9.5

$$\phi(x,h)=e^{2xh-h^2}=\sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)h^n$$

For LHS we have

$$\frac{\partial\phi}{\partial x}=e^{2xh-h^2}(2h)=2h\phi$$

$$\frac{\partial^2\phi}{\partial x^2}=e^{2xh-h^2}(2h)(2h)=(2h)^2\phi$$

$$\frac{\partial\phi}{\partial h}=e^{2xh-h^2}(2x-2h)=(2x-2h)\phi$$

Substitution gives

$$\frac{\partial^2\phi}{\partial x^2}-2x\frac{\partial\phi}{\partial x}+2h\frac{\partial\phi}{\partial h}=(2h)^2\phi-2x2h\phi+2h(2x-2h)\phi=0$$

Now using RHS

$$\frac{\partial\phi(x,h)}{\partial x}=\sum_{n=0}^{\infty}\frac{1}{n!}H_n'(x)h^n$$

$$\frac{\partial^2\phi(x,h)}{\partial x^2}=\sum_{n=0}^{\infty}\frac{1}{n!}H_n''(x)h^n$$

$$\frac{\partial\phi(x,h)}{\partial h}=\sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)nh^{n-1}$$

Substitution gives

$$\frac{\partial^2\phi}{\partial x^2}-2x\frac{\partial\phi}{\partial x}+2h\frac{\partial\phi}{\partial h}=\sum_{n=0}^{\infty}\frac{1}{n!}H_n''(x)h^n-2x\sum_{n=0}^{\infty}\frac{1}{n!}H_n'(x)h^n+2h\sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)nh^{n-1}$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}H_n''(x)h^n-2x\sum_{n=0}^{\infty}\frac{1}{n!}H_n'(x)h^n+2\sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)nh^n$$

$$=\sum_{n=0}^{\infty}[H_n''(x)-2xH_n'(x)+2nH_n(x)]\frac{1}{n!}h^n=0$$

Since in general for any n, $\frac{1}{n!}h^n\neq 0$

$H_n''(x)-2xH_n'(x)+2nH_n(x)=0$ where $y(x)=H_n(x)$

Start with $\phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) h^n$

Take partial with respect to x on both sides!

$$\frac{\partial \phi}{\partial x} = e^{2xh-h^2}(2h) = 2h\phi = \frac{\partial \phi(x,h)}{\partial x} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n^{'}(x) h^n$$

Therefore

$$2h\phi = \sum_{n=0}^{\infty} \frac{1}{n!} H_n^{'}(x) n h^n$$

Replace ϕ

$$2h\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) h^n = \sum_{n=0}^{\infty} \frac{1}{n!} H_n^{'}(x) h^n$$

Collect 'h's on LHS and reindex LHS $m = n+1$ or $n = m-1$

$$\sum_{m=1}^{\infty} \frac{2}{(m-1)!} H_{m-1}(x) h^m = \sum_{n=0}^{\infty} \frac{1}{n!} H_n^{'}(x) h^n$$

On RHS look at $H_0(x) = 1$ and note $H_0^{'}(x) = 0$

therefore can start index on RHS at n=1, or at this point simply call it m

$$\sum_{m=1}^{\infty} \frac{2}{(m-1)!} H_{m-1}(x) h^m = \sum_{m=1}^{\infty} \frac{1}{m!} H_m^{'}(x) h^m$$

Since same index range and powers of h we can equate terms on both sides

$$\frac{2}{(m-1)!} H_{m-1}(x) = \frac{1}{m!} H_m^{'}(x)$$

Multiple by m!

$$\frac{m!}{m!} H_m^{'}(x) = \frac{2m!}{(m-1)!} H_{m-1}(x)$$

or

$$H_m^{'}(x) = 2mH_{m-1}(x)$$

Same as book with $m = n$

Prove (b) by starting with $H_n^{'}(x) = 2nH_{n-1}(x)$ and taking second derivative and substituting for the first and second derivative into $H_n^{''}(x)h^n - 2xH_n^{'}(x) + 2H_n(x) = 0$ from first part. Finally re-indexing with n = m+1 will give the desired result. Try it!