10.18

Solve

I will take the solution a little different than book and then make the connection.

Take $u(x,t)=Ae^{mx+i\omega t} \rightarrow \frac{\partial^4}{\partial x^4}Ae^{mx+i\omega t}=m^4Ae^{mx+i\omega t}$ and $\frac{\partial^2}{\partial t^2}Ae^{mx+i\omega t}=(i\omega)^2Ae^{mx+i\omega t}$

Substitute into $a^4\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}=0 \rightarrow a^4m^4Ae^{mx+i\omega t}+(i\omega)^2Ae^{mx+i\omega t}=[a^4m^4+(i\omega)^2]Ae^{mx+i\omega t}=0$

Therefore $a^4m^4+(i\omega)^2=0$ or $m^4=-\frac{(i\omega)^2}{a^4}=\frac{\omega^2}{a^4} \rightarrow$ (we expect 4 roots) $m=\pm\frac{\sqrt{\omega}}{a},\pm\frac{i\sqrt{\omega}}{a}$

Now take $\lambda=\frac{\sqrt{\omega}}{a}$ then $m=\pm\lambda,\pm i\lambda$

and we write the solution as a superposition $u(x,t)=[Ae^{\lambda x}+Be^{-\lambda x}+\tilde{C}e^{i\lambda x}+\tilde{D}e^{-i\lambda x}]e^{i\omega t}=X(x)T(t)$

where $X(x) = Ae^{\lambda x}+Be^{-\lambda x}+\tilde{C}e^{i\lambda x}+\tilde{D}e^{-i\lambda x}$

Compare this u(x,t) to that in the book which is just simnply written as $u(x,t)=Ae^{\lambda x}e^{i\omega t}$

Next applying Euler's identity for complex exponential and taking C and D as our new constants gives $X(x) = Ae^{\lambda x}+Be^{-\lambda x}+C\cos\lambda x+D\sin\lambda x$

Next look at the boundary conditions $u(0,t)=0, u(x,L)=0, \left.\frac{\partial u}{\partial x}\right|_{x=0}=0, \left.\frac{\partial u}{\partial x}\right|_{x=L}=0$

All of them can be written in terms of X(x), for example

$u(0,t)=X(0)T(t)=0$ since in general $T(t)\neq 0 \rightarrow X(0)=0$

The same relations can be derived for all the boundary conditions, that is

$X(0)=0,\ X(L)=0,\ \left.\frac{\partial X}{\partial x}\right|_{x-0}=0, \left.\frac{\partial X}{\partial x}\right|_{x-L}=0$

Taking the derivative of $X(x) = Ae^{\lambda x} + Be^{-\lambda x} + C\cos\lambda x + D\sin\lambda x$

gives $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x} - C\lambda\sin(\lambda x) + D\lambda\cos(\lambda x)$

Next apply the boundary conditions at x = 0 $X(0) = 0$ and $\left.\frac{\partial X}{\partial x}\right|_{x=0} = 0$

$X(0) = Ae^{\lambda\cdot 0} + B\lambda e^{-\lambda\cdot 0} + C\cos(\lambda\cdot 0) + D\sin(\lambda\cdot 0) = A + B + C = 0$

$X'(0) = A\lambda e^{\lambda\cdot 0} - B\lambda e^{-\lambda\cdot 0} - C\lambda\sin(\lambda\cdot 0) + D\lambda\cos(\lambda\cdot 0) = \lambda A - \lambda B + \lambda D = 0$

Therefore $C = -A - B$ and $\lambda A - \lambda B + \lambda D = 0 \rightarrow (A - B + D)\lambda = 0 \rightarrow D = -A + B$

Substitution gives $X(x) = Ae^{\lambda x} + Be^{-\lambda x} + (-A - B)\cos\lambda x + (-A + B)\sin\lambda x =$

$A[e^{\lambda x} - \cos\lambda x - \sin\lambda x] + B[e^{-\lambda x} - \cos\lambda x + \sin\lambda x]$

Taking the derivative of $X(x)=A[e^{\lambda x} - \cos\lambda x - \sin\lambda x] + B[e^{-\lambda x} - \cos\lambda x + \sin\lambda x]$

gives $X'(x)=A[\lambda e^{\lambda x} + \lambda\sin\lambda x - \lambda\cos\lambda x] + B[-\lambda e^{-\lambda x} + \lambda\sin\lambda x + \lambda\cos\lambda x]$

Next apply the boundary conditions at x = L $X(L) = 0$ and $\left.\frac{\partial X}{\partial x}\right|_{x=L} = 0$

1. $X(L)=A[e^{\lambda\cdot L} - \cos\lambda\cdot L - \sin\lambda\cdot L] + B[e^{-\lambda\cdot L} - \cos\lambda\cdot L + \sin\lambda\cdot L] = 0$

$X'(L)=A[\lambda e^{\lambda\cdot L} + \lambda\sin\lambda\cdot L - \lambda\cos\lambda\cdot L] + B[-\lambda e^{-\lambda\cdot L} + \lambda\sin\lambda\cdot L + \lambda\cos\lambda\cdot L] = 0$

OR factoring out λ in the second equation

2. $X'(L)=A[e^{\lambda\cdot L} + \sin\lambda\cdot L - \cos\lambda\cdot L] + B[-e^{-\lambda\cdot L} + \sin\lambda\cdot L + \cos\lambda\cdot L] = 0$

Add equations 1. and 2. gives

$A[2e^{\lambda\cdot L} - 2\cos\lambda L] + B[2\ \sin\lambda L] = 0 \rightarrow B = \frac{\cos\lambda L - e^{\lambda\cdot L}}{\sin\lambda L}A$

Substitute this B into 1.

$X(L)=A[e^{\lambda\cdot L} - \cos\lambda\cdot L - \sin\lambda\cdot L] + \left[\frac{\cos\lambda L - e^{\lambda\cdot L}}{\sin\lambda L}A\right][e^{-\lambda\cdot L} - \cos\lambda\cdot L + \sin\lambda\cdot L] = 0$

Get a common denominator and since $\sin\lambda L \neq 0$ we get

$$A[e^{\lambda L} - \cos\lambda L - \sin\lambda L]\sin\lambda L + A(\cos\lambda L - e^{\lambda L})[e^{-\lambda L} - \cos\lambda L + \sin\lambda L] = 0$$

or factoring out A we get

$$[e^{\lambda L} - \cos\lambda L - \sin\lambda L]\sin\lambda L + (\cos\lambda L - e^{\lambda L})[e^{-\lambda L} - \cos\lambda L + \sin\lambda L] = 0$$

Multiplying out we get

$$e^{\lambda \cdot L}\sin\lambda L - \cos\lambda L\sin\lambda L - \sin^2\lambda L + e^{-\lambda L}\cos\lambda L - \cos^2\lambda L + \cos\lambda L\sin\lambda L - e^{\lambda L}e^{-\lambda L} + e^{\lambda L}\cos\lambda L - e^{\lambda L}\sin\lambda L = 0$$

Now simplify by canceling out terms

$$-\sin^2\lambda L + e^{-\lambda L}\cos\lambda L + e^{-\lambda L}\cos\lambda L - \cos^2\lambda L - e^{\lambda L}e^{-\lambda L} + e^{\lambda L}\cos\lambda L = 0$$

OR rearranging

$$-\sin^2\lambda L - \cos^2\lambda L - e^{\lambda L}e^{-\lambda L} + e^{\lambda L}\cos\lambda L + + e^{-\lambda L}\cos\lambda L + e^{-\lambda L}\cos\lambda = 0$$

$$-(\sin^2\lambda L + \cos^2\lambda L) - e^{\lambda L - \lambda L} + e^{\lambda L}\cos\lambda L + e^{-\lambda L}\cos\lambda L = 0$$

$$-1 - e^0 + \cos\lambda L(e^{\lambda L} + e^{-\lambda L}) = 0 \rightarrow -1 - 1 + \cos\lambda L(2\cosh\lambda L) = 0$$

Finally

$$-2 + 2\cos\lambda L\cosh\lambda L = 0 \rightarrow \cos\lambda L\cosh\lambda L = 1$$