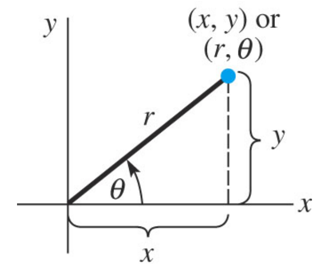
Boundary-Value Problems in Other Coordinate Systems

Polar Coordinates

- The relationship between polar and rectangular coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$



- What are $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial y^2}$ in terms of $\dfrac{\partial u}{\partial r}, \dfrac{\partial u}{\partial \theta}, \dfrac{\partial^2 u}{\partial r^2}, \dfrac{\partial^2 u}{\partial \theta^2}$

2D Laplace Equation in Polar Coordinates

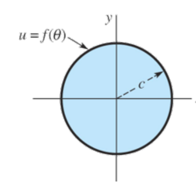
- Using the relationship between derivatives with respect to x and y and derivatives with respect to r and θ it can be shown that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Solution of 2D Laplace Equation in Polar Coordinates

- Steady-state temperature distribution in a circular plate

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad u(c,\theta) = f(\theta), \quad u(r,\theta+2\pi) = u(r,\theta)$$



$$u(r,\theta) = R(r)\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \quad \Rightarrow \quad \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \alpha^2$$

$$\Theta'' + \alpha^2\Theta = 0, \qquad \Theta(\theta+2\pi) = \Theta(\theta)$$

$$\alpha^2 = 0 \quad \Rightarrow \quad \Theta = c_1 + c_2\theta$$

$$\Theta(\theta+2\pi) = \Theta(\theta) \quad \Rightarrow \quad c_2 = 0$$

$$\alpha^2 \neq 0 \quad \Rightarrow \quad \Theta = c_1\cos\alpha\theta + c_2\sin\alpha\theta$$

$$\Theta(\theta+2\pi) = \Theta(\theta) \quad \Rightarrow \quad \alpha = n, \quad n = 1,2,3,\cdots$$

$$\Theta_0 = c_1, \qquad \Theta_n = c_1\cos n\theta + c_2\sin n\theta \quad n = 1,2,3,\cdots$$

Solution of 2D Laplace Equation in Polar Coordinates

- The equation for R is a Cauchy-Euler equation.

$$r^2R''+rR'-\alpha^2R=0$$
$$\alpha^2=0 \Rightarrow R=c_3+c_4\ln r$$
$$\alpha^2=n^2\neq 0 \Rightarrow R=c_3r^n+c_4r^{-n}$$

- In order for R to be finite at $r = 0$, c_4=0.

$$u(r,\theta)=A_0+\sum_{n=1}^{\infty}r^n(A_n\cos n\theta+B_n\sin n\theta)$$

- Applying the boundary condition at $r = c$ gives.

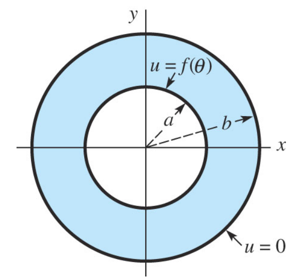
$$f(\theta)=A_0+\sum_{n=1}^{\infty}c^n(A_n\cos n\theta+B_n\sin n\theta)$$

$$A_0=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)d\theta,\quad A_n=\frac{1}{c^n\pi}\int_0^{2\pi}f(\theta)\cos n\theta\, d\theta,\quad B_n=\frac{1}{c^n\pi}\int_0^{2\pi}f(\theta)\sin n\theta\, d\theta$$

Class Exercise

- Find the steady-state temperature $u(r,\theta)$ in the circular ring shown below.

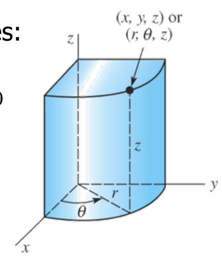
$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$
$$u(a,\theta)=f(\theta)$$
$$u(b,\theta)=0$$
$$u(r,\theta+2\pi)=u(r,\theta)$$



Cylindrical Coordinates

- Laplace Equation in cylindrical coordinates:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \quad\Rightarrow\quad \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0$$

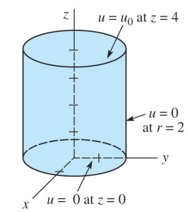


- Radial symmetry means u does not depend on θ

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

Steady-State Temperature Distribution in a Cylinder

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$
$$u(2,z)=0, \qquad 0<z<4$$
$$u(r,0)=0, \quad u(r,4)=u_0, \quad 0<r<2$$



$$u(r,z)=R(r)Z(z)$$
$$\frac{R''+R'/r}{R}=-\frac{Z''}{Z}=-\alpha^2$$
$$rR''+R'+\alpha^2 rR=0$$
$$R=c_1J_0(\alpha r)+c_2Y_0(\alpha r)$$
$$\text{R is finite at } r=0 \quad\Rightarrow\quad c_2=0$$
$$R(2)=0 \quad\Rightarrow\quad J_0(\alpha 2)=0 \quad \alpha_n, n=1,2,3,\cdots \text{ are roots of this equation.}$$

Steady-State Temperature Distribution in a Cylinder

- The solution to equation for Z is

$$Z'' - \alpha_n^2 Z = 0$$

$$Z = c_3 \cosh \alpha z + c_4 \sinh \alpha z$$

$$Z(0) = 0 \quad \Rightarrow \quad c_3 = 0 \quad \Rightarrow \quad Z_n = c_4 \sinh \alpha_n z$$

- The solution for u is

$$u(r,z) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n z J_0(\alpha_n r)$$

- The last boundary condition at $z = 4$ gives

$$u_0 = \sum_{n=1}^{\infty} A_n \sinh 4\alpha_n J_0(\alpha_n r)$$

$$A_n \sinh 4\alpha_n = \frac{2u_0}{2^2 J_1^2(2\alpha_n)} \int_0^2 r J_0(\alpha_n r)\, dr$$