Johns Hopkins
Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

- **First we will look at some properties of Fourier transform**
- **Next we will look at some properties of Laplace transform**
- **Finally we will look at the convolution function**

Some properties of Fourier transform

Fourier transform of the derivative of a function

$$\begin{aligned}\mathcal{F}\left[f'(t)\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(t)\,e^{-i\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega t} f(t)\right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\omega\, e^{-i\omega t} f(t)\,dt \\ &= i\omega\tilde{f}(\omega),\end{aligned}$$

Similarly $F\{\mathrm{f}''(\mathrm{t})\} = -\omega^2\tilde{\mathrm{f}}(\omega)$

We also can easily construct the following identities

$$F\{\mathrm{e}^{\alpha \mathrm{t}}\mathrm{f}(\mathrm{t})\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\alpha\mathrm{t}}\mathrm{f}(\mathrm{t})\mathrm{e}^{\mathrm{i}\omega\mathrm{t}}\,\mathrm{dt} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{f}(\mathrm{t})\mathrm{e}^{\mathrm{i}(\omega+\alpha\mathrm{t})\mathrm{t}}\,\mathrm{dt} = \tilde{\mathrm{f}}(\omega+\alpha\mathrm{t})$$

$$F\{\mathrm{f}(\mathrm{t}+\mathrm{a})\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{f}(\mathrm{t})\mathrm{e}^{\mathrm{i}\omega(\mathrm{t}+\mathrm{a})}\,\mathrm{dt} = \mathrm{e}^{\mathrm{i}\omega\mathrm{a}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{f}(\mathrm{t})\mathrm{e}^{\mathrm{i}\omega\mathrm{t}}\,\mathrm{dt} = \mathrm{e}^{\mathrm{i}\omega\mathrm{a}}\tilde{\mathrm{f}}(\omega)$$

$$F\{\mathrm{f}(\mathrm{at})\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{f}(\mathrm{at})\mathrm{e}^{\mathrm{i}\omega\mathrm{t}}\,\mathrm{dt} = \frac{1}{\mathrm{a}}\tilde{\mathrm{f}}(\frac{\omega}{\mathrm{a}})$$

Fourier of even and odd functions and the Fourier sine and cosine transform

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)(\cos\omega t - i\sin\omega t)\, dt$$
$$= \frac{-2i}{\sqrt{2\pi}} \int_{0}^{\infty} f(t)\sin\omega t\, dt,$$

First, use Euler identity, then take f(t) as an odd function, since cosine is an even function Then $f(t)\cos\omega t$ is odd and this part of the Is zero integral

For $\tilde{f}(\omega)$ an odd function we have following derivation

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega = \frac{2i}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{f}(\omega)\sin\omega t\, d\omega$$
$$= \frac{2}{\pi} \int_{0}^{\infty} d\omega\, \sin\omega t \left\{ \int_{0}^{\infty} f(u)\sin\omega u\, du \right\}.$$

Which enables us to define the Fourier sine transform $\boxed{\tilde{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t)\, \sin\omega t\, dt}$

Similarly we can define the Fourier cosine transform.

Some properties of the Laplace transform - shift theorems

$$L\{e^{at}f(t)\} = \bar{f}(s-a) \text{ or } L^{-1}\{\bar{f}(s-a)\} = e^{at}f(t)$$

The derivation follows

Start with $\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$ and let s-a replace s giving

$$\bar{f}(s-a) = \int_0^\infty e^{-(s-a)t} f(t)dt = \int_0^\infty e^{-st} e^{at} f(t)dt$$

As an example, find the inverse Laplace transform of the following function

$$\bar{f}(s-3) = \frac{1}{[(s-3)^2 + 4^2}$$

Take a = 3 and note that since for $\bar{f}(s) = \dfrac{1}{s^2+4^2}$ we have $\dfrac{1}{4} L^{-1}\{\dfrac{4}{s^2+4^2}\} = \dfrac{1}{4}\sin 4t$

This gives $L^{-1}\{\bar{f}(s-3)\} = e^{3t}\dfrac{1}{4}\sin 4t = \dfrac{1}{4}e^{3t}\sin 4t$

Another Laplace transform shift theorem

$$L\{f(t-b)H(t-b)\} = e^{-as}\overline{f}(s) \text{ or } f(t-b)H(t-b) = L^{-1}\{e^{-as}\overline{f}(s)\}$$

The derivation follows

Start with $e^{-bs}\overline{f}(s) = e^{-bs}\int_0^\infty e^{-s\tau}f(\tau)d\tau = \int_0^\infty e^{-bs}e^{-s\tau}f(\tau)d\tau = \int_0^\infty e^{-s(b+\tau)}f(\tau)d\tau$

And make the substitution $b+\tau = t$ giving

$$\int_{\tau=0}^{\infty} e^{-s(b+\tau)}f(\tau)d\tau = \int_{t=b+0=b}^{\infty} e^{-s(t)}f(t-b)dt = \int_0^\infty e^{-st}f(t-b)H(t-b)dt$$

As an example, find the Laplace transform of the following function

$$f(t-b)H(t-b) = e^{-2(t-3)}H(t-3)$$

Which is an exponential function shifted 3 units right and zero value for t less than 3!

Since $L\{e^{-2t}\} = \dfrac{1}{s+2}$ we get $L\{e^{-2(t-3)}H(t-3)\} = e^{-3s}\dfrac{1}{s+2}$

Laplace transform of the derivative

Start with the following integral $L\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt$

The derivation follows where integration by parts is used and details are included.

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du, \text{ Take } v = f(t),\ dv = f'(t)dt,\ u = e^{-st} \text{ and } du = -se^{-st}dt$$

Therefore $\int_0^\infty e^{-st}\ f'(t)dt = e^{-st}f(t)\Big|_0^\infty - \int_0^\infty f(t)[-se^{-st}\,dt] = -e^{-s0}f(0) + s\int_0^\infty f(t)e^{-st}\,dt$

$= -f(0) + s\tilde{f}(s)$

Therefore we have

$$L\{f'(t)\} = \int_0^\infty e^{-st}\ f'(t)dt = -f(0) + s\tilde{f}(s)$$

and for Laplace transform of the second derivative of a function, just as easily found

$$L\{f''(t)\} = \int_0^\infty e^{-st}\ f''(t)dt = -sf(0) - f'(0) + s^2\tilde{f}(s)$$

Laplace transform of the integral of a function

For the Laplace transform of the integral of a function we again use integration by parts

$$L\{\int_0^t f(x)dx\} = \int_0^\infty [\int_0^t f(x)dx]e^{-st}\,dt$$

$$\text{Let } u = \int_0^t f(x)dx, \text{ du=f(t)dt and } dv = e^{-st}dt, \; v = \frac{1}{-s}e^{-st}$$

$$\text{Then } \int_0^\infty [\int_0^t f(x)dx]e^{-st}\,dt = \int_0^t f(x)dx[\frac{1}{-s}e^{-st}]\Bigg|_0^\infty - \int_0^\infty f(t)\frac{1}{-s}e^{-st}\,dt$$

$$= \lim_{t\to\infty}[\frac{1}{-s}e^{-st}]\int_0^t f(x)dx - \int_0^0 f(x)dx[\frac{1}{-s}e^{-s0}] + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt$$

$$= \frac{1}{s}L\{f(t)\} = \tilde{f}(s)$$

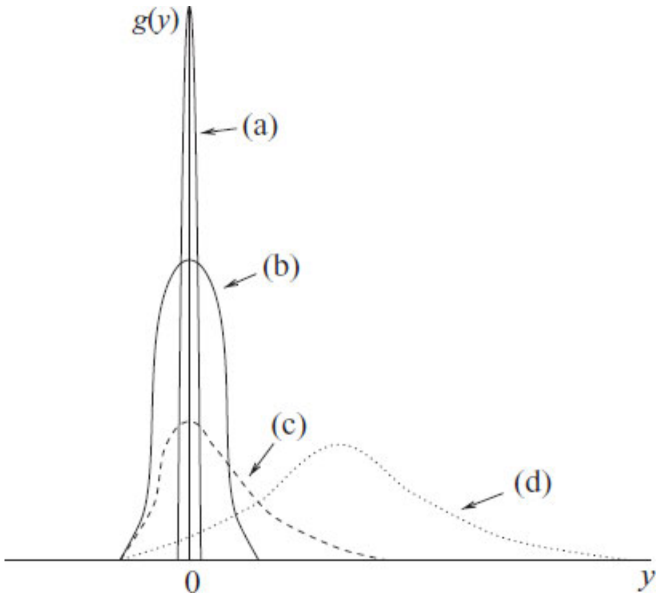
Therefore $\boxed{L\{\int_0^t f(x)dx\} = \frac{1}{s}L\{f(t)\} = \tilde{f}(s)}$

Convolution

Define the convolution $h(z) = \int_{-\infty}^{\infty} f(x)g(z-x)\,dx.$

The *convolution* is often written $f * g$ and is commutative ($f * g = g * f$), associative and distributive. The observed distribution is the convolution of the true distribution and the experimental resolution function g(y).

As a spatial example take the following



Resolution functions: (a) ideal δ-function; (b) typical unbiased resolution; (c) and (d) biases tending to shift observations to higher values than the true one.

Fourier transform of a Convolution

It is fairly straight forward to find the Fourier transform of the convolution.

Here we define, as before, the convolution $h(z) = \int_{-\infty}^{\infty} f(x) g(z-x) dx$

Then
$$\widetilde{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-ikz} \left\{ \int_{-\infty}^{\infty} f(x) g(z-x)\, dx \right\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) \left\{ \int_{-\infty}^{\infty} g(z-x)\, e^{-ikz}\, dz \right\}.$$

and letting $u = z - x$ in the second integral we have

$$\widetilde{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) \left\{ \int_{-\infty}^{\infty} g(u)\, e^{-ik(u+x)}\, du \right\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \int_{-\infty}^{\infty} g(u)\, e^{-iku}\, du$$
$$= \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi}\, \widetilde{f}(k) \times \sqrt{2\pi}\, \widetilde{g}(k) = \sqrt{2\pi}\, \widetilde{f}(k) \widetilde{g}(k).$$

Laplace transform of a Convolution

For the Laplace transform of the convolution we need to be a little more careful.

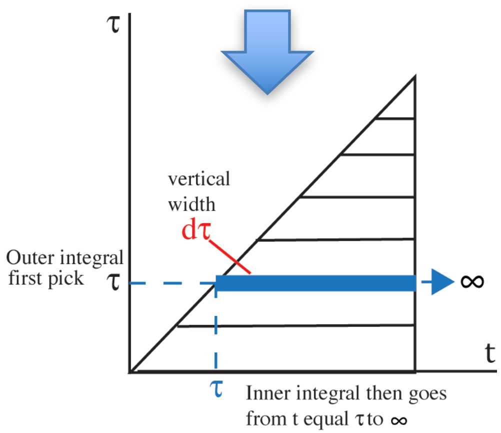
$$\tilde{f}(s) = L\{f(\tau)\} = \int_0^\infty f(\tau)e^{-s\tau}\,d\tau \text{ and } \tilde{g}(s) = L\{g(u)\} = \int_0^\infty g(u)e^{-su}\,du$$

Let $t = u + \tau$ or $u = t - \tau$, $du = dt$ (treating τ as constant in the second integral above)

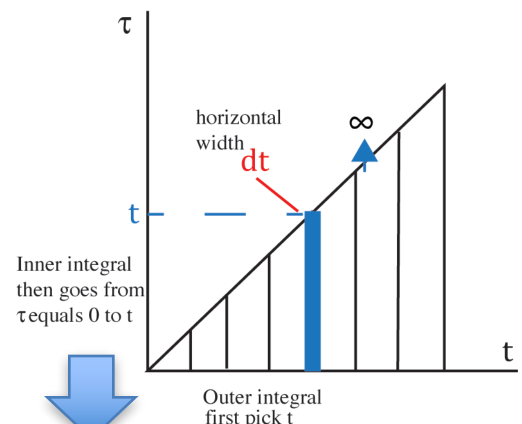
$$\text{Therefore } \tilde{g}(s) = \int_{t=0+\tau=\tau}^{t=\infty+\tau=\infty} g(t-\tau)e^{-s(t-\tau)}\,dt = e^{s\tau}\int_\tau^\infty g(t-\tau)e^{-st}\,dt$$

Next, construct the product

$$\tilde{f}(s)\tilde{g}(s) = \int_0^\infty [e^{s\tau}\int_\tau^\infty g(t-\tau)e^{-st}\,dt]f(\tau)e^{-s\tau}\,d\tau = \int_0^\infty f(\tau)\int_\tau^\infty g(t-\tau)e^{-st}\,dt\,d\tau$$



Change the order
Of integration



$$\text{Therefore} \quad \tilde{h}(s) = \tilde{f}(s)\tilde{g}(s) = \int_0^\infty e^{-st}[\int_\tau^\infty f(\tau)g(t-\tau)\,d\tau]dt = \int_0^\infty e^{-st}h(t)\,dt$$