9.15

(a)

$$P(a,x)=\frac{\gamma(a,x)}{\Gamma(a)}=\frac{\int_0^x u^{a-1}e^{-u}du}{\Gamma(a)}$$ from page 375 and 376

Now let $x \to x^2$

$$P(a,x^2)=\frac{\int_{u=0}^{u=x^2} u^{a-1}e^{-u}du}{\Gamma(a)}$$

Then let $y=\sqrt{u}$ or $u=y^2$ therefore $du=2ydy$

Substitution gives

$$P(a,x^2)=\frac{\int_{y=\sqrt{0^2}}^{y=\sqrt{x^2}} y^{2(a-1)}e^{-y^2}\,2ydy}{\Gamma(a)}=\frac{2}{\Gamma(a)}\int_0^x y^{2a-2+1}e^{-y^2}dy=\frac{2}{\Gamma(a)}\int_0^x y^{2a-1}e^{-y^2}dy$$

Note the integrals is a function of y therefore the
bounds are now functions of y

therefore the lower bound came from $0=\sqrt{0}$

and the upper bound from $x=\sqrt{x^2}$

In order to create erf function we need to get
power of y in the integral above to be zero, therefore

$2a-1=0$ or take $a=\frac{1}{2}$

Therefore

$$P(\frac{1}{2},x^2)=\frac{2}{\Gamma(\frac{1}{2})}\int_0^x y^0 e^{-y^2}dy=\frac{2}{\Gamma(\frac{1}{2})}\int_0^x e^{-y^2}\,dy$$

Note $\Gamma(\frac{1}{2})=\sqrt{\pi}$ and substitution gives

$$P(\frac{1}{2},x^2)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy=\operatorname{erf}(x)$$

(b)

Start with def. of erf function evaluated for $\frac{\sqrt{\pi}}{2}(1-i)x$, that is

$$\operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right]=\frac{2}{\sqrt{\pi}}\int_{u=0}^{u=\frac{\sqrt{\pi}}{2}(1-i)x} e^{-u^2}du$$

In order to make integral bound go from 0 to x take the variable subst.

$$t=\frac{2}{\sqrt{\pi}(1-i)}u \rightarrow u=\frac{\sqrt{\pi}}{2}(1-i)t \text{ and } du=\frac{\sqrt{\pi}}{2}(1-i)dt$$

Substitution

$$\frac{2}{\sqrt{\pi}}\int_{u=0}^{u=\frac{\sqrt{\pi}}{2}(1-i)x} e^{-u^2}du \rightarrow \frac{2}{\sqrt{\pi}}\int_{t=\frac{2}{\sqrt{\pi}(1-i)}0}^{t=\frac{2}{\sqrt{\pi}(1-i)}\frac{\sqrt{\pi}}{2}(1-i)x} e^{-\left[\frac{\sqrt{\pi}}{2}(1-i)t\right]^2}\frac{\sqrt{\pi}}{2}(1-i)dt=(1-i)\int_{t=0}^{t=x} e^{-\left[\frac{\sqrt{\pi}}{2}(1-i)t\right]^2}dt=$$

$$(1-i)\int_{t=0}^{t=x} e^{-\frac{\pi}{4}(1-i)^2t^2}dt$$

Note $(1\text{-}i)^2=1-2i+i^2=1\text{-}2i+1=\text{-}2i$ therefore

$$\operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right]=(1-i)\int_{t=0}^{t=x} e^{-\frac{\pi}{4}((-2i)t^2}dt=(1-i)\int_{t=0}^{t=x} e^{i\frac{\pi}{2}t^2}dt$$

Now $e^{i\frac{\pi}{2}t^2}=\cos\frac{\pi}{2}t^2+i\sin\frac{\pi}{2}t^2$ and substitution

$$=(1-i)\int_{t=0}^{t=x}\left(\cos\frac{\pi}{2}t^2+i\sin\frac{\pi}{2}t^2\right)dt=(1-i)[C(x)+iS(x)]$$

$\operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right]=(1-i)[C(x)+iS(x)]$ Multiply LHS by $\frac{(1+i)}{(1+i)}$

$$\operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right]=(1-i)\frac{(1+i)}{(1+i)}[C(x)+iS(x)]=\frac{2}{1+i}[C(x)+iS(x)]$$

Finally $C(x)+iS(x)=\frac{1+i}{2}\operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right]$