8.5 (a)

$(1-x^2)y''-2xy'+by=f(x)$ $-1\le x\le 1$

We note that $(1-x^2)y''-2xy'$ has the first two terms similar to that of Legendre's DE equation, therefore take (assume) solution as sum of Legendre polynomials

$$y(x)=\sum_{n=0}^{\infty}a_nP_n(x)$$

Then substitute in DE gives

$$(1-x^2)\sum_{n=0}^{\infty}a_nP_n''(x)-2x\sum_{n=0}^{\infty}a_nP_n'(x)+b\sum_{n=0}^{\infty}a_nP_n(x)=f(x)$$

The defining equation for Legendre polynomials is

$$(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0$$

or $(1-x^2)P_n''(x)-2xP_n'(x)=-n(n+1)P_n(x)$

Substitute this equation into our DE gives

$$\sum_{n=0}^{\infty}-n(n+1)P_n(x)+b\sum_{n=0}^{\infty}a_nP_n(x)=f(x) \text{ or } \sum_{n=0}^{\infty}[-n(n+1)+b]a_nP_n(x)=f(x)$$

Multiply the equation above by $P_m(x)$ and integrate from -1 to 1

$$\int_{-1}^{1}\sum_{n=0}^{\infty}[-n(n+1)+b]a_nP_n(x)P_m(x)dx=\int_{-1}^{1}f(x)P_m(x)dx$$

$$\sum_{n=0}^{\infty}[-n(n+1)+b]a_n\int_{-1}^{1}P_n(x)P_m(x)dx=\int_{-1}^{1}f(x)P_m(x)dx$$

Using orthoginality condition

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 2/(2n+1) & n=m \\ 0 & n \neq m \end{cases}$$

We get

$$[-m(m+1)+b]a_m \frac{2}{2m+1} = \int_{-1}^{1} f(x)P_m(x)dx \text{ or } a_m = \frac{2m+1}{2[-m(m+1)+b]}\int_{-1}^{1} f(x)P_m(x)dx$$

(b) Take b = 14 and $f(x)=5x^3$

$f(x)=5x^3$

can be represented in terms of Legredre polyonomials [KEY1]

with $P_1(x)=x$ and $P_3(x)=\frac{1}{2}(5x^3-3x)$

$$f(x)=5x^3=2\frac{1}{2}(5x^3-3x)+3x=2P_3(x)+3P_1(x)$$

Therefore only a_1 and a_3 are non-zero due to orthogonality [KEY2]

and for example only the m=1 term contributes to the integral term

for a_1

$$a_1 = \frac{2(1)+1}{2[-1(1+1)+14]}\int_{-1}^{1}[2P_3(x)+3P_1(x)]P_m(x)dx = \frac{3}{2(-2+14)}\left(3\frac{2}{2(1)+1}\right)=\frac{1}{4}$$

A similar calculation gives $a_3 = 1$

Therefore substitution gives

$$y(x)=\sum_{n=1,3} a_n P_n(x) = a_1P_1(x)+a_3P_3(x) = \frac{1}{4}(x)+1[\frac{1}{2}(5x^3-3x)] = \frac{5}{2}x^3 - \frac{5}{4}x$$