7.7

Lets start with equation on page 349 leading to 9.76 - note take $\nu=\sigma$ and $x=z$

Then $a_n=\frac{-1}{n(n+2\sigma)}a_{n-2}$ Note also that at the end we will use $a_0=\frac{1}{2^\sigma\Gamma(\sigma+1)}$

For n = 2 $a_2=\frac{-1}{2(2+2\sigma)}a_0$; n = 4 $a_4=\frac{-1}{4(4+2\sigma)}a_2=\frac{(-1)}{4(4+2\sigma)}\frac{-1}{2(2+2\sigma)}a_0=\frac{(-1)^2}{4\cdot2(4+2\sigma)(2+2\sigma)}a_0$

$n=6$ $a_6=\frac{-1}{6(6+2\sigma)}a_4=\frac{-1}{6(6+2\sigma)}\frac{(-1)^2}{4\cdot2(4+2\sigma)(2+2\sigma)}a_0=\frac{(-1)^3}{6\cdot4\cdot2(6+2\sigma)(4+2\sigma)(2+2\sigma)}a_0=$

$\frac{(-1)^3}{(2\cdot3)(2\cdot2)(2\cdot1)(6+2\sigma)(4+2\sigma)(2+2\sigma)}a_0=\frac{(-1)^3}{2^3 3!(6+2\sigma)(4+2\sigma)(2+2\sigma)}a_0$

In general $a_{2n}=\frac{(-1)^n}{2^n n!(2+2\sigma)(4+2\sigma)(6+2\sigma)\cdots(2n+2\sigma)}a_0$

So solution is $y_1(x,\sigma)=x^\sigma\sum_{n=0}^{\infty}a_{2n}x^{2n}=x^\sigma a_0x^0+x^\sigma\sum_{n=1}^{\infty}a_{2n}x^{2n}$

Second solution is $y_2(x)=\left[\frac{\partial}{\partial\sigma}y_1(x,\sigma)\right]_{\sigma=0}=\left[\frac{\partial}{\partial\sigma}\left(x^\sigma a_0x^0+x^\sigma\sum_{n=1}^{\infty}a_{2n}x^{2n}\right)\right]_{\sigma=0}=$

$\left[\frac{\partial}{\partial\sigma}\left(x^\sigma a_0+x^\sigma\sum_{n=1}^{\infty}a_{2n}x^{2n}\right)\right]_{\sigma=0}=\left[\frac{\partial}{\partial\sigma}x^\sigma a_0+\left(\frac{\partial}{\partial\sigma}x^\sigma\right)\sum_{n=1}^{\infty}a_{2n}x^{2n}+x^\sigma\sum_{n=1}^{\infty}\frac{\partial}{\partial\sigma}a_{2n}x^{2n}\right]_{\sigma=0}=$

Derivative in first two terms is $\frac{\partial}{\partial\sigma}x^\sigma=\frac{\partial}{\partial\sigma}e^{\ln x^\sigma}=\frac{\partial}{\partial\sigma}e^{\sigma\ln x}=e^{\sigma\ln x}(\ln x)=(\ln x)x^\sigma$

Derivative for the third term within the sum involves $\frac{\partial}{\partial\sigma}a_{2n}=\frac{\partial}{\partial\sigma}\left[\frac{(-1)^n}{2^n n!(2+2\sigma)(4+2\sigma)(6+2\sigma)\cdots(2n+2\sigma)}a_0\right]=$

$\frac{(-1)^n}{2^n n!}a_0\frac{\partial}{\partial\sigma}\left[\frac{1}{(2+2\sigma)(4+2\sigma)(6+2\sigma)\cdots(2n+2\sigma)}\right]=?$

Lets do an example n = 2 for just the derivative $\frac{\partial}{\partial\sigma}\frac{1}{(4+2\sigma)(2+2\sigma)}=\frac{-1(2)}{(2+2\sigma)(4+2\sigma)}\left[\frac{1}{2+2\sigma}+\frac{1}{4+2\sigma}\right]$

Therefore in general $\frac{(-1)(-1)^n(2)}{2^n n!(2+2\sigma)(4+2\sigma)\cdots(2n+2\sigma)}a_0\left[\frac{1}{2+2\sigma}+\frac{1}{4+2\sigma}\cdots+\frac{1}{2n+2\sigma}\right]$

Substitution gives

$$y_2(x)=\left[(\ln x)x^\sigma+(\ln x)x^\sigma\sum_{n=1}^{\infty}a_{2n}x^{2n}+x^\sigma\sum_{n=1}^{\infty}\frac{(-1)(-1)^n(2)}{2^n n!(2+2\sigma)(4+2\sigma)\cdots(2n+2\sigma)}a_0\left[\frac{1}{2+2\sigma}+\frac{1}{4+2\sigma}\cdots+\frac{1}{2n+2\sigma}\right]x^{2n}\right]_{\sigma=0}=$$

$$\left[(\ln x)x^\sigma\left(1+\sum_{n=1}^{\infty}a_{2n}x^{2n}\right)+x^\sigma\sum_{n=1}^{\infty}\frac{(-1)(-1)^n(2)}{2^n n!(2+2\sigma)(4+2\sigma)\cdots(2n+2\sigma)}a_0\left[\frac{1}{2+2\sigma}+\frac{1}{4+2\sigma}\cdots+\frac{1}{2n+2\sigma}\right]x^{2n}\right]_{\sigma=0}=$$

Note with σ=0 in the first term we have

$$y_1(x,0)=x^0\sum_{n=0}^{\infty}a_{2n}(0)x^{2n}=\sum_{n=0}^{\infty}a_{2n}(0)x^{2n}=\sum_{n=0}^{\infty}a_{2n}x^{2n}\equiv 1+\sum_{n=1}^{\infty}a_{2n}x^{2n}=y_1(x)$$

since $a_0(0)=1$ Substitution gives

$$y_2(x)=(\ln x)x^0y_1(x)+\left[x^{\sigma}\sum_{n=1}^{\infty}\frac{(-1)(-1)^n(2)}{2^n n!(2+2\sigma)(4+2\sigma)\cdots(2n+2\sigma)}a_0\left[\frac{1}{2+2\sigma}+\frac{1}{4+2\sigma}\cdots+\frac{1}{2n+2\sigma}\right]x^{2n}\right]_{\sigma=0}=$$

With $x^0=1$ and $y_1(x)=J_0(x)$ by definition (see chapter 9) in first term and also substitution of σ=0 in the second term this gives

$$y_2(x)=(\ln x)J_0(x)+x^0\sum_{n=1}^{\infty}\frac{(-1)(-1)^n(2)}{2^n n!(2)(4)\cdots(2n)}\left[\frac{1}{2}+\frac{1}{4}\cdots+\frac{1}{2n}\right]x^{2n}=$$

$$(\ln x)J_0(x)-\sum_{n=1}^{\infty}\frac{(-1)^n(2)}{2^n n!(2)(4)\cdots(2n)}\left[\frac{1}{2}+\frac{1}{4}\cdots+\frac{1}{2n}\right]x^{2n}$$

$$(\ln x)J_0(x)-\sum_{n=1}^{\infty}\frac{(-1)^n(2)}{2^n n![(2\cdot 1)(2\cdot)\cdots(2\cdot n)]}\left[\frac{1}{2\cdot 1}+\frac{1}{2\cdot 2}\cdots+\frac{1}{2\cdot n}\right]x^{2n}=$$

$$(\ln x)J_0(x)-\sum_{n=1}^{\infty}\frac{(-1)^n(2)}{2^n n![2^n n!]}\frac{1}{2}\left[\frac{1}{1}+\frac{1}{2}\cdots+\frac{1}{n}\right]x^{2n}=$$

with $\frac{1}{1}+\frac{1}{2}\cdots+\frac{1}{n}=\sum_{r=1}^{n}\frac{1}{r}$ and combining terms as needed we get

$$y_2(x)=(\ln x)J_0(x)-\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\left(\sum_{r=1}^{n}\frac{1}{r}\right)x^{2n}=(\ln x)J_0(x)-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\sum_{r=1}^{n}\frac{1}{r}\right)\frac{x^{2n}}{2^{2n}}$$

That is $y_2(x)=(\ln x)J_0(x)-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\sum_{r=1}^{n}\frac{1}{r}\right)\left(\frac{x}{2}\right)^{2n}$