5.1

Using $f(t)=\frac{\pi}{2}e^{-|t|}$ as my function (unlike start of problem in book)

I will just find $\tilde{f}(\omega)$ directly for this f(t)

Don't forget to split integral for calculation to take into account absolute value!!

$$\tilde{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}\frac{\pi}{2}e^{-(-t)}e^{-i\omega t}\,dt+\int_{0}^{\infty}\frac{\pi}{2}e^{-(t)}e^{-i\omega t}\,dt\right]=$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\int_{-\infty}^{0}e^{(1-i\omega)t}\,dt+\int_{0}^{\infty}e^{(-1-i\omega t)}\,dt\right]=\frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\left.\frac{e^{(1-i\omega)t}}{(1-i\omega)}\right|_{-\infty}^{0}+\left.\frac{e^{(-1-i\omega)t}}{(-1-i\omega)}\right|_{0}^{\infty}\right]=$$

$$\frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\left.\frac{e^{(1-i\omega)t}}{(1-i\omega)}\right|_{-\infty}^{0}+\left.\frac{e^{(-1-i\omega)t}}{(-1-i\omega)}\right|_{0}^{\infty}\right]=\frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{\left[\frac{e^{(1-i\omega)0}}{(1-i\omega)}-\frac{e^{(1-i\omega)-\infty}}{(1-i\omega)}\right]+\left[\frac{e^{(-1-i\omega)\infty}}{-(1+i\omega)}-\frac{e^{(-1-i\omega)0}}{-(1+i\omega)}\right]\right\}=$$

$$\frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{\left[\frac{1}{(1-i\omega)}-\frac{0}{(1-i\omega)}\right]+\left[\frac{0}{-(1+i\omega)}-\frac{1}{-(1+i\omega)}\right]\right\}=\frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\frac{1}{(1-i\omega)}+\frac{1}{(1+i\omega)}\right]=$$

$$\frac{1}{2}\sqrt{\frac{\pi}{2}}\left[\frac{2}{1+\omega^2}\right]=\sqrt{\frac{\pi}{2}}\frac{1}{1+\omega^2}$$

Now apply inverse Fourier transform

$$f(t)=\frac{\pi}{2}e^{-|t|}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(w)e^{i\omega t}\,d\omega=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{2}}\frac{1}{1+\omega^2}e^{i\omega t}\,d\omega=\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{1+\omega^2}e^{i\omega t}\,d\omega$$

$$\frac{1}{2}\left[\int_{-\infty}^{0}\frac{1}{1+\omega^2}e^{i\omega t}\,d\omega+\int_{0}^{\infty}\frac{1}{1+\omega^2}e^{i\omega t}\,d\omega\right]=(\text{in first integral let }\omega\to-\omega)=$$

Aside: $\displaystyle\int_{-\infty}^{0}\frac{1}{1+\omega^2}e^{i\omega t}\,d\omega=\int_{\omega=-\infty}^{\omega=0}\frac{1}{1+\omega^2}e^{i\omega t}\,d\omega\xrightarrow{\text{Chg here}}\int_{\omega=-\infty}^{\omega=-0}\frac{1}{1+[-\omega]^2}e^{-i\omega t}(-d\omega)=-\int_{\infty}^{0}\frac{1}{1+\omega^2}e^{-i\omega t}\,d\omega$ Back:

Subst into first integral above $\displaystyle\frac{1}{2}\left[-\int_{\infty}^{0}\frac{1}{1+\omega^2}e^{-i\omega t}\,d\omega+\int_{0}^{\infty}\frac{1}{1+\omega^2}e^{i\omega t}\,d\omega\right]=\frac{1}{2}\left[\int_{0}^{\infty}\frac{1}{1+\omega^2}e^{-i\omega t}\,d\omega+\int_{0}^{\infty}\frac{1}{1+\omega^2}e^{i\omega t}\,d\omega\right]=$

$$\frac{1}{2}\left[\int_{0}^{\infty}\frac{1}{1+\omega^2}(e^{i\omega t}e+^{-i\omega t})\,d\omega\right]=(\text{Euler identity})=\frac{1}{2}\int_{0}^{\infty}\frac{1}{1+\omega^2}(2\cos\omega t)\,d\omega=\int_{0}^{\infty}\frac{\cos\omega t}{1+\omega^2}\,d\omega$$

Therefore

$$\frac{\pi}{2}e^{-|t|}=\int_{0}^{\infty}\frac{\cos\omega t}{1+\omega^2}\,d\omega$$

For part (b) carry out integrations on left and right hand side
for Parseval's theoerm using given f(t) and calculated $\tilde{f}(w)$ and hint in book
That is

$$\int_{-\infty}^{\infty}\left|f(t)\right|^2 dt = ?\int_{-\infty}^{\infty}\left|\tilde{f}(\omega)\right|^2 d\omega$$

From (a) $f(t)=\frac{\pi}{2}e^{-|t|}$ and $\tilde{f}(\omega)=\sqrt{\frac{\pi}{2}}\frac{1}{1+\omega^2}$

Substitute

$$\int_{-\infty}^{\infty}\left(\frac{\pi}{2}e^{-t}\right)^2 dt \ = ?\int_{-\infty}^{\infty}\left[\sqrt{\frac{\pi}{2}}\frac{1}{1+\omega^2}\right]^2 d\omega$$

Since both even functions

$$\int_{-\infty}^{\infty}\left(\frac{\pi}{2}e^{-t}\right)^2 dt \ \rightarrow 2\int_{0}^{\infty}\left(\frac{\pi}{2}e^{-t}\right)^2 dt \ = \frac{\pi^2}{2}\int_{0}^{\infty}e^{-2t}\,dt$$

$$\int_{-\infty}^{\infty}\left[\sqrt{\frac{\pi}{2}}\frac{1}{1+\omega^2}\right]^2 d\omega \rightarrow 2\int_{0}^{\infty}\left[\sqrt{\frac{\pi}{2}}\frac{1}{1+\omega^2}\right]^2 d\omega = \pi\int_{0}^{\infty}\left[\frac{1}{1+\omega^2}\right]^2 d\omega$$

LHS $\left(\frac{\pi}{2}\right)^2\int_{0}^{\infty}e^{-2t}\,dt = \cdots = \frac{\pi^2}{4}$

RHS $\pi\int_{0}^{\infty}\left[\frac{1}{1+\omega^2}\right]^2 d\omega =$

$(\text{let } \omega=\tan\theta,\ \theta=\tan^{-1}\omega, 1+\omega^2 \rightarrow 1+\tan^2\theta=\sec^2\theta, d\omega=\sec^2\theta d\theta) =$

$$\pi\int_{0}^{\pi/2}\frac{1}{\sec^2\theta}d\theta=\pi\int_{0}^{\pi/2}\cos^2\theta d\theta = \text{ (use Integral Table)}\cdots=\frac{\pi^2}{4}$$

Thus verifying Parseval's Th. since LHS = RHS