Interactive Assignment 11

16 pages

Problems

Yves GREATTI

Problem 11.1

Solve the following first-order partial differential equations by separating the variables:

(a) $\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$

Assume a solution of the form $u(x,y) = X(x)\,Y(y)$ substitute this expression and dividing through by $u = XY$ we obtain

$$\frac{X'}{X} - x\frac{Y'}{Y} = 0$$

$$\frac{X'}{xX} = \frac{Y'}{Y}$$

LHS function of x only and RHS is a function of y only thus for λ being a constant

$$\frac{X'}{xX} = \lambda\ , \quad \frac{Y'}{Y} = \lambda$$

$$\frac{X'}{X} = \lambda x \Rightarrow \ln X = \lambda\frac{x^2}{2} + c_1$$

$$X = A\,e^{\lambda\frac{x^2}{2}}$$

$$\frac{Y'}{Y} = \lambda \Rightarrow \ln Y = \lambda y + c_2$$

$$Y = B\,e^{\lambda y}$$ and $u(x,y) = c\,e^{\lambda\left(\frac{x^2}{2}+y\right)}$

Problem 11.1 (b)

$$x \frac{\partial u}{\partial x} - 2y \frac{\partial u}{\partial y} = 0$$

Assume a solution of the form $u(x,y) = X(x)\,Y(y)$, substitute and divide through by XY gives

$$x \frac{X'}{X} = 2y \frac{Y'}{Y}$$

$$x \frac{X'}{X} = \lambda \,, \quad \frac{X'}{X} = \frac{\lambda}{x} \quad \text{thus } \ln X = \lambda \ln x + c_1$$

$$X = A x^{\lambda}$$

$$2y \frac{Y'}{Y} = \lambda$$

$$\frac{Y'}{Y} = \frac{\lambda}{2y} \quad \text{and } \ln Y = \frac{\lambda}{2} \ln y + c_2$$

$$Y = B y^{\lambda/2}$$

$$u(x,y) = A x^{\lambda} B y^{\lambda/2} = c\, x^{\lambda} y^{\lambda/2} = c\,(x^2 y)^k$$

with $k = \lambda/2$

Problem 11.2

The diffusion equation is

$$K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 y}{\partial z^2}\right) = \frac{\partial u}{\partial t}$$

With $u(x,y,z,t) = A \cos\frac{\pi x}{a} \sin\frac{\pi z}{a} \quad e^{-\frac{2K\pi^2}{a^2}t}$

$$\frac{\partial u}{\partial x} = -A \cdot \frac{\pi}{a} \sin\frac{\pi x}{a} \sin\frac{\pi z}{a} \quad e^{-\frac{2K\pi^2}{a^2}t}$$

$$\frac{\partial^2 u}{\partial x^2} = -A\left(\frac{\pi}{a}\right)^2 \cos\frac{\pi x}{a} \sin\frac{\pi z}{a} \quad e^{-\frac{2K\pi^2}{a^2}t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial z} = A \cos\frac{\pi x}{a} \left(\frac{\pi}{a}\right) \cdot \cos\frac{\pi z}{a} \quad e^{-\frac{2K\pi^2}{a^2}t}$$

$$\frac{\partial^2 u}{\partial z^2} = -A\left(\frac{\pi}{a}\right)^2 \cos\frac{\pi x}{a} \sin\frac{\pi z}{a} \quad e^{-\frac{2K\pi^2}{a^2}t}$$

thus $K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 y}{\partial z^2}\right) = K\left(-\frac{2K\pi^2}{a^2}\right) \cdot \cos\frac{\pi x}{a} \sin\frac{\pi z}{a} \quad e^{-\frac{2K\pi^2}{a^2}t}$

$$= \frac{\partial u}{\partial t}$$

Problem 11.2

Take $u(x,y,z,t) = X(x)\,Z(z)\,T(t) = A\cos\frac{\pi x}{a}\,\sin\frac{\pi z}{a}\,e^{-\frac{2k\pi^2}{a^2}t}$

We define heat flow vie Fourier law for heat flow

$Q = -k\nabla u$, k: thermal conductivity

$$Q_x = -k\frac{\partial u}{\partial x} = -kZT\frac{dX}{dx} = kZT\left(\frac{\pi}{a}\right)\sin\frac{\pi}{a}x = 0 \text{ for } x = \pm a$$

$$Q_y = -k\frac{\partial u}{\partial y} = 0 \text{ since no } y \text{ dependency}$$

$$Q_z = -k\frac{\partial u}{\partial z} = -kXT\frac{dZ}{dz} = -kXT\left(\frac{\pi}{a}\right)\cos\frac{\pi}{a}z \neq 0 \text{ for } x = \pm a$$

Therefore there is heat flow across faces in z. (no heat flows across any other faces)

Next calculate the heat flow at $(x,y,z) = (3a/4, a/4, a)$ at time $t = a^2/k\pi^2$

$$Q_z = -k\frac{\partial u}{\partial z} = -kA\left(\frac{\pi}{a}\right)\cos\frac{\pi x}{a}\cos\frac{\pi}{a}z\;e^{-\frac{2k\pi^2}{a^2}t}$$

At $(3a/4, a/4, a)$ $\quad Q_z = -kA\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}\frac{3a}{4}\right)\cos\left(\frac{\pi a}{a}\right)e^{-\left(\frac{2k\pi^2}{a^2}\frac{a^2}{k\pi^2}\right)}$

$$= -kA\left(\frac{\pi}{a}\right)\cos\left(\frac{3\pi}{4}\right)\cos(\pi)\,e^{-2}$$

$$= -kA\frac{\pi}{a}\left(-\frac{\sqrt{2}}{2}\right)(-1)\,e^{-2} = -\frac{kA\pi e^{-2}}{a\sqrt{2}}$$ heat flow into the cube across z face

Problem 11.4

Schrodinger's equation for a non-relativistic particle in a constant potential region can be written as

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=i\hbar\frac{\partial u}{\partial t}$$

Assume $u(x,y,z,t)=X(x)\,Y(y)\,Z(z)\,T(t)$

Substitute into the Schrodinger's equation

$$-\frac{\hbar^2}{2m}\left(X''YZT+XY''ZT+XYZ''T\right)=i\hbar\,XYZT'$$

Dividing through by $XYZT$, we get

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}\right)=i\hbar\frac{T'}{T} \qquad (1)$$

Separation of variables implies that: let $\frac{X''}{X}=-k_x^2 \rightarrow X=e^{ik_x x}$

similarly $Y=e^{ik_y y}$, $Z=e^{ik_z z}$. Then $XYZ=e^{ik_x x}e^{ik_y y}e^{ik_z z}$

$$=e^{i(k\cdot r)}$$

where $k=k_x i+k_y j+k_z k$

$r=xi+yj+zk$

(i,j,k): unit direction vectors

Now substitute $\frac{X''}{X}=k_x^2$ and $\frac{Y''}{Y},\frac{Z''}{Z}$ similarly into ~~the~~ equation (1)

Problem 11.4

we have $-\frac{\hbar^2}{2m}\left(-k_x^2-k_y^2-k_z^2\right)=i\hbar\frac{T'}{T}$

Given $p=\hbar k=\hbar\left(k_x,k_y,k_z\right)=\left(\hbar k_x,\hbar k_y,\hbar k_z\right)$

The previous equation can be written as

$$-\frac{hi}{2m}\left[k_x^2+k_y^2+k_z^2\right]=\frac{T'}{T}$$

$$-\frac{hi}{2m}\left[\frac{p_x^2}{\hbar^2}+\frac{p_y^2}{\hbar^2}+\frac{p_z^2}{\hbar^2}\right]=\frac{T'}{T} \text{ or } -\frac{i}{\hbar}\left[\frac{p_x^2+p_y^2+p_z^2}{2m}\right]=\frac{T'}{T}$$

Now $\frac{p_x^2+p_y^2+p_z^2}{2m}=\frac{p^2}{2m}=E=\hbar\omega$

So $-\frac{i}{\hbar}\left[\hbar\omega\right]=\frac{T'}{T} \Rightarrow \frac{dT}{T}=-i\omega\, dt$

$$T=e^{-i\omega t}$$

Therefore $u(x,y,z,t)=A\,X(x)\,Y(y)\,Z(z)\,T(t)$

$$=A\,e^{i(k\cdot r)}e^{-i\omega t}=A\,e^{i(k\cdot r-\omega t)}$$

(b) The boundary conditions for a box of side a requires that

$u(0,y,z,t)=u(a,y,z,t)=0$, same for y and z.

thus in particular $X(0)=X(a)=0$, and similar for y and z.

Y6 7/16

Problem 11.4

For $\frac{X''}{X} = -k_x^2$ with $X(0) = X(a) = 0$, since this is a periodic boundary condition we take $X(x) = A\cos k_x x + B \sin k_x x$

The boundary condition at $x=0$ implies $X(x) = A \sin k_x x$

and at $x=a$, $X(a) = A \sin k_x a = 0 \Rightarrow k_x = \frac{n_x \pi}{a}$, similar for y and z.

As before $E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{(\hbar k_x)^2 + (\hbar k_y)^2 + (\hbar k_z)^2}{2m}$

$$= \frac{\hbar^2}{2m}\left[k_x^2 + k_y^2 + k_z^2\right]$$

$$= \frac{\hbar^2 \pi^2}{2ma^2}\left[n_x^2 + n_y^2 + n_z^2\right]$$

Problem 11.9

A circular disc of radius a is heated in such a way that its perimeter $\rho = a$ has a steady temperature distribution $A + B\cos^2\phi$, where ρ and ϕ are plane polar coordinates and A and B are constants. Find the temperature $T(\rho, \phi)$ everywhere in the region $\rho < a$.

Assume separation of variables solution: $u(\rho, \phi) = P(\rho)\Phi(\phi)$

Substitution gives

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right] P(\rho)\,\Phi(\phi) = 0$$

$$\phi(\phi)\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)P(\rho) + \frac{1}{\rho^2}P(\rho)\frac{d^2}{d\phi^2}\phi(\phi) = 0$$

Dividing by $P(\rho)\,\Phi(\phi)$ and multiplying by ρ^2

$$\frac{\rho}{P(\rho)}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)P(\rho) + \frac{1}{\phi(\phi)}\frac{d^2}{d\phi^2}\phi(\phi) = 0$$

Following the same steps described in "11D Partial Differential equations II." we eventually obtain

$$u(\rho, \phi) = D_0 + \sum_{n=1}^{\infty}(A_n\cos n\phi + B_n\sin n\phi)\,C_n\rho^n$$

Problem 11.9

Next we apply boundary condition:

$$u(a,\phi) = D_0 + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi)\, C_n a^n$$

$$= A + B\cos^2\phi$$

$$= A + B\frac{(1+\cos 2\phi)}{2} = A + \frac{B}{2} + \frac{B}{2}\cos 2\phi$$

Equating left and right hand sides gives the following relations:

$$D_0 = A + \frac{B}{2}$$

$$A_2\, C_2\, a^2 = \frac{B}{2}$$

$A_n C_n a^n = 0$ for all $n \neq 2$ and $B_n C_n = 0$ for all n

So the temperature everywhere in the region $\rho < a$ is:

$$u(\rho,\phi) = A + \frac{B}{2} + \frac{B\rho^2}{2a^2}\cos 2\phi$$

Chapter 11 - Problem 11.11

The free transverse vibrations of a thick rod satisfy the equation

$$a^4 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

Obtain a solution in separated-variable form and, for a rod clamped at one end, $x=0$, and free at the other, $x=L$, show that the angular frequency of vibration ω satisfies

$$\cosh\left(\frac{\omega^{1/2} L}{a}\right) = -\sec\left(\frac{\omega^{1/2} L}{a}\right)$$

Take $u(x,t) = X(x)\, T(t)$ and substitute into the equation

$$a^4 X^{(4)} T + X T'' = 0$$

Dividing through by $X(x)\, T(t)$ gives

$$a^4 \frac{X^{(4)}}{X} + \frac{T''}{T} = 0$$

Take as separation constant ω^2 gives separate equations in X and T:

$$a^4 \frac{X^{(4)}}{X} = \omega^2$$

$$\frac{T''}{T} = -\omega^2$$

Problem 11.11

For $\frac{X^{(4)}}{X} = \frac{\omega^2}{a^4}$, we have four roots: $\pm\frac{\sqrt{\omega}}{a}$, $\pm\frac{i\sqrt{\omega}}{a}$

Now take $\lambda = \frac{\sqrt{\omega}}{a}$, we can write the solution $X(x)$ using Euler's identity and hyperbolic functions

$$X(x) = A\sin\lambda x + B\cos\lambda x + C\sinh\lambda x + D\cosh\lambda x$$

From $\frac{T''}{T} = -\omega^2$ we have $T(t) = e^{i\omega t}$

Therefore $u(x,t) = X(x)T(t) = A\,X(x)\,e^{i\omega t}$

where $X(x) = A\sin\lambda x + B\cos\lambda x + C\sinh\lambda x + D\cosh\lambda x$

Next look at the boundary conditions $u(0,t) = 0$, $u^{(2)}(L,t) = 0$
$u'(0,t) = 0$, $u^{(3)}(L,t) = 0$

All of them can be written in terms of $X(x)$, for example

$u(0,t) = 0 = X(0)\,T(t)$ since in general $T(t) \neq 0 \rightarrow X(0) = 0$

The same relations can be derived for all the other boundary conditions: $X(0) = 0$, $X'(0) = 0$, $X''(L) = 0$, $X^{(3)}(L) = 0$

16/12/16

Problem 11.11

Take the first one, $X(0)=0$, gives:

$$A\cdot 0+B\cdot 1+C\cdot 0+D\cdot 1=0 \quad \text{so } D=-B$$

So $X(x)=A\sin\lambda x+C\sinh\lambda x+B(\cos\lambda x-\cosh\lambda x)$

$$X'(x)=\lambda\left[A\cos\lambda x+C\cosh\lambda x-B(\sin\lambda x+\sinh\lambda x)\right]$$

$$X'(0)=0\rightarrow\lambda\left[A\cdot 1+C\cdot 1-B\cdot 0\right]=0 \quad \text{thus } C=-A$$

So $X(x)=A(\sin\lambda x-\sinh\lambda x)+B(\cos\lambda x-\cosh\lambda x)$

$$X'(x)=\lambda\left[A(\cos\lambda x-\cosh\lambda x)-B(\sin\lambda x+\sinh\lambda x)\right]$$

$$X''(x)=(-\lambda^2)\left[A(\sin\lambda x+\sinh\lambda x)+B(\cos\lambda x+\cosh\lambda x)\right]$$

$$X^{(3)}(x)=-\lambda^3\left[A(\cos\lambda x+\cosh\lambda x)+B(-\sin\lambda x+\sinh\lambda x)\right]$$

$X''(L)=0$ and $X^{(3)}(L)=0$ gives

$$A(\sin\lambda L+\sinh\lambda L)+B(\cos\lambda L+\cosh\lambda L)=0 \qquad (1)$$

$$-A(\cos\lambda L+\cosh\lambda L)+B(\sin\lambda L-\sinh\lambda L)=0 \qquad (2)$$

From (1): $\sin\lambda L+\sinh\lambda L=-\dfrac{B}{A}(\cos\lambda L+\cosh\lambda x)$

From (2): $\sin\lambda L-\sinh\lambda L=\dfrac{A}{B}(\cos\lambda L+\cosh\lambda L)$

Y6 13/16

Problem 11.11

Combining $\sin\lambda L + \sinh\lambda L = -\dfrac{(\cos\lambda L + \cosh\lambda L)^2}{\sin\lambda L + \sinh\lambda L}$

or $(\sin\lambda L)^2 - (\sinh\lambda L)^2 = -(\cos\lambda L + \cosh\lambda L)^2$

$= -(\cos\lambda L^2 + \cosh\lambda L^2 + 2\cos\lambda L\cosh\lambda L)$

So $2\cos\lambda L\cosh\lambda L = -((\cos\lambda L)^2 + (\sin\lambda L)^2) + (\sinh\lambda)^2 - (\cosh\lambda L)^2$

$= -(\cos\lambda L^2 + \sin\lambda L^2) - (\cosh\lambda^2 - \sinh\lambda L^2)$

we know $\cos\lambda L^2 + \sin\lambda L^2 = 1$, $\cosh\lambda^2 - \sinh\lambda L^2 = 1$

So $2\cos\lambda L\cosh\lambda L = -2 \rightarrow \cosh\lambda L = -\dfrac{1}{\cos\lambda L}$

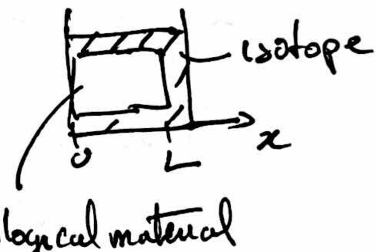
$= -\sec\lambda L$

Therefore $\cosh\left(\dfrac{\omega^{1/2}L}{a}\right) = -\sec\left(\dfrac{\omega^{1/2}L}{a}\right)$

4/6/14/16

Problem 11.16

A slice of biological material of thickness L is placed into a solution of a radioactive isotope of constant concentration C_0 at time $t=0$. For a later time t, find the concentration of radioactive ions at a depth x inside of its surfaces if the diffusion constant is K.



Diffusion equation in one dimension is

$$K \nabla^2 u(x,t) = \frac{\partial}{\partial t} u(x,t)$$

Substituting $u(x,t) = X(x)\,T(t)$ into the PDE gives

$$K X'' T = X T'$$

and dividing by XT, we get $\quad \frac{T'}{T} = K \frac{X''}{X}$

separation of variables gives

$$K \frac{X''}{X} = \frac{T'}{T} = -\lambda^2 \quad \text{or} \quad X'' + \frac{\lambda^2}{K} X = 0$$

$$T' + \lambda^2 T = 0$$

4/6/15/16

Problem 11.16

Equations in X and T gives us solutions:

$$X(x) = B \sin \frac{d}{\sqrt{K}} x + C \cos \frac{d}{\sqrt{K}} x$$
$$T(t) = A e^{-d t^2}$$

After infinite time we have isotopic decay, $\lim_{t \to \infty} T(t) = 0$, and concentration C_0 is everywhere within the biological material

So $u(x,t) = C_0 + A e^{-d^2 t} \left[B \sin \frac{d}{\sqrt{K}} x + C \cos \frac{d}{\sqrt{K}} x \right]$

which also implies that: $u(0,t) = u(L,t) = C_0$ and

$X(0) = X(L) = 0$: So $\quad B \cdot 0 + C \cdot 1 = 0$ thus $C = 0$

$B \cdot \sin \frac{d}{\sqrt{K}} L = 0$ withowout $B = 0$

so we must have $\sin \frac{d}{\sqrt{K}} L = 0$. This occurs only when $\frac{d}{\sqrt{K}} L = n\pi$

or $d = \frac{n\pi\sqrt{K}}{L}$

Combining the constants A and C into A_n we write

$$u_n(x,t) = A_n \, e^{-d_n^2 t} \sin \frac{d_n x}{\sqrt{K}}$$

Next we use the superposition property to write:

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} u_n(x,t) = C_0 + \sum_{n=1}^{\infty} A_n \cdot \sin \frac{d_n x}{\sqrt{K}} \cdot e^{-d_n^2 t}$$

Problem 11.16

At $t=0$ before any diffusion happens:

$$u(x,0) = 0 = C_0 + \sum_{n=1}^{\infty} A_n \sin \frac{\lambda_n x}{\sqrt{K}}$$

thus $\sum_{n=1}^{\infty} A_n \sin \frac{\lambda_n x}{\sqrt{K}} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right) = -C_0$

This is a Fourier series therefore

$$A_n = \frac{2}{L} \int_0^L (-C_0) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$= -\frac{2C_0}{L} \int_0^L \sin\left(\frac{n\pi}{L} x\right) dx$$

$$= \left(-\frac{2C_0}{L}\right)\left(-\frac{L}{n\pi}\right) \left[\cos \frac{n\pi}{L} x\right]_0^L$$

$$= \frac{2C_0}{n\pi}\left((-1)^n - 1\right)$$

$$= \begin{cases} -\frac{4C_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Finally $u(x,t) = C_0 - \frac{4C_0}{\pi} \sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{L} x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Kt}$

$$= C_0 \left[1 - \frac{4}{\pi} \sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{L} x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Kt} \right]$$