TAKE HOME PROJECT 3 Biomedical Engineering

Start by reading the paper by Wormersley. Then derive some results form it below (copying equations from the paper is not sufficient you **must show all work**!)

Part I. [Pts100-total, not counting extra credit]

1. [Pts40] Here is a version of the Wormersley equation for blood flow

$$\rho\frac{\partial w}{\partial t}=\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial P}{\partial z}$$

where ρ is density and μ is viscosity and $w(r,t)\equiv w_z(r,t)$ is the longitudinal velocity (along the axis) as a function of r, the distance from the center of the artery and time, t.

The pressure gradient takes the form $\frac{\partial P}{\partial z}=Ae^{int}$ and the boundary conditions are

$$\left.\frac{\partial w}{\partial r}\right|_{r=0}=0,\quad w(R)=0$$

Substitute for $\frac{\partial P}{\partial z}$ above and solve for $w(r,t)$ in the equation above by separation of variables. Take $w(r,t)=u(r)e^{int}$ and let $\nu=\frac{\mu}{\rho}$. Find the differential equation for u(r)

and solve it (see hints)!

Hint: Solutions to the resulting equation involve both a homogenous and particular solution. The particular solution is "particularly" easy to guess – think constant with respect to r. The homogenous solution requires that you make a variable substitution to get it into the form of a Bessel equation – it's almost obvious what it has to be. Do it! You will now have a general solution to this equation that is a superposition of Bessel functions of the first and second kind. Finally applying the boundary conditions, along with looking at plots of the Bessel functions of the second kind will point you to the solution in the paper, (equation 8) in terms of only Bessel functions of the first kind. Write your solution in terms of alpha (defined in paper) and the variable y. Compare your solution to equation 9 in the paper.

2. [Pts15] Find an expression for the flow rate, Q given

$$Q=2\pi\int_0^R w(r,t)r\,dr$$

Compare your answer to equation 22 in Wormersley's paper.

Hint: You will need the following Bessel identity $\int zJ_0(z)dz=zJ_1(z)$

3. [Pts35] For a constant input pressure we call it Poiseuille flow. You could solve this using the original differential equation however in this case use the Wormersley solution above at zero frequency. **However** in order to do this you will need to use L'Hospital's rule as there is an indeterminate term for your derived solution at zero frequency [Note: $J_0(0)=1,\ J_1(0)=0,\ J_2(0)=0$]. You will also need to use the following Bessel identities:

$$\frac{d}{dz}J_0(z)=-J_1(z)$$
$$\frac{d}{dz}J_n(z)=\frac{1}{2}[J_{n-1}(z)-J_{n+1}(z)]$$

Compare your answer to equation 2 in Wormersley's paper. What is your A equal to in this paper?

Hint: Don't forget to use the chain rule for differentiation when appropriate!

Another "important" hint: It will make it a little easier if you first rewrite Wormersley solution (derived in question 1) given in Wormersley's paper as equation 8 in terms of the parameter $\alpha = R\sqrt{\frac{n}{\nu}}$ (also given in the paper). Then let $\alpha \to 0$ (equivalent to letting frequency $n \to 0$) to derive Poiseuille's flow. It's here you will need to apply L'Hospital's rule but now it will be with respect to α instead of the frequency, n. It's just easier this way! Again, don't forget to use the chain rule for differentiation when appropriate!

Extra credit [Pts15]: Starting with the equation for differential equation for u(r), set n = 0 and for the remaining inhomogeneous equation put it into standard Euler form (think chapter 6 in book). Then solve for the homogeneous and particular solutions and obtain the general solution. Finally, don't forget to apply the boundary conditions from part 1. to your general solution. Compare to the solution in part 3. above.

4. [Pts10] Find an expression for the flow rate, Q for this steady state or Poiseuille's flow.

$$Q = 2\pi\int_0^R w(r)r\,dr$$

Compare your answer to equation 20 in Wormersley's paper.