Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Mid-term Exam
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Question 7

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{d} - y = x \,, y(e) = 0 \,, y'(e) = 2$$

a. This is Euler differential equation, and we make the change of variable $x = e^t$ or $t = \ln(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{d\ln x}{dx} = \frac{dy}{dt}\frac{1}{x} = \frac{1}{x}\frac{dy}{dt}$$
$$x\frac{dy}{dx} = \frac{dy}{dt}$$

And since this is a Legendre ODE with $\alpha = 1$ and $\beta = 0$, we can use the expression for the second derivative $(\alpha x + \beta)^2 \frac{d^2y}{dx^2} = \alpha^2 \frac{d}{dt}[\frac{d}{dt} - 1]y$. With $\alpha = 1$ and $\beta = 0$, we have: $\frac{d^2y}{dx^2} = \frac{d^2y}{t^2} - \frac{dy}{dt}$.

Substitute into the above equation yields:

$$(\frac{d^2y}{dt^2} - \frac{dy}{dt}) + \frac{dy}{dt} - y = e^t$$
$$\frac{d^2y}{dt^2} - y = e^t$$

b. The homogeneous equation is

$$\frac{d^2y}{dt^2} - y = 0$$

Assume a solution of the form $y(t) = Ae^{\lambda t}$ gives the characteristic equation $\lambda^2 - 1 = 0$ which has for roots $\lambda = \pm 1$ and gives for solution $y(t) = c_1 e^t + c_2 e^{-t}$.

c. The ODE to solve is:

$$\frac{d^2y}{dt^2} - y = 0$$

It is in standard form and it is defined at any point t, it is analytic, thus we take as solution

$y(t) = \sum_{t=0}^{\infty} a_n t^n$. So:

$$y'(t) = \sum_{t=0}^{\infty} n a_n t^{n-1}$$
$$y''(t) = \sum_{t=0}^{\infty} n(n-1) a_n t^{n-2}$$

by reindexing

$$y''(t) = \sum_{t=-2}^{\infty} (n+2)(n+1) a_{n+2} t^n$$
$$y''(t) = \sum_{t=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$$

Substitute into the ODE gives:

$$\sum_{t=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{t=0}^{\infty} a_n t^n = 0$$
$$\sum_{t=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] t^n = 0$$

or

$$a_{n+2} = \frac{1}{(n+2)(n+1)} a_n$$
$$a_n = \frac{1}{n(n-1)} a_{n-2}$$

Take $a_0 = a_1 = 1$ and we generate the coefficients:

. $n = 2$ then $a_2 = \frac{1}{2\cdot 1} a_0 = \frac{1}{2\cdot 1} = \frac{1}{2!}$
. $n = 3$ then $a_3 = \frac{1}{3\cdot 2} a_1 = \frac{1}{3\cdot 2} = \frac{1}{3!}$
. $n = 4$ then $a_4 = \frac{1}{4\cdot 3} a_2 = \frac{1}{4\cdot 3\cdot 2\cdot 1} = \frac{1}{4!}$

$\vdots$

. $a_n = \frac{1}{n(n-1)} a_{n-2} = \cdots = \frac{1}{n!}$

The first solution we obtain is: $y_1(t) = \sum_{t=0}^{\infty} a_n t^n = \sum_{t=0}^{\infty} \frac{t^n}{n!} = e^t$. Secondly, if we set $a_0 = 1$ and choose $a_1 = -1$, then we obtain a second independent solution:

. $n = 2$ then $a_2 = \frac{1}{2\cdot 1} a_0 = \frac{1}{2\cdot 1} = \frac{1}{2!}$
. $n = 3$ then $a_3 = \frac{1}{3\cdot 2} a_1 = -\frac{1}{3\cdot 2} = \frac{-1}{3!}$
. $n = 4$ then $a_4 = \frac{1}{4\cdot 3} a_2 = \frac{1}{4\cdot 3\cdot 2\cdot 1} = \frac{1}{4!}$
. $n = 5$ then $a_5 = \frac{1}{5\cdot 4} a_3 = \frac{-1}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{-1}{5!}$

$\vdots$

. $a_n = \frac{1}{n(n-1)} a_{n-2} = \cdots = \frac{(-1)^n}{n!}$

We have the second solution: $y_2(t) = \sum_{t=0}^{\infty} a_n t^n = \sum_{t=0}^{\infty} \frac{(-t)^n}{n!}$, recognizing the last series as e^{-t}, we can write the general solution of the homogeneous equation as

$$y_H(t) = c_1 e^t + c_2 e^{-t}$$

which is the solution we found in question b.

d. The differential equation to solve is

$$\frac{d^2y}{dt^2} - y = e^t$$

Next we use the variation of parameters method, we are looking for a solution $y_p(t) = k_1(t)e^t + k_2(t)e^{-t}$. We solve for derivatives of k's a system of two equations:

$$\begin{cases} k_1' e^t + k_2' e^{-t} & = 0 \\ k_1' e^t - k_2' e^{-t} & = e^t \end{cases}$$

Multiplying through by e^t gives:

$$\begin{cases} k_1' e^{2t} + k_2' & = 0 \\ k_1' e^{2t} - k_2' & = e^{2t} \end{cases}$$

Adding first equation to second yields $2k_1' e^{2t} = e^{2t}$ or $k_1' = \frac{1}{2}$ and $k_1 = \frac{t}{2}$. Substitute

$$\begin{aligned} k_2' &= -k_1' e^{2t} \\ &= -\frac{1}{2} e^{2t} \end{aligned}$$

integrating

$$k_2 = -\frac{e^{2t}}{4}$$

Therefore:

$$\begin{aligned} y_p(t) &= k_1(t)e^t + k_2(t)e^{-t} \\ &= \frac{t}{2}e^t - \frac{e^{2t}}{4}e^{-t} \\ &= \frac{t}{2}e^t - \frac{e^t}{4} \\ &= \frac{e^t}{2}(t - \frac{1}{2}) \end{aligned}$$

e. The general solution is: $y(t) = y_H(t) + y_p(t) = c_1 e^t + c_2 e^{-t} + \frac{e^t}{2}(t - \frac{1}{2})$, simplifying the constants, we can rewrite the general solution as $y(t) = c_1 e^t + c_2 e^{-t} + \frac{t}{2} e^t$. Plugging back $x = e^t$ or $t = \ln(x)$ gives

$$y(x) = c_1 x + \frac{c_2}{x} + \frac{x \ln x}{2}$$

f. The total solution is

$$y(x) = c_1 x + \frac{c_2}{x} + \frac{x \ln x}{2}$$
$$y'(x) = c_1 x - \frac{c_2}{x^2} + \frac{1}{2}(1 + \ln x)$$

And the initial conditions are $y(e) = 0\,, y'(e) = 2$, plugging back these into the previous equations gives

$$\begin{cases} y(e) = c_1 e + \frac{c_2}{e} + \frac{e \ln e}{2} = 0 \\ y'(e) = c_1 - \frac{c_2}{e^2} + \frac{1}{2}(1 + \ln e) = 2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 e + c_2 e^{-1} = -\frac{e}{2} \\ c_1 - c_2 e^{-2} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1 e^2 + c_2 = -\frac{e^2}{2} \\ c_1 - c_2 e^{-2} = 1 \end{cases}$$

Adding equation (1) to equation (2) leads to $2c_1 = e^2 - \frac{e^2}{2} = \frac{e^2}{2}$, $c_1 = \frac{1}{4}$,$c_2 = e^2(c_1 - 1) = \frac{3}{4}e^2$. Reporting these constants into the expression of the total solution gives:

$$y(x) = \frac{1}{4}x - \frac{3}{4}e^2\frac{1}{x} + \frac{x \ln x}{2}$$

$$y(x) = \frac{x^2 + 2x^2 \ln(x) - 3e^2}{4x}$$