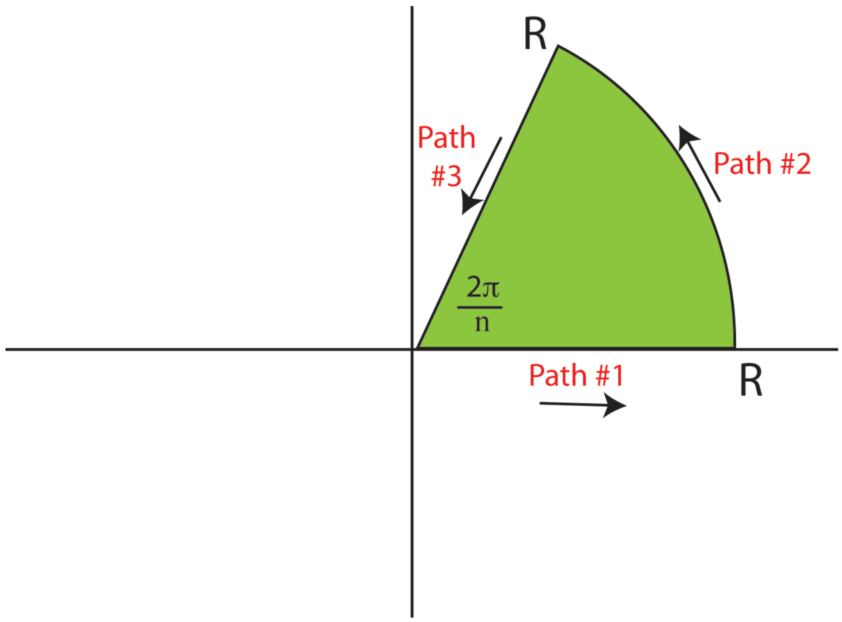
14.12



As it states evaluate the following integral around the wedge $2\pi/n$ using the residue theorem

$$\lim_{R\to\infty}\int_0^R \frac{dz}{1+z^n}$$

First what are the poles of $\frac{1}{1+z^n}$? When $z^n=-1$ or $z=(-1)^{1/n}$

Note -1 in the complex plane so eg. use equation from page 551 example

$-1=1(\cos\pi+i\sin\pi)=1e^{i\pi}\equiv 1e^{i(\pi+2k\pi)}$ Note 1 on RHS is real valued!

Therefore $z=(-1)^{1/n}=[1e^{i(\pi+2k\pi)}]^{1/n}=1^{1/n}e^{i(\pi+2k\pi)/n}=e^{i\pi(1+2k)/n}$

$k = 0,1,2,\ldots,n-1$

Now for

$k=0 \quad z=e^{i\pi/n}$ INSIDE wedge

$k=1 \; z=e^{i3\pi/n}$ OUTSIDE wedge

Therefore only consider $k = 0$ case and $z=e^{i\pi/n}$

Next set up path around wedge

Path #1: $z=t+0i\equiv t$ and $dz=dt \qquad 0\le t\le R$

Path #2: $z=Re^{it}$ and $dz=iRe^{it}dt \qquad 0\le t\le 2\pi/n$

Path #3: $z=te^{i2\pi/n}$ and $dz=e^{i2\pi/n}dt \qquad R\ge t\ge 0$ Note t goes from R to 0!!

Set up integrals (=path#1+path#2+path#3)

$$\lim_{R\to\infty}\left[\int_0^R \frac{dz}{1+z^n}\right]=\lim_{R\to\infty}\left[\int_0^R \frac{dt}{1+t^n}+\int_0^{2\pi/n}\frac{iRe^{it}dt}{1+(Re^{it})^n}+\int_R^0 \frac{e^{i2\pi/n}dt}{1+(te^{i2\pi/n})^n}\right]=\lim_{R\to\infty}\left[2\pi i\sum_j \mathbf{R}_j\right]$$

where $\mathbf{R}$ are residues of $\dfrac{1}{1+z^n}$ (DO NOT CONFUSE THE radius R with residues $\mathbf{R}_j$)

We have already calculated simple pole as $z=e^{i\pi/n}$ and residue can be calculated using equation 14.56, pg. 573, note j = 1 in sum above. therefore $\sum_{j=1}\mathbf{R}_j \to \mathbf{R}(e^{i\pi/n})$

$$\mathbf{R}(e^{i\pi/n})=\lim_{z\to e^{i\pi/n}}\frac{1}{(1+z^n)'}=\lim_{z\to e^{i\pi/n}}\frac{1}{nz^{n-1}}=\frac{1}{n(e^{i\pi/n})^{n-1}}$$

$$=\frac{1}{ne^{i\pi}e^{-i\pi/n}}\ \left\{\text{Note } e^{i\pi}=\cos\pi+i\sin\pi=-1\text{, therefore}\right\}\ =\frac{-1}{ne^{-i\pi/n}}$$

Therefore RHS is $\lim_{R\to\infty}\left[2\pi i\sum_j \mathbf{R}_j\right]=\lim_{R\to\infty}2\pi i\mathbf{R}=\lim_{R\to\infty}2\pi i\left(\dfrac{-1}{ne^{-i\pi/n}}\right)=\dfrac{-2\pi i}{ne^{-i\pi/n}}$

Back to the integrals (LHS)

$$\lim_{R\to\infty}\left[\int_0^R \frac{dt}{1+t^n}+\int_0^{2\pi/n}\frac{iRe^{it}dt}{1+(Re^{it})^n}+\int_R^0 \frac{e^{i2\pi/n}dt}{1+(te^{i2\pi/n})^n}\right]$$

For the second integral we have

$$\lim_{R\to\infty}\int_0^{2\pi/n}\frac{iRe^{it}dt}{1+(Re^{it})^n}=\lim_{R\to\infty}\int_0^{2\pi/n}\frac{ie^{it}dt}{1/R+R^{n-1}e^{itn}}\equiv\int_0^{2\pi/n}\lim_{R\to\infty}\left[\frac{ie^{it}dt}{0+R^{n-1}e^{itn}}\right]=0$$

Also reverse bounds on third integral and simplify

We thus have left the first and third integrals

$$\lim_{R\to\infty}\left[\int_0^R \frac{dt}{1+t^n}-e^{i2\pi/n}\int_0^R \frac{dt}{1+t^n e^{i2\pi}}\right]=2\pi i\mathbf{R}=\frac{-2\pi i}{ne^{-i\pi/n}}$$

Note $e^{i2\pi}=\cos 2\pi+i\sin 2\pi=1$

Therefore

$$\lim_{R\to\infty}\left[\int_0^R \frac{dt}{1+t^n}-e^{i2\pi/n}\int_0^R \frac{dt}{1+t^n}\right]=[1-e^{i2\pi/n}]\lim_{R\to\infty}\int_0^R \frac{dt}{1+t^n}=\frac{-2\pi i}{ne^{-i\pi/n}}$$

For t = x along real axis we have

$$[1-e^{i2\pi/n}]\lim_{R\to\infty}\int_0^R \frac{dt}{1+t^n} = [1-e^{i2\pi/n}]\lim_{R\to\infty}\int_0^R \frac{dx}{1+x^n} = [1-e^{i2\pi/n}]\int_0^\infty \frac{dx}{1+x^n} = \frac{-2\pi i}{ne^{-i\pi/n}}$$

Finally we have $[1-e^{i2\pi/n}]\int_0^\infty \frac{dx}{1+x^n} = \frac{-2\pi i}{ne^{-i\pi/n}}$

Multiply both sides by $e^{-i\pi/n}$ therefore $[e^{-i\pi/n} - e^{i\pi/n}]\int_0^\infty \frac{dx}{1+x^n} = \frac{-2\pi i}{n}$

or $\int_0^\infty \frac{dx}{1+x^n} = \frac{-2\pi i}{n[e^{-i\pi/n} - e^{i\pi/n}]} = \frac{2\pi i}{n[e^{i\pi/n} - e^{-i\pi/n}]}$

Using identity $\frac{e^{i\pi/n} - e^{-i\pi/n}}{2i} = \sin\left(\frac{\pi}{n}\right)$

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{2\pi i}{n[e^{i\pi/n} - e^{-i\pi/n}]} = \frac{2\pi i}{n[2i\sin\frac{\pi}{n}]} = \frac{\pi}{n}\csc\left(\frac{\pi}{n}\right)$$