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Complex Fourier series

Since cosine and sine can be related via Euler's identity, that is

$$e^{\pm 2\pi irx/L} = \cos\left(\frac{2\pi rx}{L}\right) \pm i\sin\left(\frac{2\pi rx}{L}\right)$$

Another form for a Fourier series is $f(x) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi irx/L}$

We can solve for the c_r by the usual method – multiple both sides by $e^{-2\pi ipx/L}$

and integrate over the period L, that is

$$\int_{x_0}^{x_0+L} f(x)e^{-2\pi ipx/L}dx = \sum_{r=-\infty}^{\infty} c_r \int_{x_0}^{x_0+L} e^{2\pi irx/L}e^{-2\pi ipx/L}dx =$$

$$\sum_{r=-\infty}^{\infty} c_r \int_{x_0}^{x_0+L} \left[\cos\left(\frac{2\pi rx}{L}\right) + i\sin\left(\frac{2\pi rx}{L}\right)\right]\left[\cos\left(\frac{2\pi px}{L}\right) - i\sin\left(\frac{2\pi px}{L}\right)\right]dx$$

Cross multiplying the trigonometric functions and integrating gives only contributions from cosine-cosine and sine-sine integrals of L/2 similar to that of the "standard" Fourier series. Therefore we get

$$\int_{x_0}^{x_0+L} f(x)e^{-2\pi irx/L}dx = c_r\left(\frac{L}{2}+\frac{L}{2}\right)$$

and $c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-2\pi irx/L}dx$

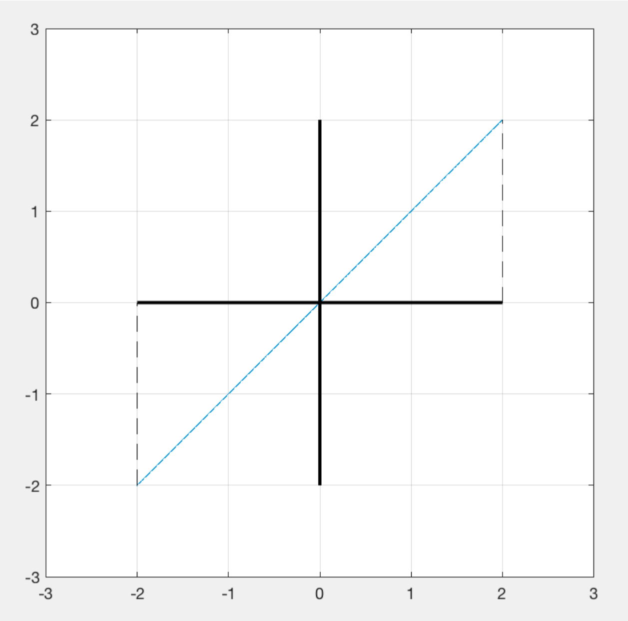
It is also easy to show that

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-2\pi irx/L}dx = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)\left[\cos\left(\frac{2\pi rx}{L}\right)+i\sin\left(\frac{2\pi rx}{L}\right)\right]dx$$

$$= \frac{1}{L}\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi rx}{L}\right)dx - i\frac{1}{L}\int_{x_0}^{x_0+L} f(x)\sin\left(\frac{2\pi rx}{L}\right)dx = \frac{a_r}{2} - i\frac{b_r}{2}$$

Similar for $c_{-r} = \frac{a_r}{2}+i\frac{b_r}{2}$

Complex Fourier series - example



$$f(x) = x \quad -2 < x < 2,\ L = 4$$

Using

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi i r x/L}$$

with

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x) e^{-2\pi i r x/L} dx$$

Therefore (with $x_0 = -2$) we have

$$c_r = \frac{1}{4}\int_{-2}^{2} x e^{-2\pi i r x/4} dx$$

This integral can be done by parts or looked up in a table of integrals

$$c_r = \frac{1}{4}\int_{-2}^{2} x e^{-2\pi i r x/4}dx = \frac{1}{4}\int_{-2}^{2} x e^{-\pi i r x/2}dx\Bigg|_{-2}^{2} = \frac{1}{4}\frac{e^{-\pi i r x/2}}{\left[-\pi i r/2\right]^2}\left(\frac{-\pi i r x}{2}-1\right)\Bigg|_{-2}^{2}$$

$$= \frac{e^{-\pi i r x/2}}{-(\pi r)^2}\left(\frac{-\pi i r x}{2}-1\right)\Bigg|_{-2}^{2} = \frac{e^{-\pi i r x/2}}{(\pi r)^2}\left(\frac{\pi i r x}{2}+1\right)\Bigg|_{-2}^{2}$$

$$= \frac{e^{-\pi i r}}{(\pi r)^2}(\pi i r+1) - \frac{e^{\pi i r}}{(\pi r)^2}(-\pi i r+1) = \frac{i}{\pi r}(e^{-\pi i r}+e^{\pi i r}) + \frac{1}{(\pi r)^2}(e^{-\pi i r}-e^{\pi i r})$$

$$= \frac{i}{\pi r}(2\underbrace{\cos\pi r}_{=(-1)^r}) + \frac{1}{(\pi r)^2}(-2i\underbrace{\sin\pi r}_{=0}) = \frac{2i(-1)^r}{\pi r}$$

Note the r=0 case must be done separately, as $c_0 = \frac{1}{4}\int_{-2}^{2} x e^{-2\pi i r 0/4}dx = \frac{1}{4}\int_{-2}^{2} x dx = 0$

Therefore we have

$$x = \sum_{r=-\infty}^{\infty} \frac{2i(-1)^r}{\pi r} e^{\pi i r x/2},\ r \neq 0$$

Aside:

Notice for our example that x is a real valued variable, but the Fourier series has complex arguments

$$x=\sum_{r=-\infty}^{\infty}\frac{2i(-1)^r}{\pi r}e^{2\pi irx/L},\ r\neq 0$$

We can fix this!

First break the sum into two pieces

$$x=\sum_{r=-\infty}^{1}\frac{2i(-1)^r}{\pi r}e^{\pi irx/2}+\sum_{1}^{\infty}\frac{2i(-1)^r}{\pi r}e^{\pi irx/2}$$

Then in the first sum replace r by $-r$ and note $(-1)^{-r}=(-1)^r$

$$x=\sum_{r=1}^{\infty}\frac{2i(-1)^{-r}}{\pi(-r)}e^{-\pi irx/2}+\sum_{1}^{\infty}\frac{2i(-1)^r}{\pi r}e^{\pi irx/2}$$

$$=\frac{2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}i\left[-e^{-\pi irx/2}+e^{\pi irx/2}\right]=\frac{2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\left[-e^{-\pi irx/2}+e^{\pi irx/2}\right]$$

REAL

$$x=\frac{2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}i\left[-e^{-\pi irx/2}+e^{\pi irx/2}\right]=\frac{2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}i\left[-2i\sin\frac{\pi rx}{2}\right]=\boxed{\frac{4}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\sin\frac{\pi rx}{2}}$$

Parseval's Theorem

Parseval's theorem essentially relates the average value over a period of the modulus of the function squared in terms of the coefficients in its Fourier series representation.

$$\frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 dx = \sum_{r=-\infty}^{\infty} |c_r|^2$$

Remember $c_r = \frac{a_r}{2} - i\frac{b_r}{2}$

$$= \left(\tfrac{1}{2}a_0\right)^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r^2 + b_r^2).$$

A more general form can be derived using the complex Fourier series

$$f(x) = \sum_{r=-\infty}^{\infty} c_r \exp\left(\frac{2\pi irx}{L}\right),$$

$$g(x) = \sum_{r=-\infty}^{\infty} \gamma_r \exp\left(\frac{2\pi irx}{L}\right),$$

Then we can write

$$f(x)g^*(x) = \sum_{r=-\infty}^{\infty} c_r g^*(x) \exp\left(\frac{2\pi irx}{L}\right).$$

Finally

$$\frac{1}{L}\int_{x_0}^{x_0+L} f(x)g^*(x)\,dx = \sum_{r=-\infty}^{\infty} c_r \frac{1}{L}\int_{x_0}^{x_0+L} g^*(x)\exp\left(\frac{2\pi irx}{L}\right)dx$$

$$= \sum_{r=-\infty}^{\infty} c_r \left[\frac{1}{L}\int_{x_0}^{x_0+L} g(x)\exp\left(\frac{-2\pi irx}{L}\right)dx\right]^*$$

$$= \sum_{r=-\infty}^{\infty} c_r \gamma_r^*,$$

This reduces to Parseval’s identity when $f(x) = g(x)$ and therefore $c_r = \gamma_r$

$$\frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 dx = \sum_{r=-\infty}^{\infty} |c_r|^2$$

$$= \left(\tfrac{1}{2}a_0\right)^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}\left(a_r^2 + b_r^2\right).$$

Finally note that this is related to the mean square error if only a finite number of terms in the Fourier series are taken, that is

$$E_N = \frac{1}{L}\int_{x_0}^{x_0+L} |f(x)|^2 \, dx - [(\frac{1}{2}a_0)^2 + \frac{1}{2}\sum_{r=1}^{N}(a_r + b_r)^2]$$

KEY: Note that in this error formula that if N goes to infinity that it is simply Parseval's Identity!