1.5

(a)

$$\begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

Property: If 2 columns/rows same then determinate is 0

Ans: By inspection let x = a then col#1 = $a \cdot$col#4

Ans: By inspection let x = b then row#2 = col#3

Ans: By inspection let x = c then row#3 = col#4

Since x appears in 3 different rows order of equation associated with determinate expansion is third order and has at most 3 roots.

(b)

$$\begin{vmatrix} x+2 & x+4 & x-3 \\ x+3 & x & x+5 \\ x-2 & x-1 & x+1 \end{vmatrix} = 0$$

Property: Subtract row 1 from row 2, row 1 from row 3 does not change value of determinate!!

This gives

$$\begin{vmatrix} x+2 & x+4 & x-3 \\ 1 & -4 & 8 \\ -4 & -5 & 4 \end{vmatrix} = 0$$ Next expand LHS determinate by row 1 gives

$(x+2)[-16+40]-(x+4)(4+32)+(x-3)(-5-16)=0 \rightarrow x=-1$

1.16

Equations in matrix form

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Use Gauss elimination on augmented matrix

subtract -3·row 1 from row2, subtract -1·row 1 from row3

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 4 & 5 & 2 \\ 1 & 3 & 4 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -2 & -4 & -1 \\ 0 & 1 & 4 & 3 \end{array}\right]$$

divide row 2 by -2 then subtract row 2 from row 3

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & \tfrac{1}{2} \\ 0 & 0 & -1 & \tfrac{3}{2} \end{array}\right]$$

Therefore $-1x_3 = \tfrac{3}{2}$, $x_2 + 2x_3 = \tfrac{1}{2}$, $1x_1 + 2x_2 + 3x_3 = 1$

Solving gives $x_3 = -\tfrac{3}{2}$, $x_2 = 1$, $x_1 = -\tfrac{3}{2}$

There are other ways of solving this. E.g.

Let

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix},\ x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Then $Ax = b$ and $x=A^{-1}b$

Find the inverse by hand.

Note all these methods can be checked using **MatLab** to solve!

1.22

Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & -2 \\ -1 & -2 & 2 \end{bmatrix}$ Solve for eigenvalues, λ by chacteristic equation $|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 1 & 3 & -1 \\ 3 & 4 & -2 \\ -1 & -2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

Matrix algebra and expansion of determinate,

then algebra to factor resulting equation gives

$(1-\lambda)(\lambda^2 - 6\lambda - 6) = 0 \to \lambda_1 = 1$ and $\lambda_{2,3} = 3 \pm \sqrt{15}$

For each eigenvalue λ_i we have eigenvector, v_i with defining equation

$$(A - \lambda_i I)v_i = 0 \text{ where } v_i = \begin{pmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{pmatrix}$$

For $\lambda_1 = 1$

$$\begin{pmatrix} 1-1 & 3 & -1 \\ 3 & 4-1 & -2 \\ -1 & -2 & 2-1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} = 0$$

Solving (e.g. by Gauss elimination again gives)

$$v_{31} = c \text{ (arbitary)},\ v_{21} = \frac{1}{3}c,\ v_{11} = \frac{1}{3}c \text{ or } \begin{pmatrix} \tfrac{1}{3}c \\ \tfrac{1}{3}c \\ c \end{pmatrix} \equiv c \begin{pmatrix} \tfrac{1}{3} \\ \tfrac{1}{3} \\ 1 \end{pmatrix}, \text{ For c = 3, } \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\text{For } \lambda_2 = 3 + \sqrt{15} \to v_2 = \begin{pmatrix} 5+\sqrt{15} \\ 7+2\sqrt{15} \\ -4-\sqrt{15} \end{pmatrix},\ \lambda_3 = 3 - \sqrt{15} \to v_2 = \begin{pmatrix} 5-\sqrt{15} \\ 7-2\sqrt{15} \\ -4+\sqrt{15} \end{pmatrix}$$

To verify orthogonal evaluate $v_i^T v_j = 0$ for $i \neq j$, i,j = 1,2,3 - e.g. I checked i = 1, j = 2 and $v_1^T v_2 = 0$

1.36

$Q = -x_1^2 - 2x_2^2 - x_3^2 + 8x_2x_3 + 6x_1x_3 + 8x_1x_2$

Let $Q = x^T Ax$ where $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$, where A is symmetrix!!!, $x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Expand above $Q = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$

Equating with first expression gives eg. $a_{11} = -1,\ 2a_{12} = 8 \rightarrow a_{12} = 4,$ etc.

Therefore $A = \begin{pmatrix} -1 & 4 & 3 \\ 4 & -2 & 4 \\ 3 & 4 & -1 \end{pmatrix}$ (notice symmetry about diagonal)

Now decompose such that $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = s^T As \rightarrow A = s\Lambda s^T$, λ eigenvalues,

s made from eigenvectors.

Therefore $Q = x^T Ax = x^T s\Lambda s^T x = (x^T s)\Lambda(s^T x) = (x')^T \Lambda x' = \lambda_1(x'_1)^2 + \lambda_2(x'_2)^2 + \lambda_3(x'_3)^2$

Find eigenvalues using determinate $|A - \lambda I| = 0$

$$\left|\begin{pmatrix} -1 & 4 & 3 \\ 4 & -2 & 4 \\ 3 & 4 & -1 \end{pmatrix} - \lambda I\right| = 0 \rightarrow \lambda^3 + 4\lambda^2 - 36\lambda + 144 = 0 \rightarrow \lambda = -4, -6, 6$$

Solve for eigenvectors $(A - \lambda I)s = 0$.Take $\lambda_2 = -4$ (labeled to match matlab code result for eig(A), otherwise do it in any order as long as matches eigenvalue order in your own matrix)

$$\begin{pmatrix} -1--4 & 4 & 3 \\ 4 & -2--4 & 4 \\ 3 & 4 & -1-4 \end{pmatrix}\begin{pmatrix} s_{12} \\ s_{22} \\ s_{32} \end{pmatrix} = 0 \rightarrow s_2 = \begin{pmatrix} s_{12} \\ s_{22} \\ s_{32} \end{pmatrix} = \begin{pmatrix} -c \\ 0 \\ c \end{pmatrix}$$ where c is a constant

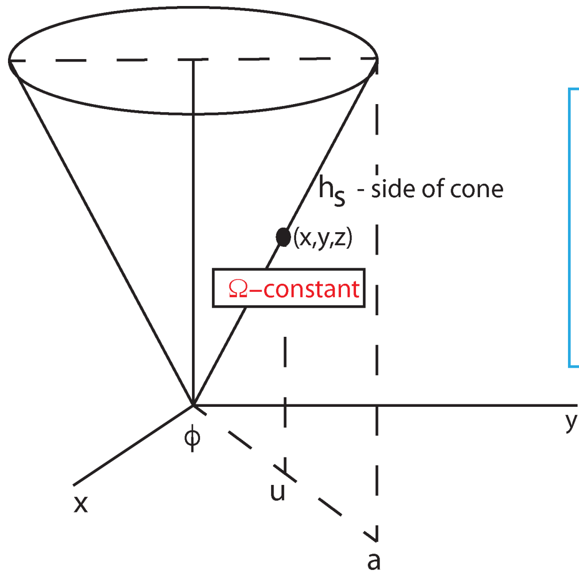
Take c = 1 and normalize $s_2 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ Then $x_2' = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -1/\sqrt{2}\,x_1 + 0 + 1/\sqrt{2}\,x_3$

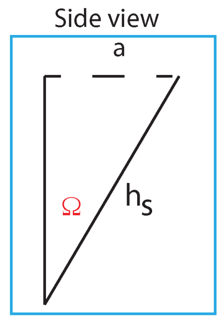
etc. for x_1' and x_3'

Matlab code to generate answer 1.36 and check with specific vector X

```
% Take X
X = [1 2 3].' % = [x1 x2 x3].'
XT = X.' % Transpose
% Original matrix A
A=[-1 4 3; 4 -2 4; 3 4 -1]
% Q using Original A and X
Q=XT*A*X
% Now decompose A into diagonal matrix D
[S,D]=eig(A)
ST = S.'
%Check
B=S*D*ST
% Calculate new coord X' = XP = S^-1*X = ST*X as in book
XP=ST*X
XPT = XP.' % Transpose
% Q using D and XP= X' = SX
Q=XPT*D*XP
```

2.6





In cylindrical coordinates

Ω - constant angle of cone, fixed

$x = u\cos\phi$

$y = u\sin\phi$

$z = u\cot\Omega$

Point on cone $= r = (x,y,z) = xi + yj + zk$, where r is vector and i,j,k are unit vectors in x, y, and z direction

$r = (u\cos\phi, u\sin\phi, u\cot\Omega)$

Note cone is mapped out as function of two variables u and ϕ

A surface element $dS = |n| du d\phi$ where n is normal to surface

Normal vector to surface can be calculated by $n = \dfrac{\partial r}{\partial u} \times \dfrac{\partial r}{\partial \phi}$

where $\times$ is cross product

$\dfrac{\partial r}{\partial u} = \cos\phi i + \sin\phi j + \cot\Omega k$, $\dfrac{\partial r}{\partial \phi} = -u\sin\phi i + u\cos\phi j + 0k$

$$n = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial u} = \begin{vmatrix} i & j & k \\ \cos\phi & \sin\phi & \cot\Omega \\ -u\sin\phi & u\cos\phi & 0 \end{vmatrix}$$, n a vector!!

$n = i(-u\cos\phi\cot\Omega) - j(u\sin\phi\cot\Omega) + k(u\cos^2\phi + u\sin^2\phi)$

$$|n| = \left[\left(-u\cos\phi\cot\Omega\right)^2 + \left(-u\sin\phi\cot\Omega\right)^2 + u^2\right]^{1/2} =$$

$$\left[u^2\cos^2\phi\cot^2\Omega + u^2\sin^2\phi\cot^2\Omega + u^2\right]^{1/2} = \left[u^2\cot^2\Omega(\cos^2\phi + \sin^2\phi) + u^2\right]^{1/2} =$$

$$\left[u^2\cot^2\Omega + u^2\right]^{1/2} = u\left(\cot^2\Omega + 1\right)^{1/2}$$

Therefore dS= $|n|\,du\,d\phi =$ dS= $u\left(\cot^2\Omega + 1\right)^{1/2} du\,d\phi$

Integration over surface of surface element dS where

u goes from 0 to a and ϕ goes from 0 to 2π, remember Ω is constant

$$S = \int_0^{2\pi}\int_0^{a} dS = \int_0^{2\pi}\int_0^{a} u\left(\cot^2\Omega + 1\right)^{1/2} du\,d\phi = \left(\cot^2\Omega + 1\right)^{1/2}\int_0^{2\pi}\int_0^{a} u\,du\,d\phi =$$

$$\left(\cot^2\Omega + 1\right)^{1/2}\int_0^{2\pi} d\phi \int_0^{a} u\,du = \left(\cot^2\Omega + 1\right)^{1/2} 2\pi\frac{a^2}{2} = \frac{1}{2}2\pi a\left[a\left(\cot^2\Omega + 1\right)^{1/2}\right]$$

Note $\cot^2\Omega + 1 = \csc^2\Omega$ or $\left(\cot^2\Omega + 1\right)^{1/2} = \csc\Omega$ and $\csc\Omega = \frac{h_s}{a}$ or $h_s = a\csc\Omega$

Finally $S = \frac{1}{2}2\pi a\left[a\csc\Omega\right] = \frac{1}{2}2\pi a h_s = \frac{1}{2}p h_s$ where p is perimeter of cone

2.11 Yes this takes a lot of algebra!!

$$\Psi(x,y,z)=\frac{zx^2}{x^2+y^2+z^2}$$

$$\nabla^2\Psi(x,y,z)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Psi(x,y,z)$$

Taking first $\frac{\partial^2}{\partial x^2}\Psi(x,y,z)=\frac{\partial^2}{\partial x^2}\frac{zx^2}{x^2+y^2+z^2}=$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\frac{zx^2}{x^2+y^2+z^2}\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(zx^2)(x^2+y^2+z^2)^{-1}\right)=$$

$$\frac{\partial}{\partial x}\left[(z2x)(x^2+y^2+z^2)^{-1}+(zx^2)(-1)(x^2+y^2+z^2)^{-2}(2x)\right]=$$

$$\frac{\partial}{\partial x}\left[\frac{2zx}{x^2+y^2+z^2}+\frac{-(zx^2)2x}{(x^2+y^2+z^2)^2}\right]=\frac{\partial}{\partial x}\left[\frac{2zx(x^2+y^2+z^2)-(zx^2)2x}{(x^2+y^2+z^2)^2}\right]=$$

$$\frac{\partial}{\partial x}\left[\frac{2zx(y^2+z^2)}{(x^2+y^2+z^2)^2}\right]=\cdots\frac{\partial^2}{\partial x^2}\Psi(x,y,z)=\frac{(y^2+z^2)(2zy^2+2z^3-6zx^2)}{(x^2+y^2+z^2)^3}=$$

Similar

$$\frac{\partial^2}{\partial y^2}\Psi(x,y,z)=\frac{(-2zx^2)(x^2-3y^2+z^2)}{(x^2+y^2+z^2)^3},\frac{\partial^2}{\partial z^2}\Psi(x,y,z)=\frac{(-2zx^2)(3x^2+3y^2-z^2)}{(x^2+y^2+z^2)^3}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Psi(x,y,z)=$$

$$\frac{(y^2+z^2)(2zy^2+2z^3-6zx^2)+(-2zx^2)(x^2-3y^2+z^2)+(-2zx^2)(3x^2+3y^2-z^2)}{(x^2+y^2+z^2)^3}=$$

$$\cdots\frac{\partial^2}{\partial y^2}\Psi(x,y,z)=\frac{2z[(y^2+z^2)(y^2+z^2-3x^2)-4x^4]}{(x^2+y^2+z^2)^3}$$

Take $r^2=x^2+y^2+z^2$ or $y^2+z^2=r^2-x^2$ and substitute

$$\frac{\partial^2}{\partial y^2}\Psi(x,y,z)=\frac{2z[(r^2-x^2)(r^2-x^2-3x^2)-4x^4]}{(r^2)^3}=\cdots=\frac{2zr^2(r^2-5x^4)}{r^6}=\frac{2z(r^2-5x^4)}{r^4}$$

$$\Psi(x,y,z)=\frac{zx^2}{x^2+y^2+z^2}\rightarrow\ \Psi(r,\theta,\phi)=?$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$r^2 = x^2+y^2+z^2$

$$\Psi(r,\theta,\phi)=\frac{r\cos\theta(r\sin\theta\cos\phi)^2}{r^2}=r\cos\theta\sin^2\theta\cos^2\phi$$

$$\nabla^2\Psi(x,y,z)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$$

Taking first $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cos\theta\sin^2\theta\cos^2\phi)=\frac{1}{r^2}(2r\cos\theta\sin^2\theta\cos^2\phi)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)=\frac{2}{r}\cos\theta\sin^2\theta\cos^2\phi$$

Similar(but a lot more work) $\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)=\frac{\cos^2\phi}{r}(4\cos^3\theta-8\sin^2\theta\cos\theta)$

(a little easier) $\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}=\frac{2\cos\theta}{r}(\sin^2\phi-\cos^2\phi)$

$$\nabla^2\Psi(r,\theta,\phi)=\frac{2}{r}\cos\theta\sin^2\theta\cos^2\phi+\frac{\cos^2\phi}{r}(4\cos^3\theta-8\sin^2\theta\cos\theta)+\frac{2\cos\theta}{r}(\sin^2\phi-\cos^2\phi)=$$

$\cdots$(a lot more algebra, using e.g. $\sin^2\theta+\cos^2\theta=1$) $\cdots\nabla^2\Psi(r,\theta,\phi)=\frac{2\cos\theta}{r}[1-5\cos^2\phi\sin^2\theta]$

Note $\cos\theta=\frac{z}{r}$ in spherical coordinates and multiple by $\frac{r^2}{r^2}$ inside bracket, substitute

$$\nabla^2\Psi(r,\theta,\phi)=\frac{2z}{r^2}[\frac{r^2}{r^2}(1-5\cos^2\phi\sin^2\theta)]=\frac{2z}{r^4}(r^2-5r^2\cos^2\phi\sin^2\theta)=$$

Finally also note that x $=r\cos\phi\sin\theta$ from spherical coordinates and substitute

$\nabla^2\Psi(x,y,z)= \ =\frac{2z}{r^4}(r^2-5x^2)$ SAME AS CARTESIAN RESULT!!!!!

3.2

$Q = Q_x i + Q_y J + Q_z k =$

$[3x^2(y+z)+y^3+z^3]i+[3y^2(z+x)+z^3+x^3]j+[3z^2(y+z)+x^3+y^3]k$

? Is it - conservative $\leftrightarrow \nabla\times Q=0$

$$\nabla\times Q = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q_x & Q_y & Q_z \end{vmatrix} = \left(\frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z}\right) i + \cdots j + \cdots k =$$

$\frac{\partial Q_z}{\partial y} = \frac{\partial Q_y}{\partial z} = 3z^2+3y^2$, therefore $\frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z} = 0$

Same for other terms, therefore $\nabla\times Q = 0$

and Q is derivable from a potential ϕ, that is $Q = \nabla\phi$

where $\phi = ?$ can be a function of x, y and z

Start with $Q = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

Equate components of Q to components of $\nabla\phi$, starting with i

$Q_x = \frac{\partial\phi}{\partial x} \rightarrow \phi = \int Q_x dx + C$, since $\frac{\partial}{\partial x}$ gives back Q_x the C (a constant with respect to x)can be a function of y and z!!!

Calculating $\phi(x,y,z) = \int [3x^2(y+z)+y^3+z^3]dx + f(y,z) = 3\frac{x^3}{3}(y+z)+x(y^3+z^3)+f(y,z)$

Therefore $\phi(x,y,z) = x^3(y+z)+x(y^3+z^3)+f(y,z) = x^3y+x^3z+xy^3+xz^3+f(y,z)$

Next take $\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}[x^3y+x^3z+xy^3+xz^3+f(y,z)] = x^3+0+x3y^2+0+\frac{\partial}{\partial y}f(y,z)$

Set equal to Q_y, $x^3+x3y^2+\frac{\partial}{\partial y}f(y,z) = 3y^2z+3y^2x+z^3+x^3 \rightarrow \frac{\partial}{\partial y}f(y,z) = 3y^2z+z^3$

$f(y,z) = \int \frac{\partial}{\partial y}f(y,z)dy + g(z) = \int (3y^2z+z^3)dy + g(z) = 3\frac{y^3}{3}z+z^3y+g(z)$

Therefore $f(y,z) = y^3z+z^3y+g(z)$

Now $\phi(x,y,z) = x^3y+x^3z+xy^3+xz^3+y^3z+z^3y+g(z) = x^3(y+z)+y^3(x+z)+z^3(x+y)+g(z)$

Finally we have $\frac{\partial\phi}{\partial z}$=Q substitution gives

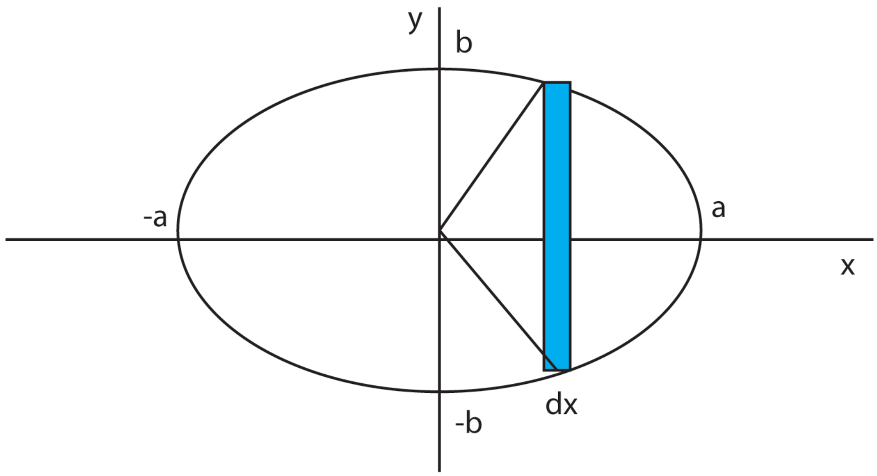
$$\frac{\partial\phi}{\partial z}[x^3y+x^3z+y^3x+y^3z+z^3x+z^3y+g(z)]= =3z^2(y+z)+x^3+y^3$$

or $0+x^3+0+y^3+3z^2x+3z^2y+g'(z)=3z^2(y+z)+x^3+y^3 \rightarrow g'(z)=0$

So $g(z)=C$

Therefore $\phi(x,y,z)=x^3y+x^3z+y^3x+y^3z+z^3x+z^3y+C$ and we can use this to calculate $\int_{(1,-1,1)}^{(2,1,2)} Q\cdot dr=\int_{(1,-1,1)}^{(2,1,2)} \nabla\phi\cdot dr=\int_{(1,-1,1)}^{(2,1,2)} d\phi=\phi(2,1,2)-\phi(1,-1,1)=54$

3.7



$$I = \oint_C y(4x^2 + y^2)dx + x(2x^2 + 3y^2)dy$$

where C is ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

KEY: Use Green's Theorem pg. 135

$$\oint_C P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

Let $P = 4x^2y + y^3$, $Q = 2x^3 + 3xy^2$

$$\frac{\partial P}{\partial y} = 4x^2 + 3y^2 \qquad \frac{\partial Q}{\partial x} = 6x^2 + 3y^2$$

$$\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial z} = (6x^2 + 3y^2) - (4x^2 + 3y^2) = 2x^2$$

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_A 2x^2 dxdy$$

The area is mapped out by solving ellipse for $y = \pm b\left[1 - \frac{x^2}{a^2}\right]^{\frac{1}{2}}$

which gives the y bounds and now x goes from -a to a, therefore

$$\iint_A 2x^2 dxdy = \int_{-a}^{a} \int_{-b\left[1-\frac{x^2}{a^2}\right]^{\frac{1}{2}}}^{b\left[1-\frac{x^2}{a^2}\right]^{\frac{1}{2}}} 2x^2 dydx = \int_{-a}^{a} 2x^2 y\Big|_{-b\left[1-\frac{x^2}{a^2}\right]^{\frac{1}{2}}}^{b\left[1-\frac{x^2}{a^2}\right]^{\frac{1}{2}}} dx = \int_{-a}^{a} 2x^2 2b\left[1-\frac{x^2}{a^2}\right]^{\frac{1}{2}} dx =$$

$$\frac{4b}{a}\int_{-a}^{a} x^2(a^2-x^2)^{\frac{1}{2}}dx$$

Look up in table? or by trig. substitution

Let $x = a\cos\theta$, $dx = -a\sin\theta d\theta$

and $(a^2-x^2)^{\frac{1}{2}} = (a^2-a^2\cos^2\theta)^{\frac{1}{2}} = a(1-\cos^2\theta)^{\frac{1}{2}} = a\sin\theta$

Note for $x = a$, $a = a\cos\theta \to \cos\theta = 1 \to \theta = 0$

$x = -a$, $-a = a\cos\theta \to \cos\theta = -1 \to \theta = \pi$

$$\frac{4b}{a}\int_{-a}^{a} x^2(a^2-x^2)^{\frac{1}{2}}dx \to \frac{4b}{a}\int_{\pi}^{0}(a^2\cos^2\theta)(a\sin\theta)(-a\sin\theta d\theta) =$$

$$-4ba^3\int_{\pi}^{0}\cos^2\theta\sin^2\theta d\theta$$

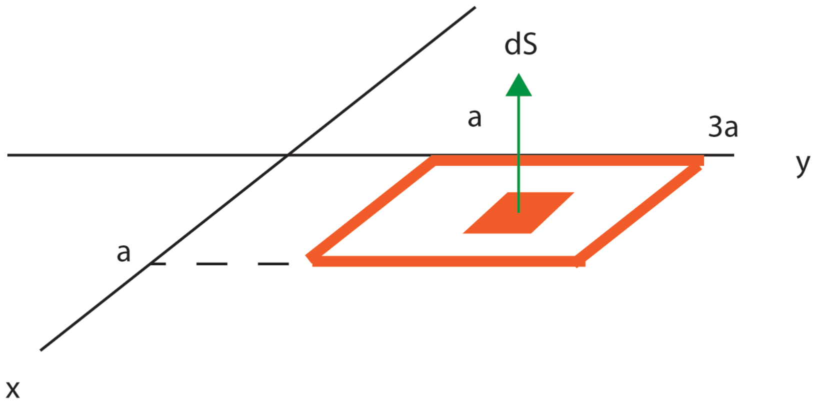
$$\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta \to \cos^2\theta\sin^2 = \left(\frac{1}{2}\sin 2\theta\right)^2$$

Substitution and table look gives

$$-\frac{4ba^3}{4}\int_{\pi}^{0}\sin^2 2\theta d\theta = -ba^3\int_{\pi}^{0}\sin^2 2\theta d\theta = -ba^3\left[\frac{\theta}{2} - \frac{1}{4\cdot 2}\sin(2\cdot 2\theta)\right]_{\pi}^{0} =$$

$$-ba^3\left[\left(\frac{0}{2}-0\right) - \left(\frac{\pi}{2}-0\right)\right] = \frac{1}{2}\pi ba^3$$

3.28



$$F = F_x i + F_y J + F_z k = F_0\left[\frac{y^3}{3a^3} + \frac{ye^{xy/a^2}}{a} + 1\right]i + F_0\left[\frac{xy^2}{a^3} + \frac{(x+y)e^{xy/a^2}}{a}\right]j + F_0\left[\frac{z}{a}e^{xy/a^2}\right]k$$

Evaluate $\oint_L F \cdot dr$ over indicated path in orange

KEY: Use Stoke's theorem $\oint_L F \cdot dr = \int_S (\nabla \times F) \cdot dS$, dS is an area element with direction in z (hence a vector like unit vector k in z direction - see diagram)

that is dS = kds and ds = dxdy is a scalar area element

$$\nabla \times F = \cdots i + \cdots j + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)k$$ and since i, j and k orthogonal only $k \cdot k$ component contributes!!!

That is $(\nabla \times F) \cdot dS = (\nabla \times F) \cdot kds = \left[\cdots i + \cdots j + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)k\right] \cdot kds = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)ds$

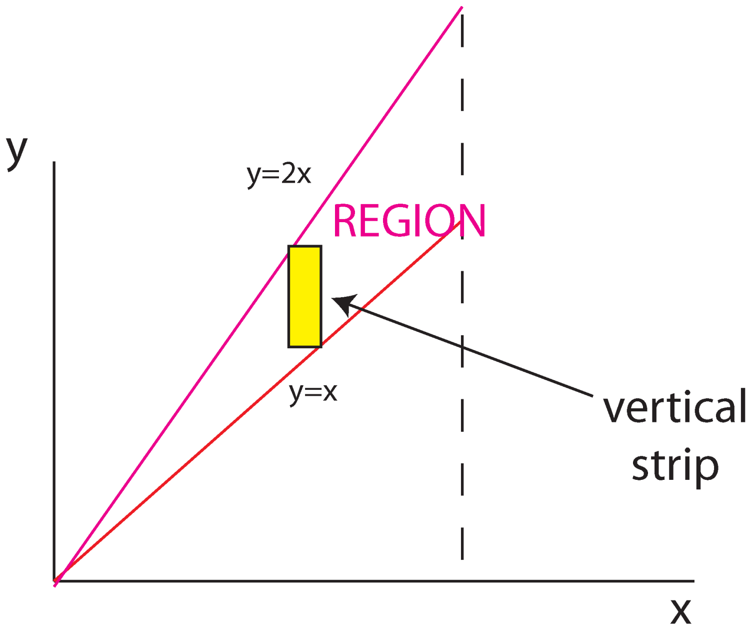
$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \cdots(\text{after some derivatives and algebra}) = F_0\frac{y^2}{a^3}e^{xy/a^2}$$

Therefore $\int_S (\nabla \times F) \cdot dS = \int_S \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)ds = \int_a^{3a}\int_0^a F_0\frac{y^2}{a^3}e^{xy/a^2}\,dx\,dy = \frac{F_0}{a^3}\int_a^{3a}\int_0^a y^2 e^{xy/a^2}\,dx\,dy =$

$$\frac{F_0}{a^3}\int_a^{3a} y^2 \left.\frac{e^{xy/a^2}}{y/a^2}\right|_0^a dy = \frac{F_0}{a^3}\int_a^{3a} ya^2\left[e^{ay/a^2} - e^{0y/a^2}\right]dy = \frac{F_0}{a}\int_a^{3a} y\left[e^{y/a} - 1\right]dy = \frac{F_0}{a}\int_a^{3a}(ye^{y/a} - y)dy$$

$\cdots$(split into two integrals use table for $ye^{y/a}$, then after some algebra)=$F_0 a(2e^3 - 4)$

(R1)



$$\int_0^2\int_x^{2x}(x+y)^2dydx=\int_0^2\int_x^{2x}(x^2+2xy+y^2)dydx=$$

$$\int_0^2(x^2y+2x\frac{y^2}{2}+\frac{y^3}{3})\Big|_x^{2x}dx=\int_0^2(x^2 2x+2x\frac{(2x)^2}{2}+\frac{(2x)^3}{3})-(x^2x+2x\frac{(x)^2}{2}+\frac{(x)^3}{3})dx=$$

$$\int_0^2(2x^3+4x^3+\frac{8x^3}{3})-(x^3+x^3+\frac{x^3}{3})dx=\int_0^2(4+\frac{7}{3})x^3dx=\frac{19}{3}\frac{x^4}{4}\Big|_0^2=\frac{19}{3}\frac{2^4}{4}=\frac{76}{3}$$

(R2)

$V = \frac{1}{3}\iint_S r\cos\phi\, dA$ where ϕ angle normal to surface = 0

Therefore $\cos\phi = 1$, also on sphere r = a - constant and $V = \frac{1}{3}\iint_S a\, dA$

Surface element in spherical coordinates $dA = rd\theta r\sin\tilde{\phi} d\tilde{\phi} = a^2 d\theta \sin\tilde{\phi} d\tilde{\phi}$

Note $\tilde{\phi}$ usual spherical coordinate with $\sim$ to not confuse with ϕ, normal angle

To map out the surface we take $\tilde{\phi}$ from 0 to π and θ from -π to π

Therefore $V = \frac{1}{3}\iint_S a\, dA = \frac{1}{3}\iint_S a(a^2 d\theta \sin\tilde{\phi} d\tilde{\phi}) = \frac{1}{3}\int_0^{\pi}\int_{-\pi}^{\pi} a^3 d\theta \sin\tilde{\phi} d\tilde{\phi} =$

$\frac{a^3}{3}\int_{-\pi}^{\pi} d\theta \int_0^{\pi} \sin\tilde{\phi} d\tilde{\phi} = \frac{a^3}{3}(2\pi)\int_0^{\pi} \sin\tilde{\phi} d\tilde{\phi} = \frac{a^3}{3}(2\pi)(-\cos\tilde{\phi})\Big|_0^{\pi} = \frac{a^3}{3}(2\pi)[-\cos\pi - (-\cos 0)] =$

$\frac{a^3}{3}(2\pi)[-(-1)-(-1)] = \frac{a^3}{3}(2\pi)2 = \frac{4\pi}{3}a^3 = \frac{a}{3}(4\pi a^2)$ where Surface area is $4\pi a^2$

(R3) This one is short for a change

$y^{(4)} + 2y^{(2)} + y = 0$

Just assume y(x)=Ae^{mx}, then $y^{(n)}(x) = m^n Ae^{mx}$

Substitute $m^4Ae^{mx} + 2m^2Ae^{mx} + Ae^{mx} = 0$

or $Ae^{mx}(m^4 + 2m^2 + 1) = 0 \rightarrow m^4 + 2m^2 + 1 = 0$

Make substitution for $m^2 = s$, and then use quadratic formula gives

four roots (call them m_i) $s = \dfrac{-2 \pm \sqrt{2}}{2}$, $m = \pm\sqrt{s}$

Then $y(x) = \sum_{i=1}^{4} A_i e^{m_i x}$

(R4) This one is also relatively short

$4y'' - 4y' - 3y = 0 \;\; y(\text{-}2) = e,\;\; y'(\text{-}2) = \dfrac{-e}{2}$

Just assume y(x)=Ae^{mx}, then $y^{(n)}(x) = m^n Ae^{mx}$

Substitute $4m^2Ae^{mx} - 4mAe^{mx} - 3Ae^{mx} = 0$

or $Ae^{mx}(4m^2 - 4m - 3) = 0 \rightarrow 4m^2 - 4m - 3 = 0 \rightarrow m = \frac{3}{2}, -\frac{1}{2}$

Then $y(x) = A_1 e^{\frac{3}{2}x} + A_2 e^{-\frac{1}{2}x}$

Now apply the initial conditions

$y(-2) = A_1 e^{\frac{3}{2}(-2)} + A_2 e^{-\frac{1}{2}(-2)} = A_1 e^{-3} + A_2 e^{1} = e$

and

$y'(x) = A_1\left(\dfrac{3}{2}\right)e^{\frac{3}{2}x} + A_2\left(\dfrac{-1}{2}\right)e^{-\frac{1}{2}x}$

$y'(-2) = A_1\left(\dfrac{3}{2}\right)e^{-3} + A_2\left(\dfrac{-1}{2}\right)e^{1} = \dfrac{-e}{2}$

These two equations for $y(-2)$ and $y'(-2)$ can be solved for

the coefficients $A_1 = 0$, $A_2 = 1$

Therefore $y(x) = e^{-\frac{1}{2}x}$

(R5)

$$\frac{dx}{dt} = -k_1 x + k_2 y \quad x(0) = 2$$
$$\frac{dy}{dt} = k_1 x - k_2 y \quad y(0) = 0$$

There are many ways to solve this, here is one

Solve the second equation for x and substitute into first

$$x \equiv x(t) = \frac{1}{k_1}\left[\frac{dy}{dt} + k_2 y\right]$$

$$\frac{d}{dt}\frac{1}{k_1}\left[\frac{dy}{dt} + k_2 y\right] = -k_1 \frac{1}{k_1}\left[\frac{dy}{dt} + k_2 y\right] + k_2 y$$

$$\rightarrow \frac{1}{k_1}\frac{d}{dt}\left[\frac{dy}{dt} + k_2 y\right] = -\left[\frac{dy}{dt} + k_2 y\right] + k_2 y$$

$$\rightarrow \frac{1}{k_1}\frac{d^2 y}{dt^2} + \frac{k_2}{k_1}\frac{dy}{dt} = -\frac{dy}{dt} - k_2 y + k_2 y$$

$$\rightarrow \frac{1}{k_1}\frac{d^2 y}{dt^2} + \frac{k_2}{k_1}\frac{dy}{dt} = -\frac{dy}{dt} \rightarrow (\text{mult. by } k_1 \text{and rewrite})$$

$$\frac{d^2 y}{dt^2} + (k_2 + k_1)\frac{dy}{dt} = 0$$

Take $y(t)\text{=}Ae^{mt} \rightarrow (\text{usual}) \rightarrow m = 0, -(k_1 + k_2)$

Therefore (answer before applying initial conditions)

$y(t)\text{=}Ae^{0t} + Be^{-(k_1+k_2)t} = A + Be^{-(k_1+k_2)t}$

Then $\dfrac{dy}{dt} = -(k_1 + k_2)Be^{-(k_1+k_2)t}$

Substitute into equation for x above

$$x(t) = \frac{1}{k_1}\left[\frac{dy}{dt} + k_2 y\right] = \frac{1}{k_1}\left[-(k_1 + k_2)Be^{-(k_1+k_2)t} + k_2(A + Be^{-(k_1+k_2)t})\right] =$$

$$\left[-1 - \frac{k_2}{k_1}\right]Be^{-(k_1+k_2)t} + \frac{k_2}{k_1}A + \frac{k_2}{k_1}Be^{-(k_1+k_2)t} =$$

Finally (answer before applying initial conditions) $x(t)\text{=}\dfrac{k_2}{k_1}A - Be^{-(k_1+k_2)t}$

Applying initial conditions $x(0)=2$ and $y(0)=0$

gives B = -A and $A=\dfrac{2k_1}{k_1+k_2}$

Finally $x(t)=\dfrac{2}{k_1+k_2}\left[k_2+k_1e^{-(k_1+k_2)t}\right]$, $y(t)=\dfrac{2k_1}{k_1+k_2}\left[1-e^{-(k_1+k_2)t}\right]$

(R6)

$$P_n(x)=\sum_{m=0}^{M}\frac{(-1)^m(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$$

$$M=\begin{cases} n/2 & \text{n even} \\ (n-1)/2 & \text{n odd} \end{cases}$$

For n = 6 $\rightarrow$ M=3

$$P_6(x)=\sum_{m=0}^{3}\frac{(-1)^m(2\cdot 6-2m)!}{2^6 m!(6-m)!(6-2m)!}x^{6-2m}=$$

$$\frac{(-1)^0(2\cdot 6-2\cdot 0)!}{2^6 0!(6-0)!(6-2\cdot 0)!}x^{6-2\cdot 0}+\frac{(-1)^1(2\cdot 6-2\cdot 1)!}{2^6 1!(6-1)!(6-2\cdot 1)!}x^{6-2\cdot 1}+\frac{(-1)^2(2\cdot 6-2\cdot 2)!}{2^6 2!(6-2)!(6-2\cdot 2)!}x^{6-2\cdot 2}+$$

$$\frac{(-1)^3(2\cdot 6-2\cdot 3)!}{2^6 3!(6-3)!(6-2\cdot 3)!}x^{6-2\cdot 3}=$$

Remember $(\text{anything})^0=1$, $0!=1$, $1!=1$ otherwise I left most other terms as is

$$P_6(x)=\frac{12!}{2^6 6!6!}x^6-\frac{10!}{2^6 5!4!}x^4+\frac{8!}{2^6 2!4!2!}x^2-\frac{6!}{2^6 3!3!}$$

I don't know about you but I am tried of typing so $P_7(x)$ is up you.

Hope this was a good review (I know this was a lot of work) of a lot of techniques you should mostly be familiar with. I believe it will be beneficial to have attempted these to see what you have retained or missed from your undergraduate course work

I will look at the work you turned in but I will not be grading it as I previously stated, however you should go over your own answers and see how you did.

If you did poorly, focus on the formal course work now and try to review a little as we move along in the next few weeks.