Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Take Home Project 2
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Question 1

(a) Please see attached separate pdf.

(b) $f(t) = C_0 e^{-\frac{t}{\tau}}$ with period T, so

$$\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} dt \\
&= \frac{2C_0}{T}(-\tau)[e^{-\frac{t}{\tau}}]_0^T \\
&= -2C_0\frac{\tau}{T}[e^{-\frac{T}{\tau}} - 1] \\
&= 2C_0\frac{\tau}{T}(1 - e^{-\frac{T}{\tau}})
\end{aligned}$$

If $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_0 \approx 2C_0\frac{\tau}{T}$.

$$\begin{aligned}
a_k &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt
\end{aligned}$$

Using integration by parts with $u = \cos\frac{2k\pi t}{T}, du = -\frac{2k\pi}{T}\sin\frac{2k\pi t}{T}$ and $dv = e^{-\frac{t}{\tau}}, v = (-\tau)e^{-\frac{t}{\tau}}$:

$$\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt = (-\tau)[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}]_0^T - \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt$$

Using again integration by parts:

$$\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt = (-\tau)[e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt$$

So

$$\begin{aligned}
(1 + (\frac{2k\pi\tau}{T}))^2\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt &= (-\tau)[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau^2}{T}[e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T}]_0^T \\
&= (-\tau)[e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T}]_0^T + 0 \\
&= \tau(1 - e^{-\frac{T}{\tau}})
\end{aligned}$$

$$\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt = \frac{\tau}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}})$$

Substituting back into the expression found for a_k yields

$$\begin{aligned} a_k &= 2C_0 \frac{\tau}{T} \frac{1}{1+(\frac{2k\pi\tau}{T})^2}(1-e^{-\frac{T}{\tau}}) \\ &= 2C_0 \frac{\tau T}{T^2+(2k\pi\tau)^2}(1-e^{-\frac{T}{\tau}}) \end{aligned}$$

With the same assumption $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_k \approx 2C_0 \frac{\tau}{T} \frac{1}{1+(\frac{2k\pi\tau}{T})^2}$. Similarly to compute b_k

$$\begin{aligned} b_k &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T}\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T}\frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T}\frac{2k\pi\tau}{T}\frac{\tau}{1+(\frac{2k\pi\tau}{T})^2}(1-e^{-\frac{T}{\tau}}) \\ &= 4C_0 k\pi \frac{\tau^2}{T^2+(2k\pi\tau)^2}(1-e^{-\frac{T}{\tau}}) \end{aligned}$$

Once again, since $e^{-\frac{T}{\tau}} \approx 0$ and $b_k \approx 4C_0(\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2}\pi k$

(c) For $k \geq 1$

$$\begin{aligned} p_k &= \frac{1}{2}(a_k^2+b_k^2) \\ &= \frac{1}{2}\left[4C_0^2(\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2} + 16C_0^2(\frac{\tau}{T})^4\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\pi^2k^2\right] \\ &= \frac{1}{2}4C_0^2(\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\left[1+4(\frac{\tau}{T})^2\pi^2k^2\right] \\ &= 2C_0^2(\frac{\tau}{T})^2\frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2}\left[1+4(\frac{\tau}{T})^2\pi^2k^2\right] \end{aligned}$$

(d)

(e)

(f) We have

$$\begin{aligned} a_k\cos(\frac{k2\pi t}{T}) + b_k\sin(\frac{k2\pi t}{T}) &= \cos(\phi_k)\cos(\frac{k2\pi t}{T}) + \sin(\phi_k)\sin(\frac{k2\pi t}{T}) \\ &= \cos(\frac{k2\pi t}{T} - \phi_k) \end{aligned}$$

where

$$\tan(\phi_k) = \frac{\sin(\phi_k)}{\cos(\phi_k)} = \frac{b_k}{a_k} = 4C_0(\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2} \pi k (2C_0 \frac{\tau}{T} \frac{1}{1+(\frac{2k\pi\tau}{T})^2})^{-1}$$

$$= 2\frac{\tau}{T}\pi k$$

$$\phi_k = \arctan(2\frac{\tau}{T}\pi k)$$

For $\frac{\tau}{T} = .1$, $\phi_1 \approx 32.14°$ and $\phi_2 \approx 51.48°$ and for $\frac{\tau}{T} = .01$, $\phi_1 \approx 3.59°$ and $\phi_2 \approx 7.16°$