Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Midterm Exam
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Question 1

a. Graph of the function attached as a separate pdf.

b. Since we have made the function $f(x)$ even using an even extension, all the b_k coefficients in its Fourier series are zero. With a period $L = 4$, we determine the remaining coefficients a_k:

$$a_k = \frac{2}{4}\int_{-2}^{2} x \cos\left(\frac{2k\pi x}{4}\right)dx$$

And since f is even now

$$\begin{aligned} a_k &= \frac{4}{4}\int_{0}^{2} x \cos\left(\frac{2k\pi x}{4}\right)dx \\ &= \int_{0}^{2} x \cos\left(\frac{k\pi x}{2}\right)dx \end{aligned}$$

Using integration by parts, for $k > 0$:

$$\begin{aligned} a_k &= \frac{2}{k\pi}[x\sin(\frac{k\pi x}{2})]_0^2 - \frac{2}{k\pi}\int_0^2 \sin(\frac{k\pi x}{2})dx \\ &= 0 - \frac{2}{k\pi}(-\frac{2}{k\pi})[\cos\frac{k\pi x}{2})]_0^2 \\ &= \frac{4}{(k\pi)^2)}[\cos(k\pi) - \cos(0)] \\ &= \frac{4}{(k\pi)^2}[(-1)^k - 1] \end{aligned}$$

Then

$$a_k = \begin{cases} -\dfrac{8}{(k\pi)^2} \text{ for odd } k \\ 0 \text{ for even } k \end{cases}$$

And $a_0 = \frac{2}{4}\int_{-2}^{2} xdx = \frac{4}{4}\int_0^2 xdx = \frac{1}{2}[x^2]_0^2 = 2$. With the coefficients a_k determined, we obtain the Fourier series for $f(x)$:

$$\begin{aligned} f(x) &= \frac{2}{2} - \sum_{k=1}^{\infty} \frac{8}{(k\pi)^2}\cos(\frac{2k\pi x}{4}) \; k \text{ odd} \\ x &= 1 - \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(\frac{(2k+1)\pi x}{2}) \end{aligned}$$

c. Applying Parseval's identity for Fourier series and using the result of part b.:

$$\frac{1}{4}\int_{-2}^{2} x^2 dx = \frac{2^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + 0)\ k \text{ odd}$$
$$\frac{2}{4}\int_{0}^{2} x^2 dx = 1 + \frac{1}{2}\sum_{k=0}^{\infty}(\frac{8}{(2k+1)^2\pi^2})^2$$
$$\frac{1}{2}[\frac{x^3}{3}]_0^2 = 1 + \frac{1}{2}\cdot\frac{64}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}$$
$$\frac{4}{3} - 1 = \frac{32}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}$$
$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^4}{32}\cdot\frac{1}{3}$$

Therefore

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

Question 2

a. Graph of the function attached as a separate pdf.

b.

$$f(t) = A\Big[H(t) - H(t-\tau)\Big]$$

c.

$$\tilde{f}(w) = F\{f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt$$

Since $f(t) = 0$ for $t \geq 0$ or $t \leq \tau$:

$$\begin{aligned}\tilde{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{0}^{\tau} A\cdot e^{-iwt}\,dt \\ &= \frac{A}{\sqrt{2\pi}}(\frac{1}{-iw})[e^{-iwt}]_0^{\tau} \\ &= \frac{iA}{w\sqrt{2\pi}}(e^{-iw\tau} - 1) \\ &= \frac{iA}{w\sqrt{2\pi}}e^{-iw\frac{\tau}{2}}(e^{-iw\frac{\tau}{2}} - e^{iw\frac{\tau}{2}})\end{aligned}$$

From Euler identity:

$$e^{-iw\frac{\tau}{2}} - e^{iw\frac{\tau}{2}} = -2i\sin w\frac{\tau}{2}$$

Therefore

$$
\begin{aligned}
\tilde{f}(w) &= \frac{2A}{w\sqrt{2\pi}} e^{-iw\frac{\tau}{2}} \sin w\frac{\tau}{2} \\
&= \sqrt{\frac{2}{\pi}} \frac{A}{w} e^{-iw\frac{\tau}{2}} \sin w\frac{\tau}{2} \\
&= A\sqrt{\frac{2}{\pi}} e^{-iw\frac{\tau}{2}} \frac{\tau}{2} \frac{\sin(w\frac{\tau}{2})}{w\frac{\tau}{2}} \\
&= \frac{A}{\sqrt{2\pi}} \tau e^{-iw\frac{\tau}{2}} \text{sinc}(w\frac{\tau}{2})
\end{aligned}
$$

d. Let $A = \frac{1}{\tau}$ then substituting in $f(t)$ from part c., gives:

$$
F\{\lim_{\tau\to 0} f(t)\} = \lim_{\tau\to 0} F\{f(t)\} = \lim_{\tau\to 0} \frac{1}{\sqrt{2\pi}} e^{-iw\frac{\tau}{2}} \frac{\sin(w\frac{\tau}{2})}{w\frac{\tau}{2}}
$$

$$
\begin{aligned}
&\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1 \text{ by Hospitals rule} \\
&\lim_{\tau\to 0} e^{-iw\frac{\tau}{2}} = \lim_{\tau\to 0} e^{0} = 1
\end{aligned}
$$

Therefore

$$
F\{\lim_{\tau\to 0} f(t)\} = \frac{1}{\sqrt{2\pi}}
$$

e. The Fourier transform of $f(t)$ as $\tau \to 0$ is the Fourier transform of a δ-function as we can expect as we "transform" the rectangular function $f(t)$ to a Dirac impulse.

Question 3

a. By definition, the Laplace transform of $g(t) = \sin(5t)$ is:

$$
\bar{g}(s) = L\{g(t)\} = \int_0^\infty \sin(5t)e^{-st}dt = \lim_{L\to\infty} \int_0^L \sin(5t)e^{-st}dt
$$

First compute $\int e^{-st} \sin at\, dt$, using integration by parts with $u = \sin at$, $u' = a\cos at$, $v' = e^{-st}$, $v = -\frac{1}{s}e^{-st}$:

$$
\int e^{-st} \sin(at)\, dt = -\frac{1}{s} e^{-st} \sin(at) + \frac{a}{s} \int e^{-st} \cos(at) dt \tag{1}
$$

Next compute $\int e^{-st} \cos(at)\, dt$, again, using integration by parts with $u = \cos at$, $u' = -a \sin at$, $v' = e^{-st}$, $v = -\frac{1}{s}e^{-st}$:

$$
\int e^{-st} \cos(at)\, dt = -\frac{1}{s} e^{-st} \cos(at) - \frac{a}{s} \int e^{-st} \sin(at)\, dt
$$

Substituting into (1):

$$\int e^{-st}\sin(at)\,dt = -\frac{1}{s}e^{-st}\sin(at) + \frac{a}{s}\left(-\frac{1}{s}e^{-st}\cos(at) - \frac{a}{s}\int e^{-st}\sin(at)\,dt\right)$$
$$= -\frac{1}{s}e^{-st}\sin(at) - \frac{a}{s^2}e^{-st}\cos(at) + \frac{a}{s^2}\int e^{-st}\sin(at)\,dt$$

thus

$$(1+\frac{a^2}{s^2})\int e^{-st}\sin(at)\,dt = -e^{-st}(\frac{1}{s}\sin(at) + \frac{a}{s^2}\cos(at))$$

Evaluating at $t=0$ and $t\to\infty$:

$$\begin{aligned}(1+\frac{a^2}{s^2})L\{\sin(at)\} &= \lim_{L\to\infty}\left[-e^{-st}(\frac{1}{s}\sin(at)+\frac{a}{s^2}\cos(at))\right]_0^L \\ &= 0-(-1(\frac{1}{s}\cdot 0+\frac{a}{s^2}\cdot 1)) \\ &= \frac{a}{s^2}\end{aligned}$$

Therefore

$$\begin{aligned}L\{\sin(at)\} &= \frac{a}{s^2}(1+\frac{a^2}{s^2})^{-1} \\ &= \frac{a}{a^2+s^2}\end{aligned}$$

Set $a=5$ and

$$L\{g(t)\} = L\{\sin(5t)\} = \frac{5}{s^2+25}$$

b. From the book, one property of the Laplace transform is $L[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}$ for $n = 1,2,3,\cdots$, take $n=1$, $L[tf(t)] = -\frac{d\bar{f}(s)}{ds}$. Set $f(t)=t\sin(5t)$ and from part b, $L\{\sin(5t)\} = \frac{5}{s^2+25}$, therefore:

$$\begin{aligned}L\{t\sin(5t)\} &= -\frac{d}{ds}\left(\frac{5}{s^2+25}\right) \\ &= -5\frac{d}{ds}\left(\frac{1}{s^2+25}\right) \\ &= -5\left(\frac{-2s}{(s^2+25)^2}\right) \\ &= \frac{10s}{(s^2+25)^2}\end{aligned}$$

c. By definition $(f*g)(t) = \int_0^t \tau e^{-(t-\tau)}d\tau = e^{-t}\int_0^t \tau e^{\tau}d\tau$. Using integration by parts:

$$\begin{aligned}\int_0^t \tau e^{\tau}d\tau &= [\tau e^{\tau}]_0^t - \int_0^t e^{\tau}d\tau \\ &= te^t - [e^{\tau}]_0^t \\ &= te^t - (e^t-1) \\ &= e^t(t-1)+1\end{aligned}$$

And

$$(f*g)(t) = e^{-t}\Big[e^{-t}(t-1)+1\Big] = e^{-t}+t-1$$

From $L\{(f*g)(t)\} = \bar{f}(s)\cdot\bar{g}(s)$, we have:

$$\begin{aligned}\bar{f}(s)\cdot\bar{g}(s) &= \frac{1}{s^2}\cdot\frac{1}{s+1}\\ &= \frac{1-s}{s^2+1}+\frac{1}{s+1}\\ &= \frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\end{aligned}$$

Therefore

$$\begin{aligned}(f*g)(t) = L^{-1}\{L\{(f*g)(t)\}\} &= L^{-1}\{\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\}\\ &= L^{-1}\{\frac{1}{s^2}\} - L^{-1}\{\frac{1}{s}\} + L^{-1}\{\frac{1}{s+1}\}\\ &= t-1+e^{-t}\\ &= e^{-t}+t-1\end{aligned}$$

Question 4

$$y''+4y'-5y = \delta(t-1)\; y(0)=0\; y'(0)=3$$

a. Taking the Laplace transform on both sides of the equation gives:

$$\begin{aligned}s^2\tilde{y}(s)-sy(0)-y'(0)+4[s\tilde{y}(s)-y(0)]-5\tilde{y}(s) &= e^{-s}\\ s^2\tilde{y}(s)-s\cdot 0-3+4[s\tilde{y}(s)-0]-5\tilde{y}(s) &= e^{-s}\\ s^2\tilde{y}(s)+4s\tilde{y}(s)-5\tilde{y}(s) &= e^{-s}+3\end{aligned}$$

Combining the terms: $(s^2+4s-5)\tilde{y}(s) = 3+e^{-s}$. Therefore

$$\tilde{y}(s) = \frac{3+e^{-s}}{s^2+4s-5}$$

b. The roots of $s^2+4s-5=0$ are -5 and 1, so we can rewrite $\tilde{y}(s)$ as $\tilde{y}(s) = \frac{3}{(s-1)(s+5)}+\frac{e^{-s}}{(s-1)(s+5)}$
Computing the fraction expansion:

$$\begin{aligned}\frac{1}{(s-1)(s+5)} &= \frac{A}{s-1}+\frac{B}{s+5}\\ &= \frac{(A+B)s+5A-B}{(s-1)(s+5)}\end{aligned}$$

Equating the powers of s on each side of the previous equation:

$$\begin{aligned}s^1 &: A+B=0\\ s^0 &: 5A-B=1\end{aligned}$$

gives $A = \frac{1}{6}$ and $B = -\frac{1}{6}$. Thus

$$\frac{1}{(s-1)(s+5)} = \frac{1}{6}\left(\frac{1}{s-1} - \frac{1}{s+5}\right)$$

So

$$\begin{aligned}\tilde{y}(s) &= 3\left[\frac{1}{6}(\frac{1}{s-1} - \frac{1}{s+5})\right] + \frac{1}{6}(\frac{e^{-s}}{s-1} - \frac{e^{-s}}{s+5}) \\ &= \frac{1}{2}(\frac{1}{s-1} - \frac{1}{s+5}) + \frac{1}{6}(\frac{e^{-s}}{s-1} - \frac{e^{-s}}{s+5})\end{aligned}$$

c. $y(t) = L^{-1}\{\tilde{y}(s)\}$ and from part b:

$$y(t) = \frac{1}{2}\left[L^{-1}\{\frac{1}{s-1}\} - L^{-1}\{\frac{1}{s+5}\}\right] + \frac{1}{6}\left[L^{-1}\{\frac{e^{-s}}{s-1}\} - L^{-1}\{\frac{e^{-s}}{s+5}\}\right]$$

$L^{-1}\{\frac{1}{s-1}\} = e^t$, $L^{-1}\{\frac{1}{s+5}\} = e^{-5t}$, and using the shift theorem:

$$L\{f(t-t_0)H(t-t_0)\} = e^{-st_0}F(s) \;\; f(t-t_0)H(t-t_0) = L^{-1}\{e^{-st_0}F(s)\}$$

So for $t_0 = 1$

$$\begin{aligned}L^{-1}\{\frac{e^{-s}}{s-1}\} &= e^{(t-1)}H(t-1) \\ L^{-1}\{\frac{e^{-s}}{s+5}\} &= e^{-5(t-1)}H(t-1)\end{aligned}$$

Plugging back these into $y(t)$ yields:

$$\begin{aligned}y(t) &= \frac{1}{2}\left[e^t - e^{-5t}\right] + \frac{1}{6}\left[e^{(t-1)}H(t-1) - e^{-5(t-1)}H(t-1)\right] \\ &= \frac{1}{2}\left[e^t - e^{-5t}\right] + \frac{1}{6}\left(e^{(t-1)} - e^{-5(t-1)}\right)H(t-1) \\ &= \frac{e^t}{2}\left(1 + \frac{1}{3e}H(t-1)\right) - \frac{e^{-5t}}{2}(1 + \frac{1}{3}e^5 H(t-1)\Big)\end{aligned}$$

Question 5

a. Let rate $r = 10\ \text{min}^{-1}$, the rate of change of A is equal to how much of A goes to B at rate r, how much of A goes to C at rate r, and how much from B goes to A at rate r, and how much from C goes to A at rate r. The transport dynamics are the same for B and C , so the system looks like:

$$\begin{aligned}\frac{dA}{dt} &= -rA - rA + rB + rC \\ \frac{dA}{dt} &= -2rA + rB + rC \\ \frac{dA}{dt} &= -20A + 10B + 10C \;\text{ with } A(0) = 20\end{aligned}$$

Similarly

$$
\begin{aligned}
\frac{dB}{dt} &= rA - rB - rB + rC \\
\frac{dB}{dt} &= rA - 2rB + rC \\
\frac{dB}{dt} &= 10A - 20B + 10C \;\text{ with } B(0) = 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{dC}{dt} &= rA + rB - rC - rC \\
\frac{dC}{dt} &= rA + rB - 2rC \\
\frac{dC}{dt} &= 10A + 10B - 20C \;\text{ with } C(0) = 0
\end{aligned}
$$

b. We take the Laplace transforms of the differential equations which gives:

$$
\begin{aligned}
s\tilde{A}(s) - A(0) &= -20\tilde{A}(s) + 10\tilde{B}(s) + 10\tilde{C}(s) \\
(s+20)\tilde{A}(s) - 10\tilde{B}(s) - 10\tilde{C}(s) &= 20 \\
s\tilde{B}(s) - B(0) &= 10\tilde{A}(s) - 20\tilde{B}(s) + 10\tilde{C}(s) \\
10\tilde{A}(s) - (s+20)\tilde{B} + 10\tilde{C}(s) &= 0 \\
s\tilde{C}(s) - C(0) &= 10\tilde{A}(s) + 10\tilde{B}(s) - 20\tilde{C}(s) \\
10\tilde{A}(s) + 10\tilde{B}(s) - (s+20)\tilde{C} &= 0
\end{aligned}
$$

c. We write the equations in matrix form:

$$
\begin{bmatrix} s+20 & -10 & -10 \\ 10 & -(s+20) & 10 \\ 10 & 10 & -(s+20) \end{bmatrix}
\begin{bmatrix} \tilde{A}(s) \\ \tilde{B}(s) \\ \tilde{C}(s) \end{bmatrix}
=
\begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}
$$

The determinant of the system is:

$$
\begin{aligned}
D = \begin{vmatrix} s+20 & -10 & -10 \\ 10 & -(s+20) & 10 \\ 10 & 10 & -(s+20) \end{vmatrix} &= s^3 + 60s^2 + 900s \\
&= s(s^2 + 60s + 900) \\
&= s(s+30)^2
\end{aligned}
$$

Solving using Cramer's rule gives:

$$
\begin{aligned}
\tilde{A}(s) &= \frac{\begin{vmatrix} 20 & -10 & -10 \\ 0 & -(s+20) & 10 \\ 0 & 10 & -(s+20) \end{vmatrix}}{D} \\
&= 20\frac{\begin{vmatrix} -(s+20) & 10 \\ 10 & -(s+20) \end{vmatrix}}{D} \\
&= \frac{20((s+20)^2 - 100)}{s(s+30)^2} \\
&= \frac{20(s^2 + 40s + 300)}{s(s+30)^2} \\
&= \frac{20(s+10)(s+30)}{s(s+30)^2} \\
&= \frac{20(s+10)}{s(s+30)} \\
&= \frac{20}{3}[\frac{1}{s} + \frac{2}{s+30}]
\end{aligned}
$$

Taking the inverse Laplace transform using the table:

$$
A(t) = \frac{20}{3}(1 + 2e^{-30t})
$$

$$
\begin{aligned}
\tilde{B}(s) &= \frac{\begin{vmatrix} s+20 & 20 & -10 \\ 10 & 0 & 10 \\ 10 & 0 & -(s+20) \end{vmatrix}}{D} \\
&= \frac{200s + 6000}{s(s+30)^2} \\
&= \frac{200(s+30)}{s(s+30)^2} \\
&= \frac{200}{s(s+30)} \\
&= \frac{200}{30}\left(\frac{1}{s} - \frac{1}{s+30}\right)
\end{aligned}
$$

Taking the inverse Laplace transform using the table:

$$
\begin{aligned}
B(t) &= \frac{200}{30}(1 - e^{-30t}) \\
&= \frac{20}{3}(1 - e^{-30t})
\end{aligned}
$$

$$\begin{aligned}
\tilde{C}(s) &= \frac{\begin{vmatrix} s+20 & -10 & 20 \\ 10 & -(s+20) & 0 \\ 10 & 10 & 0 \end{vmatrix}}{D} \\
&= \frac{200s+6000}{s(s+30)^2} \\
&= \frac{200(s+30)}{s(s+30)^2} \\
&= \frac{200}{s(s+30)} \\
&= \frac{200}{30}\left(\frac{1}{s} - \frac{1}{s+30}\right)
\end{aligned}$$

Taking the inverse Laplace transform using the table:

$$C(t) = \frac{20}{3}(1 - e^{-30t}) = B(t)$$

d. From part c:

$$A(t) = \frac{20}{3}(1 + 2e^{-30t})$$

$$B(t) = C(t) = \frac{20}{3}(1 - e^{-30t})$$

e. $\lim_{t\to\infty} e^{-30t} = 0$ therefore as $t \to \infty$, $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} B(t) = \lim_{t\to\infty} C(t) = \frac{20}{3}$ which is the equilibrium state of this system when t goes to infinity.

Question 6

$$L_0(x) = 1 \text{ and } L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \; n = 1, 2, \cdots$$

Applying the recurrence relationship

$n = 1$

$$
\begin{aligned}
L_1(x) &= \frac{e^x}{1!}\frac{d}{dx}(xe^{-x}) \\
\frac{d}{dx}(xe^{-x}) &= e^{-x} + x(-1)e^{-x} = e^{-x}(1-x) \\
L_1(x) &= e^x e^{-x}(1-x) \\
L_1(x) &= 1 - x
\end{aligned}
$$

$n = 2$

$$
\begin{aligned}
L_2(x) &= \frac{e^x}{2!}\frac{d^2}{dx^2}(x^2e^{-x}) \\
\frac{d}{dx}(x^2e^{-x}) &= \frac{d}{dx}(x(xe^{-x})) \\
&= 1(xe^{-x}) + xe^{-x}(1-x) \\
&= xe^{-x}(2-x) \\
\frac{d^2}{dx^2}(x^2e^{-x}) &= \frac{d}{dx}(xe^{-x})(2-x) + xe^{-x}\frac{d}{dx}(2-x) \\
&= e^{-x}(1-x)(2-x) + xe^{-x}(-1) \\
&= e^{-x}[(1-x)(2-x) - x] \\
&= e^{-x}(x^2 - 4x + 2) \\
L_2(x) &= \frac{e^x}{2\,1}e^{-x}(x^2 - 4x + 2) \\
&= 1 - 2x + \frac{x^2}{2}
\end{aligned}
$$

$n = 3$

$$
\begin{aligned}
L_3(x) &= \frac{e^x}{3!}\frac{d^3}{dx^3}(x^3e^{-x}) \\
\frac{d}{dx}(x^3e^{-x}) &= \frac{d}{dx}(x(x^2e^{-x})) \\
&= 1(x^2e^{-x}) + xxe^{-x}(2-x) \\
&= x^2e^{-x}(3-x) \\
\frac{d^2}{dx^2}(x^3e^{-x}) &= \frac{d}{dx}\big(\frac{d}{dx}(x^3e^{-x})\big) \\
&= \frac{d}{dx}\big(x^2e^{-x}(3-x)\big) \\
&= \frac{d}{dx}(x^2e^{-x})(3-x) + (x^2e^{-x})\frac{d}{dx}(3-x) \\
&= xe^{-x}(2-x)(3-x) + (x^2e^{-x})(-1) \\
&= xe^{-x}[(2-x)(3-x) - x] \\
&= xe^{-x}(x^2 - 6x + 6) \\
\frac{d^3}{dx^3}(x^3e^{-x}) &= \frac{d}{dx}\big(\frac{d^2}{dx^2}(x^3e^{-x})\big) \\
&= \frac{d}{dx}\big(xe^{-x}(x^2-6x+6)\big) \\
&= \frac{d}{dx}(xe^{-x})(x^2-6x+6) + xe^{-x}\frac{d}{dx}(x^2-6x+6) \\
&= e^{-x}(1-x)(x^2-6x+6) + xe^{-x}(2x-6) \\
&= e^{-x}(x^2 - 6x + 6 - x^3 + 6x^2 - 6x + 2x^2 - 6x) \\
&= e^{-x}(-x^3 + 9x^2 - 18x + 6) \\
L_3(x) &= \frac{e^x}{6}e^{-x}(-x^3 + 9x^2 - 18x + 6) \\
&= 1 - 3x + \frac{3x^2}{2} - \frac{x^3}{6}
\end{aligned}
$$

Show that the Laguerre polynomials are orthogonal on the positive axis ($0 \leq x < \infty$) w.r.t. the weight function e^{-x}. We want to show

$$\int_0^\infty L_n(x)L_k(x)e^{-x}dx = 0 \ \text{ if } n \neq k$$

From the expression of the Laguerre polynomial, $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^ne^{-x})$, a Laguerre polynomial $L_n(x)$ is a polynomial of degree n. Without loss of generality, let assume in the previous equation that $k < n$. By multiplying two polynomials of degree M and N, the result is a polynomial of degree at most $M+N$. Therefore to prove orthogonality, suffices to prove the following equation:

$$\int_0^\infty e^{-x}x^kL_n(x)dx = 0 \ \text{ for all } k < n$$

First we show that

$$\int_0^\infty \frac{d^n}{dx^n}(x^me^{-x}) = 0 \ \text{ for } n < m$$

We are applying nth derivative rules:

$$\begin{aligned}\frac{d^n}{dx^n}(x^m e^{-x}) &= \sum_{r=0}^{n} C_r^n \frac{d^r}{dx^r} x^n \frac{d^{n-r}}{dx^{n-r}} e^{-x} \\ &= \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{n!}{(n-r)!} x^{n-r} (-1)^{n-r} e^{-x} \\ &= \sum_{r=0}^{n} (-1)^{n-r} \frac{(n!)^2}{r!((n-r)!^2} x^{n-r} e^{-x}\end{aligned}$$

reindexing with $l = n - r$, we obtain

$$\frac{d^n}{dx^n}(x^m e^{-x}) = \sum_{l=0}^{m} (-1)^l \frac{(n!)^2}{(l!)^2 (n-l)!} x^l e^{-x}$$

Next

$$\int_0^\infty \frac{d^n}{dx^n}(x^m e^{-x}) = \sum_{l=0}^{m} (-1)^l \frac{(n!)^2}{(l!)^2 (n-l)!} \int_0^\infty x^l e^{-x} dx$$

Integration by parts l times gives:

$$\int_0^\infty x^l e^{-x} dx = (-1)^l l! \int_0^\infty e^{-x} dx = (-1)^l l! [-e^{-x}]_0^\infty = 0$$

Going back to our initial equation of interest, and applying integration by parts k times:

$$\begin{aligned}\int_0^\infty e^{-x} x^k L_n(x) dx &= \int_0^\infty x^k \frac{d^n}{dx^n}(x^n e^{-x}) dx \\ &= [x^k \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})]_0^\infty - k \int_0^\infty x^{k-1} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) dx\end{aligned}$$

The first term is 0 as it is a sum of terms $[x^{l+k} e^{-x}]_0^\infty = 0$ and after k iterations we are left with

$$\int_0^\infty e^{-x} x^k L_n(x) dx = (-1)^k k! \int_0^\infty \frac{d^{n-k}}{dx^{n-k}}(x^n e^{-x}) dx = 0$$

Question 7

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{d} - y = x \, , y(e) = 0 \, , y'(e) = 2$$

a. This is Euler differential equation, and we make the change of variable $x = e^t$ or $t = \ln(x)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{d\ln x}{dx} = \frac{dy}{dt}\frac{1}{x} = \frac{1}{x}\frac{dy}{dt} \\ x\frac{dy}{dx} &= \frac{dy}{dt}\end{aligned}$$

And since this is a Legendre ODE with $\alpha = 1$ and $\beta = 0$, we can use the expression for the second derivative $(\alpha x + \beta)^2 \frac{d^2y}{dx^2} = \alpha^2 \frac{d}{dt}[\frac{d}{dt} - 1]y$. With $\alpha = 1$ and $\beta = 0$, we have: $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.

Substitute into the above equation yields:

$$(\frac{d^2y}{dt^2} - \frac{dy}{dt}) + \frac{dy}{dt} - y = e^t$$
$$\frac{d^2y}{dt^2} - y = e^t$$

b. The homogeneous equation is

$$\frac{d^2y}{dt^2} - y = 0$$

Assume a solution of the form $y(t) = Ae^{\lambda t}$ gives the characteristic equation $\lambda^2 - 1 = 0$ which has for roots $\lambda = \pm 1$ and gives for solution $y(t) = c_1 e^t + c_2 e^{-t}$.

c. The ODE to solve is:

$$\frac{d^2y}{dt^2} - y = 0$$

It is in standard form and it is defined at any point t, it is analytic, thus we take as solution $y(t) = \sum_{t=0}^{\infty} a_n t^n$. So:

$$y'(t) = \sum_{t=0}^{\infty} n a_n t^{n-1}$$
$$y''(t) = \sum_{t=0}^{\infty} n(n-1) a_n t^{n-2}$$

by reindexing

$$y''(t) = \sum_{t=-2}^{\infty} (n+2)(n+1) a_{n+2} t^n$$
$$y''(t) = \sum_{t=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$$

Substitute into the ODE gives:

$$\sum_{t=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{t=0}^{\infty} a_n t^n = 0$$
$$\sum_{t=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] t^n = 0$$

or

$$a_{n+2} = \frac{1}{(n+2)(n+1)} a_n$$
$$a_n = \frac{1}{n(n-1)} a_{n-2}$$

Take $a_0 = a_1 = 1$ and we generate the coefficients:

. $n = 2$ then $a_2 = \frac{1}{2 \cdot 1} a_0 = \frac{1}{2 \cdot 1} = \frac{1}{2!}$

. $n = 3$ then $a_3 = \frac{1}{3 \cdot 2} a_1 = \frac{1}{3 \cdot 2} = \frac{1}{3!}$

. $n = 4$ then $a_4 = \frac{1}{4 \cdot 3} a_2 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{4!}$

$\vdots$

. $a_n = \frac{1}{n(n-1)} a_{n-2} = \cdots = \frac{1}{n!}$

The first solution we obtain is: $y_1(t) = \sum_{t=0}^{\infty} a_n t^n = \sum_{t=0}^{\infty} \frac{t^n}{n!} = e^t$. Secondly, if we set $a_0 = 1$ and choose $a_1 = -1$, then we obtain a second independent solution:

. $n = 2$ then $a_2 = \frac{1}{2 \cdot 1} a_0 = \frac{1}{2 \cdot 1} = \frac{1}{2!}$

. $n = 3$ then $a_3 = \frac{1}{3 \cdot 2} a_1 = -\frac{1}{3 \cdot 2} = \frac{-1}{3!}$

. $n = 4$ then $a_4 = \frac{1}{4 \cdot 3} a_2 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{4!}$

. $n = 5$ then $a_5 = \frac{1}{5 \cdot 4} a_3 = \frac{-1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{-1}{5!}$

$\vdots$

. $a_n = \frac{1}{n(n-1)} a_{n-2} = \cdots = \frac{(-1)^n}{n!}$

We have the second solution: $y_2(t) = \sum_{t=0}^{\infty} a_n t^n = \sum_{t=0}^{\infty} \frac{(-t)^n}{n!}$, recognizing the last series as e^{-t}, we can write the general solution of the homogeneous equation as

$$y_H(t) = c_1 e^t + c_2 e^{-t}$$

which is the solution we found in question b.

d. The differential equation to solve is

$$\frac{d^2 y}{dt^2} - y = e^t$$

Next we use the variation of parameters method, we are looking for a solution $y_p(t) = k_1(t) e^t + k_2(t) e^{-t}$. We solve for derivatives of k's a system of two equations:

$$\begin{cases} k_1' e^t + k_2' e^{-t} & = 0 \\ k_1' e^t - k_2' e^{-t} & = e^t \end{cases}$$

Multiplying through by e^t gives:

$$\begin{cases} k_1' e^{2t} + k_2' & = 0 \\ k_1' e^{2t} - k_2' & = e^{2t} \end{cases}$$

Adding first equation to second yields $2k_1'e^{2t} = e^{2t}$ or $k_1' = \frac{1}{2}$ and $k_1 = \frac{t}{2}$. Substitute

$$\begin{aligned} k_2' &= -k_1'e^{2t} \\ &= -\frac{1}{2}e^{2t} \end{aligned}$$

integrating

$$k_2 = -\frac{e^{2t}}{4}$$

Therefore:

$$\begin{aligned} y_p(t) &= k_1(t)e^t + k_2(t)e^{-t} \\ &= \frac{t}{2}e^t - \frac{e^{2t}}{4}e^{-t} \\ &= \frac{t}{2}e^t - \frac{e^t}{4} \\ &= \frac{e^t}{2}(t - \frac{1}{2}) \end{aligned}$$

e. The general solution is: $y(t) = y_H(t) + y_p(t) = c_1e^t + c_2e^{-t} + \frac{e^t}{2}(t - \frac{1}{2})$, simplifying the constants, we can rewrite the general solution as $y(t) = c_1e^t + c_2e^{-t} + \frac{t}{2}e^t$. Plugging back $x = e^t$ or $t = \ln(x)$ gives

$$y(x) = c_1x + \frac{c_2}{x} + \frac{x\ln x}{2}$$

f. The total solution is

$$\begin{aligned} y(x) &= c_1x + \frac{c_2}{x} + \frac{x\ln x}{2} \\ y'(x) &= c_1x - \frac{c_2}{x^2} + \frac{1}{2}(1 + \ln x) \end{aligned}$$

And the initial conditions are $y(e) = 0\,, y'(e) = 2$, plugging back these into the previous equations gives

$$\begin{cases} y(e) = c_1e + \frac{c_2}{e} + \frac{e\ln e}{2} = 0 \\ y'(e) = c_1 - \frac{c_2}{e^2} + \frac{1}{2}(1 + \ln e) = 2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1e + c_2e^{-1} = -\frac{e}{2} \\ c_1 - c_2e^{-2} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1e^2 + c_2 = -\frac{e^2}{2} \\ c_1 - c_2e^{-2} = 1 \end{cases}$$

Adding equation (1) to equation (2) leads to $2c_1 = e^2 - \frac{e^2}{2} = \frac{e^2}{2}$, $c_1 = \frac{1}{4}$,$c_2 = e^2(c_1 - 1) = \frac{3}{4}e^2$. Reporting these constants into the expression of the total solution gives:

$$\begin{aligned} y(x) &= \frac{1}{4}x - \frac{3}{4}e^2\frac{1}{x} + \frac{x\ln x}{2} \\ y(x) &= \frac{x^2 + 2x^2\ln(x) - 3e^2}{4x} \end{aligned}$$