Johns Hopkins
Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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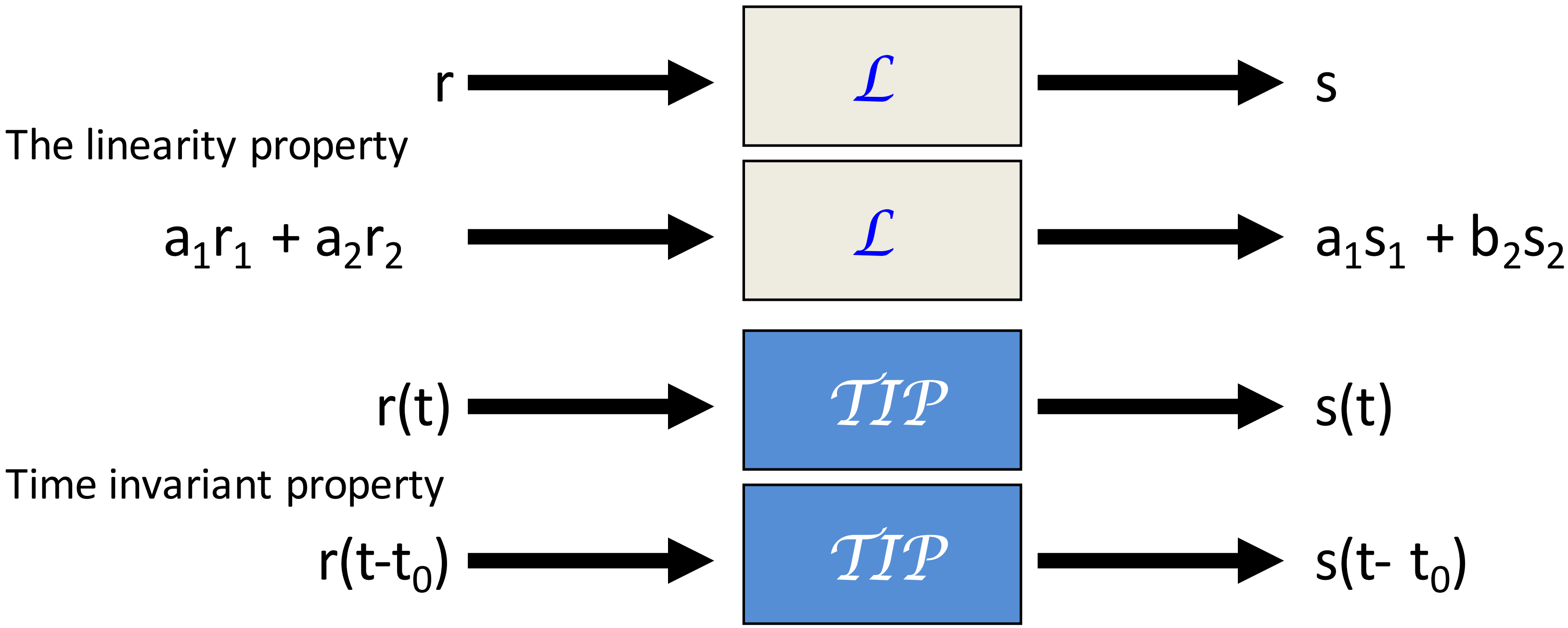
Remember the properties of a linear time invariant model



The convolution in fMRI

- **What is the mathematical definition of convolution?**
- **How is the convolution used in fMRI analysis?**
- **How do you calculate a convolution?**

What is the mathematical definition of convolution?

$$y(t) = x(t) * h(t) = \int_0^{\infty} x(\tau)h(t-\tau)\,d\tau, \quad t \geq 0$$

Note , this has been restricted to t greater than 0

How is the convolution used in fMRI analysis?

For the fMRI signal at each voxel we assume a linear time invariant model with stationary property with a response that is a convolution of the input and has a particular noise structure

$$s(\underline{x},t) = \mu(\underline{x}) + \sum_{u=-\infty}^{\infty} a(\underline{x},t-u)r(u) + \varepsilon(\underline{x},t)$$

Noise!

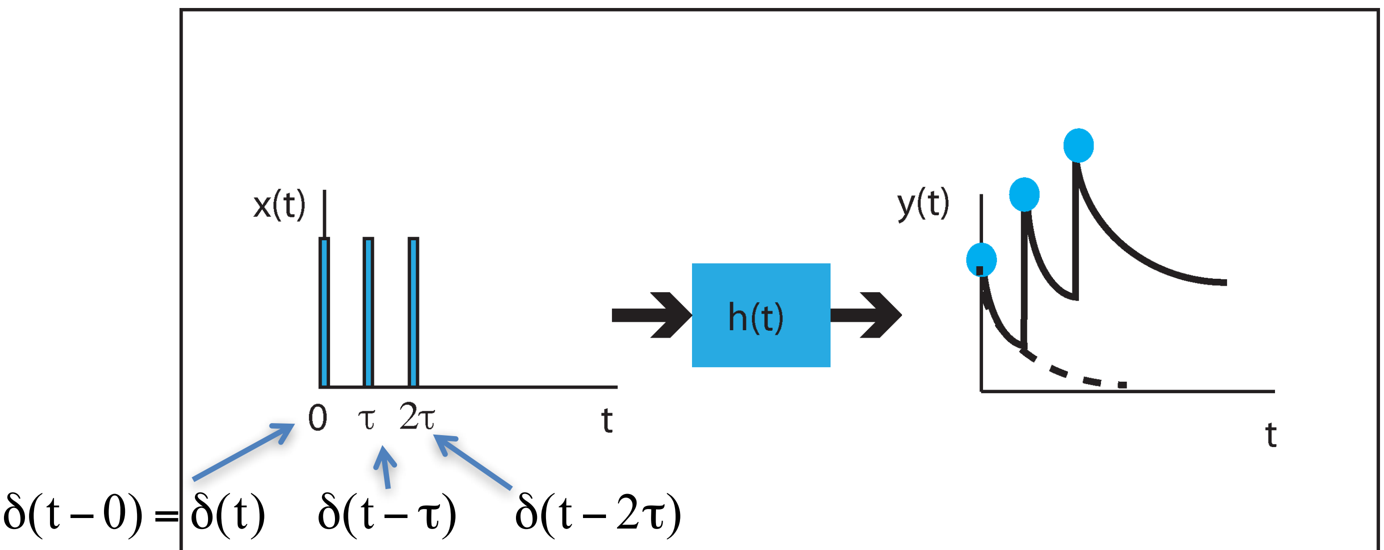
Convolution (discrete version)!

r(t) - input (experimental design)
s($\underline{x}$,t) - output (fMRI signal)
μ($\underline{x}$) - constant
a($\underline{x}$,t) – hemodynamic response function

How do you calculate a convolution?

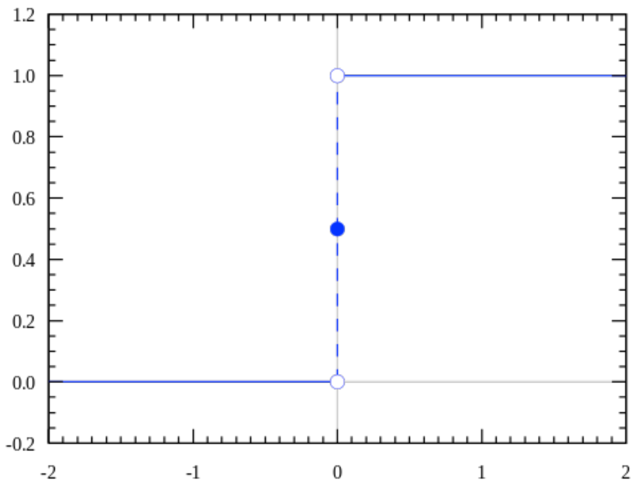
There are many ways to look at convolution.

- One methods that seems particularly applicable to fMRI is in terms of linear time-invariant (LTI) theory
- Here the system response can be looked at in it's simplest form as the response to a series of delta inputs.
- That is knowing the response to the delta input characterizes the response of the system to any input!



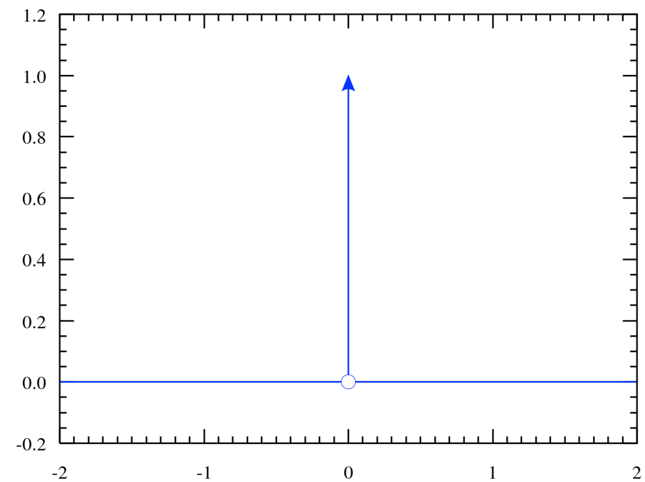
Aside: Some preliminaries

Heaviside step function



$$H(t-0) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Dirac delta function



$$\delta(t) = \delta(t-0) = \lim_{k\to 0} f_k(t-0) = \lim_{k\to 0} \frac{1}{k}[H(t-0) - H(t-(0+k))]$$

Laplace transform definition and some simple properties

Def. $\mathbf{L}\{\mathbf{y}(\mathbf{t})\} = Y(s) = \int_0^\infty y(t)e^{-st}dt$

$\mathbf{L}\{\delta(t-a)\} = e^{-sa}$

$\mathbf{L}\{y'(t)\} = sY(s) - y(0)$

Shift Th. $\mathbf{L}\{y(t-a)H(t-a)\} = e^{-sa}Y(s)$

Convolution $\mathbf{L}\{x(t) * y(t)\} = X(s)Y(s)$

Let’s look at a very simple system

Here the response function of the system is associated with the differential equation (think of it as the hemodynamic response of the brain)
The input is a delta function

The easiest way to solve is by

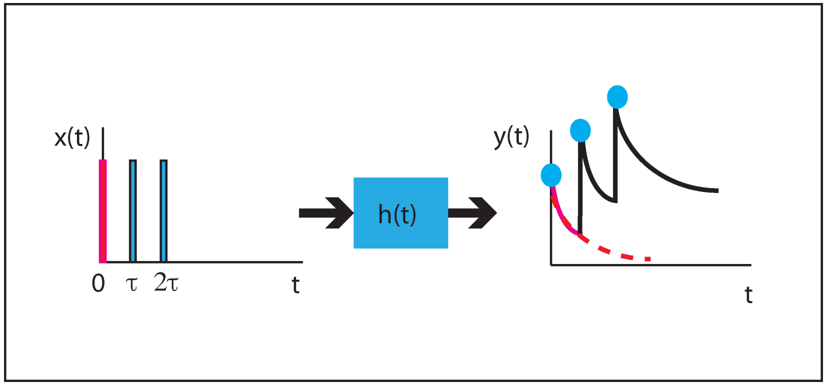
$$y' + 2y = \delta(t-a) \xrightarrow{\text{Laplace}} Y(s) = \frac{1}{s+2}e^{-as}$$

Taking the inverse Laplace gives

$$y(t) = L^{-1}\{Y(s)\} = e^{-2(t-a)}H(t-a)$$

Suppose the delta input is given at a = 0

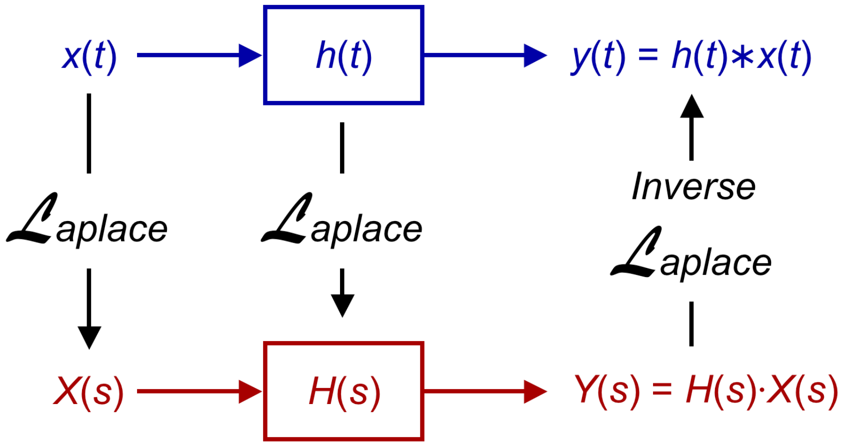
$$y(t) = e^{-2(t-0)}H(t-0) = e^{-2t}$$



Back

Laplace transform of a convolution

- We can look at any input, x(t) to this system as a continuous collection of delta functions of different magnitude
- Then the system h(t) response to the input x(t) is a convolution
- Of course this can easily be done using the Laplace transform, since the convolution in Laplace space, s is a simple multiplication of the Laplace transforms of the input, X(s) and system response, H(s)



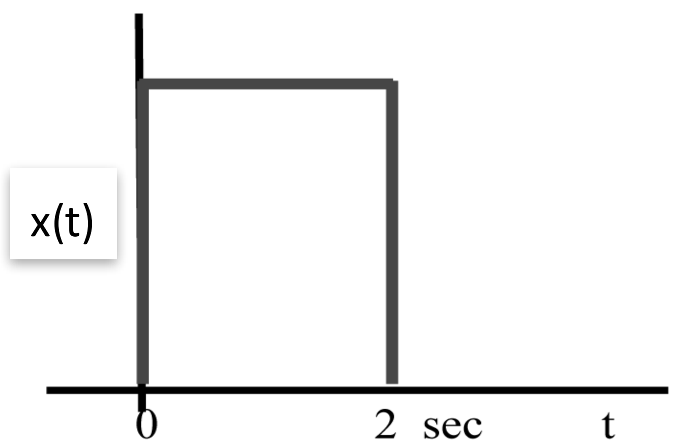
https://en.wikipedia.org/wiki/LTI_system_theory

Calculation of the convolution by Laplace Transform

In a typical functional MRI experiment that measures the BOLD response the input function often consist of input that occurs for one TR (= 2 sec). Ignoring the noise, the response function or fMRI signal to a good approximation is the convolution of the input with the hemodynamic transfer function. Given the following hemodynamic transfer function

$$\text{htf}(t) = 6.75(e^{-t/6} - e^{-t/4})$$

An example of this would be the presentation of a visual image for 2 sec (synchronized to one TR) in a fMRI experiment.



I'll start this off

$$\mathbf{L}\{x(t) * htf(t)\} = X(s)HTF(s)$$

$$htf(t) = 6.75(e^{-t/6} - e^{-t/4})$$

$$x(t) = H(t-0) - H(t-2) = 1 - H(t-2)$$

Laplace Transform

$$L\{htf(t)\} = 6.75[1/(s + {}^{1}\!/_{6}) - 1/(s + {}^{1}\!/_{4})] = HTF(s)$$

$$L\{x(t)\} = {}^{1}\!/_{s} - {}^{e^{-2s}}\!/_{s} = X(s)$$

Can you finish this?

In order to get the answer back to t (or time) you will need to take the inverse Laplace transform, $\mathbf{L}^{-1}$ using standard tables.

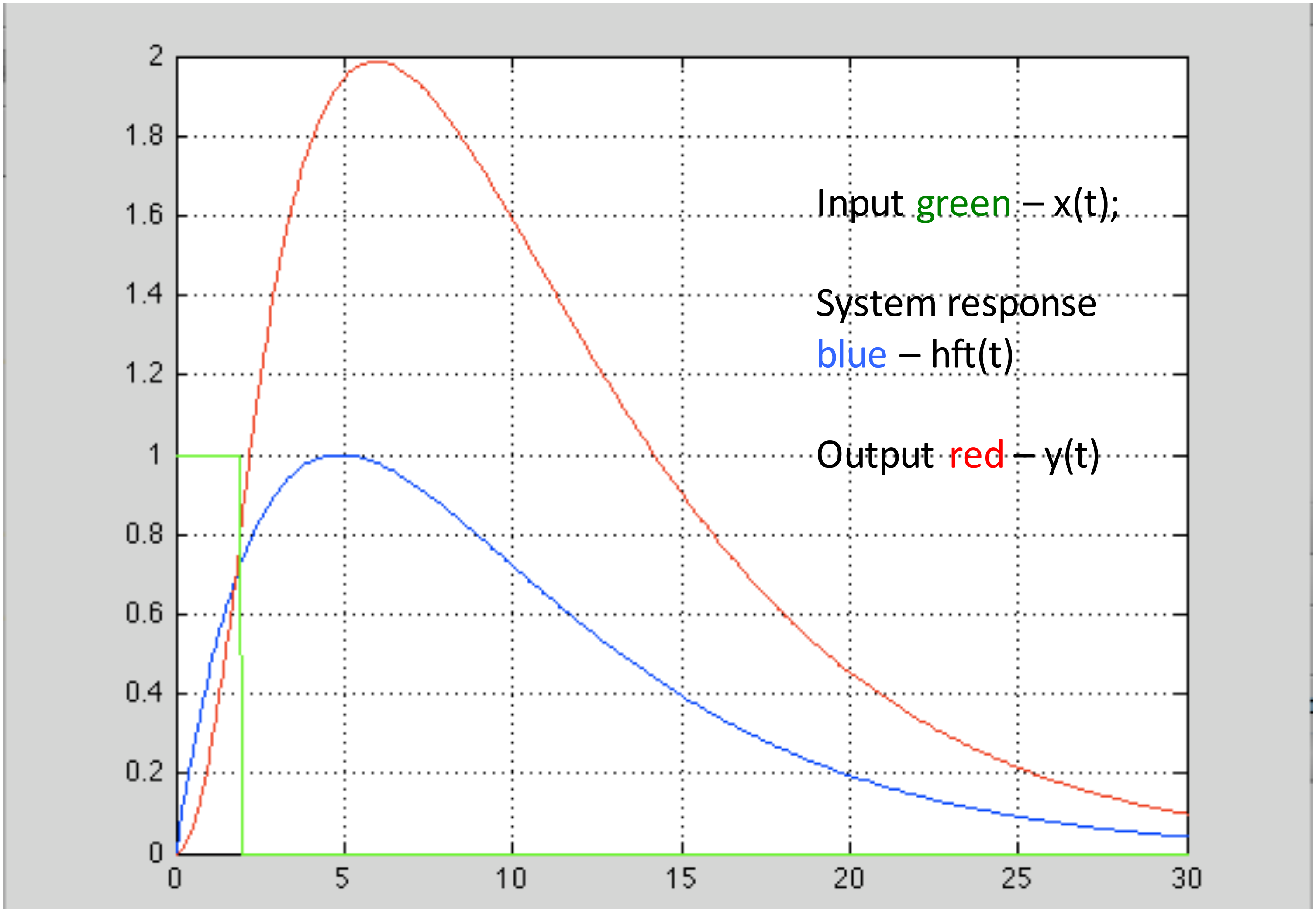
This will take a little more effort as you will need two techniques:

- Partial Fractions
- Shift theorem as used in our simple example

$$y(t) = \mathbf{L}^{-1}\{X(s)HTF(s)\}$$

$$y(t) = 6.75[(6 - 6e^{-t/6}) - (6 - 6e^{-(t-2)/6})H(t-2) - (4 - 4e^{-t/4}) + (4 - 4e^{-(t-2)/4})H(t-2)]$$

Here is what the answer looks like



By the way the delay in the response in this example is about 4 to 6 seconds!

An even easier way is using MATLAB’s conv function for the calculation of the convolution

Initialize

```
clear all
tint=0;
tfinal=10;
tstep=.01;
t=tint:tstep:tfinal;
```

Input

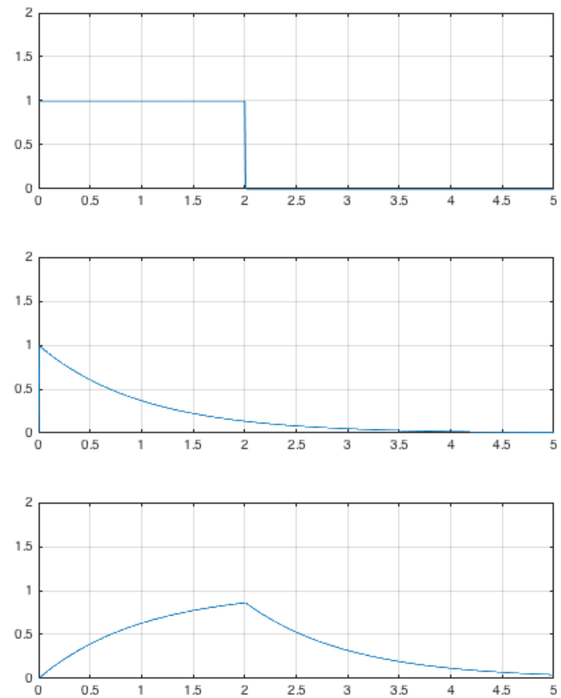
```
x=1*((t>=0)&(t<=2));
subplot(3,1,1), plot(t,x)
axis([0 5 0 2])
grid on
```

System Response

```
h= (exp(-t)).*((t>0)&(t<5));
subplot(3,1,2), plot(t,h)
axis([0 5 0 2])
 grid on
```

Output

```
t2=2*tint:tstep:2*tfinal;
y=conv(x,h)*tstep;
subplot(3,1,3),plot(t2,y)
axis([0 5 0 2])
grid on
```



- Can you do this by the Laplace transform method?
- Can you use this MATLAB technique to calculate the sample response to the problem I presented by the Laplace transform?