Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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Gamma function

Define the Gamma function $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \;\; n > 0, \text{real}$

Derive a recursive definition for the Gamma function

Letting $n \to n+1$ we have

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$$

Applying integration by parts $\int u\, dv = uv - \int v\, du$ with

$u = x^n \to du = nx^{n-1} dx$ and $dv = e^{-x} dx \to v = -e^{-x}$

$$\int_0^\infty x^n e^{-x}\, dx = x^n(-e^{-x})\Big|_0^\infty - \int_0^\infty (-e^{-x}) n x^{n-1}\, dx = -\frac{x^n}{e^x}\Bigg|_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

For $x = 0$ in the first term (lower bound) we have $\frac{x^n}{e^x} = \frac{0^n}{e^0} = \frac{0}{1} = 0$

and for the upper bound $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ (indeterminate form, using L'Hospital's rule n times)

Therefore $\int_0^\infty x^n e^{-x}\, dx = n\int_0^\infty e^{-x} x^{n-1}\, dx$

or $\Gamma(n+1) = n\Gamma(n)$

Gamma function for integer values

Starting with n = 1, $\Gamma(1)=\int_0^\infty x^{1-1}e^{-x}\,dx=\int_0^\infty e^{-x}\,dx=-e^{-x}\Big|_0^\infty=\lim_{x\to\infty}(-e^{-x})-(-e^{-0})=1$

Then n=1 in our recursive relation we have $\Gamma(1+1)=1\cdot\Gamma(1)=1\cdot1=1!$

For n = 2: $\Gamma(2+1)=2\cdot\Gamma(2)=2\cdot1=2=2!$

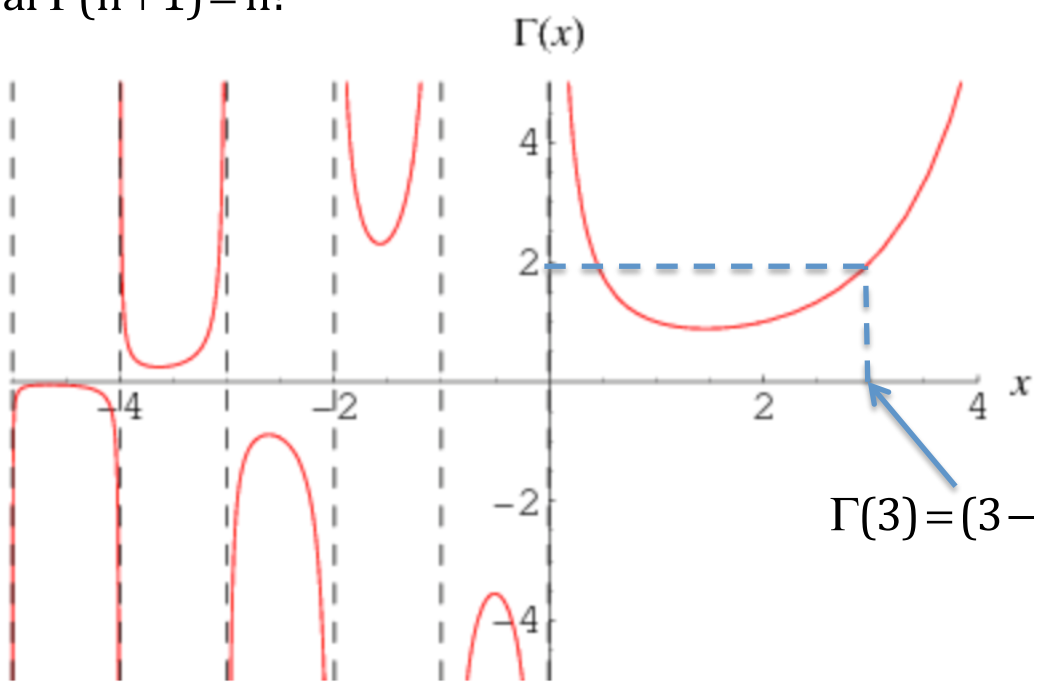
For n = 3: $\Gamma(3+1)=3\cdot\Gamma(3)=3\cdot2=3!$

For n = 4: $\Gamma(4+1)=4\cdot\Gamma(4)=4\cdot3\cdot2=4!$

$\vdots$

In general $\Gamma(n+1)=n!$

KEY: The Gamma function acts like a generalized factorial



$\Gamma(3)=(3-1)!=2!=2\cdot1=2$

http://mathworld.wolfram.com/GammaFunction.html

Gamma function for some fractional values

Start with $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad n > 0$, real

Let $x = y^2 \rightarrow dx = 2y\,dy$

Substitution gives $\Gamma(n) = \int_0^\infty (y^2)^{n-1} e^{-y^2}\, 2y\, dy = 2\int_0^\infty y^{2n-2} y e^{-y^2}\, dy = 2\int_0^\infty y^{2n-1} e^{-y^2}\, dy$

Take $n = \frac{1}{2}$

Then $\Gamma(\frac{1}{2}) = 2\int_0^\infty y^{2(1/2)-1} e^{-y^2}\, dy = 2\int_0^\infty y^0 e^{-y^2}\, dy = 2\int_0^\infty e^{-y^2}\, dy = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

Also since $\Gamma(n+1) = n\Gamma(n)$ and taking $n = \frac{1}{2}$ we have

$\Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

and since $\Gamma(n+1) \equiv n!$

We have $\boxed{\left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}}$ Similarly we have $\boxed{\left(\frac{3}{2}\right)! = \frac{3}{4}\sqrt{\pi}}$