Interactive Assignment 10

11 pages

Problems

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Problem 10.1a

Determine whether the following can be written as functions of $\rho = x^2 + 2y$ only, and hence whether they are solutions of (10.8)

(a) $u(x,y) = x^2(x^2-4) + 4y(x^2-2) + 4(y^2-1)$

$$\frac{\partial u}{\partial x} = 4x^3 - 8x + 8yx = x(4x^2 - 8 + 8y) = x\frac{\partial u}{\partial y}$$

Next let's convert $u(x,y)$ to $u(\rho)$

$$u(x,y) = x^2(x^2-4) + 4y(x^2-2) + 4(y^2-1)$$
$$= x^4 - 4x^2 + 4x^2y - 8y + 4y^2 - 1$$
$$= x^4 + 4x^2y + 4y^2 - 4(x^2+2y) - 4$$
$$= (x^2+2y)^2 - 4(x^2+2y) - 4$$
$$= \rho^2 - 4\rho - 4$$

Problem 10.2.b

Find partial differential equations satisfied by the following functions $u(x,y)$ for all arbitrary functions f and all arbitrary constants a and b

(b) $u(x,y) = (x-a)^2 + (y-b)^2$

we have $\frac{\partial u}{\partial x} = 2(x-a)$, $\frac{\partial^2 u}{\partial x^2} = 2$, $\frac{\partial u}{\partial y} = 2(y-b)$, $\frac{\partial^2 u}{\partial y^2} = 2$

Laplace's equation in two-dimension: $\nabla^2 u = 4$

Problem 10.3

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Solve the following partial differential equations for $u(x,y)$ with boundary conditions given

(a) $x\frac{\partial u}{\partial x} + xy = u$, $u = 2y$ on the line $x=1$

(b) $1 + x\frac{\partial u}{\partial y} = xu$, $u(x,0) = x$

(a) Dividing by x gives

$$\frac{\partial u}{\partial x} + y = \frac{u}{x}$$

$$\text{or } \frac{\partial u}{\partial x} - \frac{u}{x} = -y$$

Integration factor is

$$e^{-\int \frac{dx}{x}} = e^{-\ln x} = e^{\ln 1/x} = 1/x$$

Problem 10.3

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(a) Multiplying the partial differential equation by I.F.:

$$\frac{1}{x}\frac{\partial u}{\partial x} - \frac{u}{x^2} = -y/x$$

or $\frac{\partial}{\partial x}\left[\frac{u}{x}\right] = -y/x$

Integrating gives $\frac{u}{x} = -y\ln x + f(y)$

thus $u(x,y) = (-x\ln x)\,y + f(y)\,x$

for $x=1$ $u=2y$, substitutes gives $2y = f(y)$

so $f(y) = 2y$

therefore $u(x,y) = xy\,(2-\ln x)$

(b) $1 + x\frac{\partial u}{\partial y} = xu$, $u(x,0) = x$

Rewrite it to $x\frac{\partial u}{\partial y} - xu = -1$

Dividing by x through gives

$$\frac{\partial u}{\partial y} - u = -1/x$$

Integrating factor : $e^{-\int dy} = e^{-y}$

Problem 10.3 (b)

Multiplying the pde by the integrating factor yields

$$e^{-y}\frac{\partial u}{\partial y} - e^{-y}u = -\frac{e^{-y}}{x}$$

$$\frac{\partial}{\partial y}\left[e^{-y}u\right] = -e^{-y}/x$$

Integrating $\quad e^{-y}u = \frac{e^{-y}}{x} + f(x)$

$$u = \frac{1}{x} + e^{y}f(x)$$

$u(x,0) = \frac{1}{x} + f(x) = x$ gives $f(x) = x - \frac{1}{x} = \frac{x^2-1}{x}$

Therefore $u(x,y) = \frac{1}{x} + e^{y}\frac{(x^2-1)}{x} = \frac{1}{x}(1-e^{y}) + xe^{y}$

Problem 10.4(a)

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0 \quad , \quad u(x,0) = 1+\sin x$$

$A(x,y) = y$ and $B(x,y) = -x$

Using $\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}$ we have $\frac{dx}{y} = -\frac{dy}{dx}$

or $\cancel{x}\,dx = -y\,dy$

Integrating $\quad x^2 = -y^2 + C$

$$C = x^2 + y^2$$

Problem 10.4 (a)

$u(x,y) = f(p) = f(x^2+y^2)$

We require $u(x,0) = f(x^2) = 1+\sin x$

Take $u(x,y) = 1+\sin\sqrt{p} = 1+\sin\sqrt{x^2+y^2}$

This satisfies that for $y=0$ $u(x,0) = 1+\sin(x^2)^{1/2} = 1+\sin x$

Problem 10.8

Function $u(x,y)$ satisfies

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 10$$

and takes the value 3 on the line $y=4x$. Evaluate $u(2,4)$

The homogeneous equation is $2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0$

Using $\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}$ with $A(x,y)=2$

and $B(x,y)=3$

Problem 10.8

From $\frac{dx}{A} = \frac{dy}{B}$ we have $\frac{dx}{2} = \frac{dy}{3}$

or $3dx = 2dy$

Integrating $3\int dx = 2\int dy$ which gives $3x = 2y + C$

Solving for C: $C = 3x - 2y$

Therefore we take $p(x,y) = 3x - 2y$ and the solution to the homogeneous equation is: $u_h(x,y) = f(3x-2y)$

the original equation is: $2\frac{\partial u}{\partial x} + \frac{3\partial u}{\partial y} = 10$

By inspection we choose the particular solution: $u_p(x,y) = 5x$

$2\frac{\partial u_p}{\partial x} + 3\frac{\partial u_p}{\partial y} = 2 \cdot 5 + 3 \cdot 0 = 10$

Thus the general solution of the pde is:

$$u(x,y) = u_h(x,y) + u_p(x,y) = f(3x-2y) + 5x$$

Next we include the boundary condition $u(x, 4x) = 3$

which gives $f(p) + 5x = 3$

$$f(p) - p = 3$$

$$f(p) = 3 + p = 3 + (3x - 2y)$$

Problem 10.8

Therefore $u(x,y) = f(p) + 5x$

$$= 3x - 2y + 3 + 5x$$

$$= 8x - 2y + 3$$

Finally $u(2,4) = 8 \cdot 2 - 2 \cdot 4 + 3 = 11$

Problem 10.18

Like the Schrödinger equation, the equation describing the transverse vibrations of a rod

$$a^4 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

has different orders of derivatives in its various terms. Show, however, that it has solutions of exponential form $u(x,t) = A\exp(\alpha x + i\omega t)$ provided that the relation $a^4 \alpha^4 = \omega^2$ is satisfied.

Take $u(x,t) = A e^{\alpha x + i\omega t}$

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial}{\partial x}\left(A e^{i\omega t} e^{\alpha x}\right) = A \alpha \, e^{i\omega t} e^{\alpha x}$$

$$\ldots \quad \frac{\partial^4 u(x,t)}{\partial x^4} = A \alpha^4 e^{i\omega t} e^{\alpha x}$$

$$\frac{\partial u(x,t)}{\partial t} = A(i\omega)\, e^{\alpha x} e^{i\omega t}, \quad \frac{\partial^2 u(x,t)}{\partial t^2} = (-A)\omega^2 e^{\alpha x + i\omega t}$$

Plugging these into the initial equation:

$$a^4 A \alpha^4 e^{\alpha x + i\omega t} - A\omega^2 e^{\alpha x + i\omega t} = 0$$

$$(a^4 \alpha^4 - \omega^2)\, A e^{\alpha x + i\omega t} = 0$$

Problem 10.18

Discarding $A=0$, we require $a^4\lambda^4-\omega^2=0$

or $a^4\lambda^4=\omega^2$

$\lambda^4=\dfrac{\omega^2}{a^4}$; we have 4 roots $\lambda=\pm\dfrac{\sqrt{\omega}}{a}, \pm i\dfrac{\sqrt{\omega}}{a}$

And we write the solution as a superposition:

$u(x,t)=Ae^{\lambda x}e^{i\omega t}$

Next applying Euler's identity gives

$X(x)=A\sin\lambda x+B\cos\lambda x+C\sinh\lambda x+D\cosh\lambda x$

Next look at the boundary conditions $u(0,t)=0$

$u(x,L)=0$

$\left.\dfrac{\partial u}{\partial x}\right|_{x}=0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=L}=0$

Take the first one

$u(0,t)=X(0)T(t)=0$ since in general $T(t)\neq 0 \rightarrow X(0)=0$

The same relations can be derived for all the boundary conditions, that is:

$$X(0)=0, X(L)=0, \left.\frac{\partial X}{\partial x}\right|_{x=0}=0, \left.\frac{\partial X}{\partial x}\right|_{x=L}=0$$

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Problem 10.18

$$X(0) = A\cdot 0 + B\cdot 1 + C\cdot 0 + D\cdot 1 = 0 \quad \text{so } D = -B$$

thus $X(x) = A\sin\lambda x + C\sinh\lambda x + B(\cos\lambda x - \cosh\lambda x)$

$$X'(x) = \lambda\left[A\cos\lambda x + C\cosh\lambda x - B(\sin\lambda x + \sinh\lambda x)\right]$$

$$X'(0) = \lambda\left[A\cdot 1 + C\cdot 1 - B\cdot 0\right] = 0 \quad \text{thus } C = -A$$

Hence $X(x) = A(\sin\lambda x - \sinh\lambda x) + B(\cos\lambda x - \cosh\lambda x)$

$$X(L) = 0 = A(\sin\lambda L - \sinh\lambda L) + B(\cos\lambda L - \cosh\lambda L)$$

$$X'(L) = 0 = \lambda\left(A(\cos\lambda L - \cosh\lambda L) - B(\sin\lambda L + \sinh\lambda L)\right]$$

or

$$\left.\begin{aligned} A(\sin\lambda L - \sinh\lambda L) + B(\cos\lambda L - \cosh\lambda L) &= 0 \quad (1)\\ -B(\sin\lambda L + \sinh\lambda L) + A(\cos\lambda L - \cosh\lambda L) &= 0 \quad (2)\end{aligned}\right\}$$

From (1): $\sin\lambda L - \sinh\lambda L = -\frac{B}{A}(\cos\lambda L - \cosh\lambda L)$

From (2): $\sin\lambda L + \sinh\lambda L = \frac{A}{B}(\cos\lambda L - \cosh\lambda L)$

Plugging (1) into (2):

$$\sin\lambda L + \sinh\lambda L = -\frac{(\cos\lambda L - \cosh\lambda L)}{\sin\lambda L - \sinh\lambda L}(\cos\lambda L - \cosh\lambda L)$$

$$\sin\lambda L^2 - \sinh\lambda L^2 = -(\cos\lambda L - \cosh\lambda L)^2$$

Problem 10.18

$$\sin^2 \alpha L - \sinh^2 \alpha L = -(\cos^2 \alpha L + \cosh^2 \alpha L - 2\cos\alpha L \cosh \alpha L)$$

Rearranging and using $\cos^2 \alpha L + \sin^2 \alpha L = 1$

$\cosh^2 \alpha L - \sinh^2 \alpha L = 1$

$$2\cos\alpha L \cosh\alpha L = \cos^2\alpha L + \sin^2 \alpha L + \cosh^2 \alpha x - \sinh^2 \alpha L$$

$$= 2$$

therefore $\cos\alpha L \cosh \alpha L = 1$