7.1

$$(1-z^2)y''-3zy'+\lambda y=0$$

Put into standard form

$$y''+\frac{3z}{(1-z^2)}y'+\frac{\lambda}{(1-z^2)}y=0$$

$$p(z)=\frac{-3z}{(1-z^2)},\ q(z)=\frac{\lambda}{(1-z^2)}$$

$zp(z)$ and $z^2q(z)$ are analytic at $z=0$

Therefore ordinary point

Take

$$y=\sum_{n=0}^{\infty}a_n z^n\ \ y'=\sum_{n=0}^{\infty}na_n z^{n-1}\ \ y''=\sum_{n=0}^{\infty}n(n-1)a_n z^{n-2}$$

Substitution

$$(1-z^2)\sum_{n=0}^{\infty}n(n-1)a_n z^{n-2}-3z\sum_{n=0}^{\infty}na_n z^{n-1}+\lambda\sum_{n=0}^{\infty}a_n z^n=0$$

First term above

$$(1-z^2)\sum_{n=0}^{\infty}n(n-1)a_n z^{n-2}=\sum_{n=0}^{\infty}n(n-1)a_n z^{n-2}-\sum_{n=0}^{\infty}n(n-1)a_n z^n$$

Second and third term above

$$-3z\sum_{n=0}^{\infty}na_n z^{n-1}+\lambda\sum_{n=0}^{\infty}a_n z^n=\sum_{n=0}^{\infty}(-3na_n z^n+\lambda a_n z^n)=0$$

Combining

$$\sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=0}^{\infty}[-n(n-1)-3n+\lambda]a_n z^{\,n} = 0$$

and reindex leading (forst term above to match powers of z) term

Note this mean n-2 $\rightarrow$ n (call it n') so n-2 = n' or n = n'+2 then lower bound n = n'+2 = 0 or n' = -2 also change all the remaining within first sum by subst. of b by n'+2, when we end subst. replace n' with n since index doesn't care if its n' or n at this point and need to have same index symbol in all our sums

$$\sum_{n=-2}^{\infty}(n+2)(n+1)a_{n+2}z^{\,n} + \sum_{n=0}^{\infty}[-n(n-1)-3n+\lambda]a_n z^{\,n} = 0$$

However for n = -1 and -2 terms in first sum are 0 therefore start index at n = 0

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^{\,n} + \sum_{n=0}^{\infty}[-n(n-1)-3n+\lambda]a_n z^{\,n} = 0$$

Combine into single sum

$$\sum_{n=0}^{\infty}\{(n+2)(n+1)a_{n+2} + [-n(n-1)-3n+\lambda]a_n\}z^n = 0$$

Since z^n not zero in general terms in {} are zero and gives

$$(n+2)(n+1)a_{n+2} + [-n^2-2n+\lambda]a_n = 0$$

Therefore recursion relation is

$$a_{n+2} = \frac{(n^2+2n-\lambda)}{(n+2)(n+1)}a_n$$

Take $a_0 \neq 0$ and $a_1 = 0$ to generate series with even terms

and for series to terminate with $\lambda = N(N+2)$(some integer $n = N$)

This means all terms in series above point N= n would have coefficents that are 0!

Take $N = 2$ then $\lambda = 2(2+2) = 8$

Using recursion with $n = 0$ gives $a_2 = \frac{(-8)}{(2)(1)} a_0 = -4a_0$

Therefore for N= 2 we get $y(z) = U_2(z) = a_0 - 4a_0 z^2$

Take $a_0 = 0$ and $a_1 \neq 0$ to generate series with odd terms

Take $N = 3$ then $\lambda = 3(3+2) = 15$

Using recursion with $n = 0$ gives $a_3 = \frac{(1^2 + 2(1) - 15)}{(1+2)(1+1)} a_1 = \frac{-12}{6} a_1 = -2a_1$

Therefore for N= 3 we get $y(z) = U_3(z) = a_1 z - 2a_1 z^3$