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EN.585.615.81.SP21 Mathematical Methods
Take Home Project 3
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Question 1

The Wormersley equation for blood flow is:

$$\rho \frac{\partial w}{\partial t} = \frac{\mu}{r} \frac{\partial}{\partial r}(r \frac{\partial w}{\partial r}) + \frac{\partial P}{\partial z}$$

Using $\frac{\partial P}{\partial z} = A e^{int}$ and taking $w(r,t) = u(r) e^{int}$ yields: $\frac{\partial w}{\partial t} = (in) u e^{int}$, $\frac{\partial w}{\partial r} = u'(r) e^{int}$, and $\frac{\partial^2 w}{\partial r^2} = u''(r) e^{int}$, $\frac{\partial}{\partial r}(r \frac{\partial w}{\partial r}) = u'(r) e^{int} + r u''(r) e^{int}$ Therefore the Wormersley equation becomes:

$$\frac{\mu}{r} \left[u'(r) e^{int} + r u''(r) e^{int} \right] + A e^{int} = \rho (i\ n) u(r) e^{int}$$

$$\mu \frac{d^2 u(r)}{dr^2} + \frac{\mu}{r} \frac{du(r)}{dr} + A = (i\ n)\ \rho\ u(r) \text{ by dividing through } e^{int}$$

$$\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} - \frac{i\ n\ \rho}{\mu} u = -\frac{A}{\mu} \text{ by dividing through } \mu \text{ and rearranging}$$

Finally using $\nu = \frac{\mu}{\rho}$ we have:

$$\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} - \frac{i\ n}{\nu} u = -\frac{A}{\mu}$$

By simple inspection, one particular solution is a constant w.r.t. r, such as $u_p = C$, substituting it into the differential equation gives:

$$-\frac{i\ n\ \rho}{\mu} u_p = -\frac{A}{\mu}$$

thus $u_p = \frac{A}{in\rho}$ The homogeneous equation is:

$$\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} + \frac{i^3\ n}{\nu} u = 0$$

Take $\lambda^2 = \frac{i^3\ n}{\nu}$, we now have:

$$\frac{d^2u(r)}{dr^2} + \frac{1}{r}\frac{du(r)}{dr} + \lambda^2 u = 0$$
$$r^2\frac{d^2u(r)}{dr^2} + r\frac{du(r)}{dr} + (\lambda r)^2 u = 0 \quad (1)$$

Take $x = \lambda r$, then:

$$\frac{du(x)}{dr} = \frac{du(\lambda r)}{dr} = \lambda\frac{du(x)}{dx}$$
$$\frac{d^2u(x)}{dr^2} = \lambda^2\frac{d^2u(x)}{dx^2}$$

Substitute back into (1), we have

$$\lambda^2 r^2\frac{d^2u(x)}{dx^2} + \lambda r\frac{du(x)}{dx} + (\lambda r)^2 u(x) = 0$$
$$x^2\frac{d^2u(x)}{dx^2} + x\frac{du(x)}{dx} + x^2 u = 0$$

The last equation is a Bessel's equation of order 0, therefore the solution, u_h, of the homogeneous equation is a solution of a Bessel's equation of order 0:

$$u_h(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$$

And

$$u(r) = u_h(r) + u_p(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r) + \frac{A}{in\rho}$$

Now we apply the boundary conditions to our solution.

$$u'(r) = C_1 J_0'(\lambda r) + C_2 Y_0'(\lambda r)$$

We have

$$\begin{aligned}
J_0(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n} n!\Gamma(1+n)} \\
&= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \cdots \\
J_0'(x) &= -2\frac{x}{2^2} + 4\frac{x^3}{2^2 4^2} - \cdots \\
J_0'(0) &= 0 \\
\lim_{r\to 0} u'(r) &= \lim_{r\to 0} C_1 J_0'(\lambda r) + C_2 Y_0'(\lambda r) \\
&= 0 + \lim_{r\to 0} C_2 Y_0'(\lambda r)
\end{aligned}$$

Looking at the plot of $Y_0(x)$, we see that in order to have $\frac{\partial w}{\partial r}|_{r=0} = 0$ or $\frac{\partial u}{\partial r}|_{r=0} = 0$, the term in Y_0 must be discarded and we need $C_2 = 0$. Thus

$$u(r) = C_1 J_0(\lambda r) + \frac{A}{i\ n\ \rho}$$

Using the second boundary condition $w(R) = u(R) = 0$ we have $C_1 J_0(\lambda R) + \frac{A}{i\ n\ \rho} = 0$ or $C_1 = -\frac{A}{i\ n\ \rho J_0(\lambda R)}$ Putting everything back

$$\begin{aligned} u(r) &= \frac{A}{\rho\ i\ n}\left[1 - \frac{J_0(\lambda r)}{J_0(\lambda R)}\right] \\ &= \frac{A}{\rho\ i\ n}\left[1 - \frac{J_0(r\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}})}{J_0(R\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}})}\right] \end{aligned}$$

Take $\alpha = R\sqrt{\frac{\lambda}{\nu}}$ and $y = \frac{r}{R}$ then

$$\begin{aligned} J_0(r\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}}) &= J_0(\frac{r}{R}R\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}}) = J_0(\alpha y i^{\frac{3}{2}}) \\ J_0(R\sqrt{\frac{\lambda}{\nu}}i^{\frac{3}{2}}) &= J_0(\alpha i^{\frac{3}{2}}) \end{aligned}$$

Lastly

$$w(y,t) = u(r)e^{int} = \frac{A}{\rho\ i\ n}\left[1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right]e^{int}$$

Question 2

From

$$Q = 2\pi \int_0^R w(r,t) r dr$$

Make the change of variable $y = \frac{r}{R}, dy = \frac{dr}{R}$ and we have

$$Q = 2\pi \int_0^1 w(y,t) R^2 y\ dy = 2\pi R^2 \int_0^1 w\ y\ dy$$

Plugging the expression of w found in the previous question

$$
\begin{aligned}
Q &= 2\pi R^2 \frac{A}{\rho\, i\, n} \int_0^1 \left[1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right] e^{int}\, y\, dy \\
&= \frac{2\pi R^2 A}{\rho\, i\, n} e^{int} \left[\int_0^1 y\, dy - \frac{1}{J_0(\alpha i^{\frac{3}{2}})} \int_0^1 y J_0(\alpha y i^{\frac{3}{2}})\, dy\right]
\end{aligned}
$$

$\int_0^1 y\, dy = [\frac{y^2}{2}]_0^1 = \frac{1}{2}$ and we make the change of variable $s = \alpha i^{\frac{3}{2}} y, ds = \alpha i^{\frac{3}{2}} dy$ so

$$
\begin{aligned}
\int_0^1 y J_0(\alpha y i^{\frac{3}{2}})\, dy &= \int_0^{\alpha i^{\frac{3}{2}}} \frac{s}{\alpha i^{\frac{3}{2}}} J_0(s)\, \frac{1}{\alpha i^{\frac{3}{2}}}\, ds \\
&= \frac{1}{\alpha^2 i^3} \int_0^{\alpha i^{\frac{3}{2}}} s J_0(s)\, ds \\
&= \frac{\alpha i^{\frac{3}{2}}}{\alpha^2 i^3} J_1(\alpha i^{\frac{3}{2}})
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Q &= \frac{2\pi R^2 A}{\rho\, i\, n} e^{int} \left[\frac{1}{2} - \frac{\alpha i^{\frac{3}{2}}}{\alpha^2 i^3} \frac{J_1(\alpha i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right] \\
&= \frac{\pi R^2}{\rho} \frac{A}{i\, n} \left[1 - \frac{2\alpha i^{\frac{3}{2}}}{i^3 \alpha^2} \frac{J_1(\alpha i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right] e^{int}
\end{aligned}
$$

Question 3

Using the expression of the differential equation established in question (1) and with $n = 0$, we want to solve

$$
r^2 \frac{d^2 u(r)}{dr^2} + r \frac{du(r)}{dr} = -\frac{A}{\mu} r^2
$$

This is an Euler equation or Legendre ordinary differential equation $\alpha = 1, \beta = 0$, so we make the change of variable $e^t = r$ or $\ln r = t$. Then $r\frac{du}{dr} = \frac{du}{dt}$ and $r^2 \frac{d^2 y}{dr^2} = \frac{d^2 u}{dt^2} - \frac{du}{dt}$.

which yields for the ODE

$$
\begin{aligned}
\frac{d^2 u}{dt^2} - \frac{du}{dt} + \frac{du}{dt} &= -\frac{A}{\mu} e^{2t} \\
\frac{d^2 u}{dt^2} &= -\frac{A}{\mu} e^{2t}
\end{aligned}
$$

Considering the homogeneous equation and integrating twice gives $u(t) = C_1 t + C_2$ or $u(r) = C_1 \ln(r) + C_2$. Take for one particular solution of the ODE: $u_p(t) = C_3 e^{2t}, u_p'(t) = 2C_3 e^{2t}, u_p''(t) = 4C_3 e^{2t}$, substitute in the ODE gives $4C_3 e^{2t} = -\frac{A}{\mu} e^{2t}$ or $C_3 = -\frac{A}{4\mu}$ thus $u_p(t) = -\frac{A}{4\mu} e^{2t}$ or $u_p(r) = -\frac{A}{4\mu} r^2$. The total solution is

$$u(r) = -\frac{A}{4\mu} r^2 + C_1 \ln(r) + C_2$$

From this, we write $u'(r) = -\frac{A}{2\mu} r + \frac{C_1}{r}$. So to have the boundary condition $\frac{\partial w}{\partial r}|_{r=0} = 0$ or $\frac{\partial u}{\partial r}|_{r=0} = 0$, C_1 has to be zero. The second boundary condition $w(R) = 0$, or $u(R) = 0$, gives $C_2 = \frac{A}{4\mu} R^2$. Finally

$$u(r) = -\frac{A}{4\mu}(r^2 - R^2) = \frac{A}{4\mu} R^2 (1 - (\frac{r}{R})^2) = \frac{A}{4\mu} R^2 (1 - y^2)$$

which is equation (2) in Wormersley's paper with $A = \frac{p_1 - p_2}{l}$

Question 4

For Poiseuille's flow

$$w = \frac{p_1 - p_2}{4\mu l} R^2 (1 - y^2)$$

And

$$Q = 2\pi \int_0^R w(r,t) r dr$$

Make the change of variable $y = \frac{r}{R}, dy = \frac{dr}{R}$ and we have

$$\begin{aligned}
Q &= 2\pi \int_0^1 \frac{p_1 - p_2}{4\mu l} R^2 (1 - y^2) R\, y\, R\, dy \\
&= 2\pi \frac{p_1 - p_2}{4\mu l} R^4 \int_0^1 (1 - y^2)\, y\, dy \\
&= 2\pi \frac{p_1 - p_2}{4\mu l} R^4 [\frac{y^2}{2} - \frac{y^4}{4}]_0^1 \\
&= 2\pi \frac{p_1 - p_2}{4\mu l} R^4 \frac{1}{4} \\
&= \frac{p_1 - p_2}{8\mu l} \pi R^4
\end{aligned}$$