Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Mid-term Exam
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Question 1

a. Graph of the function attached as a separate pdf.

b. Since we have made the function $f(x)$ even using an even extension, all the b_k coefficients in its Fourier series are zero. With a period $L = 4$, we determine the remaining coefficients a_k:

$$a_k = \frac{2}{4} \int_{-2}^{2} x \cos(\frac{2k\pi x}{4}) dx$$

And since f is even now

$$\begin{aligned} a_k &= \frac{4}{4} \int_{0}^{2} x \cos(\frac{2k\pi x}{4}) dx \\ &= \int_{0}^{2} x \cos(\frac{k\pi x}{2}) dx \end{aligned}$$

Using integration by parts, for $k > 0$:

$$\begin{aligned} a_k &= \frac{2}{k\pi}[x \sin(\frac{k\pi x}{2})]_0^2 - \frac{2}{k\pi} \int_0^2 \sin(\frac{k\pi x}{2}) dx \\ &= 0 - \frac{2}{k\pi}(-\frac{2}{k\pi})[\cos\frac{k\pi x}{2})]_0^2 \\ &= \frac{4}{(k\pi)^2)}[\cos(k\pi) - \cos(0)] \\ &= \frac{4}{(k\pi)^2}[(-1)^k - 1] \end{aligned}$$

Then

$$a_k = \begin{cases} -\dfrac{8}{(k\pi)^2} \text{ for odd } k \\ 0 \text{ for even } k \end{cases}$$

And $a_0 = \frac{2}{4}\int_{-2}^{2} x dx = \frac{4}{4}\int_0^2 x dx = \frac{1}{2}[x^2]_0^2 = 2$. With the coefficients a_k determined, we obtain the Fourier series for $f(x)$:

$$\begin{aligned} f(x) &= \frac{2}{2} - \sum_{k=1}^{\infty} \frac{8}{(k\pi)^2} \cos(\frac{2k\pi x}{4}) \ k \text{ odd} \\ x &= 1 - \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(\frac{(2k+1)\pi x}{2}) \end{aligned}$$

c. Applying Parseval's identity for Fourier series and using the result of part b.:

$$
\begin{aligned}
\frac{1}{4}\int_{-2}^{2} x^2 dx &= \frac{2^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + 0)\ k \text{ odd} \\
\frac{2}{4}\int_{0}^{2} x^2 dx &= 1 + \frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{8}{(2k+1)^2\pi^2}\Big)^2 \\
\frac{1}{2}\Big[\frac{x^3}{3}\Big]_0^2 &= 1 + \frac{1}{2}\cdot\frac{64}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} \\
\frac{4}{3} - 1 &= \frac{32}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} \\
\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} &= \frac{\pi^4}{32}\cdot\frac{1}{3}
\end{aligned}
$$

Therefore

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

Question 2

a. Graph of the function attached as a separate pdf.

b.

$$f(t) = A\Big[H(t) - H(t-\tau)\Big]$$

c.

$$\tilde{f}(w) = F\{f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt$$

Since $f(t) = 0$ for $t \geq 0$ or $t \leq \tau$:

$$
\begin{aligned}
\tilde{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{0}^{\tau} A\cdot e^{-iwt}\,dt \\
&= \frac{A}{\sqrt{2\pi}}\Big(\frac{1}{-iw}\Big)[e^{-iwt}]_0^{\tau} \\
&= \frac{iA}{w\sqrt{2\pi}}(e^{-iw\tau} - 1) \\
&= \frac{iA}{w\sqrt{2\pi}}e^{-iw\frac{\tau}{2}}(e^{-iw\frac{\tau}{2}} - e^{iw\frac{\tau}{2}})
\end{aligned}
$$

From Euler identity:

$$e^{-iw\frac{\tau}{2}} - e^{iw\frac{\tau}{2}} = -2i\sin w\frac{\tau}{2}$$

Therefore

$$
\begin{aligned}
\tilde{f}(w) &= \frac{2A}{w\sqrt{2\pi}} e^{-iw\frac{\tau}{2}} \sin w\frac{\tau}{2} \\
&= \sqrt{\frac{2}{\pi}} \frac{A}{w} e^{-iw\frac{\tau}{2}} \sin w\frac{\tau}{2} \\
&= A\sqrt{\frac{2}{\pi}} e^{-iw\frac{\tau}{2}} \frac{\tau}{2} \frac{\sin(w\frac{\tau}{2})}{w\frac{\tau}{2}} \\
&= \frac{A}{\sqrt{2\pi}} \tau e^{-iw\frac{\tau}{2}} \text{sinc}(w\frac{\tau}{2})
\end{aligned}
$$

d. Let $A = \frac{1}{\tau}$ then substituting in $f(t)$ from part c., gives:

$$
F\{\lim_{\tau\to 0} f(t)\} = \lim_{\tau\to 0} F\{f(t)\} = \lim_{\tau\to 0} \frac{1}{\sqrt{2\pi}} e^{-iw\frac{\tau}{2}} \frac{\sin(w\frac{\tau}{2})}{w\frac{\tau}{2}}
$$

$$
\begin{aligned}
&\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1 \text{ by Hospitals rule} \\
&\lim_{\tau\to 0} e^{-iw\frac{\tau}{2}} = \lim_{\tau\to 0} e^{0} = 1
\end{aligned}
$$

Therefore

$$
F\{\lim_{\tau\to 0} f(t)\} = \frac{1}{\sqrt{2\pi}}
$$

e. The Fourier transform of $f(t)$ as $\tau \to 0$ is the Fourier transform of a δ-function as we can expect as we "transform" the rectangular function $f(t)$ to a Dirac impulse.

Question 3

a. By definition, the Laplace transform of $g(t) = \sin(5t)$ is:

$$
\bar{g}(s) = L\{g(t)\} = \int_0^\infty \sin(5t)e^{-st}dt = \lim_{L\to\infty} \int_0^L \sin(5t)e^{-st}dt
$$

First compute $\int e^{-st} \sin at\, dt$, using integration by parts with $u = \sin at$, $u' = a\cos at$, $v' = e^{-st}$, $v = -\frac{1}{s}e^{-st}$:

$$
\int e^{-st} \sin(at)\, dt = -\frac{1}{s} e^{-st} \sin(at) + \frac{a}{s} \int e^{-st} \cos(at) dt \tag{1}
$$

Next compute $\int e^{-st} \cos(at)\, dt$, again, using integration by parts with $u = \cos at$, $u' = -a\sin at$, $v' = e^{-st}$, $v = -\frac{1}{s}e^{-st}$:

$$
\int e^{-st} \cos(at)\, dt = -\frac{1}{s} e^{-st} \cos(at) - \frac{a}{s} \int e^{-st} \sin(at)\, dt
$$

Substituting into (1):

$$\int e^{-st}\sin(at)\,dt = -\frac{1}{s}e^{-st}\sin(at) + \frac{a}{s}\Big(-\frac{1}{s}e^{-st}\cos(at) - \frac{a}{s}\int e^{-st}\sin(at)\,dt \Big)$$
$$= -\frac{1}{s}e^{-st}\sin(at) - \frac{a}{s^2}e^{-st}\cos(at) + \frac{a}{s^2}\int e^{-st}\sin(at)\,dt$$

thus

$$(1+\frac{a^2}{s^2})\int e^{-st}\sin(at)\,dt = -e^{-st}(\frac{1}{s}\sin(at) + \frac{a}{s^2}\cos(at))$$

Evaluating at $t=0$ and $t\to\infty$:

$$(1+\frac{a^2}{s^2})L\{\sin(at)\} = \lim_{L\to\infty}\Big[-e^{-st}(\frac{1}{s}\sin(at)+\frac{a}{s^2}\cos(at))\Big]_0^L$$
$$= 0 - (-1(\frac{1}{s}\cdot 0 + \frac{a}{s^2}\cdot 1))$$
$$= \frac{a}{s^2}$$

Therefore

$$L\{\sin(at)\} = \frac{a}{s^2}(1+\frac{a^2}{s^2})^{-1}$$
$$= \frac{a}{a^2+s^2}$$

Set $a=5$ and

$$L\{g(t)\} = L\{\sin(5t)\} = \frac{5}{s^2+25}$$

b. From the book, one property of the Laplace transform is $L[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}$ for $n = 1,2,3,\cdots$, take $n=1$, $L[tf(t)] = -\frac{d\bar{f}(s)}{ds}$. Set $f(t) = t\sin(5t)$ and from part b, $L\{\sin(5t)\} = \frac{5}{s^2+25}$, therefore:

$$L\{t\sin(5t)\} = -\frac{d}{ds}\Big(\frac{5}{s^2+25}\Big)$$
$$= -5\frac{d}{ds}\Big(\frac{1}{s^2+25}\Big)$$
$$= -5\Big(\frac{-2s}{(s^2+25)^2}\Big)$$
$$= \frac{10s}{(s^2+25)^2}$$

c. By definition $(f*g)(t) = \int_0^t \tau e^{-(t-\tau)}d\tau = e^{-t}\int_0^t \tau e^{\tau} d\tau$. Using integration by parts:

$$\int_0^t \tau e^{\tau} d\tau = [\tau e^{\tau}]_0^t - \int_0^t e^{\tau} d\tau$$
$$= te^t - [e^{\tau}]_0^t$$
$$= te^t - (e^t - 1)$$
$$= e^t(t-1) + 1$$

And

$$(f*g)(t) = e^{-t}\Big[e^{-t}(t-1)+1\Big] = e^{-t}+t-1$$

From $L\{(f*g)(t)\} = \bar{f}(s)\cdot\bar{g}(s)$, we have:

$$\begin{aligned}\bar{f}(s)\cdot\bar{g}(s) &= \frac{1}{s^2}\cdot\frac{1}{s+1}\\ &= \frac{1-s}{s^2+1}+\frac{1}{s+1}\\ &= \frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\end{aligned}$$

Therefore

$$\begin{aligned}(f*g)(t) &= L^{-1}\{L\{(f*g)(t)\}\} = L^{-1}\{\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\}\\ &= L^{-1}\{\frac{1}{s^2}\} - L^{-1}\{\ \frac{1}{s}\} + L^{-1}\{\ \frac{1}{s+1}\}\\ &= t-1+e^{-t}\\ &= e^{-t}+t-1\end{aligned}$$

Question 4

$$y''+4y'-5y=\delta(t-1)\;y(0)=0\;y'(0)=3$$

a. Taking the Laplace transform on both sides of the equation gives:

$$\begin{aligned}s^2\tilde{y}(s)-sy(0)-y'(0)+4[s\tilde{y}(s)-y(0)]-5\tilde{y}(s) &= e^{-s}\\ s^2\tilde{y}(s)-s\cdot 0-3+4[s\tilde{y}(s)-0]-5\tilde{y}(s) &= e^{-s}\\ s^2\tilde{y}(s)+4s\tilde{y}(s)-5\tilde{y}(s) &= e^{-s}+3\end{aligned}$$

Combining the terms: $(s^2+4s-5)\tilde{y}(s) = 3+e^{-s}$. Therefore

$$\tilde{y}(s) = \frac{3+e^{-s}}{s^2+4s-5}$$

b. The roots of $s^2+4s-5=0$ are -5 and 1, so we can rewrite $\tilde{y}(s)$ as $\tilde{y}(s) = \frac{3}{(s-1)(s+5)}+\frac{e^{-s}}{(s-1)(s+5)}$
Computing the fraction expansion:

$$\begin{aligned}\frac{1}{(s-1)(s+5)} &= \frac{A}{s-1}+\frac{B}{s+5}\\ &= \frac{(A+B)s+5A-B}{(s-1)(s+5)}\end{aligned}$$

Equating the powers of s on each side of the previous equation:

$$\begin{aligned}s^1 &: A+B=0\\ s^0 &: 5A-B=1\end{aligned}$$

gives $A = \frac{1}{6}$ and $B = -\frac{1}{6}$. Thus

$$\frac{1}{(s-1)(s+5)} = \frac{1}{6}\left(\frac{1}{s-1} - \frac{1}{s+5}\right)$$

So

$$\begin{aligned}\tilde{y}(s) &= 3\left[\frac{1}{6}(\frac{1}{s-1} - \frac{1}{s+5})\right] + \frac{1}{6}(\frac{e^{-s}}{s-1} - \frac{e^{-s}}{s+5}) \\ &= \frac{1}{2}(\frac{1}{s-1} - \frac{1}{s+5}) + \frac{1}{6}(\frac{e^{-s}}{s-1} - \frac{e^{-s}}{s+5})\end{aligned}$$

c. $y(t) = L^{-1}\{\tilde{y}(s)\}$ and from part b:

$$y(t) = \frac{1}{2}\left[L^{-1}\{\frac{1}{s-1}\} - L^{-1}\{\frac{1}{s+5}\}\right] + \frac{1}{6}\left[L^{-1}\{\frac{e^{-s}}{s-1}\} - L^{-1}\{\frac{e^{-s}}{s+5}\}\right]$$

$L^{-1}\{\frac{1}{s-1}\} = e^t$, $L^{-1}\{\frac{1}{s+5}\} = e^{-5t}$, and using the shift theorem:

$$L\{f(t-t_0)H(t-t_0)\} = e^{-st_0}F(s) \;\; f(t-t_0)H(t-t_0) = L^{-1}\{e^{-st_0}F(s)\}$$

So for $t_0 = 1$

$$\begin{aligned}L^{-1}\{\frac{e^{-s}}{s-1}\} &= e^{(t-1)}H(t-1) \\ L^{-1}\{\frac{e^{-s}}{s+5}\} &= e^{-5(t-1)}H(t-1)\end{aligned}$$

Plugging back these into $y(t)$ yields:

$$\begin{aligned}y(t) &= \frac{1}{2}\left[e^t - e^{-5t}\right] + \frac{1}{6}\left[e^{(t-1)}H(t-1) - e^{-5(t-1)}H(t-1)\right] \\ &= \frac{1}{2}\left[e^t - e^{-5t}\right] + \frac{1}{6}\left(e^{(t-1)} - e^{-5(t-1)}\right)H(t-1) \\ &= \frac{e^t}{2}\left(1 + \frac{1}{3e}H(t-1)\right) - \frac{e^{-5t}}{2}(1 + \frac{1}{3}e^5 H(t-1)\Big)\end{aligned}$$

Question 7

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{d} - y = x\,, y(e) = 0\,, y'(e) = 2$$

a. This is Euler differential equation, and we make the change of variable $x = e^t$ or $t = \ln(x)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{d\ln x}{dx} = \frac{dy}{dt}\frac{1}{x} = \frac{1}{x}\frac{dy}{dt} \\ x\frac{dy}{dx} &= \frac{dy}{dt}\end{aligned}$$

And since this is a Legendre ODE with $\alpha = 1$ and $\beta = 0$, we can use the expression for the second derivative $(\alpha x+\beta)^2 \frac{d^2y}{dx^2} = \alpha^2 \frac{d}{dt}[\frac{d}{dt} - 1]y$. With $\alpha = 1$ and $\beta = 0$, we have: $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.

Substitute into the above equation yields:

$$(\frac{d^2y}{dt^2} - \frac{dy}{dt}) + \frac{dy}{dt} - y = e^t$$
$$\frac{d^2y}{dt^2} - y = e^t$$

b. The homogeneous equation is

$$\frac{d^2y}{dt^2} - y = 0$$

Assume a solution of the form $y(t) = Ae^{\lambda t}$ gives the characteristic equation $\lambda^2 - 1 = 0$ which has for roots $\lambda = \pm 1$ and gives for solution $y(t) = c_1e^t + c_2e^{-t}$.

c. The ODE to solve is:

$$\frac{d^2y}{dt^2} - y = 0$$

It is in standard form and it is defined at any point t, it is analytic, thus we take as solution $y(t) = \sum_{t=0}^{\infty} a_n t^n$. So:

$$y'(t) = \sum_{t=0}^{\infty} n a_n t^{n-1}$$
$$y''(t) = \sum_{t=0}^{\infty} n(n-1) a_n t^{n-2}$$

by reindexing

$$y''(t) = \sum_{t=-2}^{\infty} (n+2)(n+1) a_{n+2} t^n$$
$$y''(t) = \sum_{t=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$$

Substitute into the ODE gives:

$$\sum_{t=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{t=0}^{\infty} a_n t^n = 0$$
$$\sum_{t=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] t^n = 0$$

or

$$a_{n+2} = \frac{1}{(n+2)(n+1)} a_n$$
$$a_n = \frac{1}{n(n-1)} a_{n-2}$$

Take $a_0 = a_1 = 1$ and we generate the coefficients:

. $n = 2$ then $a_2 = \frac{1}{2\cdot 1}a_0 = \frac{1}{2\cdot 1} = \frac{1}{2!}$

. $n = 3$ then $a_3 = \frac{1}{3\cdot 2}a_1 = \frac{1}{3\cdot 2} = \frac{1}{3!}$

. $n = 4$ then $a_4 = \frac{1}{4\cdot 3}a_2 = \frac{1}{4\cdot 3\cdot 2\cdot 1} = \frac{1}{4!}$

$\vdots$

. $a_n = \frac{1}{n(n-1)}a_{n-2} = \cdots = \frac{1}{n!}$

The first solution we obtain is: $y_1(t) = \sum_{t=0}^{\infty} a_n t^n = \sum_{t=0}^{\infty} \frac{t^n}{n!} = e^t$. Secondly, if we set $a_0 = 1$ and choose $a_1 = -1$, then we obtain a second independent solution:

. $n = 2$ then $a_2 = \frac{1}{2\cdot 1}a_0 = \frac{1}{2\cdot 1} = \frac{1}{2!}$

. $n = 3$ then $a_3 = \frac{1}{3\cdot 2}a_1 = -\frac{1}{3\cdot 2} = \frac{-1}{3!}$

. $n = 4$ then $a_4 = \frac{1}{4\cdot 3}a_2 = \frac{1}{4\cdot 3\cdot 2\cdot 1} = \frac{1}{4!}$

. $n = 5$ then $a_5 = \frac{1}{5\cdot 4}a_3 = \frac{-1}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{-1}{5!}$

$\vdots$

. $a_n = \frac{1}{n(n-1)}a_{n-2} = \cdots = \frac{(-1)^n}{n!}$

We have the second solution: $y_2(t) = \sum_{t=0}^{\infty} a_n t^n = \sum_{t=0}^{\infty} \frac{(-t)^n}{n!}$, recognizing the last series as e^{-t}, we can write the general solution of the homogeneous equation as

$$y_H(t) = c_1 e^t + c_2 e^{-t}$$

which is the solution we found in question b.

d. The differential equation to solve is

$$\frac{d^2y}{dt^2} - y = e^t$$

Next we use the variation of parameters method, we are looking for a solution $y_p(t) = k_1(t)e^t + k_2(t)e^{-t}$. We solve for derivatives of k's a system of two equations:

$$\begin{cases} k_1' e^t + k_2' e^{-t} & = 0 \\ k_1' e^t - k_2' e^{-t} & = e^t \end{cases}$$

Multiplying through by e^t gives:

$$\begin{cases} k_1' e^{2t} + k_2' & = 0 \\ k_1' e^{2t} - k_2' & = e^{2t} \end{cases}$$

Adding first equation to second yields $2k_1'e^{2t} = e^{2t}$ or $k_1' = \frac{1}{2}$ and $k_1 = \frac{t}{2}$. Substitute

$$\begin{aligned} k_2' &= -k_1'e^{2t} \\ &= -\frac{1}{2}e^{2t} \end{aligned}$$

integrating

$$k_2 = -\frac{e^{2t}}{4}$$

Therefore:

$$\begin{aligned} y_p(t) &= k_1(t)e^t + k_2(t)e^{-t} \\ &= \frac{t}{2}e^t - \frac{e^{2t}}{4}e^{-t} \\ &= \frac{t}{2}e^t - \frac{e^t}{4} \\ &= \frac{e^t}{2}(t - \frac{1}{2}) \end{aligned}$$

e. The general solution is: $y(t) = y_H(t) + y_p(t) = c_1e^t + c_2e^{-t} + \frac{e^t}{2}(t - \frac{1}{2})$, simplifying the constants, we can rewrite the general solution as $y(t) = c_1e^t + c_2e^{-t} + \frac{t}{2}e^t$. Plugging back $x = e^t$ or $t = \ln(x)$ gives

$$y(x) = c_1x + \frac{c_2}{x} + \frac{x\ln x}{2}$$

f. The total solution is

$$\begin{aligned} y(x) &= c_1x + \frac{c_2}{x} + \frac{x\ln x}{2} \\ y'(x) &= c_1x - \frac{c_2}{x^2} + \frac{1}{2}(1 + \ln x) \end{aligned}$$

And the initial conditions are $y(e) = 0\,, y'(e) = 2$, plugging back these into the previous equations gives

$$\begin{cases} y(e) = c_1e + \frac{c_2}{e} + \frac{e\ln e}{2} = 0 \\ y'(e) = c_1 - \frac{c_2}{e^2} + \frac{1}{2}(1 + \ln e) = 2 \end{cases}$$

$$\Rightarrow \begin{cases} c_1e + c_2e^{-1} = -\frac{e}{2} \\ c_1 - c_2e^{-2} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} c_1e^2 + c_2 = -\frac{e^2}{2} \\ c_1 - c_2e^{-2} = 1 \end{cases}$$

Adding equation (1) to equation (2) leads to $2c_1 = e^2 - \frac{e^2}{2} = \frac{e^2}{2}$, $c_1 = \frac{1}{4}$,$c_2 = e^2(c_1 - 1) = \frac{3}{4}e^2$. Reporting these constants into the expression of the total solution gives:

$$y(x) = \frac{1}{4}x - \frac{3}{4}e^2\frac{1}{x} + \frac{x\ln x}{2}$$

$$y(x) = \frac{x^2 + 2x^2\ln(x) - 3e^2}{4x}$$