5.20 (c) method 1

$$\tilde{y}(s)=e^{-(\gamma+s)t_0}\frac{1}{(s+\gamma)^2+b^2}=e^{-\gamma t_0}e^{-st_0}\frac{1}{(s+\gamma)^2+b^2}$$

$$\rightarrow y(t)=e^{-\gamma t_0}L^{-1}\left\{e^{-st_0}\frac{1}{(s+\gamma)^2+b^2}\right\}\equiv e^{-\gamma t_0}L^{-1}\left\{\tilde{f}(s)\tilde{g}(s)\right\}$$

Aside: $L^{-1}\{\tilde{g}(s)\}=L^{-1}\left\{\frac{1}{(s+\gamma)^2+b^2}\right\}=\frac{1}{b}e^{-\gamma t}\sin bt$

and $L^{-1}\{\tilde{f}(s)\}=L^{-1}\{e^{-st_0}\}=\delta(t-t_0)$ Back:

The integral is delicate to evaluate (using a rigorous method)

First its not in a form we can easily work , i.e. $\delta(t-a)$ therefore make the subst.

$\tau=u-t_0\rightarrow u=\tau+t_0, d\tau=du$ then $\int_0^t e^{-\gamma\tau}\sin b\tau\ \delta(t-t_0-\tau)d\tau\rightarrow\int_{t_0}^{t+t_0}e^{-\gamma(u-t_0)}\sin b(u-t_0)\ \delta(t-u)du$

Now we have to evaluate the delta function over a finite inteval whereas its defining integral for a variable (u in this case) would be over all possible values of u, i.e. -∞ to ∞ and we would regularly have $\int_{-\infty}^{\infty}f(u)\delta(t-u)du=f(t)$

Therefore rewrite the integral as (the Heaviside functions restrict the interval!!!)

$$\int_{-\infty}^{\infty}e^{-\gamma(u-t_0)}\sin b(u-t_0)\ [H(u\text{-}t_0)-H(u\text{-}t\text{-}t_0)]\delta(t-u)du=$$

$$\int_{-\infty}^{\infty}e^{-\gamma(u-t_0)}\sin b(u-t_0)\ H(u\text{-}t_0)\delta(t-u)du-\int_{-\infty}^{\infty}e^{-\gamma(u-t_0)}\sin b(u-t_0)\ H(u\text{-}t\text{-}t_0)]\delta(t-u)du$$

$$=e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0)\ \text{-}e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t-t_0)=$$

$$e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0)\ \text{-}e^{-\gamma(t-t_0)}\sin b(t-t_0)H(-t_0)=e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0)$$

Note, the second Heaviside function is evaluated for a negtive value, but the Heaviside function is 0 for negative values, i.e. values less than 0!!! So substiution of the remaining integrated function into the expression for y(t) gives

Finally $y(t)=e^{-\gamma t_0}\frac{1}{b}e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0)=\frac{1}{b}e^{-\gamma t}\sin b(t-t_0)H(t-t_0)$

(c) method 2

$$\tilde{y}(s)=e^{-(\gamma+s)t_0}\frac{1}{(s+\gamma)^2+b^2}=e^{-\gamma t_0}e^{-st_0}\frac{1}{(s+\gamma)^2+b^2}$$

$$\rightarrow y(t)=e^{-\gamma t_0}L^{-1}\left\{e^{-st_0}\frac{1}{(s+\gamma)^2+b^2}\right\}\equiv e^{-\gamma t_0}L^{-1}\{e^{-st_0}F(s)\}$$

The shift theorem section given in module M03 lecture and pdf can be written as follows for this case:

$$L\left\{f(t-t_0)H(t-t_0)\right\}=e^{-st_0}F(s) \text{ or } f(t-t_0)H(t-t_0)=L^{-1}\{e^{-st_0}F(s)\}$$

where $L^{-1}\{F(s)\}=\ L^{-1}\left\{\frac{1}{(s+\gamma)^2+b^2}\right\}=\frac{1}{b}e^{-\gamma t}\sin bt=f(t)$

Therefore $L^{-1}\{e^{-st_0}F(s)\}=\ L^{-1}\left\{e^{-st_0}\frac{1}{(s+\gamma)^2+b^2}\right\}=\frac{1}{b}e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0)$

And we have

$$y(t)=e^{-\gamma t_0}\frac{1}{b}e^{-\gamma(t-t_0)}\sin b(t-t_0)H(t-t_0)=\frac{1}{b}e^{-\gamma t}\sin b(t-t_0)H(t-t_0)$$