

Professor Rio  
 EN.585.615.81.SP21 Mathematical Methods  
 Take Home Project 3  
 Johns Hopkins University  
 Student: Yves Greatti

## Question 1

The Womersley equation for blood flow is:

$$\rho \frac{\partial w}{\partial t} = \frac{\mu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) + \frac{\partial P}{\partial z}$$

Using  $\frac{\partial P}{\partial z} = A e^{int}$  and taking  $w(r, t) = u(r) e^{int}$  yields:  $\frac{\partial w}{\partial t} = (in) u e^{int}$ ,  $\frac{\partial w}{\partial r} = u'(r) e^{int}$ , and  $\frac{\partial^2 w}{\partial r^2} = u''(r) e^{int}$ ,  $\frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) = u'(r) e^{int} + r u''(r) e^{int}$  Therefore the Womersley equation becomes:

$$\frac{\mu}{r} \left[ u'(r) e^{int} + r u''(r) e^{int} \right] + A e^{int} = \rho (in) u(r) e^{int}$$

$$\mu \frac{d^2 u(r)}{dr^2} + \frac{\mu}{r} \frac{du(r)}{dr} + A = (in) \rho u(r) \text{ by dividing through } e^{int}$$

$$\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} - \frac{in \rho}{\mu} u = -\frac{A}{\mu} \text{ by dividing through } \mu \text{ and rearranging}$$

Finally using  $\nu = \frac{\mu}{\rho}$  we have:

$$\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} - \frac{in}{\nu} u = -\frac{A}{\mu}$$

By simple inspection, one particular solution is a constant w.r.t.  $r$ , such as  $u_p = C$ , substituting it into the differential equation gives:

$$-\frac{in \rho}{\mu} u_p = -\frac{A}{\mu}$$

thus  $u_p = \frac{A}{in \rho}$  The homogeneous equation is:

$$\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} + \frac{i^3 n}{\nu} u = 0$$

Take  $\lambda^2 = \frac{i^3 n}{\nu}$ , we now have:

$$\begin{aligned}\frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} + \lambda^2 u &= 0 \\ r^2 \frac{d^2 u(r)}{dr^2} + r \frac{du(r)}{dr} + (\lambda r)^2 u &= 0 \quad (1)\end{aligned}$$

Take  $x = \lambda r$ , then:

$$\begin{aligned}\frac{du(x)}{dr} &= \frac{du(\lambda r)}{dr} = \lambda \frac{du(x)}{dx} \\ \frac{d^2 u(x)}{dr^2} &= \lambda^2 \frac{d^2 u(x)}{dx^2}\end{aligned}$$

Substitute back into (1), we have

$$\begin{aligned}\lambda^2 r^2 \frac{d^2 u(x)}{dx^2} + \lambda r \frac{du(x)}{dx} + (\lambda r)^2 u(x) &= 0 \\ x^2 \frac{d^2 u(x)}{dx^2} + x \frac{du(x)}{dx} + x^2 u &= 0\end{aligned}$$

The last equation is a Bessel's equation of order 0, therefore the solution,  $u_h$ , of the homogeneous equation is a solution of a Bessel's equation of order 0:

$$u_h(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$$

And

$$u(r) = u_h(r) + u_p(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r) + \frac{A}{in\rho}$$

Now we apply the boundary conditions to our solution.

$$u'(r) = C_1 J'_0(\lambda r) + C_2 Y'_0(\lambda r)$$

We have

$$\begin{aligned}J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n! \Gamma(1+n)} \\ &= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \dots \\ J'_0(x) &= -2 \frac{x}{2^2} + 4 \frac{x^3}{2^2 4^2} - \dots \\ J'_0(0) &= 0 \\ \lim_{r \rightarrow 0} u'(r) &= \lim_{r \rightarrow 0} C_1 J'_0(\lambda r) + C_2 Y'_0(\lambda r) \\ &= 0 + \lim_{r \rightarrow 0} C_2 Y'_0(\lambda r)\end{aligned}$$

Looking at the plot of  $Y_0(x)$ , we see that in order to have  $\frac{\partial w}{\partial r}|_{r=0} = 0$  or  $\frac{\partial u}{\partial r}|_{r=0} = 0$ , the term in  $Y_0$  must be discarded and we need  $C_2 = 0$ . Thus

$$u(r) = C_1 J_0(\lambda r) + \frac{A}{i n \rho}$$

Using the second boundary condition  $w(R) = u(R) = 0$  we have  $C_1 J_0(\lambda R) + \frac{A}{i n \rho} = 0$  or  $C_1 = -\frac{A}{i n \rho J_0(\lambda R)}$  Putting everything back

$$\begin{aligned} u(r) &= \frac{A}{\rho i n} \left[ 1 - \frac{J_0(\lambda r)}{J_0(\lambda R)} \right] \\ &= \frac{A}{\rho i n} \left[ 1 - \frac{J_0(r \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}})}{J_0(R \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}})} \right] \end{aligned}$$

Take  $\alpha = R \sqrt{\frac{\lambda}{\nu}}$  and  $y = \frac{r}{R}$  then

$$\begin{aligned} J_0(r \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}}) &= J_0\left(\frac{r}{R} R \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}}\right) = J_0(\alpha y i^{\frac{3}{2}}) \\ J_0(R \sqrt{\frac{\lambda}{\nu}} i^{\frac{3}{2}}) &= J_0(\alpha i^{\frac{3}{2}}) \end{aligned}$$

Lastly

$$w(y, t) = u(r) e^{int} = \frac{A}{\rho i n} \left[ 1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})} \right] e^{int}$$

## Question 2

From

$$Q = 2\pi \int_0^R w(r, t) r dr$$

Make the change of variable  $y = \frac{r}{R}$ ,  $dy = \frac{dr}{R}$  and we have

$$Q = 2\pi \int_0^1 w(y, t) R^2 y dy = 2\pi R^2 \int_0^1 w y dy$$

Plugging the expression of  $w$  found in the previous question

$$\begin{aligned} Q &= 2\pi R^2 \frac{A}{\rho i n} \int_0^1 \left[ 1 - \frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})} \right] e^{int} y dy \\ &= \frac{2\pi R^2 A}{\rho i n} e^{int} \left[ \int_0^1 y dy - \frac{1}{J_0(\alpha i^{\frac{3}{2}})} \int_0^1 y J_0(\alpha y i^{\frac{3}{2}}) dy \right] \end{aligned}$$

$\int_0^1 y dy = [\frac{y^2}{2}]_0^1 = \frac{1}{2}$  and we make the change of variable  $s = \alpha i^{\frac{3}{2}} y, ds = \alpha i^{\frac{3}{2}} dy$   
so

$$\begin{aligned} \int_0^1 y J_0(\alpha y i^{\frac{3}{2}}) dy &= \int_0^{\alpha i^{\frac{3}{2}}} \frac{s}{\alpha i^{\frac{3}{2}}} J_0(s) \frac{1}{\alpha i^{\frac{3}{2}}} ds \\ &= \frac{1}{\alpha^2 i^3} \int_0^{\alpha i^{\frac{3}{2}}} s J_0(s) ds \\ &= \frac{\alpha i^{\frac{3}{2}}}{\alpha^2 i^3} J_1(\alpha i^{\frac{3}{2}}) \end{aligned}$$

Therefore

$$\begin{aligned} Q &= \frac{2\pi R^2 A}{\rho i n} e^{int} \left[ \frac{1}{2} - \frac{\alpha i^{\frac{3}{2}}}{\alpha^2 i^3} \frac{J_1(\alpha i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})} \right] \\ &= \frac{\pi R^2 A}{\rho i n} \left[ 1 - \frac{2\alpha i^{\frac{3}{2}}}{i^3 \alpha^2} \frac{J_1(\alpha i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})} \right] e^{int} \end{aligned}$$