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Mathematical Methods for Applied Biomedical Engineering
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What do partial differential equations look like

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$ one dimensional wave equation

$$k\frac{\partial^2 u}{\partial x^2} + f(x,t) = \sigma\rho\frac{\partial u}{\partial t}$$ one dimensional diffusion equation with source term

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ two dimensional diffusion equation "steady state" - Laplace's

$$\nabla^2 u = \rho(r)$$ three dimensional diffusion with source - Poisson's

$$-\frac{\hbar}{2m}\nabla u + V(r)u = i\hbar\frac{\partial u}{\partial t}$$ Schrödinger's equation (quantum mechanics)

A look at some general forms of solutions to partial differential equations

Take two solutions that at first look dissimilar

$$u_1(x,y) = x^4 + 4(x^2y + y^2 + 1)$$
$$u_2(x,y) = \sin x^2 \cos 2y + \cos x^2 \sin 2y$$

However letting $p(x,y) = x^2 + 2y$

We have

$$u_1(x,y) = x^4 + 4(x^2y + y^2 + 1) \rightarrow \; x^4 + 4x^2y + 4y^2 + 4 = (x^2 + 2y)^2 + 4 = p^2 + 4 = f_1(p)$$
$$u_2(x,y) = \sin x^2 \cos 2y + \cos x^2 \sin 2y = \sin(x^2 + 2y) = \sin(p) = f_2(p)$$

Therefore expressing both functions in terms of the variable p we see that they are quit similar in that both are a function of p(x,y). In fact let's take some partial derivatives to see how similar!

Since $u(x,y) = f(p(x,y))$ we can write (via chain rule) $\dfrac{\partial u}{\partial x} = \dfrac{\partial f(p)}{\partial p}\dfrac{\partial p}{\partial x}$

First

For $p(x,y) = x^2 + 2y$ we then have for $u_1(x,y) = p^2 + 4$

Take $\dfrac{\partial u_1}{\partial x} = \dfrac{\partial f(p)}{\partial p}\dfrac{\partial p}{\partial x} = (2p)(2x) = 4px = 4(x^2 + 2y)x$

and $\dfrac{\partial u_1}{\partial y} = \dfrac{\partial f(p)}{\partial p}\dfrac{\partial p}{\partial y} = (2p)(2) = 4p = 4(x^2 + 2y)$

Therefore $\boxed{\dfrac{\partial u_1}{\partial x} = x\dfrac{\partial u_1}{\partial y}}$

Next

For $p(x,y) = x^2 + 2y$ we then have for $u_2(x,y) = \sin p$

Take $\dfrac{\partial u_1}{\partial x} = \dfrac{\partial f(p)}{\partial p}\dfrac{\partial p}{\partial x} = \cos(p)(2x) = 2\cos(p)x = 2\cos(x^2 + 2y)x$

and $\dfrac{\partial u_1}{\partial y} = \dfrac{\partial f(p)}{\partial p}\dfrac{\partial p}{\partial y} = \cos(p)(2) = 2\cos(p) = 2\cos(x^2 + 2y)$

Therefore $\boxed{\dfrac{\partial u_2}{\partial x} = x\dfrac{\partial u_2}{\partial y}}$

KEY: Therefore we see both these solutions solve the same partial differential equation!

A quick look at methods to solve some very simple equations

Example 1

$$\frac{\partial u}{\partial y} + y^2 u = 0 \text{ where } u = u(x,y)$$

Since we have only have derivatives with respect to y we can rewrite

this $\frac{du}{dy} = -y^2 u$ where implicitly u is a function of x and y!

Here f(x) is a constant with respect to integration by y
We also use ln(f(x) to give a "nice" form to the finally solution!

Therefore $\frac{du}{dy} = -y^2 u$ or $\frac{du}{u} = -y^2 dy$

Integrating gives $\int \frac{du}{u} = \int -y^2\, dy \rightarrow \ln(u) = -\frac{y^3}{3} + \ln f(x)$

Therefore $\ln(u) - \ln f(x) = -\frac{y^3}{3}$ or $\ln\left[\frac{u}{f(x)}\right] = -\frac{y^3}{3}$

Taking the antilog of exponential on both sides gives $\frac{u}{f(x)} = e^{-\frac{y^3}{3}}$

Finally $\boxed{u(x,y) = f(x) e^{-\frac{y^3}{3}}}$

Example 2

$$x\frac{\partial u}{\partial x}+3u=x^2 \text{ where } u = u(x,y)$$

Dividing by x gives $\frac{\partial u}{\partial x}+\frac{3}{x}u=x$

Working on the left hand side and treating u(x,y) as and function of only x again we can find an integrating factor with the following technque

$$\boxed{e^{\int f(x)dx}\left[\frac{du}{dx}+f(x)u(x)\right]=\frac{d}{dx}\left[e^{\int f(x)dx}u(x)\right]}$$

Therefore in our case with f(x) = $\frac{3}{x}$ we find the integration factor

$$e^{\int \frac{3}{x}dx}=e^{3\ln x}=\left(e^{\ln x}\right)^3=x^3$$

Multiplying our differential equation by this gives

$$x^3\left[\frac{du}{dx}+\frac{3}{x}u=x\right]\rightarrow x^3\frac{du}{dx}+3x^2u=x^4\rightarrow\frac{d}{dx}\left[x^3u\right]=x^4$$

Integrating gives $x^3u=\frac{x^5}{5}+f(y)$ or $\boxed{u(x,y)=\frac{1}{x^3}\left[\frac{x^5}{5}+f(y)\right]=\frac{x^2}{5}+\frac{f(y)}{x^3}}$