Yves GREATTI

Chapter 8 - Problem 8.2

Write the homogeneous Sturm-Liouville eigenvalue equation for which $y(a)=y(b)=0$ as $\mathcal{L}(y;\lambda)=(py')'+qy+\lambda\rho y=0$ where $p(x)$, $q(x)$ and $\rho(x)$ are continuously differentiable functions.

The general form of the Sturm-Liouville equation is:

$$p(x)\frac{d^2y}{dx^2}+r(x)\frac{dy}{dx}+q(x)y+\lambda\rho(x)y=0$$

$$\text{where } r(x)=\frac{dp(x)}{dx}$$

rewrite it:

$$p\,y''(x)+p'y'+qy+\lambda\rho y=0$$

$$\Rightarrow (py')'+qy+\lambda\rho y=0$$

or $\mathcal{L}(y;\lambda)=0$ where $\mathcal{L}(y;\lambda)=(py')'+qy+\lambda\rho y$

Show that if $z(x)$ and $F(x)$ satisfy $\mathcal{L}(z;\lambda)=F(x)$ with $z(a)=z(b)=0$ then

$$\int_a^b y(x)\,F(x)\,dx=0$$

Substitute $F(x)=\mathcal{L}(z;\lambda)=(pz')'+qz+\lambda\rho z$ into $\int_a^b y\,F(x)\,dx$

$$\int_a^b y\,F(x)\,dx=\int_a^b y\left[(pz')'+qz+\lambda\rho z\right]dx$$

Chapter 8 - Problem 8.2

Split the integral in two parts:

$$\int_a^b y\,(pz')'\,dx + \int_a^b [qy + \lambda\rho y]\,z\,dx$$

Integration by parts yields for the first term:

$$\int_a^b y\,(pz')'\,dx = [\,y\,(pz')]_a^b - \int_a^b y' p z'\,dx$$

$$= y(b)\,(pz')(b) - y(a)\,(pz')(a) - \int_a^b y' p z'\,dx$$

$$= 0\cdot(pz')(b) - 0\cdot(pz')\,(a) - \int_a^b y' p z'\,dx$$

$$= -\int_a^b y' p z'\,dx$$

$$\int_a^b y' p z'\,dx = [(py')\,z]_a^b - \int_a^b (py')'\,z\,dx$$

$$= (py')(b)\,z(b) - (py')(a)\,z(a) - \int_a^b (py')'\,z\,dx$$

$$= (py')(b)\cdot 0 - (py')\,(a)\cdot 0 - \int_a^b (py')'\,z\,dx$$

$$= -\int_a^b (py')'\,z\,dx$$

then $\int_a^b y\,(pz')'\,dx = \int_a^b (py')'\,z\,dx$

Therefore $\int_a^b y\,F(x)\,dx = \int_a^b (py')'\,z\,dx + \int_a^b [qy + \lambda\rho y]\,z\,dx$

$$= \int_a^b [(py')' + qy + \lambda\rho y]\,z\,dx$$

Chapter. 8 - Problem 8. 2

Since y is a solution $L(y;\lambda) = (py')' + qy + \lambda\rho y = 0$

Therefore $\int_a^b y\, F(x)\, dx = \int_a^b 0 . z\, dx = 0$

Demonstrate the validity of this general result by direct calculation for the specific case in which $p(x) = \rho(x) = 1$, $q(x) = 0$, $a = -1$, $b = 1$ and $z(x) = 1 - x^2$.

With $p(x) = \rho(x) = 1$, $q(x) = 0$, the Sturm-Liouville equation becomes: $y'' + \lambda y = 0$

The characteristic equation is: $\alpha^2 + \lambda = 0$ which has for roots $\alpha = \pm i\sqrt{\lambda}$ and the solution $y(x)$ of this equation is:

$$y(x) = A \cos\sqrt{\lambda}\, x + B \sin\sqrt{\lambda}\, x$$

Given the conditions $y(-1) = y(1) = 0$ yields:

$$\begin{cases} A \cos\sqrt{\lambda} + B \sin\sqrt{\lambda} = 0 \\ A \cos(-\sqrt{\lambda}) + B \sin -\sqrt{\lambda} = 0 \end{cases}$$

or

$$\begin{cases} A \cos(\sqrt{\lambda}) + B \sin\sqrt{\lambda} = 0 & (1) \\ A \cos(\sqrt{\lambda}) - B \sin\sqrt{\lambda} = 0 & (2) \end{cases}$$

YG

Chapter 8 - Problem 8.2

Add (1) and (2) together gives: $2A\cos\sqrt{\lambda} = 0$

If $A = 0$ then $y(x) = B\sin\sqrt{\lambda}\, x$

Now $L(z;\lambda) = (z')' + \lambda z = z'' + \lambda z$

$= -2 + \lambda(1-x^2) = -\lambda x^2 + \lambda - 2$

Next $\int_{-1}^{1} y(x)\,F(x)\,dx = \int_{-1}^{1} y(x)\,L(z;\lambda)\,dx$

$= \int_{-1}^{1} y(x)\,(-\lambda x^2 + \lambda - 2)\,dx$

$= \int_{-1}^{1} B\sin(\sqrt{\lambda}x)\,(-\lambda x^2 + \lambda - 2)\,dx$

$= B\left[\int_{-1}^{1} (-\lambda x^2)\sin(\sqrt{\lambda}x) + \int_{-1}^{1} (\lambda - 2)\sin(\sqrt{\lambda}x)\,dx\right]$

$= B\left[(-\lambda)\int_{-1}^{1} \sin(\sqrt{\lambda}x)\,x^2\,dx + (\lambda-2)\int_{-1}^{1} \sin\sqrt{\lambda}x\,dx\right]$

$= B\left[(-\lambda)\cdot 0 + (\lambda - 2)\cdot 0\right]$

$= 0$

Since the functions $\sin\sqrt{\lambda}x\, x^2$ and $\sin\sqrt{\lambda}\,x$ are odd.

YG

Chapter 8 - Problem 8.2

From $2A\cos(\sqrt{\lambda})=0$ if A is non-zero then $\sqrt{\lambda}=\frac{\pi}{2}+k\pi$

with $k=0,\pm 1,\pm 2,\dots$

And thus $y(x)=A\cos\sqrt{\lambda}x$ with $\lambda=\left(\frac{2k+1}{2}\pi\right)^2$

We have to evaluate then:

$$\int_{-1}^{1}y(x)F(x)\,dx=\int_{-1}^{1}A\cos(\sqrt{\lambda}x)\left(-\lambda x^2+(\lambda-2)\right)dx$$
$$=A\left[(-\lambda)\int_{-1}^{1}\cos(\sqrt{\lambda}x)x^2\,dx+(\lambda-2)\int_{-1}^{1}\cos\sqrt{\lambda}x\,dx\right]$$

$$\int_{-1}^{1}\cos(\sqrt{\lambda}x)\,dx=\frac{1}{\sqrt{\lambda}}\left[\sin\sqrt{\lambda}x\right]_{-1}^{1}=\frac{2}{\sqrt{\lambda}}\sin\sqrt{\lambda}$$
$$=\frac{2}{\sqrt{\lambda}}\sin\left(\frac{\pi}{2}+k\pi\right)$$
$$=\frac{2\cdot(-1)^k}{\sqrt{\lambda}}$$

$$\int_{-1}^{1}\cos(\sqrt{\lambda}x)x^2\,dx=\frac{1}{\sqrt{\lambda}}\left[\sin(\sqrt{\lambda}x)x^2\right]_{-1}^{1}-\frac{1}{\sqrt{\lambda}}\int_{-1}^{1}\sin(\sqrt{\lambda}x)(2x)\,dx$$

$$\frac{1}{\sqrt{\lambda}}\left[\sin(\sqrt{\lambda}x)x^2\right]_{-1}^{1}=\frac{1}{\sqrt{\lambda}}\left(2\sin\sqrt{\lambda}\right)=\frac{2\cdot(-1)^k}{\sqrt{\lambda}}$$

$$\int_{-1}^{1}\sin(\sqrt{\lambda}x)\,dx=\frac{1}{\sqrt{\lambda}}\left[\left(-\cos(\sqrt{\lambda}x)\right)\cdot x\right]_{-1}^{1}+\frac{1}{\sqrt{\lambda}}\int_{-1}^{1}\cos(\sqrt{\lambda}x)\,dx$$

Chapter 8 - Problem 8.2

$$\int_{-1}^{1} \sin(\sqrt{\lambda}x)x\,dx = 0 + \frac{1}{\sqrt{\lambda}} \int_{-1}^{1} \cos(\sqrt{\lambda}x)\,dx$$

$$= \frac{1}{(\sqrt{\lambda})^2} \left[\sin(\sqrt{\lambda}x)\right]_{-1}^{1} = \frac{2}{\lambda}\cdot(-1)^k$$

Thus $\int \cos(\sqrt{\lambda}x)\, x^2 dx = \frac{2(-1)^k}{\sqrt{\lambda}} - \frac{4}{\lambda\sqrt{\lambda}}\cdot(-1)^k$

And

$$\int_{-1}^{1} y(x)\,F(x)\,dx = A\left[(-\lambda)\left(\frac{2\cdot(-1)^k}{\sqrt{\lambda}} - \frac{4}{\lambda\sqrt{\lambda}}(-1)^k\right) + (\lambda-2)\cdot\frac{2\cdot(-1)^k}{\sqrt{\lambda}}\right]$$

$$= A\left[-2\frac{\lambda}{\sqrt{\lambda}}(-1)^k + 2\frac{\lambda}{\sqrt{\lambda}}(-1)^k + \frac{4}{\sqrt{\lambda}}(-1)^k - \frac{4\cdot(-1)^k}{\sqrt{\lambda}}\right]$$

$$= A\cdot 0 = 0$$

Chapter 8 - Problem 8.5

Use the Properties of Legendre Polynomials to solve the following problems.

(a) Find the solutions of $(1-x^2)y''-2xy'+by=f(x)$, valid in the range $1 \leq x \leq -1$ and finite at $x=0$ in terms of Legendre Polynomials.

Assuming the solution is of the form $y(x)=\sum_{l=0}^{\infty} a_l P_l(x)$

it verifies the equation:

$$(1-x^2)\sum_{l=0}^{\infty} a_l P_l''(x) - 2x\sum_{l=0}^{\infty} a_l P_l'(x) + b\sum_{l=0}^{\infty} a_l P_l(x) = f(x)$$

Grouping terms together for same coefficient a_l gives:

$$\sum_{l=0}^{\infty} a_l \left[(1-x^2)P_l''(x) - 2x P_l'(x) + b P_l(x)\right] = f(x)$$

Expanding LHS:

$$\sum_{l=0}^{\infty} a_l \left[\left((1-x^2)P_l''(x) - 2x P_l'(x) + l(l+1)P_l(x)\right) - l(l+1)P_l(x) + bP_l(x)\right] = f(x)$$

The first three terms together is the Legendre's differential equation for the Legendre Polynomials and it is zero.

Chapter 8 - Problem 8.5

Therefore we are left with:

$$\sum_{l=0}^{\infty} a_l \left[b - l(l+1)\right] P_l(x) = f(x)$$

Multiplying through by $P_k(x)$ and integrating from -1 to 1 yields:

$$\sum_{l=0}^{\infty} a_l \left[b - l(l+1)\right] \int_{-1}^{1} P_k(x) P_l(x)\,dx = \int_{-1}^{1} P_k(x) f(x)\,dx$$

Using orthogolization property and the normalization property:

$$\int_{-1}^{1} P_k(x) P_l(x) = \begin{cases} 0 & k \neq l \\ \dfrac{2}{2k+1} & k = l \end{cases}$$

we obtain : $a_k \left(b - k(k+1)\right) = \left(\int_{-1}^{1} f(x) P_k(x)\,dx\right) \times \dfrac{2k+1}{2}$

or $a_k = \dfrac{2k+1}{2\left(b - k(k+1)\right)} \displaystyle\int_{-1}^{1} f(x) P_k(x)\,dx$

$$a_k = \frac{k + 1/2}{b - k(k+1)} \int_{-1}^{1} f(x) P_k(x)\,dx$$

Chapter 8 - Problem 8.5

(b) If $b=14$ and $f(x)=5x^3$, find the explicit solution and verify it by direct substitution.

Using Wolfram Alpha web page about Legendre Polynomial,

$5x^3 = 3P_1(x) + 2P_3(x)$.

So the solution $y(x)$ is a linear combination of $P_1(x)$ and $P_3(x)$.

Note first that $\int_{-1}^{1} x^n \, dx = 0$ for any odd power of x.

Using the relation established in part (a), we need to compute only a_1 and a_3 since $f(x)$ is only a linear combination of those Legendre polynomials.

Therefore we compute:

$$a_1 = \frac{2\times 1+1}{2(14-1\cdot(1+1))} \int_{-1}^{1} 5x^3 \cdot P_1(x) \, dx$$

$$a_3 = \frac{2\times 3+1}{2(14-3(3+1))} \int_{-1}^{1} 5x^3 P_3(x) \, dx$$

Chapter 8 - Problem 8-5

$$a_1 = \frac{2\times 1+1}{2(14-2)} \times 5 \int_{-1}^{1} x^3 \cdot x \, dx$$

$$a_1 = \frac{3}{2(14-2)} \times 5 \times \frac{2}{5} = 3/12 = 1/4$$

$$a_3 = \frac{2\times 3+1}{2(14-12)} \times 5 \int_{-1}^{1} x^3 \cdot 1/2 \, (5x^3 - 3x) \, dx$$

$$= \frac{6+1}{2\times 2} \times \frac{5}{2} \times \left[\frac{5x^7}{7} - \frac{3}{5}x^5\right]_{-1}^{1} = \frac{7\times 5}{8} \times \frac{8}{35} = 1$$

The solution is $y(x) = \frac{P_1(x)}{4} + P_3(x)$

$$= x/4 + 1/2\,(5x^3 - 3x)$$

$$= 5/2\, x^3 - 5/4\, x$$

Substitute $y(x)$ in the initial equation:

$$(1-x^2)\,y'' - 2x\,y' + 14\,y = (1-x^2)\,15x - 2x\,(15/2\,x^2 - 5/4) + 14\,(5/2\,x^3 - 5/4\,x)$$

Adding together coefficients for same power of x yields

$$(15 - 15 + 35)\,x^3 + (15 + 5/2 - 35/2)\,x = 5x^3$$

Chapter 8 - Problem 8.6

Starting from the linearly independent functions $1, x, x^2, x^3 \dots$, in the range $0 \leq x < \infty$, find the first three orthogonal functions ϕ_0, ϕ_1 and ϕ_2, w.r.t the weight function $\rho(x) = e^{-x}$. By comparing your answers with the Laguerre polynomials generated by the recurrence relation (9-112), deduce the form of $\phi_3(x)$.

Starting with $\phi_0 = 1$

$$\langle \phi_0 | \phi_0 \rangle = \int_0^\infty 1 \times 1 \times e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1$$

$$\Rightarrow \langle \phi_0 | \phi_0 \rangle^{1/2} = 1 \text{ and } \hat{\phi}_0 = 1/1 = 1$$

Following now Gram-Schmidt procedure:

$$\phi_1 = y_1 - \hat{\phi}_0 \langle \hat{\phi}_0 | y_1 \rangle$$

$$\langle \hat{\phi}_0, y_1 \rangle = \int_0^\infty 1 \cdot x \cdot e^{-x} dx = \left[x(-e^{-x})\right]_0^\infty - \int_0^\infty (-e^{-x}) dx$$

$$= 0 + \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1$$

Thus $\phi_1 = x - 1 \cdot 1 = x - 1$

$$\langle \phi_1, \phi_1 \rangle = \int_0^\infty (x-1)^2 \cdot e^{-x} dx$$

YG

Chapter 8 - Problem 8.6

$$\int_0^\infty (x-1)^2 e^{-x} dx = \int_{-1}^\infty u^2 e^{-(u+1)} du$$ by change of variable $u = x-1$, $du = dx$

$$= \left[u^2 (-e^{-(u+1)}) \right]_{-1}^\infty + 2\int_{-1}^\infty u \cdot e^{-(u+1)} du$$

$$= 1 + 2\int_{-1}^\infty u \cdot e^{-(u+1)} du$$

$$\int_{-1}^\infty u \cdot e^{-(u+1)} du = \left[u(-e^{-(u+1)}) \right]_{-1}^\infty + \int_{-1}^\infty e^{-(u+1)} du$$

$$= -1 + \left[-e^{-(u+1)} \right]_{-1}^\infty = -1 + 1 = 0$$

thus $\int_0^\infty (x-1)^2 e^{-x} dx = 1$

And $\hat{\phi}_1 = \dfrac{\phi_1}{\langle \phi_1 | \phi_1 \rangle^{1/2}} = \dfrac{x-1}{1} = x-1$

$$\phi_2 = y_2 - \hat{\phi}_1 \langle \hat{\phi}_1, y_2 \rangle - \hat{\phi}_0 \langle \hat{\phi}_0 | y_2 \rangle$$

$$\langle \hat{\phi}_0 | y_2 \rangle = \int_0^\infty 1 \cdot x^2 e^{-x} dx = \int_0^\infty x^2 e^{-x} dx = 0 + 2\int_0^\infty x e^{-x} dx$$

by integration by parts

$$= 2 \times \int_0^\infty x e^{-x} dx = 2 \times 1 = 2$$

$$\langle \hat{\phi}_1 | y_2 \rangle = \int_0^\infty (x-1) x^2 e^{-x} dx = \int_0^\infty x^3 e^{-x} dx - \int_0^\infty x^2 e^{-x} dx$$

Using the expression $\dfrac{1}{n!} \int_0^\infty x^n e^{-x} dx = 1$

Chapter 8 - Problem 8.6

$\int_0^\infty x^2 e^{-x} dx = 2! = 2$ $\qquad$ $\int_0^\infty x^3 e^{-x} dx = 3! = 6$

Substituting into $\langle \hat{\phi}_1 | y_2 \rangle = 6 - 2 = 4$

thus $\phi_2 = x^2 - (x-1) \cdot 4 - 1 \cdot 2 = x^2 - 4x + 4 - 2$
$= x^2 - 4x + 2$

$$\langle \phi_2, \phi_2 \rangle = \int_0^\infty (x^2 - 4x + 2)^2 e^{-x} dx$$
$$= \int_0^\infty (x^4 - 8x^3 + 20x^2 - 16x + 4) e^{-x} dx.$$

$\int_0^\infty x^4 e^{-x} dx = 4! = 24$ $\quad$ $\int_0^\infty x^3 e^{-x} dx = 3! = 6$ $\quad$ $\int_0^\infty x^2 e^{-x} dx = 2! = 2$

$\int_0^\infty 1 \cdot e^{-x} dx = 1! = 1$

$\Rightarrow \langle \phi_2 | \phi_2 \rangle = 24 - 8 \times 6 + 20 \times 2 - 16 \times 1 + 4 = 4$

And $\hat{\phi}_2 = \dfrac{\phi_2}{\langle \phi_2 | \phi_2 \rangle^{1/2}} = 1/2 \, (x^2 - 4x + 2)$

The recurrence relation of the Laguerre polynomials is:

$$(n+1) L_{n+1}(x) = (2n+1-x) L_n(x) - n L_{n-1}(x)$$

The first few Laguerre polynomials are given in page 378.

$L_0(x) = 1 = \phi_0(x)$

$L_1(x) = -x + 1 = -\phi_1(x)$

$L_2(x) = 1/2 \, [x^2 - 4x + 2] = \phi_2(x)$

YG

Chapter 8 - Problem 8.6

We see that the first three ϕ_i $_{i=0,1,2}$ polynomials are the same as L_i $_{i=0,1,2}$ but alternating in sign, therefore:

$$\phi_3(x) = -L_3(x) = -1/6\,[-x^3+9x^2-18x+6]$$
$$= 1/6\,[x^3-9x^2+18x-6]$$
$$= \frac{x^3}{6} - \frac{3}{2}x^2 + 3x - 1$$