Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Final Exam
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Question 1

a. $f(x) = x$ is odd on $[-\pi, \pi]$ therefore its Fourier coefficients a_n are 0 and we need to find its b_n coefficients:

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{4}{2\pi} \int_{0}^{\pi} x \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) dx \end{aligned}$$

Using integration by parts:

$$\begin{aligned} \int_0^\pi x \sin(nx) dx &= [x(-\frac{\cos(nx)}{n})]_0^\pi + \int_0^\pi 1 \cdot \frac{\cos(nx)}{n} dx \\ &= (-\frac{\pi}{n}) \cos(n\pi) + \frac{1}{n} [\sin(nx)]_0^\pi \\ &= \frac{(-1)^{n+1}\pi}{n} \end{aligned}$$

Thus $b_n = \frac{2}{\pi} \frac{(-1)^{n+1}\pi}{n} = \frac{(-1)^{n+1}2}{n}$ and the Fourier series of x, on $[-\pi, \pi]$, is:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

b. If we integrate terms by terms the previous expression, the Fourier series of x over $[-\pi, \pi]$, we have:

$$\begin{aligned} \frac{x^2}{2} &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-\frac{\cos(nx)}{n}) + c \;\; c \text{constant of integration} \\ x^2 &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + c \text{ with } 2c \to c \\ &= c + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) \end{aligned}$$

c. $f(x) = x^2$ is an even function, by Fourier Series for even function over symmetric range, we have:

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos{(\frac{2\pi n x}{2\pi})} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \text{ (1)}$$

where

$$\begin{aligned} a_0 &= \frac{4}{2\pi}\int_0^{\pi} x^2 dx \\ &= \frac{2}{\pi}[\frac{x^3}{3}]_0^{\pi} \\ &= \frac{2}{3}\pi^2 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{4}{2\pi}\int_0^{\pi} x^2 \cos{(\frac{2\pi n x}{2\pi})} dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx \\ \int_0^{\pi} x^2\cos(nx)dx &= [x^2\ \frac{\sin(nx)}{n}]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin(nx)dx \\ &= 0 - \frac{2}{n}\frac{(-1)^{n+1}\pi}{n} \\ a_n &= \frac{2}{\pi}\frac{(-1)^n 2\ \pi}{n^2} \\ &= (-1)^n\frac{4}{n^2} \end{aligned}$$

Substituting for a_n in (1):

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

d. Fourier series of x^2 using integration terms by terms or calculating directly match, as required, by taking $c = \frac{\pi^2}{3}$ since x is a piecewise smooth function on the specified range.

Question 2

Consider the differential equation:

$$z\frac{d^2y}{dy^2} + y = 0$$

a. We put the equation in standard form:

$$\frac{d^2y}{dy^2} + \frac{1}{z}y = 0$$

$z\ p(z) = 0$ and $z^2 q(z) = z$ therefore 0 is a regular singular point.

Question 3

Question 4

Question 5

Question 6