10.3

(a)

$$x\frac{\partial u}{\partial x}+xy=u$$

Put in standard form

$$\frac{\partial u}{\partial x}-\frac{1}{x}u=-y$$

Use an integration factor approach

That is factor = $e^{\int -\frac{1}{x}dx}=e^{-\ln x}=(e^{\ln x})^{-1}=x^{-1}=\frac{1}{x}$

Now use it on DE. above

$$\frac{1}{x}\frac{\partial u}{\partial x}-(\frac{1}{x})^2 u=-\frac{y}{x}$$

$$\frac{\partial}{\partial x}(x^{-1}u)=-\frac{y}{x}$$

Now integrate both sides with respect to x
remember for the indefinite integral constant that
it will be a function of y!

$$\int\frac{\partial}{\partial x}(x^{-1}u)dx=\int-\frac{y}{x}dx$$

$$x^{-1}u=-y\int\frac{dx}{x}=-y\ln(x)+f(y)$$

Multiply both sides by x gives

$u(x,y)=-yx\ln(x)+xf(y)$

Next apply the condition that $u(x,y)=2y$ when $x=1$
to solve for f(y). Therefore

$2y=-y\cdot1\cdot\ln(1)+1\cdot f(x)=-y\cdot1\cdot0+f(y)$

Therefore $f(y)=2y$ and finally

$u(x,y)=-yx\ln(x)+2xy$

(b)

$$1+x\frac{\partial u}{\partial y}=xu,\ u(x,0)=x$$

In standard form

$$\frac{\partial u}{\partial y}-u=-\frac{1}{x}$$

Again use an integrating factor approach, therefore

$$e^{\int -1dy}=e^{-y}$$

Multiply the DE by the integration factor gives

$$e^{-y}\frac{\partial u}{\partial y}-e^{-y}u=-\frac{1}{x}e^{-y}$$

$$\frac{\partial}{\partial y}[e^{-y}u]=-\frac{1}{x}e^{-y}$$

Integrate both sides

$$\int\frac{\partial}{\partial y}[e^{-y}u]dy=\int-\frac{1}{x}e^{-y}\,dy$$

Therefore

$$e^{-y}u=-\frac{1}{x}\int e^{-y}\,dy=-\frac{1}{x}(-e^{-y})+f(x)$$

Multiple by e^{y}

$$u(x,y)=\frac{1}{x}+e^{y}f(x)$$

Now apply $u(x,0)=x$ to solve for $f(x)$

$$u(x,0)=\frac{1}{x}+e^{0}f(x)=\frac{1}{x}+f(x)=x$$

Therefore

$f(x)=x-\dfrac{1}{x}$ and finally

$$u(x,y)=\frac{1}{x}+e^{y}\left(x-\frac{1}{x}\right)$$