Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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This module presents methods to solve differential equations with constant coefficients using Laplace transforms but begins by introducing some general terminology

A short general introduction to differential equations

The order of a differential equation is associated with its highest-order derivative.
An ordinary differential equation (ODE) has only integer powers of derivatives.
Linear nth-order differential equations have the following form (note many of the most Important equations in engineering or physics of this form are only second order)

$$a_n(x)\frac{d^n y}{dx^n}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x)$$

where n is an integer. When f(x)=0 we call this a homogenous ODE and inhomogeneous if f(x) is not equal to 0.

The n-constants are evaluated using n initial conditions.

The solution to a homogenous n-order ODE consist of a linear combination of n linearly independent solution. That is $y_c(x)=\sum_{i=1}^{n} c_i y_i(x)$ often called the complementary solution

The solution to the inhomogeneous differential equations is called the particular solution, typical represented by $y_p(x)$ and the general or total solution is a combination of the homogeneous and particular solution.

Review: Laplace transform of the derivative

We previously derived the following Laplace transform of the derivative of a function

$$L\{f'(t)\} = \int_0^\infty e^{-st}\ f'(t)dt = -f(0) + s\tilde{f}(s)$$

As we noted before the Laplace transform of the second derivative of a function can also be easily derived by using integration by parts.

$$L\{f''(t)\} = \int_0^\infty e^{-st}\ f''(t)dt = s^2\tilde{f}(s) - sf(0) - f'(0)$$

To evaluate the integral let $\begin{array}{l} dv = f''(t)dt \quad v=f'(t) \\ u = e^{-st} \quad du=-se^{-st}dt \end{array}$ This gives

$$\int_0^\infty e^{-st}\ f''(t)dt = f'(t)e^{-st}\Big|_0^\infty - \int_0^\infty -se^{-st}\ f'(t)dt =$$

$$\lim_{t\to\infty} f'(t)e^{-st} - f'(0)e^{-s0} + s\int_0^\infty e^{-st}\ f'(t)dt = -f'(0) + s[-f(0) + s\tilde{f}(s)]$$

In general

$$\boxed{L\{f^{(n)}(t)\} = s^{(n)}\tilde{f}(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

An example of using the Laplace transform to solve a second order differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{-x} \quad y(0) = 2,\ y'(0) = 1$$

Note that this is a second order inhomogeneous ODE with two initial conditions. Of particular importance is that the Laplace methodology presented solves for both the complementary and particular solutions together and incorporates the two required initial conditions to completely determine the general solution.

Take the Laplace transform of the differential equation
(note, for the right hand side refer to a Laplace transform table).

$$s^2\tilde{y}(s) - sy(0) - y'(0) - 3[s\tilde{y}(s) - y(0)] + 2\tilde{y} = 2\left(\frac{1}{s+1}\right)$$

Simplifying gives

$$\tilde{y}(s) = \left[\frac{2}{s+1} + 2s - 5\right] \Big/ \left[(s-2)(s-1)\right] = \frac{2s^2 - 3s - 3}{(s+1)(s-2)(s-1)}$$

To finish solving this problem we need to take the inverse Laplace transform of this expression and to do this we need use partial fractions to put this expression into one we can look up in a Laplace transform Table.

$$\frac{2s^2 - 3s - 3}{(s+1)(s-2)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$$

Finding a common denominator gives the following expression

$$\frac{2s^2-3s-3}{(s+1)(s-2)(s-1)}=\frac{A(s-1)(s-2)+B(s+1)(s-2)+C(s+1)(s-1)}{(s+1)(s-2)(s-1)}$$

Next equate powers of s in the numerator on left and right hand side giving us the following set of three equations for the parameters A, B and C.

$$\begin{aligned} s^2 &: A+B+C=2 \\ s^1 &: -3A-B=-3 \\ s^0 &: 2A-2B-C=-3 \end{aligned}$$

In matrix form $\begin{pmatrix} 1 & 1 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & -1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix}=\begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$

Fixed sign element (2,2)

Use Gaussian elimination on the augmented matrix (or solve by your method of choice).

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -3 & -1 & 0 & -3 \\ 2 & -2 & -1 & -3 \end{array}\right)=\cdots=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3/2 & 3/2 \\ 0 & 0 & 1 & -1/3 \end{array}\right)$$

This gives $A=1/3$, B=2 and C=$-1/3$

Therefore $\tilde{y}(s)=\frac{1/3}{s+1}+\frac{2}{s-1}+\frac{-1/3}{s-2}$ and taking the inverse Laplace transform

using a Table gives $\boxed{y(x)=\frac{1}{3}e^{-1x}+2e^{1x}-\frac{1}{3}e^{2x}}$

Another example of using the Laplace transform to solve a set of coupled differential equations

Let's look at some details for solving this example of coupled differential equations.

The mathematical problem is laid out as follows for two electrical circuits having charges q_1 and q_2 and essentially no resistance:

$$L\frac{d^2q_1}{dx^2}+M\frac{d^2q_2}{dx^2}+Gq_1=0 \qquad q_1(0)=0,\ q_1^{'}(0)=0$$

$$M\frac{d^2q_1}{dx^2}+L\frac{d^2q_2}{dx^2}+Gq_2=0 \qquad q_2(0)=CV_0,\ q_2^{'}(0)=0$$

Initial conditions

L – self inductance, C – capacitance, M – mutual induction
initial charge for the second circuit is $CV_0 = V_0/G$

Taking the Laplace transform for both equations gives
(use previous form for Laplace transform for a first and second derivative)

$$L[s^2\tilde{q}_1(s)-sq_1(0)-q_1'(0)]+M[s^2\tilde{q}_2(s)-sq_2(0)-q_2'(0)]+G\tilde{q}_1(s)=0$$

$$M[s^2\tilde{q}_1(s)-sq_1(0)-q_1'(0)]+L[s^2\tilde{q}_2(s)-sq_2(0)-q_2'(0)]+G\tilde{q}_2(s)=0$$

Substitute initial conditions and collecting terms gives

$$[Ls^2+G]\tilde{q}_1(s)+Ms^2\tilde{q}_2(s)=sMV_0/G$$

$$Ms^2\tilde{q}_1(s)+[Ls^2+G]\tilde{q}_2(s)=sLV_0/G$$

So at this point we have a set of two coupled linear equations in

$$[Ls^2+G]\tilde{q}_1(s)+Ms^2\tilde{q}_2(s)=sMV_0/G$$
$$Ms^2\tilde{q}_1(s)+[Ls^2+G]\tilde{q}_2(s)=sLV_0/G$$

We can solve for $\tilde{q}_1(s)$, $\tilde{q}_2(s)$ a number of ways. One of the most routine ways is using Cramer's rule for solving a system of linear equations using determinates. So first put the equations in matrix form:

$$\begin{pmatrix} Ls^2+G & Ms^2 \\ Ms^2 & Ls^2+G \end{pmatrix}\begin{pmatrix} \tilde{q}_1(s) \\ \tilde{q}_2(s) \end{pmatrix}=\begin{pmatrix} sMV_0/G \\ sLV_0/G \end{pmatrix}$$

Solving for $\tilde{q}_1(s)$, $\tilde{q}_2(s)$ Cramer's rule initially gives

$$\tilde{q}_1(s)=\frac{\begin{vmatrix} sMV_0/G & Ms^2 \\ sLV_0/G & Ls^2+G \end{vmatrix}}{D},\ \tilde{q}_2(s)=\frac{\begin{vmatrix} Ls^2+G & sMV_0/G \\ Ms^2 & sLV_0/G \end{vmatrix}}{D}$$

$$\text{where } D=\begin{vmatrix} Ls^2+G & Ms^2 \\ Ms^2 & Ls^2+G \end{vmatrix}=(Ls^2+G)^2-(Ms^2)^2=(L^2-M^2)s^2+2LGs^2+G^2=$$

(and the not so obvious factorization) = $[(L+M)s^2+G][(L-M)s^2+G]$

Next let's just solve for $\tilde{q}_1(s)$ as the book does

$$\tilde{q}_1(s)=\frac{\begin{vmatrix} sMV_0/G & Ms^2 \\ sLV_0/G & Ls^2+G \end{vmatrix}}{D}=\frac{(Ls^2+G)sMV_0/G-(Ms^2)sLV_0/G}{D}=\frac{sMV_0}{D}$$

$$D=[(L+M)s^2+G][(L-M)s^2+G]$$

But we are not finished yet! This is the solution in s space and we need the solution in t (or time) space so we have to take the inverse Laplace transform.
However we need put this in a nice form before we look it up in a Laplace transform Table.

USE PARTIAL FRACTIONS (for quadratic factors in the denominator)

$$\tilde{q}_1(s)=\frac{sMV_0}{[(L+M)s^2+G][(L-M)s^2+G]}=\frac{As+B}{(L+M)s^2+G}+\frac{Cs+D}{(L-M)s^2+G}$$

Do the usual, find a common denominator and equate powers of s in the numerator

$$s^3: A(L-M)+C(L+M)=0$$
$$s^2:\ B(L-M)+D(L+M)=0$$
$$s^1:\ AG+CG=V_0M$$
$$s^0:\ BG+DG=0$$

Now we will be efficient in solving these (you could do this via matrix math by route) but we can solve them by hand with a little insight!

$$s^3 : A(L-M)+C(L+M)=0$$
$$s^2 :\ B(L-M)+D(L+M)=0$$
$$s^1 :\ AG+CG=V_0M$$
$$s^0 :\ BG+DG=0$$

Starting with the last equation we see that D = -B

Then substitute this in the second equation gives -2BM=0 (M cannot be zero) so B = 0 and therefore D = 0

That leaves equations one and three to solve for A and C and this gives

$$A=\frac{V_0(M+L)}{2G}-C,\ C=\frac{-V_0(L-M)}{2G}$$

Substitution gives

$$\tilde{q}_1(s)=\frac{(\frac{V_0(M+L)}{2G})s}{(L+M)s^2+G}+\frac{(\frac{-V_0(L-M)}{2G})s}{(L-M)s^2+G}=\frac{V_0(M+L)s}{2G[(L+M)s^2+G]}+\frac{-V_0(L-M)s}{2G[(L-M)s^2+G]}$$

Then making a convenient substitution in the first term $G=\omega_1^2(L+M)$

and in the second term $G=\omega_2^2(L-M)$, in the brackets

(implicit in this is that the frequencies are related) we get

$$=\frac{V_0(M+L)}{2G}\frac{s}{[(M+L)s^2+(M+L)\omega_1^2]}+\frac{-V_0(L-M)}{2G}\frac{s}{[(L-M)s^2+(L-M)\omega_2^2]}$$

Next cancel out the common terms and set G = 1/C implict in the initial conditions

$$\tilde{q}_1(s) = \frac{V_0(M+L)}{2G(L+M)}\frac{s}{[s^2+\omega_1^2]} - \frac{V_0(L-M)}{2G(L-M)}\frac{s}{[s^2+\omega_2^2]} = \frac{V_0 C}{2}\left\{\frac{s}{[s^2+\omega_1^2]} - \frac{s}{[s^2+\omega_2^2]}\right\}$$

Finally taking the inverse Laplace transform gives

$$q_1(t) = L^{-1}\{\tilde{q}_1(s)\} = \frac{V_0 C}{2}\left\{L^{-1}\frac{s}{[s^2+\omega_1^2]} - L^{-1}\frac{s}{[s^2+\omega_2^2]}\right\} =$$

$$\frac{V_0 C}{2}\left\{\cos\omega_1 t - \cos\omega_2 t\right\}$$

Where the current can be simply constructed by taking the rate of change of the charge $q_1(t)$, that is

$$\boxed{i_1(t) = \dot{q}_1(t) = \frac{V_0 C}{2}\left\{-\omega_1 \sin\omega_1 t + \omega_2 \sin\omega_2 t\right\}}$$