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First an examples of separable and inseparable functions

KEY: The technique of separation of variable requires that our solution can be expressed as a product of functions of separated or separate variables!

Here is an example of a separable and inseparable functions

$$u_1(x,y,z,t) = xyz^2 \sin bt \text{ is separable } u_1(x,y,z,t) = X(x)Y(y)Z(z)T(t)$$

$$\text{where } X(x) = x,\ Y(y) = y,\ Z(z) = z^2 \text{ and } T(t) = \sin bt$$

$$u_2(x,y,z,t) = xy + zt \text{ is not separable } u_2(x,y,z,t) \neq X(x)Y(y)Z(z)T(t)$$

Solution of the wave equation in multiple dimensions using separation of variables

Starting with our multi-dimensional wave equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]u=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Take $u(x,y,z,t)=X(x)Y(y)Z(z)T(t)$ and substitute into our equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]X(x)Y(y)Z(z)T(t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}X(x)Y(y)Z(z)T(t) \quad \text{or}$$

$$Y(y)Z(z)T(t)\frac{d^2}{dx^2}X(x)+X(x)Z(z)T(t)\frac{d^2}{dy^2}Y(y)+X(x)Y(y)T(t)\frac{\partial^2}{\partial z^2}Z(z)$$

$$=X(x)Y(y)Z(z)\frac{1}{c^2}\frac{\partial^2}{\partial t^2}T(t)$$

Dividing every term by $X(x)Y(y)Z(z)T(t)$ gives

$$\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)+\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)+\frac{1}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z)=\frac{1}{c^2}\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}T(t)$$

KEY1: The last equation has all four variables separated into into four parts each only a function of x, y, z or t respectively.

KEY2: Since each part is independent of the variables in the other part a choice that makes this work is if each of these terms is a constant and the sum of the three constants on the LHS is equal to a constant on the RHS!

Therefore let $-l^2 + -m^2 + -n^2 = -\mu^2$ where

$$\frac{1}{X(x)}\frac{d^2}{dx^2}X(x) = -l^2,\ \frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y) = -m^2 + \frac{1}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z) = -n^2,\ \frac{1}{c^2}\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}T(t) = -\mu^2$$

Taking for example the equation in the variable x gives

Multipling $\frac{1}{X(x)}\frac{d^2}{dx^2}X(x) = -l^2$ by $X(x)$

gives $\frac{d^2}{dx^2}X(x) = -l^2 X(x) \rightarrow \frac{d^2}{dx^2}X(x) + l^2 X(x) = 0$

Since l^2 is strictly positive the solution is $\boxed{X(x) = Ae^{-ilx} + Be^{ilx}}$

Using Euler's identity $e^{\pm iy} = \cos(y) \pm i\sin(y)$ gives

$$X(x) = A[\cos(lx) - i\sin(lx)] + B[\cos(lx) + i\sin(lx)] = (A+B)\cos(lx) + (A-B)i\sin(lx)$$

Taking $A' = A + B$ and $B' = (A-B)i$ gives

$$\boxed{X(x) = A'\cos(lx) + B'\sin(lx)}$$

Of course all the equations in x, y and z have the same form for their solution

$$
\begin{aligned}
X(x) &= Ae^{-ilx} + Be^{ilx} & & X(x) = A'\cos(lx) + B'\sin(lx) \\
Y(x) &= Ce^{-imy} + De^{imy} & \rightarrow \quad & Y(y) = C'\cos(my) + D'\sin(my) \\
Z(z) &= Ee^{-inz} + Fe^{inz} & & Z(z) = E'\cos(nz) + F'\sin(nz)
\end{aligned}
$$

However for T(t) we have a slightly different form of the differential equation

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{d^2}{dt^2}T(t) = -\mu^2 \rightarrow \frac{d^2}{dt^2}T(t) + c^2\mu^2 T(t) = 0$$

Therefore the solution is

$$T(t) = Ge^{-ic\mu t} + He^{lc\mu t} \rightarrow T(t) = G'\cos(c\mu t) + H'\sin(c\mu t)$$

Taking the positive exponential in variables x, y and z and the negative in t
we have the following solution

$$u(x,y,z,t) = X(x)Y(y)Z(z)T(t) = Ae^{ilx}De^{imy}Fe^{inz}Ge^{-ic\mu t} = K(e^{ilx}e^{imy}e^{inz}e^{-ic\mu t})$$

Finally with $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and wave number $\mathbf{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$ with $\hat{i}$, $\hat{j}$, $\hat{k}$ unit vectors

in x, y and z direction and with $\mu = |\mathbf{k}| = \dfrac{2\pi}{\lambda}$ where λ is the wavelenght

and $c\mu = \omega$ the angular frequency of the wave

$$\boxed{u(x,y,z,t) = Ke^{ilx+imy+inz-ic\mu t} = Ke^{ilx+imy+inz-ic\mu t} = Ke^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} = Ke^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}}$$

Solution of the Laplace's equation using separation of variables

Starting with Laplace's equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Take $u(x,y)=X(x)Y(y)$ and substitute into our equation

$$\frac{\partial^2}{\partial x^2}X(x)Y(y)+\frac{\partial^2}{\partial y^2}X(x)Y(y)=0 \;\rightarrow\; Y(y)\frac{d^2}{dx^2}X(x)+X(x)\frac{d^2}{dy^2}Y(y)=0$$

Then dividing by $X(x)Y(y)$ gives $\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)+\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=0$

Take as the separation constant $\lambda^2>0$ gives

$$\frac{1}{X(x)}\frac{d^2}{dx^2}X(x)=-\frac{1}{Y(y)}\frac{d^2}{dy^2}Y(y)=\lambda^2$$

Finally we get separate equations in X and Y

$$\boxed{\frac{d^2}{dx^2}X(x)=\lambda^2X(x) \text{ and } \frac{d^2}{dy^2}Y(y)=-\lambda^2Y(y)}$$

For $\frac{d^2}{dx^2}X(x) = \lambda^2 X(x)$ we immediately see that $X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

and here we can use the definitions for the hyperbolic functions, that is

$\sinh\lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{2}$ and $\cosh\lambda x = \frac{e^{\lambda x} + e^{-\lambda x}}{2}$ to reformulate our solution as

$$X(x) = c_1(\cosh\lambda x + \sinh\lambda x) + c_2(\cosh\lambda x - \sinh\lambda x) = (c_1 + c_2)\cosh\lambda x + (c_1 - c_2)\sinh\lambda x$$

Therefore we can write $\boxed{X(x) = A\sinh\lambda x + B\cos\lambda x}$

As we have set our separation constant up the equation for the variable y is slightly different and its corresponding solution very different in it's behaviour.

For $\frac{d^2}{dy^2}Y(x) = -\lambda^2 Y(x)$ we immediately see that $Y(y) = c_3 e^{i\lambda y} + c_4 e^{-i\lambda y}$

Using Euler's identity we have $e^{\pm i\lambda y} = \cos\lambda y \pm i\sin\lambda y$ we get the reformated solution

$$Y(y) = c_3(\cos\lambda y + i\sin\lambda y) + c_4(\cos\lambda y - i\sin\lambda y) = (c_3 + c_4)\cos\lambda y + i(c_3 - c_4)\sin\lambda y$$

Therefore $\boxed{Y(y) = C\sin\lambda y + D\cos\lambda y}$

Finally putting these two solutions together gives us

$$\boxed{u(x,y) = X(x)Y(y) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(C\sin\lambda y + D\cos\lambda y)}$$

or

$$\boxed{u(x,y) = X(x)Y(y) = (A\sinh\lambda x + B\cos\lambda x)(C\sin\lambda y + D\cos\lambda y)}$$

Either solution can be valid depending on the constraints of the problem and in fact the forms for X(x) and Y(y) may be reversed depending on these same boundary conditions

Solution of the diffusion equation

Let's look at the one spatial dimension diffusion equation.
In this case we have also not included a production term, therefore

KEY as $t \to \infty$ we expect $u(x,t) \to 0$

Given $K\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ take $u(x,t) = X(x)T(t)$

and substitute into our equation $K\frac{\partial^2}{\partial x^2}X(x)T(t) = \frac{\partial}{\partial t}X(x)T(t)$

Then dividing by K and X(x)T(t) gives $\frac{1}{X(x)}\frac{d^2}{dx^2}X(x) = \frac{1}{KT(t)}\frac{d}{dt}T(t)$

Take the separation constant to be strictly negative, that is $-\lambda^2$

Take as the separation constant $\lambda^2 > 0$ gives $\frac{1}{X(x)}\frac{d^2}{dx^2}X(x) = \frac{1}{KT(t)}\frac{d}{dt}T(t) = -\lambda^2$

We are already familar with the solution for $X(x) = A\cos\lambda x + B\sin\lambda x$

For the temporal equation we have $\frac{d}{dt}T(t) + \lambda^2 KT(t) = 0$ or $\frac{d}{dt}T(t) = -\lambda^2 KT(t)$

This equation, which is fairly well known, is that of expeonential decay

Its solution is $T(t)=Ce^{-\lambda^2 t}$

Therefore $u(x,t) = X(x)T(t) = (A\cos\lambda x + B\sin\lambda x)Ce^{-\lambda^2 Kt}$

Obviously as

$t \to \infty \quad u(x,t) \to 0$