Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Take Home Project 3
Johns Hopkins University
Student: Yves Greatti

Question 1

The Wormersley equation for blood flow is:

$$\rho \frac{\partial w}{\partial t} = \frac{\mu}{r} \frac{\partial}{\partial r} (r \frac{\partial w}{\partial r}) + \frac{\partial P}{\partial z}$$

Using $\frac{\partial P}{\partial z} = Ae^{int}$ and taking $w(r,t) = u(r)e^{int}$ yields: $\frac{\partial w}{\partial t} = (in)ue^{int}$, $\frac{\partial w}{\partial r} = u'(r)e^{int}$, and $\frac{\partial^2 w}{\partial r^2} = u''(r)e^{int}$, $\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r}) = u'(r)e^{int} + ru''(r)e^{int}$ Therefore the Wormersley equation becomes:

$$\frac{\mu}{r}\left[u'(r)e^{int} + ru''(r)e^{int}\right] + Ae^{int} = \rho(i\,n)u(r)e^{int}$$

$$\mu \frac{d^2u(r)}{dr^2} + \frac{\mu}{r}\frac{du(r)}{dr} + A = (i\,n)\,\rho\,u(r) \text{ by dividing through } e^{int}$$

$$\frac{d^2u(r)}{dr^2} + \frac{1}{r}\frac{du(r)}{dr} - \frac{i\,n\,\rho}{\mu}u = -\frac{A}{\mu} \text{ by dividing through } \mu \text{ and rearranging}$$

Finally using $\nu = \frac{\mu}{\rho}$ we have:

$$\frac{d^2u(r)}{dr^2} + \frac{1}{r}\frac{du(r)}{dr} - \frac{i\,n}{\nu}u = -\frac{A}{\mu}$$

By simple inspection, one particular solution is a constant w.r.t. r, such as $u_p = C$, substituting it into the differential equation gives:

$$-\frac{i\,n\,\rho}{\mu}u_p = -\frac{A}{\mu}$$

thus $u_p = \frac{A}{in\rho}$ The homogeneous equation is:

$$\frac{d^2u(r)}{dr^2} + \frac{1}{r}\frac{du(r)}{dr} + \frac{i^3\,n}{\nu}u = 0$$

Take $\lambda^2 = \frac{i^3\ n}{\nu}$, we now have:

$$\frac{d^2u(r)}{dr^2} + \frac{1}{r}\frac{du(r)}{dr} + \lambda^2 u = 0$$
$$r^2\frac{d^2u(r)}{dr^2} + r\frac{du(r)}{dr} + (\lambda r)^2 u = 0 \quad (1)$$

Take $x = \lambda r$, then:

$$\frac{du(x)}{dr} = \frac{du(\lambda r)}{dr} = \lambda\frac{du(x)}{dx}$$
$$\frac{d^2u(x)}{dr^2} = \lambda^2\frac{d^2u(x)}{dx^2}$$

Substitute back into (1), we have

$$\lambda^2 r^2\frac{d^2u(x)}{dx^2} + \lambda r\frac{du(x)}{dx} + (\lambda r)^2 u(x) = 0$$
$$x^2\frac{d^2u(x)}{dx^2} + x\frac{du(x)}{dx} + x^2 u = 0$$

The last equation is a Bessel's equation of order 0, therefore the solution, u_h, of the homogeneous equation is a solution of a Bessel's equation of order 0:

$$u_h(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$$

And

$$u(r) = u_h(r) + u_p(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r) + \frac{A}{in\rho}$$

Now we apply the boundary conditions to our solution.

$$u'(r) = C_1 J_0'(\lambda r) + C_2 Y_0'(\lambda r)$$

We have

$$\begin{aligned}
J_0(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n} n!\Gamma(1+n)} \\
&= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \cdots \\
J_0'(x) &= -2\frac{x}{2^2} + 4\frac{x^3}{2^2 4^2} - \cdots \\
J_0'(0) &= 0 \\
\lim_{r\to 0} u'(r) &= \lim_{r\to 0} C_1 J_0'(\lambda r) + C_2 Y_0'(\lambda r) \\
&= 0 + \lim_{r\to 0} C_2 Y_0'(\lambda r)
\end{aligned}$$

Looking at the plot of $Y_0(x)$, we see that in order to have $\frac{\partial w}{\partial r}|_{r=0} = 0$ or $\frac{\partial u}{\partial r}|_{r=0} = 0$, the term in Y_0 must be discarded and we need $C_2 = 0$. Thus

$$u(r) = C_1 J_0(\lambda r) + \frac{A}{i\;n\;\rho}$$

Using the second boundary condition $w(R) = u(R) = 0$ we have $C_1 J_0(\lambda R) + \frac{A}{i\;n\;\rho} = 0$ or $C_1 = -\frac{A}{i\;n\;\rho J_0(\lambda R)}$ Putting everything back

$$\begin{aligned} u(r) &= \frac{A}{\rho\;i\;n}\left[1 - \frac{J_0(\lambda r)}{J_0(\lambda R)}\right] \\ &= \frac{A}{\rho\;i\;n}\left[1 - \frac{J_0(r\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}})}{J_0(R\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}})}\right] \end{aligned}$$

Take $\alpha = R\sqrt{\frac{\lambda}{\nu}}$ and $y = \frac{r}{R}$ then

$$\begin{aligned} &J_0(r\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}}) = J_0(\frac{r}{R}R\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}}) = J_0(\alpha y i^{\frac{3}{2}}) \\ &J_0(R\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}}) = J_0(\alpha i^{\frac{3}{2}}) \end{aligned}$$

Lastly

$$w(y,t) = u(r)e^{int} = \frac{A}{\rho\;i\;n}\left[1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right]e^{int}$$

Question 2

From

$$Q = 2\pi \int_0^R w(r,t) r dr$$

Make the change of variable $y = \frac{r}{R}, dy = \frac{dr}{R}$ and we have

$$Q = 2\pi \int_0^1 w(y,t) R^2 y\;dy = 2\pi R^2 \int_0^1 w\;y\;dy$$

Plugging the expression of w found in the previous question

$$
\begin{aligned}
Q &= 2\pi R^2 \frac{A}{\rho\, i\, n} \int_0^1 \left[1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right] e^{int}\, y\, dy \\
&= \frac{2\pi R^2 A}{\rho\, i\, n} e^{int} \left[\int_0^1 y\, dy - \frac{1}{J_0(\alpha i^{\frac{3}{2}})} \int_0^1 y J_0(\alpha y i^{\frac{3}{2}})\, dy\right]
\end{aligned}
$$

$\int_0^1 y\, dy = [\frac{y^2}{2}]_0^1 = \frac{1}{2}$ and we make the change of variable $s = \alpha i^{\frac{3}{2}} y, ds = \alpha i^{\frac{3}{2}} dy$ so

$$
\begin{aligned}
\int_0^1 y J_0(\alpha y i^{\frac{3}{2}})\, dy &= \int_0^{\alpha i^{\frac{3}{2}}} \frac{s}{\alpha i^{\frac{3}{2}}} J_0(s)\, \frac{1}{\alpha i^{\frac{3}{2}}}\, ds \\
&= \frac{1}{\alpha^2 i^3} \int_0^{\alpha i^{\frac{3}{2}}} s J_0(s)\, ds \\
&= \frac{\alpha i^{\frac{3}{2}}}{\alpha^2 i^3} J_1(\alpha i^{\frac{3}{2}})
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Q &= \frac{2\pi R^2 A}{\rho\, i\, n} e^{int} \left[\frac{1}{2} - \frac{\alpha i^{\frac{3}{2}}}{\alpha^2 i^3} \frac{J_1(\alpha i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right] \\
&= \frac{\pi R^2}{\rho} \frac{A}{i\, n} \left[1 - \frac{2\alpha i^{\frac{3}{2}}}{i^3 \alpha^2} \frac{J_1(\alpha i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right] e^{int}
\end{aligned}
$$

Question 3

Start with the equation for $w(y,t)$ established in question 1:

$$
w(y,t) = \frac{A}{\rho\, i\, n} \left[1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right] e^{int}
$$

Substituting into the previous equation n with $\alpha = R\sqrt{\frac{n}{\nu}}, n = \nu \frac{\alpha}{R}^2$

$$
w(y,t) = \frac{A\, R^2}{i\rho\nu} \left[\frac{J_0(\alpha i^{\frac{3}{2}}) - J_0(\alpha y i^{\frac{3}{2}})}{\alpha^2 J_0(\alpha i^{\frac{3}{2}})}\right] e^{i \frac{\nu\, t}{R^2} \alpha^2}
$$

Let $B = \alpha y i^{\frac{3}{2}}, C = \alpha i^{\frac{3}{2}}$ and $D = i\frac{\nu\, t}{R^2}\alpha^2$, rewrite the previous equation

$$
w(y,t) = \frac{A\, R^2}{i\rho\nu} \left[\frac{J_0(C) - J_0(B)}{\alpha^2 J_0(C)}\right] e^{D}
$$

When $n \to 0, \alpha \to 0$ and we have the indeterminate form for $w(y,t) = \frac{A\ R^2}{i\rho\nu}(\frac{1-1}{0\cdot 1})\cdot 1 = \frac{0}{0}$. Therefore we apply L'Hospital's rule, compute the derivatives of numerator and denominator and taking the limit $\alpha \to 0$:

$$\begin{aligned}
\frac{d}{d\alpha}(J_0(C) - J_0(B))e^D &= \frac{d}{d\alpha}(J_0(C) - J_0(B))\,e^D + (J_0(C) - J_0(B))\frac{d}{d\alpha}e^D \\
\frac{d}{d\alpha}(J_0(C) - J_0(B)) &= -i^{\frac{3}{2}}J_1(C) + i^{\frac{3}{2}}yJ_1(B) \\
&= i^{\frac{3}{2}}(yJ_1(B) - J_1(C))e^D \\
\frac{d}{d\alpha}e^D &= \frac{2i\nu t}{R^2}\alpha e^D
\end{aligned}$$

So

$$\begin{aligned}
\frac{d}{d\alpha}(J_0(C) - J_0(B))e^D &= \left(i^{\frac{3}{2}}(yJ_1(B) - J_1(C)) + (J_0(C) - J_0(B))\frac{2i\nu t}{R^2}\alpha\right)e^D \\
\frac{d}{d\alpha}\alpha^2 J_0(C) &= 2\alpha J_0(C) + \alpha^2 i^{\frac{3}{2}}(-\,J_1(C)) \\
&= \alpha(2J_0(C) - i^{\frac{3}{2}}\alpha J_1(C))
\end{aligned}$$

And

$$\begin{aligned}
\lim_{\alpha\to 0}\frac{d}{d\alpha}(J_0(C) - J_0(B))e^D &= \left(i^{\frac{3}{2}}(y\cdot 0 - 0) + (0-0)\frac{2i\nu t}{R^2}\cdot 0\right)1 = 0 \\
\lim_{\alpha\to 0}\frac{d}{d\alpha}\alpha^2 J_0(C) &= 0\cdot(2\cdot 1 - i^{\frac{3}{2}}\cdot 0\cdot 0) = 0
\end{aligned}$$

We still have the indeterminate form $\frac{0}{0}$, so we apply one more time L'Hospital's rule

$$\begin{aligned}
\frac{d^2}{d\alpha^2}(J_0(C)-J_0(B))e^D &= \left(i^{\frac{3}{2}}(y\frac{d}{d\alpha}J_1(B)-\frac{d}{d\alpha}J_1(C))+(\frac{d}{d\alpha}J_0(C)-\frac{d}{d\alpha}J_0(B))\frac{2i\nu t}{R^2}\alpha\right.\\
&\left.+(J_0(C)-J_0(B))\frac{2i\nu t}{R^2}\right)e^D\\
&\left(i^{\frac{3}{2}}(yJ_1(B)-J_1(C))+(J_0(C)-J_0(B))\frac{2i\nu t}{R^2}\alpha\right)\frac{2i\nu t}{R^2}\alpha e^D\\
&=\left(i^{\frac{3}{2}}(i^{\frac{3}{2}}y^2\frac{J_0(B)-J_2(B)}{2}-i^{\frac{3}{2}}\frac{J_0(C)-J_2(C)}{2})+\right.\\
&\left.(i^{\frac{3}{2}}yJ_1(B)-i^{\frac{3}{2}}J_1(C))\frac{2i\nu t}{R^2}\alpha+(J_0(C)-J_0(B))\frac{2i\nu t}{R^2}\right)e^D+\\
&\left(i^{\frac{3}{2}}(yJ_1(B)-J_1(C))+(J_0(C)-J_0(B))\frac{2i\nu t}{R^2}\alpha\right)\frac{2i\nu t}{R^2}\alpha e^D\\
&=\left(\frac{i^3y^2}{2}(J_0(B)-J_2(B))-\frac{i^3}{2}(J_0(C)-J_2(C))+\right.\\
&\left.(i^{\frac{3}{2}}yJ_1(B)-i^{\frac{3}{2}}J_1(C))\frac{2i\nu t}{R^2}\alpha+(J_0(C)-J_0(B))\frac{2i\nu t}{R^2}\right)e^D+\\
&\left(i^{\frac{3}{2}}(yJ_1(B)-J_1(C))+(J_0(C)-J_0(B))\frac{2i\nu t}{R^2}\alpha\right)\frac{2i\nu t}{R^2}\alpha e^D\\
\frac{d^2}{d\alpha^2}\alpha^2J_0(C) &= 2J_0(C)-i^{\frac{3}{2}}\alpha J_1(C)+\alpha(2\frac{d}{d\alpha}J_0(C)\\
&-i^{\frac{3}{2}}J_1(C)-i^{\frac{3}{2}}\alpha\frac{d}{d\alpha}J_1(C))\\
&=2J_0(C)-i^{\frac{3}{2}}\alpha J_1(C)+\alpha(2i^{\frac{3}{2}}(-J_1(C))\\
&-i^{\frac{3}{2}}J_1(C)-i^3\alpha\frac{J_0(C)-J_2(C)}{2})\\
&=2J_0(C)-i^{\frac{3}{2}}J_1(C)-3i^{\frac{3}{2}}J_1(C)\alpha-\frac{i^3}{2}(J_0(C)-J_2(C))\alpha
\end{aligned}$$

Next we are going to take the second derivative of the numerator of our initial expression of $w(y,t)$ when $\alpha \to 0$ using the expression above:

$$\begin{aligned}
\lim_{\alpha\to 0}\frac{d^2}{d\alpha^2}(J_0(C)-J_0(B))e^D = \Bigg(&\frac{i^3y^2}{2}(1-0)-\frac{i^3}{2}(1-0)+\\
&(i^{\frac{3}{2}}y\cdot 0 - i^{\frac{3}{2}}\cdot 0)\frac{2i\nu t}{R^2}\cdot 0+(1-1)\frac{2i\nu t}{R^2}\Bigg)\cdot 1+\\
&\left(i^{\frac{3}{2}}(y\cdot 0-0)+(1-1)\frac{2i\nu t}{R^2}\cdot 0\right)\frac{2i\nu t}{R^2}\cdot 0\cdot 1\\
&=\frac{i^3}{2}(y^2-1)=\frac{i}{2}(1-y^2)
\end{aligned}$$

Similarly for the limit of $\alpha \to 0$ of the second derivative of the denominator of $w(y,t)$

$$\begin{aligned}
\lim_{\alpha\to 0}\frac{d^2}{d\alpha^2}\alpha^2 J_0(C) &= 2\cdot 1 - i^{\frac{3}{2}}\cdot 0 - 3i^{\frac{3}{2}}\cdot 0\cdot 0 - \frac{i^3}{2}(1-0)\cdot 0\\
&= 2
\end{aligned}$$

Therefore for a constant input pressure, we have

$$w = \frac{A\ R^2}{i\rho\nu}\frac{i}{4}(1-y^2) = \frac{A}{4\mu}R^2(1-y^2)$$

which the equation 2 in Wormersley's paper with $A = \frac{p_1-p_2}{l}$

Using the expression of the differential equation established in question (1) and with $n=0$, we want to solve

$$r^2\frac{d^2u(r)}{dr^2}+r\frac{du(r)}{dr} = -\frac{A}{\mu}r^2$$

This is an Euler equation or Legendre ordinary differential equation $\alpha = 1, \beta = 0$, so we make the change of variable $e^t = r$ or $\ln r = t$. Then $r\frac{du}{dr} = \frac{du}{dt}$ and $r^2\frac{d^2y}{dr^2} = \frac{d^2u}{dt^2} - \frac{du}{dt}$.

which yields for the ODE

$$\begin{aligned}
\frac{d^2u}{dt^2}-\frac{du}{dt}+\frac{du}{dt} &= -\frac{A}{\mu}e^{2t}\\
\frac{d^2u}{dt^2} &= -\frac{A}{\mu}e^{2t}
\end{aligned}$$

Considering the homogeneous equation and integrating twice gives $u(t) = C_1 t + C_2$ or $u(r) = C_1 \ln(r) + C_2$. Take for one particular solution of the ODE: $u_p(t) = C_3 e^{2t}, u_p'(t) = 2C_3 e^{2t}, u_p''(t) = 4C_3 e^{2t}$, substitute in the ODE gives $4C_3 e^{2t} = -\frac{A}{\mu} e^{2t}$ or $C_3 = -\frac{A}{4\mu}$ thus $u_p(t) = -\frac{A}{4\mu} e^{2t}$ or $u_p(r) = -\frac{A}{4\mu} r^2$. The total solution is

$$u(r) = -\frac{A}{4\mu} r^2 + C_1 \ln(r) + C_2$$

From this, we write $u'(r) = -\frac{A}{2\mu} r + \frac{C_1}{r}$. So to have the boundary condition $\frac{\partial w}{\partial r}|_{r=0} = 0$ or $\frac{\partial u}{\partial r}|_{r=0} = 0$, C_1 has to be zero. The second boundary condition $w(R) = 0$, or $u(R) = 0$, gives $C_2 = \frac{A}{4\mu} R^2$. Finally

$$u(r) = -\frac{A}{4\mu}(r^2 - R^2) = \frac{A}{4\mu} R^2 (1 - (\frac{r}{R})^2) = \frac{A}{4\mu} R^2 (1 - y^2)$$

which is equation (2) in Wormersley's paper with $A = \frac{p_1 - p_2}{l}$

Question 4

For Poiseuille's flow

$$w = \frac{p_1 - p_2}{4\mu l} R^2 (1 - y^2)$$

And

$$Q = 2\pi \int_0^R w(r,t) r dr$$

Make the change of variable $y = \frac{r}{R}, dy = \frac{dr}{R}$ and we have

$$\begin{aligned} Q &= 2\pi \int_0^1 \frac{p_1 - p_2}{4\mu l} R^2 (1 - y^2) R\ y\ R\ dy \\ &= 2\pi \frac{p_1 - p_2}{4\mu l} R^4 \int_0^1 (1 - y^2)\ y\ dy \\ &= 2\pi \frac{p_1 - p_2}{4\mu l} R^4 [\frac{y^2}{2} - \frac{y^4}{4}]_0^1 \\ &= 2\pi \frac{p_1 - p_2}{4\mu l} R^4 \frac{1}{4} \\ &= \frac{p_1 - p_2}{8\mu l} \pi R^4 \end{aligned}$$