Johns Hopkins
Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
EN. 585.409

JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

Using Rodriquez formula to evaluate Legendre function orthogonal condition

The Legendre polynomials are defined over the interval [-1,1] with a weighting function $\rho(x) = 1$

Therefore the defining integral is

$$\int_{-1}^{1} P_\ell(x) P_k(x)\,dx$$

Let's investigate the two important cases, $\ell = k$ and $\ell \neq k$

For the case $\ell = k$ we have (using Rodriquez defining equation)

$$\int_{-1}^{1} P_\ell(x) P_\ell(x)\,dx = \int_{-1}^{1} [\frac{1}{\ell! 2^\ell} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell][\frac{1}{\ell! 2^\ell} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell]\,dx =$$

$$\left(\frac{1}{\ell! 2^\ell}\right)^2 \int_{-1}^{1} [\frac{d^\ell}{dx^\ell}(x^2-1)^\ell][\frac{d^\ell}{dx^\ell}(x^2-1)^\ell]\,dx$$

$$\int_{-1}^{1} P_\ell(x) P_\ell(x)\,dx = \int_{-1}^{1} [\frac{1}{\ell! 2^\ell} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell][\frac{1}{\ell! 2^\ell} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell]dx =$$

$$\left(\frac{1}{\ell! 2^\ell}\right)^2 \int_{-1}^{1} [\frac{d^\ell}{dx^\ell}(x^2-1)^\ell][\frac{d^\ell}{dx^\ell}(x^2-1)^\ell]dx$$

Using integration by parts

$$u = \frac{d^\ell}{dx^\ell}(x^2-1)^\ell \rightarrow du = \frac{d^{\ell+1}}{dx^{\ell+1}}(x^2-1)^\ell dx$$

$$dv = \frac{d^\ell}{dx^\ell}(x^2-1)^\ell dx \rightarrow v = \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell$$

Therefore (using our standard form for integration by parts $\int u\,dv = uv - \int v\,du$):

$$\left(\frac{1}{\ell! 2^\ell}\right)^2 \int_{-1}^{1} \left[\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right]\left[\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right]dx =$$

$$\left(\frac{1}{\ell! 2^\ell}\right)^2 \left\{\left[\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right]\left[\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right]\Bigg|_{-1}^{1} - \int_{-1}^{1} \frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \frac{d^{\ell+1}}{dx^{\ell+1}}(x^2-1)^\ell\,dx\right\}$$

$\ell-1$ derivatives of $(x^2-1)^\ell$ at -1 or 1 gives $\left[\dfrac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\right]_{-1}^{1}=0$

Therefore $\displaystyle\int_{-1}^{1}P_\ell(x)P_\ell(x)dx=\left(\frac{1}{\ell!2^\ell}\right)^2\int_{-1}^{1}\left[\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right]\left[\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right]dx=$

$$\left(\frac{1}{\ell!2^\ell}\right)^2\left\{-\int_{-1}^{1}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell\frac{d^{\ell+1}}{dx^{\ell+1}}(x^2-1)^\ell dx\right\}$$

and using integration by parts $\ell-1$ times gives

$$\int_{-1}^{1}P_\ell(x)P_\ell(x)dx=\left(\frac{1}{\ell!2^\ell}\right)^2\left\{(-1)^\ell\int_{-1}^{1}(x^2-1)^\ell\frac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell dx\right\}$$

Let's look at part of the integrand next, that is $\dfrac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell$

Note for $\ell=1$ we have $\dfrac{d^2}{dx^2}(x^2-1)^1=\dfrac{d}{dx}2x=2=2\cdot1=2!$

Note for $\ell=2$ we have $\dfrac{d^4}{dx^4}(x^2-1)^2=\dfrac{d^4}{dx^4}(x^4-2x^2+1)=\cdots=4\cdot3\cdot2\cdot1=4!$

etc. Therefore $\boxed{\dfrac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell=(2\ell)!}$

Substitution gives

$$\int_{-1}^{1} P_\ell(x)P_\ell(x)dx = \left(\frac{1}{\ell!2^\ell}\right)^2 \left\{(-1)^\ell \int_{-1}^{1}(x^2-1)^\ell (2\ell)!dx\right\} =$$

$$\left(\frac{1}{\ell!2^\ell}\right)^2 (2\ell)!\int_{-1}^{1}(-1)^\ell (x^2-1)^\ell dx = \frac{(2\ell)!}{(\ell!2^\ell)^2}\int_{-1}^{1}(1-x^2)^\ell dx$$

Let $K_\ell = \int_{-1}^{1}(1-x^2)^\ell dx$ then $\int_{-1}^{1} P_\ell(x)P_\ell(x)dx = \frac{(2\ell)!}{(\ell!2^\ell)^2}K_\ell$

Using integration by parts on K_ℓ we have

$$u = (1-x^2)^\ell \rightarrow du = \ell(1-x^2)^{\ell-1}(-2x)dx$$

$$dv = dx \rightarrow v = x$$

Therefore $K_\ell = \int_{-1}^{1}(1-x^2)^\ell dx = (1-x^2)^\ell x\Big|_{-1}^{1} - \int_{-1}^{1} x\ell(1-x^2)^{\ell-1}(-2x)dx$

Again note $(1-x^2)^\ell x\Big|_{-1}^{1} = 0$ therefore

$$K_\ell = \int_{-1}^{1} 2x^2\ell(1-x^2)^{\ell-1} dx$$

Next a key observation is that $\boxed{2x^2\ell \equiv 2\ell - 2\ell(1-x^2)}$ Therefore

$$K_\ell = \int_{-1}^{1} 2x^2\ell(1-x^2)^{\ell-1}\,dx = K_\ell = \int_{-1}^{1} [2\ell - 2\ell(1-x^2)](1-x^2)^{\ell-1}\,dx =$$

$$\int_{-1}^{1} 2\ell(1-x^2)^{\ell-1}\,dx - \int_{-1}^{1} [2\ell(1-x^2)](1-x^2)^{\ell-1}\,dx =$$

$$2\ell\int_{-1}^{1} (1-x^2)^{\ell-1}\,dx - 2\ell\int_{-1}^{1} (1-x^2)(1-x^2)^{\ell-1}\,dx =$$

$$2\ell\int_{-1}^{1} (1-x^2)^{\ell-1}\,dx - 2\ell\int_{-1}^{1} (1-x^2)^{\ell}\,dx = 2\ell K_{\ell-1} - 2\ell K_\ell$$

That is $K_\ell = 2\ell K_{\ell-1} - 2\ell K_\ell$ or $K_\ell = \dfrac{2\ell}{1+2\ell} K_{\ell-1}$

Now for $\ell = 0$ use the original integral definition for $K_0 = \int_{-1}^{1} (1-x^2)^0\,dx = \int_{-1}^{1} 1\,dx = 2$

Then for $\ell = 1$ $K_1 = \dfrac{2(1)}{1+2(1)} K_0 = \dfrac{2}{3}(2)$

$$\ell = 2 \quad K_2 = \frac{2(2)}{1+2(2)} K_1 = \frac{4}{5}\frac{2}{3}(2) = \frac{4}{5}\left(\frac{4}{4}\right)\frac{2}{3}\left(\frac{2}{2}\right)(2) = \frac{2^7}{5!} = \frac{2^5\cdot 2\cdot 2}{(2\cdot 2+1)!} = \frac{2^{2\cdot 2+1}\cdot 2\cdot 2}{(2\cdot 2+1)!} = \frac{2^{2\cdot 2+1}\cdot 2!\cdot 2!}{(2\cdot 2+1)!}$$

Inferred pattern from looking at $\ell = 3,\ 4,\ \cdots$

Therefore $\boxed{K_\ell = \dfrac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)!}}$

Finally substitution gives

$$\int_{-1}^{1} P_\ell(x)P_\ell(x)dx = \frac{(2\ell)!}{(\ell!2^\ell)^2}K_\ell = \frac{(2\ell)!}{(\ell!2^\ell)^2}\frac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)!} =$$

$$\int_{-1}^{1} P_\ell(x)P_\ell(x)dx = \frac{(2\ell)!}{(\ell!2^\ell)^2}\frac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)!} = \frac{2^{2\ell+1}(\ell!)^2}{2^{2\ell}(\ell!)^2}\frac{(2\ell)!}{(2\ell+1)!} = \boxed{\frac{2}{2\ell+1}}$$

For the case $\ell \neq k$ we refer back to the original D.E. defining Legendre polynomials

$$(1-x^2)y''+-2xy'+\ell(\ell+1)y=0 \to [(1-x^2)y']'+\ell(\ell+1)y=$$

Since $y \equiv P_k(x)$ substitution gives $[(1-x^2)P_\ell'(x)]'dx \;\; [(1-x^2)P_\ell'(x)]'+\ell(\ell+1)P_\ell(x)=0$

Next multiply by $P_k(x)$ and integrate from -1 to 1

$$\int_{-1}^{1} P_k(x)[[(1-x^2)P_\ell'(x)]'+\ell(\ell+1)P_k(x)P_\ell(x)]dx=$$

$$\int_{-1}^{1} P_k(x)[(1-x^2)P_\ell'(x)]'dx+\int_{-1}^{1} \ell(\ell+1)P_k(x)P_\ell(x)dx=0$$

Using integration by parts on the first integral

Let $dv=[(1-x^2)P_\ell'(x)]'dx \to v=(1-x^2)P_\ell'(x)$ and $u=P_k(x) \to du=P_k'(x)dx$

Then $\int_{-1}^{1} P_k(x)[(1-x^2)P_\ell'(x)]'dx=P_k(x)(1-x^2)P_\ell'(x)\Big|_{-1}^{1}-\int_{-1}^{1}(1-x^2)P_\ell'(x)P_k'(x)dx$

where the first term is zero for either -1 or 1 leaving us with the integral term

Substitution gives

$$-\int_{-1}^{1}(1-x^2)P_\ell'(x)P_k'(x)dx+\int_{-1}^{1}\ell(\ell+1)P_k(x)P_\ell(x)dx=0$$

OR $\int_{-1}^{1}(1-x^2)P_\ell'(x)P_k'(x)dx=\int_{-1}^{1}\ell(\ell+1)P_k(x)P_\ell(x)dx=0$

Fixed

Starting again but using k as index we have $[(1-x^2)P_k'(x)]'+k(k+1)P_k(x)=0$

and after a similar proceedure we get $-\int_{-1}^{1}(1-x^2)P_\ell'(x)P_k'(x)dx+\int_{-1}^{1}k(k+1)P'_k(x)P'_\ell(x)dx=0$

OR $\int_{-1}^{1}k(k+1)P'_k(x)P'_\ell(x)dx=\int_{-1}^{1}(1-x^2)P_\ell'(x)P_k'(x)dx$ Fixed

Previously we had $\int_{-1}^{1}\ell(\ell+1)P_k'(x)P_\ell'(x)dx=\int_{-1}^{1}(1-x^2)P_\ell'(x)P_k'(x)dx$

Subtraction of these two results gives

$$\int_{-1}^{1}k(k+1)P_k(x)P_\ell(x)dx-\int_{-1}^{1}\ell(\ell+1)P_k(x)P_\ell(x)dx=0$$

or $\int_{-1}^{1}[k(k+1)-\ell(\ell+1)]P_k(x)P_\ell(x)dx=[k(k+1)-\ell(\ell+1)]\int_{-1}^{1}P_k(x)P_\ell(x)dx=0$

or since $k\neq\ell\rightarrow[k(k+1)-\ell(\ell+1)]\neq 0$ therefore $\boxed{\int_{-1}^{1}P_k(x)P_\ell(x)dx=0}$

Therefore we have the following **orthogonality condition** for Legendre polynomials!

$$\boxed{\int_{-1}^{1}P_\ell(x)P_k(x)dx=\begin{cases}\ell\neq k & 0\\ \ell=k & \dfrac{2}{2\ell+1}\end{cases}}$$