11.11

Take $u(x,t)=Ae^{mx+i\omega t} \rightarrow \frac{\partial^4}{\partial x^4}Ae^{mx+i\omega t}=m^4 Ae^{mx+i\omega t}$ and $\frac{\partial^2}{\partial t^2}Ae^{mx+i\omega t}=(i\omega)^2 Ae^{mx+i\omega t}$

Substitute into $a^4\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}=0 \rightarrow a^4 m^4 Ae^{mx+i\omega t}+(i\omega)^2 Ae^{mx+i\omega t}=[a^4 m^4+(i\omega)^2]Ae^{mx+i\omega t}=0$

Therefore $a^4 m^4+(i\omega)^2=0$ or $m^4=-\frac{(i\omega)^2}{a^4}=\frac{\omega^2}{a^4}\rightarrow$ (we expect 4 roots) $m=\pm\frac{\sqrt{\omega}}{a},\pm\frac{i\sqrt{\omega}}{a}$

Now take $\lambda=\frac{\sqrt{\omega}}{a}$ then $m=\pm\lambda,\pm i\lambda$

and we write the solution as a superposition $u(x,t)=[\tilde{A}e^{i\lambda x}+\tilde{B}e^{-i\lambda x}+\overleftrightarrow{C}e^{\lambda x}+\tilde{D}e^{-\lambda x}]e^{i\omega t}=$

$=[\tilde{A}e^{i\frac{\sqrt{\omega}}{a}x}+\tilde{B}e^{-i\frac{\sqrt{\omega}}{a}x}+\overleftrightarrow{C}e^{\frac{\sqrt{\omega}}{a}x}+\tilde{D}e^{-\frac{\sqrt{\omega}}{a}x}]e^{i\omega t}=X(x)T(t)$ where $X(x)=\tilde{A}e^{i\frac{\sqrt{\omega}}{a}x}+\tilde{B}e^{-i\frac{\sqrt{\omega}}{a}x}+\overleftrightarrow{C}e^{\frac{\sqrt{\omega}}{a}x}+\tilde{D}e^{-\frac{\sqrt{\omega}}{a}x}$

Now using our Euler identity and definition for hyperbolic functions we have

$$X(x)=A\sin\frac{\sqrt{\omega}}{a}x+B\cos\frac{\sqrt{\omega}}{a}x+C\sinh\frac{\sqrt{\omega}}{a}x+D\cosh\frac{\sqrt{\omega}}{a}x$$

At the clamped end $u(0,t)=0$ and $\left.\frac{\partial u}{\partial x}\right|_{x=0}=0$ and at free end $\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L}=\left.\frac{\partial^3 u}{\partial x^3}\right|_{x=L}=0$

as usual leads to $X(0)=0$ and $\left.\frac{dX}{dx}\right|_{x=0}=0$ and $\left.\frac{d^2X}{dx^2}\right|_{x=L}=\left.\frac{d^3X}{dx^3}\right|_{x=L}=0$

Let's apply them

$$X(0)=A\sin\frac{\sqrt{\omega}}{a}0+B\cos\frac{\sqrt{\omega}}{a}0+C\sinh\frac{\sqrt{\omega}}{a}0+D\cosh\frac{\sqrt{\omega}}{a}0=B+D=0\rightarrow D=-B$$

$$\frac{\partial X}{\partial x}=A\frac{\sqrt{\omega}}{a}\cos\frac{\sqrt{\omega}}{a}x-B\frac{\sqrt{\omega}}{a}\sin\frac{\sqrt{\omega}}{a}x+C\frac{\sqrt{\omega}}{a}\cosh\frac{\sqrt{\omega}}{a}x+D\frac{\sqrt{\omega}}{a}\sinh\frac{\sqrt{\omega}}{a}x$$

Then $\left.\frac{\partial X}{\partial x}\right|_{x=0}=A\frac{\sqrt{\omega}}{a}\cos\frac{\sqrt{\omega}}{a}0-B\frac{\sqrt{\omega}}{a}\sin\frac{\sqrt{\omega}}{a}0+C\frac{\sqrt{\omega}}{a}\cosh\frac{\sqrt{\omega}}{a}0+D\frac{\sqrt{\omega}}{a}\sinh\frac{\sqrt{\omega}}{a}0=$

$A\frac{\sqrt{\omega}}{a}+C\frac{\sqrt{\omega}}{a}=0\rightarrow C=-A$

Therefore $X(x)=A\sin\frac{\sqrt{\omega}}{a}x+B\cos\frac{\sqrt{\omega}}{a}x-A\sinh\frac{\sqrt{\omega}}{a}x-B\cosh\frac{\sqrt{\omega}}{a}x$

as usual leads to $X(0)=0$ and $\left.\frac{dX}{dx}\right|_{x=0}=0$ and $\left.\frac{d^2X}{dx^2}\right|_{x=L}=\left.\frac{d^3X}{dx^3}\right|_{x=L}=0$

Let's apply the free end boundary conditions

Start with derivatives

$$X^{(2)}(x) = -A\frac{\omega}{a^2}\sin\frac{\sqrt{\omega}}{a}x - B\frac{\omega}{a^2}\cos\frac{\sqrt{\omega}}{a}x - A\frac{\omega}{a^2}\sinh\frac{\sqrt{\omega}}{a}x - B\frac{\omega}{a^2}\cosh\frac{\sqrt{\omega}}{a}x$$

$$X^{(3)}(x) = -A\left(\frac{\sqrt{\omega}}{a}\right)^3\cos\frac{\sqrt{\omega}}{a}x - B\left(\frac{\sqrt{\omega}}{a}\right)^3\cos\frac{\sqrt{\omega}}{a}x - A\left(\frac{\sqrt{\omega}}{a}\right)^3\cosh\frac{\sqrt{\omega}}{a}x - B\left(\frac{\sqrt{\omega}}{a}\right)^3\sinh\frac{\sqrt{\omega}}{a}x$$

Then $\left.\frac{\partial^2X}{\partial x^2}\right|_{x=L} = -A\frac{\omega}{a^2}\sin\frac{\sqrt{\omega}}{a}L - B\frac{\omega}{a^2}\cos\frac{\sqrt{\omega}}{a}L - A\frac{\omega}{a^2}\sinh\frac{\sqrt{\omega}}{a}L - B\frac{\omega}{a^2}\cosh\frac{\sqrt{\omega}}{a}L = 0$

$$\left.\frac{d^3X}{dx^3}\right|_{x=L} = -A\left(\frac{\sqrt{\omega}}{a}\right)^3\cos\frac{\sqrt{\omega}}{a}L + B\left(\frac{\sqrt{\omega}}{a}\right)^3\sin\frac{\sqrt{\omega}}{a}L - A\left(\frac{\sqrt{\omega}}{a}\right)^3\cosh\frac{\sqrt{\omega}}{a}L - B\left(\frac{\sqrt{\omega}}{a}\right)^3\sinh\frac{\sqrt{\omega}}{a}L = 0$$

Which gives for the second order $A\left[\frac{\omega}{a^2}\sin\frac{\sqrt{\omega}}{a}L + \frac{\omega}{a^2}\sinh\frac{\sqrt{\omega}}{a}L\right] = -B\left[\frac{\omega}{a^2}\cos\frac{\sqrt{\omega}}{a}L + \frac{\omega}{a^2}\cosh\frac{\sqrt{\omega}}{a}L\right] \rightarrow$

$A\left[\sin\frac{\sqrt{\omega}}{a}L + \sinh\frac{\sqrt{\omega}}{a}L\right] = -B\left[\cos\frac{\sqrt{\omega}}{a}L + \cosh\frac{\sqrt{\omega}}{a}L\right]$ factoring out common term $\frac{\omega}{a^2}$

and for the third order $-A\left[\left(\frac{\sqrt{\omega}}{a}\right)^3\cos\frac{\sqrt{\omega}}{a}L + \left(\frac{\sqrt{\omega}}{a}\right)^3\cosh\frac{\sqrt{\omega}}{a}L\right] = -B\left[\left(\frac{\sqrt{\omega}}{a}\right)^3\sin\frac{\sqrt{\omega}}{a}L - \left(\frac{\sqrt{\omega}}{a}\right)^3\sinh\frac{\sqrt{\omega}}{a}L\right] \rightarrow$

$-A\left[\cos\frac{\sqrt{\omega}}{a}L + \cosh\frac{\sqrt{\omega}}{a}L\right] = -B\left[\sin\frac{\sqrt{\omega}}{a}L - \sinh\frac{\sqrt{\omega}}{a}L\right]$ factoring out common term $\left(\frac{\sqrt{\omega}}{a}\right)^3$

Cross multiplying these two equations gives, that is LHS equation 1 x RHS equation 2 , etc.

$$A\left[\sin\frac{\sqrt{\omega}}{a}L + \sinh\frac{\sqrt{\omega}}{a}L\right]\left\{-B\left[\sin\frac{\sqrt{\omega}}{a}L - \sinh\frac{\sqrt{\omega}}{a}L\right]\right\} =$$

$$-B\left[\cos\frac{\sqrt{\omega}}{a}L + \cosh\frac{\sqrt{\omega}}{a}L\right]\left\{-A\left[\cos\frac{\sqrt{\omega}}{a}L + \cosh\frac{\sqrt{\omega}}{a}L\right]\right\}$$

Cross multiplying these two equations gives, that is LHS equation 1 x RHS equation 2 , etc.

$$A\left[\sin\frac{\sqrt{\omega}}{a}L+\sinh\frac{\sqrt{\omega}}{a}L\right]\left\{-B\left[\sin\frac{\sqrt{\omega}}{a}L-\sinh\frac{\sqrt{\omega}}{a}L\right]\right\}=$$

$$-B\left[\cos\frac{\sqrt{\omega}}{a}L+\cosh\frac{\sqrt{\omega}}{a}L\right]\left\{-A\left[\cos\frac{\sqrt{\omega}}{a}L+\cosh\frac{\sqrt{\omega}}{a}L\right]\right\}$$

OR

$$-AB\left[\sin^2\frac{\sqrt{\omega}}{a}L-\sinh^2\frac{\sqrt{\omega}}{a}L\right]=AB\left[\cos^2\frac{\sqrt{\omega}}{a}L+2\cos\frac{\sqrt{\omega}}{a}L\cosh\frac{\sqrt{\omega}}{a}L+\cosh^2\frac{\sqrt{\omega}}{a}L\right]$$

Factoring out AB and distribute negative in LHS gives

$$-\sin^2\frac{\sqrt{\omega}}{a}L+\sinh^2\frac{\sqrt{\omega}}{a}L=\cos^2\frac{\sqrt{\omega}}{a}L+2\cos\frac{\sqrt{\omega}}{a}L\cosh\frac{\sqrt{\omega}}{a}L+\cosh^2\frac{\sqrt{\omega}}{a}L$$

$$\sinh^2\frac{\sqrt{\omega}}{a}L-\cosh^2\frac{\sqrt{\omega}}{a}L=\sin^2\frac{\sqrt{\omega}}{a}L+\cos^2\frac{\sqrt{\omega}}{a}L+2\cos\frac{\sqrt{\omega}}{a}L\cosh\frac{\sqrt{\omega}}{a}L$$

Use identities $\sinh^2\frac{\sqrt{\omega}}{a}L-\cosh^2\frac{\sqrt{\omega}}{a}L=-1,\ \sin^2\frac{\sqrt{\omega}}{a}L+\cos^2\frac{\sqrt{\omega}}{a}L=1$

$$-1=1+2\cos\frac{\sqrt{\omega}}{a}L\cosh\frac{\sqrt{\omega}}{a}L\rightarrow-2=2\cos\frac{\sqrt{\omega}}{a}L\cosh\frac{\sqrt{\omega}}{a}L\rightarrow-1=\cos\frac{\sqrt{\omega}}{a}L\cosh\frac{\sqrt{\omega}}{a}L$$

Finally $\dfrac{-1}{\cos\frac{\sqrt{\omega}}{a}L}=\cosh\frac{\sqrt{\omega}}{a}L\rightarrow-\sec\frac{\sqrt{\omega}}{a}L=\cosh\frac{\sqrt{\omega}}{a}L$