7.8

$zy'' - 2y' + zy = 0$

Put into standard form

$y'' - \frac{2}{z}y' + y = 0$

$p(z) = -\frac{2}{z},\ q(z) = 1$

$zp(z)$ and $z^2q(z)$ are analytic at $z = 0$

Therefore regular singular point

Take

$$y = z^{\sigma}\sum_{n=0}^{\infty} a_n z^{\,n} = \sum_{n=0}^{\infty} a_n z^{n+\sigma} \text{ and therefore } y' = \sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2}$$

Substitution

$$z\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} - 2\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1} + z\sum_{n=0}^{\infty} a_n z^{\,n+\sigma} = 0$$

Simplify

$$\sum_{n=0}^{\infty}[(n+\sigma)(n+\sigma-1) - 2(n+\sigma)]a_n z^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n z^{\,n+\sigma+1} = 0$$

Take term with lowest power of z, that is first sum with $n = 0$, then since each power of z term must be equal to zero we have

$$[(\sigma)(\sigma-1) - 2(\sigma)]a_0 z^{\,\sigma-1} = 0$$

Now $a_0 \neq 0$ and $z^{\,\sigma-1} \neq 0$ therefore $(\sigma)(\sigma-1) - 2(\sigma) = 0$ and $\sigma = 0, 3$

Next go back to sums above, that is

$$\sum_{n=0}^{\infty}[(n+\sigma)(n+\sigma-1) - 2(n+\sigma)]a_n z^{n+\sigma-1} + \sum_{n=2}^{\infty} a_{n-2} z^{\,n+\sigma-1} = 0$$

IMPORTANT: Ignore terms with n = 0, 1 in sum one (remember each power of z sums to 0 BUT $\sigma = 0,\ 3$ makes n = 0 term in first sum 0 - because thats how we found values for $\sigma$. Secondly for n = 1 we can take $a_1 = 0$ and therefore start sum one at n = 2, thus matching powers of z in second sum! We then have

$$\sum_{n=2}^{\infty}\{[(n+\sigma)(n+\sigma-1) - 2(n+\sigma)]a_n + a_{n-2}\}z^{\,n+\sigma-1} = 0$$

Therefore for $n \geq 2$ each power of z term is zero and we have

$$\{[(n+\sigma)(n+\sigma-1)-2(n+\sigma)]a_n + a_{n-2}\}z^{n+\sigma-1} = 0$$

Since $z^{n+\sigma-1} \neq 0$ we have $[(n+\sigma)(n+\sigma-1)-2(n+\sigma)]a_n + a_{n-2} = 0$

Simplifing gives

$$a_n = \frac{-a_{n-2}}{(n+\sigma)(n+\sigma-3)} \quad n \geq 2 \text{ or alternatively } a_{n+2} = \frac{-a_n}{(n+\sigma+2)(n+\sigma-1)} \quad n \geq 0$$

Now taking $\sigma = 0$, $a_0$ as parameter and $a_1 = 0$

$$\text{For } n = 0 \;\; a_2 = \frac{-a_0}{(2)(-1)} = \frac{a_0}{2} \text{ or } a_{2\cdot 1} = \frac{(-1)^2 1 a_0}{(2\cdot 1)1} = \frac{(-1)^{1+1}(2\cdot 1-1)a_0}{(2\cdot 1)!}$$

Skip n=1 since $a_1 = 0$ and therefore skip all odd values of $a_n$ since they will be 0

$$\text{For } n = 2 \;\; a_4 = \frac{-a_2}{(4)(1)} = \frac{-a_0}{4\cdot 2} \text{ or } a_{2\cdot 2} = \frac{(-1)^3 3a_0}{(2\cdot 2)3(2\cdot 1)1} = \frac{(-1)^{2+1}(2\cdot 2-1)a_0}{(2\cdot 2)!}$$

Also do n = 4 and 6 where we have

$$a_6 = \frac{-a_4}{(6)(3)} = \frac{-(-1)a_0}{(6)(3)(2\cdot 2)(2\cdot 1)} \text{ or } a_{2\cdot 3} = \frac{(-1)^4 5a_0}{(2\cdot 3)5(2\cdot 2)3(2\cdot 1)1} = \frac{(-1)^{3+1}(2\cdot 3-1)a_0}{(2\cdot 3)!}$$

$$a_8 = \cdots = a_{2\cdot 4} = \frac{(-1)^{4+1}(2\cdot 4-1)a_0}{(2\cdot 4)7(2\cdot 3)5(2\cdot 2)3(2\cdot 1)}$$

And yes in order to get the above I was comparing with the book solution $y_2$ where $\sigma = 0$ Therefore an explicit formula for $a_{2n}$ is

$$a_{2n} = \frac{(-1)^{n+1}(2n-1)a_0}{2n!}$$

and we have

$$y_2(z) = z^0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)a_0}{2n!} z^{2n} = a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)}{2n!} z^{2n}$$

Now for $\sigma = 3$ $a_n = \dfrac{-a_{n-2}}{(n+\sigma)(n+\sigma-3)} \rightarrow a_n = \dfrac{-a_{n-2}}{(n+3)(n+3-3)} = \dfrac{-a_{n-2}}{(n+3)n}$ $n \geq 2$

Take $a_0 = 1$ and $a_1 = 0$ (therefore only need to look at even indexed terms)

For $n = 2$ $a_2 = \dfrac{-a_{2-2}}{(2+3)2} = \dfrac{-1}{5\cdot 2}a_0$

For $n = 4$ $a_4 = \dfrac{-a_{4-2}}{(4+3)4} = \dfrac{-a_2}{7\cdot 4}$ or $a_4 = \dfrac{-a_2}{(4+3)4} = \dfrac{-1}{7\cdot 4}\left(\dfrac{-a_0}{5\cdot 2}\right) = \dfrac{1}{7\cdot 5\cdot 4\cdot 2}a_0$

$n = 6$ $a_6 = \dfrac{-a_{6-2}}{(6+3)6} = \dfrac{-a_4}{9\cdot 6}$ or $a_6 = \dfrac{-a_4}{9\cdot 6} = \dfrac{-1}{9\cdot 6}\left(\dfrac{1\cdot a_0}{7\cdot 5\cdot 4\cdot 2}\right) = \dfrac{-1}{9\cdot 7\cdot 6\cdot 5\cdot 4\cdot 2}a_0$

Looking at result we need we re-factor and change indexing

(formula in book starts with n = 1) and increments by 1

Lets look at $a_2 = \dfrac{-1}{5\cdot 2}a_0 \rightarrow a_{2\bullet 1} = \dfrac{(-1)^1}{5\cdot 2}a_0 = \dfrac{(-1)(4\cdot 3)}{5\cdot 4\cdot 3\cdot 2}a_0 = \dfrac{(-1)^{1+2}2(1+1)}{(2\cdot 1+3)!}3a_0$;Note new n =1

$a_4 = \dfrac{1}{7\cdot 5\cdot 4\cdot 2}a_0 \rightarrow$ $a_{2\bullet 2} = \dfrac{(-1)^2(6\cdot 3)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}a_0 = \dfrac{(-1)^2(6)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}3a_0 = \dfrac{(-1)^{2+2}2(2+1)}{(2\cdot 2+3)!}3a_0$;Note new n =2

$a_6 = \dfrac{-1}{9\cdot 7\cdot 6\cdot 5\cdot 4\cdot 2}a_0 \rightarrow a_{2\bullet 3} = \dfrac{(-1)^3(8\cdot 3)}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}a_0 = \cdots = \dfrac{(-1)^{3+2}2(3+1)}{(2\cdot 3+3)!}3a_0$;Note new n =3

So in general $a_{2\bullet n} \equiv a_{2n} = \dfrac{(-1)^{n+2}2(n+1)}{(2\cdot n+3)!}3a_0$ ; n=1,2,3,$\cdots$

Substitution gives $y_1(z) = z^3\sum_{n=0}^{\infty} a_{2n}z^{2n} = z^3\sum_{n=0}^{\infty}\dfrac{(-1)^{n+2}2(n+1)}{(2\cdot n+3)!}3a_0z^{2n} = \sum_{n=0}^{\infty}\dfrac{(-1)^{n+2}2(n+1)}{(2\cdot n+3)!}3a_0z^{2n+3}$

To match book reindex with 2n+3 $\rightarrow$2m+1 OR m = n+1 and in particular for n = 0 $\rightarrow m = 1$

(just a different index but equivalent terms)

Therefore $y_1(z) = \sum_{n=0}^{\infty}\dfrac{(-1)^{n+1+1}2(n+1)}{(2n+3)!}3a_0z^{2n+3} \rightarrow y_1(z) = 3a_0\sum_{m=1}^{\infty}\dfrac{(-1)^{m+1}2m}{(2m+1)!}z^{2m+1}$

Note in this case my m index is like n in book for $y_1(z)$

\

(b)- part 1

Look at
$3a_0(\sin z - z\cos z)$

Note the following Taylor expansions

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

Then

$$3a_0(\sin z - z\cos z) = 3a_0 \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)!} z^{2n+1} - z\frac{(-1)^n}{(2n)!} z^{2n} \right] = 3a_0 \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)!} - \frac{(-1)^n}{(2n)!} \right] z^{2n+1}$$

$$= 3a_0 \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)!} - \frac{(-1)^n(2n+1)}{(2n+1)(2n)!} \right] z^{2n+1} = 3a_0 \sum_{n=0}^{\infty} \left[ \frac{(-1)^n - (-1)^n(2n+1)}{(2n+1)!} \right] z^{2n+1}$$

$$= 3a_0 \sum_{n=0}^{\infty} \frac{(-1)^n[1-(2n+1)]}{(2n+1)!} z^{2n+1} = 3a_0 \sum_{n=0}^{\infty} \frac{(-1)^n 2n}{(2n+1)!} z^{2n+1}$$

$$= 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^n 2n}{(2n+1)!} z^{2n+1}$$ reindex since $n = 0$ term does not contribute to sum.

Then this sum is equal to $y_1(z)$

Now find $y_2(z)$ using Wronskian method pg. 286

(b)- part 2

Take $y_1(z) = \sin z - z\cos z$ don't include $3a_0$ it's just a constant

Then

$$y_2(z) = y_1(z)\int^z \frac{1}{[y_1(u)]^2} e^{-\int^u p(v)dv} du$$

from original D.E. $p(v) = \frac{-2}{v}$ and $e^{-\int^u p(v)dv} = e^{\int^u \frac{2}{v}dv} = e^{2\ln v}\Big|^u = u^2$

Next evaluate

$$\int^z \frac{1}{[y_1(z)]^2} e^{-\int^u p(v)dv} du = \int^z \frac{1}{[\sin u - u\cos u]^2} u^2 du$$

Next do as the book says write $u^2$ as $\left(\frac{u}{\sin u}\right)\sin u$ and substitute

$$\int^z \left(\frac{u}{\sin u}\right) \frac{u\sin u}{[\sin u - u\cos u]^2}\, du$$

Now integrate by parts, let $s = \frac{u}{\sin u} = u(\sin u)^{-1}$ therefore $ds = [(\sin u)^{-1} + (-1)(\sin u)^{-2}(\cos u)]du$

$$= \frac{(\sin u) - u(\cos u)}{\sin^2 u} du$$

and let $dt = \frac{u\sin u}{[\sin u - u\cos u]^2} du$ therefore $t = \int^z \frac{u\sin u}{[\sin u - u\cos u]^2} du$

Note $\frac{d}{du}(\sin u - u\cos u)^{-1} = \frac{-u\sin u}{[\sin u - u\cos u]^2}$

Therefore $t = \int^z \frac{d}{du}(\sin u - u\cos u)^{-1} du = -(\sin u - u\cos u)^{-1}$

Using integration by parts $\int s\,dt = st - \int t\,ds$ and relations above we have

$$\int^{z}\left(\frac{u}{\sin u}\right)\frac{u\sin u}{[\sin u - u\cos u]^2}\,du = \left(\frac{u}{\sin u}\right)[-(\sin u - u\cos u)^{-1}]\Big|^{z} - \int^{z}[-(\sin u - u\cos u)^{-1}]\frac{(\sin u)-u(\cos u)}{\sin^2 u}du$$

$$= \frac{-z}{\sin z(\sin z - z\cos z)} + \int^{z}\frac{1}{\sin^2 u}du = \frac{-z}{\sin z(\sin z - z\cos z)} - \cot z$$

Finally

$$y_2(z) = y_1(z)\int^{z}\frac{1}{[y_1(u)]^2}e^{-\int^{u}p(v)dv}du \text{ and from work above } \int^{z}\frac{1}{[y_1(u)]^2}e^{-\int^{u}p(v)dv} = \frac{-z}{\sin z(\sin z - z\cos z)} - \cot z$$

and original $y_1(z) = \sin z - z\cos z$

We have

$$y_2(z) = (\sin z - z\cos z)\left[\frac{-z}{\sin z(\sin z - z\cos z)} - \cot z\right] = \frac{-z}{\sin z} - (\sin z - z\cos z)\cot z$$

$$= \frac{-z}{\sin z} - (\sin z - z\cos z)\frac{\cos z}{\sin z} = \frac{-z}{\sin z} - \cos z + z\frac{\cos^2 z}{\sin z} = \frac{-z(1-\cos^2 z)}{\sin z} - \cos z = \frac{-z\sin^2 z}{\sin z} - \cos z$$

and

$y_2(z) = -z\sin z - \cos z$

(c)

$$W(z) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$y_1(z) = \sin z - z\cos z$

$y_2(z) = -z\sin z - \cos z$

Gives

$y_1'(z) = z\sin z$

$y_2'(z) = -z\cos z$

Therefore

$$W(z) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (\sin z - z\cos z)(-z\cos z) - (z\sin z)(-z\sin z - \cos z) = \cdots = z^2$$