

Chapter 4 18 pages

Problems

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Chapter 4. Problem 4-4

By moving the origin of  $t$  to the center of an interval in which  $f(t) = +1$ , i.e. by changing to a new independent variable  $t' = t - \frac{1}{4} T$ , express the square-wave function in the example in Section 4.2 as a cosine series.

Calculate the Fourier coefficients involved (a) directly and (b) by changing the variable in result (4.10)

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Using the new independent variable  $t'$ , the square wave is now defined by:

$$f(t') = \begin{cases} -1 & \text{for } -3T/4 \leq t' < -T/4 \\ +1 & \text{for } -T/4 \leq t' \leq T/4 \end{cases}$$

The function  $f(t')$  is even thus the Fourier coefficients

$$b_r = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin\left(\frac{2\pi r t}{T}\right) dt = 0$$

as is the average of the square-wave function over  $[-T/2, T/2]$  which is easily seen be 0.

Chapter 4 - Problem 4.4For  $r > 1$ 

$$ar = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos\left(\frac{2\pi t}{T}\right) dt$$

$$\text{let } \omega = \frac{2\pi r}{T} \quad ar = \frac{2}{T} \left[ \int_{-T/2}^{-T/4} (-1) \cos(\omega t) dt + \int_{-T/4}^{T/4} \cos(\omega t) dt \right. \\ \left. + \int_{T/4}^{T/2} (-1) \cos(\omega t) dt \right]$$

$$\int_{-T/2}^{-T/4} \cos(\omega t) dt = \frac{1}{\omega} [8m\omega t]_{-T/2}^{-T/4} = \frac{1}{\omega} [8m\omega T/4 + 8m\omega T/2]$$

Similarly we find:

$$\int_{-T/4}^{T/4} \cos(\omega t) dt = 1/\omega [8m\omega t]_{-T/4}^{T/4} = \frac{2}{\omega} 8m\omega T/4$$

$$\int_{T/4}^{T/2} \cos(\omega t) dt = \frac{1}{\omega} [8m\omega t]_{T/4}^{T/2} = \frac{1}{\omega} [8m\omega T/2 - 8m\omega T/4]$$

Putting these integrals back into the expression of  $ar$ :

$$ar = \frac{2}{T} \cdot \frac{1}{\omega} \cdot [8m\omega T/4 - 8m\omega T/2 + 2 \cdot 8m\omega T/4 - 8m\omega T/4 + 8m\omega T/4]$$

$$ar = \frac{2}{T} \times \frac{T}{2\pi r} \times 4 \cdot 8m\omega \frac{T}{4} = \frac{4}{\pi r} 8m\left(\frac{2\pi r}{T} - \frac{T}{4}\right) \\ = \frac{4}{\pi r} 8m\left(\frac{\pi r}{2}\right)$$

Therefore the Fourier series for the square wave function centered at 0 is:

$$\parallel f(t) = \sum_{r=1}^{\infty} a_r \cos \frac{2\pi r t}{T} \text{ with } a_r = \frac{4}{\pi r} \sin \left( \frac{\pi r}{2} \right)$$

All the even  $a_r$  coefficients are zero and for  $r=2p+1$

$$a_{2p+1} = \frac{4(-1)^p}{\pi(2p+1)}$$

$$\text{and } f(t) = \sum_{p=0}^{\infty} \frac{4(-1)^p}{\pi(2p+1)} \cos \frac{2\pi(2p+1)t}{T}$$

$$= \frac{4}{\pi} \sum_{p=1}^{\infty} \frac{(-1)^p}{2p+1} \cos \frac{2\pi(2p+1)t}{T}$$

$$= \frac{4}{\pi} \left( \cos \omega t - \frac{\cos 3\omega t}{3} + \frac{1}{5} \cos 5\omega t - \dots \right)$$

$$\text{where } \omega = 2\pi/T$$

(b) Using the result (4.10) and making the change of variable

$$t' = t + T/4$$

$$f(t) = \frac{4}{\pi} \left( \sin \omega(t + T/4) + \frac{1}{3} \sin 3\omega(t + T/4) + \dots \right)$$

$$= \frac{4}{\pi} \left[ \sin(\omega t' + \omega T/4) + \frac{1}{3} \sin(3\omega t' + 3\omega T/4) + \dots \right]$$

$$\omega \frac{T}{4} = \frac{2\pi}{T} \cdot \frac{T}{4} = \frac{\pi}{2}$$

$$f(t) = \frac{4}{\pi} \left[ \sin(\omega t + \pi/2) + \frac{1}{3} \sin(3\omega t + 3\pi/2) + \frac{1}{5} \sin(5\omega t + 5\pi/2) + \dots \right]$$

$$f(t) = \frac{4}{\pi} \left[ \cos \omega t - \frac{1}{3} \cos 3\omega t + \frac{1}{5} \cos 5\omega t + \dots \right]$$

### Chapter 4-Problem 4.6

For the function  $f(x) = 1-x \quad 0 \leq x \leq 1$

- Find (a) the Fourier sine series and  
 (b) the Fourier cosine series.

Which would be better for numerical evaluation?

Relate your answer to the relevant periodic approximations

(a) The odd extension of  $f(x)$ :

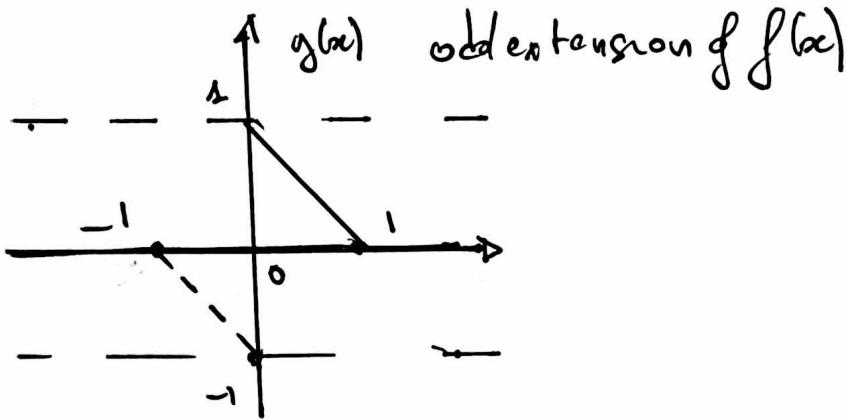
$$g(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq 1 \\ -f(-x) & \text{if } -1 \leq x \leq 0 \end{cases}$$

$$= \begin{cases} 1-x & 0 \leq x \leq 1 \\ -1-x & -1 \leq x \leq 0 \end{cases}$$

# Chapter 4 - Problem 4.6

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$g(x)$  is odd and has only Fourier coefficient  $b_r$  in the Fourier series expansion of  $f(x)$ :

$$b_r = \frac{2}{\pi} \times 2 \times \int_0^1 f(x) \sin\left(\frac{\pi r x}{2}\right) dx$$

$$= 2 \times \int_0^1 (1-x) \sin(\pi r x) dx$$

And  $\int_0^1 (1-x) \sin(\pi r x) dx = -\frac{1}{\pi r} [(1-x) \cos(\pi r x)]_0^1$

$$+ \frac{1}{\pi r} \int_0^1 1 \cos(\pi r x) dx$$

(Integration by parts)

$$= -\frac{1}{\pi r} [0 - \cos(0)] - \frac{1}{\pi r} [\sin(\pi r x)]_0^1$$

$$= \frac{1}{\pi r}$$

$$b_r = \frac{2}{\pi r}$$

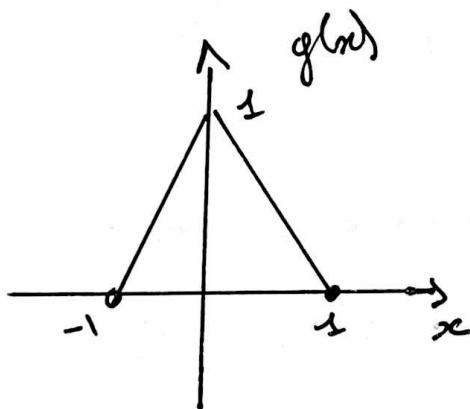
Chapter 4. Problem 4.6

So, we get the Fourier sine series for this function:

$$\| f(x) = \sum_{k=1}^{\infty} \frac{2}{\pi k} \sin(\pi k x)$$

(b) The even extension of this function

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq 1 \\ f(-x) & -1 \leq x \leq 0 \end{cases} = \begin{cases} 1-x & 0 \leq x \leq 1 \\ 1+x & \text{for } -1 \leq x \leq 0 \end{cases}$$



$g(x)$  is even, and only the  $a_0$  coefficients in its Fourier series expansion are non-zero:

$$a_0 = \frac{2}{L} \times \int_{-1}^1 (1-x) dx = \frac{2}{2} + 2 + \int_0^1 (1-x) dx$$

with  $L=2$

$$a_0 = 2 + (-\frac{1}{2}) \times [(1-x)^2]_0^1 = 1$$

Chapter 4 - Problem 4.6

$$a_r = \frac{2}{2} \times 2 \times \int_0^1 (1-x) \cos\left(\frac{2\pi r x}{2}\right) dx$$

$$a_r = 2 \times \int_0^1 (1-x) \cos(\pi r x) dx$$

$$\begin{aligned} \int_0^1 (1-x) \cos(\pi r x) dx &= \frac{1}{\pi r} \left[ (1-x) \sin(\pi r x) \right]_0^1 + \frac{1}{\pi r} \int_0^1 \sin(\pi r x) dx \\ &= \frac{1}{\pi^2 r^2} \left[ -\cos(\pi r x) \right]_0^1 = \frac{1}{\pi^2 r^2} \left[ (-1)^r - 1 \right] \\ &= \frac{1}{\pi^2 r^2} (1 - (-1)^r) \end{aligned}$$

thus  $a_r = \frac{1}{\pi^2 r^2} (1 - (-1)^r)$  and  $a_r = \begin{cases} \frac{1}{\pi^2 r^2} & r \text{ odd} \\ 0 & r \text{ even} \end{cases}$

The Fourier Series (Fourier cosine series) is then:

$$f(x) = 1/2 + \sum_{\substack{r=1 \\ \text{odd}}}^{\infty} \frac{1}{\pi^2 r^2} \cos(\pi r x)$$

Note the terms on  $1/r^2$  for the even extension compare to the term in  $1/r$  for the odd extension; the convergence to the solution is faster for the even extension. The Fourier sine series is also discontinuous at  $x=0$  which can lead to numerical instabilities. The Fourier cosine series by comparison does not have such issues.

Chapter 4- Problem 4-8

The function  $y(x) = x \sin x$  for  $0 \leq x \leq \pi$  is to be represented by a Fourier series of period  $2\pi$  that is even or odd. By sketching the function and considering its derivative, determine which series will have the more rapid convergence.

Find the full expression for the better of these two series, showing that the convergence is  $\sim n^{-3}$  and that alternate terms are missing.

See attached plot. of even and odd extensions of the function.

Note the derivative of the even extension at  $x = \pi$  is undefined since the graph has different slopes on the left of  $x = \pi$  and on the right.

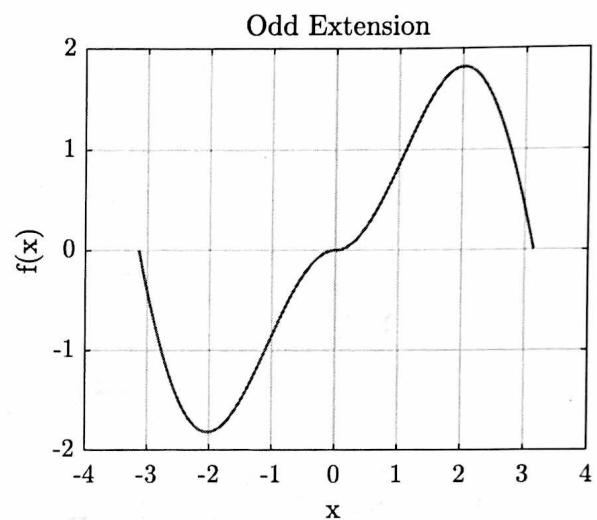
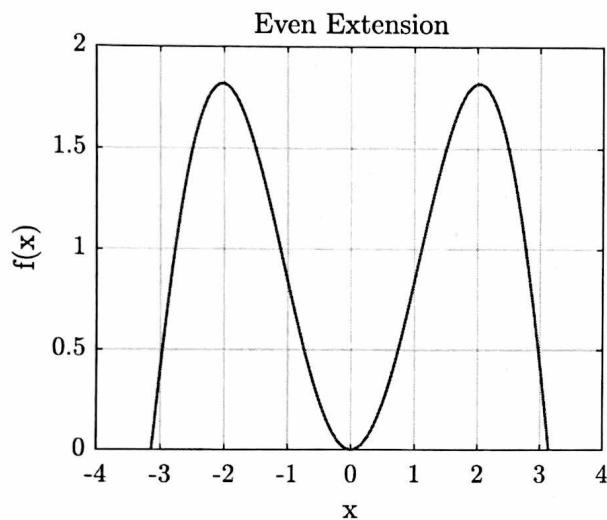
However the odd extension, & the derivative exist at  $x = \pi$ , the function is smooth at  $\pi$ , and we take the

odd extension:

$$f(x) = \begin{cases} x \sin x & x \geq 0 \\ -x \sin x & x < 0 \end{cases}$$

for  $x \in [-\pi, \pi]$ ,  $T = 2\pi$

Graph of the function:



## Chapter 4 - Problem 4.8

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The Fourier coefficients of the odd extension are:

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \cdot 2 \cdot \int_0^\pi x \sin(n) \sin(nx) dx \\
 &= \frac{1}{\pi} \int_0^\pi x [\cos(n-1)x - \cos(n+1)x] dx \\
 &= \frac{1}{\pi} \left[ \frac{(-1)^{n-1}}{(n-1)^2} - \frac{(-1)^{n+1}}{(n+1)^2} \right] \\
 &= \frac{1}{\pi} (-1) [(-1)^{n-1}] \cdot \frac{4n}{(n^2-1)^2} \quad \text{for } n \neq 1
 \end{aligned}$$

$$\begin{aligned}
 b_1 &= \frac{2}{\pi} \int_0^\pi x \sin(x) \sin(x) dx = \frac{2}{\pi} \int_0^\pi x \sin^2(x) dx \\
 &= \frac{1}{\pi} \int_0^\pi x (1 - \cos 2x) dx = \frac{1}{\pi} \left[ \frac{x^2}{2} - \frac{\sin 2x}{2} \right]_0^\pi \\
 &= \frac{1}{\pi} \frac{\pi^2}{2} = \pi/2 \\
 b_n &= \begin{cases} \frac{1}{\pi} (-1) [(-1)^{n-1}] \frac{4n}{(n^2-1)^2} & n-\text{even} \\ 0 & n-\text{odd} \end{cases}
 \end{aligned}$$

Therefore the convergence is  $\approx \frac{n}{(n^2-1)^2} = O(n^{-3})$ .

for n even.

Chapter 4 - Problem 4.9

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Find the Fourier coefficients in the expansion of  
 $f(x) = e^x$  over the range  $-1 < x < 1$ . What value  
 will the expansion have when  $x=2$ ?

$$a_0 = \frac{2}{2} \times \int_{-1}^1 e^x dx = [e^x]_{-1}^1 \times \frac{e^1 - e^{-1}}{2} = \sinh(1) \times 2$$

$$\begin{aligned} a_n &= \frac{2}{2} \times \int_{-1}^1 e^x \cos\left(\frac{2\pi n x}{2}\right) dx \\ &= \int_{-1}^1 e^x \cos(\pi n x) dx \end{aligned}$$

By Integral table:  $\int e^{ax} \cos nx dx = \frac{e^{ax}}{a^2+n^2} (\alpha \cos nx + n \sin nx)$

Therefore

$$\begin{aligned} a_n &= \frac{1}{1+(n\pi)^2} \left[ e^{2x} (\cos n\pi x + n\pi \sin(n\pi x)) \right]_{-1}^1 \\ &= \frac{1}{1+(n\pi)^2} [e^1 \cos(n\pi) - e^{-1} \cos(n\pi)] \\ &= 2 \frac{\sinh(n)}{1+(n\pi)^2} \times \cos n\pi = 2 \times \frac{(-1)^n \sinh(n)}{1+(n\pi)^2} \end{aligned}$$

## Chapter 4 - Problem 4-9

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We now find the Fourier coefficients  $b_n$ :

$$b_n = \frac{1}{2} \times \int_{-1}^1 e^x \sin(n\pi x) dx = \int_{-1}^1 e^x \sin(n\pi x) dx$$

Using an integral table:

$$\int e^{ax} \sin(nx) dx = \frac{e^{ax}}{a^2 + n^2} (a \sin nx - n \cos nx)$$

$$\begin{aligned} \Rightarrow b_n &= \frac{1}{1+(n\pi)^2} \left[ e^x (\sin n\pi x - n\pi \cos n\pi x) \right]_{-1}^1 \\ &= 2 \times \cos(n\pi) \frac{e^{-1} - e^1}{2} \times n\pi \times \frac{1}{1+(n\pi)^2} \\ &= -2 \times \sinh(1) \frac{n\pi}{1+(n\pi)^2} (-1)^n = \frac{2n\pi \sinh(1) (-1)^{n+1}}{1+(n\pi)^2} \end{aligned}$$

The Fourier series expansion of  $f(x) = e^x$  on  $(-1, 1)$  is then:

$$\begin{aligned} e^x &= 2 \frac{\sinh(1)}{2} + 2 \sinh(1) \times \sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2} [\cos n\pi x - n\pi \sin(n\pi x)] \\ &= \sinh(1) \left[ 1 + 2 \times \sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2} [\cos n\pi x - n\pi \sin(n\pi x)] \right] \end{aligned}$$

When we expand  $f(x) = e^x$  over the range  $-1 < x < 3$ , we have extended the same function in the range  $-1 < x < 3$  with the same period  $P=2$ . Therefore at  $x=2$ ,  $f(2) = f(2-2) = f(0) = 1$ .

Problem 4.10 Chapter 4

By integrating term by term the Fourier series found in the previous question and using the Fourier series for  $f(x) = x$  found in section 4.6, show that  $\int e^x dx = e^x + C$ .

Why is it not possible to show that  $\frac{d}{dx} e^x = e^x$  by differentiating the Fourier series of  $f(x) = e^x$  in a similar manner?

We found in the previous question that the Fourier series expansion of  $f(x) = e^x$  on  $(-1, 1)$  is:

$$e^x = \sinh(1) \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{4(n\pi)^2} [\cos(n\pi x) - n\pi \sin(n\pi x)] \right]$$

Integrating term by term, we have:

$$\begin{aligned} \int e^x dx &= \sinh(1) \int dx + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{4(n\pi)^2} \int (\cos(n\pi x) - n\pi \sin(n\pi x)) dx \\ &= \sinh(1)x + 2\sinh(1) \sum_{n=1}^{\infty} \frac{(-1)^n}{4(n\pi)^2} \left[ \frac{\sin(n\pi x)}{n\pi} + (\cos(n\pi x)) \right] + C \end{aligned}$$

$C = \text{constant}$

Chapter 4 - Problem 4.10

Using the result of section 4.6, on the range  $-1 \leq x \leq 1$

we have:

$$\begin{aligned} x &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin(\pi n x) \\ &= (-2) \sum_{n=1}^{\infty} (-1)^n \left[ \frac{\pi n}{1+(\pi n)^2} + \frac{1}{\pi n (1+(\pi n)^2)} \right] \sin(\pi n x) \end{aligned}$$

Substituting this expression of  $x$  back into the integral we just expressed, we have:

$$\begin{aligned} \int e^x dx &= 2 \sinh(x) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+(\pi n)^2} \cos(\pi n x) \\ &\quad + 2 \sinh(x) \sum_{n=1}^{\infty} (-1)^n \left[ \frac{-\pi n}{1+(\pi n)^2} - \frac{1}{\pi n (1+(\pi n)^2)} \right] \sin(\pi n x) \\ &\quad + C \\ \Rightarrow \int e^x dx &= 2 \sinh(x) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+(\pi n)^2} [\cos(\pi n x) - \pi n \sin(\pi n x)] + C \\ &= e^x + C \end{aligned}$$

Chapter 4 - Problem 4.10

When we differentiate term by term the Fourier series of  $\exp$ , we obtain:

$$\begin{aligned} & \frac{d}{dx} \left[ \sinh(x) \left[ 1 + 2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2} [\cos(n\pi x) - n\pi \sin(n\pi x)] \right] \right] \\ &= 2 \sinh(x) \sum_{n=1}^{\infty} \left[ \frac{(-1)^n}{1+(n\pi)^2} (n\pi) (-\sin(n\pi x)) + \frac{(n\pi)^2}{1+(n\pi)^2} (-1)^{n+1} \cos(n\pi x) \right] \\ &= 2 \sinh(x) \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n+1}}{1+(n\pi)^2} n\pi \sin(n\pi x) - \frac{(n\pi)^2 (-1)^n}{1+(n\pi)^2} \cos(n\pi x) \right] \end{aligned}$$

$$\text{let } S(x) = \sum_{n=1}^{\infty} \frac{(n\pi)^2 (-1)^n}{1+(n\pi)^2} \cos(n\pi x)$$

$\lim_{n \rightarrow \infty} \frac{(n\pi)^2}{1+(n\pi)^2} = 1$  the term  $\frac{(n\pi)^2}{1+\pi^2 n^2}$  does not go to zero

as  $n \rightarrow \infty$  therefore  $\sum_{n=1}^{\infty} \frac{(n\pi)^2 (-1)^n}{1+(n\pi)^2} \cos(n\pi x)$  does not

converge and the derivative does not exist.

Chapter 4. Problem 4.14

Show that the Fourier series for the function  $y(x) = |x|$   
in the range  $-\pi \leq x \leq \pi$  is

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos((2m+1)x)}{(2m+1)^2}$$

By integrating this equation term by term from 0 to  $x$ ,  
find the function  $g(x)$  whose Fourier series is

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)x)}{(2m+1)^3}$$

Deduce the value of the sums of the series

$$1. \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$$

The function  $y(x)$  can be described as

$$y(x) = \begin{cases} -x & ; x \in [-\pi, 0] \\ x & ; x \in [0, \pi] \end{cases}$$

$y(-x) = y(x)$  so  $y(x)$  is even and the Fourier coefficients  
 $b_r = 0$ , and we need to find the coefficients  $a_r$ ,  $r=0, 1, 2, \dots$

$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} y(x) dx$$

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$$= \frac{2}{\pi} \int_0^{\pi} y(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \pi$$

$$a_r = \frac{2}{2\pi} \times 2 \times \int_0^{\pi} x \cos\left(\frac{2\pi r}{2\pi} x\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos(rx) dx$$

$$= \frac{2}{\pi} \left[ \frac{rx \sin(rx) + \cos(rx)}{r^2} \right]_0^{\pi} \text{ by integration by part}$$

$$= \frac{2}{\pi} \times \frac{(-1)^{r-1}}{r^2} = \begin{cases} -\frac{4}{\pi r^2} & r \text{ is odd} \\ 0 & r \text{ is even} \end{cases}$$

The Fourier series of  $y(x)$  in  $[-\pi, \pi]$  is then:

$$y(x) = \frac{\pi}{2} + \sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} a_r \cos\left(\frac{2\pi r}{2\pi} x\right)$$

$$y(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} \frac{\cos(rx)}{r^2}$$

$$\| y(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos((2m+1)x)}{(2m+1)^2}$$

Chapter 4 - Problem 4.14

$$\int_0^x y(x) dx = \int_0^x \left[ \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2} \right] dx$$

$$\int_0^x x dx = \int_0^x \frac{\pi}{2} dx - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \int_0^x \cos(2m+1)x dx$$

$$\Rightarrow \frac{x^2}{2} = \frac{\pi}{2} x - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^3} [\sin(2m+1)x]_0^x$$

$$\Rightarrow \frac{x^2}{2} = \frac{\pi}{2} x - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{(2m+1)^3}$$

$$\Rightarrow \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{(2m+1)^3} = \frac{\pi}{2} x - \frac{x^2}{2} = \frac{x}{2} (\pi - x)$$

Take  $x = \frac{\pi}{2}$  and we find:

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)\frac{\pi}{2}}{(2m+1)^3} = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(m\pi + \frac{\pi}{2})}{(2m+1)^3}$$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} = \frac{\pi}{4} \times \frac{\pi}{2+2} \times \left(\pi - \frac{\pi}{2}\right)$$

$$= \frac{\pi^3}{32}$$

$$\Rightarrow 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = \frac{\pi^3}{32}$$