

Interactive Assignment 6

Problems

7.7 — P⁴

7.8 — P⁵

7.12 — P¹³

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Chapter 7 - Problem 7.7

Use the derivative method to obtain, as a second (independent) solution of Bessel's equation for the case when $\nu=0$, the following expression:

$$J_0(z) \ln z - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sum_{r=1}^n \frac{1}{r} \right) \left(\frac{z}{2}\right)^{2n}$$

given that the first solution is $J_0(z)$, as specified by 9.76

The recurrence relation (9.75) leads to coefficient a_n :

$$a_n = \frac{(-1)}{n(n+2s)} a_{n-2}$$

$$\text{For } n=2 \quad a_2 = \frac{(-1)}{2(2+2s)} a_0 \quad n=4 \quad a_4 = \frac{-1}{4(4+2s)} a_2 = \frac{(-1)^2}{4 \cdot 2(4+2s)(2+2s)} a_0$$

$$n=6 \quad a_6 = \frac{-1}{6(6+2s)} a_4 = \dots = \frac{(-1)^3}{6 \cdot 4 \cdot 2 (6+2s)(4+2s)(2+2s)} a_0$$

$$\text{In general} \quad a_{2n} = \frac{(-1)^n}{2^n n! (2+2s)(4+2s)(6+2s)\dots(2n+2s)} a_0$$

$$\text{Thus } g_1(x) = x^s \sum_{n=0}^{\infty} a_{2n} x^{2n} = x^s a_0 x^0 + x^s \sum_{n=1}^{\infty} a_{2n} x^{2n}$$

Using the derivative method, the second solution is:

$$\begin{aligned}
 y_2(x) &= \left[\frac{\partial}{\partial \zeta} y_1(x, \zeta) \right]_{\zeta=0} \text{ for double radical roots} \\
 &= \left[\frac{\partial}{\partial \zeta} \left(x^\zeta a_0 x^0 + x^\zeta \sum_{n=1}^{\infty} a_{2n} x^{2n} \right) \right]_{\zeta=0} \\
 &= \left[\frac{\partial}{\partial \zeta} \left(x^\zeta a_0 + x^\zeta \sum_{n=1}^{\infty} a_{2n} x^{2n} \right) \right]_{\zeta=0} \\
 &= \left[\frac{\partial}{\partial \zeta} x^\zeta a_0 + \left(\frac{\partial}{\partial \zeta} x^\zeta \right) \sum_{n=1}^{\infty} a_{2n} x^{2n} + x^\zeta \sum \frac{\partial}{\partial \zeta} a_{2n} x^{2n} \right]
 \end{aligned}$$

Note that $\frac{\partial}{\partial \zeta} x^\zeta = \frac{\partial}{\partial \zeta} e^{\zeta \ln x} = \ln x \ e^{\zeta \ln x} = (\ln x) x^\zeta$

And $\frac{\partial}{\partial \zeta} a_{2n} = \frac{\partial}{\partial \zeta} \left[\frac{(-1)^n}{2^n n! (2+2\zeta)(4+2\zeta)\dots(2n+2\zeta)} a_0 \right]$

Take the case for example $n=2$

$$\frac{\partial}{\partial \zeta} \frac{1}{(4+2\zeta)(2+2\zeta)} = \frac{(-1)(2)}{(2+2\zeta)(4+2\zeta)} \left[\frac{1}{2+2\zeta} + \frac{1}{4+2\zeta} \right]$$

Therefore $\frac{\partial}{\partial \zeta} a_{2n} = \frac{(-1) (-1)^n (2)}{2^n n! (2+2\zeta)(4+2\zeta)\dots(2n+2\zeta)} \left[\frac{1}{2+2\zeta} + \frac{1}{4+2\zeta} + \dots + \frac{1}{2n+2\zeta} \right]$

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Substitution back into $y_2(x)$ gives:

$$\begin{aligned} y_2(x) &= \left[(\ln x) x^{\zeta} + (\ln x) x^{\zeta} \sum_{n=1}^{\infty} a_{2n} x^{2n} \right. \\ &\quad \left. + x^{\zeta} \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n (2)}{2^n n! (2+2\zeta) (4+2\zeta) \dots (2n+2\zeta)} a_0 \left[\frac{1}{2+2\zeta} + \frac{1}{4+2\zeta} + \dots + \frac{1}{2n+2\zeta} \right] x^{2n} \right] \\ &= \left[(\ln x) x^{\zeta} \left(1 + \sum_{n=1}^{\infty} a_{2n} x^{2n} \right) \right. \\ &\quad \left. + x^{\zeta} \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n (2)}{2^n n! (2+2\zeta) (4+2\zeta) \dots (2n+2\zeta)} a_0 \left[\frac{1}{2+2\zeta} + \frac{1}{4+2\zeta} + \dots + \frac{1}{2n+2\zeta} \right] x^{2n} \right] \end{aligned}$$

Note that with $\zeta = 0$

$$y_1(x, 0) = x^0 \sum_{n=0}^{\infty} a_{2n} x^{2n} = a_0 + \sum_{n=1}^{\infty} a_{2n} x^{2n}$$

And from the expression of a_{2n} : $a_{2n}(0) = \frac{1}{n(n+2\Delta)}$

$$\text{and } a_0 = \frac{1}{2^\zeta \Gamma(\zeta+1)}$$

$$\Rightarrow a_0(0) = 1 \text{ for } \zeta = 0$$

$$\text{Thus } 1 + \sum_{n=1}^{\infty} a_{2n}(0) x^{2n} = y_1(x).$$

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Coming back to $y_2(x)$:

$$y_2(x) = (\ln x) x^0 y_1(x) + \left[x^0 \sum_{n=1}^{\infty} \frac{(-1)(-1)^n (2)}{2^n n! (2+2n)(4+2n)\dots(2n+2n)} a_n \right.$$

$$\left. \left[\frac{1}{2+2n} + \frac{1}{4+2n} + \dots + \frac{1}{2n+2n} \right] x^{2n} \right]_{n=0}$$

$x^0 = 1$ and $y_1(x) = J_0(x)$ gives by substitution with $a=0$

$$y_2(x) = (\ln x) J_0(x) + x^0 \sum_{n=1}^{\infty} \frac{(-1)(-1)^n (2)}{2^n n! (2)(4)\dots(2n)} \left[\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n} \right] x^{2n}$$

$$= (\ln x) J_0(x) - \sum_{n=1}^{\infty} \frac{(-1)^n (2)}{2^n n! (2)(4)\dots(2n)} \left[\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n} \right] x^{2n}$$

The term in bracket could be expressed as:

$$\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n} = \frac{1}{2} \left[1 + \frac{1}{2} + \dots + \frac{1}{n} \right] = 1/2 \sum_{r=1}^n 1/r$$

Leading to:

$$y_2(x) = (\ln x) J_0(x) - \sum_{n=1}^{\infty} \frac{(-1)^n (2)}{2^n n! (2)(4)\dots(2n)} \frac{1}{2} \cdot \left(\sum_{r=1}^n 1/r \right) x^{2n}$$

$$= (\ln x) J_0(x) - \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n (n!)^2} \left(\sum_{r=1}^n 1/r \right) x^{2n}$$

$$= (\ln x) J_0(x) - \sum_{n=2}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sum_{r=1}^n 1/r \right) \left(\frac{x}{2} \right)^{2n}$$

Chapter 7 - Problem 7-8

Consider a series solution of the equation:

$$z^2 y'' - 2z y' + y = 0 \quad (*)$$

about its regular singular point.

- (a) Show that its indicial equation has roots that differ by an integer but that the two roots nevertheless generate linearly independent solutions

$$y_1(z) = 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n z^{2n+1}}{(2n+1)!}$$

$$y_2(z) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n-1) z^{2n}}{2n!}$$

Putting (*) in its standard form:

$$y'' - \frac{2}{z} y' + y = 0$$

$z p(z) = -2$ and $z^2 q(z) = z^2$ are finite at the singular point $z=0$, it is a regular singular point and we expect to find a solution of the form: $y = z^\alpha \sum_{n=0}^{\infty} a_n z^n$

Substituting it in the previous equation, gives:

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n z^{n+\alpha-2} - \frac{2}{z} \sum_{n=0}^{\infty} (n+\alpha) a_n z^{n+\alpha-1} + \sum_{n=0}^{\infty} a_n z^{n+\alpha} = 0$$

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Dividing through by $z^{\sigma-2}$, gives:

$$\sum_{n=0}^{\infty} (n+\varsigma) (n+\varsigma-1) a_n z^n - 2 \sum_{n=0}^{\infty} (n+\varsigma) a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+2} = 0$$

Collecting coefficients of same power of z together and simplifying yields:

$$\sum_{n=0}^{\infty} (n+\varsigma) (n+\varsigma-1-2) a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+2} = 0$$

$$\sum_{n=0}^{\infty} (n+\varsigma) (n+\varsigma-3) a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+2} = 0$$

Rearranging the second sum: $\sum_{n=0}^{\infty} (n+\varsigma) (n+\varsigma-3) a_n z^n + \sum_{n=2}^{\infty} a_{n-2} z^n = 0$

$$\text{if } \varsigma(\varsigma-3) a_0 + (\varsigma+1)(\varsigma+1-3) a_1 z + \sum_{n=2}^{\infty} [(n+\varsigma)(n+\varsigma-3) a_n + a_{n-2}] z^n = 0$$

If we set $z=0$ the all terms in the sum and in coefficient a_n disappear, and we obtain the indicial equation:

$$\varsigma(\varsigma-3)=0$$

which has roots $\varsigma=0$ and $\varsigma=3$ and the roots differ by an integer 3.

If we are looking for a solution for $z \neq 0$ then, we obtain the recurrence relation:

$$(n+\varsigma)(n+\varsigma-3) a_n + a_{n-2} = 0$$

Chapter 7 - Problem 7.3

We choose the largest root $\omega = 3$, the recurrence relation is:

$$(n+3) n a_n = -a_{n-2} \quad n \geq 2$$

$$a_n = -\frac{a_{n-2}}{n(n+3)}$$

Setting $a_1 = 0$ and considering only even coefficients since odd coefficients vanish, one solution is:

$$g_1(z) = z^3 \sum_{n=0}^{\infty} a_n z^n \text{ for } n \text{ even}$$

$$= z^3 \left(a_0 - \frac{a_0}{2 \times 5} z^2 + \frac{a_0}{2 \times 4 \times 5 \times 7} z^4 + \dots + \frac{(-1)^n}{2 \times 4 \times \dots \times 2n (2n+3)} z^{2n} \right) + \dots$$

$$= a_0 z^3 - \frac{a_0}{2 \times 5} z^5 + \frac{a_0}{2 \times 4 \times 5 \times 7} z^7 + \dots + \frac{(-1)^n}{2 \times 4 \times \dots \times 2n (2n+3)} z^{2n+3} + \dots$$

$$= \frac{3 \times 2}{3!} a_0 z^3 - \frac{a_0 \times 3 \times 4}{5!} z^5 + \frac{a_0 \times 3 \times 4 \times 5 \times 6}{7!} z^7 + \dots$$

by reindexing with $n' = n+1$

$$= \frac{3 \times 2}{3!} a_0 z^3 - \frac{a_0 \times 3 \times 2 \times 2}{5!} z^5 + \dots + \frac{(-1)^{n+1+1} 3 \times 2 (n+1)}{2 \times 4 \times \dots \times 2n (2(n+1)+1)!} z^{2(n+1)+1}$$

$$= 3a_0 \frac{z^3}{3!} - \frac{3 \times 2 \times 2}{5!} z^5 + \dots + \frac{(-1)^{n'+1} 3 \times 2 \times 1'}{(2n'+1)!} z^{2n'+1}$$

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Therefore

$$y_1(z) = 3a_0 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n z^{2n+1}}{(2n+1)!}$$

To obtain the second solution, we set $\zeta = 0$ in the recurrence relation, which gives:

$$n(n-3)a_n = -a_{n-2}$$

$$a_n = -\frac{a_{n-2}}{n(n-3)}$$

Setting $a_1 = 0$ makes all the odd coefficients to vanish, we are left with the second solution $y_2(z) = z^0 \sum_{n=0}^{\infty} a_n z^n$
for n even

$$a_0 \text{ non zero, we have: } a_2 = -\frac{a_0}{2(-1)} = \frac{a_0}{2} = \frac{(-1)^2 (2 \times 1 - 1)}{(2 \times 1)!} a_0$$

$$a_4 = -\frac{a_2}{4+1} = -\frac{a_0}{4 \times 2} = \frac{(-1)^{2+1} \times 3}{4!} a_0$$

$$\therefore a_{2n} = \frac{(-1)^{n+1} (2n-1)}{2n!} a_0$$

Therefore

$$y_2(z) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n-1)}{2n!} z^{2n}$$

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(b) Show that $y_1(z)$ is equal to $3a_0(\sin z - z \cos z)$ by expanding the sinusoidal function. Then, using the Wronskian method, find an expression for $y_2(z)$ in terms of sinusoids.

We have $y_1(z) = 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n z^{2n+1}}{(2n+1)!}$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n z^{2n+1}}{(2n+1)!} &= - \sum_{n=0}^{\infty} \frac{(-1)^n 2^n z^{2n+1}}{(2n+1)!} = - \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{2^n!} - \frac{1}{2^{n+1}!} \right] z^{2n+1} \\ &= - \left[z \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n!} z^{2n} - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \right] \\ &= - [z \cos z - \sin z] = \sin z - z \cos z \end{aligned}$$

Thus: $y_1(z) = 3a_0(\sin z - z \cos z)$

The initial equation in standard form is:

$$y'' - \frac{2}{z} y' + y = 0 \quad \text{and } p(z) = -2/z$$

Using the Wronskian method

$$y_2(z) = y_1(z) \int^z \frac{e^{-\int^u p(v) dv}}{[y_1, y_2]^2} du$$

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$$\text{First : } e^{-\int^u \rho(v) dv} = e^{-\int(-2/v) dv} = e^{2 \int^u \frac{dv}{v}}$$

$$= e^{2 \ln u} = e^{\ln u^2} = u^2$$

Substituting this into the expression of $g_2(z)$ gives:

$$g_2(z) = g_1(z) \int^z \frac{a^2}{[g_1(u)]^2} du = 3a_0(\sin z - z \cos z) \int^z \frac{u^2}{[3a_0(\sin u - u \cos u)]^2} du$$

$$= \frac{3a_0}{g_{10}^2} (\sin z - z \cos z) \int^z \frac{u^2}{(\sin u - u \cos u)^2} du$$

$$\int^z \frac{u^2}{(\sin u - u \cos u)^2} du = \int^z \frac{u}{\sin u} \frac{u \sin u}{(\sin u - u \cos u)^2} du$$

$$\text{Note that } \frac{d}{du} \sin u - u \cos u = \cos u - \cos u + u \sin u = u \sin u$$

Integration by parts gives:

$$\int^z \frac{u^2}{(\sin u - u \cos u)^2} du = \left[\frac{u}{\sin u} \left(-\frac{1}{\sin u - u \cos u} \right) \right]^z + \int \frac{1}{\sin u - u \cos u} \frac{\sin u - u \cos u}{(\sin u)^2} du$$

$$= -\frac{z}{\sin z (\sin z - z \cos z)} + \int \frac{du}{(\sin u)^2}$$

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Therefore

$$\begin{aligned}
 \int \frac{z}{(8\mu u - u \cos u)^2} du &= -\frac{z}{\sin z (\sin z - z \cos z)} - \frac{\cos z}{\sin z} \\
 &= -\frac{\cos z \sin z - z \cos^2 z + z}{\sin z (\sin z - z \cos z)} \\
 &= -\frac{\cos z \sin z - z \cos^2 z + z \cos^2 z + z \sin^2 z}{\sin z (\sin z - z \cos z)} \\
 &= -\frac{\cos z \sin z + z \sin^2 z}{\sin z (\sin z - z \cos z)} = -\frac{\cos z + z \sin z}{\sin z - z \cos z} \\
 &= \frac{\cos z + z \sin z}{z \cos z - \sin z}
 \end{aligned}$$

And $y_2(z) = \frac{1}{3a_0} (\sin z - z \cos z) \frac{\cos z + z \sin z}{z \cos z - \sin z}$

$$= A \cdot (\cos z + z \sin z) \quad A = \text{constant} = 1/3a_0$$

- c) Confirm that the two solutions are linearly independent by showing that their Wronskian is equal to $-z^2$,

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The Wronskian is $W = y_1 y'_2 - y_2 y'_1$

$$y_1(z) = 3a_0(\sin z - 2\cos z)$$

$$y'_1(z) = 3 \cancel{z} \sin z + 3\cos z$$

$$y_2(z) = -\frac{1}{3a_0} (\cos z + z\sin z) \text{ then } y'_2(z) = \frac{-1}{3a_0} z\cos z$$

$$\text{Therefore } W = 3a_0 \left(-\frac{1}{3a_0}\right) (\sin z - 2\cos z) z\cos z$$

$$= \left(-\frac{1}{3a_0}\right) (\cos z + z\sin z) 3a_0 z\sin z$$

$$= -\frac{1}{3} z\cos z (\sin z - 2\cos z) + z\sin z (z\sin z + \cos z)$$

$$= z^2$$

For $z \neq 0$, $W \neq 0$ and the solutions $y_1(z)$ and $y_2(z)$ are linearly independent.

Chapter 7 - Problem 7.12

Prove that the Legendre equation,

$$z \frac{d^2y}{dz^2} + (1-z) \frac{dy}{dz} + \lambda y = 0$$

has polynomial solutions $L_N(z)$ if λ is a non-negative integer N , and determine the recurrence relationship for the polynomial coefficients.

Hence show that an expression for $L_N(z)$, normalized in such a way that

$$L_N(0) = N!$$

$$L_N(z) = \sum_{n=0}^N \frac{(-1)^n (N!)^2}{(N-n)! (n!)^2} z^n$$

Evaluate $L_3(z)$ explicitly.

Dividing through by z to put the equation in normal form,
we obtain

$$y'' + \frac{1-z}{z} y' + \lambda \frac{y}{z} = 0 \quad (1)$$

With the standard form, we identify $p(z) = 1-z/z$ and $q(z) = \lambda/z$
 $z=0$ is a singular point, but since $zp(z) = 1-z$ and $z^2q(z) = \lambda z$
are finite for $z=0$, it is a regular singular point of the equation above.

We substitute the Frobenius series $y = z^\alpha \sum_{n=0}^{\infty} a_n z^n$ into equation
(1), and we obtain:

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n z^{n+\alpha-2} + (1-\alpha) \sum_{n=0}^{\infty} (\alpha+n) a_n z^{n+\alpha-1} + \alpha \sum_{n=0}^{\infty} a_n z^{n+\alpha-1} = 0$$

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Dividing through by z^{G-2} gives:

$$\sum_{n=0}^{\infty} (n+\varsigma) (n+\varsigma-1) a_n z^n + (1-\lambda) \sum_{n=0}^{\infty} (n+\varsigma) a_n z^n + d \sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

Collecting coefficients in some power of z :

$$\sum_{n=0}^{\infty} [(n+\varsigma)(n+\varsigma-1) + (n+\varsigma)] a_n z^n + \sum_{n=0}^{\infty} [-(n+\varsigma) + d] a_n z^{n+1} = 0$$

Reindexing the second summation in z^n :

$$\sum_{n=0}^{\infty} (n+\varsigma) (n+\varsigma-1+1) a_n z^n + \sum_{n=1}^{\infty} [-(n-1+\varsigma) + d] a_{n-1} z^n = 0$$

Simplifying the first summation yields:

$$\sum_{n=0}^{\infty} (n+\varsigma)^2 a_n z^n + \sum_{n=1}^{\infty} [-(n-1+\varsigma) + d] a_{n-1} z^n = 0$$

$$\text{or } \varsigma^2 a_0 z^0 + \sum_{n=1}^{\infty} [(n+\varsigma)^2 a_n + [-(n-1+\varsigma) + d] a_{n-1}] z^n = 0$$

If we set $z=0$ then all the terms in the sum vanish, and we obtain the indicial equation: $\varsigma^2 a_0 = 0$

If we discard the trivial solution $z=0$ then $a_0 \neq 0$ since the recurrence relation shows that coefficients are dependent on a_0 . This implies that $\varsigma=0$ is a double root.

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Demanding that the coefficients of z^n vanish separately, we obtain the recurrence relation:

$$a_n = \frac{(n-1) + \zeta - 1}{(n+\zeta)^2} a_{n-1} \quad \text{for } n > 1$$

$$\text{Substitute } \zeta = 0 \text{ gives: } a_n = \frac{n-1-1}{n^2} a_{n-1}; \quad n > 1$$

For a given $\lambda = NT$, $a_{N+1} = \frac{N+1-1-N}{(N+1)^2} a_N = 0$, and all subsequent terms $n > N+1$ are zeros. Therefore the solution is a polynomial $f(z) = \sum_{n=0}^N a_n z^n$

Expanding each coefficient gives:

$$a_1 = \frac{1-1-N}{1^2} a_0 = \frac{(-1)^1 N}{1!^2} a_0$$

$$a_2 = \frac{1-N}{2^2} a_1 = \frac{(-1)^2 N(N-1)}{(2!)^2} a_0$$

$$\vdots \\ a_n = \frac{(-1)^n N(N-1) \dots (N-(n-1))}{(n!)^2} a_0$$

$$a_n = \frac{(-1)^n N(N-1) \dots (N-n+1)(N-n) \dots 1}{(N-n)! (n!)^2} a_0$$

$$a_n = \frac{(-1)^n n!}{(N-n)! (n!)^2} a_0$$

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We want $L_N(z)$ normalized such that $L_N(0) = N!$

$$L_N(0) = a_0 z^0 = a_0 \Rightarrow a_0 = N!$$

Plugging back in the expression of the coefficients a_n :

$$a_n = \frac{(-1)^n N!}{(N-n)! (n!)^2} \cdot N! = \frac{(-1)^n (N!)^2}{(N-n)! (n!)^2}$$

$$\text{And therefore } L_N(z) = \sum_{n=0}^{N-1} \frac{(-1)^n (N!)^2}{(N-n)! (n!)^2} z^n$$

Thus

$$\begin{aligned} L_3(z) &= 3! + \frac{(-1)^1 (3!)^2}{(3-1)! (1!)^2} z + \frac{(-1)^2 (3!)^2}{(3-2)! (2!)^2} z^2 + \frac{(-1)^3 (3!)^2}{(3-3)! (3!)^2} z^3 \\ &= 3 + 2z + \frac{(-1)(3 \times 2 \times 1)}{2! 1!} z^2 + \frac{(3 \times 2 \times 1)^2}{1! (2+1)^2} z^2 + \frac{(-1)(3 \times 2 \times 1)}{0! \times (3!)^2} z^3 \\ &= 6 - 18z + 9z^2 - z^3 \end{aligned}$$