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JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

Series solution of ODE at ordinary point

We will be looking at second order ODEs, however the techniques can be generalized to higher order equations. Furthermore we will look at the solutions at $z=z_0=0$. A shift of z can accommodate other non-zero point, that is we could make the variable substitution $\tilde{z}=z-z_0$ in our differential equation.

For a point which is ordinary we can represent the solution in terms of an analytic function, that is

$$\boxed{y(z)=\sum_{n=0}^{\infty} a_n z^n}$$

This representation is good for some radius of convergence, that is where R can be infinity $|z|<R$

where the a_ns are constants.

KEY: Note that for second order ODEs this solution will include two arbitrary constants and the series solution method will generate two linearly independent solutions!

Since we will be dealing with second order ODEs we will need to investigate the first and second derivatives of this expression for y(z).

$$y'(z)=\sum_{n=0}^{\infty} n a_n z^{n-1} \equiv \sum_{n=1}^{\infty} n a_n z^{n-1}$$

$$y''(z)=\sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} \equiv \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$$

KEY: It is often necessary to re-index these series. Since n is an index it can be replaced with any other index symbol or shifted provided that the series generates the same terms.

Lets re-index the previous differentiated series so that the index for the powers of z are the same a s that for the original series!

Ignore the n = -1 term
since its contribution is 0
so start index at n = 0

Let $n \to n+1$ in the first derivative series

$$y'(z)=\sum_{n=0}^{\infty} na_n z^{n-1} \to \sum_{n+1=0}^{\infty}(n+1)a_{n+1}z^{(n+1)-1}=\sum_{n=-1}^{\infty}(n+1)a_{n+1}z^{n}\equiv\sum_{n=0}^{\infty}(n+1)a_{n+1}z^{n}$$

Let $n \to n+2$ in the first derivative series

Ignore the n = -2, -1 terms
since there contribution is 0
so start index at n = 0

$$y''(z)=\sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} \to \sum_{n+2=0}^{\infty}(n+2)(n+1)a_{n+2}z^{(n+2)-2}=\sum_{n=-2}^{\infty}(n+2)(n+1)a_{n+2}z^{n}\equiv$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^{n}$$

An example of a series solution of an ODE at an ordinary point

Here is a familiar differential equations $y''+y=0$

If the independent variable is time and we rearrange the terms we get $\frac{d^2y}{dt^2}=-y$

We recognize this as the equation for a spring system with spring constant k = 1 (so keep this in mind as we solve this ODE by series solution).

Substitution of the previous constructed derivatives and original series into our ODE gives

$$y''+y=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}z^n+\sum_{n=0}^{\infty}a_n z^n=0$$

Combining the sums (since index, powers of z and range of the indices are the same) gives

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}z^n+a_n z^n]=\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+a_n]z^n=0$$

KEY: z or correspondingly z^n cannot equal 0 otherwise we have a degenerate solution, correct but of no interest

Therefore for each (or any power of z), that is for all n: $\boxed{(n+2)(n+1)a_{n+2}+a_n=0}$

Rearranging terms in our recursion relationship for the constants a_n we get

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

Start with n = 0 this gives $a_2 = \frac{-a_0}{(2)(1)} \equiv \frac{(-1)a_0}{2!}$

For n = 1 $a_3 = \frac{-a_1}{(3)(2)} \equiv \frac{-a_1}{3!}$

For n =2 $a_4 = \frac{-a_2}{(4)(3)} \equiv \frac{-1}{(4)(3)} a_2 = \frac{-1}{(4)(3)}\left(\frac{-1a_0}{2!}\right) = \frac{a_0}{4!}$

In general there are two distinct patterns (or explicit formulas) to generate constants

$$a_{2n} = \frac{a_0(-1)^n}{2n!}$$

$$a_{2n+1} = \frac{a_1(-1)^n}{(2n+1)!}$$

Remember that we said that a second order ODE would have two arbitrary constants and we see them in the expressions above, that is a_0 and a_1. Furthermore for a second order ODE the series solution would generate two linearly independent solutions.

So for one case set $a_0=1$ and $a_1=0$ and since $a_{2n+1} = \frac{a_n(-1)^n}{(2n+1)!}$ we have $a_{2n+1} = 0$

and also $a_{2n} = \frac{(-1)^n}{2n!}$

Using all this information we get

$$\boxed{y_1(z)} = a_0 + 0z^1 + a_2 z^2 + 0z^3 + a_4 z^4 + \cdots = a_0 + a_2 z^2 + a_4 z^4 + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} z^{2n}}$$

For our other independent solution take $a_0=0$ and $a_1=1$, then $a_{2n} = \frac{0(-1)^n}{2n!} = 0$

and also $a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$

So our second solution is

$$\boxed{y_2(z)} = 0 + a_1 z^1 + 0z^2 + a_3 z^3 + 0z^4 + \cdots = a_1 z + a_3 z^3 + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}}$$

Aside: However while these answers are completely correct, remember we said this ODE was the solution to a spring system. These systems give harmonic solutions with sine and cosine functional forms. So let's go a little further.

Let's look at the Taylor series for the cosine function. The generating formula for a Taylor series near z_0=0 is

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n \text{ and at } z_0=0 \text{ we have } f(z)=\sum_{n=0}^{\infty}a_n z^n,\ a_n=\left.\frac{f^{(n)}(z)}{n!}\right|_{z=z_0=0}$$

Setting $f(z)=\cos z$

then

$$a_0=\left.\frac{f^{(0)}(z)}{0!}\right|_{z=z_0=0}=\left.\frac{\cos z}{0!}\right|_{z=z_0=0}=\frac{\cos 0}{0!}=\frac{1}{1}=1$$

$$a_1=\left.\frac{f^{(1)}(z)}{1!}\right|_{z=z_0=0}=\left.\frac{-\sin z}{1!}\right|_{z=z_0=0}=\frac{-\sin 0}{1!}=0$$

$$a_2=\left.\frac{f^{(2)}(z)}{2!}\right|_{z=z_0=0}=\frac{-\cos z}{2!}=\frac{-\cos 0}{2!}=\frac{-1}{2!}$$

$$\vdots$$

Therefore $f(z)=\cos z=\sum_{n=0}^{\infty}a_n z^n=1z^0+0z^1+\frac{-1}{2!}z^2+0z^3+\frac{-1}{4!}z^4+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}$

The same as our previous series solution $y_1(z)$, as it should be.
$y_2(z)$ is the same as Taylor expansion for sinz.