Interactive Assigment 9

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Problems

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Problem 9.1

For $l=0$ $Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$ and $|Y_0^0(\theta,\varphi)|^2 = \frac{1}{4\pi} = \frac{2\times 0+1}{4\pi}$

For $l=1$ $|Y_1^{-1}(\theta,\phi)|^2 = \left|\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{+i\phi}\right|^2$

$$= \frac{3}{8\pi} \sin^2\theta$$

$$|Y_1^0(\theta,\phi)|^2 = \left|\left(\frac{3}{4\pi}\right)^{1/2} \cos\theta\right|^2 = \frac{3}{4\pi}\cos^2\theta$$

$$|Y_1^1(\theta,\phi)|^2 = \left|\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}\right|^2$$

$$= \frac{3}{8\pi} \sin^2\theta$$

$$|Y_1^{-1}(\theta,\phi)|^2 + |Y_1^0(\theta,\phi)|^2 + |Y_1^1(\theta,\phi)|^2 =$$

$$2\times\frac{3}{8\pi}\sin^2\theta + \frac{3}{4\pi}\cos^2\theta = \frac{3}{\cancel{8}4\pi}(\sin^2\theta+\cos^2\theta) = \frac{3}{4\pi} = \frac{2\times 1+1}{4\pi}$$

For $l=2$

$$|Y_2^{-2}|^2 = |Y_2^2|^2 = \frac{15}{32\pi}\sin^4\theta$$

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Problem 9.1

$$|Y_2^{\pm 1}|^2 = \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

$$|Y_2^0|^2 = \frac{5}{16\pi}(3\cos^2\theta - 1)^2 = \frac{5}{16\pi}(9\cos^4\theta - 6\cos^2\theta + 1)$$

In terms of power of $\sin\theta$ and $\cos\theta$, we have:

$\sin^4\theta$: $\frac{2\times 15}{32\pi} = \frac{15}{16\pi}$

$\cos^4\theta$: $\frac{5\times 9}{16\pi}$

$\sin^2\theta\cos^2\theta$: $\frac{2\times 15}{8\pi}$

$\cos^2\theta$: $-\frac{5\times 6}{16\pi} = -\frac{5\times 3}{8\pi}$

1: $\frac{5}{16\pi}$

Take term in $\cos^2\theta$: $1 \times \frac{5\times 3}{8\pi}\cos^2\theta = \frac{5\times 3}{8\pi}(\cos^2\theta + \sin^2\theta)\cos^2\theta$

We now have these coefficients in term of power of $\sin\theta$ and $\cos\theta$:

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Problem 9.1

(1) $\sin^4\theta$: $\frac{15}{16\pi}$

(2) $\cos^4\theta$: $\frac{5\times 9}{16\pi} - \frac{5\times 3}{8\pi} = \frac{5\times 3}{16\pi}$

(3) $\sin^2\theta\cos^2\theta$: $\frac{2\times 15}{8\pi} - \frac{5\times 3}{8\pi} = \frac{15}{8\pi}$

(4) $\frac{1}{4}$: $\frac{5}{16\pi}$

$$\frac{15}{16\pi}\left(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta\right) = \frac{15}{16\pi} \qquad (1)+(2)+2\times(3)$$

[which is $(\sin^2\theta+\cos^2\theta)^2$]

And: $(1)+(2)+2\times(3)+(4) = \frac{15}{16\pi} + \frac{5}{16\pi} = \frac{20}{16\pi} = \frac{2\times 2+1}{4\pi}$

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Problem 9.3

Use the generating function for the Legendre polynomials $P_n(x)$ to show that

$$\int_0^1 P_{2n+1}(x)\,dx = \frac{(-1)^n\ (2n)!}{2^{2n+1} n!\ (n+1)!}$$

and that, except for the case $n=0$,

$$\int_0^1 P_{2n}(x)\,dx = 0$$

From $\quad G(x,h) = (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,h^n$

Integrating both sides between 0 and 1 gives

$$\int_0^1 (1-2xh+h^2)^{-1/2}\,dx = \int_0^1 \Big(\sum_{n=0}^{\infty} P_n(x)\,h^n\Big)\,dx$$

$$= \sum_{n=0}^{\infty} \Big(\int_0^1 P_n(x)\,dx\Big)\,h^n$$

$$\int_0^1 (1-2xh+h^2)^{-1/2}\,dx = -\frac{1}{h}\Big[(1+h^2-2hx)^{1/2}\Big]_0^1$$

$$= -\frac{1}{h}\Big[(1+h^2-2h)^{1/2} - (1+h^2)^{1/2}\Big]$$

Problem 3.3

So

$$\int_0^1 (1-2hx+h^2)^{-1/2}dx = -\frac{1}{h}\left[(h-1)^{2/2} - (1+h^2)^{1/2}\right]$$

$$(1+x)^{1/2} = \sum_{m=0}^{\infty} C_m^{1/2} x^m$$
$$= 1 + \sum_{m=1}^{\infty} C_m^{1/2} x^m$$

Next $\int_0^1 (1-2hx+h^2)^{-1/2}dx = -\frac{1}{h}\left[(1-h)^{2/2} - (1+h^2)^{1/2}\right]$

$$= -\frac{1}{h}\left[1-h-\left(1+\sum_{m=1}^{\infty} C_m^{1/2} h^{2m}\right)\right]$$

$$= -\frac{1}{h}\left[-h-\sum_{m=1}^{\infty} C_m^{1/2} h^{2m}\right]$$

$$= 1 + \sum_{m=1}^{\infty} C_m^{1/2} h^{2m-1}$$

Equating the coefficients of h^m we obtain that all the ~~exa~~ integrals of the even Legendre polynomials are zero except for h^0: $\int_0^1 P_0(x)\,dx = 1$

$$\int_0^1 P_{2m}(x)\,dx = 0 \quad \text{for } m \neq 0.$$

This leaves

$$\int_0^1 P_{2m-1}(x)\,dx = C_m^{1/2}$$

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Problem 9.3

Reindexing $n \to n+1$ we obtain

$$\int_0^1 P_{2n+1}(x)\,dx = C_{n+1}^{1/2}$$

$$= \frac{(-1)^{n+1-1}\, 2(n+1)-2\,!}{2^{2(n+1)-1}(n+1)!\,(n+1-1)!}$$

$$= \frac{(-1)^{n}\; 2(n+1-1)\,!}{2^{2n+2-1}\,(n+1)!\; n!}$$

$$= \frac{(-1)^{n}\; 2n\,!}{2^{2n+1}\; n!\;(n+1)!}$$

Problem 9.5

The Hermite polynomials $H_n(x)$ may be defined by

$$\phi(x,h) = \exp(2xh - h^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, h^n$$

$$\frac{\partial \phi(x,h)}{\partial x} = \frac{\partial}{\partial x} \exp(2xh-h^2) = 2h\, e^{2xh-h^2}$$

$$\frac{\partial^2 \phi(x,h)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} \exp(2xh-h^2)\right) = 4h^2\, e^{2xh-h^2}$$

$$\frac{\partial \phi(x,h)}{\partial h} = \frac{\partial}{\partial h} \exp(2xh-h^2) = 2(x-h)\, e^{2xh-h^2}$$

So

$$\frac{\partial^2 \phi}{\partial x^2} - 2x \frac{\partial \phi}{\partial x} + 2h \frac{\partial \phi}{\partial h}$$

$$= 4h^2 e^{2xh-h^2} - 4xh\, e^{2xh-h^2} + 4h(x-h)\, e^{2xh-h^2}$$

$$= \left[4h^2 - 4xh + 4xh - 4h^2\right] e^{2xh-h^2}$$

$$= 0 \cdot e^{2xh-h^2} = 0$$

thus $$\frac{\partial^2 \phi}{\partial x^2} - 2x \frac{\partial \phi}{\partial x} + 2h \frac{\partial \phi}{\partial h} = 0 \qquad (1)$$

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Problem 9.5

$$\frac{\partial}{\partial x} \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, h^n = \sum_{n=0}^{\infty} \frac{1}{n!} H_n'(x)\, h^n$$

$$\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, h^n = \sum_{n=0}^{\infty} \frac{1}{n!} H_n''(x)\, h^n$$

Next $$\frac{\partial}{\partial h} \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, h^n = \sum_{n=0}^{\infty} \frac{1}{n!}\, n\, H_n(x)\, h^{n-1}$$

Substitute back into equation (a):

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n''(x)\, h^n - 2x \sum_{n=0}^{\infty} \frac{1}{n!} H_n'(x)\, h^n + 2h \sum_{n=0}^{\infty} \frac{1}{n!}\, n\, H_n(x)\, h^{n-1} = 0$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left[H_n''(x) - 2x\, H_n'(x) + 2n\, H_n(x) \right] h^n = 0$$

Since this last equation is verified for any power of h^n

we obtain : $H_n''(x) - 2x\, H_n'(x) + 2n\, H_n(x) = 0$ for any $n \geq 0$

$H_n(x)$ satisfy the Hermite equation:

$$y'' - 2x\, y' + 2n\, y = 0$$

Problem 9.5

We have $\frac{\partial \phi}{\partial x} - 2h\,\phi = 2he^{2xh-h^2} - 2he^{2xh-h^2} = 0.$

And $\frac{\partial \phi}{\partial x} = \sum_{n=0}^{\infty} \frac{1}{n!} H'_n(x)\, h^n$

But $H_0(x) = 1$ and $H'_0(x) = 0$

Reindexing we have $\frac{\partial \phi}{\partial x} = \sum_{n=1}^{\infty} \frac{1}{n!} H'_n(x)\, h^n$

Substitute back in the equation above yields

$$\sum_{n=1}^{\infty} \frac{1}{n!} H'_n(x)\, h^n - 2h \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, h^n = 0$$

Reindexing the second sum gives

$$\sum_{n=1}^{\infty} \frac{1}{n!} H'_n(x)\, h^n - 2 \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, h^{n+1} = 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n!} H'_n(x)\, h^n - 2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_{n-1}(x)\, h^n = 0$$

For $n \geq 1$, ~~readh~~ collecting the terms and since the equation is verified for any power of h:

$$\sum_{n=1}^{\infty} \left[\frac{1}{n!} H'_n(x) - 2 \frac{1}{n-1!} H_{n-1}(x)\right] h^n = 0$$

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Problem 5.3

So $\frac{1}{n!} H'_n(x) - \frac{2}{(n-1)!} H_{n-1}(x) = 0$

$$\frac{1}{n!}\left[H'_n(x) - 2n H_{n-1}(x)\right] = 0$$

a) $$H'_n(x) = 2n\, H_{n-1}(x)$$

b) $H_{n+1}(x)$ satisfies the Hermite equation:

$$H''_{n+1}(x) - 2x H'_{n+1}(x) + 2(n+1) H_{n+1}(x) = 0$$

From (a) : $H'_{n+1}(x) = 2(n+1) H_n(x)$

And differentiating $H''_{n+1}(x) = 2(n+1) H'_n(x)$

Substituting back into the Hermite equation above gives:

$$2(n+1) H'_n(x) - 2x\left(2(n+1) H_n(x)\right) + 2(n+1) H_{n+1}(x) = 0$$

Dividing through by $2(n+1)$ yields

$$H'_n(x) - 2x H_n(x) + H_{n+1}(x) = 0$$

but $H'_n(x) = 2n\, H_{n-1}(x)$

therefore $H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0$

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Problem 3.8

The generating function for the polynomial is

$$G(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n$$

With $H_0(x) = 1$ we have

$$\frac{\partial}{\partial h} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n = \sum_{n=1}^{\infty} \frac{H_n(x)}{n!} n\, h^{n-1} = \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} h^{n-1}$$

$$= \sum_{n=0}^{\infty} \frac{H_{n+1}(x)}{n!} h^n$$

and $H_1(x) = \left.\frac{\partial G(x,h)}{\partial h}\right|_{h=0}$

similarly $\frac{\partial^2}{\partial h^2} \sum_{n=0}^{\infty} \frac{H_n(x)\, h^n}{n!} = \sum_{n=0}^{\infty} \frac{H_{n+2}(x)}{n!} h^n$

$$H_2(x) = \left.\frac{\partial^2 G(x,h)}{\partial h^2}\right|_{h=0}$$

In General $H_n(x) = \left.\frac{\partial^n G(x,h)}{\partial h^n}\right|_{h=0}$

$$\frac{\partial G(x,h)}{\partial h} = (2x - 2h)\, e^{2xh-h^2}$$

$$= 2(x-h)\, e^{h(2x-h)}$$

$$H_1(x) = \left.\frac{\partial G(x,h)}{\partial h}\right|_{h=0} = 2x$$

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$$\frac{\partial^2 G(x,h)}{\partial h^2} = \frac{\partial}{\partial h}\left[\frac{\partial G(x,h)}{\partial h}\right]$$

$$= 2\left[- e^{h(2x-h)} + 2(x-h)^2 e^{h(2x-h)}\right]$$

$$= 2 e^{h(2x-h)} \left[-1 + 2(x-h)^2\right]$$

So

$$H_2(x) = \left.\frac{\partial^2 G(x,h)}{\partial h^2}\right|_{h=0} = 2\left[-1 + 2x^2\right] = 4x^2 - 2$$

$$\frac{\partial^3 G(x,h)}{\partial h^3} = \frac{\partial}{\partial h}\left[\frac{\partial^2 G(x,h)}{\partial h^2}\right]$$

$$= 2\left[-4 \times (x-h) e^{h(2x-h)} + (-1+2(x-h)^2) \times 2(x-h) e^{h(2x-h)}\right]$$

$$= 4(x-h) e^{h(2x-h)} \left[-2 + (-1+2(x-h)^2)\right]$$

$$= 4(x-h) e^{h(2x-h)} (-3 + 2(x-h)^2)$$

$$= 4(x-h)(-3+2(x-h)^2) e^{h(2x-h)}$$

So $H_3(x) = \left.\frac{\partial^3 G(x,h)}{\partial h^3}\right|_{h=0} = 4x(-3+2x^2) = -12x + 8x^3$

$$\frac{\partial^4 G(x,h)}{\partial h^4} = \frac{\partial}{\partial h}\left[\frac{\partial^3 G(x,h)}{\partial h^3}\right]$$

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$$\frac{\partial^4 G(x,h)}{\partial h^4} = 4e^{h(2x-h)}\Big[(-1)\left(-3+2(x-h)^2\right)+(x-h)(-4)\times(x-h)$$
$$+(x-h)\left(-3+2(x-h)^2\right)\times 2(x-h)\Big]$$

$$\vdots$$

$$= 4e^{h(2x-h)}\left[3-12(x-h)^2+4(x-h)^4\right]$$

And $H_4(x) = \left.\frac{\partial^4 G(x,h)}{\partial h^4}\right|_{h=0} = 4\times(3-12x^2+4x^4)$

$$= 16x^4 - 48x^2 + 12$$

(b) $\int_{-\infty}^{\infty} e^{-x^2} H_2(x)\, H_3(x)\, dx$

$$= \int_{-\infty}^{\infty} e^{-x^2}(4x^2-2)(-12x+8x^3)\, dx$$

$$= 8\int_{-\infty}^{\infty} e^{-x^2}(3x-8x^3+4x^5)\, dx$$

We have to consider 3 integrals:

$$\int_{-\infty}^{\infty} e^{-x^2} x\, dx, \quad \int_{-\infty}^{\infty} e^{-x^2} x^3\, dx, \quad \int_{-\infty}^{\infty} e^{-x^2} x^5\, dx$$

Since $e^{-x^2}x$, $e^{-x^2}x^3$ and $e^{-x^2}x^5$ are all odd functions and the interval $(-\infty,\infty)$ is symmetric about 0, all these integrals are zero and so: $\int_{-\infty}^{\infty} e^{-x^2} H_2(x)\, H_3(x)\, dx = 0$

Problem 9.8

Next we need to compute:

$$\int_{-\infty}^{\infty} e^{-x^2} H_2(x)\, H_4(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} (4x^2-2) \times 4 \times (3-12x^2+4x^4)\, dx$$

$$= 8\int_{-\infty}^{\infty} e^{-x^2} (2x^2-1) \; (4x^4-12x^2+3)\, dx$$

$$= 8\int_{-\infty}^{\infty} e^{-x^2} (8x^6-28x^4+18x^2-3)\, dx$$

We have four integrals to compute:

$$\int_{-\infty}^{\infty} e^{-x^2} x^6 dx, \int_{-\infty}^{\infty} e^{-x^2} x^4 dx, \int_{-\infty}^{\infty} e^{-x^2} x^2 dx \text{ and } \int_{-\infty}^{\infty} e^{-x^2} dx$$

Applying: $\displaystyle\int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = \frac{(2n)!\sqrt{\pi}}{2^{2n}\, n!}$

we find:

$$\int_{-\infty}^{\infty} x^6 e^{-x^2} dx = \frac{3\times 5}{2^3}\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{2^2}$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \text{ and } \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

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Problem 9.8

$$8\times\int_{-\infty}^{\infty} e^{-x^2}x^6\,dx - 28\int_{-\infty}^{\infty} e^{-x^2}x^4\,dx + 18\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx - 3\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= 8\times\frac{3\times5\sqrt{\pi}}{2^3} - \frac{28\times3\sqrt{\pi}}{2^2} + 18\times\frac{\sqrt{\pi}}{2} - 3\sqrt{\pi}$$

$$= \sqrt{\pi}\,(15-21+9-3) = \sqrt{\pi}\cdot 0 = 0$$

So $\int_{-\infty}^{\infty} e^{-x^2}H_2(x)\,H_4(x)\,dx = 0$

For (i) and (ii) we did obtain the expected values $2^p p!\sqrt{\pi}\,\delta_{pq} = 0$ when $p \neq q$.

Next we need to calculate

$$\int_{-\infty}^{\infty} e^{-x^2}H_3(x)\,H_3(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2}(8x^3-12x)(8x^3-12x)\,dx$$

$$= 16\int_{-\infty}^{\infty} e^{-x^2}(2x^3-3x)^2\,dx = 16\int_{-\infty}^{\infty} e^{-x^2}x^2(2x^2-3)^2\,dx$$

$$= 16\int_{-\infty}^{\infty} e^{-x^2}(4x^6-12x^4+9x^2)\,dx$$

Reusing the previous integral values for $\int_{-\infty}^{\infty} e^{-x^2}x^6\,dx$, $\int_{-\infty}^{\infty} e^{-x^2}x^4\,dx$ and $\int_{-\infty}^{\infty} e^{-x^2}x^2\,dx$ we find:

Problem 9.8

$$\int_{-\infty}^{\infty} e^{-x^2} H_3(x)\, H_3(x)\, dx = 2^4 \left[4 \times \frac{3\times 5}{2^3} - 12 \times \frac{3}{2^2} + \frac{9}{9} \right] \sqrt{\pi}$$

$$= 2^4 \cdot 3 \sqrt{\pi} = 2^3 \cdot 3! \sqrt{\pi}$$

Problem 9.15

(a)

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

change of variable $u = \sqrt{t}$, $du = \frac{1}{2} t^{-1/2} dt$

$$erf(x^2) = \frac{2}{\sqrt{\pi}} \int_0^{x^2} e^{-t} \frac{t^{-1/2}}{2} dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{x^2} e^{-t} t^{-1/2} dt$$

$$= \frac{1}{\Gamma(1/2)} \int_0^{x^2} t^{\frac{1}{2}-1} e^{-t} dt$$

$$= P(1/2, x^2)$$

(b) $C(x) = \int_0^x \cos\left(\frac{\pi}{2} t^2\right) dt$, $S(x) = \int_0^x \sin\left(\frac{\pi}{2} t^2\right) dt$

$$erf\left[\frac{\sqrt{\pi}}{2}(1-i)x\right] = \frac{2}{\sqrt{\pi}} \int_0^{\frac{\sqrt{\pi}}{2}(1-i)x} e^{-u^2} du$$

let make the change of variable $u = \frac{1}{2}\sqrt{\pi}(1-i)s$

$du = \frac{1}{2}\sqrt{\pi}(1-i)ds$

and $s = \frac{2}{\sqrt{\pi}(1-i)} u$

$$u^2 = \frac{1}{4} \times \pi (-2i) s^2 = -\frac{i\pi}{2} s^2$$

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So $\text{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right] = \frac{2}{\sqrt{\pi}}\int_0^x e^{\frac{i\pi}{2}s^2}\frac{\sqrt{\pi}}{2}(1-i)\,ds$

$$= (1-i)\int_0^x e^{i\frac{\pi}{2}s^2}\,ds$$

$$= (1-i)\left[\int_0^x \cos\frac{\pi}{2}s^2\,ds + i\int_0^x \sin\frac{\pi}{2}s^2\,ds\right]$$

Multiplying both sides by $\frac{1+i}{2}$:

$$\frac{1+i}{2}\,\text{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right] = \int_0^x \cos\frac{\pi}{2}s^2\,ds + i\int_0^x \sin\frac{\pi}{2}s^2\,ds$$

$$= C(x) + i\,S(x)$$

From part a: $C(x) + i\,S(x) = \frac{1+i}{2}\,\text{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right]$

$$= \frac{(1+i)}{2} \times P\left(1/2, \left[\frac{\sqrt{\pi}}{2}(1-i)x\right]^2\right)$$

$$= \frac{(1+i)}{2}\,P\left(1/2, -i\frac{\pi}{2}x^2\right)$$