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JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

The Fourier transform

Let's start out looking at the complex Fourier series again.

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi i r x/L} \qquad c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x) e^{-2\pi i r x/L} dx$$

Remember it applies to functions that are periodic or can be extended to be periodic.

How do we deal with non-periodic functions?

An informal derivation of the Fourier transform

First writing our complex Fourier series as a function of t and taking as the primary period –T/2 to T/2 we have

$$f(t) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi i r t/T} = \sum_{r=-\infty}^{\infty} c_r e^{i\omega_r t} \quad \text{where} \quad \omega_r = \frac{2\pi r}{T}$$

and

$$c_r = \frac{1}{T}\int_{-T/2}^{T/2} f(x) e^{-i\omega_r t}\,dt$$

Next take $\omega_{r+1} - \omega_r = \frac{2\pi(r+1)}{T} - \frac{2\pi r}{T} = \boxed{\frac{2\pi}{T} \equiv \Delta\omega}$

Note that T and $\Delta\omega$ are inversely proportional to each other

Substitution for T using the expression highlighted above in the integrand of c_r With u as it's variable and substitution of c_r in the complex Fourier series gives

$$f(t) = \sum_{r=-\infty}^{\infty} c_r e^{i\omega_r r} = \sum_{r=-\infty}^{\infty} \left[\underbrace{\boxed{\frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(u) e^{-i\omega_r u}\,du}}_{c_r} \right] e^{i\omega_r t}$$

Now comes the important part – Let T go to infinity, that is the function f(t) will not be periodic!

Note that as , that $T \to \infty$ then $\Delta\omega \to d\omega,\ \omega_r \to \omega,\ \sum \to \int$

$$f(t) = \sum_{r=-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} f(u) e^{-i\omega_r u} du \right] e^{i\omega_r t} \to \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du e^{i\omega t} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du \right] e^{i\omega t} d\omega$$

Let $\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ then $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega$

We then identify $\boxed{\tilde{f}(\omega) = F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt}$ as the Fourier transform of f(t)

$\boxed{f(t) = F^{-1}\{\tilde{f}(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega}$ as the inverse Fourier transform

Note that the Fourier transform of the function only exist provided that $\int_{-\infty}^{\infty} |f(t)| dt \leq \infty$

Fourier transform – an example

Find the Fourier transform of the exponential decay function
$f(t) = 0$ for $t < 0$ and $f(t) = A\,e^{-\lambda t}$ for $t \geq 0$ ($\lambda > 0$).

Using the definition and separating the integral into two parts

$$
\begin{aligned}
\widetilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} (0)\, e^{-i\omega t}\, dt + \frac{A}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda t}\, e^{-i\omega t}\, dt \\
&= 0 + \frac{A}{\sqrt{2\pi}} \left[-\frac{e^{-(\lambda + i\omega)t}}{\lambda + i\omega} \right]_0^{\infty} \\
&= \frac{A}{\sqrt{2\pi}(\lambda + i\omega)},
\end{aligned}
$$

The inverse Fourier transform is not so easily performed as it requires contour integration involving complex variables.

The Laplace transform

In fact there are my types of integral transforms.
One that most students have seen is the Laplace transform.

It is similar to the Fourier transform.
but instead of having a complex exponential
it has a real regularization factor, that is $i\omega \rightarrow s$,
no normalization factor and the integral bounds go from 0 to ∞

$$\tilde{f}(s) = L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$$

However the inverse Laplace transform is not so simply defined.
It is called the Bromwich integral and requires integration in the complex domain. We will see it toward the end of this semester.

Some Laplace transforms

First the Laplace transform has the linearity property

$$\mathcal{L}[af_1(t) + bf_2(t)] = a\mathcal{L}[f_1(t)] + b\mathcal{L}[f_2(t)] = a\bar{f}_1(s) + b\bar{f}_2(s).$$

Next let's derive the Laplace transforms for a few simple but interesting functions

$$L\{1\} = \int_0^\infty 1e^{-st}dt = \frac{1}{-s}e^{-st}\Big|_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right] = \frac{1}{s}$$

Similar to above

$$L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{-(s-a)}e^{-(s-a)t}\Big|_0^\infty = \cdots = \frac{1}{s-a}$$

By **integration by parts** we can find the Laplace transform of t^n

$$L\{t^n\}=\int_0^\infty t^n e^{-st}dt=\left.\frac{t^n}{-s}e^{-st}\right|_0^\infty-\int_0^\infty \frac{nt^{n-1}}{-s}e^{-st}dt$$

For the first term the form is ∞/∞ We can use **L'Hospitals rule**, that says that the limit of the ratio of the derivatives of the functions that make up the quotient behave the same as the limit of the ratio of the original functions.

$$\lim_{t\to\infty}\frac{t^n}{-s}e^{-st}-\frac{0^n}{-s}e^{-s0}=\lim_{t\to\infty}\frac{t^n}{-se^{st}}=\lim_{t\to\infty}\frac{nt^{n-1}}{-s(se^{st})}=\cdots=\lim_{t\to\infty}\frac{n(n-1)\cdots 1}{-s^{n+1}e^{st}}=\frac{n(n-1)\cdots 1}{-s^{n+1}}\lim_{t\to\infty}\frac{1}{e^{st}}=0$$

Therefore $$L\{t^n\}=\int_0^\infty t^n e^{-st}dt=\int_0^\infty \frac{nt^{n-1}}{s}e^{-st}dt=\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt=\frac{n}{s}L\{t^{n-1}\}$$

And in general $$L\{1\}=\frac{1}{s}$$

$$L\{t^1\}=\frac{1}{s}L\{t^{1-1}\}=\frac{1}{s}L\{t^0\}=\frac{1}{s}L\{1\}=\frac{1}{s}\frac{1}{s}=\frac{1}{s^2}$$

$$L\{t^2\}=\frac{2}{s}L\{t^{2-1}\}=\frac{2}{s}L\{t^1\}=\frac{2}{s}L\{t^1\}=\frac{2}{s}\frac{1}{s^2}=\frac{2}{s^3}$$

$$\vdots$$

$$L\{t^n\}=\frac{n!}{s^{n+1}}$$

A variation of a previous Laplace transform, identity from complex analysis (we will see more of this later in the semester) and the linearity property allows us to find the following Laplace transforms.

First $L\{e^{at}\} = \frac{1}{s-a}$ Then letting a = ib gives $L\{e^{ibt}\} = \frac{1}{s-ib}$

Next normalize $L\{e^{ibt}\} = \frac{1}{s-ib}\frac{s+ib}{s+ib} = \frac{s+ib}{s^2+b^2}$

Finally using the Euler identity from complex analysis $e^{ibt} = \cos bt + i\sin bt$

Allows us to write $L\{e^{ibt}\} = L\{\boxed{\cos bt} + i\boxed{\sin bt}\} = \boxed{\frac{s}{s^2+b^2}} + i\boxed{\frac{b}{s^2+b^2}}$

Finally the linearity property lets us make the association for the real and Imaginary parts of the expression on the left and right hand sides, that is

$$\boxed{L\{\cos bt\} = \frac{s}{s^2+b^2} \qquad L\{\sin bt\} = \frac{b}{s^2+b^2}}$$

Short Table of Laplace transforms

$f(t)$	$\bar{f}(s)$	s_0
c	c/s	0
ct^n	$cn!/s^{n+1}$	0
$\sin bt$	$b/(s^2+b^2)$	0
$\cos bt$	$s/(s^2+b^2)$	0
e^{at}	$1/(s-a)$	a
$t^n e^{at}$	$n!/(s-a)^{n+1}$	a
$\sinh at$	$a/(s^2-a^2)$	$\lvert a\rvert$
$\cosh at$	$s/(s^2-a^2)$	$\lvert a\rvert$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2]$	a
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2]$	a
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$	0
$t^{-1/2}$	$(\pi/s)^{1/2}$	0
$\delta(t-t_0)$	e^{-st_0}	0
$H(t-t_0) = \begin{cases} 1 & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$	e^{-st_0}/s	0

Inverse Laplace transforms

Inverse Laplace transforms are general done by looking the pattern up in a Table of Laplace transforms (see previous slide).

In addition we also often need a result from algebra called partial fraction expansions. Here is a short list of partial fraction formulas.

Factor in denominator	*Term in partial fraction*
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\ldots+\dfrac{A_k}{(ax+b)^k},\ k=1,2,3,\ldots$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k},\ k=1,2,3,\ldots$

https://medium.com/@dheerajdhobley/all-about-partial-fractions-e827115ea039#.tde5lzjrv

Partial fraction – an example

As an example write $\frac{x+14}{(2x+3)(x-1)}$ as a partial fraction

Start with $$\frac{x+14}{(2x+3)(x-1)}=\frac{A_1}{2x+3}+\frac{A_2}{x-1}$$

$$\frac{x+14}{(2x+3)(x-1)}=\frac{A_1\boxed{(x-1)}}{(2x+3)\boxed{(x-1)}}+\frac{A_2\boxed{(2x+3)}}{\boxed{(2x+3)}(x-1)}$$

Equating the coefficients associated with the powers of x in this case the constant and x!

$$x+14=A_1(x-1)+A_2(2x+3)$$
$$x=A_1x+A_2 2x=(A_1+2A_2)x \text{ and } 14=-A_1+3A_2$$

This gives two equations in two unknowns $$\boxed{\begin{aligned}1&=A_1+2A_2\\14&=-A_1+3A_2\end{aligned}}$$

Which can be easily solve for $A_1=-5,\ A_2=3$

Therefore $$\frac{x+14}{(2x+3)(x-1)}=\frac{-5}{2x+3}+\frac{3}{x-1}$$

Inverse Laplace transform– an example

$$L^{-1}\left\{\frac{s+3}{s(s+1)}\right\}=?$$

First use partial fractions

$$\frac{s+3}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}$$

that is $s+3=A(s+1)+Bs=(A+B)s+A$

or equating powers of s gives $A+B=1$ and $A=3$

$\therefore\ B=2$

Substitution of the above and applying the linearity property we get

$$L^{-1}\left\{\frac{3}{s}+\frac{2}{s+1}\right\}=L^{-1}\left\{\frac{3}{s}\right\}+L^{-1}\left\{\frac{2}{s+1}\right\}=3L^{-1}\left\{\frac{1}{s}\right\}+2L^{-1}\left\{\frac{1}{s+1}\right\}$$

Then looking in our table of Laplace transforms we see $L^{-1}\left\{\frac{1}{s}\right\}=1$ and $L^{-1}\left\{\frac{1}{s+1}\right\}=e^{-1t}$

Therefore $L^{-1}\left\{\frac{s+3}{s(s+1)}\right\}=3(1)+2(e^{-t})=3+2e^{-t}$