Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Take Home Project 2
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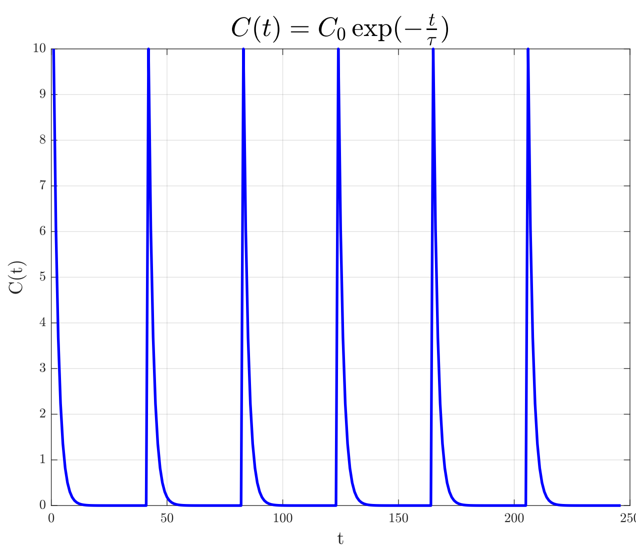
Question 1



(a)

(b) $f(t) = C_0 e^{-\frac{t}{\tau}}$ with period T, so

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} dt \\ &= \frac{2C_0}{T}(-\tau)[e^{-\frac{t}{\tau}}]_0^T \\ &= -2C_0\frac{\tau}{T}[e^{-\frac{T}{\tau}} - 1] \\ &= 2C_0\frac{\tau}{T}(1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

If $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_0 \approx 2C_0\frac{\tau}{T}$.

$$\begin{aligned} a_k &= \frac{2}{T}\int_0^T C_0 e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \\ &= \frac{2C_0}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt \end{aligned}$$

Using integration by parts with $u = \cos\frac{2k\pi t}{T}, du = -\frac{2k\pi}{T}\sin\frac{2k\pi t}{T}$ and $dv = e^{-\frac{t}{\tau}}, v = (-\tau)e^{-\frac{t}{\tau}}$:

$$\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt = (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T - \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt$$

Using again integration by parts:

$$\int_0^T e^{-\frac{t}{\tau}} \sin\frac{2k\pi t}{T} dt = (-\tau)[e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau}{T}\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt$$

So

$$\begin{aligned} (1 + (\frac{2k\pi\tau}{T}))^2\int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt &= (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T + \frac{2k\pi\tau^2}{T}[e^{-\frac{t}{\tau}}\sin\frac{2k\pi t}{T}]_0^T \\ &= (-\tau)[e^{-\frac{t}{\tau}}\cos\frac{2k\pi t}{T}]_0^T + 0 \\ &= \tau(1 - e^{-\frac{T}{\tau}}) \\ \int_0^T e^{-\frac{t}{\tau}} \cos\frac{2k\pi t}{T} dt &= \frac{\tau}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

Substituting back into the expression found for a_k yields

$$\begin{aligned} a_k &= 2C_0\frac{\tau}{T}\frac{1}{1 + (\frac{2k\pi\tau}{T})^2}(1 - e^{-\frac{T}{\tau}}) \\ &= 2C_0\frac{\tau T}{T^2 + (2k\pi\tau)^2}(1 - e^{-\frac{T}{\tau}}) \end{aligned}$$

With the same assumption $\tau \ll T$ then $e^{-\frac{T}{\tau}} \approx 0$ and $a_k \approx 2C_0 \frac{\tau}{T} \frac{1}{1+(\frac{2k\pi\tau}{T})^2}$. Similarly to compute b_k

$$\begin{aligned}
b_k &= \frac{2}{T} \int_0^T C_0 e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \int_0^T e^{-\frac{t}{\tau}} \sin \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \int_0^T e^{-\frac{t}{\tau}} \cos \frac{2k\pi t}{T} dt \\
&= \frac{2C_0}{T} \frac{2k\pi\tau}{T} \frac{\tau}{1+(\frac{2k\pi\tau}{T})^2} (1 - e^{-\frac{T}{\tau}}) \\
&= 4C_0 k\pi \frac{\tau^2}{T^2 + (2k\pi\tau)^2} (1 - e^{-\frac{T}{\tau}})
\end{aligned}$$

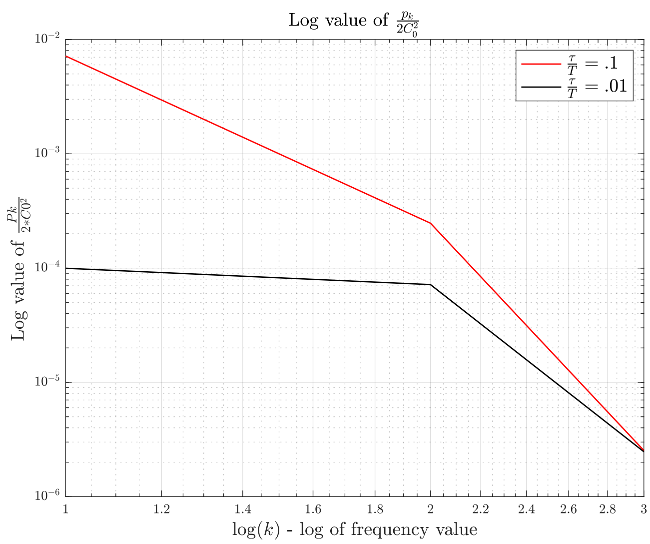
Once again, since $e^{-\frac{T}{\tau}} \approx 0$ and $b_k \approx 4C_0 (\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2} \pi k$

(c) For $k \geq 1$

$$\begin{aligned}
p_k &= \frac{1}{2}(a_k^2 + b_k^2) \\
&= \frac{1}{2}\left[4C_0^2 (\frac{\tau}{T})^2 \frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2} + 16C_0^2 (\frac{\tau}{T})^4 \frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2} \pi^2 k^2 \right] \\
&= \frac{1}{2} 4C_0^2 (\frac{\tau}{T})^2 \frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2 \right] \\
&= 2C_0^2 (\frac{\tau}{T})^2 \frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2 \right]
\end{aligned}$$

(d) We have

$$\frac{p_k}{2\,C_0^2} = (\frac{\tau}{T})^2 \frac{1}{(1+(\frac{2k\pi\tau}{T})^2)^2} \left[1 + 4(\frac{\tau}{T})^2 \pi^2 k^2 \right]$$



Looking at the plot as $\frac{\tau}{T}$ decreases, the power p_k decreases. For a greater $\frac{\tau}{T}$, the power starts at a higher value until an inflection point corresponding to frequency of ≈ 100 (10^2). Also for a higher $\frac{\tau}{T}$, the steepest the decrease in power. Eventually the two curves combine in one curve around a frequency of ≈ 1000 (10^3).

(e) As the pulse τ, becomes narrower, the power decreases linearly.

(f) We have

$$\begin{aligned} a_k \cos(\frac{k2\pi t}{T}) + b_k \sin(\frac{k2\pi t}{T}) &= \cos(\phi_k)\cos(\frac{k2\pi t}{T}) + \sin(\phi_k)\sin(\frac{k2\pi t}{T}) \\ &= \cos(\frac{k2\pi t}{T} - \phi_k) \end{aligned}$$

where

$$\begin{aligned}\tan(\phi_k) = \frac{\sin(\phi_k)}{\cos(\phi_k)} = \frac{b_k}{a_k} &= 4C_0(\frac{\tau}{T})^2 \frac{1}{1+(\frac{2k\pi\tau}{T})^2}\pi k(2C_0\frac{\tau}{T}\frac{1}{1+(\frac{2k\pi\tau}{T})^2})^{-1} \\ &= 2\frac{\tau}{T}\pi k \\ \phi_k &= \arctan(2\frac{\tau}{T}\pi k)\end{aligned}$$

For $\frac{\tau}{T} = .1$, $\phi_1 \approx 32.14°$ and $\phi_2 \approx 51.48°$ and for $\frac{\tau}{T} = .01$, $\phi_1 \approx 3.59°$ and $\phi_2 \approx 7.16°$

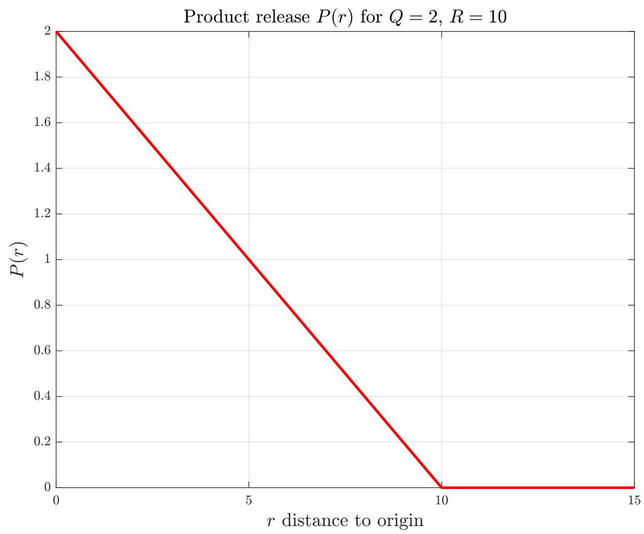
Question 2

(a) One simple way to describe $P(r)$ is to define it as $P(r) = Ar + B$ with the conditions:

$$\begin{aligned}A \cdot 0 + B = Q \\ A \cdot R + B = 0\end{aligned}$$

which gives $A = -\frac{Q}{R}$ and $B = Q$. So

$$P(r) = \begin{cases} Q(1-\frac{r}{R}) & \text{for } 0 \leq r \leq R \\ 0 & \text{for } r > R \end{cases}$$



(b) Since we assume no angular dependence: $\nabla^2 C = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dC}{dr})$, the differential equation is now:

$$\begin{aligned}\frac{D}{r^2}\frac{d}{dr}(r^2\frac{dC(r)}{dr}) + P(r) = 0\\ \frac{d}{dr}(r^2\frac{dC(r)}{dr}) = -\frac{r^2}{D}P(r)\end{aligned}$$

(c) Inside the cell $P(r) = Q(1 - \frac{r}{R})$, so we have to solve the differential equation

$$\begin{aligned}\frac{d}{dr}(r^2\frac{dC(r)}{dr}) &= -\frac{r^2}{D}Q(1 - \frac{r}{R})\\ &= \frac{Q}{DR}r^2(r - R)\\ &= \frac{Q}{DR}r^3 - \frac{Q}{D}r^2\end{aligned}$$

Integrating once

$$
\begin{aligned}
r^2 \frac{dC(r)}{dr} &= \frac{Q}{4DR} r^4 - \frac{Q}{3D} r^3 + A \\
\frac{dC(r)}{dr} &= \frac{Q}{4DR} r^2 - \frac{Q}{3D} r + \frac{A}{r^2}
\end{aligned}
$$

Integrating again

$$C_i(r) = \frac{Q}{12DR} r^3 - \frac{Q}{6D} r^2 - \frac{A}{r} + B \quad A, B\text{:constants}, C_i\text{:inside cell concentration}$$

Outside the cell $P(r) = 0$ and the we want to solve the differential equation

$$\frac{d}{dr}(r^2 \frac{dC(r)}{dr}) = 0$$

Which by integration gives

$$
\begin{aligned}
r^2 \frac{dC(r)}{dr} &= C_1 \\
\frac{dC(r)}{dr} &= \frac{C_1}{r^2} \\
C_o(r) &= -\frac{C_1}{r} + C_2 \quad C_1, C_2\text{:constants}, C_o\text{:outside cell concentration}
\end{aligned}
$$

(d) Applying the boundary conditions

(i)

$$\lim_{r \to 0} C_i(r) = \lim_{r \to 0} \frac{Q}{12DR} r^3 - \frac{Q}{6D} r^2 - \frac{A}{r} + B$$

since $\lim_{r \to 0} C_i(r) = \frac{1}{r} = \infty$ therefore to have finite concentration $C_i(r)$ at $r = 0$ we need $A = 0$

(ii)

$$\lim_{r \to \infty} C_o(r) = \lim_{r \to \infty} \left(-\frac{C_1}{r} + C_2 \right) = C_2$$

The concentration goes to zero at infinity implies $C_2 = 0$

(iii) We have now for $C_i(r)$ and $C_o(r)$:

$$
\begin{aligned}
C_i(r) &= \frac{Q}{12DR} r^3 - \frac{Q}{6D} r^2 + B \\
C_o(r) &= -\frac{C_1}{r}
\end{aligned}
$$

$C_i(R) = C_o(R)$ and $\frac{dC_i(r)}{dr} = \frac{dC_o(r)}{dr}|_{r=R}$ yields

$$
\begin{aligned}
\frac{Q}{12DR} R^3 - \frac{Q}{6D} R^2 + B &= -\frac{C_1}{R} \\
\frac{Q}{4D} R - \frac{Q}{3D} R &= \frac{C_1}{R^2}
\end{aligned}
$$

Rearranging

$$
\begin{aligned}
-\frac{Q}{12D}R^2 + B &= -\frac{C_1}{R} \\
-\frac{Q}{12D}R &= \frac{C_1}{R^2}
\end{aligned}
$$

which gives

$$
\begin{aligned}
B &= \frac{Q}{6D}R^2 \\
C_1 &= -\frac{Q}{12D}R^3
\end{aligned}
$$

substituting back

$$
\begin{aligned}
C_i(r) &= \frac{Q}{12DR}r^3 - \frac{Q}{6D}r^2 + \frac{Q}{6D}R^2 \\
C_o(r) &= \frac{Q}{12D}R^3\frac{1}{r}
\end{aligned}
$$

(e) The concentration maximum happens within the cell since $P(r)$ has maximum value Q at $r = 0$ and then it is zero for $r > R$. We are looking for the value of r for which $\frac{dC_i(r)}{dr} = 0$:

$$
\frac{dC_i(r)}{dr} = \frac{Q}{4DR}r^2 - \frac{Q}{3D}r = \frac{Q}{D}r(\frac{r}{4R} - \frac{1}{3})
$$

Discarding the solution $r = 0$ we are left that concentration maximum is for $r = \frac{4}{3}R$ and it is

$$
\begin{aligned}
C_M &= \frac{Q}{12DR}(\frac{4}{3})^3R^3 - \frac{Q}{6D}(\frac{4}{3})^2R^2 + \frac{Q}{6D}R^2 \\
&= \frac{Q}{6D}R^2\left[\frac{4^3}{2\ 3^3} - \frac{4^2}{3^2} + 1\right] \\
&= \frac{11}{162}\frac{Q}{D}R^2
\end{aligned}
$$

Inside the cell

$$
\begin{aligned}
C_i(r) &= \frac{Q}{6D}(\frac{1}{2}\frac{r^3}{R} - r^2 + R^2) \\
\frac{C_i(r)}{C_M} &= \frac{Q}{6D}\frac{162}{11}\frac{D}{Q}R^{-2}(\frac{1}{2}\frac{r^3}{R} - r^2 + R^2) \\
&= \frac{162}{6\ \ 11}\left[\frac{1}{2}(\frac{r}{R})^3 - (\frac{r}{R})^2 + 1\right] \\
&= \frac{27}{11}\left[\frac{1}{2}(\frac{r}{R})^3 - (\frac{r}{R})^2 + 1\right]
\end{aligned}
$$

And outside the cell

$$
\begin{aligned}
C_o(r) &= \frac{Q}{12D}R^3\frac{1}{r} \\
\frac{C_o(r)}{C_M} &= \frac{Q}{12D}\frac{162}{11}\frac{D}{Q}R^{-2}R^3\frac{1}{r} \\
&= \frac{27}{22}\frac{R}{r}
\end{aligned}
$$

When the diffusion constant is doubled, the curve $\frac{C_i(r)}{C_M}$ stays the same.