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Differentiation and Integration of Fourier series

It's relatively straight forward to differentiate or integrate a Fourier series to indirectly find expressions for other functions.

Here is an example starting with the Fourier series previously derived.

For $f(x) = x^2$ from the previous example we have $a_0 = \frac{8}{3}$ $a_r = \frac{16}{\pi^2 r^2}(-1)^r$ $b_r = 0$

The Fourier series is $$x^2 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}(-1)^r \cos\left(\frac{\pi r x}{2}\right)$$

Integration of Fourier series - example

Start by using integration with respect to the variable x

$$\int x^2 dx = \int \left[\frac{4}{3} + \frac{16}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} (-1)^r \cos\left(\frac{\pi r x}{2} \right) \right] dx$$

Only functions of x are to be integrated

This gives

$$\frac{x^3}{3} = \frac{4}{3}x + \frac{16}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} (-1)^r \int \cos\left(\frac{\pi r x}{2} \right) dx$$

$$= \frac{4}{3}x + \frac{16}{\pi^2} \sum_{r=1}^{\infty} \frac{1}{r^2} (-1)^r \frac{2}{\pi r} \sin\left(\frac{\pi r x}{2} \right) + C$$

$$= \frac{4}{3}x + \frac{32}{\pi^3} \sum_{r=1}^{\infty} \frac{1}{r^3} (-1)^r \sin\left(\frac{\pi r x}{2} \right) + C$$

Differentiation of Fourier series – an example

We can also differentiate both sides of the original series.

$$\frac{d}{dx}x^2 = \frac{d}{dx}\left[\frac{4}{3} + \frac{16}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}(-1)^r \cos\left(\frac{\pi r x}{2}\right)\right]$$

This gives

$$2x = \frac{16}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}(-1)^r \frac{d}{dx}\cos\left(\frac{\pi r x}{2}\right)$$

$$= \frac{16}{\pi^2}\sum_{r=1}^{\infty}\frac{1}{r^2}(-1)^r \frac{\pi r}{2}\left[-\sin\left(\frac{\pi r x}{2}\right)\right]$$

$$= \frac{-8}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\sin\left(\frac{\pi r x}{2}\right)$$

Only functions of x
are to be differentiated

Therefore the Fourier series for x can be written as follows

$$x = \frac{-4}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\sin\left(\frac{\pi r x}{2}\right)$$

We can the expression for the Fourier series of x in the integrated Fourier series Expression we first derived. Thus

$$\frac{x^3}{3} = \frac{4}{3}\boxed{\left[\frac{-4}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\sin\left(\frac{\pi r x}{2}\right)\right]} + \frac{32}{\pi^3}\sum_{r=1}^{\infty}\frac{1}{r^3}(-1)^r\sin\left(\frac{\pi r x}{2}\right) + C$$

Combining terms and solving for x^3 gives

$$x^3 = \frac{16}{\pi^3}\sum_{r=1}^{\infty}\frac{(-1)^r[6-\pi^2 r^2]}{r^3}\sin\left(\frac{\pi r x}{2}\right) + C$$