Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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We will again focus on second order ODEs, where we have obtained one solution and wish to find another.

We will look at two methods of obtaining a second linearly independent solution:

- Wronskian methodology
- Derivative methodology

The Wronskian methodology

As we have already mentioned we will be looking at second order ODEs, so we start with the usual representation where we have one solution, $y_1(z)$.

$$y''+p(z)y'+q(z)y=0$$

This method depends on using the Wronskian to obtain this second solution

$$\boxed{W(z)=\begin{vmatrix} y_1(z) & y_2(z) \\ y_1'(z) & y_2'(z) \end{vmatrix}=y_1(z)y_2'(z)-y_1'(z)y_2(z)}$$

Dividing by $[y_1(z)]^2$ gives $\dfrac{W(z)}{[y_1(z)]^2}=\dfrac{y_1(z)y_2'(z)-y_1'(z)y_2(z)}{[y_1(z)]^2}=\boxed{\dfrac{y_2'(z)}{y_1(z)}-y_1'(z)\dfrac{y_2(z)}{[y_1(z)]^2}}$

Note $\dfrac{d}{dz}\dfrac{1}{y_1(z)}=-\dfrac{1}{[y_1(z)]^2}$

Then it follows $\dfrac{d}{dz}\dfrac{y_2(z)}{y_1(z)}\equiv\dfrac{d}{dz}\left[y_2(z)\dfrac{1}{y_1(z)}\right]=\left[\dfrac{d}{dz}y_2(z)\right]\dfrac{1}{y_1(z)}+y_2(z)\left[\dfrac{d}{dz}\dfrac{1}{y_1(z)}\right]$

$$=\boxed{y_2'(z)\frac{1}{y_1(z)}+y_2(z)\left[\frac{-1}{(y_1(z))^2}\right]}$$

Therefore **KEY Equation** $\boxed{\dfrac{W(z)}{[y_1(z)]^2}=\dfrac{d}{dz}\dfrac{y_2(z)}{y_1(z)}}$

Integrating $\int^{z} \frac{d}{du}\left[\frac{y_2(u)}{y_1(u)}\right]du = \int^{z} \frac{W(u)}{[y_1(u)]^2}du$ gives $\frac{y_2(z)}{y_1(z)} = \int^{z} \frac{W(u)}{[y_1(u)]^2}du$

Now substituting for the Wronskian as a integral involving p(z) previously derived, that is

$$W(u) = e^{-\int^{u} p(v)dv}$$

This gives $\frac{y_2(z)}{y_1(z)} = \int^{z} \frac{e^{-\int^{u} p(v)dv}}{[y_1(u)]^2}du$ and solving $\boxed{y_2(z) = y_1(z)\int^{z} \frac{e^{-\int^{u} p(v)dv}}{[y_1(u)]^2}du}$

An example of finding a second solution using the Wronskian method

Previously we looked at the following differential equation $z(z-1)y''+3zy'+y=0$

Where the standard form is $y''+\frac{3}{(z-1)}y'+\frac{1}{z(z-1)}y=0$ and $p(z)=\frac{3}{z-1}$

The first solution was $y_1(z)=\frac{z}{(1-z)^2}$

Now we are ready to find the second solution (note most of the time the functions are not so easy to integrate) using

$$y_2(z)=y_1(z)\int^z \frac{e^{-\int^u p(v)dv}}{[y_1(u)]^2}du$$

First let's look at the integral $e^{-\int^u p(v)dv}$ with $p(v)=\frac{3}{v-1}$ this gives

$$e^{-\int^u \frac{3}{v-1}dv}=e^{-3\int^u \frac{dv}{v-1}}=e^{-3\ln(u-1)}=\left[e^{\ln(u-1)}\right]^{-3}=(u-1)^{-3}$$

Substitution and integration gives into the solution for $y_2(z)$ gives

$$y_2(z)=y_1(z)\int^z \frac{(u-1)^{-3}}{[y_1(u)]^2}du=\frac{z}{(1-z)^2}\int^z \frac{(u-1)^{-3}}{\left[\frac{u}{(1-u)^2}\right]^2}du=\frac{z}{(1-z)^2}\int^z \frac{(1-u)^4}{u^2}\frac{1}{(u-1)^3}du=$$

$$\frac{z}{(1-z)^2}\int^z \frac{-(1-u)}{u^2}du=\frac{z}{(1-z)^2}\int^z \frac{-1}{u^2}+\frac{1}{u}du=\frac{z}{(1-z)^2}\left[\frac{1}{z}+\ln z\right]=\boxed{\frac{1}{(1-z)^2}+\frac{z\ln z}{(1-z)^2}}$$

Finally, factoring gives (we will use this form as a comparison later)

$$\boxed{y_2(z) = \frac{z}{(1-z)^2}\left[\ln z + \frac{1}{z}\right]}$$

The derivative methodology

Let's start by looking at our second order ODE written in operator form, that is

$$y''+p(z)y'+q(z)y=\boxed{\mathcal{L}y(z,\sigma)=0}$$ where we would write in general $y(z,\sigma)=z^{\sigma}\sum_{n=0}^{\infty}a_n(\sigma)z^n$

KEY: Where this solution encompasses both solutions since it allows for the two values derived from the indicial equation!

Now for a quadratic indicial equation we have two roots and that means that the following expression is also equal to 0, that is the coefficient associated with our sum above when n = 0 (also a_0 = 1 as usual) and we have the roots of the indicial equation, that is

$$a_0(\sigma-\sigma_1)(\sigma-\sigma_2)z^{0+\sigma}=(1)(\sigma-\sigma_1)(\sigma-\sigma_2)z^{\sigma}=0$$

Equating these two expression for the case of a double root $\sigma_1=\sigma_2$ gives

$$\boxed{\mathcal{L}y(z,\sigma)=a_0(\sigma-\sigma_1)^2z^{\sigma}}$$

Taking a derivative with respect to σ and evaluating it at $\sigma=\sigma_1$

KEY

Since the partial derivative term occupies the same position as in our original operator defined ODE it is also a solution!

$$\left.\frac{\partial}{\partial\sigma}\mathcal{L}y(z,\sigma)\right|_{\sigma=\sigma_1}=\boxed{\mathcal{L}\left[\frac{\partial}{\partial\sigma}y(z,\sigma)\right]\Bigg|_{\sigma=\sigma_1}}=\left.\frac{\partial}{\partial\sigma}(\sigma-\sigma_1)^2z^{\sigma}\right|_{\sigma=\sigma_1}=$$

$$\left[\frac{\partial}{\partial\sigma}(\sigma-\sigma_1)^2z^{\sigma}+(\sigma-\sigma_1)^2\frac{\partial}{\partial\sigma}z^{\sigma}\right]\Bigg|_{\sigma=\sigma_1}=\left[2(\sigma-\sigma_1)z^{\sigma}+(\sigma-\sigma_1)^2z^{\sigma}\ln z\right]\Big|_{\sigma=\sigma_1}\boxed{=0}$$

Therefore $$y_2(z) = \left[\frac{\partial}{\partial \sigma} y(z,\sigma) \right] \Bigg|_{\sigma=\sigma_1}$$ produces a second solution (for double indicial roots)

For indicial roots that differ by an integer we have the following expression to generate a second independent solution

$$y_2(z) = \left[\frac{\partial}{\partial \sigma} (\sigma - \sigma_2) y(z,\sigma) \right] \Bigg|_{\sigma=\sigma_2}$$

An example of finding a second solution using the derivative method

Going back again to our previously investigated ODE $z(z-1)y''+3zy'+y=0$

The first solution was $y_1(z)=\dfrac{z}{(1-z)^2}$ and this solution was found setting $\sigma=\sigma_1=1$

However we need a general solution in terms of z and σ therefore we need

$$y(z,\sigma)=z^{\sigma}\sum_{n=0}^{\infty}a_n(\sigma)z^n$$

where we previously only had a recursive expression for a_n, that is

$$a_n(\sigma)=\frac{n+\sigma}{n+\sigma-1}a_{n-1}$$

KEY: We need to find an explicit form for $a_n(\sigma)$ and therefore an expression for $y(z,\sigma)$

$$\text{For } n=1 \text{ we have } a_1=\frac{1+\sigma}{1+\sigma-1}a_{1-1}=\frac{1+\sigma}{\sigma}a_0=\frac{1+\sigma}{\sigma}(1)=\frac{1+\sigma}{\sigma}$$

$$n=2 \text{ we have } a_2=\frac{2+\sigma}{2+\sigma-1}a_{2-1}=\frac{2+\sigma}{1+\sigma}a_1=\frac{2+\sigma}{1+\sigma}\left[\frac{1+\sigma}{\sigma}\right]=\frac{2+\sigma}{\sigma}$$

$$n=3 \text{ we have } a_3=\frac{3+\sigma}{3+\sigma-1}a_{2-1}=\frac{3+\sigma}{2+\sigma}a_1=\frac{3+\sigma}{2+\sigma}\left[\frac{2+\sigma}{\sigma}\right]=\frac{3+\sigma}{\sigma}$$

$$\vdots$$

Therefore an explicit form is $\boxed{a_n=\dfrac{n+\sigma}{\sigma}}$

Therefore using $y(z,\sigma)=z^{\sigma}\sum_{n=0}^{\infty}\left(\frac{n+\sigma}{\sigma}\right)z^{n}$ with $\sigma_2=0$ we have

$$y_2(z)=\left[\frac{\partial}{\partial\sigma}(\sigma-0)y(z,\sigma)\right]\Bigg|_{\sigma=0}=\left[\frac{\partial}{\partial\sigma}\sigma z^{\sigma}\sum_{n=0}^{\infty}\left(\frac{n+\sigma}{\sigma}\right)z^{n}\right]\Bigg|_{\sigma=0}=\left[\frac{\partial}{\partial\sigma}z^{\sigma}\sum_{n=0}^{\infty}(n+\sigma)z^{n}\right]\Bigg|_{\sigma=0}=$$

$$\left[\frac{\partial z^{\sigma}}{\partial\sigma}\left(\sum_{n=0}^{\infty}(n+\sigma)z^{n}\right)+z^{\sigma}\left(\frac{\partial}{\partial\sigma}\sum_{n=0}^{\infty}(n+\sigma)z^{n}\right)\right]\Bigg|_{\sigma=0}=\left[\frac{\partial z^{\sigma}}{\partial\sigma}\left(\sum_{n=0}^{\infty}(n+\sigma)z^{n}\right)+z^{\sigma}\left(\sum_{n=0}^{\infty}\frac{\partial}{\partial\sigma}(n+\sigma)z^{n}\right)\right]\Bigg|_{\sigma=0}=$$

$$\left[(z^{\sigma}\ln z)\left(\sum_{n=0}^{\infty}(n+\sigma)z^{n}\right)+z^{\sigma}\left(\sum_{n=0}^{\infty}z^{n}\right)\right]\Bigg|_{\sigma=0}=(z^{0}\ln z)\left(\sum_{n=0}^{\infty}(n+0)z^{n}\right)+z^{0}\left(\sum_{n=0}^{\infty}z^{n}\right)=\ln z\sum_{n=0}^{\infty}nz^{n}+\sum_{n=0}^{\infty}z^{n}$$

From our previous lecture we have $\frac{z}{(1-z)^2}=\sum_{n=0}^{\infty}nz^{n},\quad \frac{1}{1-z}=\sum_{n=0}^{\infty}z^{n}$

$$\boxed{y_2(z)=\ln z\left[\frac{z}{(1-z)^2}\right]+\frac{1}{1-z}=\frac{z}{(1-z)^2}[\ln z+\frac{1-z}{z}]=\frac{z}{(1-z)^2}[\ln z+\frac{1}{z}]-\frac{z}{(1-z)^2}}$$

Comparison to the Wronskian solution for $y_2(z)$ we see that this is the same except for the addition of a term of the form of our first solution, that is

$$y_1(z)=\frac{z}{(1-z)^2}$$