Interactive Assigment 12

9 pages

Problems

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Problem 14.1

Find an analytic function of $z=x+i$

Let $f(z)=u(x,y)+iv(x,y)$ where $v(x,y)=(y\cos y+x\sin y)e^x$

If f is analytic, it verifies Cauchy-Riemann equations,

in particular $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-[\sin y+(y\cos y+x\sin y)]e^x$

$$=-(y\cos y+(x+1)\sin y)e^x$$

Thus $u=-\int(y\cos y+(x+1)\sin y)e^x dy+f(x)$

$$=(-e^x)\left[\int y\cos y\, dy+(x+1)\int\sin y\, dy\right]+f(x)$$

$$=(-e^x)(\cos y+y\sin y-(x+1)\cos y)+f(x)$$

$$=e^x(x\cos y-y\sin y)+f(x)$$

Now apply the second Cauchy-Riemann equation

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

So $e^x(x\cos y-y\sin y)+e^x\cos y+f'(x)=\cancel{e^x}((x+1)\cos y-y\sin y)e^x$

or $e^x((x+1)\cos y-y\sin y)+f'(x)=((x+1)\cos y-y\sin y)e^x$

Problem 14.1

Therefore $f'(x)=0 \rightarrow f(x)=C$, C: constant

We can take $C=0$. And $u(x,y)=(x\cos y - y\sin y)e^x$

Finally $f(z)=u+iv=e^x(x\cos y-y\sin y+iy\cos y+ix\sin y)$

$$=e^x(x+iy)(\cos y+i\sin y)$$

$$=e^x(\cos y+i\sin y)(x+iy)$$

$$=e^x e^{iy} z = e^{x+iy} z = e^z z$$

$$=z e^z$$

Problem 14.3ac

Find the radii of convergence of the following Taylor series

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{z^n}{\ln n}$

Radius of convergence is defined by

$$\frac{1}{R}=\lim_{n\to\infty}\frac{1}{[\ln n]^{1/n}}=\lim_{n\to\infty}\frac{1}{\ln(n)^{1/n}} \quad \text{for } n\geq 2$$

$$(\ln n)^{1/n}=e^{\frac{1}{n}\ln(\ln n)}$$

Problem 14.3a

$\lim_{n\to\infty} \frac{\ln(\ln n)}{n}$ is indeterminate, of the form: $\frac{\infty}{\infty}$

Apply L'Hopital's rule $(\ln \ln n)' = \frac{1}{n \ln n}$

So $\lim_{n\to\infty} \frac{\ln \ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n\ln n}}{1} = \lim_{n\to\infty} \frac{1}{n \ln n} = 0$

Hence $\lim_{n\to\infty} \ln(n)^{1/n} = e^0 = 1 \rightarrow \lim_{n\to\infty} \frac{1}{\ln(n)^{1/n}} = 1$

Therefore $R=1$ and series converges for $-1 < z < 1$

For $z = -1$ $\quad S(z) = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

$\lim_{n\to\infty} \frac{1}{\ln n} = 0$ and $\left\{\frac{1}{\ln n}\right\}$ is a decreasing sequence, by the alternating series test, $S(z)$ converges at $z=-1$

For $z=1$ $\quad$ Since $\ln n \leq n$ for $n \leq 2$, we have $1/\ln n \geq 1/n$ so the series diverges by comparison with the harmonic series $\sum 1/n$

$\therefore$ Convergence: $-1 \leq z < 1$

(c) $\quad f(z) = \sum_{n=1}^{\infty} z^n \, n^{\ln n}$

Radius of convergence R: $\quad \frac{1}{R} = \lim_{n\to\infty} (n^{\ln n})^{1/n}$

Problem 14.3c YG 4/9

$$(n^{\ln n})^{1/n} = e^{\frac{1}{n}\ln(n^{\ln n})}$$

$$\ln(n^{\ln n}) = \ln\left(e^{\ln(n)\ln(n)}\right) = \ln e^{2\ln n} = 2\ln n$$

By L'Hospital's rule:

$$\lim_{n\to\infty} \frac{\ln(n^{\ln n})}{n} = \lim_{n\to\infty} \frac{2\ln n}{n} = \lim_{n\to\infty} \frac{2/n}{1} = 0$$

So $\lim_{n\to\infty} (n^{\ln n})^{1/n} = e^{0} = 1$ and $R = 1$

Therefore convergence for: $-1 < z < 1$

At $z=1$ since $\lim_{n\to\infty} n^{\ln n} = \infty$ the series diverges

(The series also diverges for $z=-1$)

Problem 14.5ac

Determine the types of singularities (if any) possessed by the following functions at $z=0$ and $z=\infty$

(a) $f(z)=\frac{1}{z-2}$

$f(0)=-1/2$ finite so 0 is a regular point

Next $f(1/z)=\frac{1}{1/z-2}=\frac{z}{1-2z}$

$\lim_{z\to\infty} f(1/z)=\lim_{\substack{\xi\to 0\\ \xi=1/z}} f(1/\xi)=\lim_{\xi\to 0}\frac{\xi}{1-2\xi}=0$ finite

$z=\infty$ not singular

(c) $f(z)=\sinh(1/z)$

Taylor series expansion of $\sinh(z)$ at $z=0$

$$\sinh z=z+\frac{z^3}{3!}+\dots=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$$

$$\text{so } \sinh 1/z=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\frac{1}{z^{2n+1}}$$

$$\text{Therefore } \lim_{z\to 0} z^m \sinh(1/z)=\lim_{z\to 0} z^m\sum_{n=0}^{\infty}\frac{1}{(2n+1)!\,z^{2n+1}}$$

$$=\lim_{z\to 0}\sum_{n=0}^{\infty}\frac{z^m}{(2n+1)!\,z^{2n+1}}$$

$1/6$ $6/9$

Problem 14.15 ac

For large enough n, $2n+1 > m$ and denominator is 0 so Y_0 is undefined and we have an essential singularity for $z=0$

For $z=\infty$ we have to look at $\lim_{z\to\infty} \sinh(1/z) = \sinh(0) = 0$ finite so $z=\infty$ is not singular.

Problem 14.6 ac

Identify the zeros, poles and essential singularities of the following functions

(a) $\tan z$

Zeros of $\tan(z)$ are zeros of $\sin(z)$ since $\tan z = \frac{\sin z}{\cos z}$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i} = 0 \iff e^{iz} = e^{-iz} \iff e^{2iz} = 1$$

$$\iff 2iz = 2in\pi, \quad n \in \mathbb{N}$$

$$\iff z = n\pi$$

Therefore $z = n\pi$ $(n = 0, \pm 1, \pm 2, \dots)$ are zeros of $\tan z$

We have singularities when $\cos z = 0$, $\cos z = \frac{e^{iz}+e^{-iz}}{2} = 0$

So $\quad e^{iz}+e^{-iz} = 0 \iff e^{iz} = -e^{-iz} \iff e^{2iz} = -1 = e^{i(2n+1)\pi}$

$$\iff 2iz = i(2n+1)\pi$$

$$\iff z = \frac{\pi}{2} + n\pi \quad n \text{ integer}$$

So points $z_n = \frac{\pi}{2} + n\pi$ are poles of $\tan z$.

Using the limit definition for poles we have

$$\lim_{z \to z_n} (z - z_n) \tan z = \lim_{z \to z_n} (z - z_n) \frac{\sin z}{\cos z}$$

$\lim_{z \to z_n} z - z_n = 0$ and $\lim_{z \to z_n} \cos z = 0$. So $\lim_{z \to z_n} (z - z_n) \frac{\sin z}{\cos z}$ is of the form $\frac{0}{0}$

And we use L'Hospitals rule.

Problem 14.6 ac

$$\lim_{z \to z_n} (z-z_n) \frac{\sin z}{\cos z} = \lim_{z \to z_n} \frac{(1) \sin z + (z-z_n) \cos z}{\sin z}$$

$$= \frac{\sin z_n + (z_n - z_n) \cos z_n}{\sin (z_n)} = 1$$

Therefore $z_n = \frac{\pi}{2} + n\pi$, $n \in \mathbb{N}$ are poles of order 1, or simple poles of $\tan z$

Taylor expansion of $\tan z$ about 0:

$$\tan z = z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \dots$$

Next look at $z = 1/\xi$ as $\xi \to 0$

$$\lim_{\xi \to 0} (\xi - 0)^n \left[\frac{1}{\xi} + \frac{1}{3} \frac{1}{\xi^3} + \frac{2}{15} \frac{1}{\xi^5} + \dots \right]$$

$= \lim_{\xi \to 0} \xi^n \left[\frac{1}{\xi} + \frac{1}{3} \frac{1}{\xi^3} + \frac{2}{15} \frac{1}{\xi^5} + \dots \right]$ is not finite for some larger n and $\tan 1/\xi$ has an essential singularity at $\xi = 0$ which is equivalent to $\tan z$ having essential singularity at $z = \infty$.

Problem 14.6(c)

c) $f(z) = \exp(1/z)$

First for $z \neq 0$ $\quad e^{z}e^{-z} = e^{0} = 1$ so $\exp(z) \neq 0$, $z \in \mathbb{C}$

If $\zeta = 1/z$ is defined then there is no such ζ that $f(\zeta) = 0$, $f(z)$ does not have any zeros.

Taylor series expansion of e^{z} for any $z \in \mathbb{C}$: $e^{z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{n}$

so $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^{n}}$

$\lim_{z \to 0} (z-0)^{m} e^{1/z} = \lim_{z \to 0} z^{m} \sum_{n=0}^{\infty} \frac{1}{n!\, z^{n}}$, there is no large enough m such that the quantity is finite therefore 0 is an essential singularity of $e^{1/z}$.