18

Partial differential equations: general and particular solutions

In this chapter and the next the solution of differential equations of types typically encountered in the physical sciences and engineering is extended to situations involving more than one independent variable. A partial differential equation (PDE) is an equation relating an unknown function (the dependent variable) of two or more variables to its partial derivatives with respect to those variables. The most commonly occurring independent variables are those describing position and time, and so we will couch our discussion and examples in notation appropriate to them.

As in other chapters we will focus our attention on the equations that arise most often in physical situations. We will restrict our discussion, therefore, to linear PDEs, i.e. those of first degree in the dependent variable. Furthermore, we will discuss primarily second-order equations. The solution of first-order PDEs will necessarily be involved in treating these, and some of the methods discussed can be extended without difficulty to third- and higher-order equations. We shall also see that many ideas developed for ordinary differential equations (ODEs) can be carried over directly into the study of PDEs.

In this chapter we will concentrate on general solutions of PDEs in terms of arbitrary functions and the particular solutions that may be derived from them in the presence of boundary conditions. We also discuss the existence and uniqueness of the solutions to PDEs under given boundary conditions.

In the next chapter the methods most commonly used in practice for obtaining solutions to PDEs subject to given boundary conditions will be considered. These methods include the separation of variables, integral transforms and Green's functions. This division of material is rather arbitrary and really has been made only to emphasise the general usefulness of the latter methods. In particular, it will be readily apparent that some of the results of the present chapter are in fact solutions in the form of separated variables, but arrived at by a different approach.

18.1 Important partial differential equations

Most of the important PDEs of physics are second-order and linear. In order to gain familiarity with their general form, some of the more important ones will now be briefly discussed. These equations apply to a wide variety of different physical systems.

Since, in general, the PDEs listed below describe three-dimensional situations, the independent variables are $\mathbf{r}$ and t, where $\mathbf{r}$ is the position vector and t is time. The actual variables used to specify the position vector $\mathbf{r}$ are dictated by the coordinate system in use. For example, in Cartesian coordinates the independent variables of position are x, y and z, whereas in spherical polar coordinates they are r, θ and ϕ. The equations may be written in a coordinate-independent manner, however, by the use of the Laplacian operator ∇^2.

18.1.1 The wave equation

The wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{18.1}$$

describes as a function of position and time the displacement from equilibrium, $u(\mathbf{r}, t)$, of a vibrating string or membrane or a vibrating solid, gas or liquid. The equation also occurs in electromagnetism, where u may be a component of the electric or magnetic field in an elecromagnetic wave or the current or voltage along a transmission line. The quantity c is the speed of propagation of the waves.

► *Find the equation satisfied by small transverse displacements $u(x, t)$ of a uniform string of mass per unit length ρ held under a uniform tension T, assuming that the string is initially located along the x-axis in a Cartesian coordinate system.*

Figure 18.1 shows the forces acting on an elemental length Δs of the string. If the tension T in the string is uniform along its length then the net upward vertical force on the element is

$$\Delta F = T \sin\theta_2 - T \sin\theta_1.$$

Assuming that the angles θ_1 and θ_2 are both small, we may make the approximation $\sin\theta \approx \tan\theta$. Since at any point on the string the slope $\tan\theta = \partial u/\partial x$, the force can be written

$$\Delta F = T\left[\frac{\partial u(x+\Delta x, t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] \approx T\frac{\partial^2 u(x,t)}{\partial x^2}\Delta x,$$

where we have used the definition of the partial derivative to simplify the RHS.

This upward force may be equated, by Newton's second law, to the product of the mass of the element and its upward acceleration. The element has a mass $\rho\,\Delta s$, which is approximately equal to $\rho\,\Delta x$ if the vibrations of the string are small, and so we have

$$\rho\,\Delta x\,\frac{\partial^2 u(x,t)}{\partial t^2} = T\frac{\partial^2 u(x,t)}{\partial x^2}\Delta x.$$

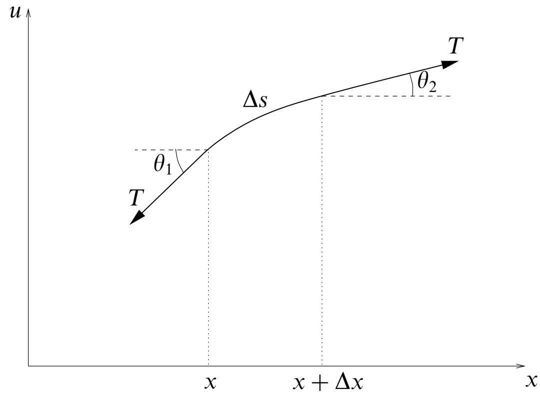

Figure 18.1 The forces acting on an element of a string under uniform tension T.

Dividing both sides by Δx we obtain, for the vibrations of the string, the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

where $c^2 = T/\rho$. ◀

The longitudinal vibrations of an elastic rod obey a very similar equation to that derived in the above example, namely

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2};$$

here ρ is the mass per unit volume and E is Young's modulus.

The wave equation can be generalised slightly. For example, in the case of the vibrating string, there could also be an external upward vertical force $f(x,t)$ per unit length acting on the string at time t. The transverse vibrations would then satisfy the equation

$$T\frac{\partial^2 u}{\partial x^2} + f(x,t) = \rho\frac{\partial^2 u}{\partial t^2},$$

which is clearly of the form 'upward force per unit length = mass per unit length × upward acceleration'.

Similar examples, but involving two or three spatial dimensions rather than one, are provided by the equation governing the transverse vibrations of a stretched membrane subject to an external vertical force density $f(x,y,t)$,

$$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x,y,t) = \rho(x,y)\frac{\partial^2 u}{\partial t^2},$$

where ρ is the mass per unit area of the membrane and T is the tension.

18.1.2 The diffusion equation

The diffusion equation

$$\kappa\nabla^2 u = \frac{\partial u}{\partial t} \tag{18.2}$$

describes the temperature u in a region containing no heat sources or sinks; it also applies to the diffusion of a chemical that has a concentration $u(\mathbf{r}, t)$. The constant κ is called the diffusivity. The equation is clearly second-order in the three spatial variables, but first order in time.

►*Derive the equation satisfied by the temperature $u(\mathbf{r}, t)$ at time t for a material of uniform thermal conductivity k, specific heat capacity s and density ρ. Express the equation in Cartesian coordinates.*

Let us consider an arbitrary volume V lying within the solid and bounded by a surface S (this may coincide with the surface of the solid if so desired). At any point in the solid the rate of heat flow per unit area in any given direction $\hat{\mathbf{r}}$ is proportional to minus the component of the temperature gradient in that direction and so is given by $(-k\nabla u)\cdot\hat{\mathbf{r}}$. The total flux of heat *out* of the volume V per unit time is given by

$$\begin{aligned} -\frac{dQ}{dt} &= \iint_S (-k\nabla u)\cdot\hat{\mathbf{n}}\, dS \\ &= \iiint_V \nabla\cdot(-k\nabla u)\, dV, \end{aligned} \tag{18.3}$$

where Q is the total heat energy in V at time t and $\hat{\mathbf{n}}$ is the outward-pointing unit normal to S; note that we have used the divergence theorem to convert the surface integral into a volume integral.

We can also express Q as a volume integral over V,

$$Q = \iiint_V s\rho u\, dV,$$

and its rate of change is then given by

$$\frac{dQ}{dt} = \iiint_V s\rho\frac{\partial u}{\partial t}\, dV, \tag{18.4}$$

where we have taken the derivative with respect to time inside the integral (see section 5.12).

Comparing (18.3) and (18.4), and remembering that the volume V is arbitrary, we obtain the three-dimensional diffusion equation

$$\kappa\nabla^2 u = \frac{\partial u}{\partial t},$$

where the diffusion coefficient $\kappa = k/(s\rho)$. To express this equation in Cartesian coordinates, we simply write ∇^2 in terms of x, y and z to obtain

$$\kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial u}{\partial t}. \blacktriangleleft$$

The diffusion equation just derived can be generalised to

$$k\nabla^2 u + f(\mathbf{r}, t) = s\rho\frac{\partial u}{\partial t}.$$

The second term, $f(\mathbf{r}, t)$, represents a varying density of heat sources throughout the material but is often not required in physical applications. In the most general case, k, s and ρ may depend on position $\mathbf{r}$, in which case the first term becomes $\nabla \cdot (k\nabla u)$. However, in the simplest application the heat flow is one-dimensional with no heat sources, and the equation becomes (in Cartesian coordinates)

$$\frac{\partial^2 u}{\partial x^2} = \frac{s\rho}{k}\frac{\partial u}{\partial t}.$$

18.1.3 Laplace's equation

Laplace's equation,

$$\nabla^2 u = 0, \tag{18.5}$$

may be obtained by setting $\partial u/\partial t = 0$ in the diffusion equation (18.2), and describes (for example) the *steady-state* temperature distribution in a solid in which there are no heat sources – i.e. the temperature distribution after a long time has elapsed.

Laplace's equation also describes the gravitational potential in a region containing no matter or the electrostatic potential in a charge-free region. Further, it applies to the flow of an incompressible fluid with no sources, sinks or vortices; in this case u is the velocity potential, from which the velocity is given by $v = \nabla u$.

18.1.4 Poisson's equation

Poisson's equation,

$$\nabla^2 u = \rho(\mathbf{r}), \tag{18.6}$$

describes the same physical situations as Laplace's equation, but in regions containing matter, charges or sources of heat or fluid. The function $\rho(\mathbf{r})$ is called the source density and in physical applications usually contains some multiplicative physical constants. For example, if u is the electrostatic potential in some region of space, in which case ρ is the density of electric charge, then $\nabla^2 u = -\rho(\mathbf{r})/\epsilon_0$, where ϵ_0 is the permittivity of free space. Alternatively, u might represent the gravitational potential in some region where the matter density is given by ρ; then $\nabla^2 u = 4\pi G\rho(\mathbf{r})$, where G is the gravitational constant.

18.1.5 Schrödinger's equation

The Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2 u + V(\mathbf{r})u = i\hbar\frac{\partial u}{\partial t}, \tag{18.7}$$

describes the quantum mechanical wavefunction $u(\mathbf{r}, t)$ of a non-relativistic particle of mass m; $\hbar$ is Planck's constant divided by 2π. Like the diffusion equation it is second order in the three spatial variables and first order in time.

18.2 General form of solution

Before turning to the methods by which we may hope to solve PDEs such as those listed in the previous section, it is instructive, as for ODEs in chapter 14, to study how PDEs may be formed from a set of possible solutions. Such a study can provide an indication of how equations obtained not from possible solutions but from physical arguments might be solved.

For definiteness let us suppose we have a set of functions involving two independent variables x and y. Without further specification this is of course a very wide set of functions, and we could not expect to find a useful equation that they all satisfy. However, let us consider a type of function $u_i(x, y)$ in which x and y appear in a particular way, such that u_i can be written as a function (however complicated) *of a single variable* p, itself a simple function of x and y.

Let us illustrate this by considering the three functions

$$\begin{aligned} u_1(x, y) &= x^4 + 4(x^2y + y^2 + 1), \\ u_2(x, y) &= \sin x^2 \cos 2y + \cos x^2 \sin 2y, \\ u_3(x, y) &= \frac{x^2 + 2y + 2}{3x^2 + 6y + 5}. \end{aligned}$$

These are all fairly complicated functions of x and y and a single differential equation of which each one is a solution is not obvious. However, if we observe that in fact each can be expressed as a function of the variable $p = x^2 + 2y$ alone (with no other x or y involved) then a great simplification takes place. Written in terms of p the above equations become

$$\begin{aligned} u_1(x, y) &= (x^2 + 2y)^2 + 4 = p^2 + 4 = f_1(p), \\ u_2(x, y) &= \sin(x^2 + 2y) = \sin p = f_2(p), \\ u_3(x, y) &= \frac{(x^2 + 2y) + 2}{3(x^2 + 2y) + 5} = \frac{p + 2}{3p + 5} = f_3(p). \end{aligned}$$

Let us now form, for each u_i, the partial derivatives $\partial u_i/\partial x$ and $\partial u_i/\partial y$. In each case these are (writing both the form for general p and the one appropriate to our particular case, $p = x^2 + 2y$)

$$\begin{aligned} \frac{\partial u_i}{\partial x} &= \frac{df_i(p)}{dp}\frac{\partial p}{\partial x} = 2xf_i', \\ \frac{\partial u_i}{\partial y} &= \frac{df_i(p)}{dp}\frac{\partial p}{\partial y} = 2f_i', \end{aligned}$$

for $i = 1, 2, 3$. All reference to the form of f_i can be eliminated from these equations by cross-multiplication, obtaining

$$\frac{\partial p}{\partial y}\frac{\partial u_i}{\partial x} = \frac{\partial p}{\partial x}\frac{\partial u_i}{\partial y},$$

or, for our specific form, $p = x^2 + 2y$,

$$\frac{\partial u_i}{\partial x} = x\frac{\partial u_i}{\partial y}. \tag{18.8}$$

It is thus apparent that not only are the three functions u_1, u_2 u_3 solutions of the PDE (18.8) but so also is *any arbitrary function* $f(p)$ of which the argument p has the form $x^2 + 2y$.

18.3 General and particular solutions

In the last section we found that the first-order PDE (18.8) has as a solution *any* function of the variable $x^2 + 2y$. This points the way for the solution of PDEs of other orders, as follows. It is *not* generally true that an nth-order PDE can always be considered as resulting from the elimination of n arbitrary *functions* from its solution (as opposed to the elimination of n arbitrary *constants* for an nth-order ODE, see section 14.1). However, given specific PDEs we can try to solve them by seeking combinations of variables in terms of which the solutions may be expressed as arbitrary functions. Where this is possible we may expect n combinations to be involved in the solution.

Naturally, the exact functional form of the solution for any particular situation must be determined by some set of boundary conditions. For instance, if the PDE contains two independent variables x and y then for complete determination of its solution the boundary conditions will take a form equivalent to specifying $u(x, y)$ along a suitable continuum of points in the xy-plane (usually along a line).

We now discuss the general and particular solutions of first- and second-order PDEs. In order to simplify the algebra, we will restrict our discussion to equations containing just two independent variables x and y. Nevertheless, the method presented below may be extended to equations containing several independent variables.

18.3.1 First-order equations

Although most of the PDEs encountered in physical contexts are second order (i.e. they contain $\partial^2 u/\partial x^2$ or $\partial^2 u/\partial x \partial y$, etc.), we now discuss first-order equations to illustrate the general considerations involved in the form of the solution and in satisfying any boundary conditions on the solution.

The most general first-order linear PDE (containing two independent variables) is of the form

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C(x,y)u = R(x,y), \tag{18.9}$$

where $A(x,y)$, $B(x,y)$, $C(x,y)$ and $R(x,y)$ are given functions. Clearly, if either $A(x,y)$ or $B(x,y)$ is zero then the PDE may be solved straightforwardly as a first-order linear ODE (as discussed in chapter 14), the only modification being that the arbitrary constant of integration becomes an *arbitrary function* of x or y respectively.

►*Find the general solution* $u(x,y)$ *of*

$$x\frac{\partial u}{\partial x} + 3u = x^2.$$

Dividing through by x we obtain

$$\frac{\partial u}{\partial x} + \frac{3u}{x} = x,$$

which is a linear equation with integrating factor (see subsection 14.2.4)

$$\exp\left(\int \frac{3}{x}\,dx\right) = \exp(3\ln x) = x^3.$$

Multiplying through by this factor we find

$$\frac{\partial}{\partial x}(x^3u) = x^4,$$

which, on integrating with respect to x, gives

$$x^3u = \frac{x^5}{5} + f(y),$$

where $f(y)$ is an *arbitrary function* of y. Finally, dividing through by x^3, we obtain the solution

$$u(x,y) = \frac{x^2}{5} + \frac{f(y)}{x^3}. \blacktriangleleft$$

When the PDE contains partial derivatives with respect to both independent variables then, of course, we cannot employ the above procedure but must seek an alternative method. Let us for the moment restrict our attention to the special case in which $C(x,y) = R(x,y) = 0$ and, following the discussion of the previous section, look for solutions of the form $u(x,y) = f(p)$ where p is some, at present unknown, combination of x and y. We then have

$$\frac{\partial u}{\partial x} = \frac{df(p)}{dp}\frac{\partial p}{\partial x},$$
$$\frac{\partial u}{\partial y} = \frac{df(p)}{dp}\frac{\partial p}{\partial y},$$

which, when substituted into the PDE (18.9), give

$$\left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right]\frac{df(p)}{dp} = 0.$$

This removes all reference to the actual form of the function $f(p)$ since for non-trivial p we must have

$$A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0. \tag{18.10}$$

Let us now consider the necessary condition for $f(p)$ to remain constant as x and y vary; this is that p itself remains constant. Thus for f to remain constant implies that x and y must vary in such a way that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0. \tag{18.11}$$

The forms of (18.10) and (18.11) are very alike, and become the same if we require that

$$\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}. \tag{18.12}$$

By integrating this expression the form of p can be found.

►*For*

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0, \tag{18.13}$$

find (i) the solution that takes the value $2y+1$ *on the line* $x=1$, *and (ii) a solution that has the value* 4 *at the point* $(1,1)$.

If we seek a solution of the form $u(x,y) = f(p)$, we deduce from (18.12) that $u(x,y)$ will be constant along lines of (x,y) that satisfy

$$\frac{dx}{x} = \frac{dy}{-2y},$$

which on integrating gives $x = cy^{-1/2}$. Identifying the constant of integration c with $p^{1/2}$ (to avoid fractional powers), we conclude that $p = x^2y$. Thus the general solution of the PDE (18.13) is

$$u(x,y) = f(x^2y),$$

where f is an arbitrary function.

We must now find the particular solutions that obey each of the imposed boundary conditions. For boundary condition (i) a little thought shows that the particular solution required is

$$u(x,y) = 2(x^2y) + 1 = 2x^2y + 1. \tag{18.14}$$

For boundary condition (ii) some obviously acceptable solutions are

$$u(x,y) = x^2y + 3,$$
$$u(x,y) = 4x^2y,$$
$$u(x,y) = 4.$$

Each is a valid solution (the freedom of choice of form arises from the fact that u is specified at only one point $(1,1)$, and not along a continuum (say), as in boundary condition (i)). All three are particular examples of the general solution, which may be written, for example, as

$$u(x,y) = x^2y + 3 + g(x^2y),$$

where $g = g(x^2y) = g(p)$ is an arbitrary function subject only to $g(1) = 0$. For this example, the forms of g corresponding to the particular solutions listed above are $g(p) = 0$, $g(p) = 3p - 3$, $g(p) = 1 - p$. ◀

As mentioned above, in order to find a solution of the form $u(x,y) = f(p)$ we require that the original PDE contains no term in u, but only terms containing its partial derivatives. If a term in u is present, so that $C(x,y) \neq 0$ in (18.9), then the procedure needs some modification, since we cannot simply divide out the dependence on $f(p)$ to obtain (18.10). In such cases we look instead for a solution of the form $u(x,y) = h(x,y)f(p)$. We illustrate this method in the following example.

▶*Find the general solution of*

$$x\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} - 2u = 0. \qquad (18.15)$$

We seek a solution of the form $u(x,y) = h(x,y)f(p)$, with the consequence that

$$\frac{\partial u}{\partial x} = \frac{\partial h}{\partial x}f(p) + h\frac{df(p)}{dp}\frac{\partial p}{\partial x},$$
$$\frac{\partial u}{\partial y} = \frac{\partial h}{\partial y}f(p) + h\frac{df(p)}{dp}\frac{\partial p}{\partial y}.$$

Substituting these expressions into the PDE (18.15) and rearranging, we obtain

$$\left(x\frac{\partial h}{\partial x} + 2\frac{\partial h}{\partial y} - 2h\right)f(p) + \left(x\frac{\partial p}{\partial x} + 2\frac{\partial p}{\partial y}\right)h\frac{df(p)}{dp} = 0.$$

The first factor in parentheses is just the original PDE with u replaced by h. Therefore, if h is *any* solution of the PDE, *however simple*, this term will vanish, to leave

$$\left(x\frac{\partial p}{\partial x} + 2\frac{\partial p}{\partial y}\right)h\frac{df(p)}{dp} = 0,$$

from which, as in the previous case, we obtain

$$x\frac{\partial p}{\partial x} + 2\frac{\partial p}{\partial y} = 0.$$

From (18.11) and (18.12) we see that $u(x,y)$ will be constant along lines of (x,y) that satisfy

$$\frac{dx}{x} = \frac{dy}{2},$$

which integrates to give $x = c\exp(y/2)$. Identifying the constant of integration c with p we find $p = x\exp(-y/2)$. Thus the general solution of (18.15) is

$$u(x,y) = h(x,y)f(x\exp(-\tfrac{1}{2}y)),$$

where $f(p)$ is any arbitrary function of p and $h(x,y)$ is any solution of (18.15).

If we take, for example, $h(x,y) = \exp y$, which clearly satisfies (18.15), then the general solution is

$$u(x,y) = (\exp y) f(x \exp(-\tfrac{1}{2}y)).$$

Alternatively, $h(x,y) = x^2$ also satisfies (18.15) and so the general solution to the equation can also be written

$$u(x,y) = x^2 g(x \exp(-\tfrac{1}{2}y)),$$

where g is an arbitrary function of p; clearly $g(p) = f(p)/p^2$. ◀

18.3.2 Inhomogeneous equations and problems

Let us discuss in a more general form the particular solutions of (18.13) found in the second example of the previous subsection. It is clear that, so far as this equation is concerned, if $u(x,y)$ is a solution then so is any multiple of $u(x,y)$ or any linear sum of separate solutions $u_1(x,y) + u_2(x,y)$. However, when it comes to fitting the boundary conditions this is not so.

For example, although $u(x,y)$ in (18.14) satisfies the PDE and the boundary condition $u(1,y) = 2y + 1$, the function $u_1(x,y) = 4u(x,y) = 8xy + 4$, whilst satisfying the PDE, takes the value $8y+4$ on the line $x = 1$ and so does not satisfy the required boundary condition. Likewise the function $u_2(x,y) = u(x,y)+f_1(x^2y)$, for arbitrary f_1, satisfies (18.13) but takes the value $u_2(1,y) = 2y + 1 + f_1(y)$ on the line $x = 1$, and so is not of the required form unless f_1 is identically zero.

Thus we see that when treating the superposition of solutions of PDEs two considerations arise, one concerning the equation itself and the other connected to the boundary conditions. The *equation* is said to be homogeneous if the fact that $u(x,y)$ is a solution implies that $\lambda u(x,y)$, for any constant λ, is also a solution. However, the *problem* is said to be homogeneous if, in addition, the boundary conditions are such that if they are satisfied by $u(x,y)$ then they are also satisfied by $\lambda u(x,y)$. The last requirement itself is referred to as that of *homogeneous boundary conditions*.

For example, the PDE (18.13) is homogeneous but the general first-order equation (18.9) would not be homogeneous unless $R(x,y) = 0$. Furthermore, the boundary condition (i) imposed on the solution of (18.13) in the previous subsection is not homogeneous though, in this case, the boundary condition

$$u(x,y) = 0 \quad \text{on the line } y = 4x^{-2}$$

would be, since $u(x,y) = \lambda(x^2y - 4)$ satisfies this condition for any λ and, being a function of x^2y, satisfies (18.13).

The reason for discussing the homogeneity of PDEs and their boundary conditions is that in linear PDEs there is a close parallel to the complementary-function and particular-integral property of ODEs. The general solution of an inhomogeneous problem can be written as the sum of *any* particular solution of the problem and the general solution of the corresponding homogeneous problem (as for ODEs, we require that the particular solution is not already contained in the general solution of the homogeneous problem). Thus, for example, the general solution of

$$\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} + au = f(x, y), \tag{18.16}$$

subject to, say, the boundary condition $u(0, y) = g(y)$, is given by

$$u(x, y) = v(x, y) + w(x, y),$$

where $v(x, y)$ is any solution (however simple) of (18.16) such that $v(0, y) = g(y)$ and $w(x, y)$ is the general solution of

$$\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y} + aw = 0, \tag{18.17}$$

with $w(0, y) = 0$. If the boundary conditions are sufficiently specified then the only possible solution of (18.17) will be $w(x, y) \equiv 0$ and $v(x, y)$ will be the complete solution by itself.

Alternatively, we may begin by finding the general solution of the inhomogeneous equation (18.16) *without* regard for any boundary conditions; it is just the sum of the general solution to the homogeneous equation and a particular integral of (18.16), both without reference to the boundary conditions. The boundary conditions can then be used to find the appropriate particular solution from the general solution.

We will not discuss at length general methods of obtaining particular integrals of PDEs but merely note that some of those methods available for ordinary differential equations can be suitably extended.†

►*Find the general solution of*

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 3x. \tag{18.18}$$

Hence find the most general particular solution (i) which satisfies $u(x, 0) = x^2$ *and (ii) which has the value* $u(x, y) = 2$ *at the point* $(1, 0)$.

This equation is inhomogeneous, and so let us first find the general solution of (18.18) without regard for any boundary conditions. We begin by looking for the solution of the corresponding homogeneous equation ((18.18) with the RHS equal to zero) of the form $u(x, y) = f(p)$. Following the same procedure as that used in the solution of (18.13) we find that $u(x, y)$ will be constant along lines of (x, y) that satisfy

$$\frac{dx}{y} = \frac{dy}{-x} \quad \Rightarrow \quad \frac{x^2}{2} + \frac{y^2}{2} = c.$$

Identifying the constant of integration c with $p/2$, we find that the general solution of the

† See for example Piaggio, *Differential Equations* (Bell, 1954), p. 175 *et seq.*

homogeneous equation is $u(x,y) = f(x^2+y^2)$ for arbitrary function f. Now by inspection a particular integral of (18.18) is $u(x,y) = -3y$, and so the general solution to (18.18) is

$$u(x,y) = f(x^2+y^2) - 3y.$$

Boundary condition (i) requires $u(x,0) = f(x^2) = x^2$, i.e. $f(z) = z$, and so the particular solution in this case is

$$u(x,y) = x^2 + y^2 - 3y.$$

Similarly, boundary condition (ii) requires $u(1,0) = f(1) = 2$. One possibility is $f(z) = 2z$, and if we make this choice, then one way of writing the most general particular solution is

$$u(x,y) = 2x^2 + 2y^2 - 3y + g(x^2+y^2),$$

where g is any arbitrary function for which $g(1) = 0$. Alternatively, a simpler choice would be $f(z) = 2$, leading to

$$u(x,y) = 2 - 3y + g(x^2+y^2). \blacktriangleleft$$

Although we have discussed the solution of inhomogeneous problems only for first-order equations, the general considerations hold true for linear PDEs of higher order.

18.3.3 Second-order equations

As noted in section 18.1, second-order linear PDEs are of great importance in describing the behaviour of many physical systems. As in our discussion of first-order equations, for the moment we shall restrict our discussion to equations with just two independent variables; extensions to a greater number of independent variables are straightforward.

The most general second-order linear PDE (containing two independent variables) has the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = R(x,y), \tag{18.19}$$

where $A, B, \ldots, F$ and $R(x,y)$ are given functions of x and y. Because of the nature of the solutions to such equations, they are usually divided into three classes, a division of which we will make further use in subsection 18.6.2. The equation (18.19) is called *hyperbolic* if $B^2 > 4AC$, *parabolic* if $B^2 = 4AC$ and *elliptic* if $B^2 < 4AC$. Clearly, if A, B and C are functions of x and y (rather than just constants) then the equation might be of different types in different parts of the xy-plane.

Equation (18.19) obviously represents a very large class of PDEs, and it is usually impossible to find closed-form solutions to most of these equations. Therefore, for the moment we shall consider only homogeneous equations, with $R(x,y) = 0$, and make the further (greatly simplifying) restriction that, throughout the remainder of this section, $A, B, \ldots, F$ are not functions of x and y but merely constants.

We now tackle the problem of solving some types of second-order PDE with constant coefficients by seeking solutions that are arbitrary functions of particular combinations of independent variables, just as we did for first-order equations.

Following the discussion of the previous section, we can hope to find such solutions only if all the terms of the equation involve the same total number of differentiations, i.e. all terms are of the same order, although the number of differentiations with respect to the individual independent variables may be different. This means that in (18.19) we require the constants D, E and F to be identically zero (we have, of course, already assumed that $R(x, y)$ is zero), so that we are now considering only equations of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0, \tag{18.20}$$

where A, B and C are constants. We note that both the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,$$

and the two-dimensional Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

are of this form, but that the diffusion equation,

$$\kappa\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0,$$

is not, since it contains a first-order derivative.

Since all the terms in (18.20) involve two differentiations, by assuming a solution of the form $u(x, y) = f(p)$, where p is some unknown function of x and y (or t), we may be able to obtain a common factor $d^2f(p)/dp^2$ as the only appearance of f on the LHS. Then, because of the zero RHS, all reference to the form of f can be cancelled out.

We can gain some guidance on suitable forms for the combination $p = p(x, y)$ by considering $\partial u/\partial x$ when u is given by $u(x, y) = f(p)$, for then

$$\frac{\partial u}{\partial x} = \frac{df(p)}{dp}\frac{\partial p}{\partial x}.$$

Clearly differentiation of this equation with respect to x (or y) will not lead to a single term on the RHS, containing f only as $d^2f(p)/dp^2$, unless the factor $\partial p/\partial x$ is a constant so that $\partial^2 p/\partial x^2$ and $\partial^2 p/\partial x \partial y$ are necessarily zero. This shows that p must be a linear function of x. In an exactly similar way p must also be a linear function of y, i.e. $p = ax + by$.

If we assume a solution of (18.20) of the form $u(x, y) = f(ax+by)$, and evaluate the terms ready for substitution into (18.20), we obtain

$$\frac{\partial u}{\partial x} = a\frac{df(p)}{dp}, \qquad \frac{\partial u}{\partial y} = b\frac{df(p)}{dp},$$

$$\frac{\partial^2 u}{\partial x^2} = a^2\frac{d^2 f(p)}{dp^2}, \qquad \frac{\partial^2 u}{\partial x \partial y} = ab\frac{d^2 f(p)}{dp^2}, \qquad \frac{\partial^2 u}{\partial y^2} = b^2\frac{d^2 f(p)}{dp^2},$$

which on substitution give

$$\left(Aa^2 + Bab + Cb^2\right)\frac{d^2 f(p)}{dp^2} = 0. \tag{18.21}$$

This is the form we have been seeking, since now a solution independent of the form of f can be obtained if we require that a and b satisfy

$$Aa^2 + Bab + Cb^2 = 0.$$

From this quadratic, two values for the ratio of the two constants a and b are obtained,

$$b/a = [-B \pm (B^2 - 4AC)^{1/2}]/2C.$$

If we denote these two ratios by λ_1 and λ_2 then *any* functions of the two variables

$$p_1 = x + \lambda_1 y, \qquad p_2 = x + \lambda_2 y$$

will be solutions of the original equation (18.20). The omission of the constant factor a from p_1 and p_2 is of no consequence since this can always be absorbed into the particular form of any chosen function; only the *relative* weighting of x and y in p is important.

Since p_1 and p_2 are in general different, we can thus write the general solution of (18.20) as

$$u(x, y) = f(x + \lambda_1 y) + g(x + \lambda_2 y), \tag{18.22}$$

where f and g are arbitrary functions.

Finally, we note that the alternative solution $d^2 f(p)/dp^2 = 0$ to (18.21) leads only to the trivial solution $u(x, y) = kx + ly + m$, for which all second derivatives are individually zero.

► *Find the general solution of the one-dimensional wave equation*

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

This equation is (18.20) with $A = 1$, $B = 0$ and $C = -1/c^2$, and so the values of λ_1 and λ_2 are the solutions of

$$1 - \frac{\lambda^2}{c^2} = 0,$$

namely $\lambda_1 = -c$ and $\lambda_2 = c$. This means that arbitrary functions of the quantities

$$p_1 = x - ct, \qquad p_2 = x + ct$$

will be satisfactory solutions of the equation and that the general solution will be

$$u(x,t) = f(x - ct) + g(x + ct), \tag{18.23}$$

where f and g are arbitrary functions. This solution is discussed further in section 18.4. ◀

The method used to obtain the general solution of the wave equation may also be applied straightforwardly to Laplace's equation.

▶ *Find the general solution of the two-dimensional Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{18.24}$$

Following the established procedure, we look for a solution that is a function $f(p)$ of $p = x + \lambda y$, where from (18.24) λ satisfies

$$1 + \lambda^2 = 0.$$

This requires that $\lambda = \pm i$, and satisfactory variables p are $p = x \pm iy$. The general solution required is therefore, in terms of arbitrary functions f and g,

$$u(x, y) = f(x + iy) + g(x - iy).$$ ◀

It will be apparent from the last two examples that the nature of the appropriate linear combination of x and y depends upon whether $B^2 > 4AC$ or $B^2 < 4AC$. This is exactly the same criterion as determines whether the PDE is hyperbolic or elliptic. Hence as a general result, hyperbolic and elliptic equations of the form (18.20), given the restriction that the constants A, B and C are real, have as solutions functions whose arguments have the form $x+\alpha y$ and $x+i\beta y$ respectively, where α and β themselves are real.

The one case not covered by this result is that in which $B^2 = 4AC$, i.e. a parabolic equation. In this case λ_1 and λ_2 are not different and only one suitable combination of x and y results, namely

$$u(x, y) = f(x - (B/2C)y).$$

To find the second part of the general solution we try, in analogy with the corresponding situation for ordinary differential equations, a solution of the form

$$u(x, y) = h(x, y)g(x - (B/2C)y).$$

Substituting this into (18.20) and using $A = B^2/4C$ results in

$$\left(A\frac{\partial^2 h}{\partial x^2} + B\frac{\partial^2 h}{\partial x \partial y} + C\frac{\partial^2 h}{\partial y^2}\right) g = 0.$$

Therefore we require $h(x, y)$ to be any solution of the original PDE. There are several simple solutions of this equation, but as only one is required we take the simplest non-trivial one, $h(x, y) = x$, to give the general solution of the parabolic equation

$$u(x, y) = f(x - (B/2C)y) + xg(x - (B/2C)y). \tag{18.25}$$

We could, of course, have taken $h(x, y) = y$, but this only leads to a solution that is already contained in (18.25).

►*Solve*

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

subject to the boundary conditions $u(0, y) = 0$ *and* $u(x, 1) = x^2$.

From our general result, functions of $p = x + \lambda y$ will be solutions provided

$$1 + 2\lambda + \lambda^2 = 0,$$

i.e. $\lambda = -1$ and the equation is parabolic. The general solution is therefore

$$u(x, y) = f(x - y) + xg(x - y).$$

The boundary condition $u(0, y) = 0$ implies $f(p) \equiv 0$, whilst $u(x, 1) = x^2$ yields

$$xg(x - 1) = x^2,$$

which gives $g(p) = p + 1$, Therefore the particular solution required is

$$u(x, y) = x(p + 1) = x(x - y + 1). \blacktriangleleft$$

To reinforce the material discussed above we will now give alternative derivations of the general solutions (18.22) and (18.25) by expressing the original PDE in terms of new variables before solving it. The actual solution will then become almost trivial; but, of course, it will be recognised that suitable new variables could hardly have been guessed if it were not for the work already done. This does not detract from the validity of the derivation to be described, only from the likelihood that it would be discovered by inspection.

We start again with (18.20) and change to new variables

$$\zeta = x + \lambda_1 y, \qquad \eta = x + \lambda_2 y.$$

With this change of variables, we have from the chain rule that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta},$$
$$\frac{\partial}{\partial y} = \lambda_1 \frac{\partial}{\partial \zeta} + \lambda_2 \frac{\partial}{\partial \eta}.$$

Using these and the fact that

$$A + B\lambda_i + C\lambda_i^2 = 0 \qquad \text{for } i = 1, 2,$$

equation (18.20) becomes

$$[2A + B(\lambda_1 + \lambda_2) + 2C\lambda_1\lambda_2]\frac{\partial^2 u}{\partial \zeta \partial \eta} = 0.$$

Then, providing the factor in brackets does not vanish, for which the required condition is easily shown to be $B^2 \neq 4AC$, we obtain

$$\frac{\partial^2 u}{\partial \zeta \partial \eta} = 0,$$

which has the successive integrals

$$\frac{\partial u}{\partial \eta} = F(\eta), \quad u(\zeta, \eta) = f(\eta) + g(\zeta).$$

This solution is just the same as (18.22),

$$u(x, y) = f(x + \lambda_2 y) + g(x + \lambda_1 y).$$

If the equation is parabolic (i.e. $B^2 = 4AC$), we instead use the new variables

$$\zeta = x + \lambda y, \qquad \eta = x,$$

and recalling that $\lambda = -(B/2C)$ we can reduce (18.20) to

$$A \frac{\partial^2 u}{\partial \eta^2} = 0.$$

Two straightforward integrations give as the general solution

$$u(\zeta, \eta) = \eta g(\zeta) + f(\zeta),$$

which in terms of x and y has exactly the form of (18.25),

$$u(x, y) = xg(x + \lambda y) + f(x + \lambda y).$$

Finally, as hinted at in subsection 18.3.2 with reference to first-order linear PDEs, some of the methods used to find particular integrals of linear ODEs can be suitably modified to find particular integrals of PDEs of higher order. In simple cases, however, an appropriate solution may often be found by inspection.

►*Find the general solution of*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6(x + y).$$

Following our previous methods and results, the complementary function is

$$u(x, y) = f(x + iy) + g(x - iy),$$

and only a particular integral remains to be found. By inspection a particular integral of the equation is $u(x, y) = x^3 + y^3$, and so the general solution can be written

$$u(x, y) = f(x + iy) + g(x - iy) + x^3 + y^3. \blacktriangleleft$$

18.4 The wave equation

We have already found that the general solution of the one-dimensional wave equation is

$$u(x,t) = f(x-ct) + g(x+ct), \tag{18.26}$$

where f and g are arbitrary functions. However, the equation is of such general importance that further discussion will not be out of place.

Let us imagine that $u(x,t) = f(x-ct)$ represents the displacement of a string at time t and position x. It is clear that all positions x and times t for which $x-ct =$ constant will have the same instantaneous displacement. But $x - ct =$ constant is exactly the relation between the time and position of an observer travelling with speed c along the positive x-direction. Consequently this moving observer sees a constant displacement of the string, whereas to a stationary observer, the initial profile $u(x,0)$ moves with speed c along the x-axis as if it were a rigid system. Thus $f(x-ct)$ represents a wave form of constant shape travelling along the positive x-axis with speed c, the actual form of the wave depending upon the function f. Similarly, the term $g(x+ct)$ is a constant wave form travelling with speed c in the negative x-direction. The general solution (18.23) represents a superposition of these.

If the functions f and g are the same then the complete solution (18.23) represents identical progressive waves going in opposite directions. This may result in a wave pattern whose profile does not progress, described as a *standing wave*. As a simple example, suppose both $f(p)$ and $g(p)$ have the form†

$$f(p) = g(p) = A\cos(kp + \epsilon).$$

Then (18.23) can be written as

$$\begin{aligned} u(x,t) &= A[\cos(kx - kct + \epsilon) + \cos(kx + kct + \epsilon)] \\ &= 2A\cos(kct)\cos(kx + \epsilon). \end{aligned}$$

The important thing to notice is that the shape of the wave pattern, given by the factor in x, is the same at all times but that its amplitude $2A\cos(kct)$ depends upon time. At some points x that satisfy

$$\cos(kx + \epsilon) = 0$$

there is no displacement at any time; such points are called *nodes*.

So far we have not imposed any boundary conditions on the solution (18.26). The problem of finding a solution to the wave equation that satisfies given boundary conditions is normally treated using the method of separation of variables

† In the usual notation, k is the wave number ($= 2\pi/\text{wavelength}$) and $kc = \omega$, the angular frequency of the wave.

discussed in the next chapter. Nevertheless, we now consider *D'Alembert's solution* $u(x,t)$ of the wave equation subject to initial conditions (boundary conditions) in the following general form:

$$\text{initial displacement, } u(x,0) = \phi(x); \quad \text{initial velocity, } \frac{\partial u(x,0)}{\partial t} = \psi(x).$$

The functions $\phi(x)$ and $\psi(x)$ are given and describe the displacement and velocity of each part of the string at the (arbitrary) time $t = 0$.

It is clear that what we need are the particular forms of the functions f and g in (18.26) that lead to the required values at $t = 0$. This means that

$$\phi(x) = u(x,0) = f(x-0) + g(x+0), \tag{18.27}$$

$$\psi(x) = \frac{\partial u(x,0)}{\partial t} = -cf'(x-0) + cg'(x+0), \tag{18.28}$$

where it should be noted that $f'(x-0)$ stands for $df(p)/dp$ evaluated, after the differentiation, at $p = x - c \times 0$; likewise for $g'(x+0)$.

Looking on the above two left-hand sides as functions of $p = x \pm ct$, but everywhere evaluated at $t = 0$, we may integrate (18.28) between an arbitrary (and irrelevant) lower limit p_0 and an indefinite upper limit p to obtain

$$\frac{1}{c}\int_{p_0}^{p} \psi(q)\,dq + K = -f(p) + g(p),$$

the constant of integration K depending on p_0. Comparing this equation with (18.27), with x replaced by p, we can establish the forms of the functions f and g as

$$f(p) = \frac{\phi(p)}{2} - \frac{1}{2c}\int_{p_0}^{p} \psi(q)\,dq - \frac{K}{2}, \tag{18.29}$$

$$g(p) = \frac{\phi(p)}{2} + \frac{1}{2c}\int_{p_0}^{p} \psi(q)\,dq + \frac{K}{2}. \tag{18.30}$$

Adding (18.29) with $p = x - ct$ to (18.30) with $p = x + ct$ gives as the solution to the original problem

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(q)\,dq, \tag{18.31}$$

in which we notice that all dependence on p_0 has disappeared.

Each of the terms in (18.31) has a fairly straightforward physical interpretation. In each case the factor $1/2$ represents the fact that only half a displacement profile that starts at any particular point on the string travels towards any other position x, the other half travelling away from it. The first term $\frac{1}{2}\phi(x-ct)$ arises from the initial displacement at a distance ct to the left of x; this travels forward arriving at x at time t. Similarly, the second contribution is due to the initial displacement at a distance ct to the right of x. The interpretation of the final term is a little less obvious. It can be viewed as representing the accumulated transverse displacement at position x due to the passage past x of all parts of the initial motion whose effects can reach x within a time t, both backward and forward travelling.

The extension to the three-dimensional wave equation of solutions of the type we have so far encountered presents no serious difficulty. In Cartesian coordinates the three-dimensional wave equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{18.32}$$

In close analogy with the one-dimensional case we try solutions that are functions of linear combinations of all four variables,

$$p = lx + my + nz + \mu t.$$

It is clear that a solution $u(x, y, z, t) = f(p)$ will be acceptable provided that

$$\left(l^2 + m^2 + n^2 - \frac{\mu^2}{c^2}\right)\frac{d^2 f(p)}{dp^2} = 0.$$

Thus, as in the one-dimensional case, f can be arbitrary provided that

$$l^2 + m^2 + n^2 = \mu^2/c^2.$$

Using an obvious normalisation, we take $\mu = \pm c$ and l, m, n as three numbers such that

$$l^2 + m^2 + n^2 = 1.$$

In other words (l, m, n) are the Cartesian components of a unit vector $\hat{\mathbf{n}}$ that points along the direction of propagation of the wave. The quantity p can be written in terms of vectors as the scalar expression $p = \hat{\mathbf{n}} \cdot \mathbf{r} \pm ct$, and the general solution of (18.32) is then

$$u(x, y, z, t) = u(\mathbf{r}, t) = f(\hat{\mathbf{n}} \cdot \mathbf{r} - ct) + g(\hat{\mathbf{n}} \cdot \mathbf{r} + ct), \tag{18.33}$$

where $\hat{\mathbf{n}}$ is *any* unit vector. It would perhaps be more transparent to write $\hat{\mathbf{n}}$ explicitly as one of the arguments of u.

18.5 The diffusion equation

One important class of second-order PDEs, which we have not yet considered in detail, is that in which the second derivative with respect to one variable appears, but only the first derivative with respect to another (usually time). This is exemplified by the one-dimensional diffusion equation

$$\kappa \frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{18.34}$$

in which κ is a constant with the dimensions length^2 $\times$ time^{-1}. The physical constants that go to make up κ in a particular case depend upon the nature of the process (e.g. solute diffusion, heat flow, etc.) and the material being described.

With (18.34) we cannot hope to repeat successfully the method of subsection 18.3.3, since now $u(x,t)$ is differentiated a different number of times on the two sides of the equation; any attempted solution in the form $u(x,t) = f(p)$ with $p = ax + bt$ will lead only to an equation in which the form of f cannot be cancelled out. Clearly we must try other methods.

Solutions may be obtained by using the standard method of separation of variables discussed in the next chapter. Alternatively, a simple solution is also given if both sides of (18.34), as it stands, are separately set equal to a constant α (say), so that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\alpha}{\kappa}, \qquad \frac{\partial u}{\partial t} = \alpha.$$

These equations have the general solutions

$$u(x,t) = \frac{\alpha}{2\kappa}x^2 + xg(t) + h(t) \quad \text{and} \quad u(x,t) = \alpha t + m(x)$$

respectively, and may be made compatible with each other if $g(t)$ is taken as constant, $g(t) = g$ (where g could be zero), $h(t) = \alpha t$ and $m(x) = (\alpha/2\kappa)x^2 + gx$. An acceptable solution is thus

$$u(x,t) = \frac{\alpha}{2\kappa}x^2 + gx + \alpha t + \text{constant}. \tag{18.35}$$

Let us now return to seeking solutions of equations by combining the independent variables in particular ways. Having seen that a linear combination of x and t will be of no value, we must search for other possible combinations. It has been noted already that κ has the dimensions length^2 $\times$ time^{-1} and so the combination of variables

$$\eta = \frac{x^2}{\kappa t}$$

will be dimensionless. Let us see if we can satisfy (18.34) with a solution of the form $u(x,t) = f(\eta)$. Evaluating the necessary derivatives we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{df(\eta)}{d\eta}\frac{\partial \eta}{\partial x} = \frac{2x}{\kappa t}\frac{df(\eta)}{d\eta}, \\
\frac{\partial^2 u}{\partial x^2} &= \frac{2}{\kappa t}\frac{df(\eta)}{d\eta} + \left(\frac{2x}{\kappa t}\right)^2 \frac{d^2 f(\eta)}{d\eta^2}, \\
\frac{\partial u}{\partial t} &= -\frac{x^2}{\kappa t^2}\frac{df(\eta)}{d\eta}.
\end{aligned}$$

Substituting these expressions into (18.34) we find that the new equation can be written entirely in terms of η,

$$4\eta\frac{d^2f(\eta)}{d\eta^2}+(2+\eta)\frac{df(\eta)}{d\eta}=0.$$

This is a straightforward ODE, which can be solved (using a minimum of explanation) as follows. Writing $f'(\eta)=df(\eta)/d\eta$, etc., we have

$$\begin{aligned}\frac{f''(\eta)}{f'(\eta)}&=-\frac{1}{2\eta}-\frac{1}{4}\\ \Rightarrow\quad \ln[\eta^{1/2}f'(\eta)]&=-\frac{\eta}{4}+c\\ \Rightarrow\quad f'(\eta)&=\frac{A}{\eta^{1/2}}\exp\left(\frac{-\eta}{4}\right)\\ \Rightarrow\quad f(\eta)&=A\int_{\eta_0}^{\eta}\mu^{-1/2}\exp\left(\frac{-\mu}{4}\right)d\mu.\end{aligned}$$

If we now write this in terms of a slightly different variable

$$\zeta=\frac{\eta^{1/2}}{2}=\frac{x}{2(\kappa t)^{1/2}},$$

then $d\zeta=\frac{1}{4}\eta^{-1/2}\,d\eta$, and the solution to (18.34) is given by

$$u(x,t)=f(\eta)=g(\zeta)=B\int_{\zeta_0}^{\zeta}\exp(-v^2)\,dv. \tag{18.36}$$

Here B is a constant and it should be noticed that x and t appear on the RHS only in the indefinite upper limit ζ, and then only in the combination $xt^{-1/2}$. If ζ_0 is chosen as zero then $u(x,t)$ is, to within a constant factor,† the error function $\text{erf}[x/2(\kappa t)^{1/2}]$, which is tabulated in many reference books. Only non-negative values of x and t are to be considered here, so that $\zeta\geq\zeta_0$.

Let us try to determine what kind of (say) temperature distribution and flow this represents. For definiteness we take $\zeta_0=0$. Firstly, since $u(x,t)$ in (18.36) depends only upon the product $xt^{-1/2}$, it is clear that all points x at times t such that $xt^{-1/2}$ has the same value have the same temperature. Put another way, at any specific time t the region having a particular temperature has moved along the positive x-axis a distance proportional to the square root of t. This is a typical *diffusion* process.

Notice that, on the one hand, at $t=0$, the variable $\zeta\to\infty$ and u becomes quite independent of x (except perhaps at $x=0$); the solution then represents a uniform spatial temperature distribution. On the other hand, at $x=0$, $u(x,t)$ is identically zero for all t.

† Take $B=2\pi^{-1/2}$ to give the usual error function normalised such that $\text{erf}(\infty)=1$. See the Appendix.

►*An infrared laser delivers a pulse of (heat) energy E to a point P on a large insulated sheet of thickness b, thermal conductivity k, specific heat s and density ρ. The sheet is initially at a uniform temperature. If $u(r,t)$ is the excess temperature a time t later, at a point that is a distance r ($\gg b$) from P, then show that a suitable expression for u is*

$$u(r,t) = \frac{\alpha}{t}\exp\left(-\frac{r^2}{2\beta t}\right), \tag{18.37}$$

where α and β are constants. (Note that we use r instead of ρ to denote the radial coordinate in plane polars so as to avoid confusion with the density.)

Further, (i) show that $\beta = 2k/(s\rho)$; (ii) show that the excess heat energy in the sheet is independent of t, and hence evaluate α; and (iii) show that the total heat flow past any circle of radius r is E.

The equation to be solved is the heat diffusion equation

$$k\nabla^2 u(\mathbf{r},t) = s\rho\frac{\partial u(\mathbf{r},t)}{\partial t}.$$

Since we only require the solution for $r \gg b$ we can treat the problem as two-dimensional with obvious circular symmetry. Thus only the r-derivative term in the expression for $\nabla^2 u$ is non-zero, giving

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = s\rho\frac{\partial u}{\partial t}, \tag{18.38}$$

where now $u(\mathbf{r},t) = u(r,t)$.

(i) Substituting the given expression (18.37) into (18.38) we obtain

$$\frac{2k\alpha}{\beta t^2}\left(\frac{r^2}{2\beta t} - 1\right)\exp\left(-\frac{r^2}{2\beta t}\right) = \frac{s\rho\alpha}{t^2}\left(\frac{r^2}{2\beta t} - 1\right)\exp\left(-\frac{r^2}{2\beta t}\right),$$

from which we find that (18.37) is a solution, provided $\beta = 2k/(s\rho)$.

(ii) The excess heat in the system at any time t is

$$\begin{aligned} b\rho s\int_0^\infty u(r,t)2\pi r\,dr &= 2\pi b\rho s\alpha\int_0^\infty \frac{r}{t}\exp\left(-\frac{r^2}{2\beta t}\right)dr \\ &= 2\pi b\rho s\alpha\beta. \end{aligned}$$

The excess heat is therefore independent of t and must be equal to the total heat input E, implying that

$$\alpha = \frac{E}{2\pi b\rho s\beta} = \frac{E}{4\pi bk}.$$

(iii) The total heat flow past a circle of radius r is

$$\begin{aligned} -2\pi rbk\int_0^\infty \frac{\partial u(r,t)}{\partial r}dt &= -2\pi rbk\int_0^\infty \frac{E}{4\pi bkt}\left(\frac{-r}{\beta t}\right)\exp\left(-\frac{r^2}{2\beta t}\right)dt \\ &= E\left[\exp\left(-\frac{r^2}{2\beta t}\right)\right]_0^\infty = E \qquad \text{for all } r. \end{aligned}$$

As we would expect, all the heat energy E deposited by the laser will eventually flow past a circle of any given radius r. ◄

18.6 Characteristics and the existence of solutions

So far in this chapter we have discussed how to find general solutions to various types of first- and second-order linear PDE. Moreover, given a set of boundary conditions we have shown how to find the particular solution (or class of solutions) that satisfies them. For first-order equations, for example, we found that if the value of $u(x, y)$ is specified along some curve in the xy-plane then the solution to the PDE is in general unique, but that if $u(x, y)$ is specified at only a single point then the solution is not unique: there exists a class of particular solutions all of which satisfy the boundary condition. In this section and the next we make more rigorous the notion of the types of boundary condition that cause a PDE to have a unique solution, a class of solutions, or no solution at all.

18.6.1 First-order equations

Let us consider the general first-order PDE (18.9) but now write it as

$$A(x, y)\frac{\partial u}{\partial x} + B(x, y)\frac{\partial u}{\partial y} = F(x, y, u). \tag{18.39}$$

Suppose we wish to solve this PDE subject to the boundary condition that $u(x, y) = \phi(s)$ is specified along some curve C in the xy-plane that is described parametrically by the equations $x = x(s)$ and $y = y(s)$, where s is the arc length along C. The variation of u along C is therefore given by

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{d\phi}{ds}. \tag{18.40}$$

We may then solve the two (inhomogeneous) simultaneous linear equations (18.39) and (18.40) for $\partial u/\partial x$ and $\partial u/\partial y$, *unless* the determinant of the coefficients vanishes (see section 8.18), i.e. unless

$$\begin{vmatrix} dx/ds & dy/ds \\ A & B \end{vmatrix} = 0.$$

At each point in the xy-plane this equation determines a set of curves called *characteristic curves* (or just *characteristics*), which thus satisfy

$$B\frac{dx}{ds} - A\frac{dy}{ds} = 0,$$

or, multiplying through by ds/dx and dividing through by A,

$$\frac{dy}{dx} = \frac{B(x, y)}{A(x, y)}. \tag{18.41}$$

However, we have, already met (18.41) in subsection 18.3.1 on first-order PDEs, where solutions of the form $u(x, y) = f(p)$, where p is some combination of x and y, were discussed. Comparing (18.41) with (18.12) we see that the characteristics are merely those curves along which p is constant.

Since the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ may be evaluated provided the boundary curve C does *not* lie along a characteristic, defining $u(x, y) = \phi(s)$ along C is sufficient to specify the solution to the original problem (equation plus boundary conditions) near the curve C, in terms of a Taylor expansion about C. Therefore the characteristics can be considered as the curves along which information about the solution $u(x, y)$ 'propagates'. This is best understood by using an example.

►*Find the general solution of*

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0 \qquad (18.42)$$

that takes the value $2y + 1$ *on the line* $x = 1$ between $y = 0$ and $y = 1$.

We solved this problem in subsection 18.3.1 for the case where $u(x, y)$ takes the value $2y + 1$ along the *entire* line $x = 1$. We found then that the general solution to the equation (ignoring boundary conditions) is of the form

$$u(x, y) = f(p) = f(x^2y),$$

for some arbitrary function f. Hence the characteristics of (18.42) are given by $x^2y = c$ where c is a constant; some of these curves are plotted in figure 18.2 for various values of c. Furthermore, we found that the particular solution for which $u(1, y) = 2y + 1$ *for all* y was given by

$$u(x, y) = 2x^2y + 1.$$

In the present case the value of x^2y is fixed by the boundary conditions only between $y = 0$ and $y = 1$. However, since the characteristics are curves along which x^2y, and hence $f(x^2y)$, remains constant, the solution is determined everywhere along any characteristic that intersects the line segment denoting the boundary conditions. Thus $u(x, y) = 2x^2y + 1$ is the particular solution that holds in the shaded region in figure 18.2 (corresponding to $0 \le c \le 1$).

Outside this region, however, the solution is not precisely specified, and any function of the form

$$u(x, y) = 2x^2y + 1 + g(x^2y)$$

will satisfy both the equation and the boundary condition, provided $g(p) = 0$ for $0 \le p \le 1$. ◄

In the above example the boundary curve was not itself a characteristic and furthermore it crossed each characteristic *once only*. For a general boundary curve C this may not be the case. Firstly, if C is itself a characteristic (or is just a single point) then information about the solution cannot 'propagate' away from C, and so the solution remains unspecified everywhere except on C.

The second possibility is that C (although not a characteristic itself) crosses some characteristics more than once, as in figure 18.3. In this case specifying the value of $u(x, y)$ along the curve PQ determines the solution along all the characteristics that intersect it. Therefore, also specifying $u(x, y)$ along QR can *overdetermine* the problem solution and generally results in there being no solution.

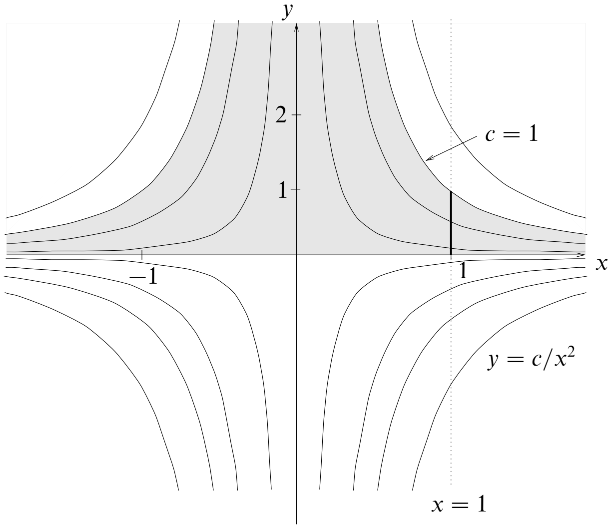

Figure 18.2 The characteristics of equation (18.42). The shaded region shows where the solution to the equation is defined, given the imposed boundary condition at $x = 1$ between $y = 0$ and $y = 1$, shown as a bold vertical line.

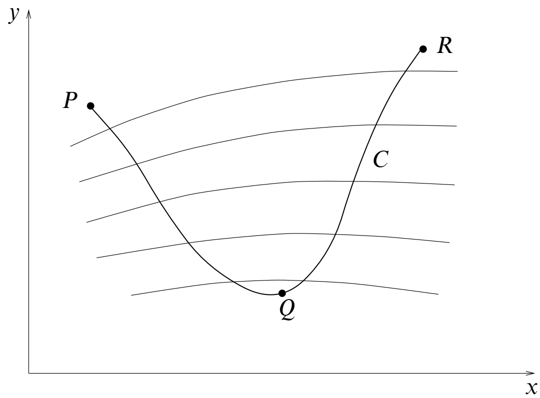

Figure 18.3 A boundary curve C that crosses characteristics more than once.

18.6.2 Second-order equations

The concept of characteristics can be extended naturally to second- (and higher-) order equations. In this case let us write the general second-order linear PDE (18.19) as

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \qquad (18.43)$$

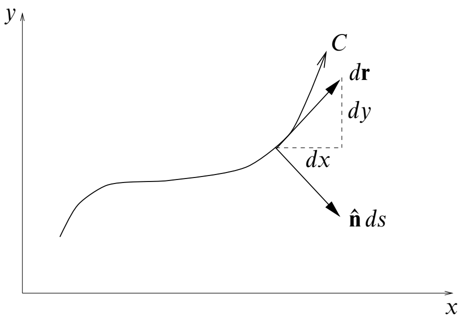

Figure 18.4 A boundary curve C and its tangent and unit normal at a given point.

For second-order equations we might expect that relevant boundary conditions would involve specifying u, or some of its first derivatives, or both, along a suitable set of boundaries bordering or enclosing the region over which a solution is sought. Three common types of boundary condition occur and are associated with the names of Dirichlet, Neumann and Cauchy. They are as follows.

(i) *Dirichlet*: The value of u is specified at each point of the boundary.

(ii) *Neumann*: The value of $\partial u/\partial n$, the *normal derivative* of u, is specified at each point of the boundary. Note that $\partial u/\partial n = \nabla u \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the normal to the boundary at each point.

(iii) *Cauchy*: Both u and $\partial u/\partial n$ are specified at each point of the boundary.

Let us consider for the moment the solution of (18.43) subject to the Cauchy boundary conditions, i.e. u and $\partial u/\partial n$ are specified along some boundary curve C in the xy-plane defined by the parametric equations $x = x(s)$, $y = y(s)$, s being the arc length along C (see figure 18.4). Let us suppose that along C we have $u(x, y) = \phi(s)$ and $\partial u/\partial n = \psi(s)$. At any point on C the vector $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ is a tangent to the curve and $\hat{\mathbf{n}}\,ds = dy\,\mathbf{i} - dx\,\mathbf{j}$ is a vector normal to the curve. Thus on C we have

$$\frac{\partial u}{\partial s} \equiv \nabla u \cdot \frac{d\mathbf{r}}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{d\phi(s)}{ds},$$
$$\frac{\partial u}{\partial n} \equiv \nabla u \cdot \hat{\mathbf{n}} = \frac{\partial u}{\partial x}\frac{dy}{ds} - \frac{\partial u}{\partial y}\frac{dx}{ds} = \psi(s).$$

These two equations may then be solved straightforwardly for the first partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ along C. Using the chain rule to write

$$\frac{d}{ds} = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y},$$

we may differentiate the two first derivatives $\partial u/\partial x$ and $\partial u/\partial y$ along the boundary to obtain the pair of equations

$$\frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) = \frac{dx}{ds}\frac{\partial^2 u}{\partial x^2} + \frac{dy}{ds}\frac{\partial^2 u}{\partial x \partial y},$$
$$\frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) = \frac{dx}{ds}\frac{\partial^2 u}{\partial x \partial y} + \frac{dy}{ds}\frac{\partial^2 u}{\partial y^2}.$$

We may now solve these two equations, together with the original PDE (18.43), for the second partial derivatives of u, *except* where the determinant of their coefficients equals zero,

$$\begin{vmatrix} A & B & C \\ \dfrac{dx}{ds} & \dfrac{dy}{ds} & 0 \\ 0 & \dfrac{dx}{ds} & \dfrac{dy}{ds} \end{vmatrix} = 0.$$

Expanding out the determinant,

$$A\left(\frac{dy}{ds}\right)^2 - B\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) + C\left(\frac{dx}{ds}\right)^2 = 0.$$

Multiplying through by $(ds/dx)^2$ we obtain

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0, \qquad (18.44)$$

which is the ODE for the curves in the xy-plane along which the second partial derivatives of u *cannot* be found.

As for the first-order case, the curves satisfying (18.44) are called characteristics of the original PDE. These characteristics have tangents at each point given by (when $A \neq 0$)

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \qquad (18.45)$$

Clearly, when the original PDE is hyperbolic ($B^2 > 4AC$), equation (18.45) defines two families of real curves in the xy-plane; when the equation is parabolic ($B^2 = 4AC$) it defines one family of real curves; and when the equation is elliptic ($B^2 < 4AC$) it defines two families of complex curves. Furthermore, when A, B and C are constants, rather than functions of x and y, the equations of the characteristics will be of the form $x + \lambda y =$ constant, which is reminiscent of the form of solution discussed in subsection 18.3.3.

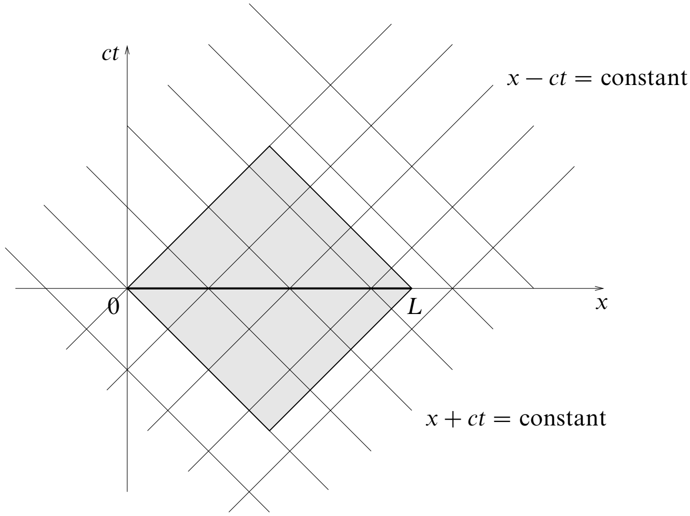

Figure 18.5 The characteristics for the one-dimensional wave equation. The shaded region indicates the region over which the solution is determined by specifying Cauchy boundary conditions at $t = 0$ on the line segment $x = 0$ to $x = L$.

►*Find the characteristics of the one-dimensional wave equation*

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

This is a hyperbolic equation with $A = 1$, $B = 0$ and $C = -1/c^2$. Therefore from (18.44) the characteristics are given by

$$\left(\frac{dx}{dt}\right)^2 = c^2,$$

and so the characteristics are the straight lines $x - ct =$ constant and $x + ct =$ constant. ◄

The characteristics of second-order PDEs can be considered as the curves along which *partial* information about the solution $u(x, y)$ ‘propagates’. Consider a point in the space that has the independent variables as its coordinates; if either or both of the two characteristics which pass through the point does not intersect the curve along which the boudary conditions are specified then the solution will not be determined at that point.In particular, if the equation is hyperbolic, so that we obtain two families of real characteristics in the xy-plane, then Cauchy boundary conditions propagate partial information concerning the solution along the characteristics, belonging to each family, that intersect the boundary curve C. The solution u is then specified in the region common to these two families of characteristics. For instance, the characteristics of the hyperbolic one-dimensional wave equation in the last example are shown in figure 18.5. By specifying Cauchy

Equation type	Boundary	Conditions
hyperbolic	open	Cauchy
parabolic	open	Dirichlet or Neumann
elliptic	closed	Dirichlet or Neumann

Table 18.1 The appropriate boundary conditions for different types of partial differential equation.

boundary conditions u and $\partial u/\partial t$ on the line segment $t = 0$, $x = 0$ to L, the solution is specified in the shaded region.

As in the case of first-order PDEs, however, problems can arise. For example, if for a hyperbolic equation the boundary curve intersects any characteristic more than once then Cauchy conditions along C can overdetermine the problem, resulting in there being no solution. In this case either the boundary curve C must be altered, or the boundary conditions on the offending parts of C must be relaxed to Dirichlet or Neumann conditions.

The general considerations involved in deciding which boundary conditions are appropriate for a particular problem are complex, and we do not discuss them any further here.† We merely note that whether the various types of boundary condition are appropriate (in that they give a solution that is unique, sometimes to within a constant, and is well defined) depends upon the type of second-order equation under consideration and on whether the region of solution is bounded by a closed or an open curve (or a surface if there are more than two independent variables). Note that part of a closed boundary may be at infinity if conditions are imposed on u or $\partial u/\partial n$ there.

It may be shown that the appropriate boundary-condition and equation-type pairings are as given in table 18.1.

For example, Laplace's equation $\nabla^2 u = 0$ is elliptic and thus requires either Dirichlet or Neumann boundary conditions on a closed boundary which, as we have already noted, may be at infinity if the behaviour of u is specified there (most often u or $\partial u/\partial n \to 0$ at infinity).

18.7 Uniqueness of solutions

Although we have merely stated the appropriate boundary types and conditions for which, in the general case, a PDE has a unique, well-defined solution, sometimes to within an additive constant, it is often important to be able to prove that a unique solution is obtained.

† For a discussion the reader is referred, for example, to Morse and Feshbach, *Methods of Theoretical Physics, Part I* (McGraw-Hill, 1953) chapter 6.

As an extremely important example let us consider Poisson's equation in three dimensions,

$$\nabla^2 u(\mathbf{r}) = \rho(\mathbf{r}), \tag{18.46}$$

with either Dirichlet or Neumann conditions on a closed boundary appropriate to such an elliptic equation; for brevity, in (18.46), we have absorbed any physical constants into ρ. We aim to show that, to within an unimportant constant, the solution of (18.46) is *unique* if either the potential u or its normal derivative $\partial u/\partial n$ is specified on all surfaces bounding a given region of space (including, if necessary, a hypothetical spherical surface of indefinitely large radius on which u or $\partial u/\partial n$ is prescribed to have an arbitrarily small value). Stated more formally this is as follows.

Uniqueness theorem. *If u is real and its first and second partial derivatives are continuous in a region V and on its boundary S, and $\nabla^2 u = \rho$ in V and either $u = f$ or $\partial u/\partial n = g$ on S, where ρ, f and g are prescribed functions, then u is unique (at least to within an additive constant).*

►*Prove the uniqueness theorem for Poisson's equation.*

Let us suppose on the contrary that two solutions $u_1(\mathbf{r})$ and $u_2(\mathbf{r})$ both satisfy the conditions given above, and denote their difference by the function $w = u_1 - u_2$. We then have

$$\nabla^2 w = \nabla^2 u_1 - \nabla^2 u_2 = \rho - \rho = 0,$$

so that w satisfies Laplace's equation in V. Furthermore, since either $u_1 = f = u_2$ or $\partial u_1/\partial n = g = \partial u_2/\partial n$ on S, we must have either $w = 0$ or $\partial w/\partial n = 0$ on S.

If we now use Green's first theorem, (11.19), for the case where both scalar functions are taken as w we have

$$\int_V \left[w\nabla^2 w + (\nabla w)\cdot(\nabla w) \right] dV = \int_S w\frac{\partial w}{\partial n}\, dS.$$

However, either condition, $w = 0$ or $\partial w/\partial n = 0$, makes the RHS vanish whilst the first term on the LHS vanishes since $\nabla^2 w = 0$ in V. Thus we are left with

$$\int_V |\nabla w|^2\, dV = 0.$$

Since $|\nabla w|^2$ can never be negative, this can only be satisfied if

$$\nabla w = \mathbf{0},$$

i.e. if w, and hence $u_1 - u_2$, is a constant in V.

If Dirichlet conditions are given then $u_1 \equiv u_2$ on (some part of) S and hence $u_1 = u_2$ everywhere in V. For Neumann conditions, however, u_1 and u_2 can differ throughout V by an arbitrary (but unimportant) constant. ◄

The importance of this uniqueness theorem lies in the fact that if a solution to Poisson's (or Laplace's) equation that fits the given set of Dirichlet or Neumann conditions can be found by any means whatever, then that solution is the correct one, since only one exists. This result is the mathematical justification for the *method of images*, which is discussed more fully in the next chapter.

We also note that often the same general method, used in the above example for proving the uniqueness theorem for Poisson's equation, can be employed to prove the uniqueness (or otherwise) of solutions to other equations and boundary conditions.

18.8 Exercises

18.1 Determine whether the following can be written as functions of $p = x^2 + 2y$ only, and hence whether they are solutions of (18.8):

(a) $x^2(x^2 - 4) + 4y(x^2 - 2) + 4(y^2 - 1)$;
(b) $x^4 + 2x^2y + y^2$;
(c) $[x^4 + 4x^2y + 4y^2 + 4]/[2x^4 + x^2(8y + 1) + 8y^2 + 2y]$.

18.2 Find partial differential equations satisfied by the following functions $u(x, y)$ for all arbitrary functions f and all arbitrary constants a and b:

(a) $u(x, y) = f(x^2 - y^2)$;
(b) $u(x, y) = (x - a)^2 + (y - b)^2$;
(c) $u(x, y) = y^n f(y/x)$;
(d) $u(x, y) = f(x + ay)$.

18.3 Solve the following partial differential equations for $u(x, y)$ with the boundary conditions given:

$$\text{(a) } x\frac{\partial u}{\partial x} + xy = u, \quad u = 2y \text{ on the line } x = 1;$$
$$\text{(b) } 1 + x\frac{\partial u}{\partial y} = xu, \quad u(x, 0) = x.$$

18.4 Find the most general solutions $u(x, y)$ of the following equations consistent with the boundary conditions stated:

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0, \quad u(x, 0) = 1 + \sin x$;

(b) $i\dfrac{\partial u}{\partial x} = 3\dfrac{\partial u}{\partial y}, \quad u = (4 + 3i)x^2$ on the line $x = y$;

(c) $\sin x \sin y\dfrac{\partial u}{\partial x} + \cos x \cos y\dfrac{\partial u}{\partial y} = 0, \quad u = \cos 2y$ on $x + y = \pi/2$;

(d) $\dfrac{\partial u}{\partial x} + 2x\dfrac{\partial u}{\partial y} = 0, \quad u = 2$ on the parabola $y = x^2$.

18.5 Find solutions of

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} = 0$$

for which (a) $u(0, y) = y$, (b) $u(1, 1) = 1$.

18.6 Find the most general solutions $u(x, y)$ of the following equations consistent with the boundary conditions stated:

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u = x^2$ on the line $y = 0$;

(b) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u(1,0) = 2;$

(c) $y^2\dfrac{\partial u}{\partial x} + x^2\dfrac{\partial u}{\partial y} = x^2y^2(x^3+y^3),$ no boundary conditions.

18.7 Solve

$$\sin x\frac{\partial u}{\partial x} + \cos x\frac{\partial u}{\partial y} = \cos x$$

subject to (a) $u(\pi/2, y) = 0$, (b) $u(\pi/2, y) = y(y+1)$.

18.8 A function $u(x, y)$ satisfies

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 10,$$

and takes the value 3 on the line $y = 4x$. Evaluate $u(2, 4)$.

18.9 If $u(x, y)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$

and $u = -x^2$ and $\partial u/\partial y = 0$ for $y = 0$ and all x, find the value of $u(0, 1)$.

18.10 (a) Solve the previous question if the boundary condition is $u = \partial u/\partial y = 1$ when $y = 0$ for all x.
(b) In which region of the xy-plane would u be determined if the boundary condition were $u = \partial u/\partial y = 1$ when $y = 0$ for all $x > 0$?

18.11 In those cases in which it is possible to do so, evaluate $u(2, 2)$, where $u(x, y)$ is the solution of

$$2y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 2xy(2y^2 - x^2)$$

that satisfies the (separate) boundary conditions given below.

(a) $u(x, 1) = x^2$ for all x.
(b) $u(x, 1) = x^2$ for $x \geq 0$.
(c) $u(x, 1) = x^2$ for $0 \leq x \leq 3$.
(d) $u(x, 0) = x$ for $x \geq 0$.
(e) $u(x, 0) = x$ for all x.
(f) $u(1, \sqrt{10}) = 5$.
(g) $u(\sqrt{10}, 1) = 5$.

18.12 Solve

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 14,$$

subject to $u = 2x + 1$ and $\partial u/\partial y = 4 - 6x$, both on the line $y = 0$.

18.13 By changing the independent variables in the previous question to

$$\xi = x + 2y \qquad \text{and} \qquad \eta = x + 3y,$$

show that it must be possible to write $14(x^2 + 5xy + 6y^2)$ in the form

$$f_1(x + 2y) + f_2(x + 3y) - (x^2 + y^2),$$

and determine the forms of $f_1(z)$ and $f_2(z)$.

18.14 Solve

$$\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = x(2y + 3x).$$

18.15 Find the most general solution of $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = x^2y^2$.

18.16 An infinitely long string on which waves travel at speed c has an initial displacement

$$y(x) = \begin{cases} \sin(\pi x/a), & -a \le x \le a, \\ 0, & |x| > a. \end{cases}$$

It is released from rest at time $t = 0$, and its subsequent displacement is described by $y(x,t)$.

By expressing the initial displacement as one explicit function incorporating Heaviside step functions, find an expression for $y(x,t)$ at a general time $t > 0$. In particular, determine the displacement as a function of time (a) at $x = 0$, (b) at $x = a$, and (c) at $x = a/2$.

18.17 The non-relativistic Schrödinger equation (18.7) is similar to the diffusion equation in having different orders of derivatives in its various terms; this precludes solutions that are arbitrary functions of particular linear combinations of variables. However, since exponential functions do not change their forms under differentiation, solutions in the form of exponential functions of combinations of the variables may still be possible.

Consider the Schrödinger equation for the case of a constant potential, i.e. for a free particle, and show that it has solutions of the form $A\exp(lx + my + nz + \lambda t)$ where the only requirement is that

$$-\frac{\hbar^2}{2m}\left(l^2 + m^2 + n^2\right) = i\hbar\lambda.$$

In particular, identify the equation and wavefunction obtained by taking λ as $-iE/\hbar$, and l, m and n as $ip_x/\hbar, ip_y/\hbar$ and $ip_z/\hbar$ respectively, where E is the energy and $\mathbf{p}$ the momentum of the particle; these identifications are essentially the content of the de Broglie and Einstein relationships.

18.18 Like the Schrödinger equation of the previous question, the equation describing the transverse vibrations of a rod,

$$a^4\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0,$$

has different orders of derivatives in its various terms. Show, however, that it has solutions of exponential form $u(x,t) = A\exp(\lambda x + i\omega t)$ provided that the relation $a^4\lambda^4 = \omega^2$ is satisfied.

Use a linear combination of such allowed solutions, expressed as the sum of sinusoids and hyperbolic sinusoids of λx, to describe the transverse vibrations of a rod of length L clamped at both ends. At a clamped point both u and $\partial u/\partial x$ must vanish; show that this implies that $\cos(\lambda L)\cosh(\lambda L) = 1$, thus determining the frequencies ω at which the rod can vibrate.

18.19 An incompressible fluid of density ρ and negligible viscosity flows with velocity v along a thin straight tube, perfectly light and flexible, of cross-section A and held under tension T. Assume that small transverse displacements u of the tube are governed by

$$\frac{\partial^2 u}{\partial t^2} + 2v\frac{\partial^2 u}{\partial x \partial t} + \left(v^2 - \frac{T}{\rho A}\right)\frac{\partial^2 u}{\partial x^2} = 0.$$

(a) Show that the general solution consists of a superposition of two waveforms travelling with different speeds.
(b) The tube initially has a small transverse displacement $u = a\cos kx$ and is suddenly released from rest. Find its subsequent motion.

18.20 A sheet of material of thickness w, specific heat capacity c and thermal conductivity k is isolated in a vacuum, but its two sides are exposed to fluxes of radiant heat of strengths J_1 and J_2. Ignoring short-term transients, show that the temperature difference between its two surfaces is steady at $(J_2 - J_1)w/2k$, whilst their average temperature increases at a rate $(J_2 + J_1)/cw$.

18.21 In an electrical cable of resistance R and capacitance C per unit length, voltage signals obey the equation $\partial^2 V/\partial x^2 = RC\partial V/\partial t$. This has solutions of the form given in (18.36) and also of the form $V = Ax + D$.

(a) Find a combination of these that represents the situation after a steady voltage V_0 is applied at $x = 0$ at time $t = 0$.
(b) Obtain a solution describing the propagation of the voltage signal resulting from application of the signal $V = V_0$ for $0 < t < T$, $V = 0$ otherwise, to the end $x = 0$ of an infinite cable.
(c) Show that for $t \gg T$ the maximum signal occurs at a value of x proportional to $t^{1/2}$ and has a magnitude proportional to t^{-1}.

18.22 The daily and annual variations of temperature at the surface of the earth may be represented by sine-wave oscillations with equal amplitudes and periods of 1 day and 365 days respectively. Assume that for (angular) frequency ω the temperature at depth x in the earth is given by $u(x,t) = A\sin(\omega t + \mu x)\exp(-\lambda x)$, where λ and μ are constants.

(a) Use the diffusion equation to find the values of λ and μ.
(b) Find the ratio of the depths below the surface at which the amplitudes have dropped to 1/20 of their surface values.
(c) At what time of year is the soil coldest at the greater of these depths, assuming that the smoothed annual variation in temperature at the surface has a minimum on February 1st?

18.23 Consider each of the following situations in a qualitative way and determine the equation type, the nature of the boundary curve and the type of boundary conditions involved.

(a) a conducting bar given an initial temperature distribution and then thermally isolated;
(b) two long conducting concentric cylinders on each of which the voltage distribution is specified;
(c) two long conducting concentric cylinders on each of which the charge distribution is specified;
(d) a semi-infinite string the end of which is made to move in a prescribed way.

18.24 *This example gives a formal demonstration that the type of a second-order PDE (elliptic, parabolic or hyperbolic) cannot be changed by a new choice of independent variable. The algebra is somewhat lengthy, but straightforward.*

If a change of variable $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ is made in (18.19), so that it reads

$$A'\frac{\partial^2 u}{\partial \xi^2} + B'\frac{\partial^2 u}{\partial \xi \partial \eta} + C'\frac{\partial^2 u}{\partial \eta^2} + D'\frac{\partial u}{\partial \xi} + E'\frac{\partial u}{\partial \eta} + F'u = R'(\xi,\eta),$$

show that

$$B'^2 - 4A'C' = (B^2 - 4AC)\left[\frac{\partial(\xi,\eta)}{\partial(x,y)}\right]^2.$$

Hence deduce the conclusion stated above.

18.25 The Klein–Gordon equation (which is satisfied by the quantum-mechanical wavefunction $\Phi(\mathbf{r})$ of a relativistic spinless particle of non-zero mass m) is

$$\nabla^2\Phi - m^2\Phi = 0\,.$$

Show that the solution for the scalar field $\Phi(\mathbf{r})$ in any volume V bounded by a surface S is unique if either Dirichlet or Neumann boundary conditions are specified on S.

18.9 Hints and answers

18.1 (a) Yes, $p^2 - 4p - 4$; (b) no, $(p-y)^2$; (c) yes, $(p^2+4)/(2p^2+p)$.

18.2 (a) $y(\partial u/\partial x) + x(\partial u/\partial y) = 0$; (b) $(\partial u/\partial x)^2 + (\partial u/\partial y)^2 = 4u$; (c) $x(\partial u/\partial x) + y(\partial u/\partial y) = nu$; (d) $(\partial u/\partial y)(\partial^2 u/\partial x^2) = (\partial u/\partial x)(\partial^2 u/\partial x \partial y)$, or with x and y reversed.

18.3 Each equation is effectively an ordinary differential equation but with a function of the non-integrated variable as the constant of integration; (a) $u = xy(2 - \ln x)$; (b) $u = x^{-1}(1 - e^y) + xe^y$.

18.4 (a) $p = x^2 + y^2$, $u = \sin(x^2+y^2)^{1/2} + 1$; (b) $p = 3x + iy$, $u = (3x+iy)^{1/2}/2$; (c) $p = \sin x \cos y$, $u = 2\sin x \cos y - 1$; (d) $p = y - x^2$, $u = y - x^2 + 2$.

18.5 (a) $(y^2 - x^2)^{1/2}$; (b) $1 + f(y^2 - x^2)$ where $f(0) = 0$.

18.6 (a) $p = x^2 + y^2$, particular integral $u = -3y$, $u = x^2 + y^2 - 3y$; (b) $u = x^2 + y^2 - 3y + 1 + g(x^2 + y^2)$ where $g(1) = 0$; (c) $(x^6 + y^6)/6 + g(x^3 - y^3)$.

18.7 $u = y + f(y - \ln(\sin x))$; (a) $u = \ln(\sin x)$; (b) $u = y + [y - \ln(\sin x)]^2$.

18.8 $u = f(3x - 2y) + 2(x + y)$; $f(p) = 3 + 2p$; $u = 8x - 2y + 3$ and $u(2,4) = 11$.

18.9 General solution is $u(x,y) = f(x+y) + g(x + y/2)$. Show that $2p = -g'(p)/2$, and hence $g(p) = k - 2p^2$, whilst $f(p) = p^2 - k$, leading to $u(x,y) = -x^2 + y^2/2$; $u(0,1) = 1/2$.

18.10 (a) $u(x,y) = 2(x+y) - 2(x + y/2) + 1 = y + 1$; $u(0,1) = 2$; (b) in the sector $-\pi/4 \le \theta \le \pi/2 + \phi$, where $\tan\phi = 1/2$ and θ is measured from the positive x-axis.

18.11 $p = x^2 + 2y^2$; $u(x,y) = f(p) + x^2y^2/2$.

(a) $u(x,y) = (x^2 + 2y^2 + x^2y^2 - 2)/2$. $u(2,2) = 13$. The line $y = 1$ cuts each characteristic in zero or two distinct points, but this causes no difficulty with the given boundary conditions.
(b) As in (a).
(c) The solution is defined over the space between the ellipses $p = 2$ and $p = 11$; $(2,2)$ lies on $p = 12$, and so $u(2,2)$ is undetermined.
(d) $u(x,y) = (x^2 + 2y^2)^{1/2} + x^2y^2/2$; $u(2,2) = 8 + \sqrt{12}$.
(e) The line $y = 0$, cuts each characteristic in two distinct points. No differentiable form of $f(p)$ gives $f(\pm a) = \pm a$ respectively, and so there is no solution.
(f) The solution is only specified on $p = 21$, and so $u(2,2)$ is undetermined.
(g) The solution is specified on $p = 12$, and so $u(2,2) = 5 + \frac{1}{2}(4)(4) = 13$.

18.12 $u(x,y) = f(x+2y) + g(x+3y) + x^2 + y^2$, leading to $u = 1 + 2x + 4y - 6xy - 8y^2$.

18.13 The equation becomes $\partial^2 f/\partial\xi\partial\eta = -14$, with solution $f(\xi,\eta) = f(\xi) + g(\eta) - 14\xi\eta$, which can be compared with the answer from the previous question; $f_1(z) = 10z^2$ and $f_2(z) = 5z^2$.

18.14 $u = f(y - 3x) + g(x) + x^2y^2/2$.

18.15 $u(x,y) = f(x+iy) + g(x-iy) + (1/12)x^4(y^2 - (1/15)x^2)$. In the last term, x and y may be interchanged.

18.16
$$y(x,t) = \frac{1}{2}\sin[\pi(x-ct)/a][H(x-ct+a) - H(x-ct-a)] + \frac{1}{2}\sin[\pi(x+ct)/a][H(x+ct+a) - H(x+ct-a)].$$

(a) zero at all times; (b) $\frac{1}{2}\sin(\pi ct/a)$ for $0 \le t \le 2a/c$, and 0 otherwise;
(c) $\cos(\pi ct/a)$ for $0 \le t \le a/2c$, $\frac{1}{2}\cos(\pi ct/a)$ for $a/2c \le t \le 3a/2c$, and 0 otherwise.

18.17 $E = p^2/(2m)$, the relationship between energy and momentum for a non-relativistic particle; $u(\mathbf{r},t) = A\exp[i(\mathbf{p}\cdot\mathbf{r} - Et)/\hbar]$, a plane wave of wave number $\mathbf{k} = \mathbf{p}/\hbar$ and angular frequency $\omega = E/\hbar$ travelling in the direction $\mathbf{p}/p$.

18.18 $\lambda = \pm\omega^{1/2}/a$ or $\pm i\omega^{1/2}/a$; $u(x,t) = \exp(i\omega t)[A\sin\lambda x + B\cos\lambda x + C\sinh\lambda x + D\cosh\lambda x]$, with $C = -A$ and $D = -B$. The conditions at $x = L$ and consistency establish the quoted result.

18.19 (a) $c = v \pm \alpha$ where $\alpha^2 = T/\rho A$;
(b) $u(x,t) = a\cos[k(x-vt)]\cos(k\alpha t) - (va/\alpha)\sin[k(x-vt)]\sin(k\alpha t)$.

18.20 Use the first form of solution given in (18.35).

18.21 (a) $V_0\left[1 - (2/\sqrt{\pi})\int^{\frac{1}{2}x(CR/t)^{1/2}}\exp(-v^2)\,dv\right]$; (b) consider as V_0 applied at $t = 0$ and continued and $-V_0$ at $t = T$ and continued;

$$V(x,t) = \frac{2V_0}{\sqrt{\pi}}\int_{\frac{1}{2}x(CR/t)^{1/2}}^{\frac{1}{2}x[CR/(t-T)]^{1/2}} \exp\left(-v^2\right)\,dv;$$

(c) For $t \gg T$, maximum at $x = [2t/(CR)]^{1/2}$ with value

$$\frac{V_0 T\exp(-1/2)}{(2\pi)^{1/2}t}.$$

18.22 (a) $\lambda = -\mu = [\omega/(2\kappa)]^{1/2}$, where κ is the diffusion constant; (b) $x_\mathrm{a} = (365)^{1/2}x_\mathrm{d}$; (c) only the annual variation is significant at this depth and has a phase $\mu_\mathrm{a}x_\mathrm{a} = \ln 20$ behind the surface. Thus the coldest day is 1 February + $(365\ln 20)/(2\pi)$ days $\approx$ 23 July.

18.23 (a) Parabolic, open, Dirichlet $u(x,0)$ given, Neumann $\partial u/\partial x = 0$ at $x = \pm L/2$ for all t;
(b) elliptic, closed, Dirichlet;
(c) elliptic, closed, Neumann $\partial u/\partial n = \sigma/\epsilon_0$;
(d) hyperbolic, open, Cauchy.

18.24

$$A' = A\left(\frac{\partial\xi}{\partial x}\right)^2 + B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + C\left(\frac{\partial\xi}{\partial y}\right)^2,$$
$$B' = 2A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right) + 2C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}, \qquad \text{etc.}$$

18.25 Follow an argument similar to that in section 18.7 and argue that the additional term $\int m^2|w|^2\,dV$ must be zero, and hence that $w = 0$ everywhere.

19

Partial differential equations: separation of variables and other methods

In the previous chapter we demonstrated the methods by which general solutions of some partial differential equations (PDEs) may be obtained in terms of arbitrary functions. In particular, solutions containing the independent variables in definite combinations were sought, thus reducing the effective number of them.

In the present chapter we begin by taking the opposite approach, namely that of trying to keep the independent variables as separate as possible, using the method of separation of variables. We then consider integral transform methods by which one of the independent variables may be eliminated, at least from differential coefficients. Finally, we discuss the use of Green's functions in solving inhomogeneous problems.

19.1 Separation of variables: the general method

Suppose we seek a solution $u(x, y, z, t)$ to some PDE (expressed in Cartesian coordinates). Let us attempt to obtain one that has the product form†

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t). \tag{19.1}$$

A solution that has this form is said to be *separable* in x, y, z and t, and seeking solutions of this form is called the method of *separation of variables*.

As simple examples we may observe that, of the functions

$$\text{(i) } xyz^2 \sin bt, \qquad \text{(ii) } xy + zt, \qquad \text{(iii) } (x^2 + y^2)z \cos \omega t,$$

(i) is completely separable, (ii) is inseparable in that no single variable can be separated out from it and written as a multiplicative factor, whilst (iii) is separable in z and t but not in x and y.

† It should be noted that the conventional use here of upper-case (capital) letters to denote the functions of the corresponding lower-case variable is intended to enable an easy correspondence between a function and its argument to be made.

When seeking PDE solutions of the form (19.1), we are requiring not that there is no connection at all between the functions X, Y, Z and T (for example, certain parameters may appear in two or more of them), but only that the X does not depend upon y, z, t, that Y does not depend on x, z, t and so on.

For a general PDE it is likely that a separable solution is impossible, but certainly some common and important equations do have useful solutions of this form and we will illustrate the method of solution by studying the three-dimensional wave equation

$$\nabla^2 u(\mathbf{r}) = \frac{1}{c^2}\frac{\partial^2 u(\mathbf{r})}{\partial t^2}. \tag{19.2}$$

We will work in Cartesian coordinates for the present and assume a solution of the form (19.1); the solutions in alternative coordinate systems, e.g. spherical or cylindrical polars, are considered in section 19.3. Expressed in Cartesian coordinates (19.2) takes the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}; \tag{19.3}$$

substituting (19.1) gives

$$\frac{d^2 X}{dx^2} YZT + X\frac{d^2 Y}{dy^2} ZT + XY\frac{d^2 Z}{dz^2} T = \frac{1}{c^2} XYZ\frac{d^2 T}{dt^2},$$

which can also be written as

$$X''YZT + XY''ZT + XYZ''T = \frac{1}{c^2} XYZT'', \tag{19.4}$$

where in each case the primes refer to the *ordinary* derivative with respect to the independent variable upon which the function depends. This emphasises the fact that each of the functions X, Y, Z and T has only one independent variable and thus its only derivative is its total derivative. For the same reason, in each term in (19.4) three of the four functions are unaltered by the partial differentiation and behave exactly as constant multipliers.

If we now divide (19.4) throughout by $u = XYZT$ we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^2}\frac{T''}{T}. \tag{19.5}$$

This form shows the particular characteristic that is the basis of the method of separation of variables, namely that of the four terms the first is a function of x only, the second of y only, the third of z only and the RHS a function of t only and yet there is an equation connecting them. This can only be so for all x, y, z and t if *each* of the terms does not in fact, despite appearances, depend upon the corresponding independent variable but *is equal to a constant*, the four constants being such that (19.5) is satisfied.

Since there is only one equation to be satisfied and four constants involved, there is considerable freedom in the values they may take. For the purposes of our illustrative example let us make the choice of $-l^2$, $-m^2$, $-n^2$, for the first three constants. The constant associated with $c^{-2}T''/T$ must then have the value $-\mu^2 = -(l^2 + m^2 + n^2)$.

Having recognised that each term of (19.5) is individually equal to a constant (or parameter), we can now replace (19.5) by four separate ordinary differential equations (ODEs),

$$\frac{X''}{X} = -l^2, \qquad \frac{Y''}{Y} = -m^2, \qquad \frac{Z''}{Z} = -n^2, \qquad \frac{1}{c^2}\frac{T''}{T} = -\mu^2. \tag{19.6}$$

The important point to notice is not the simplicity of the equations (19.6) (the corresponding ones for a general PDE are usually far from simple) but that, by the device of assuming a separable solution, a *partial* differential equation (19.3), containing derivatives with respect to the four independent variables all in one equation, has been reduced to four *separate ordinary* differential equations (19.6). The ordinary equations are connected through four constant parameters that satisfy an algebraic relation. These constants are called *separation constants*.

The general solutions of the equations (19.6) can be deduced straightforwardly and are

$$\begin{aligned} X(x) &= A\exp(ilx) + B\exp(-ilx) \\ Y(y) &= C\exp(imy) + D\exp(-imy) \\ Z(z) &= E\exp(inz) + F\exp(-inz) \\ T(t) &= G\exp(ic\mu t) + H\exp(-ic\mu t), \end{aligned} \tag{19.7}$$

where $A, B, \ldots, H$ are constants, which may be determined if boundary condtions are imposed on the solution. Depending on the geometry of the problem and any boundary conditions, it is sometimes more appropriate to write the solutions (19.7) in the alternative form

$$\begin{aligned} X(x) &= A'\cos lx + B'\sin lx \\ Y(y) &= C'\cos my + D'\sin my \\ Z(z) &= E'\cos nz + F'\sin nz \\ T(t) &= G'\cos(c\mu t) + H'\sin(c\mu t), \end{aligned} \tag{19.8}$$

for some different set of constants $A', B', \ldots, H'$. Clearly the choice of how best to represent the solution depends on the problem being considered.

As an example, suppose that we take as particular solutions the four functions

$$X(x) = \exp(ilx), \qquad Y(y) = \exp(imy),$$
$$Z(z) = \exp(inz), \qquad T(t) = \exp(-ic\mu t).$$

This gives a particular solution of the original PDE (19.3)

$$\begin{aligned} u(x, y, z, t) &= \exp(ilx)\exp(imy)\exp(inz)\exp(-ic\mu t) \\ &= \exp[i(lx + my + nz - c\mu t)], \end{aligned}$$

which is a special case of the solution (18.33) obtained in the previous chapter and represents a plane wave of unit amplitude propagating in a direction given by the vector with components l, m, n in a Cartesian coordinate system. In the conventional notation of wave theory, l, m and n are the components of the wave-number vector $\mathbf{k}$, whose magnitude is given by $k = 2\pi/\lambda$, where λ is the wavelength of the wave; $c\mu$ is the angular frequency ω of the wave. This gives the equation in the form

$$\begin{aligned} u(x, y, z, t) &= \exp[i(k_x x + k_y y + k_z z - \omega t)] \\ &= \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \end{aligned}$$

and makes the exponent dimensionless.

The method of separation of variables can be applied to many commonly occurring PDEs encountered in physical applications.

►*Use the method of separation of variables to obtain for the one-dimensional diffusion equation*

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{19.9}$$

a solution that tends to zero as $t \to \infty$ for all x.

Here we have only two independent variables x and t and we therefore assume a solution of the form

$$u(x, t) = X(x)T(t).$$

Substituting this expression into (19.9) and dividing through by $u = XT$ (and also by κ) we obtain

$$\frac{X''}{X} = \frac{T'}{\kappa T}.$$

Now, arguing exactly as above that the LHS is a function of x only and the RHS is a function of t only, we conclude that each side must equal a constant, which, anticipating the result and noting the imposed boundary condition, we will take as $-\lambda^2$. This gives us two ordinary equations,

$$X'' + \lambda^2 X = 0, \tag{19.10}$$

$$T' + \lambda^2 \kappa T = 0, \tag{19.11}$$

which have the solutions

$$\begin{aligned} X(x) &= A\cos\lambda x + B\sin\lambda x, \\ T(t) &= C\exp(-\lambda^2 \kappa t). \end{aligned}$$

Combining these to give the assumed solution $u = XT$ yields (absorbing the constant C into A and B)

$$u(x, t) = (A\cos\lambda x + B\sin\lambda x)\exp(-\lambda^2 \kappa t). \tag{19.12}$$

In order to satisfy the boundary condition $u \to 0$ as $t \to \infty$, $\lambda^2\kappa$ must be > 0. Since κ is real and > 0, this implies that λ is a real non-zero number and that the solution is sinusoidal in x and is not a disguised hyperbolic function; this was our reason for choosing the separation constant as $-\lambda^2$. ◀

As a final example we consider Laplace's equation in Cartesian coordinates; this may be treated in a similar manner.

►*Use the method of separation of variables to obtain a solution for the two-dimensional Laplace equation,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{19.13}$$

If we assume a solution of the form $u(x, y) = X(x)Y(y)$ then, following the above method, and taking the separation constant as λ^2, we find

$$X'' = \lambda^2 X, \qquad Y'' = -\lambda^2 Y.$$

Taking λ^2 as > 0, the general solution becomes

$$u(x, y) = (A\cosh\lambda x + B\sinh\lambda x)(C\cos\lambda y + D\sin\lambda y), \tag{19.14}$$

An alternative form, in which the exponentials are written explicitly, may be useful for other geometries or boundary conditions:

$$u(x, y) = [A\exp\lambda x + B\exp(-\lambda x)](C\cos\lambda y + D\sin\lambda y), \tag{19.15}$$

with different constants A and B.

If $\lambda^2 < 0$ then the roles of x and y interchange. The particular combination of sinusoidal and hyperbolic functions and the values of λ allowed will be determined by the geometrical properties of any specific problem, together with any prescribed or necessary boundary conditions. ◀

We note here that a particular case of the solution (19.14) links up with the 'combination' result $u(x, y) = f(x + iy)$ of the previous chapter (equations (18.24) and following), namely that if $A = B$, and $D = iC$ then the solution is the same as $f(p) = AC\exp\lambda p$ with $p = x + iy$.

19.2 Superposition of separated solutions

It will be noticed in the previous two examples that there is considerable freedom in the values of the separation constant λ, the only essential requirement being that λ has the *same* value in both parts of the solution, i.e. the part depending on x and the part depending on y (or t). This is a general feature for solutions in separated form, which, if the original PDE has n independent variables, will contain $n - 1$ separation constants. All that is required in general is that we associate the correct function of one independent variable with the appropriate functions of the others, the correct function being the one with the same values of the separation constants.

If the original PDE is linear (as are the Laplace, Schrödinger, diffusion and wave equations) then mathematically acceptable solutions can be formed by superposing solutions corresponding to different allowed values of the separation constants. To take a two-variable example: if

$$u_{\lambda_1}(x, y) = X_{\lambda_1}(x)Y_{\lambda_1}(y)$$

is a solution of a linear PDE obtained by giving the separation constant the value λ_1 then the superposition

$$u(x, y) = a_1X_{\lambda_1}(x)Y_{\lambda_1}(y) + a_2X_{\lambda_2}(x)Y_{\lambda_2}(y) + \cdots = \sum_i a_iX_{\lambda_i}(x)Y_{\lambda_i}(y), \tag{19.16}$$

is also a solution for any constants a_i, provided that the λ_i are the allowed values of the separation constant λ given the imposed boundary conditions. Note that if the boundary conditions allow any of the separation constants to be zero then the form of the general solution is normally different and must be deduced by returning to the separated ordinary differential equations. We will encounter this behaviour in section 19.3.

The value of the superposition approach is that a boundary condition, say that $u(x, y)$ takes a particular form $f(x)$ when $y = 0$, might be met by choosing the constants a_i such that

$$f(x) = \sum_i a_iX_{\lambda_i}(x)Y_{\lambda_i}(0).$$

In general, this will be possible provided that the functions $X_{\lambda_i}(x)$ form a complete set – as do the sinusoidal functions of Fourier series or the spherical harmonics that we shall discuss in subsection 19.3.2.

►*A semi-infinite rectangular metal plate occupies the region* $0 \le x \le \infty$ *and* $0 \le y \le b$ *in the xy-plane. The temperature at the far end of the plate and along its two long sides is fixed at* $0\,^\circ$C. *If the temperature of the plate at* $x = 0$ *is also fixed and is given by* $f(y)$, *find the steady-state temperature distribution u(x,y) of the plate. Hence find the temperature distribution if* $f(y) = u_0$, *where* u_0 *is a constant.*

The physical situation is illustrated in figure 19.1. With the notation we have used several times before, the two-dimensional heat diffusion equation satisfied by the temperature $u(x, y, t)$ is

$$\kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t},$$

with $\kappa = k/(s\rho)$. In this case, however, we are asked to find the steady-state temperature, which corresponds to $\partial u/\partial t = 0$, and so we are led to consider the (two-dimensional) Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We saw that assuming a separable solution of the form $u(x, y) = X(x)Y(y)$ led to solutions such as (19.14) or (19.15), or equivalent forms with x and y interchanged. In the current problem we have to satisfy the boundary conditions $u(x, 0) = 0 = u(x, b)$ and so a solution that is sinusoidal in y seems appropriate. Furthermore, since we require $u(\infty, y) = 0$ it is best to write the x-dependence of the solution explicitly in terms of

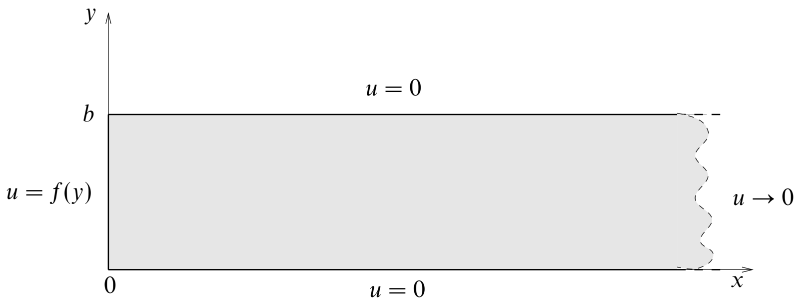

Figure 19.1 A semi-infinite metal plate whose edges are kept at fixed temperatures.

exponentials rather than of hyperbolic functions. We therefore write the separable solution in the form (19.15) as

$$u(x, y) = [A \exp \lambda x + B \exp(-\lambda x)](C \cos \lambda y + D \sin \lambda y).$$

Applying the boundary conditions, we see firstly that $u(\infty, y) = 0$ implies $A = 0$ if we take $\lambda > 0$. Secondly, since $u(x, 0) = 0$ we may set $C = 0$, which, if we absorb the constant D into B, leaves us with

$$u(x, y) = B \exp(-\lambda x) \sin \lambda y.$$

But, using the condition $u(x, b) = 0$, we require $\sin \lambda b = 0$ and so the constant λ is constrained to equal $n\pi/b$, where n is any positive integer.

Using the principle of superposition (19.16), the general solution satisfying the given boundary conditions can therefore be written

$$u(x, y) = \sum_{n=1}^{\infty} B_n \exp(-n\pi x/b) \sin(n\pi y/b), \qquad (19.17)$$

for some constants B_n. Notice that in the sum in (19.17) we have omitted negative values of n since they would lead to exponential terms that diverge as $x \to \infty$. The $n = 0$ term is also omitted since it is identically zero. Using the remaining boundary condition $u(0, y) = f(y)$ we see that the constants B_n must satisfy

$$f(y) = \sum_{n=1}^{\infty} B_n \sin(n\pi y/b). \qquad (19.18)$$

This is clearly a Fourier sine series expansion of $f(y)$ (see chapter 12). For (19.18) to hold, however, the continuation of $f(y)$ outside the region $0 \le y \le b$ must be an odd periodic function with period $2b$ (see figure 19.2). We also see from figure 19.2 that if the original function $f(y)$ does not equal zero at either of $y = 0$ and $y = b$ then its continuation has a discontinuity at the corresponding point(s); nevertheless, as discussed in chapter 12, the Fourier series will converge to the mid-points of these jumps and hence tend to zero in this case. If, however, the top and bottom edges of the plate were held not at $0\,^\circ$C but at some other non-zero temperature, then, in general, the final solution would possess discontinuities at the corners $x = 0$, $y = 0$ and $x = 0$, $y = b$.

Bearing in mind these technicalities, the coefficients B_n in (19.18) are given by

$$B_n = \frac{2}{b} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy. \qquad (19.19)$$

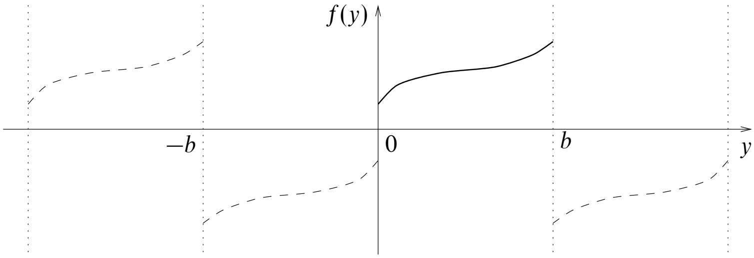

Figure 19.2 The continuation of $f(y)$ for a Fourier sine series.

Therefore, if $f(y) = u_0$ (i.e. the temperature of the side at $x = 0$ is constant along its length), (19.19) becomes

$$\begin{aligned} B_n &= \frac{2}{b}\int_0^b u_0 \sin\left(\frac{n\pi y}{b}\right)\, dy \\ &= \left[-\frac{2u_0}{b}\frac{b}{n\pi}\cos\left(\frac{n\pi y}{b}\right)\right]_0^b \\ &= -\frac{2u_0}{n\pi}[(-1)^n - 1] = \begin{cases} 4u_0/n\pi & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

Therefore the required solution is

$$u(x,y) = \sum_{n \text{ odd}} \frac{4u_0}{n\pi}\exp\left(-\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right). \blacktriangleleft$$

In the above example the boundary conditions meant that one term in each part of the separable solution could be immediately discarded, making the problem much easier to solve. Sometimes, however, a little ingenuity is required in writing the separable solution in such a way that certain parts can be neglected immediately.

►*Suppose that the semi-infinite rectangular metal plate in the previous example is replaced by one that in the x-direction has finite length a. The temperature of the right-hand edge is fixed at* $0\,^\circ$C *and all other boundary conditions remain as before. Find the steady-state temperature in the plate.*

As in the previous example, the boundary conditions $u(x,0) = 0 = u(x,b)$ suggest a solution that is sinusoidal in y. In this case, however, we require $u = 0$ on $x = a$ (rather than at infinity) and so a solution in which the x-dependence is written in terms of hyperbolic functions, such as (19.14), rather than exponentials is more appropriate. Moreover, since the constants in front of the hyperbolic functions are, at this stage, arbitrary, we may write the separable solution in the most convenient way that ensures that the condition $u(a,y) = 0$ is straightforwardly satisfied. We therefore write

$$u(x,y) = [A\cosh\lambda(a-x) + B\sinh\lambda(a-x)](C\cos\lambda y + D\sin\lambda y).$$

Now the condition $u(a,y) = 0$ is easily satisfied by setting $A = 0$. As before the conditions $u(x,0) = 0 = u(x,b)$ imply $C = 0$ and $\lambda = n\pi/b$ for integer n. Superposing the solutions for different n we then obtain

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh[n\pi(a-x)/b]\,\sin(n\pi y/b), \tag{19.20}$$

for some constants B_n. We have omitted negative values of n in the sum (19.20) since the relevant terms are already included in those obtained for positive n. Again the $n = 0$ term is identically zero. Using the final boundary condition $u(0, y) = f(y)$ as above we find that the constants B_n must satisfy

$$f(y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi a/b) \sin(n\pi y/b),$$

and, remembering the caveats discussed in the previous example, the B_n are therefore given by

$$B_n = \frac{2}{b\sinh(n\pi a/b)} \int_0^b f(y)\sin(n\pi y/b)\,dy. \tag{19.21}$$

For the case where $f(y) = u_0$, following the working of the previous example gives (19.21) as

$$B_n = \frac{4u_0}{n\pi\sinh(n\pi a/b)} \quad \text{for } n \text{ odd}, \qquad B_n = 0 \quad \text{for } n \text{ even}. \tag{19.22}$$

The required solution is thus

$$u(x,y) = \sum_{n\ \text{odd}} \frac{4u_0}{n\pi\sinh(n\pi a/b)} \sinh[n\pi(a-x)/b] \sin\left(n\pi y/b\right).$$

We note that, as required, in the limit $a \to \infty$ this solution tends to the solution of the previous example. ◄

Often the principle of superposition can be used to write the solution to problems with more complicated boundary conditions as the sum of solutions to problems that each satisfy only some part of the boundary condition but when added togther satisfy all the conditions.

►*Find the steady-state temperature in the (finite) rectangular plate of the previous example, subject to the boundary conditions* $u(x,b) = 0$, $u(a,y) = 0$ *and* $u(0,y) = f(y)$ *as before, but now in addition* $u(x,0) = g(x)$.

Figure 19.3(c) shows the imposed boundary conditions for the metal plate. Although we could find a solution to this problem using the methods presented above, we can arrive at the answer almost immediately by using the principle of superposition and the result of the previous example.

Let us suppose the required solution $u(x,y)$ is made up of two parts:

$$u(x,y) = v(x,y) + w(x,y),$$

where $v(x,y)$ is the solution satisfying the boundary conditions shown in figure 19.3(a),

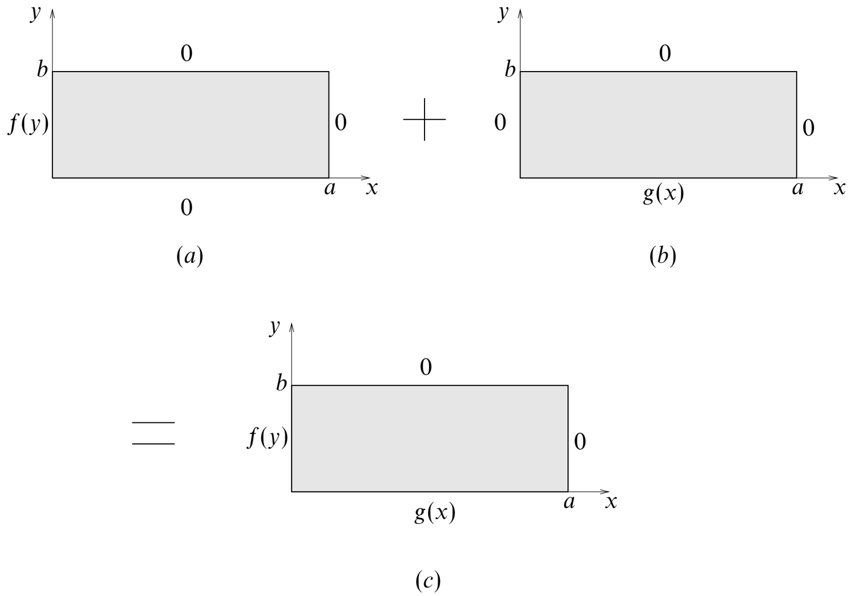

Figure 19.3 Superposition of boundary conditions for a metal plate.

whilst $w(x, y)$ is the solution satisfying the boundary conditions in figure 19.3(b). It is clear that $v(x, y)$ is simply given by the solution to the previous example,

$$v(x, y) = \sum_{n \text{ odd}} B_n \sinh\left[\frac{n\pi(a-x)}{b}\right] \sin\left(\frac{n\pi y}{b}\right),$$

where B_n is given by (19.21). Moreover, by symmetry, $w(x, y)$ must be of the same form as $v(x, y)$ but with x and a interchanged with y and b respectively, and with $f(y)$ in (19.21) replaced by $g(x)$. Therefore the required solution can be written down immediately without further calculation as

$$u(x, y) = \sum_{n \text{ odd}} B_n \sinh\left[\frac{n\pi(a-x)}{b}\right] \sin\left(\frac{n\pi y}{b}\right) + \sum_{n \text{ odd}} C_n \sinh\left[\frac{n\pi(b-y)}{a}\right] \sin\left(\frac{n\pi x}{a}\right),$$

the B_n being given by (19.21) and C_n by

$$C_n = \frac{2}{a \sinh(n\pi b/a)} \int_0^a g(x) \sin(n\pi x/a)\, dx.$$

Clearly, this method may be extended to cases in which three or four sides of the plate have non-zero boundary conditions. ◀

As a final example of the usefulness of the principle of superposition we now consider a problem that illustrates how to deal with inhomogeneous boundary conditions by a suitable change of variables.

►*A bar of length L is initially at a temperature of* $0\,^\circ$C. *One end of the bar* $(x = 0)$ *is held at* $0\,^\circ$C *and the other is supplied with heat at a constant rate per unit area of H. Find the temperature distribution within the bar after a time t.*

With our usual notation, the heat diffusion equation satisfied by the temperature $u(x,t)$ is

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

with $\kappa = k/(s\rho)$, where k is the thermal conductivity of the bar, s is its specific heat capacity and ρ its density.

The boundary conditions can be written as

$$u(x,0) = 0, \qquad u(0,t) = 0, \qquad \frac{\partial u(L,t)}{\partial x} = \frac{H}{k},$$

the last of which is inhomogeneous. In general, inhomogeneous boundary conditions can cause difficulties and it is usual to attempt a transformation of the problem into an equivalent homogeneous one. To this end, let us assume that the solution to our problem takes the form

$$u(x,t) = v(x,t) + w(x),$$

where the function $w(x)$ is to be suitably determined. In terms of v and w the problem becomes

$$\kappa\left(\frac{\partial^2 v}{\partial x^2} + \frac{d^2 w}{dx^2}\right) = \frac{\partial v}{\partial t},$$
$$v(x,0) + w(x) = 0,$$
$$v(0,t) + w(0) = 0,$$
$$\frac{\partial v(L,t)}{\partial x} + \frac{dw(L)}{dx} = \frac{H}{k}.$$

There are several ways of choosing $w(x)$ so as to make the new problem straightforward. Using some physical insight, however, it is clear that ultimately (at $t = \infty$), when all transients have died away, the end $x = L$ will attain a temperature u_0 such that $ku_0/L = H$ and there will be a constant temperature gradient $u(x,\infty) = u_0 x/L$. We therefore choose

$$w(x) = \frac{Hx}{k}.$$

Since the second derivative of $w(x)$ is zero, v satisfies the diffusion equation and the boundary conditions on v are now

$$v(x,0) = -\frac{Hx}{k}, \qquad v(0,t) = 0, \qquad \frac{\partial v(L,t)}{\partial x} = 0,$$

which are homogeneous in x.

From (19.12) a separated solution for the one-dimensional diffusion equation is

$$v(x,t) = (A\cos\lambda x + B\sin\lambda x)\exp(-\lambda^2\kappa t),$$

corresponding to a separation constant $-\lambda^2$. If we restrict λ to be real then all these solutions are transient ones decaying to zero as $t \to \infty$. These are just what is needed for adding to $w(x)$ to give the correct solution as $t \to \infty$. In order to satisfy $v(0,t) = 0$, however, we require $A = 0$. Furthermore, since

$$\frac{\partial v}{\partial x} = B\exp(-\lambda^2\kappa t)\lambda\cos\lambda x,$$

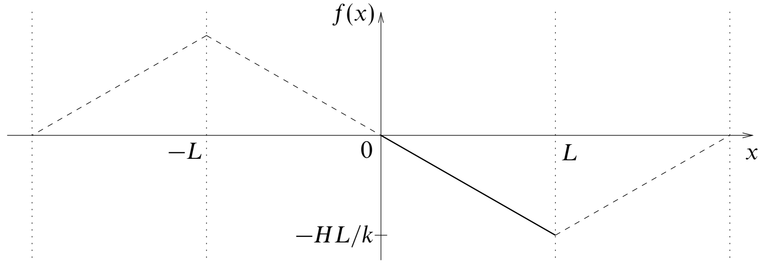

Figure 19.4 The appropriate continuation for a Fourier series containing only sine terms.

in order to satisfy $\partial v(L,t)/\partial x = 0$ we require $\cos\lambda L = 0$, and so λ is restricted to take the values

$$\lambda = \frac{n\pi}{2L},$$

where n is an odd non-negative integer, i.e. $n = 1, 3, 5, \ldots$.

Thus, to satisfy the boundary condition $v(x,0) = -Hx/k$, we must have

$$\sum_{n \text{ odd}} B_n \sin\left(\frac{n\pi x}{2L}\right) = -\frac{Hx}{k},$$

in the range $x = 0$ to $x = L$. In this case we must be more careful about the continuation of the function $-Hx/k$ for which the Fourier sine series is needed. We want a series that is odd in x (sine terms only) and continuous as $x = 0$ and $x = L$ (no discontinuities, since the series must converge at the end-points). This leads to a continuation of the function as shown in figure 19.4, with a period of $L' = 4L$. Following the discussion of section 12.3, since this continuation is odd about $x = 0$ and even about $x = L'/4 = L$ it can indeed be expressed as a Fourier sine series containing only odd-numbered terms.

The corresponding Fourier series coefficients are found to be

$$B_n = \frac{-8HL}{k\pi^2}\frac{(-1)^{(n-1)/2}}{n^2} \qquad \text{for } n \text{ odd},$$

and thus the final formula for $u(x,t)$ is

$$u(x,t) = \frac{Hx}{k} - \frac{8HL}{k\pi^2}\sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(\frac{n\pi x}{2L}\right) \exp\left(-\frac{kn^2\pi^2 t}{4L^2 s\rho}\right),$$

giving the temperature for all positions $0 \le x \le L$ and for all times $t \ge 0$. ◀

We note that in all the above examples the boundary conditions restricted the separation constant(s) to an infinite number of *discrete* values, usually integers. If, however, the boundary conditions allow the separation constant(s) λ to take a *continuum* of values then the summation in (19.16) is replaced by an integral over λ. This is discussed further in connection with integral transform methods in section 19.4.

19.3 Separation of variables in polar coordinates

So far we have considered the solution of PDEs only in Cartesian coordinates, but many systems in two and three dimensions are more naturally expressed in some form of polar coordinates, in which full advantage can be taken of any inherent symmetries. For example, the potential associated with an isolated point charge has a very simple expression, $q/(4\pi\epsilon_0 r)$, when polar coordinates are used, but involves all three coordinates and square roots, when Cartesians are employed. For these reasons we now turn to the separation of variables in plane polar, cylindrical polar and spherical polar coordinates.

Most of the PDEs we have considered so far have involved the operator ∇^2, e.g. the wave equation, the diffusion equation, Schrödinger's equation and Poisson's equation (and of course Laplace's equation). It is therefore appropriate that we recall the expressions for ∇^2 when expressed in polar coordinate systems. From chapter 10, in plane polars, cylindrical polars and spherical polars respectively we have

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}, \tag{19.23}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}, \tag{19.24}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{19.25}$$

Of course the first of these may be obtained from the second by taking z to be identically zero.

19.3.1 Laplace's equation in polar coordinates

The simplest of the equations containing ∇^2 is Laplace's equation,

$$\nabla^2 u(\mathbf{r}) = 0. \tag{19.26}$$

Since it contains most of the essential features of the other more complicated equations we will consider its solution first.

Laplace's equation in plane polars

Suppose that we need to find a solution of (19.26) that has a prescribed behaviour on the circle $\rho = a$ (e.g. if we are finding the shape taken up by a circular drumskin when its rim is slightly deformed from being planar). Then we may seek solutions of (19.26) that are separable in ρ and ϕ (measured from some arbitrary radius as $\phi = 0$) and hope to accommodate the boundary condition by examining the solution for $\rho = a$.

Thus, writing $u(\rho, \phi) = P(\rho)\Phi(\phi)$ and using the expression (19.23), Laplace's equation (19.26) becomes

$$\frac{\Phi}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial P}{\partial \rho}\right) + \frac{P}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$

Now, employing the same device as previously, that of dividing through by $u = P\Phi$ and multiplying through by ρ^2, results in the separated equation

$$\frac{\rho}{P}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial P}{\partial \rho}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$

Following our earlier argument, since the first term on the RHS is a function of ρ only, whilst the second term depends only on ϕ, we obtain the two *ordinary* equations

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) = n^2 \tag{19.27}$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2, \tag{19.28}$$

where we have taken the separation constant to have the form n^2 for later convenience; for the present n is a general (complex) number.

Let us first consider the case in which $n \neq 0$. The second equation, (19.28), then has the general solution

$$\Phi(\phi) = A\exp(in\phi) + B\exp(-in\phi). \tag{19.29}$$

Equation (19.27), on the other hand, is the homogeneous equation

$$\rho^2 P'' + \rho P' - n^2 P = 0,$$

which must be solved either by trying a power solution in ρ or by making the substitution $\rho = \exp t$ as described in subsection 15.2.1 and so reducing it to an equation with constant coefficients. Carrying out this procedure we find

$$P(\rho) = C\rho^n + D\rho^{-n}. \tag{19.30}$$

Returning to the solution (19.29) of the azimuthal equation (19.28), we can see that if Φ, and hence u, is to be single-valued and so not change when ϕ increases by 2π then n must be an integer. Mathematically, other values of n are permissible, but for the description of real physical situations it is clear that this limitation must be imposed. Having thus restricted the possible values of n in one part of the solution, the same limitations must be carried over into the radial part (19.30). Thus we may write a particular solution of the two-dimensional Laplace equation as

$$u(\rho, \phi) = (A\cos n\phi + B\sin n\phi)(C\rho^n + D\rho^{-n}),$$

where A, B, C, D are arbitrary constants and n is any integer.

We have not yet, however, considered the solution when $n = 0$. In this case, the solutions of the separated ordinary equations (19.28) and (19.27) respectively are easily shown to be

$$\Phi(\phi) = A\phi + B,$$
$$P(\rho) = C \ln \rho + D.$$

But, in order that $u = P\Phi$ is single-valued, we require $A = 0$ and so the solution for $n = 0$ is simply (absorbing B into C and D)

$$u(\rho, \phi) = C \ln \rho + D.$$

Superposing the solutions for the different allowed values of n, we can write the general solution to Laplace's equation in plane polars as

$$u(\rho, \phi) = (C_0 \ln \rho + D_0) + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi)(C_n \rho^n + D_n \rho^{-n}), \tag{19.31}$$

where n can take only integer values. Negative values of n have been omitted from the sum since they are already included in the terms obtained for positive n. We note that, since $\ln \rho$ is singular at $\rho = 0$, whenever we solve Laplace's equation in a region containing the origin, C_0 must be identically zero.

►*A circular drumskin has a supporting rim at $\rho = a$. If the rim is twisted so that it is displaced vertically by a small amount $\epsilon(\sin\phi + 2\sin 2\phi)$, where ϕ is the azimuthal angle with respect to a given radius, find the resulting displacement $u(\rho, \phi)$ over the entire drumskin.*

The transverse displacement of a circular drumskin is usually described by the two-dimensional wave equation. In this case, however, there is no time dependence and so $u(\rho, \phi)$ solves the two-dimensional Laplace equation, subject to the imposed boundary condition.

Referring to (19.31), since we wish to find a solution that is finite everywhere inside $\rho = a$, we require $C_0 = 0$ and $D_n = 0$ for all $n > 0$. Now the boundary condition at the rim requires

$$u(a, \phi) = D_0 + \sum_{n=1}^{\infty} C_n a^n (A_n \cos n\phi + B_n \sin n\phi) = \epsilon(\sin\phi + 2\sin 2\phi).$$

Firstly we see that we require $D_0 = 0$ and $A_n = 0$ for all n. Furthermore, we must have $C_1 B_1 a = \epsilon$, $C_2 B_2 a^2 = 2\epsilon$ and $B_n = 0$ for $n > 2$. Hence the appropriate shape for the drumskin (valid over the whole skin, not just the rim) is

$$u(\rho, \phi) = \frac{\epsilon\rho}{a} \sin\phi + \frac{2\epsilon\rho^2}{a^2} \sin 2\phi = \frac{\epsilon\rho}{a}\left(\sin\phi + \frac{2\rho}{a}\sin 2\phi\right). \blacktriangleleft$$

Laplace's equation in cylindrical polars

Passing to three dimensions, we now consider the solution of Laplace's equation in cylindrical polar coordinates,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2}+\frac{\partial^2 u}{\partial z^2}=0. \tag{19.32}$$

We note here that, even when considering a cylindrical physical system, if there is no dependence of the physical variables on z (i.e. along the length of the cylinder) then the problem may be treated using two-dimensional plane polars, as discussed above.

For the more general case, however, we proceed as previously by trying a solution of the form

$$u(\rho,\phi,z)=P(\rho)\Phi(\phi)Z(z),$$

which on substitution into (19.32) and division through by $u = P\Phi Z$ gives

$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2}+\frac{1}{Z}\frac{d^2 Z}{dz^2}=0.$$

The last term depends only on z and the first and second (taken together) only on ρ and ϕ. Taking the separation constant to be k^2, we find

$$\frac{1}{Z}\frac{d^2 Z}{dz^2}=k^2,$$

$$\frac{1}{P\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2}+k^2=0.$$

The first of these equations has the straightforward solution

$$Z(z)=E\exp(-kz)+F\exp kz.$$

Multiplying the second equation through by ρ^2, we obtain

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+k^2\rho^2=0,$$

in which the second term depends only on Φ and the other terms only on ρ. Taking the second separation constant to be m^2, we find

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2, \tag{19.33}$$

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+(k^2\rho^2-m^2)P=0. \tag{19.34}$$

The equation in the azimuthal angle ϕ has the very familiar solution

$$\Phi(\phi)=C\cos m\phi+D\sin m\phi.$$

As in the two-dimensional case, single-valuedness of u requires that m is an integer. However, in the particular case $m = 0$ the solution is

$$\Phi(\phi) = C\phi + D.$$

This form is appropriate to a solution with axial symmetry ($C = 0$) or one that is multivalued, but manageably so, such as the magnetic scalar potential associated with a current I (in which case $C = I/(2\pi)$ and D is arbitrary).

Finally the ρ-equation (19.34) may be transformed into Bessel's equation of order m by writing $\mu = k\rho$. This has the solution

$$P(\rho) = AJ_m(k\rho) + BY_m(k\rho).$$

The properties of these functions were investigated in chapter 16 and will not be pursued here. We merely note that $Y_m(k\rho)$ is singular at $\rho = 0$, and so when seeking solutions to Laplace's equation in cylindrical coordinates within some region containing the $\rho = 0$ axis, we require $B = 0$.

The complete separated-variable solution in cylindrical polars of Laplace's equation $\nabla^2 u = 0$ is thus

$$u(\rho, \phi, z) = [AJ_m(k\rho) + BY_m(k\rho)][C\cos m\phi + D\sin m\phi][E\exp(-kz) + F\exp kz]. \tag{19.35}$$

Of course we may use the principle of superposition to build up more general solutions by adding together solutions of the form (19.35) for all allowed values of the separation constants k and m.

►*A semi-infinite solid cylinder of radius a has its curved surface held at $0\,^\circ$C and its base held at a temperature T_0. Find the steady-state temperature distribution in the cylinder.*

The physical situation is shown in figure 19.5. The steady-state temperature distribution $u(\rho, \phi, z)$ must satisfy Laplace's equation subject to the imposed boundary conditions. Let us take the cylinder to have its base in the $z = 0$ plane and to extend along the positive z-axis. From (19.35), in order that u is finite everywhere in the cylinder we immediately require $B = 0$ and $F = 0$. Furthermore, since the boundary conditions, and hence the temperature distribution, are axially symmetric we require $m = 0$, and so the general solution must be a superposition of solutions of the form $J_0(k\rho)\exp(-kz)$ for all allowed values of the separation constant k.

The boundary condition $u(a, \phi, z) = 0$ restricts the allowed values of k since we must have $J_0(ka) = 0$. The zeroes of Bessel functions are given in most books of mathematical tables, and we find that, to two decimal places,

$$J_0(x) = 0 \qquad \text{for } x = 2.40,\ 5.52,\ 8.65,\ \ldots.$$

Writing the allowed values of k as k_n for $n = 1, 2, 3, \ldots$ (so, for example, $k_1 = 2.40/a$), the required solution takes the form

$$u(\rho, \phi, z) = \sum_{n=1}^{\infty} A_n J_0(k_n\rho)\exp(-k_n z).$$

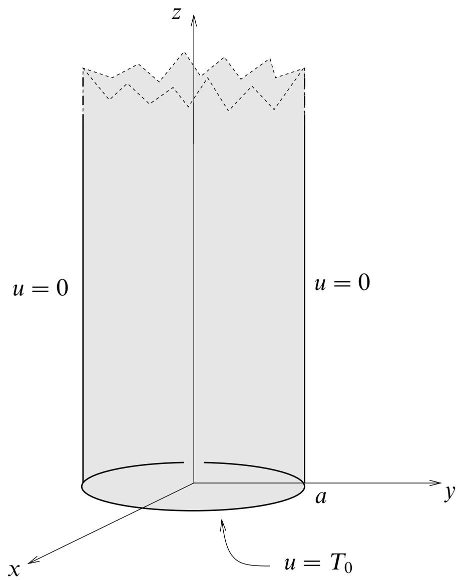

Figure 19.5 A uniform metal cylinder whose curved surface is kept at 0°C and whose base is held at a temperature T_0.

By imposing the remaining boundary condition $u(\rho, \phi, 0) = T_0$, the coefficients A_n can be found in a similar way to Fourier coefficients but this time by exploiting the orthogonality of the Bessel functions, as discussed in chapter 16. From this boundary condition we require

$$u(\rho, \phi, 0) = \sum_{n=1}^{\infty} A_n J_0(k_n \rho) = T_0.$$

If we multiply this expression by $\rho J_0(k_r \rho)$ and integrate from $\rho = 0$ to $\rho = a$, and use the orthogonality of the Bessel functions $J_0(k_n \rho)$, then the coefficients are given by (16.81) as

$$A_n = \frac{2T_0}{a^2 J_1^2(k_n a)} \int_0^a J_0(k_n \rho) \rho \, d\rho. \tag{19.36}$$

The integral on the RHS can be evaluated using the recurrence relation (16.68) of chapter 16,

$$\frac{d}{dz}[z J_1(z)] = z J_0(z),$$

which on setting $z = k_n \rho$ yields

$$\frac{1}{k_n} \frac{d}{d\rho}[k_n \rho J_1(k_n \rho)] = k_n \rho J_0(k_n \rho).$$

Therefore the integral in (19.36) is given by

$$\int_0^a J_0(k_n \rho) \rho \, d\rho = \left[\frac{1}{k_n} \rho J_1(k_n \rho) \right]_0^a = \frac{1}{k_n} a J_1(k_n a),$$

and the coefficients A_n may be expressed as

$$A_n = \frac{2T_0}{a^2 J_1^2(k_n a)} \left[\frac{a J_1(k_n a)}{k_n} \right] = \frac{2T_0}{k_n a J_1(k_n a)}.$$

The steady-state temperature in the cylinder is then given by

$$u(\rho, \phi, z) = \sum_{n=1}^{\infty} \frac{2T_0}{k_n a J_1(k_n a)} J_0(k_n \rho) \exp(-k_n z). \blacktriangleleft$$

We note that if, in the above example, the base of the cylinder were not kept at a uniform temperature T_0, but instead had some fixed temperature distribution $T(\rho, \phi)$, then the solution of the problem would become more complicated. In such a case, the required temperature distribution $u(\rho, \phi, z)$ is in general *not* axially symmetric, and so the separation constant m is not restricted to be zero but may take any integer value. The solution will then take the form

$$u(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{nm}\rho)(C_{nm} \cos m\phi + D_{nm} \sin m\phi) \exp(-k_{nm} z),$$

where the separation constants k_{nm} are such that $J_m(k_{nm} a) = 0$, i.e. $k_{nm} a$ is the nth zero of the mth-order Bessel function. At the base of the cylinder we would then require

$$u(\rho, \phi, 0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{nm}\rho)(C_{nm} \cos m\phi + D_{nm} \sin m\phi) = T(\rho, \phi). \tag{19.37}$$

The coefficients C_{nm} could be found by multiplying (19.37) by $J_q(k_{rq}\rho) \cos q\phi$, integrating with respect to ρ and ϕ over the base of the cylinder and exploiting the orthogonality of the Bessel functions and of the trigonometric functions. The D_{nm} could be found in a similar way by multiplying (19.37) by $J_q(k_{rq}\rho) \sin q\phi$.

Laplace's equation in spherical polars

We now come to an eequation that is very widely applicable in physical science, namely $\nabla^2 u = 0$ in spherical polar coordinates:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \phi^2} = 0. \tag{19.38}$$

Our method of procedure will be as before; we try a solution of the form

$$u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi).$$

Substituting this in (19.38), dividing through by $u = R\Theta\Phi$ and multiplying by r^2, we obtain

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi \sin^2\theta} \frac{d^2\Phi}{d\phi^2} = 0. \tag{19.39}$$

The first term depends only on r and the second and third terms (taken together) only on θ and ϕ. Thus (19.39) is equivalent to the two equations

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda, \tag{19.40}$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = -\lambda. \tag{19.41}$$

Equation (19.40) is a homogeneous equation,

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \lambda R = 0,$$

which can be reduced by the substitution $r = \exp t$ (and writing $R(r) = S(t)$) to

$$\frac{d^2S}{dt^2} + \frac{dS}{dt} - \lambda S = 0.$$

This has the straightforward solution

$$S(t) = A\exp\lambda_1 t + B\exp\lambda_2 t,$$

and so the solution to the radial equation is

$$R(r) = Ar^{\lambda_1} + Br^{\lambda_2},$$

where $\lambda_1 + \lambda_2 = -1$ and $\lambda_1\lambda_2 = -\lambda$. We can thus take λ_1 and λ_2 as given by ℓ and $-(\ell+1)$; λ then has the form $\ell(\ell+1)$. (It should be noted that at this stage nothing has been either assumed or proved about whether ℓ is an integer.)

Hence we have obtained some information about the first factor in the separated-variable solution, which will now have the form

$$u(r,\theta,\phi) = \left[Ar^{\ell} + Br^{-(\ell+1)}\right]\Theta(\theta)\Phi(\phi), \tag{19.42}$$

where Θ and Φ must satisfy (19.41) with $\lambda = \ell(\ell+1)$.

The next step is to take (19.41) further. Multiplying through by $\sin^2\theta$ and substituting for λ, it too takes a separated form:

$$\left[\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta\right] + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \tag{19.43}$$

Taking the separation constant as m^2, the equation in the azimuthal angle ϕ has the same solution as in cylindrical polars, namely

$$\Phi(\phi) = C\cos m\phi + D\sin m\phi.$$

As before, single-valuedness of u requires that m is an integer; for $m = 0$ we again have $\Phi(\phi) = C\phi + D$.

Having settled the form of $\Phi(\phi)$, we are left only with the equation satisfied by $\Theta(\theta)$, which is

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\ell(\ell+1)\sin^2\theta=m^2. \tag{19.44}$$

A change of independent variable from θ to $\mu=\cos\theta$ will reduce this to a form for which solutions are known, and of which some study has been made in chapter 16. Putting

$$\mu=\cos\theta, \qquad \frac{d\mu}{d\theta}=-\sin\theta, \qquad \frac{d}{d\theta}=-(1-\mu^2)^{1/2}\frac{d}{d\mu},$$

the equation for $M(\mu)\equiv\Theta(\theta)$ reads

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-\mu^2}\right]M=0. \tag{19.45}$$

This equation is the *associated Legendre equation*, which was mentioned in subsection 17.5.2 in the context of Sturm–Liouville equations.

We recall that for the case $m=0$, (19.45) reduces to Legendre's equation, which was studied at length in chapter 16, and has the solution

$$M(\mu)=EP_\ell(\mu)+FQ_\ell(\mu). \tag{19.46}$$

We have not solved (19.45) explicitly for general m, but the solutions were given in subsection 17.5.2 and are the associated Legendre functions $P_\ell^m(\mu)$ and $Q_\ell^m(\mu)$, where

$$P_\ell^m(\mu)=(1-\mu^2)^{|m|/2}\frac{d^{|m|}}{d\mu^{|m|}}P_\ell(\mu), \tag{19.47}$$

and similarly for $Q_\ell^m(\mu)$. We then have

$$M(\mu)=EP_\ell^m(\mu)+FQ_\ell^m(\mu); \tag{19.48}$$

here m must be an integer, $0\leq|m|\leq\ell$. We note that if we require solutions to Laplace's equation that are finite when $\mu=\cos\theta=\pm1$ (i.e. on the polar axis where $\theta=0,\pi$), then we must have $F=0$ in (19.46) and (19.48) since $Q_\ell^m(\mu)$ diverges at $\mu=\pm1$.

It will be remembered that one of the important conditions for obtaining finite polynomial solutions of Legendre's equation is that ℓ is an integer ≥ 0. This condition therefore applies also to the solutions (19.46) and (19.48) and is reflected back into the radial part of the general solution given in (19.42).

Now that the solutions of each of the three ordinary differential equations governing R, Θ and Φ have been obtained, we may assemble a complete separated-variable solution of Laplace's equation in spherical polars. It is

$$u(r,\theta,\phi) = (Ar^{\ell} + Br^{-(\ell+1)})(C\cos m\phi + D\sin m\phi)[EP_{\ell}^{m}(\cos\theta) + FQ_{\ell}^{m}(\cos\theta)], \tag{19.49}$$

where the three bracketted factors are connected only through the *integer* parameters ℓ and m, $0 \le |m| \le \ell$. As before, a general solution may be obtained by superposing solutions of this form for the allowed values of the separation constants ℓ and m. As mentioned above, if the solution is required to be finite on the polar axis then $F = 0$ for all ℓ and m.

►*An uncharged conducting sphere of radius a is placed at the origin in an initially uniform electrostatic field E. Show that it behaves as an electric dipole.*

The uniform field, taken in the direction of the polar axis, has a electrostatic potential

$$u = -Ez = -Er\cos\theta,$$

where u is arbitrarily taken as zero at $z = 0$. This satisfies Laplace's equation $\nabla^2 u = 0$, as must the potential v when the sphere is present; for large r the asymptotic form of v must still be $-Er\cos\theta$.

Since the problem is clearly axially symmetric we have immediately that $m = 0$, and since we require v to be finite on the polar axis we must have $F = 0$ in (19.49). Therefore the solution must be of the form

$$v(r,\theta,\phi) = \sum_{\ell=0}^{\infty}(A_{\ell}r^{\ell} + B_{\ell}r^{-(\ell+1)})P_{\ell}(\cos\theta).$$

Now the $\cos\theta$-dependence of v for large r indicates that the (θ,ϕ)-dependence of $v(r,\theta,\phi)$ is given by $P_1^0(\cos\theta) = \cos\theta$. Thus the r-dependence of v must also correspond to an $\ell = 1$ solution, and the most general such solution (outside the sphere, i.e. for $r \ge a$) is

$$v(r,\theta,\phi) = (A_1 r + B_1 r^{-2})P_1(\cos\theta).$$

The asymptotic form of v for large r immediately gives $A_1 = -E$ and so yields the solution

$$v(r,\theta,\phi) = \left(-Er + \frac{B_1}{r^2}\right)\cos\theta.$$

Since the sphere is conducting, it is an equipotential region and so v must not depend on θ for $r = a$. This can only be the case if $B_1/a^2 = Ea$, thus fixing B_1. The final solution is therefore

$$v(r,\theta,\phi) = -Er\left(1 - \frac{a^3}{r^3}\right)\cos\theta.$$

Since a dipole of moment p gives rise to a potential $p/(4\pi\epsilon_0 r^2)$, this result shows that the sphere behaves as a dipole of moment $4\pi\epsilon_0 a^3 E$, because of the charge distribution induced on its surface; see figure 19.6. ◄

Often the boundary conditions are not so easily met, and it is necessary to use the mutual orthogonality of the associated Legendre functions (and the trigonometric functions) to obtain the coefficients in the general solution.

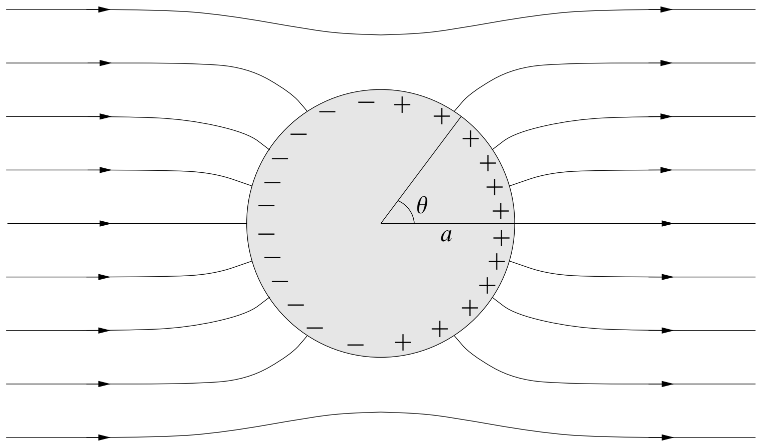

Figure 19.6 Induced charge and field lines associated with a conducting sphere placed in an initially uniform electrostatic field.

►*A hollow split conducting sphere of radius a is placed at the origin. If one half of its surface is charged to a potential v_0 and the other half is kept at zero potential, find the potential v inside and outside the sphere.*

Let us choose the top hemisphere to be charged to v_0 and the bottom hemisphere to be at zero potential, with the plane in which the two hemispheres meet perpendicular to the polar axis; this is shown in figure 19.7. The boundary condition then becomes

$$v(a, \theta, \phi) = \begin{cases} v_0 & \text{for } 0 < \theta < \pi/2 \quad (0 < \cos\theta < 1), \\ 0 & \text{for } \pi/2 < \theta < \pi \quad (-1 < \cos\theta < 0). \end{cases} \tag{19.50}$$

The problem is clearly axially symmetric and so we may set $m = 0$. Also, we require the solution to be finite on the polar axis and so it cannot contain $Q_\ell(\cos\theta)$. Therefore the general form of the solution to (19.38) is

$$v(r, \theta, \phi) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta). \tag{19.51}$$

Inside the sphere (for $r < a$) we require the solution to be finite at the origin and so $B_\ell = 0$ for all ℓ in (19.51). Imposing the boundary condition at $r = a$ we must then have

$$v(a, \theta, \phi) = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta),$$

where $v(a, \theta, \phi)$ is also given by (19.50). Exploiting the mutual orthogonality of the Legendre polynomials, the coefficients in the Legendre polynomial expansion are given by (16.48) as (writing $\mu = \cos\theta$)

$$\begin{aligned} A_\ell a^\ell &= \frac{2\ell+1}{2} \int_{-1}^{1} v(a, \theta, \phi) P_\ell(\mu)\, d\mu \\ &= \frac{2\ell+1}{2} v_0 \int_{0}^{1} P_\ell(\mu)\, d\mu, \end{aligned}$$

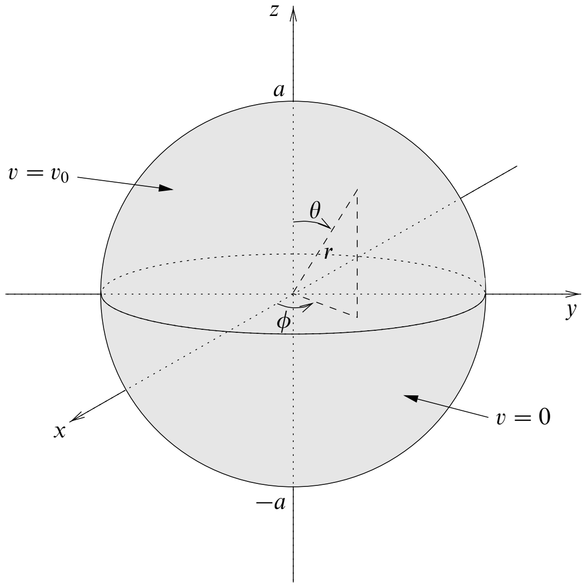

Figure 19.7 A hollow split conducting sphere with its top half charged to a potential v_0 and its bottom half at zero potential.

where in the last line we have used (19.50). The integrals of the Legendre polynomials are easily evaluated (see exercise 17.7) and we find

$$A_0 = \frac{v_0}{2}, \qquad A_1 = \frac{3v_0}{4a}, \qquad A_2 = 0, \qquad A_3 = -\frac{7v_0}{16a^3}, \qquad \cdots,$$

so that the required solution inside the sphere is

$$v(r, \theta, \phi) = \frac{v_0}{2}\left[1 + \frac{3r}{2a}P_1(\cos\theta) - \frac{7r^3}{8a^3}P_3(\cos\theta) + \cdots\right].$$

Outside the sphere (for $r > a$) we require the solution to be bounded as r tends to infinity and so in (19.51) we must have $A_\ell = 0$ for all ℓ. In this case, by imposing the boundary condition at $r = a$ we require

$$v(a, \theta, \phi) = \sum_{\ell=0}^{\infty} B_\ell a^{-(\ell+1)} P_\ell(\cos\theta),$$

where $v(a, \theta, \phi)$ is given by (19.50). Following the above argument the coefficients in the expansion are given by

$$B_\ell a^{-(\ell+1)} = \frac{2\ell + 1}{2} v_0 \int_0^1 P_\ell(\mu)\, d\mu,$$

so that the required solution outside the sphere is

$$v(r, \theta, \phi) = \frac{v_0 a}{2r}\left[1 + \frac{3a}{2r}P_1(\cos\theta) - \frac{7a^3}{8r^3}P_3(\cos\theta) + \cdots\right]. \blacktriangleleft$$

In the above example, on the equator of the sphere (i.e. at $r = a$ and $\theta = \pi/2$) the potential is given by

$$v(a, \pi/2, \phi) = v_0/2,$$

i.e. mid-way between the potentials of the top and bottom hemispheres. This is so because a Legendre polynomial expansion of a function behaves in the same way as a Fourier series expansion, in that it converges to the average of the two values at any discontinuities present in the original function.

If the potential on the surface of the sphere had been given as a function of θ *and* ϕ, then we would have had to consider a double series summed over ℓ and m (for $-\ell \leq m \leq \ell$), since, in general, the solution would not have been axially symmetric.

19.3.2 Spherical harmonics

When obtaining solutions in spherical polar coordinates of $\nabla^2 u = 0$, we found that, for solutions that are finite on the polar axis, the angular part of the solution was given by

$$\Theta(\theta)\Phi(\phi) = P_\ell^m(\cos\theta)(C\cos m\phi + D\sin m\phi).$$

This general form is sufficiently common that particular functions of θ and ϕ called *spherical harmonics* are defined and tabulated. The spherical harmonics $Y_\ell^m(\theta, \phi)$ are defined for $m \geq 0$ by

$$Y_\ell^m(\theta, \phi) = (-1)^m \left[\frac{2\ell + 1}{4\pi}\frac{(\ell - m)!}{(\ell + m)!}\right]^{1/2} P_\ell^m(\cos\theta)\exp(im\phi). \qquad (19.52)$$

For values of $m < 0$ the relation

$$Y_\ell^{-|m|}(\theta, \phi) = (-1)^{|m|}\left[Y_\ell^{|m|}(\theta, \phi)\right]^*$$

defines the spherical harmonic, the asterisk denoting complex conjugation. Since they contain as their θ-dependent part the solution P_ℓ^m to the associated Legendre equation, which is a Sturm–Liouville equation (see chapter 17), the Y_ℓ^m are mutually orthogonal when integrated from -1 to $+1$ over $d(\cos\theta)$. Their mutual orthogonality with respect to ϕ ($0 \leq \phi \leq 2\pi$) is even more obvious. The numerical factor in (19.52) is chosen to make the Y_ℓ^m an orthonormal set, that is

$$\int_{-1}^{1}\int_0^{2\pi} \left[Y_\ell^m(\theta, \phi)\right]^* Y_{\ell'}^{m'}(\theta, \phi)\, d\phi\, d(\cos\theta) = \delta_{\ell\ell'}\delta_{mm'}.$$

In addition, the spherical harmonics form a complete set in that any reasonable function (i.e. one that is likely to be met in a physical situation) of θ and ϕ can be expanded as a sum of such functions,

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\theta, \phi), \tag{19.53}$$

the constants $a_{\ell m}$ being given by

$$a_{\ell m} = \int_{-1}^{1} \int_{0}^{2\pi} \left[Y_\ell^m(\theta, \phi)\right]^* f(\theta, \phi)\, d\phi\, d(\cos\theta). \tag{19.54}$$

This is in exact analogy with a Fourier series and is a particular example of the general property of Sturm–Liouville solutions.

The first few spherical harmonics $Y_\ell^m(\theta, \phi) \equiv Y_\ell^m$ are as follows:

$$\begin{aligned}
Y_0^0 &= \sqrt{\tfrac{1}{4\pi}}, & Y_1^0 &= \sqrt{\tfrac{3}{4\pi}}\cos\theta, \\
Y_1^{\pm 1} &= \mp\sqrt{\tfrac{3}{8\pi}}\sin\theta\exp(\pm i\phi), & Y_2^0 &= \sqrt{\tfrac{5}{16\pi}}(3\cos^2\theta - 1), \\
Y_2^{\pm 1} &= \mp\sqrt{\tfrac{15}{8\pi}}\sin\theta\cos\theta\exp(\pm i\phi), & Y_2^{\pm 2} &= \sqrt{\tfrac{15}{32\pi}}\sin^2\theta\exp(\pm 2i\phi).
\end{aligned}$$

19.3.3 Other equations in polar coordinates

The development of the solutions of $\nabla^2 u = 0$ carried out in the previous subsection can be employed to solve other equations in which the ∇^2 operator appears. Since we have discussed the general method in some depth already, only an outline of the solutions will be given here.

Let us first consider the wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{19.55}$$

and look for a separated solution of the form $u = F(\mathbf{r})T(t)$, so that initially we are separating only the spatial and time dependences. Substituting this form into (19.55) and taking the separation constant as k^2 we obtain

$$\nabla^2 F + k^2 F = 0, \qquad \frac{d^2 T}{dt^2} + k^2 c^2 T = 0. \tag{19.56}$$

The second equation has the simple solution

$$T(t) = A\exp(i\omega t) + B\exp(-i\omega t), \tag{19.57}$$

where $\omega = kc$; this may also be expressed in terms of sines and cosines, of course. The first equation in (19.56) is referred to as *Helmholtz's equation*; we discuss it below.

We may treat the diffusion equation

$$\kappa\nabla^2 u = \frac{\partial u}{\partial t}$$

in a similar way. Separating the spatial and time dependences by assuming a solution of the form $u = F(\mathbf{r})T(t)$, and taking the separation constant as k^2, we find

$$\nabla^2 F + k^2 F = 0, \qquad \frac{dT}{dt} + k^2 \kappa T = 0.$$

Just as in the case of the wave equation, the spatial part of the solution satisfies Helmholtz's equation. It only remains to consider the time dependence, which has the simple solution

$$T(t) = A \exp(-k^2 \kappa t).$$

Helmholtz's equation is clearly of central importance in the solutions of the wave and diffusion equations. It can be solved in polar coordinates in much the same way as Laplace's equation, and indeed reduces to Laplace's equation when $k = 0$. Therefore, we will merely sketch the method of its solution in each of the three polar coordinate systems.

Helmholtz's equation in plane polars

In two-dimensional plane polar cooordinates Helmholtz's equation takes the form

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial F}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \phi^2} + k^2 F = 0.$$

If we try a separated solution of the form $F(\mathbf{r}) = P(\rho)\Phi(\phi)$, and take the separation constant as m^2, we find

$$\frac{d^2\Phi}{d\phi^2} + m^2 \phi = 0,$$
$$\frac{d^2 P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right) P = 0.$$

As for Laplace's equation, the angular part has the familiar solution (if $m \neq 0$)

$$\Phi(\phi) = A \cos m\phi + B \sin m\phi,$$

or an equivalent form in terms of complex exponentials. The radial equation differs from that found in the solution of Laplace's equation, but by making the substitution $\mu = k\rho$ it is easily transformed into Bessel's equation of order m (discussed in chapter 16), and has the solution

$$P(\rho) = CJ_m(k\rho) + DY_m(k\rho),$$

where Y_m is a Bessel function of the second kind, which is infinite at the origin and is not to be confused with a spherical harmonic (these are written with a superscript as well as a subscript).

Putting the two parts of the solution together we have

$$F(\rho, \phi) = [A \cos m\phi + B \sin m\phi][CJ_m(k\rho) + DY_m(k\rho)]. \qquad (19.58)$$

Clearly, for solutions of Helmholtz's equation that are required to be finite at the origin, we must set $D = 0$.

►*Find the four lowest frequency modes of oscillation of a circular drumskin of radius a whose circumference is held fixed in a plane.*

The transverse displacement $u(\mathbf{r}, t)$ of the drumskin satisfies the two-dimensional wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

with $c^2 = T/\sigma$, where T is the tension of the drumskin and σ is its mass per unit area. From (19.57) and (19.58) a separated solution of this equation, in plane polar coordinates, that is finite at the origin is

$$u(\rho, \phi, t) = J_m(k\rho)(A\cos m\phi + B\sin m\phi)\exp(\pm i\omega t),$$

where $\omega = kc$. Since we require the solution to be single-valued we must have m as an integer. Furthermore, if the drumskin is clamped at its outer edge $\rho = a$ then we also require $u(a, \phi, t) = 0$. Thus we need

$$J_m(ka) = 0,$$

which in turn restricts the allowed values of k. The zeroes of Bessel functions can be obtained from most books of tables, and the first few are

$$\begin{aligned}
J_0(x) &= 0 \quad \text{for } x \approx 2.40,\ 5.52,\ 8.65, \ldots,\\
J_1(x) &= 0 \quad \text{for } x \approx 3.83,\ 7.02,\ 10.17, \ldots,\\
J_2(x) &= 0 \quad \text{for } x \approx 5.14,\ 8.42,\ 11.62 \ldots.
\end{aligned}$$

The smallest value of x for which any of the Bessel functions is zero is $x \approx 2.40$, which occurs for $J_0(x)$. Thus the lowest-frequency mode has $k = 2.40/a$ and angular frequency $\omega = 2.40c/a$. Since $m = 0$ for this mode, the shape of the drumskin is

$$u \propto J_0\left(2.40\frac{\rho}{a}\right);$$

this is illustrated in figure 19.8.

Continuing in the same way the next three modes are given by

$$\begin{aligned}
\omega &= 3.83\frac{c}{a}, \qquad u \propto J_1\left(3.83\frac{\rho}{a}\right)\cos\phi, \qquad J_1\left(3.83\frac{\rho}{a}\right)\sin\phi;\\
\omega &= 5.14\frac{c}{a}, \qquad u \propto J_2\left(5.14\frac{\rho}{a}\right)\cos 2\phi, \qquad J_2\left(5.14\frac{\rho}{a}\right)\sin 2\phi;\\
\omega &= 5.52\frac{c}{a}, \qquad u \propto J_0\left(5.52\frac{\rho}{a}\right).
\end{aligned}$$

These modes are also shown in figure 19.8. We note that the second and third frequencies have *two* corresponding modes of oscillation; these frequencies are therefore two-fold degenerate. ◄

Helmholtz's equation in cylindrical polars

Generalising the above method to three-dimensional cylindrical polars is straightforward, and following a similar procedure to that used for Laplace's equation

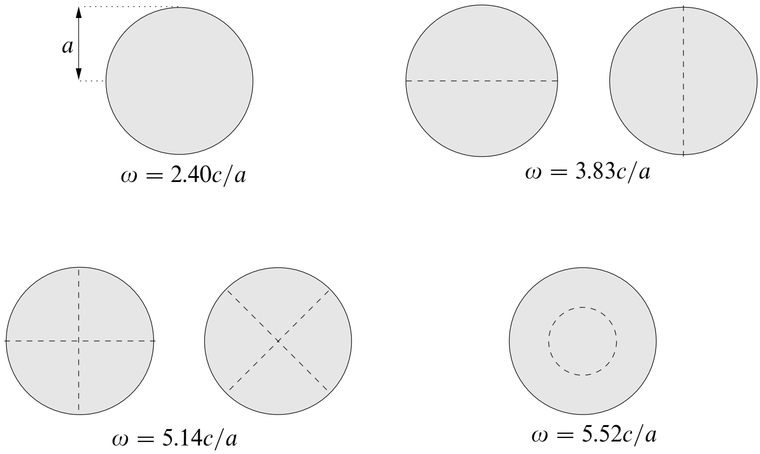

Figure 19.8 For a circular drumskin of radius a, the modes of oscillation with the four lowest frequencies. The dotted lines indicate the nodes, where the displacement of the drumskin is always zero.

we find the separated solution of Helmholtz's equation takes the form

$$F(\rho,\phi,z) = \left[AJ_m\left(\sqrt{k^2-\alpha^2}\,\rho\right) + BY_m\left(\sqrt{k^2-\alpha^2}\,\rho\right)\right] \\ \times (C\cos m\phi + D\sin m\phi)[E\exp(i\alpha z) + F\exp(-i\alpha z)],$$

where α and m are separation constants. We note that the angular part of the solution is the same as for Laplace's equation in cylindrical polars.

Helmholtz's equation in spherical polars

In spherical polars, we find again that the angular parts of the solution $\Theta(\theta)\Phi(\phi)$ are identical to those of Laplace's equation in this coordinate system, i.e. they are the spherical harmonics $Y_\ell^m(\theta,\phi)$, and so we shall not discuss them further.

The radial equation in this case is given by

$$r^2R'' + 2rR' + [k^2r^2 - \ell(\ell+1)]R = 0, \tag{19.59}$$

which has an additional term k^2r^2R compared with the radial equation for the Laplace solution. The equation (19.59) looks very much like Bessel's equation and can in fact be reduced to it by writing $R(r) = r^{-1/2}S(r)$. The function $S(r)$ then satisfies

$$r^2S'' + rS' + \left[k^2r^2 - \left(\ell + \tfrac{1}{2}\right)^2\right]S = 0,$$

which, after changing the variable to $\mu = kr$, is Bessel's equation of order $\ell + \frac{1}{2}$ and has as its solutions $S(\mu) = J_{\ell+1/2}(\mu)$ and $Y_{\ell+1/2}(\mu)$. The separated solution to

Helmholtz's equation in spherical polars is thus

$$F(r,\theta,\phi) = r^{-1/2}[AJ_{\ell+1/2}(kr) + BY_{\ell+1/2}(kr)](C\cos m\phi + D\sin m\phi) \times [EP_\ell^m(\cos\theta) + FQ_\ell^m(\cos\theta)]. \tag{19.60}$$

For solutions that are finite at the origin we require $B = 0$, and for solutions that are finite on the polar axis we require $F = 0$.

It is worth mentioning that the solutions proportional to $r^{-1/2}J_{\ell+1/2}(kr)$ when suitably normalised are called *spherical Bessel functions* and are denoted by $j_\ell(kr)$:

$$j_\ell(\mu) = \sqrt{\frac{\pi}{2\mu}}J_{\ell+1/2}(\mu).$$

They are trigonometric functions of μ (as discussed in chapter 16), and for $\ell = 0$ and $\ell = 1$ are given by

$$j_0(\mu) = \sin\mu,$$
$$j_1(\mu) = \frac{\sin\mu}{\mu} - \cos\mu.$$

The second, linearly-independent, solution of (19.59), $n_\ell(\mu)$, is derived from $Y_{\ell+1/2}(\mu)$ in a similar way.

As mentioned at the beginning of this subsection, the separated solution of the wave equation in spherical polars is the product of the time-dependent part (19.57) and a spatial part (19.60). It will be noticed that, although this solution corresponds to a solution of definite frequency $\omega = kc$, the zeroes of the radial function $j_\ell(kr)$ are not equally spaced in r, except for the case $\ell = 0$ involving $j_0(kr)$, and so there is no precise wavelength associated with the solution.

To conclude this subsection, let us mention briefly the Schrödinger equation for the electron in a hydrogen atom, the nucleus of which is taken at the origin and is assumed massive compared with the electron. Under these circumstances the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2 u - \frac{e^2}{4\pi\epsilon_0}\frac{u}{r} = i\hbar\frac{\partial u}{\partial t}.$$

For a 'stationary-state' solution, for which the energy is a constant E and the time-dependent factor T in u is given by $T(t) = A\exp(-iEt/\hbar)$, the above equation is similar to, but not quite the same as, the Helmholtz equation.† However, as with the wave equation, the angular parts of the solution are identical to those for Laplace's equation and are expressed in terms of spherical harmonics.

The important point to note is that for *any* equation involving ∇^2, provided θ and ϕ do not appear in the equation other than as part of ∇^2, a separated-variable

† For the solution by series of the r-equation in this case the reader may consult, e.g., Schiff, *Quantum Mechanics* (McGraw-Hill, 1955) p. 82.

solution in spherical polars will always lead to spherical harmonic solutions. This is the case for the Schrödinger equation describing an atomic electron whenever the potential is central, i.e. whenever $V(\mathbf{r})$ is in fact $V(r)$.

19.3.4 Solution by expansion

It is sometimes possible to use the uniqueness theorem discussed in the last chapter, together with the results of the last few subsections, in which Laplace's equation (and other equations) were considered in polar coordinates, to obtain solutions of such equations appropriate to particular physical situations.

We will illustrate the method for Laplace's equation in spherical polars and first assume that the required solution of $\nabla^2 u = 0$ can be written as a superposition in the normal way:

$$u(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(Ar^{\ell} + Br^{-(\ell+1)})P_{\ell}^{m}(\cos\theta)(C\cos m\phi + D\sin m\phi). \tag{19.61}$$

Here, all the constants A, B, C, D may depend upon ℓ and m, and we have assumed that the required solution is finite on the polar axis. As usual, boundary conditions of a physical nature will then fix or eliminate some of the constants; for example, u finite at the origin implies all $B = 0$, or axial symmetry implies that only $m = 0$ terms are present.

The essence of the method is then to find the remaining constants by determining u at values of r, θ, ϕ for which it can be evaluated *by other means*, e.g. by direct calculation on an axis of symmetry. Once the remaining constants have been fixed by these special considerations to have particular values, the uniqueness theorem can be invoked to establish that they must have these values in general.

►*Calculate the gravitational potential at a general point in space due to a uniform ring of matter of radius a and total mass M.*

Everywhere except on the ring the potential $u(\mathbf{r})$ satisfies the Laplace equation, and so if we use polar coordinates with the normal to the ring as polar axis, as in figure 19.9, a solution of the form (19.61) can be assumed.

We expect the potential $u(r,\theta,\phi)$ to tend to zero as $r \to \infty$, and also to be finite at $r = 0$. At first sight this might seem to imply that all A and B, and hence u, must be identically zero, an unacceptable result. In fact, what it means is that different expressions must apply to different regions of space. On the ring itself we no longer have $\nabla^2 u = 0$ and so it is not surprising that the form of the expression for u changes there. Let us therefore take two separate regions.

In the region $r > a$

(i) we must have $u \to 0$ as $r \to \infty$, implying that all $A = 0$, and
(ii) the system is axially symmetric and so only $m = 0$ terms appear.

With these restrictions we can write as a trial form

$$u(r,\theta,\phi) = \sum_{\ell=0}^{\infty} B_{\ell} r^{-(\ell+1)} P_{\ell}^{0}(\cos\theta). \tag{19.62}$$

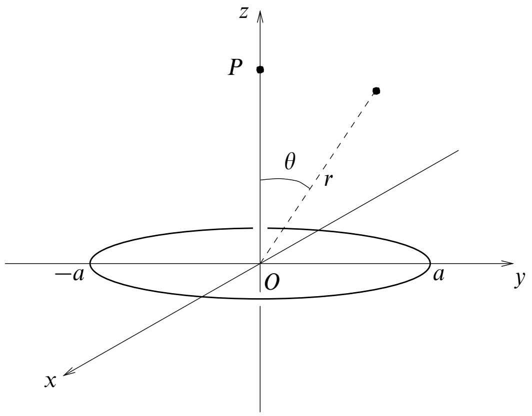

Figure 19.9 The polar axis Oz is taken as normal to the plane of the ring of matter and passing through its centre.

The constants B_ℓ are still to be determined; this we do by calculating *directly* the potential where this can be done simply – in this case, on the polar axis.

Considering a point P on the polar axis at a distance z $(> a)$ from the plane of the ring (taken as $\theta = \pi/2$), all parts of the ring are at a distance $(z^2 + a^2)^{1/2}$ from it. The potential at P is thus straightforwardly

$$u(z, 0, \phi) = -\frac{GM}{(z^2 + a^2)^{1/2}}, \tag{19.63}$$

where G is the gravitational constant. This must be the same as (19.62) for the particular values $r = z$, $\theta = 0$, and ϕ undefined. Since $P_\ell^0(\cos\theta) = P_\ell(\cos\theta)$ with $P_\ell(1) = 1$, putting $r = z$ in (19.62) gives

$$u(z, 0, \phi) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{z^{\ell+1}}. \tag{19.64}$$

However, expanding (19.63) for $z > a$ (as it applies to this region of space) we obtain

$$u(z, 0, \phi) = -\frac{GM}{z}\left[1 - \frac{1}{2}\left(\frac{a}{z}\right)^2 + \frac{3}{8}\left(\frac{a}{z}\right)^4 - \cdots\right],$$

which on comparison with (19.64) gives†

$$\begin{aligned} B_0 &= -GM, \\ B_{2\ell} &= -\frac{GMa^{2\ell}(-1)^\ell(2\ell-1)!!}{2^\ell \ell!} \qquad \text{for } \ell \geq 1, \\ B_{2\ell+1} &= 0. \end{aligned} \tag{19.65}$$

We now conclude the argument by saying that if a solution for a general point (r, θ, ϕ) exists at all, which of course we very much expect on physical grounds, then it must be (19.62) with the B_ℓ given by (19.65). This is so because thus defined it is a function with no arbitrary constants and which satisfies all the boundary conditions, and the uniqueness

† $(2\ell - 1)!! = 1 \times 3 \times \cdots \times (2\ell - 1)$.

theorem states that there is only one such function. The expression for the potential in the region $r > a$ is therefore

$$u(r, \theta, \phi) = -\frac{GM}{r}\left[1 + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}(2\ell - 1)!!}{2^{\ell}\ell!}\left(\frac{a}{r}\right)^{2\ell} P_{2\ell}(\cos\theta)\right].$$

The expression for $r < a$ can be found in a similar way. The finiteness of u at $r = 0$ and the axial symmetry give

$$u(r, \theta, \phi) = \sum_{\ell=0}^{\infty} A_{\ell} r^{\ell} P_{\ell}^{0}(\cos\theta).$$

Comparing this expression for $r = z$, $\theta = 0$ with the $z < a$ expansion of (19.63), which is valid for any z, establishes $A_{2\ell+1} = 0$, $A_0 = -GM/a$ and

$$A_{2\ell} = -\frac{GM}{a^{2\ell+1}}\frac{(-1)^{\ell}(2\ell - 1)!!}{2^{\ell}\ell!},$$

so that the final expression valid, and convergent, for $r < a$ is thus

$$u(r, \theta, \phi) = -\frac{GM}{a}\left[1 + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}(2\ell - 1)!!}{2^{\ell}\ell!}\left(\frac{r}{a}\right)^{2\ell} P_{2\ell}(\cos\theta)\right].$$

It is easy to check that the solution obtained has the expected physical value for large r and for $r = 0$ and is continuous at $r = a$. ◄

19.3.5 Separation of variables for inhomogeneous equations

So far our discussion of the method of separation of variables has been limited to the solution of homogeneous equations such as the Laplace equation and the wave equation. The solutions of inhomogeneous PDEs are usually obtained using the Green's function methods to be discussed below in section 19.5. However, as a final illustration of the usefulness of the separation of variables, we now consider its application to the solution of inhomogeneous equations.

Because of the added complexity in dealing with inhomogeneous equations, we shall restrict our discussion to the solution of Poisson's equation,

$$\nabla^2 u = \rho(\mathbf{r}), \tag{19.66}$$

in spherical polar coordinates, although the general method can accommodate other coordinate systems and equations. In physical problems the RHS of (19.66) usually contains some multiplicative constant(s). If u is the electrostatic potential in some region of space in which ρ is the density of electric charge then $\nabla^2 u = -\rho(\mathbf{r})/\epsilon_0$. Alternatively, u might represent the gravitational potential in some region where the matter density is given by ρ, so that $\nabla^2 u = 4\pi G\rho(\mathbf{r})$.

We will simplify our discussion by assuming that the required solution u is finite on the polar axis and also that the system possesses axial symmetry about that axis – in which case ρ does not depend on the azimuthal angle ϕ. The key to the method is then to assume a separated form for both the solution u *and* the density term ρ.

From the discussion of Laplace's equation, for systems with axial symmetry only $m = 0$ terms appear, and so the angular part of the solution can be expressed in terms of Legendre polynomials $P_\ell(\cos\theta)$. Since these functions form an orthogonal set let us expand both u and ρ in terms of them:

$$u = \sum_{\ell=0}^{\infty} R_\ell(r) P_\ell(\cos\theta), \tag{19.67}$$

$$\rho = \sum_{\ell=0}^{\infty} F_\ell(r) P_\ell(\cos\theta), \tag{19.68}$$

where the coefficients $R_\ell(r)$ and $F_\ell(r)$ in the Legendre polynomial expansions are functions of r. Since in any particular problem ρ is given, we can find the coefficients $F_\ell(r)$ in the expansion in the usual way (see subsection 16.6.2). It then only remains to find the coefficients $R_\ell(r)$ in the expansion of the solution u.

Writing ∇^2 in spherical polars and substituting (19.67) and (19.68) into (19.66) we obtain

$$\sum_{\ell=0}^{\infty} \left[\frac{P_\ell(\cos\theta)}{r^2} \frac{d}{dr}\left(r^2 \frac{dR_\ell}{dr} \right) + \frac{R_\ell}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP_\ell(\cos\theta)}{d\theta} \right) \right] = \sum_{\ell=0}^{\infty} F_\ell(r) P_\ell(\cos\theta). \tag{19.69}$$

However, if, in equation (19.44) of our discussion of the angular part of the solution to Laplace's equation, we set $m = 0$ we conclude that

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dP_\ell(\cos\theta)}{d\theta} \right) = -\ell(\ell+1) P_\ell(\cos\theta).$$

Substituting this into (19.69), we find that the LHS is greatly simplified and we obtain

$$\sum_{\ell=0}^{\infty} \left[\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dR_\ell}{dr} \right) - \frac{\ell(\ell+1)R_\ell}{r^2} \right] P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} F_\ell(r) P_\ell(\cos\theta).$$

This relation is most easily satisfied by equating terms on both sides for each value of ℓ separately, so that for $\ell = 0, 1, 2, \ldots$ we have

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{dR_\ell}{dr} \right) - \frac{\ell(\ell+1)R_\ell}{r^2} = F_\ell(r). \tag{19.70}$$

This is an ODE in which $F_\ell(r)$ is given, and it can therefore be solved for $R_\ell(r)$. The solution to Poisson's equation, u, is then obtained by making the superposition (19.67).

►*In a certain system, the electric charge density ρ is distributed as follows:*

$$\rho = \begin{cases} Ar\cos\theta & \textit{for } 0 \le r < a, \\ 0 & \textit{for } r \ge a. \end{cases}$$

Find the electrostatic potential inside and outside the charge distribution, given that both the potential and its radial derivative are continuous everywhere.

The electrostatic potential u satisfies

$$\nabla^2 u = \begin{cases} -(A/\epsilon_0) r\cos\theta & \text{for } 0 \le r < a, \\ 0 & \text{for } r \ge a. \end{cases}$$

For $r < a$ the RHS can be written $-(A/\epsilon_0) r P_1(\cos\theta)$, and the coefficients in (19.68) are simply $F_1(r) = -(Ar/\epsilon_0)$ and $F_\ell(r) = 0$ for $\ell \neq 1$. Therefore we need only calculate $R_1(r)$, which satisfies (19.70) for $\ell = 1$:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR_1}{dr}\right) - \frac{2R_1}{r^2} = -\frac{Ar}{\epsilon_0}.$$

This can be rearranged to give

$$r^2 R_1'' + 2rR_1' - 2R_1 = -\frac{Ar^3}{\epsilon_0},$$

where the prime denotes differentiation with respect to r. The LHS is homogeneous and the equation can be reduced by the substitution $r = \exp t$, and writing $R_1(r) = S(t)$, to

$$\ddot{S} + \dot{S} - 2S = -\frac{A}{\epsilon_0}\exp 3t, \tag{19.71}$$

where the dots indicate differentiation with respect to t.

This is an inhomogeneous second-order ODE with constant coefficients and can be straightforwardly solved by the methods of subsection 15.2.1 to give

$$S(t) = c_1 \exp t + c_2 \exp(-2t) - \frac{A}{10\epsilon_0}\exp 3t.$$

Recalling that $r = \exp t$ we find

$$R_1(r) = c_1 r + c_2 r^{-2} - \frac{A}{10\epsilon_0} r^3.$$

Since we are interested in the region $r < a$ we must have $c_2 = 0$ for the solution to remain finite. Thus inside the charge distribution the electrostatic potential has the form

$$u_1(r,\theta,\phi) = \left(c_1 r - \frac{A}{10\epsilon_0} r^3\right) P_1(\cos\theta). \tag{19.72}$$

Outside the charge distribution (for $r \ge a$), however, the electrostatic potential obeys Laplace's equation, $\nabla^2 u = 0$, and so given the symmetry of the problem and the requirement that $u \to \infty$ as $r \to \infty$ the solution must take the form

$$u_2(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta). \tag{19.73}$$

We can now use the boundary conditions at $r = a$ to fix the constants in (19.72) and (19.73). The requirement of continuity of the potential and its radial derivative at $r = a$ imply that

$$u_1(a,\theta,\phi) = u_2(a,\theta,\phi),$$
$$\frac{\partial u_1}{\partial r}(a,\theta,\phi) = \frac{\partial u_2}{\partial r}(a,\theta,\phi).$$

Clearly $B_\ell = 0$ for $\ell \neq 1$; carrying out the necessary differentiations and setting $r = a$ in (19.72) and (19.73) we obtain the simultaneous equations

$$c_1 a - \frac{A}{10\epsilon_0} a^3 = \frac{B_1}{a^2},$$
$$c_1 - \frac{3A}{10\epsilon_0} a^2 = -\frac{2B_1}{a^3}$$

which may be solved to give $c_1 = Aa^2/(6\epsilon_0)$ and $B_1 = Aa^5/(15\epsilon_0)$. Since $P_1(\cos\theta) = \cos\theta$, the electrostatic potentials inside and outside the charge distribution are given respectively by

$$u_1(r,\theta,\phi) = \frac{A}{\epsilon_0}\left(\frac{a^2 r}{6} - \frac{r^3}{10}\right)\cos\theta, \qquad u_2(r,\theta,\phi) = \frac{Aa^5}{15\epsilon_0}\frac{\cos\theta}{r^2}. \blacktriangleleft$$

19.4 Integral transform methods

In the method of separation of variables our aim was to keep the independent variables in a PDE as separate as possible. We now discuss the use of integral transforms in solving PDEs, a method by which one of the independent variables can be eliminated from the differential coefficients. It will be assumed that the reader is familiar with Laplace and Fourier transforms and their properties, as discussed in chapter 13.

The method consists simply of transforming the PDE into one containing derivatives with respect to a smaller number of variables. Thus, if the original equation has just two independent variables, it may be possible to reduce the PDE into a soluble ODE. The solution obtained can then (where possible) be transformed back to give the solution of the original PDE. As we shall see, boundary conditions can usually be incorporated in a natural way.

Which sort of transform to use, and the choice of the variable(s) with respect to which the transform is to be taken, is a matter of experience; we illustrate this in the example below. In practice, transforms can be taken with respect to each variable in turn, and the transformation that affords the greatest simplification can be pursued further.

►*A semi-infinite tube of constant cross-section contains initially pure water. At time $t = 0$, one end of the tube is put into contact with a salt solution and maintained at a concentration u_0. Find the total amount of salt that has diffused into the tube after time t, if the diffusion constant is κ.*

The concentration $u(x,t)$ at time t and distance x from the end of the tube satisfies the diffusion equation

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{19.74}$$

which has to be solved subject to the boundary conditions $u(0,t) = u_0$ for all t and $u(x,0) = 0$ for all $x > 0$.

Since we are interested only in $t > 0$, the use of the Laplace transform is suggested.

Furthermore, it will be recalled from chapter 13 that one of the major virtues of Laplace transformations is the possibility they afford of replacing derivatives of functions by simple multiplication by a scalar. If the derivative with respect to time were so removed, equation (19.74), would contain only differentiation with respect to a single variable. Let us therefore take the Laplace transform of (19.74) with respect to t:

$$\int_0^\infty \kappa \frac{\partial^2 u}{\partial x^2} \exp(-st)\, dt = \int_0^\infty \frac{\partial u}{\partial t} \exp(-st)\, dt.$$

On the LHS the (double) differentiation is with respect to x, whereas the integration is with respect to the independent variable t. Therefore the derivative can be taken outside the integral. Denoting the Laplace transform of $u(x,t)$ by $\bar{u}(x,s)$ and using result (13.57) to rewrite the transform of the derivative on the RHS (or by integrating directly by parts), we obtain

$$\kappa \frac{\partial^2 \bar{u}}{\partial x^2} = s\bar{u}(x,s) - u(x,0).$$

But from the boundary condition $u(x,0) = 0$ the last term on the RHS vanishes, and the solution is immediate:

$$\bar{u}(x,s) = A \exp\left(\sqrt{\frac{s}{\kappa}}\, x\right) + B \exp\left(-\sqrt{\frac{s}{\kappa}}\, x\right),$$

where the constants A and B may depend on s.

We require $u(x,t) \to 0$ as $x \to \infty$ and so we must also have $\bar{u}(\infty, s) = 0$; consequently we require that $A = 0$. The value of B is determined by the need for $u(0,t) = u_0$ and hence that

$$\bar{u}(0,s) = \int_0^\infty u_0 \exp(-st)\, dt = \frac{u_0}{s}.$$

We thus conclude that the appropriate expression for the Laplace transform of $u(x,t)$ is

$$\bar{u}(x,s) = \frac{u_0}{s} \exp\left(-\sqrt{\frac{s}{\kappa}}\, x\right). \tag{19.75}$$

To obtain $u(x,t)$ from this result requires the inversion of this transform – a task that is generally difficult and requires a contour integration. This is discussed in chapter 20, but for completeness we note that the solution is

$$u(x,t) = u_0 \left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right)\right],$$

where $\operatorname{erf}(x)$ is the error function discussed in the Appendix. (The more complete sets of mathematical tables list this inverse Laplace transform.)

In the present problem, however, an alternative method is available. Let $w(t)$ be the amount of salt that has diffused into the tube in time t; then

$$w(t) = \int_0^\infty u(x,t)\, dx,$$

and its transform is given by

$$\begin{aligned}\bar{w}(s) &= \int_0^\infty dt\ \exp(-st) \int_0^\infty u(x,t)\, dx \\ &= \int_0^\infty dx \int_0^\infty u(x,t) \exp(-st)\, dt \\ &= \int_0^\infty \bar{u}(x,s)\, dx.\end{aligned}$$

Substituting for $\bar{u}(x,s)$ from (19.75) into the last integral and integrating, we obtain

$$\bar{w}(s) = u_0 \kappa^{1/2} s^{-3/2}.$$

This expression is much simpler to invert, and referring to the table of standard Laplace transforms (table 13.1) we find

$$w(t) = 2(\kappa/\pi)^{1/2} u_0 t^{1/2},$$

which is thus the required expression for the amount of diffused salt at time t. ◀

The above example shows that in some circumstances the use of a Laplace transformation can greatly simplify the solution of a PDE. However, it will have been observed that (as with ODEs) the easy elimination of some derivatives is usually paid for by the introduction of a difficult inverse transformation. This problem, although still present, is less severe for Fourier transformations.

▶*An infinite metal bar has an initial temperature distribution $f(x)$ along its length. Find the temperature distribution at a later time t.*

We are interested in values of x from $-\infty$ to ∞, which suggests Fourier transformation with respect to x. Assuming that the solution obeys the boundary conditions $u(x,t) \to 0$ and $\partial u/\partial x \to 0$ as $|x| \to \infty$, we may Fourier-transform the one-dimensional diffusion equation (19.74) to obtain

$$\frac{\kappa}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial x^2} \exp(-ikx)\, dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} u(x,t) \exp(-ikx)\, dx,$$

where on the RHS we have taken the partial derivative with respect to t outside the integral. Denoting the Fourier transform of $u(x,t)$ by $\widetilde{u}(k,t)$, and using equation (13.28) to rewrite the Fourier transform of the second derivative on the LHS, we then have

$$-\kappa k^2 \widetilde{u}(k,t) = \frac{\partial \widetilde{u}(k,t)}{\partial t}.$$

This first-order equation has the simple solution

$$\widetilde{u}(k,t) = \widetilde{u}(k,0) \exp(-\kappa k^2 t),$$

where the initial conditions give

$$\begin{aligned} \widetilde{u}(k,0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) \exp(-ikx)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx = \widetilde{f}(k). \end{aligned}$$

Thus we may write the Fourier transform of the solution as

$$\widetilde{u}(k,t) = \widetilde{f}(k) \exp(-\kappa k^2 t) = \sqrt{2\pi}\, \widetilde{f}(k) \widetilde{G}(k,t), \tag{19.76}$$

where we have defined the function $\widetilde{G}(k,t) = (\sqrt{2\pi})^{-1} \exp(-\kappa k^2 t)$. Since $\widetilde{u}(k,t)$ can be written as the product of two Fourier transforms, we can use the convolution theorem, subsection 13.1.7, to write the solution as

$$u(x,t) = \int_{-\infty}^{\infty} G(x - x', t) f(x')\, dx',$$

where $G(x,t)$ is the Green's function for this problem (see subsection 15.2.5). This function is the inverse Fourier transform of $\widetilde{G}(k,t)$ and is thus given by

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-\kappa k^2 t)\exp(ikx)\,dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\kappa t\left(k^2 - \frac{ix}{\kappa t}k\right)\right]\,dk.$$

Completing the square in the integrand we find

$$G(x,t) = \frac{1}{2\pi}\exp\left(-\frac{x^2}{4\kappa t}\right)\int_{-\infty}^{\infty} \exp\left[-\kappa t\left(k - \frac{ix}{2\kappa t}\right)^2\right]\,dk$$
$$= \frac{1}{2\pi}\exp\left(-\frac{x^2}{4\kappa t}\right)\int_{-\infty}^{\infty} \exp\left(-\kappa t k'^2\right)\,dk'$$
$$= \frac{1}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right),$$

where in the second line we have made the substitution $k' = k - ix/(2\kappa t)$, and in the last line we have used the standard result for the integral of a Gaussian, given in subsection 6.4.2. (Strictly speaking the change of variable from k to k' shifts the path of integration off the real axis, since k' is complex for real k, and so results in a complex integral, as will be discussed in chapter 20. Nevertheless, in this case the path of integration can be shifted back to the real axis without affecting the value of the integral.)

Thus the temperature in the bar at a later time t is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-x')^2}{4\kappa t}\right] f(x')\,dx', \tag{19.77}$$

which may be evaluated (numerically if necessary) when the form of $f(x)$ is given. ◀

As we might expect from our discussion of Green's functions in chapter 15, we see from (19.77) that, if the initial temperature distribution is $f(x) = \delta(x-a)$, i.e. a 'point' source at $x = a$, then the temperature distribution at later times is simply given by

$$u(x,t) = G(x-a,t) = \frac{1}{\sqrt{4\pi\kappa t}}\exp\left[-\frac{(x-a)^2}{4\kappa t}\right].$$

The temperature at several later times is illustrated in figure 19.10, which shows that the heat diffuses out from its initial position; the width of the Gaussian increases as $\sqrt{t}$, a dependence on time which is characteristic of diffusion processes.

The reader may have noticed that in both examples using integral transforms the solutions have been obtained in closed form – albeit in one case in the form of an integral. This differs from the infinite series solutions usually obtained via the separation of variables. It should be noted that this behaviour is a result of the infinite range in x rather than of the transform method itself. In fact the method of separation of variables would yield the same solutions, since in the infinite-range case the separation constant is not restricted to take on an infinite set of discrete values but may have any real value, with the result that the sum over λ becomes an integral, as mentioned at the end of section 19.2.

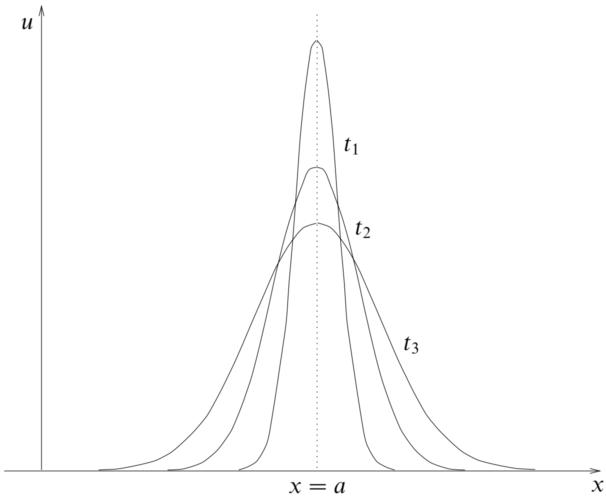

Figure 19.10 Diffusion of heat from a point source in a metal bar: the curves show the temperature u at position x for various different times $t_1 < t_2 < t_3$. The area under the curves remains constant, since the total heat energy is conserved.

►*An infinite metal bar has an initial temperature distribution $f(x)$ along its length. Find the temperature distribution at a later time t using the method of separation of variables.*

This is the same problem as in the previous example, but we now seek a solution by separating variables. From (19.12) a separated solution for the one-dimensional diffusion equation is

$$u(x,t) = [A\exp(i\lambda x) + B\exp(-i\lambda x)]\exp(-\kappa\lambda^2 t),$$

where $-\lambda^2$ is the separation constant. Since the bar is infinite we do not require the solution to take a given form at any finite value of x (for instance at $x = 0$) and so there is no restriction on λ other than its being real. Therefore instead of the superposition of such solutions in the form of a sum over allowed values of λ we have an integral over all λ,

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(\lambda)\exp(-\kappa\lambda^2 t)\exp(i\lambda x)\,d\lambda, \tag{19.78}$$

where in taking λ from $-\infty$ to ∞ we need include only one of the complex exponentials; we have taken a factor $1/\sqrt{2\pi}$ out of $A(\lambda)$ for convenience. We can see from (19.78) that the expression for $u(x,t)$ has the form of an inverse Fourier transform (where λ is the transform variable). Therefore, Fourier-transforming both sides and using the Fourier inversion theorem, we find

$$\widetilde{u}(\lambda,t) = A(\lambda)\exp(-\kappa\lambda^2 t).$$

Now the initial boundary condition requires

$$u(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(\lambda)\exp(i\lambda x)\,d\lambda = f(x),$$

from which, using the Fourier inversion theorem once more, we see that $A(\lambda) = \widetilde{f}(\lambda)$. Therefore we have

$$\widetilde{u}(\lambda, t) = \widetilde{f}(\lambda) \exp(-\kappa\lambda^2 t),$$

which is identical to (19.76) in the previous example (but with k replaced by λ), and hence leads to the same result. ◀

19.5 Inhomogeneous problems – Green's functions

In chapters 15 and 17 we encountered Green's functions and found them a useful tool for solving inhomogeneous linear ODEs. We now discuss their usefulness in solving inhomogeneous linear PDEs.

For the sake of brevity we shall again denote a linear PDE by

$$\mathcal{L}u(\mathbf{r}) = \rho(\mathbf{r}), \tag{19.79}$$

where $\mathcal{L}$ is a linear partial differential operator. For example, in Laplace's equation we have $\mathcal{L} = \nabla^2$, whereas for Helmholtz's equation $\mathcal{L} = \nabla^2 + k^2$. Note that we have not specified the dimensionality of the problem, and (19.79) may, for example, represent Poisson's equation in two or three (or more) dimensions. The reader will also notice that for the sake of simplicity we have not included any time dependence in (19.79). Nevertheless, the following discussion can be generalised to include it.

As we discussed in subsection 18.3.2, a problem is inhomogeneous if the fact that $u(\mathbf{r})$ is a solution does *not* imply that any constant multiple $\lambda u(\mathbf{r})$ is also a solution. This inhomogeneity may derive from either the PDE itself or from the boundary conditions imposed on the solution.

In our discussion of Green's function solutions of inhomogeneous ODEs (see subsection 15.2.5) we dealt with inhomogeneous boundary conditions by making a suitable change of variable such that in the new variable the boundary conditions were homogeneous. In an analogous way, as illustrated in the final example of section 19.2, it is usually possible to make a change of variables in PDEs to transform between inhomogeneity of the boundary conditions and inhomogeneity of the equation. Therefore let us assume for the moment that the boundary conditions imposed on the solution $u(\mathbf{r})$ of (19.79) are homogeneous. This most commonly means that if we seek a solution to (19.79) in some region V then on the surface S that bounds V the solution obeys the conditions $u(\mathbf{r}) = 0$ or $\partial u/\partial n = 0$, where $\partial u/\partial n$ is the normal derivative of u at the surface S.

We shall discuss the extension of the Green's function method to the direct solution of problems with inhomogeneous boundary conditions in subsection 19.5.2, but we first highlight how the Green's function approach to solving ODEs can be simply extended to PDEs for homogeneous boundary conditions.

19.5.1 Similarities with Green's functions for ODEs

As in the discussion of ODEs in chapter 15, we may consider the Green's function for a system described by a PDE as the response of the system to a 'unit impulse' or 'point source'. Thus if we seek a solution to (19.79) that satisfies some homogeneous boundary conditions on $u(\mathbf{r})$ then the Green's function $G(\mathbf{r}, \mathbf{r}_0)$ for the problem is a solution of

$$\boxed{\mathcal{L}G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0),} \qquad (19.80)$$

where $\mathbf{r}_0$ lies in V. The Green's function $G(\mathbf{r}, \mathbf{r}_0)$ must also satisfy the imposed (homogeneous) boundary conditions.

It is understood that in (19.80) the $\mathcal{L}$ operator expresses differentiation with respect to $\mathbf{r}$ as opposed to $\mathbf{r}_0$. Also, $\delta(\mathbf{r} - \mathbf{r}_0)$ is the Dirac delta function (see chapter 13) of dimension appropriate for the problem; it may be thought of as representing a unit-strength point source at $\mathbf{r} = \mathbf{r}_0$.

Following an analogous argument to that given in subsection 15.2.5 for ODEs, if the boundary conditions on $u(\mathbf{r})$ are homogeneous then a solution to (19.79) that satisfies the imposed boundary conditions is given by

$$\mathcal{L}y(x) = \int_a^b [\mathcal{L}G(x,z)]\, f(z)\, dz = f(x).$$

$$u(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r}_0)\, dV(\mathbf{r}_0), \qquad (19.81)$$

where the integral on $\mathbf{r}_0$ is over some appropriate 'volume'. In two or more dimensions, however, the task of finding directly a solution to (19.80) that satisfies the imposed boundary conditions on S can be a difficult one, and we return to this in the next subsection.

An alternative approach is to follow a similar argument to that presented in chapter 17 for ODEs and so to construct the Green's function for (19.79) as a superposition of eigenfunctions of the operator $\mathcal{L}$, provided $\mathcal{L}$ is Hermitian. By analogy with an ordinary differential operator, a partial differential operator is Hermitian if it satisfies

$$\int_V v^*(\mathbf{r})\mathcal{L}w(\mathbf{r})\, dV = \left[\int_V w^*(\mathbf{r})\mathcal{L}v(\mathbf{r})\, dV\right]^*,$$

where the asterisk denotes complex conjugation and v and w are arbitrary functions obeying the imposed (homogeneous) boundary condition on the solution of $\mathcal{L}u(\mathbf{r}) = 0$.

The eigenfunctions $u_n(\mathbf{r})$, $n = 0, 1, 2, \ldots$, of $\mathcal{L}$ satisfy

$$\mathcal{L}u_n(\mathbf{r}) = \lambda_n u_n(\mathbf{r}),$$

where λ_n are the corresponding eigenvalues, which are all real for an Hermitian operator $\mathcal{L}$. Furthermore, each eigenfunction must obey any imposed (homogeneous) boundary conditions. Using an argument analogous to that given in chapter 17, the Green's function for the problem is given by

$$\boxed{G(\mathbf{r}, \mathbf{r}_0) = \sum_{n=0}^{\infty} \frac{u_n(\mathbf{r})u_n^*(\mathbf{r}_0)}{\lambda_n}.} \tag{19.82}$$

From (19.82) we see immediately that the Green's function (irrespective of how it is found) enjoys the property

$$G(\mathbf{r}, \mathbf{r}_0) = G^*(\mathbf{r}_0, \mathbf{r}).$$

Thus, if the Green's function is real then it is symmetric in its two arguments.

Once the Green's function has been obtained, the solution to (19.79) is again given by (19.81). For PDEs this approach can become very cumbersome, however, and so we shall not pursue it further here.

19.5.2 General boundary-value problems

As mentioned above, often inhomogeneous boundary conditions can be dealt with by making an appropriate change of variables, such that the boundary conditions in the new variables are homogeneous although the equation itself is generally inhomogeneous. In this section, however, we extend the use of Green's functions to problems with inhomogeneous boundary conditions (and equations). This provides a more consistent and intuitive approach to the solution of such *boundary-value problems.*

For definiteness we shall consider Poisson's equation

$$\nabla^2 u(\mathbf{r}) = \rho(\mathbf{r}), \tag{19.83}$$

but the material of this section may be extended to other linear PDEs of the form (19.79). Clearly, Poisson's equation reduces to Laplace's equation for $\rho(\mathbf{r}) = 0$ and so our discussion is equally applicable to this case.

We wish to solve (19.83) in some region V bounded by a surface S, which may consist of several disconnected parts. As stated above, we shall allow the possibility that the boundary conditions on the solution $u(\mathbf{r})$ may be inhomogeneous on S, although as we shall see this method reduces to those discussed above in the special case that the boundary conditions are in fact homogeneous.

The two common types of inhomogeneous boundary condition for Poisson's equation are (as discussed in subsection 18.6.2):

(i) Dirichlet conditions, in which $u(\mathbf{r})$ is specified on S, and
(ii) Neumann conditions, in which $\partial u/\partial n$ is specified on S.

In general, specifying *both* Dirichlet *and* Neumann conditions on S overdetermines the problem and leads to there being no solution.

The specification of the surface S requires some further comment, since S may have several disconnected parts. If we wish to solve Poisson's equation

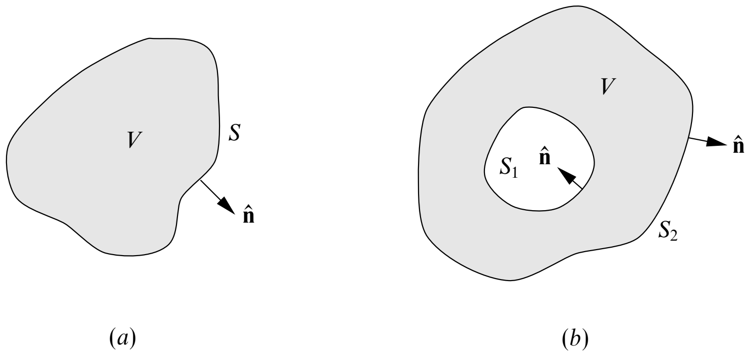

Figure 19.11 Surfaces used for solving Poisson's equation in different regions V.

inside some closed surface S then the situation is straightforward and is shown in figure 19.11(*a*). If, however, we wish to solve Poisson's equation in the gap between two closed surfaces (for example in the gap between two concentric conducting cylinders) then the volume V is bounded by a surface S that has two disconnected parts S_1 and S_2, as shown in figure 19.11(*b*); the direction of the normal to the surface is always taken as pointing *out* of the volume V. A similar situation arises when we wish to solve Poisson's equation *outside* some closed surface S_1. In this case the volume V is infinite but is treated formally by taking the surface S_2 as a large sphere of radius R and letting R tend to infinity.

In order to solve (19.83) subject to either Dirichlet or Neumann boundary conditions on S, we first remind ourselves of Green's second theorem, equation (11.20), which states that for two scalar functions $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ defined in some volume V bounded by a surface S

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{\mathbf{n}}\, dS, \tag{19.84}$$

where on the RHS it is common to write, for example, $\nabla\psi\cdot\hat{\mathbf{n}}\,dS$ as $(\partial\psi/\partial n)\,dS$. The expression $\partial\psi/\partial n$ stands for $\nabla\psi\cdot\hat{\mathbf{n}}$, the rate of change of ψ in the direction of the unit outward normal $\hat{\mathbf{n}}$ to the surface S.

The Green's function for Poisson's equation (19.83) must satisfy

$$\nabla^2 G(\mathbf{r},\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0), \tag{19.85}$$

where $\mathbf{r}_0$ lies in V. (As mentioned above, we may think of $G(\mathbf{r},\mathbf{r}_0)$ as the solution to Poisson's equation for a unit-strength point source located at $\mathbf{r}=\mathbf{r}_0$.) Let us for the moment impose no boundary conditions on $G(\mathbf{r},\mathbf{r}_0)$.

If we now let $\phi = u(\mathbf{r})$ and $\psi = G(\mathbf{r},\mathbf{r}_0)$ in Green's theorem (19.84) then we obtain

$$\boxed{\begin{aligned}\int_V \left[u(\mathbf{r})\nabla^2 G(\mathbf{r},\mathbf{r}_0) - G(\mathbf{r},\mathbf{r}_0)\,\nabla^2 u(\mathbf{r})\right]\,dV(\mathbf{r})&\\ = \int_S \left[u(\mathbf{r})\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} - G(\mathbf{r},\mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n}\right]\,dS(\mathbf{r}),&\end{aligned}}$$

where we have made explicit that the volume and surface integrals are with respect to $\mathbf{r}$. Using (19.83) and (19.85) the LHS can be simplified to give

$$\begin{aligned}\int_V [u(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0) - G(\mathbf{r},\mathbf{r}_0)\rho(\mathbf{r})]\,dV(\mathbf{r})&\\ = \int_S \left[u(\mathbf{r})\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} - G(\mathbf{r},\mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n}\right]\,dS(\mathbf{r}),&\end{aligned} \qquad (19.86)$$

Since $\mathbf{r}_0$ lies within the volume V,

$$\int_V u(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0)\,dV(\mathbf{r}) = u(\mathbf{r}_0),$$

and thus rearranging (19.86) the solution to Poisson's equation (19.83) can be written

$$\boxed{u(\mathbf{r}_0) = \int_V G(\mathbf{r},\mathbf{r}_0)\rho(\mathbf{r})\,dV(\mathbf{r}) + \int_S \left[u(\mathbf{r})\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} - G(\mathbf{r},\mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n}\right]\,dS(\mathbf{r}). \qquad (19.87)}$$

Clearly, we can interchange the roles of $\mathbf{r}$ and $\mathbf{r}_0$ in (19.87) if we wish. (Remember also that, for a real Green's function, $G(\mathbf{r},\mathbf{r}_0) = G(\mathbf{r}_0,\mathbf{r})$.)

Equation (19.87) is *central* to the extension of the Green's function method to problems with inhomogeneous boundary conditions, and we next discuss its application to both Dirichlet and Neumann boundary-value problems. But, before doing so, we also note that if the boundary condition on S is in fact homogeneous, so that $u(\mathbf{r}) = 0$ or $\partial u(\mathbf{r})/\partial n = 0$ on S, then demanding that the Green's function $G(\mathbf{r},\mathbf{r}_0)$ also obeys the same boundary condition causes the surface integral in (19.87) to vanish, and we are left with the familiar form of solution given in (19.81). The extension of (19.87) to a PDE other than Poisson's equation is discussed in exercise 19.30.

19.5.3 Dirichlet problems

In a Dirichlet problem we require the solution $u(\mathbf{r})$ of Poisson's equation (19.83) to take specific values on some surface S that bounds V, i.e. we require that $u(\mathbf{r}) = f(\mathbf{r})$ on S where f is a given function.

If we seek a Green's function $G(\mathbf{r},\mathbf{r}_0)$ for this problem it must clearly satisfy (19.85), but we are free to choose the boundary conditions satisfied by $G(\mathbf{r},\mathbf{r}_0)$ in such a way as to make the solution (19.87) as simple as possible. From (19.87), we see that by choosing

$$\boxed{G(\mathbf{r}, \mathbf{r}_0) = 0 \quad \text{for } \mathbf{r} \text{ on } S} \tag{19.88}$$

the second term in the surface integral vanishes. Since $u(\mathbf{r}) = f(\mathbf{r})$ on S, (19.87) then becomes

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) + \int_S f(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n}\, dS(\mathbf{r}). \tag{19.89}$$

Thus we wish to find the *Dirichlet Green's function* that

(i) satisfies (19.85) and hence is singular at $\mathbf{r} = \mathbf{r}_0$, and
(ii) obeys the boundary condition $G(\mathbf{r}, \mathbf{r}_0) = 0$ for $\mathbf{r}$ on S.

In general, it is, difficult to obtain this function directly, and so it is useful to separate these two requirements. We therefore look for a solution of the form

$$G(\mathbf{r}, \mathbf{r}_0) = F(\mathbf{r}, \mathbf{r}_0) + H(\mathbf{r}, \mathbf{r}_0),$$

where $F(\mathbf{r}, \mathbf{r}_0)$ satisfies (19.85) and has the required singular character at $\mathbf{r} = \mathbf{r}_0$ but does not necessarily obey the boundary condition on S, whilst $H(\mathbf{r}, \mathbf{r}_0)$ satisfies the corresponding homogeneous equation (i.e. Laplace's equation) inside V but is adjusted in such a way that the sum $G(\mathbf{r}, \mathbf{r}_0)$ equals zero on S. The Green's function $G(\mathbf{r}, \mathbf{r}_0)$ is still a solution of (19.85) since

$$\nabla^2 G(\mathbf{r}, \mathbf{r}_0) = \nabla^2 F(\mathbf{r}, \mathbf{r}_0) + \nabla^2 H(\mathbf{r}, \mathbf{r}_0) = \nabla^2 F(\mathbf{r}, \mathbf{r}_0) + 0 = \delta(\mathbf{r} - \mathbf{r}_0).$$

The function $F(\mathbf{r}, \mathbf{r}_0)$ is called the *fundamental solution* and will clearly take different forms depending on the dimensionality of the problem. Let us first consider the fundamental solution to (19.85) in three dimensions.

►*Find the fundamental solution to Poisson's equation in three dimensions that tends to zero as* $|\mathbf{r}| \to \infty$.

We wish to solve

$$\nabla^2 F(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0) \tag{19.90}$$

in three dimensions, subject to the boundary condition $F(\mathbf{r}, \mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$. Since the problem is spherically symmetric about $\mathbf{r}_0$, let us consider a large sphere S of radius R centred on $\mathbf{r}_0$, and integrate (19.90) over the enclosed volume V. We then obtain

$$\int_V \nabla^2 F(\mathbf{r}, \mathbf{r}_0)\, dV = \int_V \delta(\mathbf{r} - \mathbf{r}_0)\, dV = 1, \tag{19.91}$$

since V encloses the point $\mathbf{r}_0$. However, using the divergence theorem,

$$\int_V \nabla^2 F(\mathbf{r}, \mathbf{r}_0)\, dV = \int_S \nabla F(\mathbf{r}, \mathbf{r}_0) \cdot \hat{\mathbf{n}}\, dS, \tag{19.92}$$

where $\hat{\mathbf{n}}$ is the unit normal to the large sphere S at any point.

Since the problem is spherically symmetric about $\mathbf{r}_0$, we expect that

$$\boxed{F(\mathbf{r},\mathbf{r}_0) = F(|\mathbf{r}-\mathbf{r}_0|) = F(r),}$$

i.e. F has the same value everywhere on S. Thus, evaluating the surface integral in (19.92) and equating it to unity from (19.91), we have†

$$4\pi r^2 \frac{dF}{dr}\bigg|_{r=R} = 1.$$

Integrating this expression we obtain

$$F(r) = -\frac{1}{4\pi r} + \text{constant},$$

but, since we require $F(\mathbf{r},\mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$, the constant must be zero. The fundamental solution in three dimensions is consequently given by

$$F(\mathbf{r},\mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}_0|}. \tag{19.93}$$

This is clearly also the full Green's function for Poisson's equation subject to the boundary condition $u(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$. ◀

Using (19.93) we can write down the solution of Poisson's equation to find, for example, the electrostatic potential $u(\mathbf{r})$ due to some distribution of electric charge $\rho(\mathbf{r})$. The electrostatic potential satisfies

$$\nabla^2 u(\mathbf{r}) = -\frac{\rho}{\epsilon_0},$$

where $u(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$. Since the boundary condition on the surface at infinity is homogeneous the surface integral in (19.89) vanishes, and using (19.93) we recover the familiar solution

$$u(\mathbf{r}_0) = \int \frac{\rho(\mathbf{r})}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_0|}\, dV(\mathbf{r}), \tag{19.94}$$

where the volume integral is over all space.

We can develop an analogous theory in two dimensions. As before the fundamental solution satisfies

$$\nabla^2 F(\mathbf{r},\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0), \tag{19.95}$$

where $\delta(\mathbf{r}-\mathbf{r}_0)$ is now the two-dimensional delta function. Following an analogous method to that used in the previous example, we find the fundamental solution in two dimensions to be given by

$$F(\mathbf{r},\mathbf{r}_0) = \frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}_0| + \text{constant}. \tag{19.96}$$

† A vertical bar to the right of an expression is a common alternative notation to enclosing the expression in square brackets; as usual, the subscript shows the value of the variable at which the expression is to be evaluated.

From the form of the solution we see that in two dimensions we cannot apply the condition $F(\mathbf{r}, \mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$, and in this case the constant does not necessarily vanish.

We now return to the task of constructing the full Dirichlet Green's function. To do so we wish to add to the fundamental solution a solution of the homogeneous equation (in this case Laplace's equation) such that $G(\mathbf{r}, \mathbf{r}_0) = 0$ on S, as required by (19.89) and its attendant conditions. The appropriate Green's function is constructed by adding to the fundamental solution 'copies' of itself that represent 'image' sources at different locations *outside* V. Hence this approach is called the *method of images*.

In summary, if we wish to solve Poisson's equation in some region V subject to Dirichlet boundary conditions on its surface S then the procedure and argument are as follows.

(i) To the single source $\delta(\mathbf{r} - \mathbf{r}_0)$ inside V add image sources *outside* V

$$\sum_{n=1}^{N} q_n \delta(\mathbf{r} - \mathbf{r}_n) \qquad \text{with } \mathbf{r}_n \text{ outside } V,$$

where the positions $\mathbf{r}_n$ and the strengths q_n of the image sources are to be determined as described in step (iii) below.

(ii) Since all the image sources lie outside V, the fundamental solution corresponding to each source satisfies Laplace's equation *inside* V. Thus we may add the fundamental solutions $F(\mathbf{r}, \mathbf{r}_n)$ corresponding to each image source to that corresponding to the single source inside V, obtaining the Green's function

$$G(\mathbf{r}, \mathbf{r}_0) = F(\mathbf{r}, \mathbf{r}_0) + \sum_{n=1}^{N} q_n F(\mathbf{r}, \mathbf{r}_n).$$

(iii) Now adjust the positions $\mathbf{r}_n$ and strengths q_n of the image sources so that the required boundary conditions are satisfied on S. For a Dirichlet Green's function we require $G(\mathbf{r}, \mathbf{r}_0) = 0$ for $\mathbf{r}$ on S.

(iv) The solution to Poisson's equation subject to the Dirichlet boundary condition $u(\mathbf{r}) = f(\mathbf{r})$ on S is then given by (19.89).

In general it is very difficult to find the correct positions and strengths for the images, i.e. to make them such that the boundary conditions on S are satisfied. Nevertheless, it is possible to do so for certain problems that have simple geometry. In particular, for problems in which the boundary S consists of straight lines (in two dimensions) or planes (in three dimensions), positions of the image points can be deduced simply by imagining the boundary lines or planes to be mirrors in which the single source in V (at $\mathbf{r}_0$) is reflected.

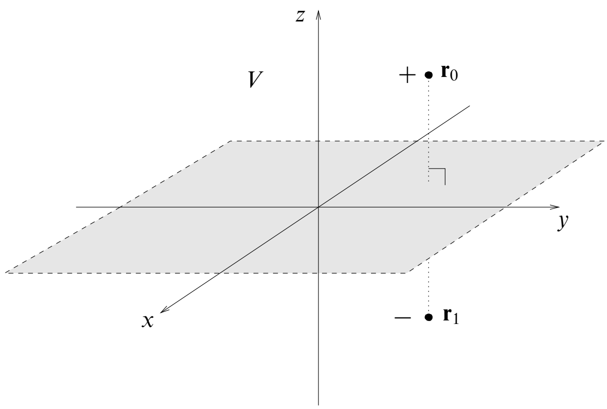

Figure 19.12 The arrangement of images for solving Laplace's equation in the half-space $z > 0$.

►*Solve Laplace's equation $\nabla^2 u = 0$ in three dimensions in the half-space $z > 0$, given that $u(\mathbf{r}) = f(\mathbf{r})$ on the plane $z = 0$.*

The surface S bounding V consists of the xy-plane and the surface at infinity. Therefore, the Dirichlet Green's function for this problem must satisfy $G(\mathbf{r}, \mathbf{r}_0) = 0$ on $z = 0$ and $G(\mathbf{r}, \mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$. Thus it is clear in this case that we require one image source at a position $\mathbf{r}_1$ that is the reflection of $\mathbf{r}_0$ in the plane $z = 0$, as shown in figure 19.12 (so that $\mathbf{r}_1$ lies in $z < 0$, outside the region in which we wish to obtain a solution). It is also clear that the strength of this image should be -1.

Therefore by adding the fundamental solutions corresponding to the original source and its image we obtain the Green's function

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} + \frac{1}{4\pi|\mathbf{r} - \mathbf{r}_1|}, \tag{19.97}$$

where $\mathbf{r}_1$ is the reflection of $\mathbf{r}_0$ in the plane $z = 0$, i.e. if $\mathbf{r}_0 = (x_0, y_0, z_0)$ then $\mathbf{r}_1 = (x_0, y_0, -z_0)$. Clearly $G(\mathbf{r}, \mathbf{r}_0) \to 0$ as $|\mathbf{r}| \to \infty$ as required. Also $G(\mathbf{r}, \mathbf{r}_0) = 0$ on $z = 0$, and so (19.97) is the desired Dirichlet Green's function.

The solution to Laplace's equation is then given by (19.89) with $\rho(\mathbf{r}) = 0$,

$$u(\mathbf{r}_0) = \int_S f(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n}\, dS(\mathbf{r}). \tag{19.98}$$

Clearly the surface at infinity makes no contribution to this integral. The outward-pointing unit vector normal to the xy-plane is simply $\hat{\mathbf{n}} = -\mathbf{k}$ (where $\mathbf{k}$ is the unit vector in the z-direction), and so

$$\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} = -\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial z} = -\mathbf{k} \cdot \nabla G(\mathbf{r}, \mathbf{r}_0).$$

We may evaluate this normal derivative by writing the Green's function (19.97) explicitly in terms of x, y and z (and x_0, y_0 and z_0) and calculating the partial derivative with respect to z directly. It is usually quicker, however, to use the fact that†

$$\nabla|\mathbf{r}-\mathbf{r}_0| = \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}; \tag{19.99}$$

thus

$$\nabla G(\mathbf{r},\mathbf{r}_0) = \frac{\mathbf{r}-\mathbf{r}_0}{4\pi|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mathbf{r}-\mathbf{r}_1}{4\pi|\mathbf{r}-\mathbf{r}_1|^3}.$$

Since $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{r}_1 = (x_0, y_0, -z_0)$ the normal derivative is given by

$$\begin{aligned} -\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial z} &= -\mathbf{k}\cdot\nabla G(\mathbf{r},\mathbf{r}_0) \\ &= -\frac{z-z_0}{4\pi|\mathbf{r}-\mathbf{r}_0|^3} + \frac{z+z_0}{4\pi|\mathbf{r}-\mathbf{r}_1|^3}. \end{aligned}$$

Therefore on the surface $z = 0$, writing out the dependence on x, y and z explicitly, we have

$$-\left.\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial z}\right|_{z=0} = \frac{2z_0}{4\pi[(x-x_0)^2+(y-y_0)^2+z_0^2]^{3/2}}.$$

Inserting this expression into (19.98) we obtain the solution

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x,y)}{[(x-x_0)^2+(y-y_0)^2+z_0^2]^{3/2}}\,dx\,dy. \blacktriangleleft$$

An analogous procedure may be applied in two-dimensional problems. For example, in solving Poisson's equation in two dimensions in the half-space $x > 0$ we again require just one image charge, of strength $q_1 = -1$, at a position $\mathbf{r}_1$ that is the reflection of $\mathbf{r}_0$ in the line $x = 0$. Since we require $G(\mathbf{r},\mathbf{r}_0) = 0$ when $\mathbf{r}$ lies on $x = 0$, the constant in (19.96) must equal zero, and so the Dirichlet Green's function is

$$G(\mathbf{r},\mathbf{r}_0) = \frac{1}{2\pi}\left(\ln|\mathbf{r}-\mathbf{r}_0| - \ln|\mathbf{r}-\mathbf{r}_1|\right).$$

Clearly $G(\mathbf{r},\mathbf{r}_0)$ tends to zero as $|\mathbf{r}| \to \infty$. If, however, we wish to solve the two-dimensional Poisson equation in the quarter space $x > 0$, $y > 0$, then more image points are required.

† Since $|\mathbf{r}-\mathbf{r}_0|^2 = (\mathbf{r}-\mathbf{r}_0)\cdot(\mathbf{r}-\mathbf{r}_0)$ we have $\nabla|\mathbf{r}-\mathbf{r}_0|^2 = 2(\mathbf{r}-\mathbf{r}_0)$, from which we obtain

$$\nabla(|\mathbf{r}-\mathbf{r}_0|^2)^{1/2} = \frac{1}{2}\frac{2(\mathbf{r}-\mathbf{r}_0)}{(|\mathbf{r}-\mathbf{r}_0|^2)^{1/2}} = \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}.$$

Note that this result holds in two *and* three dimensions.

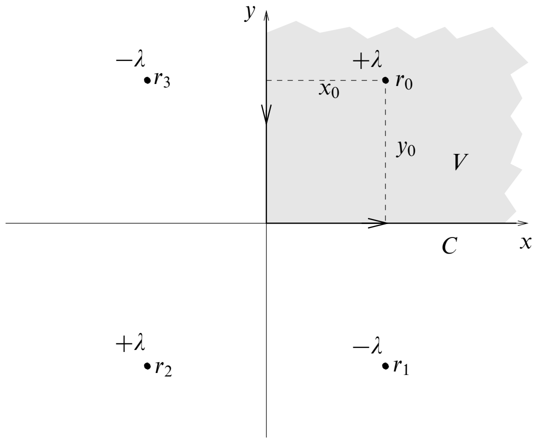

Figure 19.13 The arrangement of images for finding the force on a line charge situated in the (two-dimensional) quarter-space $x > 0$, $y > 0$, when the planes $x = 0$ and $y = 0$ are earthed.

►*A line charge in the z-direction of charge density λ is placed at some position $\mathbf{r}_0$ in the quarter-space $x > 0$, $y > 0$. Calculate the force per unit length on the line charge due to the presence of thin earthed plates along $x = 0$ and $y = 0$.*

Here we wish to solve Poisson's equation

$$\nabla^2 u = -\frac{\lambda}{\epsilon_0}\delta(\mathbf{r} - \mathbf{r}_0)$$

in the quarter space $x > 0$, $y > 0$. It is clear that we require three image line charges with positions and strengths as shown in figure 19.13 (all of which lie outside the region in which we seek a solution). The boundary condition that the electrostatic potential u is zero on $x = 0$ and $y = 0$ (shown as the 'curve' C in figure 19.13) is then automatically satisfied, and so this system of image charges is directly equivalent to the original situation of a single line charge in the presence of the earthed plates along $x = 0$ and $y = 0$. Thus the electrostatic potential is simply equal to the Dirichlet Green's function

$$u(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}_0) = -\frac{\lambda}{2\pi\epsilon_0}\left(\ln|\mathbf{r} - \mathbf{r}_0| - \ln|\mathbf{r} - \mathbf{r}_1| + \ln|\mathbf{r} - \mathbf{r}_2| - \ln|\mathbf{r} - \mathbf{r}_3|\right),$$

which equals zero on C and on the 'surface' at infinity.

The force on the line charge at $\mathbf{r}_0$, therefore, is simply that due to the three line charges at $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. The elecrostatic potential due to a line charge at $\mathbf{r}_i$, $i = 1$, 2 or 3, is given by the fundamental solution

$$u_i(\mathbf{r}) = \mp\frac{\lambda}{2\pi\epsilon_0}\ln|\mathbf{r} - \mathbf{r}_i| + c,$$

the upper or lower sign being taken according to whether the line charge is positive or negative respectively. Therefore the force per unit length on the line charge at $\mathbf{r}_0$, due to the one at $\mathbf{r}_i$, is given by

$$-\lambda\nabla u_i(\mathbf{r})\bigg|_{\mathbf{r}=\mathbf{r}_0} = \pm\frac{\lambda^2}{2\pi\epsilon_0}\frac{\mathbf{r}_0 - \mathbf{r}_i}{|\mathbf{r}_0 - \mathbf{r}_i|^2}.$$

Adding the contributions from the three image charges shown in figure 19.13, the total force experienced by the line charge at $\mathbf{r}_0$ is

$$\mathbf{F} = \frac{\lambda^2}{2\pi\epsilon_0}\left(-\frac{\mathbf{r}_0 - \mathbf{r}_1}{|\mathbf{r}_0 - \mathbf{r}_1|^2} + \frac{\mathbf{r}_0 - \mathbf{r}_2}{|\mathbf{r}_0 - \mathbf{r}_2|^2} - \frac{\mathbf{r}_0 - \mathbf{r}_3}{|\mathbf{r}_0 - \mathbf{r}_3|^2}\right),$$

where, from the figure, $\mathbf{r}_0 - \mathbf{r}_1 = 2y_0\mathbf{j}$, $\mathbf{r}_0 - \mathbf{r}_2 = 2x_0\mathbf{i} + 2y_0\mathbf{j}$ and $\mathbf{r}_0 - \mathbf{r}_3 = 2x_0\mathbf{i}$. Thus, in terms of x_0 and y_0, the total force on the line charge due to the charge induced on the plates is given by

$$\begin{aligned}\mathbf{F} &= \frac{\lambda^2}{2\pi\epsilon_0}\left(-\frac{1}{2y_0}\mathbf{j} + \frac{2x_0\,\mathbf{i} + 2y_0\,\mathbf{j}}{4x_0^2 + 4y_0^2} - \frac{1}{2x_0}\mathbf{i}\right) \\ &= -\frac{\lambda^2}{4\pi\epsilon_0(x_0^2 + y_0^2)}\left(\frac{y_0^2}{x_0}\mathbf{i} + \frac{x_0^2}{y_0}\mathbf{j}\right). \blacktriangleleft\end{aligned}$$

Further generalisations are possible. For instance, solving Poisson's equation in the two-dimensional strip $-\infty < x < \infty$, $0 < y < b$ requires an infinite series of image points.

So far we have considered problems in which the boundary S consists of straight lines (in two dimensions) or planes (in three dimensions), in which simple reflection of the source at $\mathbf{r}_0$ in these boundaries fixes the positions of the image points. For more complicated (curved) boundaries this is no longer possible, and finding the appropriate position(s) and strength(s) of the image source(s) requires further work.

►*Use the method of images to find the Dirichlet Green's function for solving Poisson's equation outside a sphere of radius a centred at the origin.*

We need to find a solution of Poisson's equation valid outside the sphere of radius a. Since an image point $\mathbf{r}_1$ cannot lie in this region, it must be located within the sphere. The Green's function for this problem is therefore

$$G(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} - \frac{q}{4\pi|\mathbf{r} - \mathbf{r}_1|},$$

where $|\mathbf{r}_0| > a$, $|\mathbf{r}_1| < a$ and q is the strength of the image which we have yet to determine. Clearly, $G(\mathbf{r}, \mathbf{r}_0) \to 0$ on the surface at infinity.

By symmetry we expect the image point $\mathbf{r}_1$ to lie on the same radial line as the original source, $\mathbf{r}_0$, as shown in figure 19.14, and so $\mathbf{r}_1 = k\mathbf{r}_0$ where $k < 1$. However, for a Dirichlet Green's function we require $G(\mathbf{r} - \mathbf{r}_0) = 0$ on $|\mathbf{r}| = a$, and the form of the Green's function suggests that we need

$$|\mathbf{r} - \mathbf{r}_0| \propto |\,\mathbf{r} - \mathbf{r}_1| \qquad \text{for all } |\mathbf{r}| = a. \tag{19.100}$$

Referring to figure 19.14, if this relationship is to hold over the whole surface of the sphere, then it must certainly hold for the points A and B. We thus require

$$\frac{|\mathbf{r}_0| - a}{a - |\mathbf{r}_1|} = \frac{|\mathbf{r}_0| + a}{a + |\mathbf{r}_1|},$$

which reduces to $|\mathbf{r}_1| = a^2/|\mathbf{r}_0|$. Therefore the image point must be located at the position

$$\mathbf{r}_1 = \frac{a^2}{|\mathbf{r}_0|^2}\mathbf{r}_0.$$

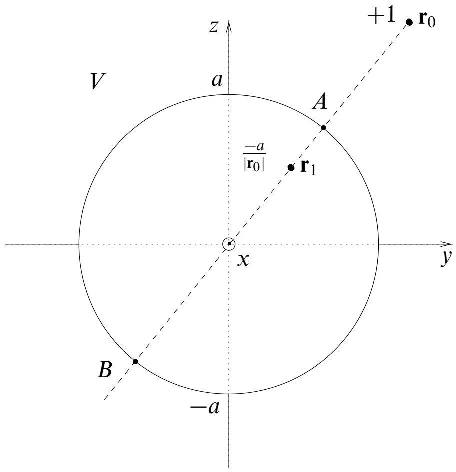

Figure 19.14 The arrangement of images for solving Poisson's equation outside a sphere of radius a centred at the origin. For a charge $+1$ at $\mathbf{r}_0$, the image point $\mathbf{r}_1$ is given by $(a/|\mathbf{r}_0|)^2\mathbf{r}_0$ and the strength of the image charge is $-a/|\mathbf{r}_0|$.

It may now be checked that, for this location of the image point, (19.100) is satisfied over the whole sphere. Using the geometrical result

$$\begin{aligned}|\mathbf{r}-\mathbf{r}_1|^2 &= |\mathbf{r}|^2 - \frac{2a^2}{|\mathbf{r}_0|^2}\mathbf{r}\cdot\mathbf{r}_0 + \frac{a^4}{|\mathbf{r}_0|^2} \\ &= \frac{a^2}{|\mathbf{r}_0|^2}\left(|\mathbf{r}_0|^2 - 2\mathbf{r}\cdot\mathbf{r}_0 + a^2\right) \qquad \text{for } |\mathbf{r}| = a, \end{aligned} \qquad (19.101)$$

we see that, on the surface of the sphere,

$$|\mathbf{r}-\mathbf{r}_1| = \frac{a}{|\mathbf{r}_0|}|\mathbf{r}-\mathbf{r}_0| \qquad \text{for } |\mathbf{r}| = a. \qquad (19.102)$$

Therefore, in order that $G = 0$ at $|\mathbf{r}| = a$, the strength of the image charge must be $-a/|\mathbf{r}_0|$. Consequently, the Dirichlet Green's function for the exterior of the sphere is

$$G(\mathbf{r},\mathbf{r}_0) = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}_0|} + \frac{a/|\mathbf{r}_0|}{4\pi\,|\mathbf{r}-(a^2/|\mathbf{r}_0|^2)\mathbf{r}_0|}.$$

For a less formal treatment of the same problem see exercise 19.24. ◀

If we seek solutions to Poisson's equation in the *interior* of a sphere then the above analysis still holds, but $\mathbf{r}$ and $\mathbf{r}_0$ are now inside the sphere and the image $\mathbf{r}_1$ lies outside it.

For two-dimensional Dirichlet problems outside the circle $|\mathbf{r}| = a$, we are led by arguments similar to those employed previously to use the same image point as in the three-dimensional case, namely

$$\mathbf{r}_1 = \frac{a^2}{|\mathbf{r}_0|^2}\mathbf{r}_0. \qquad (19.103)$$

As illustrated below, however, it is usually necessary to take the image strength as -1 in two-dimensional problems.

►*Solve Laplace's equation in the two-dimensional region $|\mathbf{r}| \leq a$, subject to the boundary condition $u = f(\phi)$ on $|\mathbf{r}| = a$.*

In this case we wish to find the Dirichlet Green's function in the interior of a disc of radius a, so the image charge must lie outside the disc. Taking the strength of the image to be -1, we have

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}_0| - \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}_1| + c,$$

where $\mathbf{r}_1 = (a^2/|\mathbf{r}_0|^2)\mathbf{r}_0$ lies outside the disc, and c is a constant that includes the strength of the image charge and does not necessarily equal zero.

Since we require $G(\mathbf{r}, \mathbf{r}_0) = 0$ when $|\mathbf{r}| = a$, the value of the constant c is determined, and the Dirichlet Green's function for this problem is given by

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi} \left(\ln |\mathbf{r} - \mathbf{r}_0| - \ln \left| \mathbf{r} - \frac{a^2}{|\mathbf{r}_0|^2}\mathbf{r}_0 \right| - \ln \frac{|\mathbf{r}_0|}{a} \right). \qquad (19.104)$$

Using plane polar coordinates, the solution to the boundary-value problem can be written as a line integral around the circle $\rho = a$:

$$\begin{aligned} u(\mathbf{r}_0) &= \int_C f(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n}\, dl \\ &= \int_0^{2\pi} f(\mathbf{r}) \left. \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \rho} \right|_{\rho = a} a\, d\phi. \end{aligned} \qquad (19.105)$$

The normal derivative of the Green's function (19.104) is given by

$$\begin{aligned} \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \rho} &= \frac{\mathbf{r}}{|\mathbf{r}|} \cdot \nabla G(\mathbf{r}, \mathbf{r}_0) \\ &= \frac{\mathbf{r}}{2\pi |\mathbf{r}|} \cdot \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^2} - \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^2} \right). \end{aligned} \qquad (19.106)$$

Using the fact that $\mathbf{r}_1 = (a^2/|\mathbf{r}_0|^2)\mathbf{r}_0$ and the geometrical result (19.102), we find that

$$\left. \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \rho} \right|_{\rho = a} = \frac{a^2 - |\mathbf{r}_0|^2}{2\pi a |\mathbf{r} - \mathbf{r}_0|^2}.$$

In plane polar coordinates, $\mathbf{r} = \rho \cos\phi\, \mathbf{i} + \rho \sin\phi\, \mathbf{j}$ and $\mathbf{r}_0 = \rho_0 \cos\phi_0\, \mathbf{i} + \rho_0 \sin\phi_0\, \mathbf{j}$, and so

$$\left. \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial \rho} \right|_{\rho = a} = \left(\frac{1}{2\pi a} \right) \frac{a^2 - \rho_0^2}{a^2 + \rho_0^2 - 2a\rho_0 \cos(\phi - \phi_0)}.$$

On substituting into (19.105), we obtain

$$u(\rho_0, \phi_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - \rho_0^2) f(\phi)\, d\phi}{a^2 + \rho_0^2 - 2a\rho_0 \cos(\phi - \phi_0)}, \qquad (19.107)$$

which is the solution to the problem. ◄

19.5.4 Neumann problems

In a Neumann problem we require the normal derivative of the solution of Poisson's equation to take on specific values on some surface S that bounds V, i.e. we require $\partial u(\mathbf{r})/\partial n = f(\mathbf{r})$ on S, where f is a given function. As we shall see, much of our discussion of Dirichlet problems can be immediately taken over into the solution of Neumann problems.

As we proved in section 18.7 of the previous chapter, specifying Neumann boundary conditions determines the relevant solution of Poisson's equation to within an (unimportant) additive constant. Unlike Dirichlet conditions, Neumann conditions impose a self-consistency requirement. In order for a solution u to exist, it is necessary that the following consistency condition holds:

$$\int_S f\, dS = \int_S \nabla u \cdot \hat{\mathbf{n}}\, dS = \int_V \nabla^2 u\, dV = \int_V \rho\, dV, \qquad (19.108)$$

where we have used the divergence theorem to convert the surface integral into a volume integral. As a physical example, the integral of the normal component of an electric field over a surface bounding a given volume cannot be chosen arbitrarily when the charge inside the volume has already been specified (Gauss's theorem).

Let us again consider (19.87), which is central to our discussion of Green's functions in inhomogeneous problems. It reads

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) + \int_S \left[u(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0) \frac{\partial u(\mathbf{r})}{\partial n} \right] dS(\mathbf{r}).$$

As always, the Green's function must obey

$$\nabla^2 G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0),$$

where $\mathbf{r}_0$ lies in V. In the solution of Dirichlet problems in the previous subsection, we chose the Green's function to obey the boundary condition $G(\mathbf{r}, \mathbf{r}_0) = 0$ on S and, in a similar way, we might wish to choose $\partial G(\mathbf{r}, \mathbf{r}_0)/\partial n = 0$ in the solution of Neumann problems. However, in general this is *not* permitted since the Green's function must obey the consistency condition

$$\int_S \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n}\, dS = \int_S \nabla G(\mathbf{r}, \mathbf{r}_0) \cdot \hat{\mathbf{n}}\, dS = \int_V \nabla^2 G(\mathbf{r}, \mathbf{r}_0)\, dV = 1.$$

The simplest permitted boundary condition is therefore

$$\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} = \frac{1}{A} \qquad \text{for } \mathbf{r} \text{ on } S,$$

where A is the area of the surface S; this defines a *Neumann Green's function.*

If we require $\partial u(\mathbf{r})/\partial n = f(\mathbf{r})$ on S, the solution to Poisson's equation is given by

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r},\mathbf{r}_0)\rho(\mathbf{r})\,dV(\mathbf{r}) + \frac{1}{A}\int_S u(\mathbf{r})\,dS(\mathbf{r}) - \int_S G(\mathbf{r},\mathbf{r}_0)f(\mathbf{r})\,dS(\mathbf{r})$$
$$= \int_V G(\mathbf{r},\mathbf{r}_0)\rho(\mathbf{r})\,dV(\mathbf{r}) + \langle u(\mathbf{r})\rangle_S - \int_S G(\mathbf{r},\mathbf{r}_0)f(\mathbf{r})\,dS(\mathbf{r}), \qquad (19.109)$$

where $\langle u(\mathbf{r})\rangle_S$ is the average of u over the surface S and is a freely specifiable constant. For Neumann problems in which the volume V is bounded by a surface S at infinity, we do not need the $\langle u(\mathbf{r})\rangle_S$ term. For example, if we wish to solve a Neumann problem outside the unit sphere centred at the origin then $r > a$ is the region V throughout which we require the solution; this region may be considered as being bounded by two disconnected surfaces, the surface of the sphere and a surface at infinity. By requiring that $u(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$, the term $\langle u(\mathbf{r})\rangle_S$ becomes zero.

As mentioned above, much of our discussion of Dirichlet problems can be taken over into the solution of Neumann problems. In particular, we may use the method of images to find the appropriate Neumann Green's function.

►*Solve Laplace's equation in the two-dimensional region $|\mathbf{r}| \le a$ subject to the boundary condition $\partial u/\partial n = f(\phi)$ on $|\mathbf{r}| = a$, with $\int_0^{2\pi} f(\phi)\,d\phi = 0$ as required by the consistency condition (19.108).*

Let us assume, as in Dirichlet problems with this geometry, that a single image charge is placed outside the circle at

$$\mathbf{r}_1 = \frac{a^2}{|\mathbf{r}_0|^2}\mathbf{r}_0,$$

where $\mathbf{r}_0$ is the position of the source inside the circle (see equation (19.103)). Then, from (19.102), we have the useful geometrical result

$$|\mathbf{r} - \mathbf{r}_1| = \frac{a}{|\mathbf{r}_0|}|\mathbf{r} - \mathbf{r}_0| \qquad \text{for } |\mathbf{r}| = a. \qquad (19.110)$$

Leaving the strength q of the image as a parameter, the Green's function has the form

$$G(\mathbf{r},\mathbf{r}_0) = \frac{1}{2\pi}\left(\ln|\mathbf{r} - \mathbf{r}_0| + q\ln|\mathbf{r} - \mathbf{r}_1| + c\right). \qquad (19.111)$$

Using plane polar coordinates, the radial (i.e. normal) derivative of this function is given by

$$\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial \rho} = \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\nabla G(\mathbf{r},\mathbf{r}_0)$$
$$= \frac{\mathbf{r}}{2\pi|\mathbf{r}|}\cdot\left[\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^2} + \frac{q(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^2}\right].$$

Using (19.110), on the circumference of the circle $\rho = a$ the radial derivative is

$$\left.\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial \rho}\right|_{\rho=a} = \frac{1}{2\pi|\mathbf{r}|}\left[\frac{|\mathbf{r}|^2 - \mathbf{r}\cdot\mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^2} + \frac{q|\mathbf{r}|^2 - q(a^2/|\mathbf{r}_0|^2)\mathbf{r}\cdot\mathbf{r}_0}{(a^2/|\mathbf{r}_0|^2)|\mathbf{r} - \mathbf{r}_0|^2}\right]$$
$$= \frac{1}{2\pi a}\frac{1}{|\mathbf{r} - \mathbf{r}_0|^2}\left[|\mathbf{r}|^2 + q|\mathbf{r}_0|^2 - (1+q)\mathbf{r}\cdot\mathbf{r}_0\right],$$

where we have set $|\mathbf{r}|^2 = a^2$ in the second term on the RHS, but not in the first. If we take $q = 1$, the radial derivative simplifies to

$$\left.\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial \rho}\right|_{\rho=a} = \frac{1}{2\pi a},$$

or $1/L$ where L is the length of the circumference, and so (19.111) with $q = 1$ is the required Neumann Green's function.

Since $\rho(\mathbf{r}) = 0$, the solution to our boundary-value problem is now given by (19.109) as

$$u(\mathbf{r}_0) = \langle u(\mathbf{r})\rangle_C - \int_C G(\mathbf{r},\mathbf{r}_0) f(\mathbf{r})\, dl(\mathbf{r}),$$

where the integral is around the circumference of the circle C. In plane polar coordinates $\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j}$ and $\mathbf{r}_0 = \rho_0\cos\phi_0\,\mathbf{i} + \rho_0\sin\phi_0\,\mathbf{j}$, and again using (19.110) we find that on C the Green's function is given by

$$\begin{aligned}
G(\mathbf{r},\mathbf{r}_0)|_{\rho=a} &= \frac{1}{2\pi}\left[\ln|\mathbf{r}-\mathbf{r}_0| + \ln\left(\frac{a}{|\mathbf{r}_0|}|\mathbf{r}-\mathbf{r}_0|\right) + c\right] \\
&= \frac{1}{2\pi}\left(\ln|\mathbf{r}-\mathbf{r}_0|^2 + \ln\frac{a}{|\mathbf{r}_0|} + c\right) \\
&= \frac{1}{2\pi}\left\{\ln\left[a^2 + \rho_0^2 - 2a\rho_0\cos(\phi-\phi_0)\right] + \ln\frac{a}{\rho_0} + c\right\}. \qquad (19.112)
\end{aligned}$$

Since $dl = a\,d\phi$ on C, the solution to the problem is given by

$$u(\rho_0,\phi_0) = \langle u\rangle_C - \frac{a}{2\pi}\int_0^{2\pi} f(\phi)\ln[a^2+\rho_0^2 - 2a\rho_0\cos(\phi-\phi_0)]\,d\phi.$$

The contributions of the final two terms terms in the Green's function (19.112) vanish because $\int_0^{2\pi} f(\phi)\,d\phi = 0$. The average value of u around the circumference, $\langle u\rangle_C$, is a freely specifiable constant as we would expect for a Neumann problem. This result should be compared with the result (19.107) for the corresponding Dirichlet problem, but it should be remembered that in the one case $f(\phi)$ is a potential, and in the other the gradient of a potential. ◀

19.6 Exercises

19.1 Solve the following first-order partial differential equations by separating the variables:

$$\text{(a)}\ \frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0; \qquad \text{(b)}\ x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0.$$

19.2 A conducting cube has as its six faces the planes $x = \pm a$, $y = \pm a$ and $z = \pm a$, and contains no internal heat sources. Verify that the temperature distribution

$$u(x,y,z,t) = A\cos\frac{\pi x}{a}\sin\frac{\pi z}{a}\exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right)$$

obeys the appropriate diffusion equation. Across which faces is there heat flow? What is the direction and rate of heat flow at the point $(3a/4, a/4, a)$ at time $t = a^2/(\kappa\pi^2)$?

19.3 The wave equation describing the transverse vibrations of a stretched membrane under tension T and having a uniform surface density ρ is

$$T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho\frac{\partial^2 u}{\partial t^2}.$$

Find a separable solution appropriate to a membrane stretched on a frame of length a and width b, showing that the natural angular frequencies of such a membrane are

$$\omega^2 = \frac{\pi^2 T}{\rho}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right),$$

where n and m are any positive integers.

19.4 Schrödinger's equation for a non-relativistic particle in a constant potential region can be taken as

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = i\hbar\frac{\partial u}{\partial t}.$$

(a) Find a solution, separable in the four independent variables, that can be written in the form of a plane wave,

$$\psi(x, y, z, t) = A\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)].$$

Using the relationships associated with de Broglie ($\mathbf{p} = \hbar\mathbf{k}$) and Einstein ($E = \hbar\omega$), show that the separation constants must be such that

$$p_x^2 + p_y^2 + p_z^2 = 2mE.$$

(b) Obtain a different separable solution describing a particle confined to a box of side a (ψ must vanish at the walls of the box). Show that the energy of the particle can only take the quantised values

$$E = \frac{\hbar^2\pi^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2),$$

where n_x, n_y, n_z are integers.

19.5 Denoting the three terms of ∇^2 in spherical polars by ∇_r^2, ∇_θ^2, ∇_ϕ^2 in an obvious way, evaluate $\nabla_r^2 u$, etc. for the two functions given below and verify that, in each case, although the individual terms are not necessarily zero their sum $\nabla^2 u$ is zero. Identify the corresponding values of ℓ and m.

(a) $u(r,\theta,\phi) = \left(Ar^2 + \dfrac{B}{r^3}\right)\dfrac{3\cos^2\theta - 1}{2}$.

(b) $u(r,\theta,\phi) = \left(Ar + \dfrac{B}{r^2}\right)\sin\theta\exp i\phi$.

19.6 Prove that the expression given in equation (19.47) for the associated Legendre function $P_\ell^m(\mu)$ satisfies the appropriate equation, (19.45), as follows.

(a) Evaluate $dP_\ell^m(\mu)/d\mu$ and $d^2P_\ell^m(\mu)/d\mu^2$ using the forms given in (19.47) and substitute them into (19.45).
(b) Differentiate Legendre's equation m times using Leibniz' theorem.
(c) Show that the equations obtained in (a) and (b) are multiples of each other, and hence that the validity of (b) implies that of (a).

19.7 Use the expressions at the end of subsection 19.3.2 to verify for $\ell = 0, 1, 2$ that

$$\sum_{m=-\ell}^{\ell} |Y_\ell^m(\theta,\phi)|^2 = \frac{2\ell+1}{4\pi}$$

and so is independent of the values of θ and ϕ. This is true for any ℓ, but a general proof is more involved. This result helps to reconcile intuition with the apparently arbitrary choice of polar axis in a general quantum mechanical system.

19.8 Express the function

$$f(\theta,\phi) = \sin\theta[\sin^2(\theta/2)\cos\phi + i\cos^2(\theta/2)\sin\phi] + \sin^2(\theta/2)$$

as a sum of spherical harmonics.

19.9 Continue the analysis of exercise 10.20, concerned with the flow of a very viscous fluid past a sphere, to find the full expression for the stream function $\psi(r,\theta)$. At the surface of the sphere $r = a$ the velocity field $\mathbf{u} = \mathbf{0}$, whilst far from the sphere $\psi \simeq (Ur^2\sin^2\theta)/2$.

Show that $f(r)$ can be expressed as a superposition of powers of r, and determine which powers give acceptable solutions. Hence show that

$$\psi(r,\theta) = \frac{U}{4}\left(2r^2 - 3ar + \frac{a^3}{r}\right)\sin^2\theta.$$

19.10 The motion of a very viscous fluid in the two-dimensional (wedge) region $-\alpha < \phi < \alpha$ can be described in (ρ,ϕ) coordinates by the (biharmonic) equation

$$\nabla^2\nabla^2\psi \equiv \nabla^4\psi = 0,$$

together with the boundary conditions $\partial\psi/\partial\phi = 0$ at $\phi = \pm\alpha$, which represents the fact that there is no radial fluid velocity close to either of the bounding walls because of the viscosity, and $\partial\psi/\partial\rho = \pm\rho$ at $\phi = \pm\alpha$, which imposes the condition that azimuthal flow increases linearly with r along any radial line. Assuming a solution in separated-variable form, show that the full expression for ψ is

$$\psi(\rho,\phi) = \frac{\rho^2}{2}\,\frac{\sin 2\phi - 2\phi\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}.$$

19.11 A circular disk of radius a is such a way that its perimeter $\rho = a$ is maintained with a temperature distribution $A + B\cos^2\phi$, where ρ and ϕ are plane polar coordinates and A and B are constants. Find the temperature $T(\rho,\phi)$ everywhere in the region $\rho < a$.

19.12 (a) Find the form of the solution of Laplace's equation in plane polar coordinates ρ, ϕ that takes the value $+1$ for $0 < \phi < \pi$ and the value -1 for $-\pi < \phi < 0$, when $\rho = a$.

(b) For a point (x, y) on or inside the circle $x^2 + y^2 = a^2$, identify the angles α and β defined by

$$\alpha = \tan^{-1}\frac{y}{a+x} \qquad \text{and} \qquad \beta = \tan^{-1}\frac{y}{a-x}.$$

Show that $u(x,y) = (2/\pi)(\alpha+\beta)$ is a solution of Laplace's equation that satisfies the boundary conditions given in (a).

(c) Deduce a Fourier series expansion for the function

$$\tan^{-1}\frac{\sin\phi}{1+\cos\phi} + \tan^{-1}\frac{\sin\phi}{1-\cos\phi}.$$

19.13 The free transverse vibrations of a thick rod satisfy the equation

$$a^4\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0.$$

Obtain a solution in separated-variable form and, for a rod clamped at one end, $x = 0$, and free at the other, $x = L$, show that the angular frequency of vibration ω satisfies

$$\cosh\left(\frac{\omega^{1/2}L}{a}\right) = -\sec\left(\frac{\omega^{1/2}L}{a}\right).$$

(At a clamped end both u and $\partial u/\partial x$ vanish, whilst at a free end, where there is no bending moment, $\partial^2 u/\partial x^2$ and $\partial^3 u/\partial x^3$ are both zero.)

19.14 A membrane is stretched between two concentric rings of radii a and b ($b > a$). If the smaller ring is transversely distorted from the planar configuration by an amount $c|\phi|$, $-\pi \le \phi \le \pi$, show that the membrane then has a shape given by

$$u(\rho, \phi) = \frac{c\pi}{2} \frac{\ln(b/\rho)}{\ln(b/a)} - \frac{4c}{\pi} \sum_{m \text{ odd}} \frac{a^m}{m^2(b^{2m} - a^{2m})} \left(\frac{b^{2m}}{\rho^m} - \rho^m \right) \cos m\phi.$$

19.15 A string of length L, fixed at its two ends, is plucked at its mid-point by an amount A and then released. Prove that the subsequent displacement is given by

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8A}{\pi^2(2n+1)^2} \sin\left[\frac{(2n+1)\pi x}{L}\right] \cos\left[\frac{(2n+1)\pi ct}{L}\right],$$

where, in the usual notation, $c^2 = T/\rho$.

Find the total kinetic energy of the string when it passes through its unplucked position, by calculating it in each mode (each n) and summing, using the result

$$\sum_0^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Confirm that the total energy is equal to the work done in plucking the string initially.

19.16 Prove that the potential for $\rho < a$ associated with a vertical split cylinder of radius a, the two halves of which ($\cos\phi > 0$ and $\cos\phi < 0$) are maintained at equal and opposite potentials $\pm V$, is given by

$$u(\rho, \phi) = \frac{4V}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{\rho}{a}\right)^{2n+1} \cos(2n+1)\phi.$$

19.17 A conducting spherical shell of radius a is cut round its equator and the two halves connected to voltages of $+V$ and $-V$. Show that an expression for the potential at the point (r, θ, ϕ) anywhere inside the two hemispheres is

$$u(r, \theta, \phi) = V \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!(4n+3)}{2^{2n+1} n!(n+1)!} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta).$$

(This is the spherical polar analogue of the previous question.)

19.18 A slice of biological material of thickness L is placed into a solution of a radioactive isotope of constant concentration C_0 at time $t = 0$. For a later time t find the concentration of radioactive ions at a depth x inside one of its surfaces if the diffusion constant is κ.

19.19 Two identical copper bars are each of length a. Initially, one is at $0\,^\circ$C and the other at $100\,^\circ$C; they are then joined together end to end and thermally isolated. Obtain in the form of a Fourier series an expression $u(x,t)$ for the temperature at any point a distance x from the join at a later time t. (Bear in mind the heat flow conditions at the free ends of the bars.)

Taking $a = 0.5$ m estimate the time it takes for one of the free ends to attain a temperature of $55\,^\circ$C. The thermal conductivity of copper is 3.8×10^2 J m^{-1} K^{-1} s^{-1}, and its specific heat capacity is 3.4×10^6 J m^{-3} K^{-1}.

19.20 A sphere of radius a and thermal conductivity k_1 is surrounded by an infinite medium of conductivity k_2 in which, far away, the temperature tends to T_∞. A distribution of heat sources $q(\theta)$ embedded in the sphere's surface establish steady temperature fields $T_1(r, \theta)$ inside the sphere and $T_2(r, \theta)$ outside it. It can be shown, by considering the heat flow through a small volume that includes part of the sphere's surface, that

$$k_1 \frac{\partial T_1}{\partial r} - k_2 \frac{\partial T_2}{\partial r} = q(\theta) \quad \text{on} \quad r = a.$$

Given that

$$q(\theta) = \frac{1}{a} \sum_{n=0}^{\infty} q_n P_n(\cos\theta),$$

find complete expressions for $T_1(r,\theta)$ and $T_2(r,\theta)$. What is the temperature at the centre of the sphere?

19.21 Using result (19.77) from the worked example in the text, find the general expression for the temperature $u(x,t)$ in the bar, given that the temperature distribution at time $t = 0$ is $u(x,0) = \exp(-x^2/a^2)$.

19.22 (a) Show that the gravitational potential due to a uniform disc of radius a and mass M, centred at the origin, is given for $r < a$ by

$$\frac{2GM}{a}\left[1 - \frac{r}{a}P_1(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) - \frac{1}{8}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + \cdots\right],$$

and for $r > a$ by

$$\frac{GM}{r}\left[1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{8}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) - \cdots\right],$$

where the polar axis is normal to the plane of the disc.

(b) Reconcile the presence of a term $P_1(\cos\theta)$, which is odd under $\theta \to \pi - \theta$, with the symmetry with respect to the plane of the disc of the physical system.

(c) Deduce that the gravitational field near an infinite sheet of matter of constant density ρ per unit area is $2\pi G\rho$.

19.23 In the region $-\infty < x,\ y < \infty$ and $-t \leq z \leq t$, a charge-density wave $\rho(\mathbf{r}) = A\cos qx$, in the x-direction, is represented by

$$\rho(\mathbf{r}) = \frac{e^{iqx}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\rho}(\alpha) e^{i\alpha z}\, d\alpha.$$

The resulting potential is represented by

$$V(\mathbf{r}) = \frac{e^{iqx}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{V}(\alpha) e^{i\alpha z}\, d\alpha.$$

Determine the relationship between $\tilde{V}(\alpha)$ and $\tilde{\rho}(\alpha)$, and hence show that the potential at the point $(x,0,0)$ is

$$\frac{A}{\pi} \int_{-\infty}^{\infty} \frac{\sin kt}{k(k^2 + q^2)}\, dk.$$

19.24 Point charges q and $-qa/b$ (with $a < b$) are placed respectively at a point P, a distance b from the origin O, and a point Q between O and P, a distance a^2/b from O. Show, by considering similar triangles QOS and SOP, where S is any point on the surface of the sphere centred at O and of radius a, that the net potential anywhere on the sphere due to the two charges is zero.

Use this result (backed up by the uniqueness theorem) to find the force with which a point charge q placed a distance b from the centre of a spherical conductor of radius a ($< b$) is attracted to the sphere (i) if the sphere is earthed, and (ii) if the sphere is uncharged and insulated.

19.25 Find the Green's function $G(\mathbf{r}, \mathbf{r}_0)$ in the half-space $z > 0$ for the solution of $\nabla^2\Phi = 0$ with Φ specified in cylindrical polar coordinates (ρ, ϕ, z) on the plane $z = 0$ by

$$\Phi(\rho, \phi, z) = \begin{cases} 1 & \text{for } \rho \leq 1, \\ 1/\rho & \text{for } \rho > 1. \end{cases}$$

Determine the variation of $\Phi(0, 0, z)$ along the z-axis.

19.26 Electrostatic charge is distributed in a sphere of radius R centred on the origin. Determine the form of the resultant potential $\phi(\mathbf{r})$ at distances much greater than R, as follows.

(a) express in the form of an integral over all space the solution of

$$\nabla^2\phi = -\frac{\rho(\mathbf{r})}{\epsilon_0};$$

(b) show that, for $r \gg r'$,

$$|\mathbf{r} - \mathbf{r}'| = r - \frac{\mathbf{r} \cdot \mathbf{r}'}{r} + \mathrm{O}\left(\frac{1}{r}\right).$$

(c) use results (a) and (b) to show that $\phi(\mathbf{r})$ has the form

$$\phi(\mathbf{r}) = \frac{M}{r} + \frac{\mathbf{d} \cdot \mathbf{r}}{r^3} + \mathrm{O}\left(\frac{1}{r^3}\right);$$

Find expressions for M and $\mathbf{d}$, and identify them physically.

19.27 Find, in the form of an infinite series the Green's function of the ∇^2 operator for the Dirichlet problem in the region $-\infty < x < \infty$, $-\infty < y < \infty$, $-c \leq z \leq c$.

19.28 Find the Green's function for the three-dimensional Neumann problem

$$\nabla^2\phi = 0 \quad \text{for } z > 0 \qquad \text{and} \qquad \frac{\partial\phi}{\partial z} = f(x, y) \quad \text{on } z = 0.$$

Determine $\phi(x, y, z)$ if

$$f(x, y) = \begin{cases} \delta(y) & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a. \end{cases}$$

19.29 (a) By applying the divergence theorem to the volume integral

$$\int_V \left[\phi(\nabla^2 - m^2)\psi - \psi(\nabla^2 - m^2)\phi\right] dV$$

obtain a Green's function expression, as the sum of a volume integral and a surface integral, for $\phi(\mathbf{r}')$ that satisfies

$$\nabla^2\phi - m^2\phi = \rho$$

in V and takes the specified form $\phi = f$ on S, the boundary of V. The Green's function $G(\mathbf{r}, \mathbf{r}')$ to be used satisfies

$$\nabla^2 G - m^2 G = \delta(\mathbf{r} - \mathbf{r}')$$

and vanishes when $\mathbf{r}$ is on S.

(b) When V is all space, $G(\mathbf{r}, \mathbf{r}')$ can be written as $G(t) = g(t)/t$ where $t = |\mathbf{r} - \mathbf{r}'|$ and $g(t)$ is bounded as $t \to \infty$. Find the form of $G(t)$.

(c) Find $\phi(\mathbf{r})$ in the half space $x > 0$ if $\rho(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_1)$ and $\phi = 0$ both on $x = 0$ and as $r \to \infty$.

19.30 Consider the PDE $\mathcal{L}u(\mathbf{r}) = \rho(\mathbf{r})$, for which the differential operator $\mathcal{L}$ is given by

$$\mathcal{L} = \nabla \cdot [p(\mathbf{r})\nabla] + q(\mathbf{r}),$$

where $p(\mathbf{r})$ and $q(\mathbf{r})$ are functions of position. By proving the generalised form of Green's theorem,

$$\int_V (\phi\mathcal{L}\psi - \psi\mathcal{L}\phi)\,dV = \oint_S p(\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{\mathbf{n}}\,dS,$$

show that the solution of the PDE is given by

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r},\mathbf{r}_0)\rho(\mathbf{r})\,dV(\mathbf{r}) + \oint_S p(\mathbf{r})\left[u(\mathbf{r})\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} - G(\mathbf{r},\mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n}\right] dS(\mathbf{r}),$$

where $G(\mathbf{r},\mathbf{r}_0)$ is the Green's function satisfying $\mathcal{L}G(\mathbf{r},\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0)$.

19.7 Hints and answers

19.1 (a) $C\exp[\lambda(x^2+2y)]$; (b) $C(x^2y)^\lambda$.

19.2 There is heat flow only across $z = \pm a$. It is into the cube at a rate of $\kappa A e^{-2}/\sqrt{2}$.

19.3 $u(x,y,t) = \sin(n\pi x/a)\sin(m\pi y/b)(A\sin\omega t + B\cos\omega t)$.

19.4 (a) $-\dfrac{\hbar^2}{2m}\dfrac{X''}{X} = \dfrac{p_x^2}{2m}$, etc., $\dfrac{i\hbar T'}{T} = E$;
(b) As in (a), but with solutions $X = A\sin(p_x x/\hbar)$, etc. with $p_x a/\hbar = n_x\pi$.

19.5 (a) $6u/r^2$, $-6u/r^2$, 0, $\ell = 2$, $m = 0$;
(b) $2u/r^2$, $(\cot^2\theta - 1)u/r^2$; $-u/(r^2\sin^2\theta)$, $\ell = 1$, $m = 1$.

19.8 The first term can contain only $\ell = 1, 2$ and $m = \pm 1$, the second only $\ell = 0, 1, 2$ and $m = 0$; $f(\theta,\phi) = (\pi)^{1/2}[Y_0^0 - 3^{-1/2}Y_1^0 - (2/3)^{1/2}Y_1^1 - (2/15)^{1/2}Y_2^{-1}]$.

19.9 Solutions of the form r^ℓ give ℓ as $-1, 1, 2, 4$. Because of the asymptotic form of ψ, an r^4 term cannot be present. The coefficients of the three remaining terms are determined by the two boundary conditions $\mathbf{u} = \mathbf{0}$ on the sphere and the form of ψ for large r.

19.10 If $\psi(\rho,\phi) = R(\rho)\Phi(\phi)$, show that $\Phi^{(4)} + 4\Phi'' = 0$ and hence that $\Phi = A + B\phi + C\cos 2\phi + D\sin 2\phi$.

19.11 Express $\cos^2\phi$ in terms of $\cos 2\phi$; $T(\rho,\phi) = A + B/2 + (B\rho^2/2a^2)\cos 2\phi$.

19.12 (a) $u(\rho,\phi) = (4/\pi)\sum_{n\ \mathrm{odd}} n^{-1}(\rho/a)^n \sin n\phi$.
(b) $\nabla^2\alpha = 0$, and $\nabla^2\beta = 0$ separately. On $\rho = a, \alpha + \beta + \pi/2 = \pi$.
(c) Equate the two forms (uniqueness theorem) and then set $\rho = a$. The Fourier series is $2\sum_{n\ \mathrm{odd}} n^{-1}\sin n\phi$.

19.13 $(A\cos mx + B\sin mx + C\cosh mx + D\sinh mx)\cos(\omega t + \epsilon)$, with $m^4a^4 = \omega^2$.

19.15 $E_n = 16\rho A^2c^2/[(2n+1)^2\pi^2 L]$; $E = 2\rho c^2A^2/L = \int_0^A [2Tv/(\frac{1}{2}L)]\,dv$.

19.17 You will need the result from exercise 17.7.

19.18 Write $C(x,t) = C_0 + \sum_1^\infty A_n\sin(n\pi x/L)f_n(t)$ where $f_n(t) \to 0$ as $t \to \infty$; $A_n = -4C_0/(n\pi)$ and $f_n(t) = \exp[-(\kappa n^2\pi^2/L^2)t]$ for n odd, and $A_n = 0$ for n even.

19.19 Since there is no heat flow at $x = \pm a$, use a series of period $4a$, $u(x,0) = 100$ for $0 < x \le 2a$, $u(x,0) = 0$ for $-2a \le x < 0$.

$$u(x,t) = 50 + \frac{200}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left[\frac{(2n+1)\pi x}{2a}\right]\exp\left[-\frac{k(2n+1)^2\pi^2 t}{4a^2 s}\right].$$

Taking only the $n = 0$ term gives $t \approx 2300$ s.

19.20 $T_1(r,\theta) = \sum_1^\infty b_m(r/a)^m P_m(\cos\theta) + q_0/k_2 + T_\infty$,
$T_2(r,\theta) = \sum_1^\infty b_m(a/r)^{m+1}P_m(\cos\theta) + aq_0/(k_2 r) + T_\infty$,
where in both cases $b_m = q_m/[mk_1 + (m+1)k_2]$; $T(0,\theta) = q_0/k_2 + T_\infty$.

19.21 $u(x,t) = [a/(a^2+4\kappa t)^{1/2}] \exp[-x^2/(a^2+4\kappa t)]$.

19.22 (a) $u(r=z,0) = 2MGa^{-2}[(a^2+z^2)^{1/2} - z]$. (b) For $\theta > \pi/2$, the factor in the square brackets is $(a^2+z^2)^{1/2} + z$. (c) Find $\partial u/\partial r$ at $\theta = 0$ for $r < a$, and let $a \to \infty$.

19.23 Fourier-transform Poisson's equation to show that $\tilde{\rho}(\alpha) = \epsilon_0(\alpha^2+q^2)\tilde{V}(\alpha)$.

19.24 (i) $q^2ab/[4\pi\epsilon_0(b^2-a^2)^2]$; (ii) $[q^2ab/(4\pi\epsilon_0)][(b^2-a^2)^{-2} - b^{-4}]$. Obtain (ii) from (i) by adding a further image charge $+qa/b$ at O, to give a net zero electrostatic flux from the sphere while maintaining its equipotential property.

19.25 Follow the worked example that includes result (19.98). For part of the explicit integration, substitute $\rho = z\tan\alpha$.

$$\Phi(0,0,z) = \frac{z(1+z^2)^{1/2} - z^2 + (1+z^2)^{1/2} - 1}{z(1+z^2)^{1/2}}.$$

19.26 (a) See equation (19.94); (c) $M = (4\pi\epsilon_0)^{-1}\int \rho(\mathbf{r}')\,dV'$ = total charge on the sphere. $\mathbf{d} = (4\pi\epsilon_0)^{-1}\int \rho(\mathbf{r}')\mathbf{r}'\,dV'$ = dipole moment of the sphere.

19.27

$$G(\mathbf{r},\mathbf{r}_0) = \frac{1}{4\pi}\sum_{n=2}^{\infty}(-1)^n\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+(-1)^n z_0 - nc)^2}} + \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+(-1)^n z_0 + nc)^2}}\right].$$

19.28

$$G(\mathbf{r},\mathbf{r}_0) = -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right].$$

$$\phi(x,y,z) = \frac{1}{2\pi}\left(\sinh^{-1}\frac{a+x}{\sqrt{y^2+z^2}} + \sinh^{-1}\frac{a-x}{\sqrt{y^2+z^2}}\right).$$

19.29 (a) As given in equation (19.89), but with $\mathbf{r}_0$ replaced by $\mathbf{r}'$.
(b) Move the origin to $\mathbf{r}'$ and integrate the defining Green's equation to obtain

$$4\pi t^2\frac{dG}{dt} - m^2\int_0^t G(t')\,4\pi t'^2\,dt' = 1,$$

leading to $G(t) = [-1/(4\pi t)]e^{-mt}$.
(c) $\phi(\mathbf{r}) = [-1/(4\pi)](p^{-1}e^{-mp} - q^{-1}e^{-mq})$, where $p = |\mathbf{r}-\mathbf{r}_1|$ and $q = |\mathbf{r}-\mathbf{r}_2|$ with $\mathbf{r}_1 = (x_1,y_1,z_1)$ and $\mathbf{r}_2 = (-x_1,y_1,z_1)$.