Z_c^f}{E}\frac{d\omega_1}{dt} + \omega_1\left(1 - \frac{Z_c^f}{E^2}\frac{dE}{dt}\right) = -\frac{Z_c^b}{E}\frac{d\omega_2}{dt} + \omega_2\left(1 + \frac{Z_c^b}{E^2}\frac{dE}{dt}\right) \tag{8.149}$$

Thus, an updated value at time level $n+1$ may be obtained for (ω_1^{n+1}) by e.g. a backward Euler dicretization of equation (8.149).

When, modelling the interaction of the heart and the cardiovacular system, one obviously have to account for the presence of the aortic valve and the fact that it is closed during diastole. The closure of the aortic valve is normally attributed to a negative pressure difference between the left ventricle and the aorta.

From equations (8.144) and (8.146) on may estimate an updated left ventricular pressure at the next time level:

$$p_v^{n+1} = E(t)\left(V_n - Q_n \Delta t - V_0\right) \tag{8.150}$$

The negative pressure difference criterion for valve closure will therefore be $p_v < p^{n+1}$. Whenever the valve is closed, there is no outflow from the heart which corresponds $Q_v = 0$ in equation (8.146). Note that despite a closed aortic valve, waves may still arrive from the periphery $\omega_2 \neq 0$, and such waves will be reflected at the closed valve. A complete reflection ($\Gamma = 1$), is a first approximation which corresponds to $\omega_1 = \omega_2$).

The pressure criteria for valve closure together with the varying elastance model as inlet condition may therefore be summarized to:

$$\omega_1 = \begin{cases} \omega_2 & \text{if } p_v > p \\ (A.35) & \text{if } p_v \leq p \end{cases} \tag{8.151}$$

8.12.3 Nonlinear wave separation

Expressions for nonlinear wave separation follow readily from equation (8.126) as ω_1 is a forward propagating Riemann invariant and ω_2 is a backward propagating Riemann invariant. The forward/backward propagating pressures and flows are found simply by setting $\omega_2 = 0$ and $\omega_1 = 0$, respectively, in equation (8.127):

$$\Delta Q_f = \frac{\omega_1}{Z_c^f + Z_c^b}, \qquad \Delta Q_b = \frac{-\omega_2}{Z_c^f + Z_c^b} \tag{8.152a}$$

$$\Delta p_f = \frac{Z_c^f \omega_1}{Z_c^f + Z_c^b}, \qquad \Delta p_b = \frac{Z_c^b \omega_2}{Z_c^f + Z_c^b} \tag{8.152b}$$

Naturally, for the more simpler case of a quiescent fluid fluid, the previously derived expressions for linear wave separation (see equation (8.108) in section 8.10) may be obtained from equation (8.127).

8.13 Fluid structure interaction for small deformations in Hookean vessel

8.13.1 The governing equations for the Hookean vessel

The averaged Cauchy equations

The Cauchy equations were derived in section 2.4 in cylindrical coordinates (see equation (2.102). We will in this section average the equations for the z-direction and r-direction over the vessel wall, assuming azimuthal symmetry (i.e. all $\partial(\cdot)/\partial\theta - \text{terms} = 0$). Based on this assumption the Cauchy equation (2.102c) in the z-direction may be integrated over the vessel wall to yield:

$$\int_{A_w} \rho_w a_z \ \mathrm{dA} = \int_{A_w} \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{zr}\right) + \frac{\partial \sigma_z}{\partial z} + \rho_w b_z\right) \ \mathrm{dA} \tag{8.153}$$

where ρ_w denotes the density of the vessel wall, which is assumed to be constant the following derivation.

By introducing the common notation that a bar-ed quantity denote the cross-sectional averaged of the same quantity without a bar, i.e. $\overline{(\cdot)} = \int_A (\cdot)\ \mathrm{dA}/A$, the first term of equation (8.153) may be reformulated to:

$$\int_{A_w} \rho_w a_z \ \mathrm{dA} = \rho_w \bar{a}_z A_w \approx \rho_w \bar{a}_z 2\pi r_i h \tag{8.154}$$

where A_w is the vessel wall area, i.e. $A_w = \pi(r_o^2 - r_i^2)$, and r_i and r_o denote inner and outer radius, respectively and h the wall thickness. For a thin walled vessels $h/r_i \ll 1$:

$$A_w = \pi(r_o^2 - r_i^2) = \pi(2r_i h + h^2) \approx 2\pi r_i h \tag{8.155}$$

and thus the approximation in equation (8.154) is valid for thin walled structures.

By expanding $dA = r\, dr d\theta$, the first term in the rhs of equation (8.153) take the form:

$$\int_0^{2\pi}\int_{r_i}^{r_o} \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{zr}\right)\ r\, dr d\theta = 2\pi r \tau_{rz}\big|_{r_i}^{r_o} = 2\pi r_i \tau_w \tag{8.156}$$

where we have assumed $\tau_{rz}|_{r=r_o} = 0$ and $\tau_{rz}|_{r=r_i} = -\tau_w$.

The second term on the rhs of equation (8.153) may similarly be written:

$$\int \frac{\partial \sigma_z}{\partial z}\ \mathrm{dA} = \overline{\frac{\partial \sigma_z}{\partial z}}\, A_w \approx \overline{\frac{\partial \sigma_z}{\partial z}}\, 2\pi r_i h \tag{8.157}$$

The last term of equation (8.153) may be treated in exactly the same manner as the lhs. Thus, by substitution of equations (8.154), (8.156), and (8.157) into equation (8.153) we get:

$$\rho_w \bar{a}_z A_w = 2\pi r_i \tau_w + \overline{\frac{\partial \sigma_z}{\partial z}}\, A_w + \rho_w \bar{b}_z A_w \tag{8.158}$$

which by neglection of body forces and assumption of a thin walled structure reduces to:

$$\rho_w \bar{a}_z\, h = \overline{\frac{\partial \sigma_z}{\partial z}}\, h + \tau_w \tag{8.159}$$

Thus, equations (8.158) and (8.159) represent cross-sectional averaged Cauchy equations in the axial direction.

In order to derive the cross-sectional averaged Cauchy equations in the radial, we proceed in the same way as above, for the Cauchy equation (2.102a) in the radial direction:

$$\int \rho a_r \ \mathrm{dA} = \int \left(\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{rz}}{\partial z} + \rho b_r \right)\ \mathrm{dA} \tag{8.160}$$

For the first and last terms of equation (8.160) we proceed in the same manner as for the axial Cauchy equation, whereas for the first term of the rhs we get:

$$\int \frac{\partial \sigma_r}{\partial r}\, r\, dr\, d\theta = 2\pi \sigma_r r\big|_{r_i}^{r_o} - \int \sigma_r\, dr\, d\theta \tag{8.161}$$

To evaluate the expression we need boundary conditions and assume $\sigma_r|_{r=r_i} = -p$ and $\sigma_r|_{r=r_o} = 0$. Further, we introduce the hat-symbol for another averaging procedure:

$$\hat{\sigma}_r = \frac{1}{2\pi h} = \int_0^{2\pi} \int_{r_i}^{r_o} \sigma_r \, dr \, d\theta \tag{8.162}$$

which by substitution into equation (8.161) yields:

$$\int \frac{\partial \sigma_r}{\partial r} \, r \, dr \, d\theta = 2\pi p r_i - \hat{\sigma}_r \, 2\pi h \tag{8.163}$$

By using the hat-convention for the second term of equation (8.160) we get:

$$\int \frac{\sigma_r - \sigma_\theta}{r} \, r \, dr \, d\theta = \hat{\sigma}_r \; 2\pi h - \hat{\sigma}_\theta \; 2\pi h \tag{8.164}$$

while the third term of equation (8.160) may be expressed as:

$$\int \frac{\partial \tau_{rz}}{\partial z} \; \mathrm{dA} = \overline{\frac{\partial \tau_{rz}}{\partial z}} \, A_w \approx \overline{\frac{\partial \tau_{rz}}{\partial z}} \, 2\pi r_i h \tag{8.165}$$

Finally, we may substitute equations (8.163), (8.164), and (8.165) into (8.160) to get:

$$\rho_w \bar{a}_r A_w = 2\pi r_i p - \hat{\sigma}_\theta \, 2\pi h + \overline{\frac{\partial \tau_{rz}}{\partial z}} \, A_w + \rho_w \bar{b}_r A_w \tag{8.166}$$

which represents a cross-sectionally averaged Cauchy equation in the radial direction. For a thin-walled structure without body forces, equation (8.166) reduce to:

$$\rho_w \bar{a}_r h = p - \frac{\hat{\sigma}_\theta \, h}{r_i} + \overline{\frac{\partial \tau_{rz}}{\partial z}} \, h \tag{8.167}$$

The constitutive equation for plane stress

In general the constitutive equations for a Hookean material in a plane stress situation have been provided previously in equation (4.33). For cylindrical coordinates (r, θ, z) we have also derived previously equation (save for some notation) that the circumferential strain $E_{\theta\theta} = u_r/r$, when u_r denote radial displacement. Thus, in engineering notation the constitutive equation for a thin walled vessel of a Hookean material, takes the form in cylindrical coordinates:

$$\sigma_z = \frac{\eta}{1 - \nu_p^2} \left(\frac{\partial u_z}{\partial z} + \nu_p \frac{u_r}{r} \right) \quad \text{and} \quad \sigma_\theta = \frac{\eta}{1 - \nu_p^2} \left(\frac{u_r}{r} + \nu_p \frac{\partial u_z}{\partial z} \right) \tag{8.168}$$

In the following we drop the symbols for averaging, and substitute equation (8.168) in the averaged Cauchy equations ((8.159) and (8.167)):

$$\rho_w a_z \, h = \frac{\eta \, h}{1 - \nu_p^2} \left(\frac{\partial^2 u_z}{\partial z^2} + \frac{\nu_p}{r} \frac{\partial u_r}{\partial z} \right) + \tau_w \tag{8.169a}$$

$$\rho_w a_r h = p - \frac{\eta \, h}{(1 - \nu_p^2) \, r_i} \left(\frac{u_r}{r} + \nu_p \frac{\partial u_z}{\partial z} \right) \tag{8.169b}$$

while assuming equation (8.168) to be a valid constitutive equation also for averaged stress/strain relations[3]. Assume further, that $a_z = \partial^2 u_z/\partial t^2$ and $a_r = \partial^2 u_r/\partial t^2$, i.e. neglect convective terms, to obtain the averaged governing equations for the thin walled vessel:

$$\frac{\partial^2 u_z}{\partial t^2} = \frac{\eta}{(1-\nu_p^2)\rho_w}\left(\frac{\partial^2 u_z}{\partial z^2} + \frac{\nu_p}{r}\frac{\partial u_r}{\partial z}\right) + \frac{\tau_w}{\rho_w\, h} \tag{8.170a}$$

$$\frac{\partial^2 u_r}{\partial t^2} = \frac{p}{\rho_w\, h} - \frac{\eta}{(1-\nu_p^2)\,\rho_w}\left(\frac{u_r}{r^2} + \frac{\nu_p}{r}\frac{\partial u_z}{\partial z}\right) \tag{8.170b}$$

where we implicitly assume that $r \approx r_i$.

8.13.2 The governing equations for the fluid

The momentum equations without convective terms:

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right) \tag{8.171a}$$

$$\frac{\partial v_r}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)\right) \tag{8.171b}$$

The equation for conservation of mass takes the form in cylindrical coordinates:

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 \tag{8.172}$$

The momentum equations

Assume solutions on the form:

$$p = \hat{p}e^{i\omega\,(t-z/c)}, \quad v_z = \hat{v}_z e^{i\omega\,(t-z/c)}, \quad v_r = \hat{v}_r e^{i\omega\,(t-z/c)} \tag{8.173}$$

which by substitution into equation (8.171a) yields:

$$i\omega\;\hat{v}_z = -\frac{1}{\rho}\left(\frac{-i\omega}{c}\right)\;\hat{p} + \frac{\nu}{r_i^2}\left(\frac{\partial^2 \hat{v}_z}{\partial y^2} + \frac{1}{y}\frac{\partial \hat{v}_z}{\partial y}\right) \tag{8.174}$$

where we have introduced the nondimensional scale $y = r/r_i$ and the Womersley parameter , previously introduced in equation (5.100),

$$\alpha = r_i\sqrt{\frac{\omega}{\nu}} \tag{8.175}$$

Rearrangement of equation (8.174) yields:

$$\frac{\partial^2 \hat{v}_z}{\partial y^2} + \frac{1}{y}\frac{\partial v_z}{\partial y} + i^3\alpha^2\hat{v}_z = \frac{i^3\alpha^2}{\rho c}\;\hat{p} \tag{8.176}$$

Now, by introducing another scale:

$$s = i^{3/2}\alpha y = i^{3/2}\alpha r/r_i \tag{8.177}$$

[3] The term $\overline{\frac{\partial \tau_{rz}}{\partial z}}$ was discarded as ???.

equation (8.176) may be transformed to an inhomogeneous Bessel equation of order zero (see equation (5.112)):

$$\frac{\partial^2 \hat{v}_z}{\partial s^2} + \frac{1}{s}\frac{\partial v_z}{\partial s} + \hat{v}_z = \frac{1}{\rho c}\,\hat{p} \tag{8.178}$$

By proceeding in the same manner for equation (8.171b) we get:

$$i\omega \hat{v}_r = -\frac{1}{\rho r_i}\frac{\partial \hat{p}}{\partial y} + \frac{\nu}{r_i^2}\left(\frac{\partial^2 \hat{v}_r}{\partial y^2} + \frac{1}{y}\frac{\partial \hat{v}_r}{\partial y} - \frac{\hat{v}_r}{y^2}\right) \tag{8.179}$$

which by rearrangement may be presented:

$$\frac{\partial^2 \hat{v}_r}{\partial y^2} + \frac{1}{y}\frac{\partial \hat{v}_r}{\partial y} + i^3\alpha^2 \hat{v}_r - \frac{\hat{v}_r}{y^2} = \frac{r_i}{\mu}\frac{\partial \hat{p}}{\partial y} \tag{8.180}$$

which also may be transformed to a Bessel equation by introducing the scale in equation (8.177), albeit of order one:

$$\frac{\partial^2 \hat{v}_r}{\partial s^2} + \frac{1}{s}\frac{\partial \hat{v}_r}{\partial s} + \left(1 - \frac{1}{s^2}\right)\hat{v}_r = \frac{i r_i}{\mu \alpha^2}\frac{\partial \hat{p}}{\partial y} = \frac{r_i}{\mu i^{3/2}\alpha}\frac{\partial \hat{p}}{\partial s} \tag{8.181}$$

For the continuity equation (8.172) we get:

$$\frac{\partial \hat{v}_r}{\partial r} + \frac{1}{r}\hat{v}_r - \frac{i\omega}{c}\hat{v}_z = 0 \tag{8.182}$$

which by introduction of the scale in equation (8.177) is transformed to:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\hat{v}_r\right) = -\frac{i^{3/2}\sqrt{\nu\omega}}{c}\hat{v}_z \tag{8.183}$$

or alternatively:

$$\frac{1}{y}\frac{\partial}{\partial y}\left(y\hat{v}_r\right) = -\frac{i r_i \omega}{c}\hat{v}_z \tag{8.184}$$

The solutions of homogeneous Bessel equations of order zero and one, like equations (8.178) and (8.181), are given by their corresponding Bessel functions of order zero and one. Thus for general solutions of equations (8.178) and (8.181) we must provide particular solutions. Assume that a particular solution $\hat{v}_z^p$ of equation (8.178) is given by:

$$\hat{v}_z^p = B_1 J_o(ks) \quad \text{with} \quad \hat{p} = A J_0(ks) \tag{8.185}$$

where k is to be determined. Substitution into equation (8.178) gives:

$$\frac{\partial^2 \hat{v}_z}{\partial s^2} + \frac{1}{s}\frac{\partial v_z}{\partial s} + \hat{v}_z = \frac{1}{\rho c}\,\hat{p} B_1 k^2 \frac{d^2}{dt^2}J_0(t) + \frac{B_1 k}{s}\frac{d}{dt}J_0(t) + B_1 J_0(ks) = \frac{A}{\rho c}J_0(ks) \tag{8.186}$$

where $t = ks$. From the useful properties of the Bessel functions given in (A.48) and (A.49) we may deduce:

$$\frac{d}{dt}J_0(t) = -J_1(t) \tag{8.187a}$$

$$\frac{d}{dt}\left(tJ_1(t)\right) = tJ_0(t) = J_1(t) + t\frac{d}{dt}J_1(t) \tag{8.187b}$$

$$\frac{d^2}{dt^2}J_0(t) = -\frac{d}{dt}J_1(t) = \frac{J_1(t)}{t} - J_0(t) \tag{8.187c}$$

Equations (8.187a) and (8.187c) may subsequently be substituted into equation (8.186) to yield:

$$
\begin{aligned}
B_1 k^2 \left(\frac{J_1(ks)}{ks} - J_0(ks) \right) - \frac{B_1 k}{s} J_1(ks) + B_1 J_0(ks) \\
= B_1(1-k^2) J_0(ks) = \frac{A}{\rho c} J_0(ks)
\end{aligned}
\tag{8.188}
$$

Thus, equation $\hat{v}_z^p$ in (8.185) is a valid particular solution provided that:

$$
B_1 = \frac{A}{(1-k^2)\rho c} \tag{8.189}
$$

and a general solution of equation (8.178) is given by:

$$
\hat{v}_z = \hat{v}_z^h + \hat{v}_z^p = \frac{C_1}{J_0(i^{3/2}\alpha)}\ J_0(s) + \frac{A}{\rho c\,(1-k^2)}\ J_0(ks) \tag{8.190}
$$

where C_1 and k are constants to be determined[4].

Similarly, we propose a particular solution for equation (8.181) of the form:

$$
\hat{v}_r^p = B_2 J_1(ks) \tag{8.191}
$$

and we deduce from equation (8.187a) and (8.187c):

$$
\frac{d^2}{dt^2} J_1 = \frac{d}{dt} J_0 - \frac{d}{dt}\left(\frac{J_1}{t} \right) = \left(\frac{1}{t^2} - 1 \right) J_1 - \frac{1}{t}\frac{d}{dt} J_1 \tag{8.192}
$$

Substitution of equations (8.191) and (8.192) into equation (8.181) yields:

$$
\begin{aligned}
B_2 k^2 \left(\frac{1}{(sk)^2} - 1 \right) J_1 - \frac{B_2 k^2}{sk}\frac{dJ_1}{dt} + \frac{B_2 k}{s}\frac{dJ_1}{dt} + \left(1 - \frac{1}{s^2} \right) B_2 J_1 \\
= B_2\ (1-k^2) J_1(ks) = \frac{r_i A}{\mu i^{3/2}\alpha}\frac{\partial \hat{p}}{\partial s} = \frac{-r_i A k}{\mu i^{3/2}\alpha}\ J_1(ks)
\end{aligned}
\tag{8.193}
$$

where we have used:

$$
\frac{\partial \hat{p}}{\partial s} = -Ak\ J_1(ks) \tag{8.194}
$$

From equation (8.193) we get that $\hat{v}_r^p$ in equation (8.191) is a particular provided that:

$$
B_2 = \frac{r_i A k}{\mu i^{3/2}\alpha (k^2 - 1)} \tag{8.195}
$$

And consequently, a general solution of equation (8.181) is:

$$
\hat{v}_r = \hat{v}_r^h + \hat{v}_r^p = \frac{C_2}{J_0(i^{3/2}\alpha)}\ J_1(s) + \frac{r_i A}{\mu i^{3/2}\alpha}\frac{k}{k^2-1}\ J_1(ks) \tag{8.196}
$$

[4] C_1 is scaled with $J_0(i^{3/2}\alpha)$ for analogy with rigid pipe solution.

Fulfillment of the continuity equation

The two general solutions given by equations (8.190) and (8.196) must also satisfy the continuity equation (8.183). Observe first that:

$$\begin{aligned}\frac{1}{s}\left(\frac{d}{ds}(\hat{v}_r s)\right) &= \frac{1}{s}\left(\frac{d}{ds}(\hat{v}_r^h s) + \frac{d}{ds}(\hat{v}_r^p s)\right)\\ &= \frac{1}{s}\frac{d}{ds}(\hat{v}_r^h s) + \frac{1}{s}\frac{d}{dt}(\hat{v}_r^p(t)t)\end{aligned} \tag{8.197}$$

From equation (8.196) we get:

$$\frac{1}{s}\frac{d}{ds}(\hat{v}_r^h s) = \frac{C_2}{J_0(i^{3/2}\alpha)}J_0(s) \tag{8.198}$$

by the use of equation (8.187b) and similarly:

$$\hat{v}_r^p(t)t = \frac{r_i A k}{\mu i^{3/2}\alpha(k^2-1)}\ J_1(t)t \tag{8.199}$$

and thus:

$$\begin{aligned}\frac{1}{s}\frac{d}{dt}(\hat{v}_r^p(t)t) &= \frac{r_i A k}{\mu i^{3/2}\alpha(k^2-1)}\ \frac{sk}{s}J_0(sk)\\ &= \frac{r_i A k^2}{\mu i^{3/2}\alpha(k^2-1)}J_0(sk)\end{aligned} \tag{8.200}$$

Substitution of equation (8.198) and (8.200) in equation (8.196) results in the following expression:

$$\frac{1}{s}\left(\frac{d}{ds}(\hat{v}_r s)\right) = \frac{C_2}{J_0(i^{3/2}\alpha)}J_0(s) + \frac{r_i A k^2}{\mu i^{3/2}\alpha(k^2-1)}J_0(ks) \tag{8.201}$$

which is the left hand side of the continuity equation (8.183). For the right hand side of the continuity equation we get by substitution of equation (8.190):

$$\begin{aligned}-\frac{i^{3/2}\sqrt{\nu\omega}}{c}\hat{v}_z &= \frac{i r_i \omega}{i^{3/2}\alpha c}\hat{v}_z\\ &= \frac{i r_i \omega}{i^{3/2}\alpha c J_0(i^{3/2}\alpha)}\ C_1\ J_0(s) + \frac{i r_i \omega}{i^{3/2}\alpha c}\ \frac{A}{\rho c\,(1-k^2)}\ J_0(ks)\end{aligned} \tag{8.202}$$

By comparing equation (8.201) and equation (8.202) we find that the ratio of C_2 and C_1 must satisfy:

$$\frac{C_2}{C_1} = \frac{i r_i \omega}{i^{3/2}\alpha c} \tag{8.203}$$

and k must fulfill the condition:

$$\frac{k^2}{\mu} = \frac{i^3\omega}{\rho c^2} \quad \Rightarrow \quad k = \pm\frac{i^{3/2}\sqrt{\nu\omega}}{c} = \pm\frac{i^{3/2} r_i \omega}{\alpha c} \tag{8.204}$$

Thus, one may argue that for physiological values of ν, ω, and c we will have $|k| \ll 1$. Further, by introducing the definition of the scale in equation (8.177) we get:

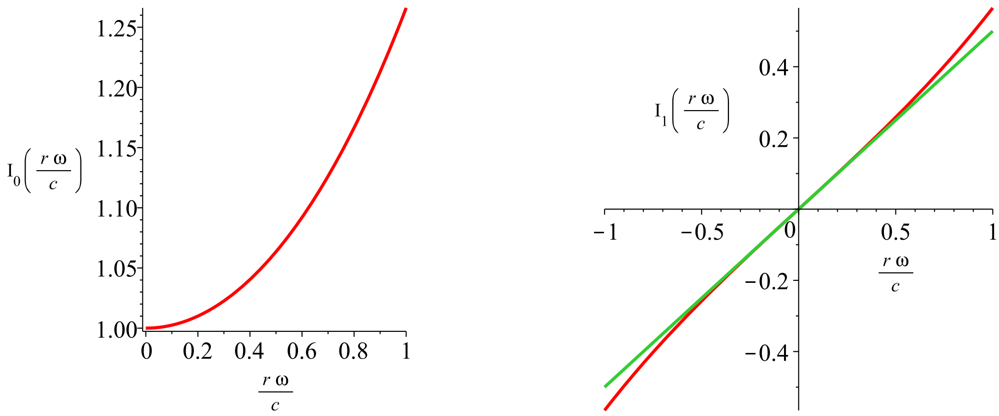
$$ks = \pm \frac{i^{3/2} r_i \omega}{\alpha c} i^{3/2} \alpha y = \mp \frac{i r_i \omega}{c} y = \mp \frac{i\omega}{c} r \tag{8.205}$$

For terms involving $J_0(ks)$ and $J_1(ks)$ one may approximate:

$$J_0(ks) = I_0(\mp \frac{r_i \omega}{c} y) \approx 1 \quad \text{and} \quad J_1(ks) = I_1(\mp \frac{r_i \omega}{c} y) \approx \mp \frac{i r_i \omega}{2c} y \tag{8.206}$$

where I_n denotes the modified Bessel function of order n (see equation (A.53) in section A.5). From equation (8.204) and (8.206) we get:

$$kJ_1(ks) \approx \pm \frac{i^{3/2} r_i \omega}{\alpha c} \mp \frac{i r_i \omega}{2c} y = -\frac{i^{3/2} r_i \omega}{\alpha c} \frac{i r_i \omega}{2c} y \tag{8.207}$$



(a) Modified Bessel function of order zero, first kind I_0.

(b) Modified Bessel function of order one, first kind I_1 (red), linear approximation (green).

Figure 8.10: Plots of modified Bessel functions of order zero and one of first kind.

The solution in the radial direction in equation (8.196) may now be simplified by first using $|k| \ll 1|$:

$$\hat{v}_r \approx \frac{C_2}{J_0(i^{3/2}\alpha)} J_1(s) - \frac{r_i A}{\mu i^{3/2} \alpha} k\, J_1(ks) \tag{8.208}$$

and then using the results in equation (8.203) and (8.207) which results in:

$$\hat{v}_r \approx \frac{i r_i \omega}{2c} \left(\frac{2C}{\alpha i^{3/2} J_0(i^{3/2}\alpha)} J_1(i^{3/2}\alpha\, y) + \frac{A}{\rho c} y \right) \tag{8.209}$$

The solution in the axial direction $\hat{v}_z$ in equation (8.190) may be simplified, by taking the approximations in equation (8.206) and $|k| \ll 1$ into account, to yield:

$$\hat{v}_z \approx \frac{C}{J_0(i^{3/2}\alpha)} \, J_0(i^{3/2}\alpha\, y) + \frac{A}{\rho c} \tag{8.210}$$

where we have dropped the subscript of C for convenience, i.e. $C = C_1$.

Boundary conditions

In order to provide boundary conditions for the structural equations given in section 8.13.1 we must evaluate the solutions in the axial and radial directions at the inner surface of the vessel:

$$\hat{v}_z(y=1) = C + \frac{A}{\rho c} \tag{8.211}$$

$$\hat{v}_r(y=1) = \frac{i r_i \omega}{2c}\left(C F_{10}(\alpha) + \frac{A}{\rho c}\right) \tag{8.212}$$

where we have introduced the Womersley function F_{10}:

$$F_{10}(\alpha) = \frac{2 J_1(i^{3/2}\alpha)}{\alpha i^{3/2}\, J_0(i^{3/2}\,\alpha)} \tag{8.213}$$

An expression for $\partial\hat{v}_z/\partial y$ must be provided to estimate the wall shear stress at the vessel wall, which is needed in equation (8.159). By differentiation of equation (8.190) we get by introducing equation (8.205):

$$\begin{aligned} \frac{\partial \hat{v}_z}{\partial y} &\approx \frac{C}{J_0(i^{3/2}\alpha)} \, \frac{d}{dy} J_0(i^{3/2}\alpha\, y) + \frac{A}{\rho c}\, \frac{d}{dy} J_0\left(\mp\frac{i r_i \omega}{c} y\right) \\ &= -\frac{C i^{3/2}\alpha}{J_0(i^{3/2}\alpha)} \, J_1(i^{3/2}\alpha\, y) - \frac{A}{\rho c}\left(\mp\frac{i r_i\omega}{c}\right) \, J_1\left(\mp\frac{i r_i \omega}{c} y\right) \end{aligned} \tag{8.214}$$

and then from equation (8.213), (8.205) and (8.214) we get:

$$\frac{\partial \hat{v}_z}{\partial y}(y=1) = -\frac{C}{2} i^3 \alpha^2 F_{10}(\alpha) + \frac{1}{2}\left(\frac{\omega r_i}{c}\right)^2 \, \frac{A}{\rho c} \tag{8.215}$$

8.13.3 Coupling of structure and fluid

Assume solutions of the governing equations (8.170) for the vessel wall on the form:

$$u_r = D e^{i\omega(t - z/c)} \quad \text{and} \quad u_z = E e^{i\omega(t-z/c)} \tag{8.216}$$

The condition that the fluid velocity and the structural velocity must be equal at the inner vessel wall, i.e. $y = 1$, is normally referred to as the kinematic condition. The mathematical representation of the kinematic condition by means of axial displacement is:

$$\frac{\partial u_z}{\partial t} = \hat{v}_z(y=1) e^{i\omega(t-z/c)} \tag{8.217}$$

which by using equation (8.211) and (8.216) is equivalent to:

$$i\omega E = C + \frac{A}{\rho c} \tag{8.218}$$

An identical approach in the radial direction yields:

$$i\omega D = \frac{i r_i \omega}{2c}\left(C F_{10}(\alpha) + \frac{A}{\rho c}\right) \tag{8.219}$$

Further, substitution of equation (8.216) into the governing equation (8.170)for the vessel wall yields:

$$-\omega^2 D = \frac{A}{\rho_w h} - \frac{\eta}{(1-\nu_p^2)\rho_w}\left(\frac{D}{r^2} - \frac{i\omega\nu_p}{rc}E\right) \tag{8.220a}$$

$$-\omega^2 E = \frac{\eta}{(1-\nu_p^2)\rho_w}\left(-\frac{\omega^2}{c^2}E - \frac{i\omega\nu_p}{rc}D\right) + \frac{\tau_w}{\rho_w h} \tag{8.220b}$$

One may argue that the wall shear stress, which in general is given by:

$$\tau_w = \mu\left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)\Big|_{r=r_i} \approx \mu\frac{\partial v_z}{\partial r}\Big|_{r=r_i} \tag{8.221}$$

as $\partial v_r/\partial z \ll \partial v_z/\partial r$ and thus:

$$\frac{\hat{\tau}_w}{\rho_w h} = \frac{\mu}{r_i}\frac{\partial \hat{v}_z}{\partial y}\Big|_{y=1} = \frac{\rho}{\rho_w}\frac{\nu}{2r_i h}\left(-Ci\alpha^2 F_{10}(\alpha) + \left(\frac{\omega r_i}{c}\right)^2\frac{A}{\rho c}\right) \tag{8.222}$$

which by substitution into equation (8.220b) yields:

$$-\omega^2 E = \frac{\eta}{(1-\nu_p^2)\rho_w}\left(-\frac{\omega^2}{c^2}E - \frac{i\omega\nu_p}{rc}D\right) + \frac{\rho}{\rho_w}\frac{\nu}{2r_i h}\left(-Ci\alpha^2 F_{10}(\alpha) + \left(\frac{\omega r_i}{c}\right)^2\frac{A}{\rho c}\right) \tag{8.223}$$

The four equations (8.218), (8.219), (8.220a), and (8.223) constitute a homogeneous algebraic equation system in the arbitrary constants A, C, D, and E, which has the matrix representation:

$$\begin{bmatrix} \frac{1}{\rho c} & 1 & 0 & -i\omega \\ \frac{i\omega r_i}{2\rho c^2} & \frac{i\omega r_i}{2c}F_{10} & -i\omega & 0 \\ \frac{1}{\rho_w h} & 0 & \omega^2 - \frac{B}{\rho_w {r_i}^2} & \frac{iB\omega\nu_p}{\rho_w r_i c} \\ \frac{\rho}{\rho_w}\frac{\nu}{2r_i h}\left(\frac{\omega r_i}{c}\right)^2\frac{1}{\rho c} & -\frac{i\rho\omega r_i F_{10}}{2\rho_w h} & \frac{-iB\omega\nu_p}{\rho_w r_i c} & \omega^2\left(1 - \frac{B}{\rho_w c^2}\right) \end{bmatrix}\begin{bmatrix} A \\ C \\ D \\ E \end{bmatrix} = 0 \tag{8.224}$$

where we for convenience have introduced the constant:

$$B = \frac{\eta}{1-\nu_p^2} \tag{8.225}$$

Further, let $\mathbf{M}$ denote the matrix in equation (8.224) and M_{ij} the element on row i and column j. The Womersley number: $\nu\alpha^2 = \omega r_i^2$ may then be used to simplify element M_{42}:

$$M_{42} = \frac{\rho}{\rho_w}\,\frac{\nu}{2r_i h} i\alpha^2 F_{10}(\alpha) = \frac{i\rho\,\omega\,r_i F_{10}}{2\rho_w h} \tag{8.226}$$

Further, one may argue that:

$$M_{32} = \omega^2 - \frac{B}{\rho_w r_i^2} \approx -\frac{B}{\rho_w r_i}^{2} \tag{8.227}$$

as B/ρ_w is proportional to the transverse wave speed in the structure (see equation (4.134)), which is assumed to be greater than the wave-speed in the coupled problem, i.e. $B/\rho_w c^2 > 1$, whereas $(\omega r_i/c)^2 \ll 1$ for physiological values.

From basic linear algebra, we know that for a non-trivial solution (i.e. unequal to zero) to exist of equation (8.224), the determinant of the matrix must be zero. Further, multiplication of the columns and rows of a matrix $\mathbf{M}$ by constants will not the change the solutions to the equation $\det(\mathbf{M}) = 0$. Thus, in the pursuit of a simple and convenient expression for the determinant of the matrix in equation (8.224), we will make all terms non-dimensional by the following recipe:

1. Multiply c1 with ρc.
2. Multiply r2 with $c/i\omega r_i$
3. Multiply c3 with r_i/c
4. Multiply r3 with r_i/c
5. Multiply c4 with $-1/i\omega$
6. Multiply r4 with $1/i\omega$

Element M_{41} will be of the order $(\omega r_i/c\alpha)^2$ and may thus be discarded. These operations leave the following expression:

$$\det(\mathbf{M}) = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1/2 & 1/2\,F_{10} & -1 & 0 \\ \dfrac{\rho\,r_i}{\rho_w h} & 0 & -\dfrac{B}{\rho_w c^2} & -\dfrac{B\nu_p}{\rho_w c^2} \\ 0 & -\dfrac{\rho\,r_i F_{10}}{2\rho_w h} & -\dfrac{B\nu_p}{\rho_w c^2} & 1 - \dfrac{B}{\rho_w c^2} \end{vmatrix} \tag{8.228}$$

to proceed further we introduce:

$$k = \rho_w h/\rho\,r_i \quad \text{and} \quad x/k = B/\rho_w c^2 \tag{8.229}$$

and multiply row 3 and four with k:

$$\det(\mathbf{M}) = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1/2 & 1/2\,F_{10} & -1 & 0 \\ 1 & 0 & -x & -\nu_p x \\ 0 & -1/2\,F_{10} & -\nu_p x & k - x \end{vmatrix} \tag{8.230}$$

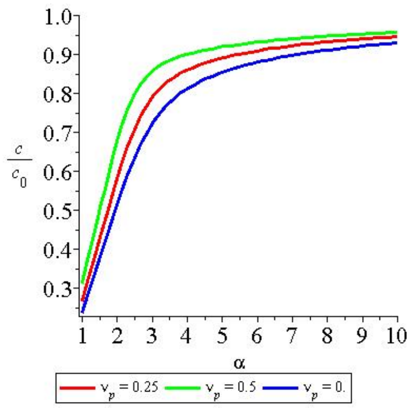


Figure 8.11: Plots of $\gamma_r = \Re(c/c_0)$ as function of α and ν_p for $k = 0.1$.

to eliminate the first elements in row 2 and row 3 we perform some linear combinations of the rows. First, we replace row 2 by row 1 - 2× row 2, second row 3 is replaced by row 1 - row 3. This will simplify the expression for the determinant by Laplacian expansion:

$$\det(\mathbf{M}) = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 - F_{10} & 2 & 1 \\ 0 & 1 & x & \nu_p x + 1 \\ 0 & -1/2\, F_{10} & -\nu_p x & k - x \end{vmatrix} = \begin{vmatrix} 1 - F_{10} & 2 & 1 \\ 1 & x & \nu_p x + 1 \\ -1/2\, F_{10} & -\nu_p x & k - x \end{vmatrix} \tag{8.231}$$

which may be evaluated to:

$$\det(\mathbf{M}) = (1 - F_{10})(1 - \nu_p^2)\ x^2 - (k(1 - F_{10}) + F_{10}\,(1/2 - 2\nu_p) + 2)\ x + F_{10} + 2k = 0 \tag{8.232}$$

i.e. a simple second order algebraic equation in x which is easy to solve. The complex solutions of equation (8.232) will be functions of ν_p, k, and α, i.e. $x = x(\nu_p k, \alpha)$, which may readily be found with e.g. Maple. From equation (8.229) one may deduce:

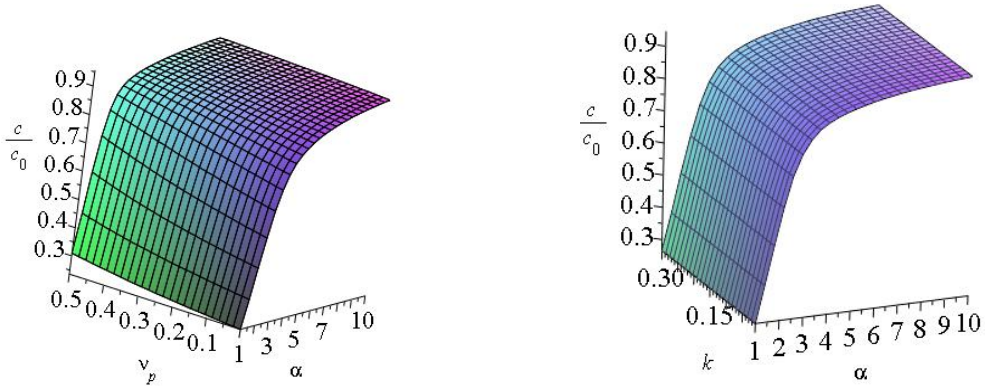
$$x = k \frac{\eta}{1 - \nu_p^2} \frac{1}{\rho_w} \frac{1}{c^2} = \frac{h}{\rho r_i} \frac{\eta}{1 - \nu_p^2} \frac{1}{c^2} \tag{8.233}$$

The Moens-Korteweg formula for the wave speed c_0 under inviscid conditions is given by equation (8.86), which by substitution and rearrangement in equation (8.233) results in:

$$\gamma = \frac{c}{c_0} = \left(\frac{(1 - \nu_p^2)\ x}{2} \right)^{-1/2} \tag{8.234}$$

The symbol γ has been introduced for the ratio of the wave speed c for the coupled problem to c_0. As x is complex, so will γ be, and we write $\gamma = \gamma_r + i\gamma_i$ for the real and complex parts. Thus, having found the solutions of equation (8.232), we may plot γ_r as a function of ν_p, k, and α (see figure 8.11).

The dependence of k and ν_p may also be illustrated in 3D, albeit some less quantitative accuracy (see figure 8.12).



(a) $\gamma_r = \Re(c/c_0)$ as function of α and ν_p for $k = 0.1$.

(b) $\gamma_r = \Re(c/c_0)$ as function of α and k for $\nu_p = 0.25$.

Figure 8.12: Plots of $\gamma_r = \Re(c/c_0)$.

APPENDIX **A**

Mathematical preliminaries

A.1 TRIGONOMETRIC RELATIONS

$$\sin 2\phi = 2\sin\phi\cos\phi, \qquad \cos 2\phi = \cos^2\phi - \sin^2\phi \tag{A.1a}$$

$$\cos^2\phi = \frac{1}{2}\left(1+\cos 2\phi\right), \qquad \sin^2\phi = \frac{1}{2}\left(1-\cos 2\phi\right) \tag{A.1b}$$

A.2 VECTORS

A vector is a coordinate invariant quantity, uniquely defined by a magnitude and a direction in space, and that obeys the parallelogram law by addition.

The magnitude of a vector $\mathbf{a}$ is denoted:

$$|\mathbf{a}| \equiv a \tag{A.2}$$

A vector $\mathbf{a}$ may be decomposed into *scalar vector components* or for short *components*, a_i, parallel to the coordinate axes, i.e. parallel to the base vectors $\mathbf{e}_i$, and then be presented in various, equivalent ways:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \equiv a_i\mathbf{e}_i \equiv (a_1, a_2, a_3) \equiv [a_i] \equiv \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{A.3}$$

Addition and subtraction of vectors are defined according to the geometric parallelogram law [see section 2.2 18], but the geometrical definition is transformed to the component form given by:

$$\mathbf{a} + \mathbf{b} = \mathbf{c} \Leftrightarrow a_i + b_i = c_i \tag{A.4}$$

The *scalar product* (or dot product) of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\mathbf{a},\mathbf{b}) \tag{A.5}$$

where $(\mathbf{a}, \mathbf{b})$ is the angle between the two vectors. The operation is commutative and distributive. For orthogonal base vectors in a coordinate system Ox we get:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{A.6}$$

where δ_{ij} is the *Kronecker delta* defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases} \tag{A.7}$$

From these fundamental definitions and properties one may show[see section 2.2 18]:

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i \tag{A.8}$$

The *vector product* or *cross product* is defined by:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \equiv |\mathbf{a}||\mathbf{b}| \sin((\mathbf{a}, \mathbf{b})\, \mathbf{e} \tag{A.9}$$

where $(\mathbf{a}, \mathbf{b})$ is the smallest angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{e}$ is a unit vector orthogonal to the plane defined by $\mathbf{a}$ and $\mathbf{b}$. By introducing the *permutation symbol*:

$$e_{ijk} = \begin{cases} 0 & \text{when two or three indices are equal} \\ 1 & \text{when the indices are cyclic permutations of 123} \\ -1 & \text{when the indices are cyclic permutations of 321} \end{cases} \tag{A.10}$$

one may also conveniently express the vector product by:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \qquad \Leftrightarrow \qquad e_{ijk}\, a_i b_j = c_k \tag{A.11}$$

A.3 Orthogonal Coordinates

In a general orthogonal coordinate system we denote the coordinates by y_i. The position $\mathbf{r}$ is then a function of y_i and time t which we denoted by $\mathbf{r} = \mathbf{r}(y_i, t)$. A one-to-one correspondence between the general orthogonal coordinates y_i and the Cartesian coordinates x_i. For convenience we denote the the general orthogonal coordinate system for the y-system.

The base vectors $\mathbf{g}_i$ (not unit vectors) of the y-system, are tangent vectors to the coordinate lines of the y-coordinates:

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial y_i} = \frac{\partial \mathbf{r}}{\partial x_k}\frac{\partial x_k}{\partial y_i} = \mathbf{e}_k \frac{\partial x_k}{\partial y_i} \quad \Leftrightarrow \quad \mathbf{e}_k = \frac{\partial y_i}{\partial x_k}\mathbf{g}_i \tag{A.12}$$

with corresponding magnitude:

$$h_i = \sqrt{\mathbf{g}_i \cdot \mathbf{g}_i} = \sqrt{\frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_i}}, \quad \text{no sum over i} \tag{A.13}$$

Further, due to orthogonality of the coordinate lines:

$$\mathbf{g}_i \cdot \mathbf{g}_j = 0 \quad \text{for} \quad i \neq j \quad \Rightarrow \frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_j} = 0 \quad \text{for} \quad i \neq j \tag{A.14}$$

From equation (A.13) and (A.14) we get the following relation:

$$\frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_j} = h_i^2\,\delta_{ij} \tag{A.15}$$

From the fundamental properties of the general orthogonal coordinate system we have:

$$\frac{\partial y_i}{\partial y_j} = \frac{\partial y_i}{\partial x_k}\frac{\partial x_k}{\partial y_j} = \delta_{ij} \tag{A.16}$$

Thus, the components in the sum in equation (A.16) may be interpreted as the product of two matrices with the components:

$$\left[\frac{\partial y_i}{\partial x_k}\right] = \left[\frac{\partial x_k}{\partial y_i}\right]^{-1} \tag{A.17}$$

Then from equation (A.15) and (A.17) we get:

$$\left(\frac{\partial y_i}{\partial x_k}\frac{\partial y_j}{\partial x_k}\right) = \left(\frac{\partial x_k}{\partial y_i}\frac{\partial x_k}{\partial y_j}\right)^{-1} = \frac{1}{h_i^2}\,\delta_{ij} \tag{A.18}$$

The del-operator is normally defined as follows in a Cartesian coordinate system:

$$\nabla() = \mathbf{e}_k\frac{\partial()}{\partial x_k} \tag{A.19}$$

A representation of the del-operator in a general orthogonal coordinate system may then be found by substitution of equation (A.12) and (A.18) into equation (A.19):

$$\nabla() = \left(\frac{\partial y_i}{\partial x_k}\,\mathbf{g}_i\right)\left(\frac{\partial()}{\partial y_j}\frac{\partial y_j}{\partial x_k}\right) = \sum_i\left(\frac{1}{h_i^2}\delta_{ij}\right)\mathbf{g}_i\frac{\partial()}{\partial y_j} = \sum_i\frac{1}{h_i^2}\mathbf{g}_i\frac{\partial()}{\partial y_i} \tag{A.20}$$

By introducing a unit base vector in the y-system:

$$\mathbf{e}_i^y = \frac{\mathbf{g}_i}{h_i} \tag{A.21}$$

the expression for the del-operator take the form:

$$\nabla() = \sum_i\frac{1}{h_i}\mathbf{e}_i^y\frac{\partial()}{\partial y_i} \tag{A.22}$$

A.3.1 Gradient, divergence and rotation in general orthogonal coordinates

The physical components of a vector $\mathbf{a}$ and a second order tensor $\mathbf{A}$ are defined in the following manner:

$$\mathbf{a} = a_k^y\mathbf{e}_k^y, \qquad \mathbf{A} = A_{kl}^y\,\mathbf{e}_k^y\otimes\mathbf{e}_l^y \tag{A.23}$$

From the expression in equation (A.20) we get for a scalar α:

$$\nabla\alpha = \sum_i\frac{1}{h_i}\mathbf{e}_i^y\frac{\partial\alpha}{\partial y_i} \tag{A.24}$$

and for the divergence of a vector $\mathbf{a}$:

$$\nabla \cdot \mathbf{a} = \sum_i \frac{1}{h_i} \mathbf{e}_i^y \frac{\partial (a_k^y \mathbf{e}_k^y)}{\partial y_i} \tag{A.25}$$

rotation of a vector $\mathbf{a}$:

$$\nabla \times \mathbf{a} = \sum_i \frac{1}{h_i} \mathbf{e}_i^y \times \frac{\partial (a_k^y \mathbf{e}_k^y)}{\partial y_i} \tag{A.26}$$

The divergence of a second order tensor $\mathbf{A}$ is:

$$\mathrm{div}\mathbf{A} = \mathbf{A} \cdot \overleftarrow{\nabla} = \sum_i \frac{1}{h_i} \frac{\partial \left(A_{kl}^y \mathbf{e}_k^y \otimes \mathbf{e}_l^y \right)}{\partial y_i} \cdot \mathbf{e}_i^y \tag{A.27}$$

This expression for the $\mathrm{div}\mathbf{A}$ in equation (A.27) may be expanded by using the chain rule and taking into account $(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} (\mathbf{b} \cdot \mathbf{c})$ and the orthogonality of the y-system (i.e. $\mathbf{e}_l^y \cdot \mathbf{e}_i^y = \delta_{li}$):

$$\mathrm{div}\mathbf{A} = \sum_i \frac{1}{h_i} \frac{\partial A_{ki}^y}{\partial y_i} \mathbf{e}_k^y + \frac{1}{h_i} A_{ki}^y \frac{\partial \mathbf{e}_k^y}{\partial y_i} + \frac{1}{h_i} A_{ki}^y \mathbf{e}_k \otimes \frac{\partial \mathbf{e}_l^y}{\partial y_i} \cdot \mathbf{e}_i^y \tag{A.28}$$

A.4 Integral Theorems

Theorem A.4.1. *Gauss' Integral Theorem*
Let V be a volume with surface A and $\mathbf{n}$ an outward unit vector on A. The for any field $f(x)$:

$$\int_V f_{,k}\, dV = \int_A f\, n_k\, dA \tag{A.29}$$

For a proof see e.g. [18, C.3].

Theorem A.4.2. *The Divergence Theorem By replacing the function $f(x)$ in by the vector components $a_i(x)$ of a vector field $\mathbf{a}(x)$ we get:*

$$\int_V a_{k,k}\, dV \quad = \textstyle\int_A a_k\, n_k\, dA \tag{A.30}$$

$$\int_V \nabla \cdot \mathbf{a}\, dV \quad = \textstyle\int_A \mathbf{a} \cdot \mathbf{n}\, dA \tag{A.31}$$

It is trivial that Eq. (A.30) holds when no summation involved, by just replacing f with a_k in Eq. (A.29). However, as the integral operator is linear, it will also hold when Einstein's summation convention is employed. Consequently, Eq. (A.30) is valid when using the summation convention too, and then Eq. (A.31) follows as the vector representation of Eq. (A.30).

Theorem A.4.3. *The First Mean Value Theorem Let $f(\mathbf{x})$ and $g(\mathbf{x})$ represent continuous field functions, such that $f, g : \mathbb{R}^3 \to \mathbb{R}$. Further, let V represent a contiguous volume, i.e. $V \in \mathbb{R}^3$. Then there exists a $\bar{\mathbf{x}} \in V$ such that:*

$$\int_V f(\mathbf{x})\, g(\mathbf{x})\, dV = f(\bar{\mathbf{x}}) \int_V g(\mathbf{x})\, dV \tag{A.32}$$

In particular, if $g(\mathbf{x}) = 1$ for all $\mathbf{x} \in V$

$$\int_V f(\mathbf{x})\, dV = f(\bar{\mathbf{x}})\, V \tag{A.33}$$

Thus, $f(\bar{\mathbf{x}})$ is the mean value of $f(\mathbf{x})$ in V.

For proofs see [51] and [18, C.3].

Theorem A.4.4. *Leibniz's rule for 1D integrals*
Let

$$F(t) = \int_{a(t)}^{b(t)} f(x,t)\, dx \tag{A.34}$$

then

$$\frac{dF(t)}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\, dx + f(x,t)\frac{\partial b(t)}{\partial t} - f(x,t)\frac{\partial a(t)}{\partial t} \tag{A.35}$$

Theorem A.4.5. *Leibniz's rule for 2D integrals*
Let us consider the integral $F(t)$ obtained by integration of the function $f(x,y,t)$ over the domain in 2D space $\{x,y\} \in A(t)$ where $A(t)$ is a function of t:

$$F(t) = \int_{A(t)} f(x,y,t)\, dA \tag{A.36}$$

then:

$$\frac{dF(t)}{dt} = \int_{A(t)} \frac{\partial f(x,y,t)}{\partial t}\, dA + \left[f(x,y,t)\frac{dA}{dt}\right]_{C(t)} = \int_{A(t)} \frac{\partial f(x,y,t)}{\partial t}\, dA + \oint_{C(t)} f\frac{dn}{dt}\, dc \tag{A.37}$$

where dA at the boundary (or contour) is expressed as $dA = dn\, dc$. Hence, (n,c) is the local coordinate, where dc is tangential and dn is orthogonal to $C(t)$.

If $\mathbf{v}$ is the velocity on $C(t)$, the velocity component of $\mathbf{v}$ in the outward normal direction $\mathbf{n}$ to $C(t)$ is:

$$v_n = \mathbf{v}\cdot\mathbf{n} = \frac{dn}{dt} \tag{A.38}$$

and thus:

$$\frac{dn}{dt}\, dc = v_n\, dc = \mathbf{v}\cdot\mathbf{n}\, dc \tag{A.39}$$

and consequently:

$$\frac{dF(t)}{dt} = \int_{A(t)} \frac{\partial f(x,y,t)}{\partial t}\, dA + \oint_{C(t)} f\, \mathbf{v}\cdot\mathbf{n}\, dc \tag{A.40}$$

Theorem A.4.6. *Leibniz's rule for 3D integrals*
Let us consider the integral $F(t)$ obtained by integration of the function $f(x,y,z,t)$ over the domain in 3D space $\{x,y,z\} \in V(t)$ where $V(t)$ is a function of t:

$$F(t) = \int_{V(t)} f(x,y,z,t)\, dV \tag{A.41}$$

The, volume $V(t)$ has the surface $A(t)$.

$$\frac{dF(t)}{dt} = \int_{V(t)} \frac{\partial f(x,y,t)}{\partial t}\, dV + \left[f(x,y,t)\frac{dV}{dt}\right]_{A(t)} = \int_{V(t)} \frac{\partial f(x,y,t)}{\partial t}\, dV + \int_{A(t)} f\frac{dn}{dt}\, dA \tag{A.42}$$

where $dV = dn\,dA$ is the area increment and dn is the increment in the direction of the outward normal to A. Note that:

$$\frac{dV}{dt} = \frac{dn}{dt}\,dA = v_n\,dA = \mathbf{v}\cdot\mathbf{n}\,dA \tag{A.43}$$

and consequently:

$$\frac{dF(t)}{dt} = \int_{V(t)} \frac{\partial f(x,y,t)}{\partial t}\,dV + \int_{A(t)} f\,\mathbf{v}\cdot\mathbf{n}\,dA \tag{A.44}$$

A.5 PROPERTIES OF BESSEL FUNCTIONS

The presentation of the Bessel functions are based on a articles in [49, 52]. The Bessel functions are solutions $y(x)$:

$$y(x) = c_1 J_n(x) + c_2 Y_n(x) \tag{A.45}$$

of Bessel's differential equation[1]:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \tag{A.46}$$

Bessel functions are also known as cylinder functions or cylindrical harmonics because they are found in the solution to Laplace's equation in cylindrical coordinates. Bessel functions of the first kind, denoted as $J_n(x)$, are solutions of Bessel's differential equation that are finite at the origin (x = 0) for non-negative integer n, and diverge as x approaches zero for negative non-integer n. The solution type (e.g. integer or non-integer) and normalization of $J_n(x)$ are defined by its properties below. It is possible to define the function by its Taylor series expansion around x = 0:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)} (\frac{x}{2})^{2m+n} \tag{A.47}$$

where $\Gamma(z)$ is the gamma function, a generalization of the factorial function to non-integer values.

The Bessel functions of the second kind, denoted by $Y_n(x)$, are also solutions of the Bessel differential equation. However, they are singular (infinite) at the origin (x = 0). Such solutions are not relevant for the applications in the this presentation, and thus we consider $c_2 = 0$ in equation (A.45).

Some useful properties of the Bessel functions of first kind are:

$$J_{-n}(x) = (-1)^n J_n(x) \tag{A.48}$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x) \tag{A.49}$$

The modified Bessel equation is very similar to equation (A.46) and may be presented:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 - \frac{n^2}{x^2}\right) y = 0 \tag{A.50}$$

[1]Note that n may in general be a complex number. Here we assume n to be an integer.

The solutions are the modified Bessel functions of the first and second kinds, and can be written:

$$y = a_1 J_n(-ix) + a_2 Y_n(-ix) \tag{A.51}$$

$$= c_1 I_n(x) + c_2 K_n(x) \tag{A.52}$$

where $I_n(x)$ are $K_n(x)$ modified Bessel functions of order n of the first and second kind, respectively.

The modified Bessel function of order n of first kind may be defined by the relation:

$$I_n(x) = i^{-n} J_n(ix) \tag{A.53}$$

APPENDIX **B**

Selected answers

Answer (Ex. 8) — A backward Euler discretization of equation (7.15) is:

$$\frac{p - p_{i-1}}{\Delta t} + \frac{1}{\tau} p = \frac{Q}{C} \tag{7.43}$$

where we have dropped subscipts for time level i for convenience, i.e. $p = p_i$). By collecting terms of p in equation (7.43) we get:

$$p \left(1 + \frac{\Delta t}{\tau}\right) = p_{i-1} + \frac{Q\,\Delta t}{C} \tag{7.44}$$

and by division by the expression in the paranthesis on the lhs of equation (7.44) we get:

$$p = p_{i-1} \frac{1}{1 + \frac{\Delta t}{\tau}} + \frac{Q\,\Delta t}{C} \frac{1}{1 + \frac{\Delta t}{\tau}} \tag{7.45}$$

which by Taylor expansion and neglection of higher order termes simplifies to:

$$p \approx p_{i-1} \left(1 - \frac{\Delta t}{\tau}\right) + \frac{Q\,\Delta t}{C} \tag{7.46}$$

which is identical with the approximation obtained by using the convolution integral in equation (7.22).

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