

Question 1

(a) See figure 1

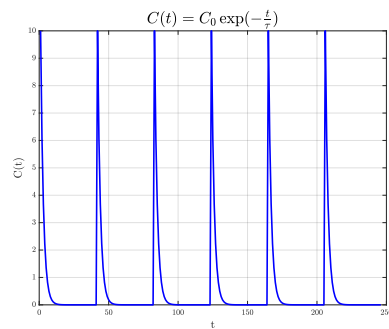


Figure 1

(b) $f(t) = C_0 e^{-\frac{t}{\tau}}$ with period T , so

$$\begin{aligned}
 a_0 &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} dt \\
 &= \frac{2C_0}{T} (-\tau) [e^{-\frac{t}{\tau}}]_0^T \\
 &= -2C_0 \frac{\tau}{T} [e^{-\frac{T}{\tau}} - 1] \\
 &= 2C_0 \frac{\tau}{T} (1 - e^{-\frac{T}{\tau}})
 \end{aligned}$$

If $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_0 \approx 2C_0 \frac{\tau}{T}$.

$$\begin{aligned}
 a_k &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt \\
 &= \frac{2C_0}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt
 \end{aligned}$$

Using integration by parts with $u = \cos \frac{2k\pi t}{T}$, $du = -\frac{2k\pi}{T} \sin \frac{2k\pi t}{T}$ and $dv = e^{-\frac{t}{\tau}}$, $v = (-\tau)e^{-\frac{t}{\tau}}$:

$$\int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt = (-\tau) \left[e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} \right]_0^T - \frac{2k\pi\tau}{T} \int_0^T e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt$$

Using again integration by parts:

$$\int_0^T e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt = (-\tau) \left[e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} \right]_0^T + \frac{2k\pi\tau}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt$$

So

$$\begin{aligned} \left(1 + \left(\frac{2k\pi\tau}{T}\right)^2\right) \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt &= (-\tau) \left[e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} \right]_0^T + \frac{2k\pi\tau^2}{T} \left[e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} \right]_0^T \\ &= (-\tau) \left[e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} \right]_0^T + 0 \\ &= \tau(1 - e^{-\frac{T}{\tau}}) \\ \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt &= \frac{\tau}{1 + \left(\frac{2k\pi\tau}{T}\right)^2} (1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

Substituting back into the expression found for a_k yields

$$\begin{aligned} a_k &= 2C_0 \frac{\tau}{T} \frac{1}{1 + \left(\frac{2k\pi\tau}{T}\right)^2} (1 - e^{-\frac{T}{\tau}}) \\ &= 2C_0 \frac{\tau T}{T^2 + (2k\pi\tau)^2} (1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

With the same assumption $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_k \approx 2C_0 \frac{\tau}{T} \frac{1}{1 + \left(\frac{2k\pi\tau}{T}\right)^2}$. Similarly to compute b_k

$$\begin{aligned} b_k &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T} \int_0^T e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \frac{\tau}{1 + \left(\frac{2k\pi\tau}{T}\right)^2} (1 - e^{-\frac{T}{\tau}}) \\ &= 4C_0 k\pi \frac{\tau^2}{T^2 + (2k\pi\tau)^2} (1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

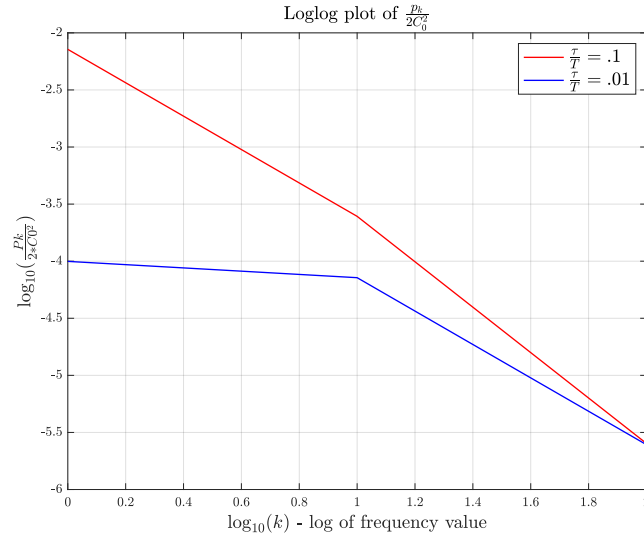
Once again, since $e^{-\frac{T}{\tau}} \approx 0$ then $b_k \approx 4C_0 \left(\frac{\tau}{T}\right)^2 \frac{1}{1 + \left(\frac{2k\pi\tau}{T}\right)^2} \pi k$

(c) For $k \geq 1$

$$\begin{aligned}
p_k &= \frac{1}{2}(a_k^2 + b_k^2) \\
&= \frac{1}{2} \left[4C_0^2 \left(\frac{\tau}{T}\right)^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} + 16C_0^2 \left(\frac{\tau}{T}\right)^4 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \pi^2 k^2 \right] \\
&= \frac{1}{2} 4C_0^2 \left(\frac{\tau}{T}\right)^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4\left(\frac{\tau}{T}\right)^2 \pi^2 k^2 \right] \\
&= 2C_0^2 \left(\frac{\tau}{T}\right)^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4\left(\frac{\tau}{T}\right)^2 \pi^2 k^2 \right]
\end{aligned}$$

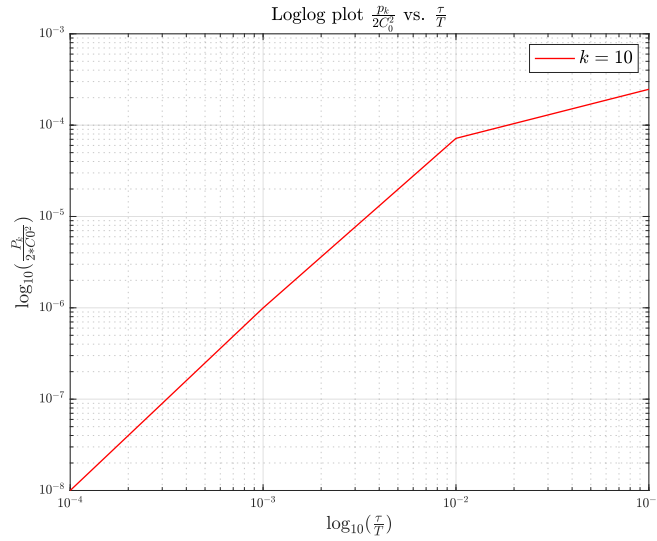
(d) We have

$$\frac{p_k}{2 C_0^2} = \left(\frac{\tau}{T}\right)^2 \frac{1}{(1 + (\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4\left(\frac{\tau}{T}\right)^2 \pi^2 k^2 \right]$$



From the plot we can see that the power $\frac{p_k}{2 C_0^2}$ decreases as the frequency increases. Power is close to 0 starting with a frequency of 10.

- (e) Looking at the plot in part (d), as the pulse τ , becomes narrower, the power decreases linearly. For a greater $\frac{\tau}{T}$, the power starts at a higher value until an inflection point corresponding to frequency of 10 ($1 = \log_{10}(10)$ on the graph). Also for a higher $\frac{\tau}{T}$, the steepest the decrease in power. Eventually the two curves combine in one curve around a frequency of (2 = $\log_{10}(100)$ on the graph). The same observations are obtained with a loglog plot of power $\frac{P_k}{2C_0^2}$ vs. $\frac{\tau}{T}$ for a frequency $k = 10$.



- (f) We have

$$\begin{aligned} a_k \cos\left(\frac{k2\pi t}{T}\right) + b_k \sin\left(\frac{k2\pi t}{T}\right) &= \cos(\phi_k) \cos\left(\frac{k2\pi t}{T}\right) + \sin(\phi_k) \sin\left(\frac{k2\pi t}{T}\right) \\ &= \cos\left(\frac{k2\pi t}{T} - \phi_k\right) \end{aligned}$$

where

$$\begin{aligned}\tan(\phi_k) &= \frac{\sin(\phi_k)}{\cos(\phi_k)} = \frac{b_k}{a_k} = 4C_0\left(\frac{\tau}{T}\right)^2 \frac{1}{1 + \left(\frac{2k\pi\tau}{T}\right)^2} \pi k \left(2C_0\frac{\tau}{T} \frac{1}{1 + \left(\frac{2k\pi\tau}{T}\right)^2}\right)^{-1} \\ &= 2\frac{\tau}{T}\pi k \\ \phi_k &= \arctan\left(2\frac{\tau}{T}\pi k\right)\end{aligned}$$

For $\frac{\tau}{T} = .1$, $\phi_1 \approx 32.14^\circ$ and $\phi_2 \approx 51.48^\circ$ and for $\frac{\tau}{T} = .01$, $\phi_1 \approx 3.59^\circ$ and $\phi_2 \approx 7.16^\circ$

If we imagine a clock with a hand that turns at constant speed, making a full turn every T seconds, and is pointing straight up at time $t = 0$ when the plasma rises to C_0 . The phase ϕ_k is then the angle from the 12 : 00 position to the current position and indicates how far is the next rise in plasma concentration to C_0 . For the same frequency (either $k = 1$ or $k = 2$), with a larger pulse we are closer to the next substance release.

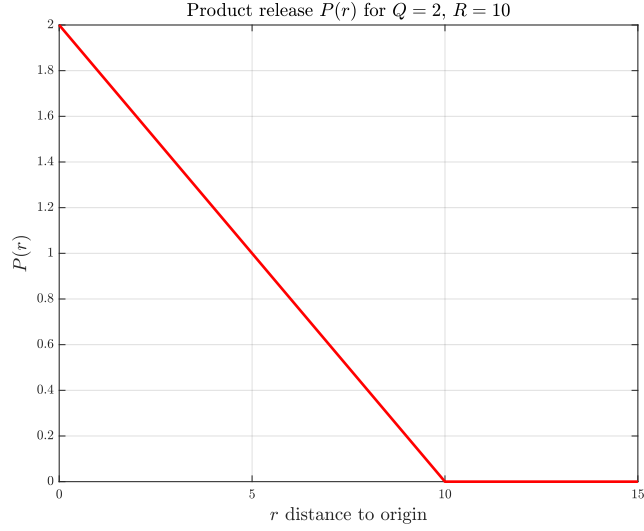
Question 2

(a) One simple way to describe $P(r)$ is to define it as $P(r) = Ar + B$ with the conditions:

$$\begin{aligned}A \cdot 0 + B &= Q \\ A \cdot R + B &= 0\end{aligned}$$

which gives $A = -\frac{Q}{R}$ and $B = Q$. So

$$P(r) = \begin{cases} Q(1 - \frac{r}{R}) & \text{for } 0 \leq r \leq R \\ 0 & \text{for } r > R \end{cases}$$



(b) Since we assume no angular dependence: $\nabla^2 C = \frac{1}{r^2} \frac{d}{dr} (r^2 \frac{dC}{dr})$, the differential equation is now:

$$\begin{aligned} \frac{D}{r^2} \frac{d}{dr} (r^2 \frac{dC(r)}{dr}) + P(r) &= 0 \\ \frac{d}{dr} (r^2 \frac{dC(r)}{dr}) &= -\frac{r^2}{D} P(r) \end{aligned}$$

(c) Inside the cell $P(r) = Q(1 - \frac{r}{R})$, so we have to solve the differential equation

$$\begin{aligned} \frac{d}{dr} (r^2 \frac{dC(r)}{dr}) &= -\frac{r^2}{D} Q(1 - \frac{r}{R}) \\ &= \frac{Q}{DR} r^2 (r - R) \\ &= \frac{Q}{DR} r^3 - \frac{Q}{D} r^2 \end{aligned}$$

Integrating once

$$r^2 \frac{dC(r)}{dr} = \frac{Q}{4DR} r^4 - \frac{Q}{3D} r^3 + A$$

$$\frac{dC(r)}{dr} = \frac{Q}{4DR} r^2 - \frac{Q}{3D} r + \frac{A}{r^2}$$

Integrating again

$$C_i(r) = \frac{Q}{12DR} r^3 - \frac{Q}{6D} r^2 - \frac{A}{r} + B \quad A, B: \text{constants}, C_i: \text{inside cell concentration}$$

Outside the cell $P(r) = 0$ and we want to solve the differential equation

$$\frac{d}{dr} \left(r^2 \frac{dC(r)}{dr} \right) = 0$$

Which by integration gives

$$r^2 \frac{dC(r)}{dr} = C_1$$

$$\frac{dC(r)}{dr} = \frac{C_1}{r^2}$$

$$C_o(r) = -\frac{C_1}{r} + C_2 \quad C_1, C_2: \text{constants}, C_o: \text{outside cell concentration}$$

(d) Applying the boundary conditions

(i)

$$\lim_{r \rightarrow 0} C_i(r) = \lim_{r \rightarrow 0} \left(\frac{Q}{12DR} r^3 - \frac{Q}{6D} r^2 - \frac{A}{r} + B \right)$$

since $\lim_{r \rightarrow 0} C_i(r) = \lim_{r \rightarrow 0} \left(-\frac{1}{r} + B \right) = \infty$ therefore to have finite concentration $C_i(r)$ at $r = 0$ we need $A = 0$

(ii)

$$\lim_{r \rightarrow \infty} C_o(r) = \lim_{r \rightarrow \infty} \left(-\frac{C_1}{r} + C_2 \right) = C_2$$

The concentration goes to zero at infinity implies $C_2 = 0$

(iii) We have now for $C_i(r)$ and $C_o(r)$:

$$C_i(r) = \frac{Q}{12DR} r^3 - \frac{Q}{6D} r^2 + B$$

$$C_o(r) = -\frac{C_1}{r}$$

$C_i(R) = C_o(R)$ and $\frac{dC_i(r)}{dr} = \frac{dC_o(r)}{dr} \big|_{r=R}$ yields

$$\frac{Q}{12DR} R^3 - \frac{Q}{6D} R^2 + B = -\frac{C_1}{R}$$

$$\frac{Q}{4D} R - \frac{Q}{3D} R = \frac{C_1}{R^2}$$

Rearranging

$$-\frac{Q}{12D}R^2 + B = -\frac{C_1}{R}$$

$$-\frac{Q}{12D}R = \frac{C_1}{R^2}$$

which gives

$$B = \frac{Q}{6D}R^2$$

$$C_1 = -\frac{Q}{12D}R^3$$

substituting back

$$C_i(r) = \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 + \frac{Q}{6D}R^2$$

$$= \frac{Q}{6D} \left[\frac{r^3}{2R} - r^2 + R^2 \right]$$

$$C_o(r) = \frac{Q}{12D}R^3 \frac{1}{r}$$

(e) We are looking for the value of r for which $\frac{dC_i(r)}{dr} = 0$:

$$\frac{dC_i(r)}{dr} = \frac{Q}{4DR}r^2 - \frac{Q}{3D}r = \frac{Q}{D}r \left(\frac{r}{4R} - \frac{1}{3} \right)$$

Discarding the solution $r = 0$ we are left that concentration maximum is for $r = \frac{4}{3}R$ and it is

$$C_M = \frac{Q}{12DR} \left(\frac{4}{3} \right)^3 R^3 - \frac{Q}{6D} \left(\frac{4}{3} \right)^2 R^2 + \frac{Q}{6D} R^2$$

$$= \frac{Q}{6D} R^2 \left[\frac{4^3}{2 \cdot 3^3} - \frac{4^2}{3^2} + 1 \right]$$

$$= \frac{11}{162} \frac{Q}{D} R^2$$

Inside the cell

$$C_i(r) = \frac{Q}{6D} \left(\frac{1}{2} \frac{r^3}{R} - r^2 + R^2 \right)$$

$$\frac{C_i(r)}{C_M} = \frac{Q}{6D} \frac{162}{11} \frac{D}{Q} R^{-2} \left(\frac{1}{2} \frac{r^3}{R} - r^2 + R^2 \right)$$

$$= \frac{162}{6 \cdot 11} \left[\frac{1}{2} \left(\frac{r}{R} \right)^3 - \left(\frac{r}{R} \right)^2 + 1 \right]$$

$$= \frac{27}{11} \left[\frac{1}{2} \left(\frac{r}{R} \right)^3 - \left(\frac{r}{R} \right)^2 + 1 \right]$$

And outside the cell

$$C_o(r) = \frac{Q}{12D} R^3 \frac{1}{r}$$

$$\frac{C_o(r)}{C_M} = \frac{Q}{12D} \frac{162}{11} \frac{D}{Q} R^{-2} R^3 \frac{1}{r}$$

$$= \frac{27}{22} \frac{R}{r}$$

When the diffusion constant is doubled, the curve $\frac{C_i(r)}{C_M}$ stays the same.

