6.31

$$\frac{d^2x}{dt^2}+\alpha\frac{dx}{dt}=\delta(t-t_0)$$ with initial conditions $x(0)=0$ and $x'(0)=0$

Solve homogenous $\frac{d^2x}{dt^2}+\alpha\frac{dx}{dt}=0$ (usual way) solution $\rightarrow x_h(t)= c_1e^{0t}+c_2e^{-\alpha t}=c_1+c_2e^{-\alpha t}$

Now comes the delicate part getting the Green's function!

Take with z (used in book pg 255) essentially equal to t_0 (using this in place of z) and using form of our homogenous solution gives

$$G(t,t_0)=\begin{cases} c_1(t_0)+c_2(t_0)e^{-\alpha t} & 0<t<t_0 \\ c_3(t_0)+c_4(t_0)e^{-\alpha t} & t>t_0 \end{cases}$$

$G(t,t_0)$ follows same conditions at initial conditions (above) therefore

$$G(t,t_0)_{t=0}=0 \text{ and } \left.\frac{\partial G(t,t_0)}{\partial t}\right|_{t=0}=0$$

Substitution gives

$$G(t,t_0)_{t=0}=c_1(t_0)+c_2(t_0)e^{-\alpha 0}=0$$

$$\left.\frac{\partial G(t,t_0)}{\partial t}\right|_{t=0}=c_2(t_0)(-e^{-\alpha 0})=0$$

So both c_1 and $c_2=0$

Therefore

$$G(t,t_0)=\begin{cases} 0 & 0<t<t_0 \\ c_3(t_0)+c_4(t_0)e^{-\alpha t} & t>t_0 \end{cases}$$

Now apply second continuity condition equation 6.67 for n = 1,2 since equation has second order derivative! Note z=t_0 and $a_n(t)=1$ coefficient of the second derivative

For second conditions, first take n = 1. Note the variable that is undegoing the limit is t not t_0!

$$\lim_{\varepsilon \to 0} \left[\frac{\partial^{1-1} G(t,t_0)}{\partial t^{1-1}}\right]_{t_0-\varepsilon}^{t_0+\varepsilon} = \lim_{\varepsilon \to 0} \left[G(t,t_0)\right]_{t_0-\varepsilon}^{t_0+\varepsilon} = \lim_{\varepsilon \to 0} \{\left[G(t,t_0)\right]_{t_0+\varepsilon} - \left[G(t,t_0)\right]_{t_0-\varepsilon}\} = 0$$

As $\varepsilon \to 0$ $G(t,t_0)_{t_0-\varepsilon} = 0$ (see above definition of $G(t,t_0) = 0$ for $0 < t < t_0$)

As $\varepsilon \to 0$ $G(t,t_0)_{t_0+\varepsilon} = c_3(t_0) + c_4(t_0)e^{-\alpha t_0}$ (see definition of $G(t,t_0) = c_3(t_0) + c_4(t_0)e^{-\alpha t}$ for $t > t_0$)

Back to limit equation gives

$$\lim_{\varepsilon \to 0} \{\left[G(t,t_0)\right]_{t_0+\varepsilon} - \left[G(t,t_0)\right]_{t_0-\varepsilon}\} = \{c_3(t_0) + c_4(t_0)e^{-\alpha t_0} - 0\} = 0$$

Therefore $c_3(t_0) + c_4(t_0)e^{-\alpha t_0} = 0$ or $c_3(t_0) = -c_4(t_0)e^{-\alpha t_0}$

and we have

$$G(t,t_0) = \begin{cases} 0 & 0 < t < t_0 \\ -c_4(t_0)e^{-\alpha t_0} + c_4(t_0)e^{-\alpha t} & t > t_0 \end{cases}$$

Now take n = 2 Note the variable that is undegoing the limit is t, not t_0.

$$\lim_{\varepsilon \to 0} \left[\frac{\partial^{2-1} G(t,t_0)}{\partial t^{2-1}}\right]_{t_0-\varepsilon}^{t_0+\varepsilon} = \lim_{\varepsilon \to 0} \left[\frac{\partial G(t,t_0)}{\partial t}\right]_{t_0-\varepsilon}^{t_0+\varepsilon} = \lim_{\varepsilon \to 0} \{\left[\frac{\partial G(t,t_0)}{\partial t}\right]_{t_0+\varepsilon} - \left[\frac{\partial G(t,t_0)}{\partial t}\right]_{t_0-\varepsilon}\} = \frac{1}{a_2(t)} = \frac{1}{1} = 1$$

As $\varepsilon \to 0$ $\left.\frac{\partial G(t,t_0)}{\partial t}\right|_{t_0-\varepsilon} = 0$ (see above definition of $G(t,t_0) = 0$ for $0 < t < t_0$)

As $\varepsilon \to 0$ $\left.\frac{\partial G(t,t_0)}{\partial t}\right|_{t_0+\varepsilon} = -\alpha c_4(t_0)e^{-\alpha t_0}$ see above definition for $G(t,t_0)$

Back limit equation gives

$$\lim_{\varepsilon \to 0} \{\left[\frac{\partial G(t,t_0)}{\partial t}\right]_{t_0+\varepsilon} - \left[\frac{\partial G(t,t_0)}{\partial t}\right]_{t_0-\varepsilon}\} = \{-\alpha c_4(t_0)e^{-\alpha t_0} - 0\} = 1$$

Therefore $-\alpha c_4(t_0)e^{-\alpha t_0} - 0 = 1$ or $c_4(t_0) = -\frac{1}{\alpha}e^{\alpha t_0}$ therefore $c_3(t_0) = -c_4(t_0)e^{-\alpha t_0} = -\left[-\frac{1}{\alpha}e^{\alpha t_0}\right]e^{-\alpha t_0} = \frac{1}{\alpha}$

$$\text{Finally } G(t,t_0) = \begin{cases} 0 & 0 < t < t_0 \\ c_3(t_0) + c_4(t_0)e^{-\alpha t} & t > t_0 \end{cases} = \begin{cases} 0 & 0 < t < t_0 \\ \frac{1}{\alpha} + \left[-\frac{1}{\alpha}e^{\alpha t_0}\right]e^{-\alpha t} & t > t_0 \end{cases}$$

$$\text{or } G(t,t_0) = \begin{cases} 0 & 0 < t < t_0 \\ \frac{1}{\alpha}\left[1 - e^{-\alpha(t-t_0)}\right] & t > t_0 \end{cases}$$

Now use the Green's function to get solution for $f(t) = Ae^{-at}$ $(t>t_0, a \neq \alpha)$

$$x(t) = \int_0^t G(t,t_0) f(t_0)dt_0 =$$

$$\int_0^t \frac{1}{\alpha}\left[1 - e^{-\alpha(t-t_0)}\right] Ae^{-at_0}dt_0 = \frac{A}{\alpha}\int_0^t \left[1 - e^{-\alpha(t-t_0)}\right] e^{-at_0}dt_0 = \frac{A}{\alpha}\int_0^t \left[e^{-at_0} - e^{-at_0}e^{-\alpha(t-t_0)}\right] dt_0 =$$

$$\frac{A}{\alpha}\int_0^t \left[e^{-at_0} - e^{-at_0}e^{-\alpha(t-t_0)}\right] dt_0 = \frac{A}{\alpha}\int_0^t e^{-at_0}dt_0 - \frac{A}{\alpha}e^{-\alpha t}\int_0^t e^{(\alpha-a)t_0}dt_0 =$$

Finish integrations with respect to t_0 gives

$$x(t) = \frac{A}{\alpha a}(1 - e^{-at}) - \frac{A}{\alpha(\alpha - a)}e^{-\alpha t}(e^{(\alpha-a)t} - 1)$$