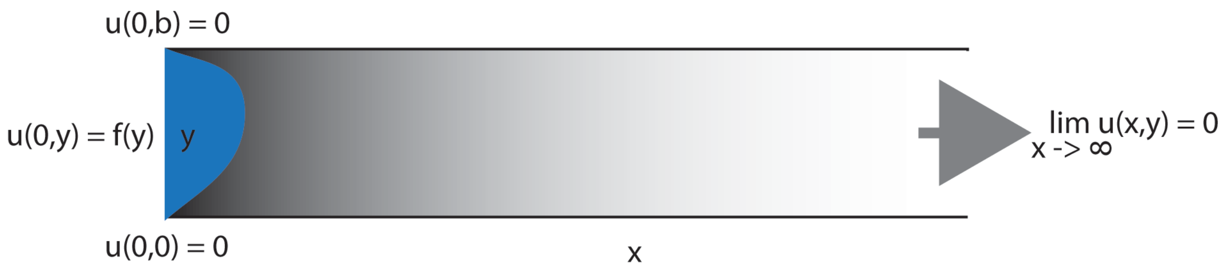
Johns Hopkins
Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
EN. 585.409

JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

Example of diffusion in a semi-infinite plate

Here is a semi-infinite plate. Given for example a constant heat source at the x = 0 boundary we are looking for a "steady" state solution!



The diffusion equation in two spatial dimensions is

$$K\nabla^2 u(x,y,t) = \frac{\partial}{\partial t} u(x,y,t)$$

For steady state assumption $u(x,y,t) \rightarrow u(x,y)$, therefore $\frac{\partial}{\partial t} u(x,y) = 0$

and with no time dependence we can write $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$

We therefore require FOUR boundary conditions, they are given as

$u(0,y) = f(y)$, $u(x,0)=0$, $u(x,b)=0$ and $\lim_{x\to\infty} u(x,y)=0$

Solution:

We will use the previous solution(same form different constant letters)

$$u(x,y) = X(x)Y(y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos\lambda y + D\sin\lambda y)$$

However we need to now take it a little further by incorporating the boundary conditions

Since $\lim_{x\to\infty} u(x,y)=0$ we need to take $A = 0$.

We are left with $u(x,y) = Be^{-\lambda x}(C\cos\lambda y + D\sin\lambda y)$

Next for u(x,0)=0 we have $u(x,0) = Be^{-\lambda x}(C\cos(\lambda \cdot 0) + D\sin(\lambda \cdot 0)) = Be^{-\lambda x}(C+0) = 0$

Therefore take C = 0 (why not B = 0) and we are left with

$$\boxed{u(x,y) = Be^{-\lambda x} D\sin\lambda y}$$

Next incorporate the boundary codition $u(x,b)=0$

$u(x,b) = Be^{-\lambda x}\ D\sin\lambda b = 0$ therefore we must take $\sin\lambda b = 0$

This occurs when $\lambda b = n\pi$, that is $\lambda = \dfrac{n\pi}{b}$

So we have $\boxed{u_n(x,y) = B_n e^{-\lambda x}\ \sin\left(\dfrac{n\pi}{b}\right)y}$

KEY: Notice that the solution is a function not only of x and y but also indexed by n

where we have combined the constants B and D into B_n and indicated that it is dependent on n!

MAIN KEY 1: Use the superposition property to get the entire solution!

So we have $u(x,y)=\sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} B_n e^{-\lambda x}\ \sin\left(\dfrac{n\pi}{b}\right)y$ **KEY:** Why doesn't the index start with n =0?

Apply the final boundary condition

$$\sum_{n=1}^{\infty} B_n e^{-\lambda 0}\ \sin\left(\frac{n\pi}{b}\right)y = f(0,y) \text{ or } f(0,y) = \sum_{n=1}^{\infty} B_n\ \sin\left(\frac{n\pi}{b}\right)y$$

MAIN KEY 2: THIS IS A FOURIER SERIES!

Therefore we can solve for the coefficients by $\boxed{B_n = \dfrac{2}{b}\int_0^b f(0,y)\sin\left(\dfrac{n\pi}{b}\right)y\,dy}$

For $f(0,y)=\mu_0$

$$B_n=\frac{2}{b}\int_0^b \mu_0 \sin\left(\frac{n\pi}{b}\right)y\,dy=\frac{-2\mu_0}{b}\frac{1}{\left(\frac{n\pi}{b}\right)}\cos\left(\frac{n\pi}{b}\right)y\Bigg|_0^b=$$

$$\frac{-2\mu_0}{b}\frac{b}{n\pi}\cos\left(\frac{n\pi}{b}\right)y\Bigg|_0^b=\frac{-2\mu_0}{n\pi}[\cos\left(\frac{n\pi}{b}\right)b-\cos\left(\frac{n\pi}{b}\right)0]=$$

Therefore

$$B_n=\frac{-2\mu_0}{n\pi}[\cos n\pi-\cos 0]=\frac{-2\mu_0}{n\pi}[(-1)^n-1]=\frac{2\mu_0}{n\pi}[1-(-1)^n]$$

This is equivalent to

$$B_n=\begin{cases}\dfrac{4\mu_0}{n\pi} & n \text{ odd}\\[2ex] 0 & n \text{ even}\end{cases}$$

Finally $\boxed{u(x,y)=\sum_{n,\,odd}^{\infty}\frac{4\mu_0}{n\pi}e^{-\lambda x}\sin\left(\frac{n\pi}{b}\right)y}$ is the solution!

Note $\lim_{x\to\infty}u(x,y)=0$ and at $x=0$ $u(0,y)=\sum_{n,\,odd}^{\infty}\frac{4\mu_0}{n\pi}\sin\left(\frac{n\pi}{b}\right)y=\mu_0$

Example of diffusion in a finite plate

Here is a finite plate. Given for example a constant heat source at the x = 0 boundary we are again looking for a "steady" state solution!

Again for steady state we write $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$

u(0,y) =f(y), u(x,0)=0, u(x,b)=0 and u(a,y)=0

Solution:

The general form of the solution needs to be slightly different to incorporate the finite boundary condition at $x = a$

KEY $u(x,y) = X(x)Y(y) = (Ae^{\lambda(x-a)} + Be^{-\lambda(x-a)})(C\cos\lambda y + D\sin\lambda y)$

Therefore the X(x) function will no longer decays to 0 as $x \to \infty$

KEY We do not have to modify the function Y(y) since the trigonometric functions can be adjusted to go to zero at the boundaries y = 0 and b.

Next apply the boundary conditions in X and Y seperately:

$u(x,0) = X(x)Y(0) = 0$ (X(x) cannot be zero in general) $\to \boxed{Y(0) = 0}$

Similar for boundaries at y = b, tha is $\boxed{Y(b) = 0}$ and at x = a, $\boxed{X(a)=0}$

Applying the boundary conditions in y first:

$$Y(0) = C\cos(\lambda\cdot 0) + D\sin(\lambda\cdot 0) = C = 0$$

Therefore $Y(y)=D\sin\lambda y$

Then $Y(b) = D\sin(\lambda\cdot b) = 0$, ($D \neq 0$, otherwise zero solution) $\rightarrow$

$$\sin(\lambda\cdot b) = 0 \rightarrow \lambda b = n\pi \rightarrow \lambda_n = \frac{n\pi}{b}$$

$$Y(y)=D_n \sin\lambda_n y \equiv D_n \sin\frac{n\pi}{b}y$$

Applying the boundary condition for x = a - implicit subscript for A, B and X(x)

$$X(a) = A_n e^{\lambda_n (a-a)} + B_n e^{-\lambda_n (a-a)} = A_n e^{0} + B_n e^{-0} = A_n + B_n = 0 \rightarrow B_n = -A_n$$

$$\text{Therefore } X(x) = A_n e^{\lambda_n (x-a)} - A_n e^{-\lambda_n (x-a)} = A_n\left[e^{\lambda_n (x-a)} - e^{-\lambda_n (x-a)}\right]$$

Using definition for hyperbolic sine function $\sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2}$

$$X(x) = 2A_n \sinh[\lambda_n (a-x)] \rightarrow A_n \sinh[\lambda_n (a-x)] \text{ (absorb 2 into } A_n)$$

Therefore we have (without A_n, B_n, D_n) $u_n(x,y) = X(x)Y(y) = \sinh[\lambda_n (a-x)]\sin\dfrac{n\pi}{b}y$

and $\boxed{u(x,y)=\sum_{n=1}^{\infty} B_n \sinh[\lambda_n (a-x)]\sin\frac{n\pi}{b}y}$ (absorbing all constants into B_n, eg. $A_n B_n D_n \rightarrow B_n$)

Finally apply the final boundary condition at x = 0

$$u(0,y)=\sum_{n=1}^{\infty} B_n \sinh[\lambda_n (a-0)]\sin\frac{n\pi}{b}y = \sum_{n=1}^{\infty} B_n \sinh(\lambda_n a)\sin\frac{n\pi}{b}y = f(y)$$

Note this a Fourier series with coefficients $B_n \sinh(\lambda_n a)$

Therefore $B_n \sinh(\lambda_n a) = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi}{b}y\,dy$ or

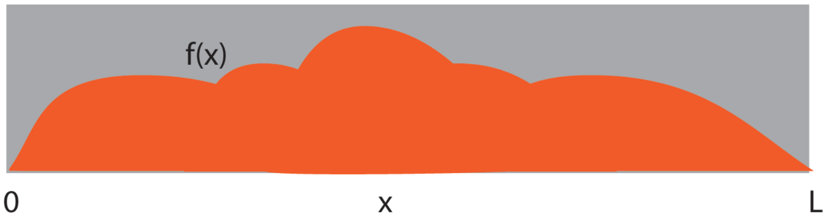
$$B_n = \frac{2}{b\sinh(\lambda_n a)}\int_0^b f(y)\sin\frac{n\pi}{b}y\,dy = \frac{2}{b\sinh(n\pi\frac{a}{b})}\int_0^b f(y)\sin\frac{n\pi}{b}y\,dy$$

Therefore the solution is

$$\boxed{u(x,y)=\sum_{n=1}^{\infty} B_n \sinh[\lambda_n (a-x)]\sin\frac{n\pi}{b}y,\; B_n = \frac{2}{b\sinh(n\pi\frac{a}{b})}\int_0^b f(y)\sin\frac{n\pi}{b}y\,dy}$$

Example of diffusion in one spatial dimension

Here is a finite rod. We are given, for example a heat profile, f(x) at t = 0 along this rod. This is called an initial condition (temporal). There is no constant heat source so we are looking at how this profile evolves in time!



The diffusion equation in one spatial dimensions is

$$\boxed{K\nabla^2 u(x,t) = \frac{\partial}{\partial t} u(x,t)}$$

Similar to previously derivations we get (can you do it?)

$$u(x,t) = X(x)T(t) = (A\cos\lambda x + B\sin\lambda x)Ce^{-\lambda^2 t}$$

We therefore require two boundary conditions, they are given as $u(0,t)=0$, $u(L,t)=0$ and $u(x,0) = f(x)$ is an initial condition required for the temporal derivative.

Solution of this diffusion equation

KEY

Let's look at the one spatial dimension diffusion equation.
In this case we have not included a production term, therefore as $t \to \infty$ we expect $u(x,t) \to 0$

Using are general solution for this situation.

$$u(x,t) = X(x)T(t) = (A\cos\lambda x + B\sin\lambda x)Ce^{-\lambda^2 t}$$

Let's apply the particular boundary and initial conditions.

Applying the boundary conditions gives

$u(0,t)=0 \;\to\; u(0,t) = X(0)T(t) = 0 \;\to X(0) = 0$

$u(L,t)=0 \;\to\; u(L,t) = X(L)T(t) = 0 \;\to X(L) = 0$

Therefore as usual

$X(0) = A\cos\lambda 0 + B\sin\lambda 0 = A = 0$

$X(L) = B\sin\lambda L = 0$ (B cannoy also be 0, therefore) $\sin\lambda L = 0 \;\to \lambda L = n\pi$

Therefore $\lambda_n = \dfrac{n\pi}{L}$ and $X_n(x) = B_n \sin\left(\dfrac{n\pi}{L}x\right)$ and $u_n(x,t) = B_n \sin\left(\dfrac{n\pi}{L}x\right)e^{-\lambda_n^2 Kt}$

Applying the superposition principle we have

KEY

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)e^{-\lambda_n^2Kt}}$$

Finally we apply the initial conditions u(0,t)=f(x), therefore

$$u(x,0))=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)e^{-\lambda_n^2K0}=\boxed{\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)=f(x)}$$

A Fourier series with $B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

One way to look at this is that we have decomposed the function f(x) in terms

KEY

of "eigenfunctions" $\sin\left(\frac{n\pi}{L}x\right)$ at t = 0 and then the solution takes each of these eigenfunctions weighted with the appropriate B_n and each one evolves in time with the appropriate temporal function (in this case an exponential also indexed as a function of n).

Therefore we see a temperature profile that starts as f(x) and decays to 0 over time where each part decays independently depending on the index, n in our representation.