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Mathematical Methods for Applied Biomedical Engineering
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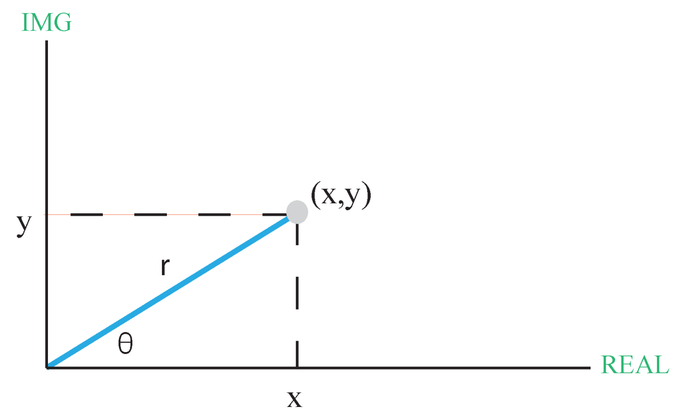
Polar form for Complex Variables

Going back to the Argand diagram and making the usual connections between the Cartesian and polar coordinates we have



$$z = x + iy$$

$$x = r\cos\theta,\ y\ =\ r\sin\theta$$

$$\tan\theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\frac{y}{x},\ -\pi \le \arg(z) = \theta \le \pi$$

$$r^2 = x^2 + y^2 \rightarrow \text{ modulus (or magnitute) } r = \sqrt{x^2 + y^2} \in \text{Real}$$

Therefore $\boxed{z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}}$

Note $|z| = |re^{i\theta}| = |r| \equiv r \text{ since } |e^{i\theta}| = 1$ as we will soon see.

In general $z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$

Power series in a complex variable

A power series in the complex variable z can be represented as

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}}$$

The series is absolutely convergent if $\sum_{n=0}^{\infty} \left|a_n e^{in\theta}\right| r^n = \sum_{n=0}^{\infty} \left|a_n\right| r^n$ is convergent

Examples of some simple power series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Let's investigate whether they converge or not!

Test for convergence of a power series

First take a look at

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}\cdots=\left(\frac{1}{2}+\frac{1}{2}\right)+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{6}\right)+\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\frac{1}{24}\right)+\cdots=$$

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

Therefore does **NOT CONVERGE**

Before the next example we introduce a few test for convergence

Limit Test $\quad \lim_{n\to\infty}|a_n|=0$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right|=\lim_{n\to\infty}\frac{|a_{n+1}||z^{n+1}|}{|a_n||z^n|}=\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}|z|<1$$

Note: When |z|=R we need to test each such case individually

Ratio Test

If we have $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\frac{1}{R}$ then we have convergence if $|z|<R$ or $-R<z<R$

where R is called the radius of convergence

Cauchy Root Test

$\sum_{n=0}^{\infty}|a_n|z^n$ converges absolutely provided $\lim_{n\to\infty}|a_n|^{\frac{1}{n}}=\frac{1}{R}$

and radius of convergence $|z|=r<R$

The next examples

For $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ we have $a_n = \frac{(-1)^n}{n}$

Then using the limit test $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0$ **CONVERGES**

For $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ using the ratio test we have

$$\lim_{n\to\infty} \frac{|a_{n+1}||z^{n+1}|}{|a_n||z^n|} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)!}\right||z^{n+1}|}{\left|\frac{1}{n!}\right||z^n|} = \lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)!}\right||z|}{\left|\frac{1}{n!}\right|} =$$

Therefore $\lim_{n\to\infty} \frac{n!|z|}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1}|z| = 0 < 1$ **CONVERGES**

Also note since $\frac{1}{R} = 0 \to R$ is $\pm\infty$, that is $|z| < \pm\infty$

or $-\infty < z < \infty$, that is it converges for all value!

Some simple functions of a complex variable

First, let's start with the Taylor expansion for the exponential function in the real domain,

that is $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ Next replacing x with iy gives us $e^{iy} = \sum_{n=0}^{\infty} \frac{1}{n!} (iy)^n = \sum_{n=0}^{\infty} \frac{1}{n!} i^n y^n$

As we have previously observed it is as relatively straight forward to generate Taylor Expansions for the sine and cosine function. They are

$$\cos y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} y^n, \ \sin y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} y^{2n+1}$$

Finally putting this all together gives us

$$e^{iy} = \sum_{n=0}^{\infty} \frac{1}{n!} i^n y^n = \frac{i^0}{0!} y^0 + \frac{i^1}{1!} y^1 + \frac{i^2}{2!} y^2 + \frac{i^3}{3!} y^3 + \frac{i^4}{4!} y^4 + \frac{i^5}{5!} y^5 + \cdots =$$

$$1 + iy + \frac{(i^2)}{2!} y^2 + \frac{i(i^2)}{3!} y^3 + \frac{(i^2)^2}{4!} y^4 + \frac{i(i^2)^2}{5!} y^5 + \cdots =$$

$$[1 - \frac{y^2}{2!} + \frac{(-1)^2}{4!} y^4 + \cdots] + i[\frac{y}{1!} + \frac{(-1)}{3!} y^3 + \frac{(-1)^2}{5!} y^5 + \cdots] =$$

$$[1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots] + i[\frac{y}{1!} - \frac{1}{3!} y^3 + \frac{1}{5!} y^5 + \cdots] = \cos y + i \sin y$$

Euler's Identity!

Now we are ready to define the complex exponential function

$$e^{z}=e^{x+iy}=e^{x}e^{iy}=e^{x}(\cos y+i\sin y)$$

Note the magnitude is $\left|e^{z}\right|=\left|e^{x+iy}\right|=\left|e^{x}e^{iy}\right|=\left|e^{x}(\cos y+i\sin y)\right|$

By definition $\left|e^{x}\cos y+ie^{x}\sin y\right|=(e^{2x}\cos^{2}y+e^{2x}\sin^{2}y)^{1/2}=(e^{2x}\cos^{2}y+e^{2x}\sin^{2}y)^{1/2}$

Finally $\boxed{\left|e^{z}\right|=(e^{2x}(\cos^{2}y+\sin^{2}y))^{1/2}=(e^{2x}(1))^{1/2}=e^{x}}$

Also if we were to plot $e^{iy}=\cos y+i\sin y$ in the Argand plane for different values of y we would quickly see that it would plot out a circle of unit radius.

KEY

Therefore in general the complex exponential functions maps out **a circle in the Argand plane** of radius e^x.

Is the complex exponential function analytic? – does it satisfy the Cauchy-Euler conditions

$$e^{z}=e^{x+iy}=e^{x}e^{iy}=e^{x}\cos y+ie^{x}\sin y$$

Therefore $u(x,y)=e^{x}\cos y$ and $v(x,y)=e^{x}\sin$

Next apply the Cauchy-Riemann conditions

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y$$

Therefore $u(x,y)=e^x \cos y$ and $v(x,y)=e^x \sin y$

Then applying the Cauchy-Riemann condition

$$\frac{\partial u}{\partial x} = \frac{\partial e^x \cos y}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial y} = \frac{\partial e^x \sin y}{\partial y} = e^x \cos y$$

and

$$\frac{\partial v}{\partial x} = \frac{\partial e^x \sin y}{\partial x} = e^x \sin y \qquad -\frac{\partial u}{\partial y} = -\frac{\partial e^x \cos y}{\partial y} = -(-e^x \sin y) = e^x \sin y$$

YES, The complex exponential function is analytic!

For non e base, that is a^z we can use the following standard “trick” from calculus

$$a^z = (e^{\ln a})^z = e^{z \ln a}$$

And we can use any relations involving base e for this case.

In the real domain, that is $x \in R$ we have Euler's identity $e^{\pm x} = \cos x \pm i \sin x$

Also

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

We can generalize these functions to be functions of a complex variable, that is

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \text{ etc.}$$

Also we can generalize the hyperbolic functions from the real to the complex domain, that is for example

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \to \cosh z = \frac{e^{z} + e^{-z}}{2}$$

Finally let's note the following relation between the function. As an example

KEY

$$\cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$