6.12

Resonant frequency of system is for no input. Note these equations if you look carefully are related to for example spring equation $\ddot{x}+x=0$ or $\ddot{x}=-x$. To find resonant frequency set the right hand sides to zero.

While it is possible to start by assuming an exponential solution (note Laplace method will not work in this case) but since it is a spring system we known solutions will be of the following form

$$x(t)=A\cos wt+B\sin wt$$
$$y(t)=C\cos wt+D\sin wt$$

Take derivatives, etc. and substitute into original equation $\begin{array}{l}\ddot{x}+2x+y=0\\ \ddot{y}+2x+3y=0\end{array}$

Collect terms for sine and cosine on one side equal to zero on the right side. Therefore coefficients must be zero since in general sine and cosine are not.

We get $\begin{array}{l}-w^2A+2A+C=0\\ -w^2C+2A+3C=0\end{array}$ Similar for B and D but we don't need to use them.

Taking the system above write in matrix form

$$-w^2A+2A+C=0$$
$$-w^2C+2A+3C=0$$
$$\begin{pmatrix} 2-w^2 & 1 \\ 2 & 3-w^2 \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To find resonant behaviour we use the determinant $\begin{bmatrix} 2-w^2 & 1 \\ 2 & 3-w^2 \end{bmatrix}=0$

Leads to $w=1,\ 2$ resonant frequencies

Back to the original system (I will use Laplace method)

$$\ddot{x} + 2x + y = \cos t$$
$$\ddot{y} + 2x + 3y = 2\cos t$$

Assume initial conditions

$$x(0) = \dot{x}(0) = 0$$
$$y(0) = \dot{y}(0) = 0$$

Take the Laplace transform

$$s^2\tilde{x}(s) + 2\tilde{x}(s) + \tilde{y}(s) = \frac{s}{1+s^2}$$
$$s^2\tilde{y}(s) + 2\tilde{x}(s) + 3\tilde{y}(s) = \frac{2s}{1+s^2}$$

Collect terms

$$(s^2+2)\tilde{x}(s) + \tilde{y}(s) = \frac{s}{1+s^2}$$
$$2\tilde{x}(s) + (s^2+3)\tilde{y}(s) = \frac{2s}{1+s^2}$$

Use Cramer's rule to solve (or any method you like)

$$\tilde{x}(s) = \frac{\begin{bmatrix} \frac{s}{1+s^2} & 1 \\ \frac{2s}{1+s^2} & s^2+3 \end{bmatrix}}{D}, \ \tilde{y}(s) = \frac{\begin{bmatrix} s^2+2 & \frac{s}{1+s^2} \\ 2 & \frac{2s}{1+s^2} \end{bmatrix}}{D}$$

$$D = \begin{bmatrix} s^2+2 & 1 \\ 2 & s^2+3 \end{bmatrix} = (s^2+4)(s^2+1)$$

Therefore

$$\tilde{x}(s) = \frac{\frac{s}{1+s^2}(s^2+3) - \frac{2s}{1+s^2}1}{(s^2+4)(s^2+1)} = \cdots = \frac{s}{(s^2+4)(s^2+1)}, \text{Similar for } \tilde{y}(s) = \frac{2s}{(s^2+4)(s^2+1)}$$

Next use partial fractions

For $\tilde{x}(s)$:

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1},\ \text{gives } A=-\frac{1}{3}, C=\frac{1}{3}, B=0, D=0$$

Therefore

$$\tilde{x}(s) = \frac{-\frac{1}{3}s}{s^2+4} + \frac{\frac{1}{3}s}{s^2+1} = -\frac{1}{3}\frac{s}{s^2+4} + \frac{1}{3}\frac{s}{s^2+1}$$

Now take the inverse Laplace transform using Table 5.1 pg212

$$x(t) = -\frac{1}{3}\cos 2t + \frac{1}{3}\cos t$$

A similar calculation gives

$$y(t) = -\frac{2}{3}\cos 2t + \frac{2}{3}\cos t$$

Therefore $y(t) = 2x(t)$, a straight line in the xy-plane. Furthermore the $\cos t$ term with frequency w=1 in the solutions is the same as that for the driving functions in the original D.E. equations.