Interactive Assignment 3

26 pages

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Problems

Chapter 5 - Problem 5.1

Find the Fourier transform of the function $f(t) = e^{-|t|}$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

$$\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt = \int_{-\infty}^{0} e^{t} e^{-i\omega t} dt + \int_{0}^{\infty} e^{-t-i\omega t} dt$$

$$= \int_{-\infty}^{0} e^{(1-i\omega)t} dt + \int_{0}^{\infty} e^{-(1+i\omega)t} dt$$

$$= \frac{1}{1-i\omega} \left[e^{(1-i\omega)t} \right]_{-\infty}^{0} - \frac{1}{1+i\omega} \left[e^{-(1+i\omega)t} \right]_{0}^{\infty}$$

$$= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}$$

$$\Rightarrow \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{2}{1+\omega^2} = \sqrt{\frac{2}{\pi}} \frac{1}{(1+\omega^2)}$$

(a) By applying Fourier's inversion theorem prove that

$$\frac{\pi}{2} \exp(|t|) = \int_{0}^{\infty} \frac{\cos \omega t}{1+\omega^2} d\omega$$

Now by the Fourier's inversion theorem:

$$\exp(|t|) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{e^{i\omega t}}{1+\omega^2} d\omega$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{1+\omega^2} d\omega$$

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Equating the real parts on the two sides of this equation: $e^{|t|} = \frac{1}{\pi}\int_{\infty}^{\infty} \frac{\cos\omega t}{1+\omega^2}\, d\omega$

The integral is symmetric in ω:

$$e^{|t|} = \frac{2}{\pi}\int_0^{\infty} \frac{\cos\omega t}{1+\omega^2}\, d\omega$$

$$\Rightarrow \frac{\pi}{2} e^{|t|} = \int_0^{\infty} \frac{\cos\omega t}{1+\omega^2}\, d\omega$$

(b) By making the substitution $\omega = \tan\theta$, demonstrate the validity of Parseval's theorem for this function.

We want to show: $\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\tilde{f}(\omega)|^2 d\omega$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{0} e^{2t} dt + \int_0^{\infty} e^{-2t} dt$$

$$= 2\int_0^{\infty} e^{-2t} dt = -\frac{2}{2}\left[e^{-2t}\right]_0^{\infty} = 1$$

$$\int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega = \int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}\right)^2 d\omega$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2}\, d\omega$$

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$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \frac{4}{\pi} \int_0^{\pi/2} \cos^2\theta \, d\theta \quad \text{with the substitution } \omega = \tan\theta$$

$$= \frac{4}{\pi} \times \frac{\pi}{4} = 1.$$

Where $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)} d\omega = 2 \times \int_0^{\infty} \frac{1}{1+\omega^2} d\omega$

since even function

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Chapter 5'. Problem 5.3

Find the Fourier transform of $H(x-a)\,e^{-bx}$, where $H(x)$ is the Heaviside function.

$$H(x-a) = \begin{cases} 1 & \text{for } x > a \\ 0 & \text{otherwise } (x<a) \end{cases}$$

let $f(x) = H(x-a)\,e^{-bx}$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\cancel{t}-a)\, e^{-bt}\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(t-a)\, e^{-(b+i\omega)t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-(b+i\omega)t}\, dt = -\frac{1}{\sqrt{2\pi}\,(b+i\omega)} \left[e^{-(b+i\omega)t} \right]_a^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-(b+i\omega)a}}{(b+i\omega)}$$

$$= \frac{(b-i\omega)\, e^{-(b+i\omega)a}}{\sqrt{2\pi}\,(b^2+\omega^2)}$$

Chapter 5. Problem 5.7

Find the Fourier transform of the unit rectangular distribution

$$f(t) = \begin{cases} 1 & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the convolution of f with itself and, without further integration, determine its transform. Deduce that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2}\, d\omega = \pi \qquad \int_{-\infty}^{\infty} \frac{\sin^4 \omega}{\omega^4}\, d\omega = \frac{2\pi}{3}$$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega t}\, dt \quad \text{by definition of } f(t)$$

$$= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{i\omega}\right) \left[e^{-i\omega t}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \times \left(-\frac{1}{i\omega}\right) \times \left(e^{-i\omega} - e^{i\omega}\right) = \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{i\omega}\right)(-2i \sin\omega)$$

$$= \frac{2}{\sqrt{2\pi}} \quad \frac{\sin\omega}{\omega}$$

$$h(z) = \int_{-\infty}^{\infty} f(x)\, f(z-x)\, dx = (f * f)(z)$$

The integrand is non zero when: $\begin{cases} -1 \le x \le 1 \\ -1 \le z - x \le 1 \end{cases} \Rightarrow \begin{cases} -1 \le x \le 1 \\ z-1 \le x \le z+1 \end{cases}$

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$(f*f)(z)$ is zero when $z-1 \geq 1$ or $z+1 \leq -1$
i.e: $z \leq -2$ or $z \geq 2 \Rightarrow (f*f)(z)=0$

Now when $z-1<-1$ and $z+1>-1$ which is $-2<z<0$

$(f*f)(z) = \int_{-1}^{z+1} dz = z+2$ which is the length of the intersection

of: $[-1,1] \cap [z-1, z+1]$

When $z-1<1$ and $z+1>1$ or $0<z<2$

$(f*f)(z) = \int_{z-1}^{1} dx = 1-(z-1) = 2-z$ which is again the length

of the intersection of $[-1,1]$ with $[z-1, z+1]$

Finally $(f*f)(z) = \begin{cases} z+2 & -2<z<0 \\ 2-z & 0<z<2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2-|z| & \text{for } |z|<2 \\ 0 & \text{otherwise} \end{cases}$

By the convolution theorem:

$$\tilde{h}(k) = \sqrt{2\pi}\, \tilde{f}(k)\, \tilde{f}(k) = \sqrt{2\pi}\, (\tilde{f}(k))^2 = \sqrt{2\pi} \cdot \frac{4}{2\pi} \cdot \frac{\sin^2 \omega}{\omega^2}$$

$$= \frac{4}{\sqrt{2\pi}} \frac{\sin^2 \omega}{\omega^2}$$

And by the Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk$$

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Expanding the previous equation on both sides:

$$\int_{-1}^{1} 1^2 \, dt = \int_{-\infty}^{\infty} \left(\frac{2}{\sqrt{2\pi}}\right)^2 \frac{\sin^2 \omega}{\omega^2} \, d\omega$$

$$2 = \frac{4}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} \, d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} \, d\omega = \pi$$

And also $\displaystyle\int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{h}(\omega)|^2 d\omega$

$$\int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-2}^{2} (2-|t|)^2 dt = 2\int_0^2 (2-t)^2 dt = 2\left[\frac{\cancel{t}^3}{3}\right]_0^2$$

$$= \frac{16}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty} |\tilde{h}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \left(\frac{4}{\sqrt{2\pi}}\right)^2 \frac{\sin^4 \omega}{\omega^4} \, d\omega = \frac{16}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^4 \omega}{\omega^4} \, d\omega = \frac{2\pi}{3}$$

Chapter 5 - Problem 5-20

Find the functions $y(t)$ whose Laplace transforms are the following

(a) $1/(s^2-s-2)$

We start by writing $\frac{1}{s^2-s-2}$ as a partial fraction

$$\frac{1}{s^2-s-2} = \frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} = \frac{(A+B)s+B-2A}{(s+1)(s-2)}$$

Equating the coefficients with the power of x:

$$\left.\begin{array}{l} A+B=0 \\ -2A+B=1 \end{array}\right\} \Rightarrow A=-1/3,\ B=1/3$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-s-2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{-1}{s+1}\right) + \frac{1}{3}\,\frac{1}{s-2}\right\}$$

$$= -1/3\ \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3}\ \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= -\frac{e^{-t}}{3} + \frac{e^{2t}}{3} = \frac{1}{3}\left[e^{2t}-e^{-t}\right]$$

$$y(t) = \frac{1}{3}e^{-t}\left(e^{3t}-1\right)$$

Chapter 5. Problem 5.20

(b) $2s/[(s+1)(s^2+4)]$

The first partial decomposition gives:

$$\frac{s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4} = \frac{(A+B)s^2+(B+C)s+4A+C}{(s+1)(s^2+4)}$$

that is to say $\begin{cases} A+B=0 \\ B+C=1 \\ 4A+C=0 \end{cases} \Rightarrow \begin{matrix} A=-1/5 \\ B=1/5 \\ C=4/5 \end{matrix}$

$$\Rightarrow \frac{s}{(s+1)(s^2+4)} = -\frac{1}{5(s+1)} + \frac{1}{5}\,\frac{s+4}{s^2+4}$$

So, we now focus on

$$\frac{s+4}{s^2+4} = \frac{s+4}{(s-2i)(s+2i)} = \frac{A}{s-2i} + \frac{B}{s+2i}$$

$$= \frac{(A+B)s+2i(A-B)}{s^2+4}$$

Identifying the coefficients on both sides of the equation:

$$\left.\begin{matrix} A+B=1 \\ 2i(A-B)=4 \end{matrix}\right\} \Rightarrow \begin{matrix} A=\dfrac{1-2i}{2} \\ B=\dfrac{1+2i}{2} \end{matrix}$$

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question (b)

we found: $\dfrac{s+4}{s^2+4} = \left(\dfrac{1-2i}{2}\right) \times \dfrac{1}{(s-2i)} + \left(\dfrac{1+2i}{2}\right) \dfrac{1}{s+2i}$

Putting all together:

$$\mathcal{L}^{-1}\left(\frac{2s}{(s+1)(s^2+4)}\right) = 2 \cdot \mathcal{L}^{-1}\left(\frac{s}{(s+1)(s^2+4)}\right)$$

$$= 2 \cdot \mathcal{L}^{-1}\left(-\frac{1}{5(s+1)} + \frac{1}{5}\,\frac{s+4}{s^2+4}\right)$$

$$= -\frac{2}{5}\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{2}{5}\,\mathcal{L}^{-1}\left(\frac{s+4}{s^2+4}\right)$$

$$= -\frac{2}{5}\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{2}{5}\,\frac{(1-2i)}{\cancel{2}}\,\mathcal{L}^{-1}\left(\frac{1}{s-2i}\right) + \frac{2}{5}\left(\frac{1+2i}{2}\right)\mathcal{L}^{-1}\left(\frac{1}{s+2i}\right)$$

$$= -\frac{2}{5}e^{-t} + \frac{1-2i}{5}e^{2it} + \frac{1+2i}{5}e^{-2it}$$

$$(1-2i)\,e^{2it} + (1+2i)\,e^{-2it} = 2\cos 2t + 4\sin 2t$$

using Euler's identity

$$\mathcal{L}^{-1}\left(\frac{2s}{(s+1)(s^2+4)}\right) = \frac{2}{5}\left(\cos 2t + 2\sin 2t - e^{-t}\right)$$

$$\Rightarrow y(t) = \frac{2}{5}\left(\cos 2t + 2\sin 2t - e^{-t}\right)$$

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(b) A quicker method is to do a partial fraction expansion of the quadratic in the denominator and use the Laplace transform:

$$\frac{2s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$= \frac{(A+B)s^2 + (B+C)s + 4A + C}{(s+1)(s^2+4)}$$

Equating the powers of s on the numerator:

$$\left.\begin{aligned} s^2&: A+B=0 \\ s^1&: B+C=2 \\ s^0&: 4A+C=0 \end{aligned}\right\} \Rightarrow \begin{aligned} &A+B-B-C=-2 \\ &\left.\begin{aligned} &A-C=-2 \\ &\text{and } C=-4A \end{aligned}\right\} \Rightarrow A+4A=-2 \end{aligned}$$

$$\Rightarrow A=-2/5,\ B=2/5\quad C=8/5$$

$$\frac{2s}{(s+1)(s^2+4)} = -2/5\,\frac{1}{s+1} + \frac{2}{5}\,\frac{s}{s^2+4} + (8/5)\,\frac{1}{s^2+4}$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos(at)\ ;\quad \mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin(at)$$

$$\Rightarrow\quad y(t) = -\frac{2}{5}e^{-t} + \frac{2}{5}\cos(2t) + \frac{4}{5}\sin(2t)$$

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(c) $e^{-(\gamma+s)t_0} / [(\gamma+s)^2+b^2]$

$$\Rightarrow \mathcal{L}^{-1}\left\{ e^{-(\gamma+s)t_0} / [(s+\gamma)^2+b^2] \right\} = e^{-\gamma t_0} \mathcal{L}^{-1}\left\{ e^{-st_0} \frac{1}{(s+\gamma)^2+b^2} \right\}$$

$$= e^{-\gamma t_0} \mathcal{L}^{-1}\left\{ \bar{f}(s)\, \bar{g}(s) \right\}$$

Note that:

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\{e^{-st_0}\} = \delta(t-t_0)$$

$$\mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s+\gamma)^2+b^2} \right\} = \frac{1}{b} e^{-\gamma t} \sin bt$$

We have to consider: $\int_{-\infty}^{\infty} e^{-\gamma z} \sin bz\, \delta(t-t_0-z)\, dz \quad (1)$

First: $\int_0^t e^{-\gamma z} \sin bz\, \delta(t-t_0-z)\, dz = \int_{t_0}^{t+t_0} e^{-\gamma(u-t_0)} \sin b(u-t_0)\, \delta(t-u)\, du$

by the change of variable $z = u - t_0$

$u = z + t_0$

$du = dz$

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We know also that: $\int_{-\infty}^{\infty} f(u)\,\delta(t-u)\,du = f(t)$

We then rewrite the first integral (I) as:

$$\int_{-\infty}^{+\infty} e^{-\gamma(u-t_0)} \sin b(u-t_0)\,[H(u-t_0)-H(u-t-t_0)]\,\delta(t-u)\,du$$

restricting it on the interval $[t_0, t_0+t]$

and taking t to $-\infty$ and $+\infty$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\gamma(u-t_0)} \sin b(u-t_0)\,[H(u-t_0)-H(u-t-t_0)]\,\delta(t-u)\,du$$

$$= \int_{-\infty}^{\infty} e^{-\gamma(u-t_0)} \sin b(u-t_0)\,H(u-t_0)\,\delta(t-u)\,du$$

$$- \int_{-\infty}^{\infty} e^{-\gamma(u-t_0)} \sin b(u-t_0)\,H(u-t-t_0)\,\delta(t-u)\,du$$

$$= e^{-\gamma(t-t_0)} \sin b(t-t_0)\,H(t-t_0) - e^{-\gamma(t-t_0)} \sin b(t-t_0)\,H(t-t-t_0)$$

The second term is zero since the Heaviside function is 0 for negative values. So we are left with

$$\mathcal{L}^{-1}\left\{e^{-(\gamma+s)t_0}/[(s+\gamma)^2+b^2]\right\} = e^{-\gamma t_0}\frac{1}{b}\,e^{-\gamma(t-t_0)} \sin b(t-t_0)\,H(t-t_0)$$

$$= \frac{1}{b}\,e^{-\gamma t} \sin b(t-t_0)\,H(t-t_0)$$

Chapter 5 - Problem 5.21 ab

Use the properties of Laplace transforms to prove the following without evaluating any Laplace integrals explicitly:

(a) $\mathcal{L}[t^{5/2}] = \frac{15}{8}\sqrt{\pi}\ s^{-7/2}$

Let $f(t) = t^{5/2}$, $f'(t) = \frac{5}{2}t^{3/2}$, $f''(t) = \frac{15}{4}t^{1/2}$

and $f(0) = f'(0) = 0$

From the linearity of the Laplace transform:

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{\tfrac{15}{4}t^{1/2}\} = \frac{15}{4}\times\mathcal{L}\{t^{1/2}\}.$$

Using the Laplace transforms table: $\mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$

$$\Rightarrow \mathcal{L}\{f''(t)\} = \frac{15\sqrt{\pi}}{8s^{3/2}}$$

The Laplace transform of the second derivative of $f(t)$:

$$\mathcal{L}\{f''(t)\} = -sf(0) - f'(0) + s^2\tilde{f}(s)$$

$$= -s\times 0 - 0 + s^2\tilde{f}(s)$$

$$= s^2\tilde{f}(s)$$

$$\Rightarrow s^2\tilde{f}(s) = \frac{15\sqrt{\pi}}{8s^{3/2}} \Rightarrow \tilde{f}(s) = \frac{15}{8}\sqrt{\pi}\ s^{-7/2}$$

$\blacksquare$

Chapter 3 - Problem 5.21 ab

(b) $\mathcal{L}[(\sinh at)/t] = \frac{1}{2}\ln[(s+a)/(s-a)]$ $s > |a|$

For $s > |a|$, let $f(t) = \sinh(at)$

From the Laplace property: $\mathcal{L}[\frac{f(t)}{t}] = \int_s^\infty \bar{f}(u)\,du$

we have, using the Laplace transforms table:

$$\bar{f}(s) = \mathcal{L}[\sinh(at)] = \frac{a}{s^2-a^2}$$

and $\mathcal{L}[\frac{f(t)}{t}] = \mathcal{L}[\frac{\sinh at}{t}] = \int_s^\infty \frac{a}{u^2-a^2}\,du$

$$= \int_s^\infty \frac{1}{2}\left[\frac{1}{u-a} - \frac{1}{u+a}\right] du$$

$$= \frac{1}{2}\left[\ln\frac{u-a}{u+a}\right]_s^\infty$$

$\lim\limits_{u\to\infty} \ln\frac{u-a}{u+a} = \lim\limits_{u\to\infty} \ln 1.$ using L'Hospital's rule

$= 0$

Therefore $\mathcal{L}[(\sinh at)/t] = \frac{1}{2}\left(-\ln\frac{s-a}{s+a}\right) = \frac{1}{2}\ln\frac{(s+a)}{(s-a)}$

$\blacksquare$

Chapter 5. Problem 5.21 ab

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(u)\,du$$

provided $\lim_{t \to 0} [f(t)/t]$ exists.

$\lim_{t \to 0} [f(t)/t]$ does exist for question b.

$$\lim_{t \to 0} \frac{f(t)}{t} = \lim_{t \to 0} \frac{\sinh(at)}{t} = \lim_{t \to 0} \frac{a\cosh(at)}{1} = a$$

using L'Hospitals rule

chapter 3. Problem 5.22

Find the solution (the so called impulse response or Green's function) of the equation

$$T\frac{dx}{dt}+x=\delta(t)$$

by proceeding as follows

(a) Show by substitution that $x(t)=A(1-e^{-t/T})H(t)$ is a solution, for which $x(0)=0$, of

$T\frac{dx}{dt}+x=AH(t)$ where $H(t)$ is the Heaviside step function. (*)

$$\frac{dx(t)}{dt}=A\frac{d}{dt}\left(1-e^{-t/T}\right)H(t)$$

$$=A\left[-\left(-\frac{1}{T}\right)e^{-t/T}H(t)+\left(1-e^{-t/T}\right)\delta(t)\right]$$

$$=A\left[\frac{1}{T}e^{-t/T}H(t)+\left(1-e^{-t/T}\right)\delta(t)\right]$$

$$T\frac{dx(t)}{dt}=Ae^{-t/T}H(t)+AT\left(1-e^{-t/T}\right)\delta(t)$$

$$T\frac{dx(t)}{dt}+x(t)=Ae^{-t/T}H(t)+AH(t)-Ae^{-t/T}H(t)$$
$$+AT\left(1-e^{-t/T}\right)\delta(t)$$

$$=AH(t)+AT\left(1-e^{-t/T}\right)\delta(t)$$

Chapter 5 - Problem 5.22

We know that $\delta(t)=0$ for $t\neq 0$

So for $t\neq 0$, the previous equation becomes

$$\tau \frac{dx(t)}{dt}+x(t)=A\,H(t)$$ which is the equation noted as (*)

For $t=0$ $\quad x(0)=A(1-e^{-0/\tau})\,H(0-0)$
$$=A(1-1).0=0$$

(b) Note that the solution for $\tau\frac{dx}{dt}+x=A\,H(t-3)$ is

$$x_3(t)=A(1-e^{-t/\tau})\,H(t-3)$$

Also note that we can write our solution for (a) as:

$$x_0(t)=A(1-e^{-t/\tau})\,H(t-0)$$

Therefore we construct a block function from $t=0$ to $t=3$ of a superposition of these two solutions using a step up at $t=0$ with $H(t-0)$ and a step down at $t=3$ with $-H(t-3)$:

$$A(1-e^{-t/\tau})\,H(t-0)-A\left(1-e^{-(t-3)/\tau}\right)H(t-3)$$

Chapter 5 – Problem 5.22

(a) To evaluate the impulse response

$$x(t)_{impulse} = \lim_{\delta \to 0} \frac{1}{\delta} \left[(1-e^{-t/T}) H(t-0) - (1-e^{-(t-\delta)/T}) H(t-\delta) \right]$$

$$= \lim \frac{1}{\delta} \left[(1-e^{-t/T}) H(t-0) - (1-e^{-t/T} e^{\delta/T}) H(t-\delta) \right]$$

$$e^{\delta/T} = 1 + \frac{\delta}{T} + \frac{\delta^2}{2T^2} + \frac{1}{3!}\left(\frac{\delta}{T}\right)^3 + \dots$$

$$\Rightarrow 1/\delta \; e^{\delta/T} = \frac{1}{\delta} + \frac{1}{T} + \frac{\delta}{2T^2} + \dots$$

$$\Rightarrow x(t)_{impulse} = \lim_{\delta \to 0} \left[\frac{1-e^{-t/T}}{\delta} H(t) - \frac{1}{\delta} \left(1 - e^{-t/T}\left(1 + \frac{\delta}{T} + \frac{\delta^2}{2T^2} + \dots\right)\right) H(t-\delta) \right]$$

$$= \lim_{\delta \to 0} \frac{1-e^{t/T}}{\delta} H(t) - \frac{1-e^{t/T}}{\delta} H(t-\delta) + \frac{e^{-t/T}}{T} H(t-\delta)$$

$$+ \frac{e^{-t/T}}{2T^2} H(t-\delta)\, O(\delta)$$

$$= \frac{e^{-t/T}}{T} H(t)$$

$$= \frac{e^{-t/T}}{T} \quad \text{for } t > 0$$

chapter 5. Problem 5.22

(d) obtain the same result ... by taking the laplace transform

$$\mathcal{L}\left\{ T\frac{dx}{dt} + x \right\} = \mathcal{L}\{\delta(t)\}$$

Using laplace transform of the first derivative:

$$T\left(-X(0) + sX(s)\right) + X(s) = e^{-s\times 0} = 1$$

$\Rightarrow$ Assuming $x(0)=0 \Rightarrow X(0)=0$

Then $$(1+sT)\,X(s) = 1$$

$$\Rightarrow X(s) = \frac{1}{1+sT} = \frac{1}{T\left(\frac{1}{T}+s\right)}$$

$$\Rightarrow \mathcal{L}^{-1}(X(s)) = x(t) = \frac{1}{T}\,e^{-t/T} \quad \text{for } t>0$$

Chapter 3 - Problem 5.26

Show that the laplace transform of $f(t-a)\,H(t-a)$ where $a \geqslant 0$, is $e^{-as}\bar{f}(s)$ and that, if $g(t)$ is a periodic function of period T, $\bar{g}(s)$ can be written as: $\displaystyle \frac{1}{1-e^{-sT}} \int_0^T e^{-st} g(t)\,dt$

First for $a \geqslant 0$ $\quad \mathcal{L}[f(t-a)\,g(t-a)] = \int_0^\infty e^{-st} f(t-a)\,H(t-a)\,dt$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Now $\int_a^\infty e^{-st} f(t-a)\,dt = \int_0^\infty e^{-s(t'+a)} f(t')\,dt'$ by change of variable $t' = t-a$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\,dt$$

$$= e^{-as}\, \mathcal{L}(f(t))$$

$$\bar{g}(s) = \int_0^\infty e^{-st} g(t)\,dt = \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} e^{-st} g(t)\,dt$$

$T \geqslant 0$ period for $g(t)$

change of variable $t = kT + u \Rightarrow \bar{g}(s) = \sum_{k=0}^{\infty} \int_0^T e^{-s(kT+u)} g(kT+u)\,du$

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T primary period of g: $g(kT+u) = g(u)$

Thus

$$\bar{g}(s) = \sum_{k=0}^{\infty} e^{-kTs} \int_0^T e^{-su} g(u)\, du$$

$\sum_{k=0}^{\infty} e^{-kTs}$ geometric series of ratio $e^{-sT} < 1$ for $s > 0$, $T > 0$

$$\Rightarrow \sum_{k=0}^{\infty} e^{-kTs} = \sum_{k=0}^{\infty} \left(e^{-Ts}\right)^k = \frac{1}{1-e^{-sT}}$$

$$\Rightarrow \bar{g}(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} g(t)\, dt$$

(a) Sketch the periodic function defined in $0 \leq t \leq T$ by

$$g(t) = \begin{cases} 2t/T & 0 \leq t < T/2 \\ 2(1-t/T) & T/2 \leq t \leq T \end{cases}$$

And using the previous result, find its Laplace transform.

See next page for the graph of g (t) taking T = 2
and $t \in [0,2]$

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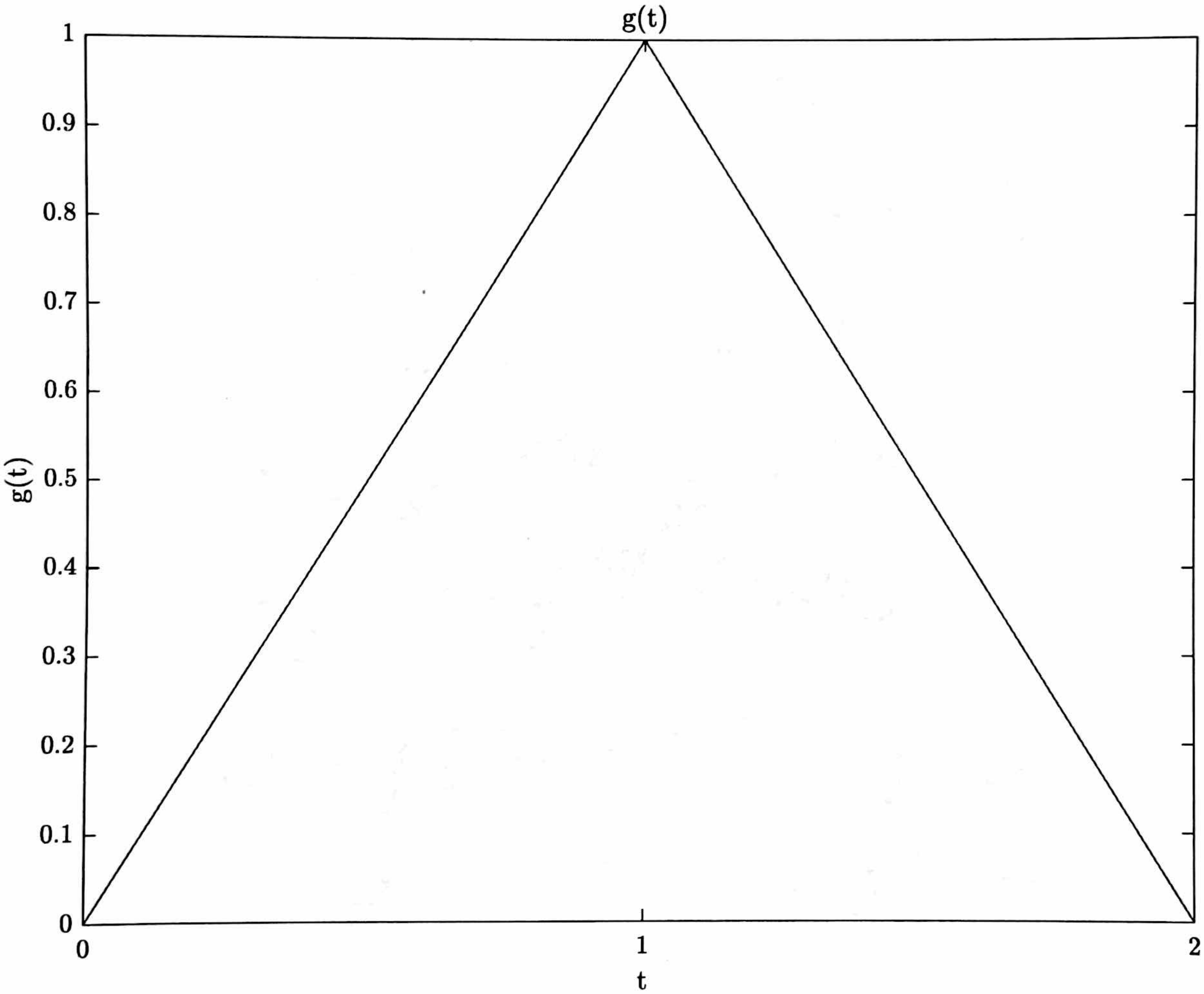
Graph of g(t) for T=2 on [0,2]



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Using the result $\bar{g}(s) = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} g(t)\, dt$

$$\int_0^T e^{-st} g(t)\, dt = \int_0^{T/2} e^{-st}\, 2t/T\, dt + \int_{T/2}^T e^{-st}\, 2(1-t/T)\, dt$$

$$\int_0^{T/2} e^{-st}\, t\, dt = \frac{1}{(-s)} \left[t e^{-st} \right]_0^{T/2} + \frac{1}{s} \int_0^{T/2} e^{-st} dt$$

$$= -1/s \left(T/2\, e^{-sT/2} - 1 \right) + \frac{1}{s} \left(-\frac{1}{s} \right) \left[e^{-st} \right]_0^{T/2}$$

$$= 1/s \left(1 - \frac{T}{2} e^{-sT/2} \right) - \frac{1}{s^2} \left(e^{-sT/2} - 1 \right)$$

$$= \frac{1}{s^2} - \frac{1}{s^2} e^{-sT/2} - \frac{T}{2s} e^{-sT/2}$$

$$= \frac{1}{s^2} \left[1 - \frac{e^{-sT/2}}{2} (sT+2) \right]$$

$$\int_{T/2}^T e^{-st} (1 - t/T)\, dt = -\frac{1}{s} \left[(1-t/T) e^{-st} \right]_{T/2}^T - \frac{1}{sT} \int_{T/2}^T e^{-st} dt$$

$$= \frac{1}{2s} e^{-sT/2} + \frac{1}{s^2 T} \left[e^{-st} \right]_{T/2}^T$$

$$= \frac{sT e^{-sT/2} + 2 \left(e^{-sT} - e^{-sT/2} \right)}{2s^2 T}$$

$$= \frac{e^{-sT}}{2s^2 T} \left(2 + e^{sT/2} (sT - 2) \right)$$

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Putting all together

$$\bar{g}(s) = \frac{1}{1-e^{-sT}}\left[\frac{2}{s^2T}\left[1 - \frac{e^{-sT/2}}{2}(sT+2)\right.\right.$$
$$\left. + \frac{2e^{-sT}}{2s^2T}\left(2 + e^{sT/2}(sT-2)\right)\right]$$

$$= \frac{1}{s^2T(1-e^{-sT})} \times \left[2 - e^{-sT/2}(sT+2) + e^{-sT}\left(2+e^{sT/2}(sT-2)\right)\right]$$

$$= \frac{1}{s^2T(1-e^{-sT})} \times \left[2 - sT\,e^{-sT/2} - 2e^{-sT/2} + 2e^{-sT} + e^{-sT/2}sT - 2e^{-sT}\right]$$

$$= \frac{1}{s^2T(1-e^{-sT})} \times \left(2 - 4e^{-sT/2} + 2e^{-sT}\right)$$

$$= \frac{2}{s^2T(1-e^{-sT})} \times \left(1 - 2e^{-sT/2} + e^{-sT}\right)$$

$$= \frac{2}{Ts^2}\;\frac{1-e^{-sT/2}}{1+e^{-sT/2}}$$

$$= \frac{2}{Ts^2}\tanh\left(\frac{sT}{4}\right)$$

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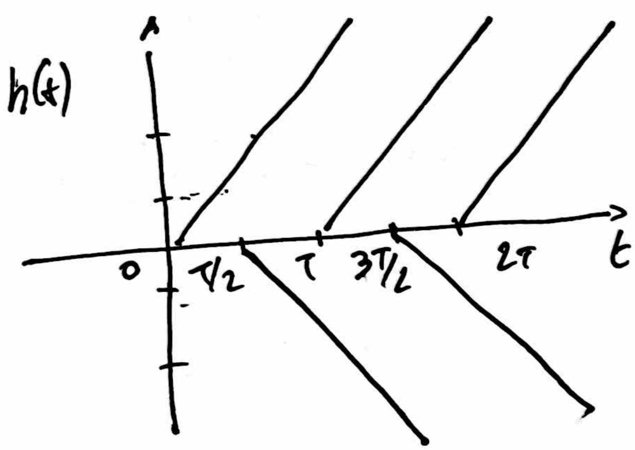
(b) Show, by sketching it, that

$$\frac{2}{T}\left[t H(t) + 2\sum_{n=1}^{\infty}(-1)^n \left(t - \frac{nT}{2}\right) H\left(t - \frac{nT}{2}\right)\right]$$

is another representation of $g(t)$ and hence derive the relationship $\tanh(x) = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}$

$$h(t) = \frac{2}{T}\left[t H(t) + 2\sum_{n=1}^{\infty}(-1)^n \left(t - \frac{nT}{2}\right) H\left(t - \frac{nT}{2}\right)\right]$$

Graph h(t)



It seems that h(t) which is also periodic is another representation of g(t)

$$\mathcal{L}\{h(t)\} = \mathcal{L}\left\{\frac{2}{T}\left[t H(t) + 2\sum_{n=1}^{\infty}(-1)^n \left(t - \frac{nT}{2}\right) H\left(t - \frac{nT}{2}\right)\right]\right\}$$

$$= \frac{2}{T}\mathcal{L}\{t H(t)\} + \frac{4}{T}\sum_{n=1}^{\infty}\mathcal{L}\left\{\left(t - \frac{nT}{2}\right) H\left(t - \frac{nT}{2}\right)\right\}$$

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Now $\mathcal{L}\{f(t-b)H(t-a)\} = e^{-sb}\tilde{f}(s)$

Therefore $\mathcal{L}\{h(t)\} = \frac{2}{T}e^{-s\times 0}\times\frac{1}{s^2} + \frac{4}{T}\sum_{n=1}^{\infty}(-1)^n e^{-s\frac{nT}{2}}\frac{1}{s^2}$

$$= \frac{2}{Ts^2}\left[1+2\sum_{n=1}^{\infty}(-1)^n e^{-\frac{snT}{2}}\right]$$

$\therefore$ $\mathcal{L}\{g(t)\}$ and $\mathcal{L}\{h(t)\}$ are the laplace transforms of the same periodic function therefore they are equal:

$$\frac{2}{Ts^2}\tanh\left(\frac{sT}{4}\right) = \frac{2}{Ts^2}\left[1+2\sum_{n=1}^{\infty}(-1)^n e^{-\frac{snT}{2}}\right]$$

$$\Rightarrow \tanh\left(\frac{sT}{4}\right) = 1+2\sum_{n=1}^{\infty}(-1)^n e^{-\frac{snT}{2}}$$

For $x = \frac{sT}{4}$ we have $\tanh(x) = 1+2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}$