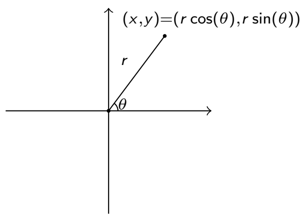
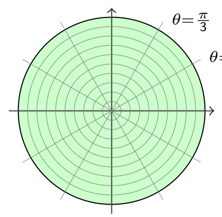
Integrals in polar coordinates

Polar coordinates

We describe points using the distance r from the origin and the angle θ anticlockwise from the x-axis.





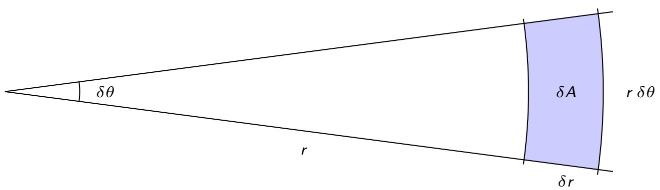
Polar coordinates are related to ordinary (cartesian) coordinates by the formulae

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$
$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan(y/x).$$

(Care is needed to choose the right value of $\arctan(y/x)$.) In the diagram on the right above, we have divided a disk into small pieces using lines of constant θ and circles of constant r.To use this kind of subdivision for integration, we need to know the area of the small pieces.

The polar area element

Consider a piece of angular width $\delta\theta$, where the radius runs from r to $r+\delta r$.



Provided that $\delta\theta$ is small this will be approximately rectangular. If we measure angles in radians (as we always will) then the length of the curved side will be $r\,\delta\theta$, and the straight side has length δr, so the area is approximately $\delta A = r\,\delta r\,\delta\theta$. In the limit this becomes $dA = r\,dr\,d\theta$, so we have the following prescription: if D is a region that is conveniently described in polar coordinates, then

$$\iint_D f(x,y)\,dA = \int_{\theta=\cdots}^{\cdots}\int_{r=\cdots}^{\cdots} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta,$$

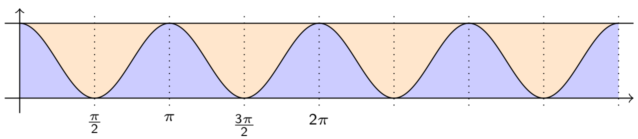
where the limits need to be filled in in accordance with the geometry of the region.

Disk integral of x^2, again

Consider again $\iint_D x^2\,dA$, where D is a disk of radius a around the origin. Here the appropriate limits are just $0 \le \theta \le 2\pi$ and $0 \le r \le a$. The integral is

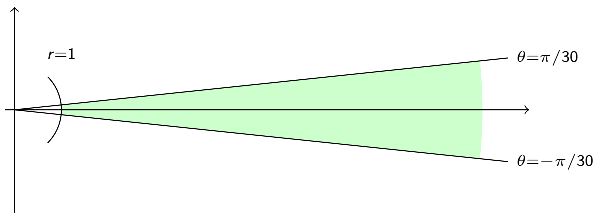
$$\begin{aligned}
\iint_D x^2\,dA &= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} r^2\cos^2(\theta)\,r\,dr\,d\theta \\
&= \int_{\theta=0}^{2\pi}\cos^2(\theta)\int_{r=0}^{a} r^3\,dr\,d\theta \\
&= \frac{a^4}{4}\int_{\theta=0}^{2\pi}\cos^2(\theta)\,d\theta = \frac{a^4}{4}\int_{\theta=0}^{2\pi}\frac{1+\cos(2\theta)}{2}\,d\theta \\
&= \frac{a^4}{4}\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} = \frac{\pi a^4}{4}
\end{aligned}$$

as before.
The following picture shows why $\int_0^{2\pi}\cos^2(\theta)\,d\theta = \pi$:



Each region has the same area, namely $\pi/4$.

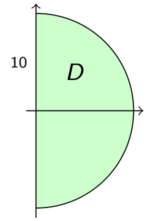
Integral over an infinite wedge



$$\iint_D \frac{1}{(x^2+y^2)^2}\,dA \iint_D \frac{1}{(x^2+y^2)^2}\,dA = \int_{\theta=-\pi/30}^{\pi/30}\int_{r=1}^{\infty}\frac{1}{r^4}\,r\,dr\,d\theta$$
$$= 2\frac{\pi}{30}\int_{r=1}^{\infty} r^{-3}\,dr = \frac{\pi}{15}\left[\frac{r^{-2}}{-2}\right]_{r=1}^{\infty}$$
$$= \frac{\pi}{15}\left(0-\frac{1}{-2}\right) = \frac{\pi}{30}$$

$(r=(x^2+y^2)^{1/2}$ $\qquad dA = r\,dr\,d\theta)$

Integral over a semicircular disk



$$I = \iint_D x^2y^2\,dA = \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{10}(r^2\cos^2(\theta))(r^2\sin^2(\theta))r\,dr\,d\theta$$
$$= \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{10}\sin^2(\theta)\cos^2(\theta)r^5\,dr\,d\theta$$

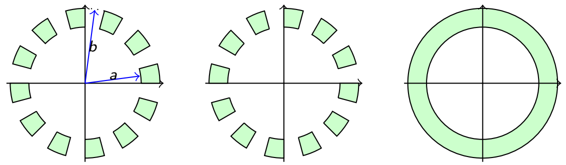
If limits are just constants, then

$$\int_{u=a}^{b}\int_{v=c}^{d} f(u)\,g(v)\,dv\,du = \left(\int_{u=a}^{b} f(u)\,du\right)\left(\int_{v=c}^{d} g(v)\,dv\right).$$

$$\int_{\theta=-\pi/2}^{\pi/2}\sin^2(\theta)\cos^2(\theta)\,d\theta = \int_{\theta=-\pi/2}^{\pi/2}\tfrac{1}{4}\sin^2(2\theta)\,d\theta = \int_{\theta=-\pi/2}^{\pi/2}\frac{1-\cos(4\theta)}{8}\,d\theta$$
$$= \left[\frac{\theta}{8}-\frac{\sin(4\theta)}{4\times 8}\right]_{\theta=-\pi/2}^{\pi/2} = \frac{\pi}{8}$$
$$\int_{r=0}^{10} r^5\,dr = \left[\frac{r^6}{6}\right]_{r=0}^{10} = \frac{1000000}{6}$$
$$I = \frac{\pi}{8}\times\frac{1000000}{6} \simeq 65449.84695.$$

A slotted rotor

Moment of inertia $I = \iint_D (x^2+y^2)dA$.



We first use a simplifying trick. Let D' be the region in the middle picture, and put $I' = \iint_{D'}(x^2+y^2)dA$. As D' is just obtained by turning D slightly, the moment of inertia will be the same, so $I' = I$. On the other hand, $2I = I + I'$ is just the integral over the simpler region D'' shown on the right. We thus have $I = \frac{1}{2}\iint_{D''}(x^2+y^2)dA$. For D'' the limits are just $0 \le \theta \le 2\pi$ and $a \le r \le b$. The integrand is

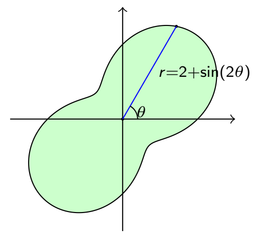
$$x^2+y^2 = (r\cos(\theta))^2 + (r\sin(\theta))^2 = r^2,$$

and the area element is $dA = r\,dr\,d\theta$. We thus have

$$I = \tfrac{1}{2}\int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3\,dr\,d\theta = \tfrac{1}{2}\int_{\theta=0}^{2\pi}\frac{b^4-a^4}{4}d\theta$$
$$= \frac{1}{2}\,\frac{b^4-a^4}{4}\,2\pi = \pi(b^4-a^4)/4.$$

Area of a curved region

The picture shows the region D given in polar coordinates by $0 \le r \le 2+\sin(2\theta)$. We would like to find the area of D, or in other words $A = \iint_D 1\,dA$. Here $dA = r\,dr\,d\theta$ as usual, and the relevant limits are $0 \le \theta \le 2\pi$ and $0 \le r \le 2+\sin(\theta)$



$$A = \int_{\theta=0}^{2\pi}\int_{r=0}^{2+\sin(2\theta)} r\,dr\,d\theta = \int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{2+\sin(2\theta)}d\theta$$
$$= \frac{1}{2}\int_{\theta=0}^{2\pi}(2+\sin(2\theta))^2\,d\theta = \frac{1}{2}\int_{\theta=0}^{2\pi}4+4\sin(2\theta)+\sin^2(2\theta)\,d\theta$$
$$= \frac{1}{2}\int_{\theta=0}^{2\pi}4+4\sin(2\theta)+\tfrac{1}{2}-\tfrac{1}{2}\cos(4\theta)\,d\theta.$$

The integral of $\sin(k\theta)$ or $\cos(k\theta)$ over a whole number of complete cycles is zero. Thus, only the terms 4 and $\frac{1}{2}$ contribute to the integral, and we have

$$A = \tfrac{1}{2}(2\pi.(4+\tfrac{1}{2})) = 9\pi/2.$$

The Gaussian integral

It is an important fact (for the theory of the normal distribution in statistics, the analysis of heat flow, the pricing of financial derivatives, and other applications) that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. We will explain one way to calculate this. Put $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. It obviously does not matter what we call the variable, so we also have $I = \int_{-\infty}^{\infty} e^{-y^2}\,dy$. We can now multiply these two expressions together to get

$$I^2 = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \iint_{\text{whole plane}} e^{-x^2-y^2}\,dA.$$

We can rewrite this using polar coordinates, noting that $x^2+y^2=r^2$ and $dA = r\,dr\,d\theta$. We get

$$I^2 = \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} r\,e^{-r^2}\,d\theta\,dr = 2\pi \int_{r=0}^{\infty} r\,e^{-r^2}\,dr.$$

We now substitute $u = r^2$, so u also runs from 0 to ∞ and $du = 2r\,dr$. The integral becomes

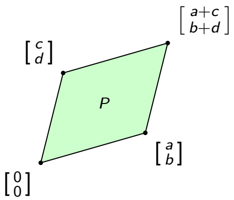
$$I^2 = 2\pi \int_{u=0}^{\infty} e^{-u}.\tfrac{1}{2}du = \pi \left[-e^{-u} \right]_{u=0}^{\infty} = \pi((-0)-(-1)) = \pi$$

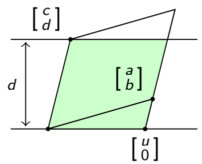
so $I = \sqrt{\pi}$ as claimed.

More general change of variables

Area of a parallelogram

The area of the parallelogram P is $|ad-bc| = \left|\det \begin{bmatrix} a & c \\ b & d \end{bmatrix}\right|$.





Indeed, P consists of the top triangle (shown in yellow) together with the middle region (shown in green). The top triangle has the same area as the bottom one, so we may as well consider the parallelogram P' consisting of the bottom triangle together with the middle region. This parallelogram has a base of length u and a perpendicular height of d, so the area is ud. Note that $\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]$ is reached from $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ by moving in the opposite direction to the vector $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$, so $\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] - t\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ for some t. By comparing the y-coordinates we see that $t = b/d$, and by looking at the x-coordinates we deduce that $u = a - bc/d$, so the area is $ud = ad - bc$. This works when $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ is anticlockwise from $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. If $\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ is clockwise from $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ it works out instead that $ad-bc<0$ and the area is $-(ad-bc)$. In all cases: the area is $|ad-bc|$.

Change of variables in a double integral

Suppose x and y can be expressed in terms of some other variables u and v. The *Jacobian matrix*:

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}.$$

For small changes δu and δv to u and v, resulting changes in x and y are

$$\delta x \simeq x_u\,\delta u + x_v\,\delta v \qquad \delta y \simeq y_u\,\delta u + y_v\,\delta v.$$

These equations can be combined as a single matrix equation:

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \begin{bmatrix} \delta u \\ \delta v \end{bmatrix} = \frac{\partial(x,y)}{\partial(u,v)} \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}.$$

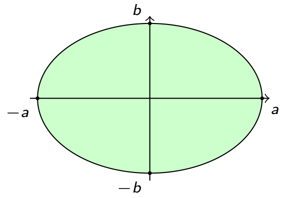
Now let the change in u vary between 0 and δu, and let the change in v vary between 0 and δv. The resulting changes in $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ then cover a small parallelogram spanned by $\begin{bmatrix} x_u \\ y_u \end{bmatrix}\delta u$ and $\begin{bmatrix} x_v \\ y_v \end{bmatrix}\delta v$, and the area of this parallelogram is $|x_u y_v - x_v y_u|\delta u\,\delta v$, or in other words $\left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right|\delta u\,\delta v$.

Using this: $$dA = \left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| du\,dv.$$

This is a key ingredient for double integrals by substitution.

Area of an ellipse

We will find the area of an ellipse E with equation $x^2/a^2 + y^2/b^2 \leq 1$ (for some $a, b > 0$). For this it is best to use a kind of distorted polar coordinates:



$$x = ar\cos(\theta) \qquad\qquad y = br\sin(\theta).$$

Then $x^2/a^2 + y^2/b^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$, so $x^2/a^2 + y^2/b^2 \leq 1$ becomes $0 \leq r \leq 1$. Partial derivatives:

$$x_r = a\cos(\theta) \qquad x_\theta = -ar\sin(\theta) \qquad y_r = b\sin(\theta) \qquad y_\theta = br\cos(\theta),$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} a\cos(\theta) & -ar\sin(\theta) \\ b\sin(\theta) & br\cos(\theta) \end{bmatrix}.$$

This means that the absolute value of the determinant is

$$|\det(J)| = |abr\cos^2(\theta) - (-abr\sin^2(\theta))| = |abr| = abr,$$

so $dA = abr\,dr\,d\theta$. We therefore have

$$\text{area} = \iint_E 1\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} abr\,dr\,d\theta = ab\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{1} d\theta = ab\int_{\theta=0}^{2\pi}\frac{1}{2}d\theta = \pi ab.$$