Chapter 4 18 pages

Problems

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Chapter 4. Problem 4.4

By moving the origin of t to the center of an interval in which $f(t) = +1$, i.e. by changing to a new independent variable $t' = t - \frac{1}{4}T$, express the square-wave function in the example in Section 4.2 as a cosine series.

Calculate the Fourier coefficients involved (a) directly and (b) by changing the variable in result (4.10)

Using the new independent variable t', the square wave is now defined by:

$$f(t') = \begin{cases} -1 & \text{for } -3T/4 \le t' < T/4 \\ +1 & \text{for } -T/4 \le t' \le T/4 \end{cases}$$

The function $f(t')$ is even thus the Fourier coefficients

$$b_r = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin\left(\frac{2\pi r t}{T}\right) dt = 0$$

as is the average of the square-wave function over $[-T/2, T/2]$ which is easily seen be 0.

Chapter 4 - Problem 4.4

For $r \geq 1$

$$a_r = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi r t}{T}\right) dt$$

Let $\omega = \frac{2\pi r}{T}$ $\quad a_r = \frac{2}{T}\left[\int_{-T/2}^{-T/4}(-1)\cos(\omega t)\,dt + \int_{-T/4}^{T/4} 1\cdot\cos(\omega t)\,dt + \int_{T/4}^{T/2}(-1)\cdot\cos(\omega t)\,dt\right]$

$$\int_{-T/2}^{-T/4}\cos(\omega t)\,dt = \frac{1}{\omega}\left[\sin\omega t\right]_{-T/2}^{-T/4} = \frac{1}{\omega}\left[-\sin\omega T/4 + \sin(\omega T/2)\right]$$

Similarly we find:

$$\int_{-T/4}^{T/4}\cos(\omega t)\,dt = 1/\omega\left[\sin(\omega T/4) + \sin\omega T/4\right] = \frac{2}{\omega}\sin(\omega T/4)$$

$$\int_{T/4}^{T/2}\cos(\omega t)\,dt = \frac{1}{\omega}\left[\sin\omega T/2 - \sin(\omega T/4)\right]$$

Putting these integrals back into the expression of ar:

$$a_r = \frac{2}{T}\,\frac{1}{\omega}\cdot\left[\sin(\omega T/4) - \sin(\omega T/2) + 2\sin(\omega T/4) - \sin(\omega T/2) + \sin(\omega T/4)\right]$$

$$a_r = \frac{2}{T}\times\frac{T}{2\pi r}\times 4\cdot\sin\left(\omega\frac{T}{4}\right) = \frac{4}{\pi r}\sin\left(\frac{2\pi r}{T}\cdot\frac{T}{4}\right)$$

$$= \frac{4}{\pi r}\sin\left(\frac{\pi r}{2}\right)$$

Therefore the Fourier series for the square wave function centred at 0 is:

$$f(t) = \sum_{r=1}^{\infty} a_r \cos \frac{2\pi r t}{T} \quad \text{with } a_r = \frac{4}{\pi r} \sin\left(\frac{\pi r}{2}\right)$$

All the even a_r coefficients are zero and for $r = 2p+1$:

$$a_{2p+1} = \frac{4(-1)^p}{\pi(2p+1)}$$

and

$$f(t) = \sum_{p=0}^{\infty} \frac{4(-1)^p}{\pi(2p+1)} \cos \frac{2\pi(2p+1)t}{T}$$

$$= \frac{4}{\pi} \sum \frac{(-1)^p}{2p+1} \cos \frac{2\pi(2p+1)t}{T}$$

$$= \frac{4}{\pi}\left(\cos \omega t - \frac{\cos 3\omega t}{3} + \frac{1}{5}\cos 5\omega t - \dots\right)$$

where $\omega = 2\pi/T$

(b) Using the result (4.10) and making the change of variable $t = t' + T/4$

$$f(t) = \frac{4}{\pi}\left(\sin \omega(t' + T/4) + \frac{1}{3}\sin 3\omega(t' + T/4) + \dots\right)$$

$$= \frac{4}{\pi}\left[\sin(\omega t' + \omega T/4) + \frac{1}{3}\sin(3\omega t' + 3\omega T/4) + \dots\right]$$

$$\omega \frac{T}{4} = \frac{2\pi}{T} \cdot \frac{T}{4} = \frac{\pi}{2}$$

$$f(t) = \frac{4}{\pi}\left[\sin(\omega t' + \pi/2) + \frac{1}{3}\sin(3\omega t' + 3\pi/2) + \frac{1}{5}\sin(5\omega t' + 5\pi/2) + \dots\right]$$

$$f(t) = \frac{4}{\pi}\left[\cos\omega t' - \frac{1}{3}\cos 3\omega t' + \frac{1}{5}\cos 5\omega t' + \dots\right]$$

Chapter 4 - Problem 4.6

For the function $f(x) = 1-x \quad 0 \leq x \leq 1$

Find (a) the Fourier sine series and

(b) the Fourier cosine series.

Which would be better for numerical evaluation?

Relate your answer to the relevant periodic continuations

(a) The odd extension of $f(x)$:

$$g(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq 1 \\ -f(-x) & \text{if } -1 \leq x \leq 0 \end{cases}$$

$$= \begin{cases} 1-x & 0 \leq x \leq 1 \\ -1-x & -1 \leq x \leq 0 \end{cases}$$

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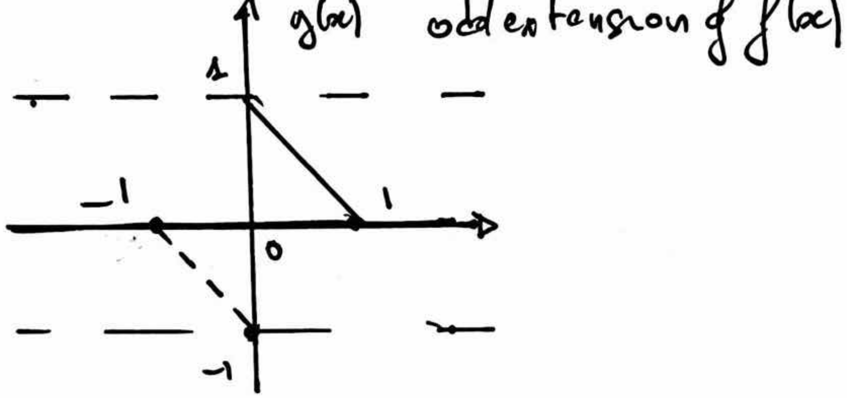
Chapter 4 - Problem 4.6



$g(x)$ is odd and has only Fourier coefficient b_r in the Fourier Series expansion of $f(x)$:

$$b_r = \frac{2}{2} \times 2 \times \int_0^1 f(x) \sin\left(\frac{2\pi r x}{2}\right) dx$$

$$= 2 \times \int_0^1 (1-x) \sin(\pi r x)\, dx$$

And $\int_0^1 (1-x) \sin(\pi r x)\, dx = -\frac{1}{\pi r}\left[(1-x)\cos(\pi r x)\right]_0^1$

$$+ \frac{1}{\pi r} \int_0^1 (-1) \cos(\pi r x)\, dx$$

(Integration by parts)

$$= -\frac{1}{\pi r}\left[0 - \cos(0)\right] - \frac{1}{(\pi r)^2}\left[\sin \pi r x\right]_0^1$$

$$= \frac{1}{\pi r}$$

$$b_r = \frac{2}{\pi r}$$

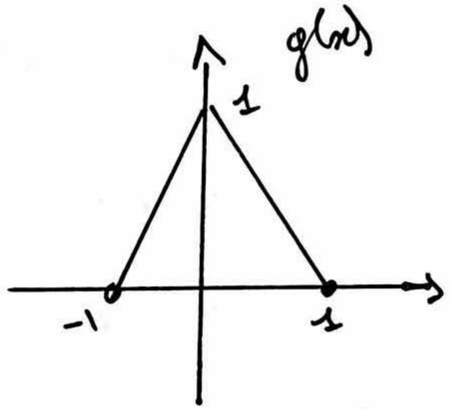
Chapter 4 - Problem 4.6

So, we get the Fourier sine series for this function:

$$\| f(x) = \sum_{r=1}^{\infty} \frac{2}{\pi r} \sin(\pi r x)$$

(b) The even extension of this function

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq 1 \\ f(-x) & -1 \leq x \leq 0 \end{cases} = \begin{cases} 1-x & 0 \leq x \leq 1 \\ 1+x & \text{for } -1 \leq x \leq 0 \end{cases}$$



g(x) is even, and only the ar coefficients in its Fourier series expansion are non-zero:

$$a_0 = \frac{2}{L} \times \int_{-1}^{1} (1-x)\, dx = \frac{2}{2} \times 2 \times \int_{0}^{1} (1-x)\, dx$$

with $L=2$

$$a_0 = 2 \times \left(-\tfrac{1}{2}\right) \times \left[(1-x)^2\right]_0^1 = 1$$

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<u>Chapter 4 – Problem 4.6</u>

$$a_r = \frac{2}{2} \times 2 \times \int_0^1 (1-x) \cos\left(\frac{2\pi r x}{2}\right) dx$$

$$a_r = 2 \times \int_0^1 (1-x) \cos(\pi r x)\, dx$$

$$\int_0^1 (1-x)\cos(\pi r x)\, dx = \frac{1}{\pi r}\left[(1-x)\sin(\pi r x)\right]_0^1 + \frac{1}{\pi r}\int_0^1 \sin(\pi r x)\, dx$$

$$= \frac{1}{\pi^2 r^2}\left[-\cos(\pi r x)\right]_0^1 = \frac{1}{\pi^2 r^2}\left[-(-1)^r - 1\right]$$

$$= \frac{1}{\pi^2 r^2}\left(1 - (-1)^r\right)$$

$$\Rightarrow a_r = \frac{2}{\pi^2 r^2}\left(1 - (-1)^r\right)$$

$$a_r = \begin{cases} \frac{4}{\pi^2 r^2} & r \text{ odd} \\ 0 & r \text{ even} \end{cases}$$

The Fourier cosine series is then

$$f(x) = \frac{1}{2} + \sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} \frac{4}{\pi^2 r^2} \cos(\pi r x)$$

The Fourier sine series is discontinuous at $x=0$ which can lead to numerical difficulties. The Fourier cosine series by comparison, does not present such issues and therefore would be better for numerical evaluation.

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Chapter 4- Problem 4.8

The function $y(x)=x\sin x$ for $0\le x\le\pi$ is to be represented by a Fourier series of period 2π that is even or odd. By sketching the function and considering its derivative, determine which series will have the more rapid convergence. Find the full expression for the better of these two series, showing that the convergence $\sim n^{-3}$ and that alternate terms are missing.

See attached plot. of even and odd extensions of the function.

Note the derivative of the even extension at $x=\pi$ is undefined since the graph has different slopes on the left of $x=\pi$ and on the right.

However the odd extension, the derivative exists at $x=\pi$, the function is smooth at π, and we take the odd extension:

$$f(x)=\begin{cases} x\sin x & x\geq 0 \\ -x\sin x & x<0 \end{cases}$$

for $x\in[-\pi,\pi]$, $T=2\pi$

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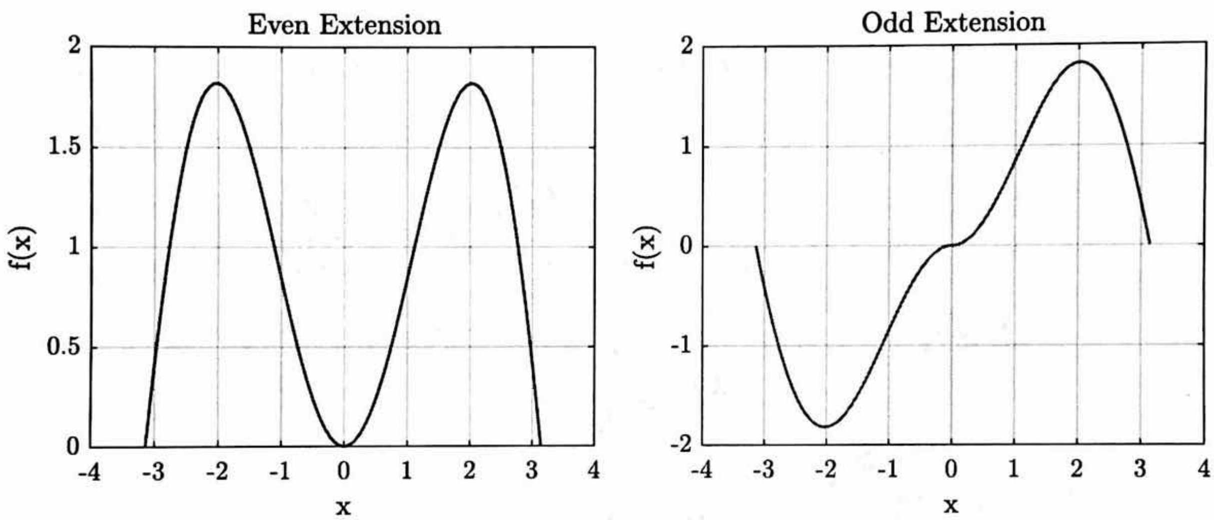
chapter 4 - Problem 4.8

Graph of the function:



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Chapter 4 - Problem 4.8

The Fourier coefficients of the odd extension are:

$$b_n = \frac{2}{2\pi} \cdot 2 \cdot \int_0^{\pi} x \sin(x) \sin(nx)\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \left[\cos(n-1)x - \cos(n+1)x \right] dx$$

$$= \frac{1}{\pi} \left[- \frac{(-1)^n + 1}{(n-1)^2} - \frac{(-1)^n + 1}{(n+1)^2} \right]$$

$$= \frac{1}{\pi} (-1) \left[(-1)^n + 1 \right] \cdot \frac{4n}{(n^2-1)^2} \quad \text{for } n \neq 1$$

$$b_1 = \frac{2}{\pi} \int_0^{\pi} x \sin(x) \sin(x)\, dx = \frac{2}{\pi} \int_0^{\pi} x \sin^2(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x (1 - \cos 2x)\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} - \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \frac{\pi^2}{2} = \pi/2$$

$$b_n = \begin{cases} \frac{1}{\pi} (-1) \left[1 + (-1)^n \right] \frac{4n}{(n^2-1)^2} & n\text{-even} \\ 0 & n\text{-odd} \end{cases}$$

Therefore the convergence is $\sim \frac{n}{(n^2-1)^2} = o(n^{-3})$.

for n even.

Chapter 4 - Problem 4.9 Yves Greatti

Find the Fourier coefficients in the expansion of $f(x) = \exp x$ over the range $-1 < x < 1$. What value will the expansion have when $x = 2$?

$$a_0 = \frac{2}{2} \times \int_{-1}^{1} e^x \, dx = [e^x]_{-1}^{1} = 2 \times \frac{e^1 - e^{-1}}{2} = \sinh(1) \times 2$$

$$a_n = \frac{2}{2} \times \int_{-1}^{1} e^x \cos\left(\frac{2\pi n x}{2}\right) dx$$

$$= \int_{-1}^{1} e^x \cos(\pi n x) \, dx$$

By Integral table: $\int e^{ax} \cos nx \, dx = \frac{e^{ax}}{a^2+n^2} (a \cos nx + n \sin nx)$

Therefore

$$a_n = \frac{1}{1+(n\pi)^2} \left[e^x \left(\cos n\pi x + n\pi \sin(n\pi x) \right) \right]_{-1}^{1}$$

$$= \frac{1}{1+(n\pi)^2} \left[e^1 \cos(n\pi) - e^{-1} \cos n\pi \right]$$

$$= 2 \times \frac{\sinh(1)}{1+(n\pi)^2} \times \cos n\pi = 2 \times \frac{(-1)^n \sinh(1)}{1+(n\pi)^2}$$

Chapter 4 - Problem 4-9

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We now find the Fourier coefficients bn:

$$b_n = \frac{2}{2} \times \int_{-1}^{1} e^x \sin(n\pi x)\, dx = \int_{-1}^{1} e^x \sin(n\pi x)\, dx$$

Using an integral table:

$$\int e^{ax} \sin(nx)\, dx = \frac{e^{ax}}{a^2+n^2} (a \sin nx - n \cos nx)$$

$$\Rightarrow b_n = \frac{1}{1+(n\pi)^2} \left[e^x (\sin n\pi x - n\pi \cos n\pi x) \right]_{-1}^{1}$$

$$= 2 \times \cos(n\pi) \frac{e^{-1}-e^{1}}{2} \times n\pi \times \frac{1}{1+(n\pi)^2}$$

$$= -2 \times \sinh(1) \frac{n\pi}{1+(n\pi)^2}$$

The Fourier series expansion of $f(x)=e^x$ on $(-1,1)$ is then:

$$e^x = 2 \frac{\sinh(1)}{2} + 2\sinh(1) \times \sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2} [\cos n\pi x - n\pi \sin(n\pi x)]$$

$$= \sinh(1) \left[1 + 2 \times \sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2} [\cos n\pi x - n\pi \sin n\pi x] \right]$$

When we expand $f(x)=e^x$ over the range $1<x<3$, we have extended the same function in the range $1<x<3$ with the same period $P=2$ therefore at $x=2$, $f(2)=f(2-P)=f(2-2)=f(0)=1$

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Problem 4.10 Chapter 4

By integrating term by term the Fourier series found in the previous question and using the Fourier series for $f(x)=x$ found in section 4.6, show that $\int \exp x \, dx = \exp x + c$.
Why is it not possible to show that $\frac{d e^x}{dx} = e^x$ by differentiating the Fourier series of $f(x)=e^x$ in a similar manner?

We found in the previous question that the Fourier series expansion of $f(x)=e^x$ in $(-1,1)$ is:

$$e^x = \sinh(1)\left[1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2}\left[\cos(n\pi x) - n\pi \sin(n\pi x)\right]\right]$$

Integrating term by term, we have:

$$\int e^x dx = \sinh(1)\int dx + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+(n\pi)^2}\int\left(\cos(n\pi x) - n\pi\sin(n\pi x)\right)dx$$

$$= \sinh(1)\,x + 2\sinh(1)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+(n\pi)^2}\left[\frac{\sin(n\pi x)}{n\pi} + \cos(n\pi x)\right] + C$$

C: constant

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Chapter 4 - Problem 4.10

Using the result of section 4.6, on the range $-1 < x < 1$

we have:

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin(\pi n x)$$

$$= (-2) \sum_{n=1}^{\infty} (-1)^n \left[\frac{\pi n}{1+(\pi n)^2} + \frac{1}{\pi n (1+(\pi n)^2)} \right] \sin(\pi n x)$$

substituting this expression of x back into the integral we just express, we have:

$$\int e^x dx = 2\sinh(1) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2} \cos(n\pi x)$$

$$+ 2\sinh(1) \sum_{n=1}^{\infty} (-1)^n \left[\frac{-\pi n}{1+(\pi n)^2} - \frac{1}{\pi n (1+(\pi n)^2)} + \frac{1}{\pi n (1+(\pi n)^2)} \right] \sin(\pi n x)$$

$$+ C$$

$$\Rightarrow \int e^x dx = 2\sinh(1) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+(n\pi)^2} \left[\cos(n\pi x) - n\pi \sin(n\pi x) \right] + C$$

$$= e^x + C$$

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Chapter 4 - Problem 4.10

When we differentiate term by term the Fourier series of exp, we obtain:

$$\frac{d}{dx}\left[\sinh(1)\left[1+2.\sum_{n=1}^{\infty}\frac{(-1)^n}{1+(n\pi)^2}\left[\cos(n\pi x)-n\pi\sin(n\pi x)\right]\right]\right]$$

$$= 2\sinh(1)\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{1+(n\pi)^2}(n\pi)(-\sin n\pi x)+\frac{(n\pi)^2}{1+(n\pi)^2}(-1)^{n+1}\cos(n\pi x)\right]$$

$$= 2\sinh(1)\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{1+(n\pi)^2}n\pi\sin n\pi x-\frac{(n\pi)^2(-1)^n}{1+(n\pi)^2}\cos(n\pi x)\right]$$

let $S(x)=\sum_{n=1}^{\infty}\frac{(n\pi)^2(-1)^n}{1+(n\pi)^2}\cos(nx)$

$\lim_{n\to\infty}\frac{(n\pi)^2}{1+(n\pi)^2}=1$ the term $\frac{(n\pi)^2}{1+\pi^2n^2}$ does not go to zero as $n\to\infty$ therefore $\sum_{n=1}^{\infty}\frac{(n\pi)^2(-1)^n}{1+(n\pi)^2}\cos(nx)$ does not converge and the derivative does not exist.

Chapter 4. Problem 4.14

Show that the Fourier series for the function $y(x) = |x|$

in the range $-\pi \leq x < \pi$ is

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}$$

By integrating this equation term by term from 0 to x,

find the function $g(x)$ whose Fourier series is

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{(2m+1)^3}$$

Deduce the value of the sum S of the series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

The function $y(x)$ can be described as

$$y(x) = \begin{cases} -x \; ; & x \in [-\pi, 0] \\ x \; ; & x \in [0, \pi) \end{cases}$$

$y(-x) = y(x)$ so $g(x)$ is even and the Fourier coefficients $b_r = 0$, and we need to find the coefficients a_r, $r = 0, 1, 2, \ldots$

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$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} y(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} y(x)\, dx = \frac{2}{\pi} \int_0^{\pi} x\, dx = \pi$$

$$a_r = \frac{2}{2\pi} \times 2 \times \int_0^{\pi} x \cos\left(\frac{2\pi r}{2\pi} x\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos(rx)\, dx$$

$$= \frac{2}{\pi} \left[\frac{rx\sin(rx) + \cos(rx)}{r^2}\right]_0^{\pi} \quad \text{by integration by part}$$

$$= \frac{2}{\pi} \times \frac{(-1)^r - 1}{r^2} = \begin{cases} -\dfrac{4}{\pi r^2} & r \text{ is odd} \\ 0 & r \text{ is even} \end{cases}$$

. The Fourier series of $y(x)$ in $[-\pi, \pi)$ is then:

$$y(x) = \frac{\pi}{2} + \sum_{\substack{r=1 \\ r\ \text{odd}}}^{\infty} a_r \cos\left(\frac{2\pi r}{2\pi} x\right)$$

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{r=1 \\ r\ \text{odd}}}^{\infty} \frac{\cos(rx)}{r^2}$$

$$\| \; y(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\cos(2m+1)x}{(2m+1)^2}$$

Chapter 4 - Problem 4.14

$$\int_0^x y(x)\,dx = \int_0^x \left[\pi/2 - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2}\right] dx$$

$$\int_0^x x\,dx = \int_0^x \frac{\pi}{2}\,dx - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\int_0^x \cos(2m+1)x\,dx$$

$$\Rightarrow \quad \frac{x^2}{2} = \frac{\pi}{2}x - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^3}\left[\sin(2m+1)x\right]_0^x$$

$$\Rightarrow \quad \frac{x^2}{2} = \frac{\pi}{2}x - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3}$$

$$\Rightarrow \quad \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3} = \frac{\pi}{2}x - \frac{x^2}{2} = \frac{x}{2}(\pi - x)$$

Take $x = \pi/2$ and we find:

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)\pi/2}{(2m+1)^3} = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(m\pi + \pi/2)}{(2m+1)^3}$$

$$\Rightarrow \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3} = \frac{\pi}{4}\times\frac{\pi}{2\times 2}\times\left(\pi - \frac{\pi}{2}\right)$$

$$= \frac{\pi^3}{32}$$

$$\Rightarrow 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = \frac{\pi^3}{32}$$