Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
EN. 585.409

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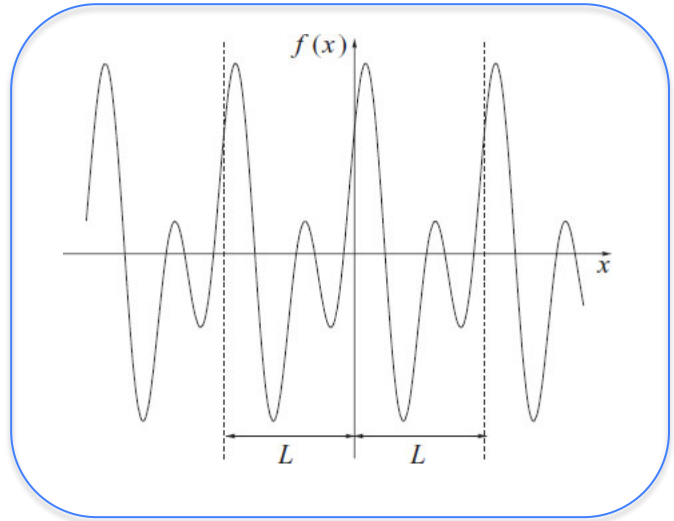
Fourier series Dirichlet conditions



- the function must be periodic;
- it must be single-valued and continuous, except possibly at a finite number of finite discontinuities;
- it must have only a finite number of maxima and minima within one period;
- the integral over one period of $|f(x)|$ must converge.

If the above conditions are satisfied then the Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous.

The value at a discontinuity is given by $\frac{1}{2}\lim_{\epsilon\to 0}[f(x_d+\epsilon)+f(x_d-\epsilon)]$

Trigonometric orthogonal conditions

$$\int_{x_0}^{x_0+L} \sin\left(\frac{2\pi r x}{L}\right) \cos\left(\frac{2\pi p x}{L}\right) dx = 0 \quad \text{for all } r \text{ and } p, \tag{4.1}$$

$$\int_{x_0}^{x_0+L} \cos\left(\frac{2\pi r x}{L}\right) \cos\left(\frac{2\pi p x}{L}\right) dx = \begin{cases} L & \text{for } r = p = 0, \\ \frac{1}{2}L & \text{for } r = p > 0, \\ 0 & \text{for } r \neq p, \end{cases} \tag{4.2}$$

$$\int_{x_0}^{x_0+L} \sin\left(\frac{2\pi r x}{L}\right) \sin\left(\frac{2\pi p x}{L}\right) dx = \begin{cases} 0 & \text{for } r = p = 0, \\ \frac{1}{2}L & \text{for } r = p > 0, \\ 0 & \text{for } r \neq p, \end{cases} \tag{4.3}$$

Derivation of a trigonometric orthogonal condition – example 1

(a) Let's look at the first orthogonal condition (4.1) from the previous slide when p = r. We therefore first evaluate the following indefinite integral

$$\int \sin\left(\frac{2\pi rx}{L}\right)\cos\left(\frac{2\pi rx}{L}\right)dx$$

$$u = \sin\left(\frac{2\pi rx}{L}\right) \quad du = \frac{2\pi r}{L}\cos\left(\frac{2\pi rx}{L}\right)dx$$

$$\int \frac{1}{\frac{2\pi r}{L}} u\,du = \frac{L}{2\pi r}\frac{1}{2}u^2$$

Therefore we can write

$$\int_{x_o}^{x_o+L} \sin\left(\frac{2\pi rx}{L}\right)\cos\left(\frac{2\pi rx}{L}\right)dx = \frac{1}{2}\frac{L}{2\pi r}\sin^2\left(\frac{2\pi rx}{L}\right)\Bigg|_{x_o}^{x_o+L}$$

$$= \frac{L}{4\pi r}\left[\sin^2\left(\frac{2\pi r(x_o+L)}{L}\right) - \sin^2\left(\frac{2\pi rx_o}{L}\right)\right]$$

Using the following trigonometric identities

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \qquad \cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

We finally have

$$\int_{x_o}^{x_o+L} \sin\left(\frac{2\pi r x}{L}\right)\cos\left(\frac{2\pi r x}{L}\right)dx = \frac{1}{2}\frac{L}{2\pi r}\sin^2\left(\frac{2\pi r x}{L}\right)\Bigg|_{x_o}^{x_o+L} = \frac{L}{4\pi r}\left[\sin^2\left(\frac{2\pi r(x_o+L)}{L}\right)-\sin^2\left(\frac{2\pi r x_o}{L}\right)\right]$$

$$= \frac{L}{4\pi r}\left\{\frac{1}{2}\left[1-\cos 2\left(\frac{2\pi r x_o}{L}+2\pi r\right)\right]-\frac{1}{2}\left[1-\cos 2\left(\frac{2\pi r x_o}{L}\right)\right]\right\}$$

$$= \frac{L}{8\pi r}\left\{1-\cos\left(\frac{4\pi r x_o}{L}+4\pi r\right)-1+\cos\left(\frac{4\pi r x_o}{L}\right)\right\} = \frac{L}{8\pi r}\left\{-\cos\left(\frac{4\pi r x_o}{L}+4\pi r\right)+\cos\left(\frac{4\pi r x_o}{L}\right)\right\}$$

$$= \frac{L}{8\pi r}\left\{-\left[\cos\left(\frac{4\pi r x_o}{L}\right)\cos 4\pi r - \sin\left(\frac{4\pi r x_o}{L}\right)\sin 4\pi r\right]+\cos\left(\frac{4\pi r x_o}{L}\right)\right\}$$

$$= \frac{L}{8\pi r}\left\{-\cos\left(\frac{4\pi r x_o}{L}\right)(1)+\sin\left(\frac{4\pi r x_o}{L}\right)(0)+\cos\left(\frac{4\pi r x_o}{L}\right)\right\} = 0$$

The case $p \neq r$ also needs to be derived.

Derivation of a trigonometric orthogonal condition – example 2

(a) Let' s look at the orthogonal condition (4.2) with p = r > 0.
First we can look up the following indefinite integral

$$\int \cos^2\left(\frac{2\pi rx}{L}\right)dx = \frac{1}{2}x - \frac{1}{4\left(\frac{2\pi r}{L}\right)}\sin 2\left(\frac{2\pi r}{L}\right)x = \frac{1}{2}x - \frac{L}{8\pi r}\sin\left(\frac{4\pi r}{L}\right)x$$

Then

$$\int_{x_o}^{x_o+L} \cos^2\left(\frac{2\pi r x}{L}\right) dx = \left[\frac{1}{2}x - \frac{L}{8\pi r}\sin\left(\frac{4\pi r}{L}\right)x\right]\Bigg|_{x_o}^{x_o+L}$$

$$= \left[\frac{1}{2}(x_o+L) - \frac{L}{8\pi r}\sin\left(\frac{4\pi r}{L}\right)(x_o+L)\right] - \left[\frac{1}{2}x_o - \frac{L}{8\pi r}\sin\left(\frac{4\pi r}{L}\right)x_o\right]$$

$$= \frac{L}{2} - \frac{L}{8\pi r}\sin\left(\frac{4\pi r x_o}{L} + 4\pi r\right) + \frac{L}{8\pi r}\sin\left(\frac{4\pi r}{L}\right)x_o$$

$$= \frac{L}{2} - \frac{L}{8\pi r}\sin\left(\frac{4\pi r x_o}{L} + 4\pi r\right) + \frac{L}{8\pi r}\sin\left(\frac{4\pi r}{L}\right)x_o$$

$$= \frac{L}{2} - \frac{L}{8\pi r}\left[\sin\left(\frac{4\pi r x_o}{L}\right)\cos 4\pi r + \cos\left(\frac{4\pi r x_o}{L}\right)\sin 4\pi r\right] + \frac{L}{8\pi r} sin\left(\frac{4\pi r}{L}\right)x_o$$

$$= \frac{L}{2} - \frac{L}{8\pi r}\left[\sin\left(\frac{4\pi r x_o}{L}\right)(1) + \cos\left(\frac{4\pi r x_o}{L}\right)(0)\right] + \frac{L}{8\pi r}\sin\left(\frac{4\pi r}{L}\right)x_o = \frac{L}{2}$$

(b) Next we will look at the orthogonal condition (4.2) with $p \neq r$.
First we can look up the following indefinite integral

$$\int cos(r\frac{2\pi}{L}x)cos(p\frac{2\pi}{L}x)dx = \frac{L}{2\pi}[\frac{sin(n\frac{2\pi}{L}x)}{2n}+\frac{sin(m\frac{2\pi}{L}x)}{2m}]$$

where n = p – r and m = p+ r. Then

$$\int_{x_o}^{x_o+L}\cos\left(\frac{2\pi rx}{L}\right)\cos\left(\frac{2\pi px}{L}\right)dx = \frac{L}{2\pi}[\frac{\sin(n\frac{2\pi}{L}x)}{2n}+\frac{\sin(m\frac{2\pi}{L}x)}{2m}]\Big|_{x_o}^{x_o+L}$$

$$=\frac{L}{2\pi}[\frac{\sin n\left(\frac{2\pi x_o}{L}+2\pi\right)}{2n}+\frac{\sin m\left(\frac{2\pi x_o}{L}+2\pi\right)}{2m}]-[\frac{\sin n\left(\frac{2\pi x_o}{L}\right)}{2n}+\frac{\sin m\left(\frac{2\pi x_o}{L}\right)}{2m}]$$

$$=\frac{L}{2\pi}[\frac{\sin n\left(\frac{2\pi x_o}{L}\right)}{2n}+\frac{\sin m\left(\frac{2\pi x_o}{L}\right)}{2m}]-[\frac{\sin n\left(\frac{2\pi x_o}{L}\right)}{2n}+\frac{\sin m\left(\frac{2\pi x_o}{L}\right)}{2m}]=0$$

Fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right) \right]$$

We solve for the a_r coefficients by multiplying both sides by a cosine function and then using our orthogonal conditions. For the b_r coefficients we use a sine function.

$$\begin{aligned}
\int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi px}{L}\right) dx = {} & \frac{a_0}{2} \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi px}{L}\right) dx \\
& + \sum_{r=1}^{\infty} a_r \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi rx}{L}\right) \cos\left(\frac{2\pi px}{L}\right) dx \\
& + \sum_{r=1}^{\infty} b_r \int_{x_0}^{x_0+L} \sin\left(\frac{2\pi rx}{L}\right) \cos\left(\frac{2\pi px}{L}\right) dx.
\end{aligned}$$

Then for the the cosine calculation we have for r = p not equal to 0

$$\int_{x_o}^{x_o+L} f(x)\cos\left(\frac{2\pi rx}{L}\right) dx = \frac{a_r}{2}L \qquad a_r = \frac{2}{L}\int_{x_o}^{x_o+L} f(x)\cos\left(\frac{2\pi rx}{L}\right)dx$$

and for r equal to 0 we have

$$a_0 = \frac{2}{L}\int_{x_o}^{x_o+L} f(x)dx$$

And for the sine calculation we have

$$\int_{x_o}^{x_o+L} f(x)\sin\left(\frac{2\pi rx}{L}\right) dx = \frac{b_r}{2}L \qquad b_r = \frac{2}{L}\int_{x_o}^{x_o+L} f(x)\sin\left(\frac{2\pi rx}{L}\right)dx$$