Johns Hopkins
Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
EN. 585.409

JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

An example of a series solution of an ODE at a regular singular point: where the indicial roots differ by an integer

Let's take as our example the ODE $z(z-1)y''+3zy'+y=0$

We are interested at the solution near or at z = 0.

In standard form $\boxed{y''+\dfrac{3z}{z(z-1)}y'+\dfrac{1}{z(z-1)}y=y''+\dfrac{3}{(z-1)}y'+\dfrac{1}{z(z-1)}y=0}$

We see that z= 0 is a singular point and identify it as a regular singular point since

$$(z-0)p(z)\Big|_{z=z_0=0}=\frac{z}{(z-1)}\Big|_{z=0}=\frac{1}{(z-1)}\Big|_{z=0}=0$$

$$(z-0)^2q(z)\Big|_{z=z_0=0}=(z)^2\frac{1}{z(z-1)}\Big|_{z=0}=\frac{z}{z-1}\Big|_{z=0}=0$$

Substitution off our proposed series solutions and it's derivatives into our original ODE gives

$$z(z-1)y''+3zy'+y=(z^2-z)\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_nz^{\sigma+n-2}+3z\sum_{n=0}^{\infty}(\sigma+n)a_nz^{\sigma+n-1}+\sum_{n=0}^{\infty}a_nz^{\sigma+n}=$$

$$\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_nz^{\sigma+n}-\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_nz^{\sigma+n-1}+\sum_{n=0}^{\infty}3(\sigma+n)a_nz^{\sigma+n}+\sum_{n=0}^{\infty}a_nz^{\sigma+n}=$$

$$\sum_{n=0}^{\infty}[(\sigma+n-1)(\sigma+n)+3(\sigma+n)+1]a_nz^{\sigma+n}-\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_nz^{\sigma+n-1}=0$$

As before to get the indicial equation we pick the term with the lowest power of z from either sum (this occurs in the second sum with n =0 and since $a_0 z^{\sigma-1} \neq 0$) we get

$$(\sigma+0-1)(\sigma+0)a_0 z^{\sigma+0-1} = (\sigma-1)(\sigma)a_0 z^{\sigma-1} = 0 \quad \rightarrow (\sigma-1)(\sigma) = 0 \rightarrow \boxed{\sigma = 0,1}$$

So here is a case where the constants $\sigma = 0,1$ **differ by an integer**.

This case allows as a possibility that the ratio of the solutions could be a constant and therefore the criteria for independence of solutions

$$c_1 y_1 + c_2 y_2 = c_1 y_1 + c_2 (k y_1) = (c_1 + c_2 k) y_1 = 0 \rightarrow c_1 + c_2 k = 0 \text{ and } c_1 = -c_2 k$$

where c_1 and c_2 could taken values not equal to zero and the solutions for y_1 and y_2 are not independent.

Let's match powers of z in the two sums by re-indexing the first sum, that is $n \rightarrow n-1$

$$\sum_{n=0}^{\infty}[(\sigma+n-1)(\sigma+n)+3(\sigma+n)+1]a_n z^{\sigma+n}-\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_n z^{\sigma+n-1}=$$

$$\sum_{n-1=0}^{\infty}[(\sigma+(n-1)-1)(\sigma+(n-1))+3(\sigma+(n-1))+1]a_{n-1}z^{\sigma+(n-1)}-\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_n z^{\sigma+n-1}=$$

$$\sum_{n=1}^{\infty}[(\sigma+n-2)(\sigma+n-1)+3(\sigma+n-1)+1]a_{n-1}z^{\sigma+n-1}-\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_n z^{\sigma+n-1}=0$$

KEY: At this point the powers of z both sums match up but the indices start at different values, however this is not a problem since as before the n = 0 term contribution in the second sum is zero for either value of the indicial constant. Let look at this quickly

The coefficient in the second sum give

$$\text{for } \sigma=0 \quad (0+n-1)(0+n)\big|_{n=0}=(n-1)n\big|_{n=0}=0$$

$$\text{for } \sigma=1 \quad (1+n-1)(1+n)\big|_{n=0}=(n)(n+1)\big|_{n=0}=0$$

Therefore we can start the index at n = 0 in the second sum without lost of information That allows us to combine the sums!

$$\sum_{n=1}^{\infty}\{[(\sigma+n-2)(\sigma+n-1)+3(\sigma+n-1)+1]a_{n-1}-(\sigma+n-1)(\sigma+n)a_n\}z^{\sigma+n-1}=0$$

Then since $z^{\sigma+n-1} \neq 0$ we have

$$[(\sigma+n-2)(\sigma+n-1)+3(\sigma+n-1)+1]a_{n-1} - (\sigma+n-1)(\sigma+n)a_n = 0,\ \ n \geq 1$$

Simplifying gives

$$[(\sigma+n-1)(\sigma+n-2+3)+1]a_{n-1} + (\sigma+n-1)(\sigma+n)a_n =$$

$$[(\sigma+n-1)(\sigma+n+1)+1]a_{n-1} + (\sigma+n-1)(\sigma+n)a_n =$$

$$[(\sigma+n)^2 - 1 + 1]a_{n-1} + (\sigma+n-1)(\sigma+n)a_n = (\sigma+n)^2 a_{n-1} + (\sigma+n-1)(\sigma+n)a_n = 0$$

And solving for a_n gives $\boxed{a_n = \dfrac{\sigma+n}{\sigma+n-1}a_{n-1}}$

Starting with $\sigma = 1$ investigate this recurrence relation setting a_0=1

For n = 1 we have $a_1 = \dfrac{1+1}{1+1-1}a_{1-1} = \dfrac{2}{1}a_0 = \dfrac{2}{1}(1) = 2$

For n = 2 we have $a_2 = \dfrac{1+2}{1+2-1}a_{2-1} = \dfrac{3}{2}a_1 = \dfrac{3}{1}(2) = 3$

For n = 3 we have $a_3 = \dfrac{1+3}{1+3-1}a_{2-1} = \dfrac{4}{3}a_1 = \dfrac{4}{3}(3) = 4$

Fixed notation

$\vdots$

Therefore the explicit form is $\boxed{a_n = n+1}$

and the solution to the ODE is $\boxed{y(z) = z^1 \sum_{n=0}^{\infty}(n+1)z^n \equiv z + 2z^2 + 3z^3 + \cdots}$

As before we can take this a little further to investigate the series solution

Let's look at the Taylor series for the sine function. $f(z) = \sum_{n=0}^{\infty} a_n z^n,\ a_n = \left.\frac{f^{(n)}(z)}{n!}\right|_{z=z_0=0}$

The generating formula for a Taylor series near z_0=0 is (previously presented)

Setting $f(z) = \frac{1}{1-z}$ then

$$a_0 = \left.\frac{f^{(0)}(z)}{0!}\right|_{z=0} = \left.\frac{1}{1-z}\right|_{z=0} = \frac{1}{1} = 1$$

$$a_1 = \left.\frac{f^{(2)}(z)}{1!}\right|_{z=z_0=0} = \frac{1}{1!}\frac{d}{dz}\left[\frac{1}{1-z}\right]_{z=0} = \frac{d}{dz}(1-z)^{-1}\Big|_{z=0} = -(1-z)^{-2}(-1)\Big|_{z=0} = 1$$

$$\vdots$$

Therefore $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ and taking a derivative for the series gives

$$\frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}\sum_{n=0}^{\infty} z^n \to \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}\frac{d}{dz}z^n = \sum_{n=0}^{\infty} nz^{n-1}$$

Finally multiplying by z gives

$$\frac{z}{(1-z)^2} = z\sum_{n=0}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty} nz^n = 0z^0 + 1z^1 + 2z^2 + 3z^3 \cdots = \boxed{z + 2z^2 + 3z^3 + \cdots = y_1(z)}$$

The same as our previous series solution, so $\boxed{y_1(z) = \frac{z}{(1-z)^2}}$

Going back to our recursive relation $\boxed{a_n = \frac{\sigma+n}{\sigma+n-1} a_{n-1}}$

Let's insert the other indicial value $\sigma = 0$

$$a_n = \frac{n}{n-1} a_{n-1} \rightarrow a_1 = \frac{1}{1-1} a_0 = \frac{1}{0} a_0$$

Now when we start with we see that the recursive formula is undefined (division by 0), therefore we can not proceed to find another solutions as we did for the first $\sigma = 1$ case!

In the next lectures you will learn how to find the second solution.