5.26

Show $L\{f(t-a)H(t-a)\} = e^{-as}\tilde{f}(t)$

$$\int_0^\infty e^{-st} f(t-a)H(t-a)dt = \text{using def. H(t-a)}=\int_0^a e^{-st} f(t-a)\cdot 0dt + \int_a^\infty e^{-st} f(t-a)\cdot 1dt =$$

$$\int_a^\infty e^{-st} f(t-a)dt$$

Now let $t-a=\hat{t}$ or $t=a+\hat{t}$ also $dt = d\hat{t}$ also the bounds change!!

$$\int_a^\infty e^{-st} f(t-a)dt \to \int_{a-a=0}^{\infty-a=\infty} e^{-s(a+\hat{t})} f(\hat{t})d\hat{t} = \int_0^\infty e^{-s(a+\hat{t})} f(\tilde{t})d\tilde{t} \to e^{-sa}\int_0^\infty e^{-st} f(t)dt = e^{-sa}\tilde{f}(t)$$

where t has replace $\hat{t}$ without change to meaning (or value)of integral

For next part the KEY is function is periodic, with period T therefore $g(t+nT)=g(t)$

$$L\{g(t)\}=\int_0^\infty g(t)e^{-st}dt$$

$$=\int_0^T g(t)e^{-st}dt+\int_T^{2T} g(t)e^{-st}dt+\int_{2T}^{3T} g(t)e^{-st}dt+\cdots$$

Substitute τ=t in first integral τ=t-T in second, τ=t-2T, etc.

$$=\int_0^T g(\tau)e^{-s\tau}d\tau+\int_0^T g(\tau+T)e^{-s(\tau+T)}d\tau+\int_0^T g(\tau+2T)e^{-s(\tau+2T)}d\tau+\cdots=$$

Now you need to do a little algebra and also use the periodicity
$g(\tau+nT)=g(\tau)$ to get the formula in book
Therefore we have

$$L\{g(t)\}=[1+e^{-sT}+e^{-2sT}+\cdots]\int_0^T g(\tau)e^{-s\tau}d\tau=[1+e^{-sT}+(e^{-sT})^2+\cdots]\int_0^T g(\tau)e^{-s\tau}d\tau=$$

$$\lim_{N\to\infty}\sum_{n=0}^{N}(e^{-sT})^n\int_0^T g(\tau)e^{-s\tau}d\tau$$

You will also need this general formula $1+x+x^2+\cdots=\sum_{n=0}^{N}x^n=\dfrac{x^{N+1}-1}{x-1}$, then

Now in our case let $x=e^{-sT}$ and N is ∞ therefore

$$L\{g(t)\}=\lim_{N\to\infty}\sum_{n=0}^{N}(e^{-sT})^n\int_0^T g(\tau)e^{-s\tau}d\tau=\text{(Finish this now using sum above, show steps)}=$$

$$\frac{1}{1-e^{-sT}}\int_0^T g(\tau)e^{-s\tau}d\tau \text{ (in book)}$$

(a) Sketch (do by hand or matlab) – hint it should look like a triangle, peak at T/2.

Then find the Laplace transform assuming its periodic (use result above)

$$g(t)=\begin{cases} 2t/T & 0\le t<T/2 \\ 2(1-t/T) & T/2<t\le T \end{cases}$$

Now do the integral $\frac{1}{1-e^{-sT}}\int_0^T g(\tau)e^{-s\tau}d\tau=$

First $\int_0^T g(\tau)e^{-s\tau}d\tau=\int_0^{T/2}\frac{2\tau}{T}e^{-s\tau}d\tau+\int_{T/2}^{T}2\left(1-\frac{\tau}{T}\right)e^{-s\tau}d\tau=$

$$\frac{2}{T}\int_0^{T/2}\tau e^{-s\tau}d\tau+2\int_{T/2}^{T}e^{-s\tau}d\tau-\frac{2}{T}\int_{T/2}^{T}\tau e^{-s\tau}d\tau=\cdots\frac{2}{Ts^2}\left(1-e^{-sT/2}\right)^2$$

Therefore

$$\tilde{g}(s)=\frac{1}{1-e^{-sT}}\int_0^T g(\tau)e^{-s\tau}d\tau=\frac{1}{1-e^{-sT}}\left[\frac{2}{Ts^2}\left(1-e^{-sT/2}\right)^2\right]=\frac{2}{Ts^2}\frac{\left(1-e^{-sT/2}\right)^2}{1-e^{-sT}}=$$

$$\frac{2}{Ts^2}\frac{\left(1-e^{-sT/2}\right)^2}{(1-e^{-sT/2})(1+e^{-sT/2})}=\frac{2}{Ts^2}\frac{1-e^{-sT/2}}{1+e^{-sT/2}}=\text{(hyperbolic tan identity)}=\frac{2}{Ts^2}\tanh\left(\frac{sT}{4}\right)$$

(b)

For this part of problem we have

$$h(t)=\frac{2}{T}\left[(t-0)H(t-0)+2\sum_{n=1}^{\infty}(-1)^n(t-\frac{1}{2}nT)H(t-\frac{1}{2}nT)\right]$$

PLOT IT FOR A COUPLE OF PERIODS, SAY 0 to 2T

Note h(t) is natuarely periodic as presented, so it looks like the function in part (a), g(t) but in that case we had to also state its periodic!

Now $L\{h(t\}=L\{\frac{2}{T}\left[(t-0)H(t-0)+2\sum_{n=1}^{\infty}(-1)^n(t-\frac{1}{2}nT)H(t-\frac{1}{2}nT)\right]\}=$

$$\frac{2}{T}L\{(t-0)H(t-0)\}+\frac{4}{T}\sum_{n=1}^{\infty}(-1)^n L\{(t-\frac{1}{2}nT)H(t-\frac{1}{2}nT)\}$$

Aside: From top pg 215 $L\{f(t-b)H(t-a)\}\equiv\int_0^{\infty}e^{-st}f(t-b)H(t-a)dt=e^{-sb}\tilde{f}(s)$

Therefore $L\{h(t\}=\frac{2}{T}e^{-s0}\frac{1}{s^2}+\frac{4}{T}\sum_{n=1}^{\infty}(-1)^n e^{-s\frac{1}{2}nT}\frac{1}{s^2}=\frac{2}{Ts^2}(1)+\frac{2}{Ts^2}2\sum_{n=1}^{\infty}(-1)^n e^{-s\frac{1}{2}nT}=$

$$\frac{2}{Ts^2}\left[1+2\sum_{n=1}^{\infty}(-1)^n e^{-s\frac{1}{2}nT}\right]$$

From (a) $\frac{2}{Ts^2}\tanh\left(\frac{sT}{4}\right)$

From (b) $\frac{2}{Ts^2}\left[1+2\sum_{n=1}^{\infty}(-1)^n e^{-s\frac{1}{2}nT}\right]$

These areLaplace transforms of essentially the same periodic function therefore they are equal to eac

$$\frac{2}{Ts^2}\tanh\left(\frac{sT}{4}\right)=\frac{2}{Ts^2}\left[1+2\sum_{n=1}^{\infty}(-1)^n e^{-s\frac{1}{2}nT}\right]\rightarrow \tanh\left(\frac{sT}{4}\right)=1+2\sum_{n=1}^{\infty}(-1)^n e^{-s\frac{1}{2}nT}$$

Let $x=\frac{sT}{4}$ gives $\tanh x=1+2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}$