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Variation of parameters is a way of solving for the particular solution to an inhomogeneous differential equation give the solution to the homogenous differential equation (also denoted the complementary solution)

Variation of parameters by example

First we need to get the complementary solution to the homogeneous equation

Let's learn how to apply the variation of parameter technique to differential equations by solving a sample problem and discussing the technique while we apply it.

$$\frac{d^2y}{dx^2}+y=f(x)=\csc x,\ y(0)=y(\pi/2)=0$$

We start by presenting the complementary solution, that is the the solution when f(x)=0.
Taking $y(x)=Ae^{\lambda x}$ is easily seen to be $\lambda=\pm i,\ y(x)=A_1e^{ix}+A_2e^{-ix}$
This can easily be reformulated using Euler's identity, that is

$$y_c(x)=c_1\sin x+c_2\cos x$$

Next we construct the particular solution for the inhomogeneous equation

The variation of parameter method involves replacing this functions with constant coefficients with variable coefficients to solve the inhomogeneous differential equations, in this case when

$$y_p(x)=k_1(x)\boxed{\sin x}+k_2(x)\cos x=k_1(x)\boxed{y_1(x)}+k_2(x)y_2(x)$$

The key then is evaluating for the two unknown functions. This is accomplished by requiring this solution to satisfy some restricting conditions when it is differentiated.

$$y_p'(x)=[k_1'(x)\sin x+k_1(x)\cos x]+[k_2'(x)\cos x+k_2(x)(-\sin x)]$$
$$=[k_1(x)\cos x+k_2(x)(-\sin x)]+[k_1'(x)\sin x+k_2'(x)\cos x]$$

At this point we have four terms that include the variable functions however **we can make a simplifying assumption** by letting

$$k_1'(x)\sin x+k_2'(x)\cos x\equiv\boxed{k_1'(x)y_1(x)+k_2'(x)y_2(x)=0}$$

First equation
For unknowns

Leaving us with $\boxed{y_p'(x)}=k_1(x)\cos x+k_2(x)(-\sin x)\equiv\boxed{k_1(x)y_1'(x)+k_2(x)y_2'(x)}$

Next take another derivative of the particular solution

$$y_p''(x) = [k_1'(x)y_1'(x) + k_1(x)y_1''(x)] + [k_2'(x)y_2'(x) + k_2(x)y_2''(x)]$$
$$= [k_1'(x)y_1'(x) + k_2'(x)y_2'(x)] + [+k_1(x)y_1''(x) + k_2(x)y_2''(x)]$$

KEY: At this point we are dealing with the same order of the derivative y_p of the solution as that for the original differential equation, so we need to combine this information. Therefore substitute this expression into this original equation $y_p''(x) + y_p(x) = \csc x$, that is

$$[k_1'(x)y_1'(x) + k_2'(x)y_2'(x)] + [k_1(x)y_1''(x) + k_2(x)y_2''(x)] + [k_1(x)y_1(x) + k_2(x)y_2(x)] = \csc x$$

Rearranging this expression using the homogeneous form of the original differential equation $\boxed{y_i''(x) + y_i(x) = 0,\ i = 1,2}$ gives us the second equation for the unknowns.

Second equation
For unknowns

$$k_1(x)\boxed{[y_1''(x) + y_1(x)]} + k_2(x)\boxed{[y_2''(x) + y_2(x)]} + [k_1'(x)y_1'(x) + k_2'(x)y_2'(x)] =$$
$$k_1(x)[0] + k_2(x)[0] + [k_1'(x)y_1'(x) + k_2'(x)y_2'(x)] = \boxed{k_1'(x)y_1'(x) + k_2'(x)y_2'(x) = \csc x}$$

Aside:
For higher order differential equations extension of the methods proceeds analogous to that presented and for each derivative (of order less than that of the differential equation) of the solution y_p similar choices for simplification need to be made as for the first derivative in this case.
Back:

So for the case of an order two differential equation with two unknown functions we get two equations to solve

$$k_1'(x)y_1(x)+k_2'(x)y_2(x) \;=k_1'(x)\sin x+k_2'(x)\cos x=0$$

$$k_1'(x)y_1'(x)+k_2'(x)y_2'(x)=k_1'(x)\cos x+k_2'(x)(-\sin x)=\csc x$$

In matrix form $\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix}\begin{pmatrix} k_1'(x) \\ k_2'(x) \end{pmatrix}=\begin{pmatrix} 0 \\ \csc x \end{pmatrix}$

Using Gaussian elimination (or any other method of your choice) you can solve for the unknown functions, that is

$$\boxed{k_1'(x)=\frac{\cos x}{\sin x}=\cot x,\;\; k_2'(x)=-\sin x\csc x=-1}$$

Integrating gives $\boxed{k_1(x)=\ln(\sin x),\;\; k_2(x)=-x}$

Finally $y_p(x)=\ln(\sin x)\sin x-x\cos x$

Note the total solution is

$$y(x)=y_c(x)+y_p(x)=c_1\sin x+c_2\cos x+\ln(\sin x)\sin x-x\cos x$$

Applying the initial conditions

$$y(0)=c_1\sin 0+c_2\cos 0+\ln(\sin 0)\sin 0-0\cos 0=c_2=0$$

$$y(\pi/2)=c_1\sin\pi/2+c_2\cos\pi/2+\ln(\sin\pi/2)\sin\pi/2-\pi/2\cos\pi/2=c_1=0$$

Therefore $y(x)=y_p(x)=\ln(\sin x)\sin x-x\cos x$