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A little more complicated example

Unlike the previous examples (last lecture) suppose we have a differential equation with partial derivative of two variables x and y. As we did previously we cannot simply treat the solution as a function of one variable for purposes of integration and have an integration factor dependent on the variable for which no derivative appears in the differential equation. In this case we develop the following ad hoc procedure.

As before we have the case where u(x,y) = f(p) and p is a function of x and y, that is p(x,y)

Take $\frac{\partial u}{\partial x} = \frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial x}$ and $\frac{\partial u}{\partial y} = \frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial y}$

Then construct the following equation

$$A(x,y)\frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial x} + B(x,y)\frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial y} = \left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right]\frac{\partial f(p)}{\partial p} = 0$$

or since in general $\frac{\partial f(p)}{\partial p} \neq 0$ simply $\boxed{A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0}$

Next consider in particular when f(p) is constant, that is

$$df = \frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial x}dx + \frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial y}dy = \left[\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy\right]\frac{\partial f(p)}{\partial p} = 0$$

Again since $\frac{\partial f(p)}{\partial p} \neq 0$ we have $df = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0$

Since we have both

$\boxed{A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0}$ and $\boxed{df = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0}$

We can equate these equations when

$\boxed{\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}}$ or $dy = \frac{B(x,y)}{A(x,y)}dx$

as can be shown next by substitution

Start with $\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = 0$ then $\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}\left[\frac{B(x,y)}{A(x,y)}dx\right] = 0$

Multiplication by A(x,y) gives

$$A(x,y)\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}B(x,y)dx = A(x,y)\frac{\partial p}{\partial x}dx + B(x,y)\frac{\partial p}{\partial y}dx$$

$$= \left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right]dx = 0 \text{ or } A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0$$

Let’s look at a couple of examples

As our first example take $x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$

Therfore $A(x,y) = x$ and $B(x,y) = -2y$

Using $\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)} \to \boxed{\frac{dx}{x} = \frac{dy}{-2y}}$

Integrating $\int \frac{dx}{x} = \int \frac{dy}{-2y}$ gives $\ln x = -\frac{1}{2}\ln y + C$

However as we have done before we will take the integration constant form that make the relationship have a "nice" look, that is let $C = \frac{1}{2}\ln c$. Therefore

$$\ln x = -\frac{1}{2}\ln y + \frac{1}{2}\ln c = \ln(c^{1/2}y^{-1/2}) \to e^{\ln x} = e^{\ln(c^{1/2}y^{-1/2})} \to x = c^{1/2}y^{-1/2}$$

Next solve for c $\quad x = c^{1/2}y^{-1/2} \to x^2 = cy^{-1} \to \boxed{c = x^2y}$

Therefore this is the general form of our solution is $u(x,y) = f(p)$ where $p(x,y) = x^2y$

For a simple case when $u(x,y) = p = x^2y$ we have $\dfrac{\partial u}{\partial x} = 2xy$ and $\dfrac{\partial u}{\partial y} = x^2$

and substitution into our original differential equation gives

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = x(2xy) - 2y(x^2) = 2x^2y - 2x^2y = 0$$

Now lets include additional information about our solution.
In case 1: Take a solution such that it takes the value 2y+1 at x = 1

Therefore its easy to guess that $\boxed{u(x,y) = f(p) = 2p+1 = 2(x^2y)+1}$

Since at $x = 1$ we have $f(p(1,y)) = 2(1^2y)+1 = 2y+1$

For case 2: Require our solution have the value 4 when x=1 and y = 1

Therefore its easy to guess that $\boxed{u(x,y) = f(p) = p+3 = x^2y+3}$

Since at $(1,1)$ we have $f(p(1,1)) = (1^2 \cdot 1)+3 = 1+3 = 4$

KEY: Note other forms of the solution are possible as long as they are functions of p(x,y)

For example in case 2 we could take $u(x,y) = f(p) = 4p = 4x^2y$

and in general we could take $u(x,y) = x^2y+3+g(x^2y)$ provided $g(1^2 \cdot 1) = 0$

As our second example let's take an inhomogeneous partial differential equation.

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 2x$$

Analogous to that for regular differential equations the solution will consist of a solution to the homogeneous equation and a particular part.

For the homogeneous equation we have $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$

Therfore $A(x,y) = y$ and $B(x,y) = -x$

Using $\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)} \rightarrow \frac{dx}{y} = \frac{dy}{-x} \rightarrow -xdx = ydy$

Integrating $\int -x\,dx = \int y\,dy$ gives $-\frac{x^2}{2} = \frac{y^2}{2} - \frac{c}{2}$

Solving for c gives $c = x^2 + y^2$

Therefore take $p(x,y) = x^2 + y^2$ and the solution to the homogeneous solution as $\boxed{u_h(x,y) = f(p) = f(x^2 + y^2)}$

Next let's look at the original equation $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 3x$

By inspection we pick the particular solution as $\boxed{u_p(x,y) = -3y}$

since $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(-3y) = 0$ and $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(-3y) = -3$

therefore $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = y(0) - x(-3) = 3x$

Finally the general solution has the form

$$\boxed{u(x,y) = u_h(x,y) + u_p(x,y) = f(x^2+y^2) - 3y}$$

Next let's include a boundary conditions (i) $u(x,0) = x^2$ or (ii) $u(1,0) = 2$

$u(x,0) = u_h(x,0) + u_p(x,0) = f(x^2+0^2) - 3(0) = x^2$

So simply take $f(x^2+y^2) = x^2+y^2$

Now we have $\boxed{\text{case (i)}\ \ u(x,y) = x^2+y^2-3y}$

Alternatively require for case (ii) $u(1,0) = 2$

Therefore we could take $\boxed{u(x,y) = f(x^2+y^2) - 3y + 2}$

provided we require $\boxed{f(1^2+0^2) = f(1) = 0}$