

## Question 1

- a. Graph of the function attached in a separate pdf.
- b. Since we have made the function  $f(x)$  even using an even extension, all the  $b_k$  coefficients in its Fourier series are zero. With a period  $L = 4$ , we determine the remaining coefficients  $a_k$ :

$$a_k = \frac{2}{4} \int_{-2}^2 x \cos\left(\frac{2k\pi x}{4}\right) dx$$

And since  $f$  is even now

$$\begin{aligned} a_k &= \frac{4}{4} \int_0^2 x \cos\left(\frac{2k\pi x}{4}\right) dx \\ &= \int_0^2 x \cos\left(\frac{k\pi x}{2}\right) dx \end{aligned}$$

Using integration by parts, for  $k > 0$ :

$$\begin{aligned} a_k &= \frac{2}{k\pi} \left[ x \sin\left(\frac{k\pi x}{2}\right) \right]_0^2 - \frac{2}{k\pi} \int_0^2 \sin\left(\frac{k\pi x}{2}\right) dx \\ &= 0 - \frac{2}{k\pi} \left( -\frac{2}{k\pi} \right) \left[ \cos\left(\frac{k\pi x}{2}\right) \right]_0^2 \\ &= \frac{4}{(k\pi)^2} [\cos(k\pi) - \cos(0)] \\ &= \frac{4}{(k\pi)^2} [(-1)^k - 1] \end{aligned}$$

Then

$$a_k = \begin{cases} -\frac{8}{(k\pi)^2} & \text{for odd } k \\ 0 & \text{for even } k \end{cases}$$

And  $a_0 = \frac{2}{4} \int_{-2}^2 x dx = \frac{4}{4} \int_0^2 x dx = \frac{1}{2} [x^2]_0^2 = 2$ . With the coefficients  $a_k$  determined, we obtain the Fourier series for  $f(x)$ :

$$\begin{aligned} f(x) &= \frac{2}{2} - \sum_{k=1}^{\infty} \frac{8}{(k\pi)^2} \cos\left(\frac{2k\pi x}{4}\right) \quad k \text{ odd} \\ x &= 1 - \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos\left(\frac{(2k+1)\pi x}{2}\right) \end{aligned}$$

c. Applying Parseval's identity for Fourier series and using the result of part b.:

$$\begin{aligned}\frac{1}{4} \int_{-2}^2 x^2 dx &= \frac{2^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + 0) \quad k \text{ odd} \\ \frac{2}{4} \int_0^2 x^2 dx &= 1 + \frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{8}{(2k+1)^2 \pi^2} \right)^2 \\ \frac{1}{2} \left[ \frac{x^3}{3} \right]_0^2 &= 1 + \frac{1}{2} \cdot \frac{64}{\pi^4} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} \\ \frac{4}{3} - 1 &= \frac{32}{\pi^4} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} \\ \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} &= \frac{\pi^4}{32} \cdot \frac{1}{3}\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

## Question 2

a. Graph of the function attached in a separate pdf.

b.

$$f(t) = A \left[ H(t) - H(t - \tau) \right]$$

c.

$$\tilde{f}(w) = F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt} dt$$

Since  $f(t) = 0$  for  $t \geq 0$  or  $t \leq \tau$ :

$$\begin{aligned}\tilde{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_0^{\tau} A \cdot e^{-iwt} dt \\ &= \frac{A}{\sqrt{2\pi}} \left( \frac{1}{-iw} \right) [e^{-iwt}]_0^{\tau} \\ &= \frac{iA}{w\sqrt{2\pi}} (e^{-iw\tau} - 1) \\ &= \frac{iA}{w\sqrt{2\pi}} e^{-iw\frac{\tau}{2}} (e^{-iw\frac{\tau}{2}} - e^{iw\frac{\tau}{2}})\end{aligned}$$

From Euler identity:

$$e^{-iw\frac{\tau}{2}} - e^{iw\frac{\tau}{2}} = -2i \sin w\frac{\tau}{2}$$

Therefore

$$\begin{aligned}
 \tilde{f}(w) &= \frac{2A}{w\sqrt{2\pi}} e^{-iw\frac{\tau}{2}} \sin w\frac{\tau}{2} \\
 &= \sqrt{\frac{2}{\pi}} \frac{A}{w} e^{-iw\frac{\tau}{2}} \sin w\frac{\tau}{2} \\
 &= A \sqrt{\frac{2}{\pi}} e^{-iw\frac{\tau}{2}} \frac{\tau}{2} \frac{\sin(w\frac{\tau}{2})}{w\frac{\tau}{2}} \\
 &= \frac{A}{\sqrt{2\pi}} \tau e^{-iw\frac{\tau}{2}} \text{sinc}(w\frac{\tau}{2})
 \end{aligned}$$

d. Let  $A = \frac{1}{\tau}$  then substituting in  $f(t)$  from part c., gives:

$$F\{\lim_{\tau \rightarrow 0} f(t)\} = \lim_{\tau \rightarrow 0} F\{f(t)\} = \lim_{\tau \rightarrow 0} \frac{1}{\sqrt{2\pi}} e^{-iw\frac{\tau}{2}} \frac{\sin(w\frac{\tau}{2})}{w\frac{\tau}{2}}$$

$$\begin{aligned}
 \lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} &= 1 \text{ by Hospitals rule} \\
 \lim_{\tau \rightarrow 0} e^{-iw\frac{\tau}{2}} &= \lim_{\tau \rightarrow 0} e^0 = 1
 \end{aligned}$$

Therefore

$$F\{\lim_{\tau \rightarrow 0} f(t)\} = \frac{1}{\sqrt{2\pi}}$$

e. The Fourier transform of  $f(t)$  as  $\tau \rightarrow 0$  is the Fourier transform of a  $\delta$ -function as we can expect since as  $\tau \rightarrow 0$  the rectangular function  $f(t) \rightarrow$  a  $\delta$ -function.

### Question 3

a. By definition, the Laplace transform of  $g(t) = \sin(5t)$  is:

$$\bar{g}(s) = L\{g(t)\} = \int_0^{\infty} \sin(5t) e^{-st} dt = \lim_{L \rightarrow \infty} \int_0^L \sin(5t) e^{-st} dt$$

First compute  $\int e^{-st} \sin at dt$ , using integration by parts with  $u = \sin at$ ,  $u' = a \cos at$ ,  $v' = e^{-st}$ ,  $v = -\frac{1}{s} e^{-st}$ :

$$\int e^{-st} \sin(at) dt = -\frac{1}{s} e^{-st} \sin(at) + \frac{a}{s} \int e^{-st} \cos(at) dt \quad (1)$$

Next compute  $\int e^{-st} \cos(at) dt$ , again, using integration by parts with  $u = \cos at$ ,  $u' = -a \sin at$ ,  $v' = e^{-st}$ ,  $v = -\frac{1}{s} e^{-st}$ :

$$\int e^{-st} \cos(at) dt = -\frac{1}{s} e^{-st} \cos(at) - \frac{a}{s} \int e^{-st} \sin(at) dt$$

Substituting into (1):

$$\begin{aligned}\int e^{-st} \sin(at) dt &= -\frac{1}{s} e^{-st} \sin(at) + \frac{a}{s} \left( -\frac{1}{s} e^{-st} \cos(at) - \frac{a}{s} \int e^{-st} \sin(at) dt \right) \\ &= -\frac{1}{s} e^{-st} \sin(at) - \frac{a}{s^2} e^{-st} \cos(at) + \frac{a}{s^2} \int e^{-st} \sin(at) dt\end{aligned}$$

thus

$$\left(1 + \frac{a^2}{s^2}\right) \int e^{-st} \sin(at) dt = -e^{-st} \left(\frac{1}{s} \sin(at) + \frac{a}{s^2} \cos(at)\right)$$

Evaluating at  $t = 0$  and  $t \rightarrow \infty$ :

$$\begin{aligned}\left(1 + \frac{a^2}{s^2}\right) L\{\sin(at)\} &= \lim_{L \rightarrow \infty} \left[ -e^{-st} \left(\frac{1}{s} \sin(at) + \frac{a}{s^2} \cos(at)\right) \right]_0^L \\ &= 0 - \left(-1 \left(\frac{1}{s} \cdot 0 + \frac{a}{s^2} \cdot 1\right)\right) \\ &= \frac{a}{s^2}\end{aligned}$$

Therefore

$$\begin{aligned}L\{\sin(at)\} &= \frac{a}{s^2} \left(1 + \frac{a^2}{s^2}\right)^{-1} \\ &= \frac{a}{a^2 + s^2}\end{aligned}$$

Set  $a = 5$  and

$$L\{g(t)\} = L\{\sin(5t)\} = \frac{5}{s^2 + 25}$$

- b. One property of the Laplace transform is  $L[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}$  for  $n = 1, 2, 3, \dots$ , take  $n = 1$ ,  $L[tf(t)] = -\frac{d\bar{f}(s)}{ds}$ . Set  $f(t) = t \sin(5t)$  and from part b,  $L\{\sin(5t)\} = \frac{5}{s^2 + 25}$ , therefore:

$$\begin{aligned}L\{t \sin(5t)\} &= -\frac{d}{ds} \left( \frac{5}{s^2 + 25} \right) \\ &= -5 \frac{d}{ds} \left( \frac{1}{s^2 + 25} \right) \\ &= -5 \left( \frac{-2s}{(s^2 + 25)^2} \right) \\ &= \frac{10s}{(s^2 + 25)^2}\end{aligned}$$

- c. By definition  $(f * g)(t) = \int_0^t \tau e^{-(t-\tau)} d\tau = e^{-t} \int_0^t \tau e^{\tau} d\tau$ . Using integration by parts:

$$\begin{aligned}\int_0^t \tau e^{\tau} d\tau &= [\tau e^{\tau}]_0^t - \int_0^t e^{\tau} d\tau \\ &= te^t - [e^{\tau}]_0^t \\ &= te^t - (e^t - 1) \\ &= e^t(t - 1) + 1\end{aligned}$$

And

$$(f * g)(t) = e^{-t} \left[ e^{-t}(t-1) + 1 \right] = e^{-t} + t - 1$$

From  $L\{(f * g)(t)\} = \bar{f}(s) \cdot \bar{g}(s)$ , we have:

$$\begin{aligned} \bar{f}(s) \cdot \bar{g}(s) &= \frac{1}{s^2} \cdot \frac{1}{s+1} \\ &= \frac{1-s}{s^2+1} + \frac{1}{s+1} \\ &= \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \end{aligned}$$

Therefore

$$\begin{aligned} (f * g)(t) &= L^{-1}\{L\{(f * g)(t)\}\} = L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right\} \\ &= L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s+1}\right\} \\ &= t - 1 + e^{-t} \\ &= e^{-t} + t - 1 \end{aligned}$$

## Question 4

$$y'' + 4y' - 5y = \delta(t-1) \quad y(0) = 0 \quad y'(0) = 3$$

a. Taking the Laplace transform on both sides of the equation gives:

$$\begin{aligned} s^2 \tilde{y}(s) - sy(0) - y'(0) + 4[s\tilde{y}(s) - y(0)] - 5\tilde{y}(s) &= e^{-s} \\ s^2 \tilde{y}(s) - s \cdot 0 - 3 + 4[s\tilde{y}(s) - 0] - 5\tilde{y}(s) &= e^{-s} \\ s^2 \tilde{y}(s) + 4s\tilde{y}(s) - 5\tilde{y}(s) &= e^{-s} + 3 \end{aligned}$$

b. Combining the terms:  $(s^2 + 4s - 5)\tilde{y}(s) = 3 + e^{-s}$ . Therefore

$$\tilde{y}(s) = \frac{3 + e^{-s}}{s^2 + 4s - 5}$$

The roots of  $s^2 + 4s - 5 = 0$  are  $-5$  and  $1$ , so we can rewrite  $\tilde{y}(s)$  as  $\tilde{y}(s) = \frac{3}{(s-1)(s+5)} + \frac{e^{-s}}{(s-1)(s+5)}$   
Computing the fraction expansion:

$$\begin{aligned} \frac{1}{(s-1)(s+5)} &= \frac{A}{s-1} + \frac{B}{s+5} \\ &= \frac{(A+B)s + 5A - B}{(s-1)(s+5)} \end{aligned}$$

Equating the powers of  $s$  on each side of the previous equation:

$$\begin{aligned} s^1 : A + B &= 0 \\ s^0 : 5A - B &= 1 \end{aligned}$$

gives  $A = \frac{1}{6}$  and  $B = -\frac{1}{6}$ . Thus

$$\frac{1}{(s-1)(s+5)} = \frac{1}{6} \left( \frac{1}{s-1} - \frac{1}{s+5} \right)$$

So

$$\begin{aligned} \tilde{y}(s) &= 3 \left[ \frac{1}{6} \left( \frac{1}{s-1} - \frac{1}{s+5} \right) \right] + \frac{1}{6} \left( \frac{e^{-s}}{s-1} - \frac{e^{-s}}{s+5} \right) \\ &= \frac{1}{2} \left( \frac{1}{s-1} - \frac{1}{s+5} \right) + \frac{1}{6} \left( \frac{e^{-s}}{s-1} - \frac{e^{-s}}{s+5} \right) \end{aligned}$$

c.  $y(t) = L^{-1}\{\tilde{y}(s)\}$  and from part b:

$$y(t) = \frac{1}{2} \left[ L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s+5}\right\} \right] + \frac{1}{6} \left[ L^{-1}\left\{\frac{e^{-s}}{s-1}\right\} - L^{-1}\left\{\frac{e^{-s}}{s+5}\right\} \right]$$

$L^{-1}\left\{\frac{1}{s-1}\right\} = e^t$ ,  $L^{-1}\left\{\frac{1}{s+5}\right\} = e^{-5t}$ , and using the shift theorem:

$$L\{f(t-t_0)H(t-t_0)\} = e^{-st_0}F(s) \quad f(t-t_0)H(t-t_0) = L^{-1}\{e^{-st_0}F(s)\}$$

So for  $t_0 = 1$

$$L^{-1}\left\{\frac{e^{-s}}{s-1}\right\} = e^{(t-1)}H(t-1)$$

$$L^{-1}\left\{\frac{e^{-s}}{s+5}\right\} = e^{-5(t-1)}H(t-1)$$

Plugging back these into  $y(t)$  yields:

$$\begin{aligned} y(t) &= \frac{1}{2} \left[ e^t - e^{-5t} \right] + \frac{1}{6} \left[ e^{(t-1)}H(t-1) - e^{-5(t-1)}H(t-1) \right] \\ &= \frac{1}{2} \left[ e^t - e^{-5t} \right] + \frac{1}{6} \left( e^{(t-1)} - e^{-5(t-1)} \right) H(t-1) \\ &= \frac{e^t}{2} \left( 1 + \frac{1}{3e} H(t-1) \right) - \frac{e^{-5t}}{2} \left( 1 + \frac{1}{3} e^5 H(t-1) \right) \\ &= \frac{1}{2} (e^t - e^{-5t}) + \frac{1}{6} (e^{t-1} - e^{-5(t-1)}) H(t-1) \end{aligned}$$

## Question 5

- a. Let rate  $r = 10 \text{ min}^{-1}$ , the rate of change of A is equal to how much of A goes to B at rate  $r$ , how much of A goes to C at rate  $r$ , and how much from B goes to A at rate  $r$ , and how much from C goes to A at rate  $r$ . The transport dynamics are the same for B and C, so the system looks like:

$$\frac{dA}{dt} = -rA - rA + rB + rC$$

$$\frac{dA}{dt} = -2rA + rB + rC$$

$$\frac{dA}{dt} = -20A + 10B + 10C \quad \text{with } A(0) = 20$$

Similarly

$$\begin{aligned}\frac{dB}{dt} &= rA - rB - rB + rC \\ \frac{dB}{dt} &= rA - 2rB + rC \\ \frac{dB}{dt} &= 10A - 20B + 10C \text{ with } B(0) = 0\end{aligned}$$

And

$$\begin{aligned}\frac{dC}{dt} &= rA + rB - rC - rC \\ \frac{dC}{dt} &= rA + rB - 2rC \\ \frac{dC}{dt} &= 10A + 10B - 20C \text{ with } C(0) = 0\end{aligned}$$

b. We take the Laplace transforms of the differential equations which gives:

$$\begin{aligned}s\tilde{A}(s) - A(0) &= -20\tilde{A}(s) + 10\tilde{B}(s) + 10\tilde{C}(s) \\ (s + 20)\tilde{A}(s) - 10\tilde{B}(s) - 10\tilde{C}(s) &= 20 \\ s\tilde{B}(s) - B(0) &= 10\tilde{A}(s) - 20\tilde{B}(s) + 10\tilde{C}(s) \\ 10\tilde{A}(s) - (s + 20)\tilde{B} + 10\tilde{C}(s) &= 0 \\ s\tilde{C}(s) - C(0) &= 10\tilde{A}(s) + 10\tilde{B}(s) - 20\tilde{C}(s) \\ 10\tilde{A}(s) + 10\tilde{B}(s) - (s + 20)\tilde{C} &= 0\end{aligned}$$

c. We write the equations in matrix form:

$$\begin{bmatrix} s + 20 & -10 & -10 \\ 10 & -(s + 20) & 10 \\ 10 & 10 & -(s + 20) \end{bmatrix} \begin{bmatrix} \tilde{A}(s) \\ \tilde{B}(s) \\ \tilde{C}(s) \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the system is:

$$\begin{aligned}D &= \begin{vmatrix} s + 20 & -10 & -10 \\ 10 & -(s + 20) & 10 \\ 10 & 10 & -(s + 20) \end{vmatrix} = s^3 + 60s^2 + 900s \\ &= s(s^2 + 60s + 900) \\ &= s(s + 30)^2\end{aligned}$$

Solving using Cramer's rule gives:

$$\begin{aligned}
\tilde{A}(s) &= \frac{\begin{vmatrix} 20 & -10 & -10 \\ 0 & -(s+20) & 10 \\ 0 & 10 & -(s+20) \end{vmatrix}}{D} \\
&= 20 \frac{\begin{vmatrix} -(s+20) & 10 \\ 10 & -(s+20) \end{vmatrix}}{D} \\
&= \frac{20((s+20)^2 - 100)}{s(s+30)^2} \\
&= \frac{20(s^2 + 40s + 300)}{s(s+30)^2} \\
&= \frac{20(s+10)(s+30)}{s(s+30)^2} \\
&= \frac{20(s+10)}{s(s+30)} \\
&= \frac{20}{3} \left[ \frac{1}{s} + \frac{2}{s+30} \right]
\end{aligned}$$

Taking the inverse Laplace transform using the table:

$$A(t) = \frac{20}{3}(1 + 2e^{-30t})$$

$$\begin{aligned}
\tilde{B}(s) &= \frac{\begin{vmatrix} s+20 & 20 & -10 \\ 10 & 0 & 10 \\ 10 & 0 & -(s+20) \end{vmatrix}}{D} \\
&= \frac{200s + 6000}{s(s+30)^2} \\
&= \frac{200(s+30)}{s(s+30)^2} \\
&= \frac{200}{s(s+30)} \\
&= \frac{200}{30} \left( \frac{1}{s} - \frac{1}{s+30} \right)
\end{aligned}$$

Taking the inverse Laplace transform using the table:

$$\begin{aligned}
B(t) &= \frac{200}{30}(1 - e^{-30t}) \\
&= \frac{20}{3}(1 - e^{-30t})
\end{aligned}$$



$$\begin{aligned}
\tilde{C}(s) &= \frac{\begin{vmatrix} s+20 & -10 & 20 \\ 10 & -(s+20) & 0 \\ 10 & 10 & 0 \end{vmatrix}}{D} \\
&= \frac{200s + 6000}{s(s+30)^2} \\
&= \frac{200(s+30)}{s(s+30)^2} \\
&= \frac{200}{s(s+30)} \\
&= \frac{200}{30} \left( \frac{1}{s} - \frac{1}{s+30} \right)
\end{aligned}$$

Taking the inverse Laplace transform using the table:

$$C(t) = \frac{20}{3}(1 - e^{-30t}) = B(t)$$

d. From part c:

$$A(t) = \frac{20}{3}(1 + 2e^{-30t})$$

$$B(t) = C(t) = \frac{20}{3}(1 - e^{-30t})$$

e.  $\lim_{t \rightarrow \infty} e^{-30t} = 0$  therefore as  $t \rightarrow \infty$ ,  $\lim_{t \rightarrow \infty} A(t) = \lim_{t \rightarrow \infty} B(t) = \lim_{t \rightarrow \infty} C(t) = \frac{20}{3}$  which is the equilibrium state of this system when  $t$  goes to infinity. The water has spread out uniformly among the three reservoirs.

## Question 6

$$L_0(x) = 1 \text{ and } L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) \quad n = 1, 2, \dots$$

Applying the recurrence relationship

$$n = 1$$

$$L_1(x) = \frac{e^x}{1!} \frac{d}{dx}(xe^{-x})$$

$$\frac{d}{dx}(xe^{-x}) = e^{-x} + x(-1)e^{-x} = e^{-x}(1 - x)$$

$$L_1(x) = e^x e^{-x}(1 - x)$$

$$L_1(x) = 1 - x$$

$$n = 2$$

$$L_2(x) = \frac{e^x}{2!} \frac{d^2}{dx^2}(x^2 e^{-x})$$

$$\begin{aligned} \frac{d}{dx}(x^2 e^{-x}) &= \frac{d}{dx}(x(xe^{-x})) \\ &= 1(xe^{-x}) + xe^{-x}(1 - x) \\ &= xe^{-x}(2 - x) \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dx^2}(x^2 e^{-x}) &= \frac{d}{dx}(xe^{-x})(2 - x) + xe^{-x} \frac{d}{dx}(2 - x) \\ &= e^{-x}(1 - x)(2 - x) + xe^{-x}(-1) \\ &= e^{-x}[(1 - x)(2 - x) - x] \\ &= e^{-x}(x^2 - 4x + 2) \end{aligned}$$

$$\begin{aligned} L_2(x) &= \frac{e^x}{2!} e^{-x}(x^2 - 4x + 2) \\ &= 1 - 2x + \frac{x^2}{2} \end{aligned}$$

$$n = 3$$

$$\begin{aligned}
L_3(x) &= \frac{e^x}{3!} \frac{d^3}{dx^3} (x^3 e^{-x}) \\
\frac{d}{dx} (x^3 e^{-x}) &= \frac{d}{dx} (x(x^2 e^{-x})) \\
&= 1(x^2 e^{-x}) + x x e^{-x} (2 - x) \\
&= x^2 e^{-x} (3 - x) \\
\frac{d^2}{dx^2} (x^3 e^{-x}) &= \frac{d}{dx} \left( \frac{d}{dx} (x^3 e^{-x}) \right) \\
&= \frac{d}{dx} (x^2 e^{-x} (3 - x)) \\
&= \frac{d}{dx} (x^2 e^{-x}) (3 - x) + (x^2 e^{-x}) \frac{d}{dx} (3 - x) \\
&= x e^{-x} (2 - x) (3 - x) + (x^2 e^{-x}) (-1) \\
&= x e^{-x} [(2 - x)(3 - x) - x] \\
&= x e^{-x} (x^2 - 6x + 6) \\
\frac{d^3}{dx^3} (x^3 e^{-x}) &= \frac{d}{dx} \left( \frac{d^2}{dx^2} (x^3 e^{-x}) \right) \\
&= \frac{d}{dx} (x e^{-x} (x^2 - 6x + 6)) \\
&= \frac{d}{dx} (x e^{-x}) (x^2 - 6x + 6) + x e^{-x} \frac{d}{dx} (x^2 - 6x + 6) \\
&= e^{-x} (1 - x) (x^2 - 6x + 6) + x e^{-x} (2x - 6) \\
&= e^{-x} (x^2 - 6x + 6 - x^3 + 6x^2 - 6x + 2x^2 - 6x) \\
&= e^{-x} (-x^3 + 9x^2 - 18x + 6) \\
L_3(x) &= \frac{e^x}{6} e^{-x} (-x^3 + 9x^2 - 18x + 6) \\
&= 1 - 3x + \frac{3x^2}{2} - \frac{x^3}{6}
\end{aligned}$$

Show that the Laguerre polynomials are orthogonal on the positive axis ( $0 \leq x < \infty$ ) w.r.t. the weight function  $e^{-x}$ . We want to show

$$\int_0^\infty L_n(x) L_k(x) e^{-x} dx = 0 \text{ if } n \neq k$$

From the expression of the Laguerre polynomial,  $L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x})$ , a Laguerre polynomial  $L_n(x)$  is a polynomial of degree  $n$ . Without loss of generality, let assume in the previous equation that  $k < n$ . By multiplying two polynomials of degree  $M$  and  $N$ , the result is a polynomial of degree  $M + N$ . Therefore to prove orthogonality, suffices to prove the following equation:

$$\int_0^\infty e^{-x} x^k L_n(x) dx = 0 \text{ for all } k < n$$

First we show that

$$\int_0^\infty \frac{d^n}{dx^n} (x^m e^{-x}) = 0 \text{ for } n < m$$

We are applying  $n$ th derivative rules:

$$\begin{aligned}\frac{d^n}{dx^n}(x^m e^{-x}) &= \sum_{r=0}^n C_r^n \frac{d^r}{dx^r} x^n \frac{d^{n-r}}{dx^{n-r}} e^{-x} \\ &= \sum_{r=0}^n \frac{n!}{r!(n-r)!} \frac{n!}{(n-r)!} x^{n-r} (-1)^{n-r} e^{-x} \\ &= \sum_{r=0}^n (-1)^{n-r} \frac{(n!)^2}{r!((n-r)!)^2} x^{n-r} e^{-x}\end{aligned}$$

reindexing with  $l = n - r$ , we obtain

$$\frac{d^n}{dx^n}(x^m e^{-x}) = \sum_{l=0}^m (-1)^l \frac{(n!)^2}{(l!)^2(n-l)!} x^l e^{-x}$$

Next

$$\int_0^\infty \frac{d^n}{dx^n}(x^m e^{-x}) = \sum_{l=0}^m (-1)^l \frac{(n!)^2}{(l!)^2(n-l)!} \int_0^\infty x^l e^{-x} dx$$

Integration by parts 1 times gives:

$$\int_0^\infty x^l e^{-x} dx = (-1)^l l! \int_0^\infty e^{-x} dx = (-1)^l l! [-e^{-x}]_0^\infty = 0$$

Going back to our initial equation of interest:

$$\begin{aligned}\int_0^\infty e^{-x} x^k L_n(x) dx &= \int_0^\infty x^k \frac{d^n}{dx^n}(x^n e^{-x}) dx \\ &= [x^k \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})]_0^\infty - k \int_0^\infty x^{k-1} \frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x}) dx\end{aligned}$$

The first term is 0 as it is a sum of terms  $[x^{l+k} e^{-x}]_0^\infty = 0$  and after applying integration by parts  $k$  times, we are left with

$$\int_0^\infty e^{-x} x^k L_n(x) dx = (-1)^k k! \int_0^\infty \frac{d^{n-k}}{dx^{n-k}}(x^n e^{-x}) dx = 0$$

## Question 7

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x, y(e) = 0, y'(e) = 2$$

- a. This is Euler differential equation, and we make the change of variable  $x = e^t$  or  $t = \ln(x)$ . Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \frac{d \ln x}{dx} = \frac{dy}{dt} \frac{1}{x} = \frac{1}{x} \frac{dy}{dt} \\ x \frac{dy}{dx} &= \frac{dy}{dt}\end{aligned}$$

And since this is a Legendre ODE with  $\alpha = 1$  and  $\beta = 0$ , we can use the expression for the second derivative  $(\alpha x + \beta)^2 \frac{d^2 y}{dx^2} = \alpha^2 \frac{d}{dt} \left[ \frac{d}{dt} - 1 \right] y$ . With  $\alpha = 1$  and  $\beta = 0$ , we have:  $x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$ .

Substitute into the above equation yields:

$$\left( \frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + \frac{dy}{dt} - y = e^t$$

$$\frac{d^2 y}{dt^2} - y = e^t$$

b. The homogeneous equation is

$$\frac{d^2 y}{dt^2} - y = 0$$

Assume a solution of the form  $y(t) = Ae^{\lambda t}$  gives the characteristic equation  $\lambda^2 - 1 = 0$  which has for roots  $\lambda = \pm 1$  and gives for solution  $y(t) = c_1 e^t + c_2 e^{-t}$ .

c. The ODE to solve is:

$$\frac{d^2 y}{dt^2} - y = 0$$

It is in standard form and it is defined at any point  $t$ , it is analytic, thus we take as solution  $y(t) = \sum_{n=0}^{\infty} a_n t^n$ . So:

$$y'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1}$$

$$y''(t) = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

by reindexing

$$y''(t) = \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} t^n$$

$$y''(t) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$$

Substitute into the ODE gives:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] t^n = 0$$

or

$$a_{n+2} = \frac{1}{(n+2)(n+1)} a_n$$

$$a_n = \frac{1}{n(n-1)} a_{n-2}$$

Take  $a_0 = a_1 = 1$  and we generate the coefficients:

$$\begin{aligned}
 & \cdot n = 2 \text{ then } a_2 = \frac{1}{2 \cdot 1} a_0 = \frac{1}{2 \cdot 1} = \frac{1}{2!} \\
 & \cdot n = 3 \text{ then } a_3 = \frac{1}{3 \cdot 2} a_1 = \frac{1}{3 \cdot 2} = \frac{1}{3!} \\
 & \cdot n = 4 \text{ then } a_4 = \frac{1}{4 \cdot 3} a_2 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{4!} \\
 & \vdots \\
 & \cdot a_n = \frac{1}{n(n-1)} a_{n-2} = \dots = \frac{1}{n!}
 \end{aligned}$$

The first solution we obtain is:  $y_1(t) = \sum_{t=0}^{\infty} a_n t^n = \sum_{t=0}^{\infty} \frac{t^n}{n!} = e^t$ . Secondly, if we set  $a_0 = 1$  and choose  $a_1 = -1$ , then we obtain a second independent solution:

$$\begin{aligned}
 & \cdot n = 2 \text{ then } a_2 = \frac{1}{2 \cdot 1} a_0 = \frac{1}{2 \cdot 1} = \frac{1}{2!} \\
 & \cdot n = 3 \text{ then } a_3 = \frac{1}{3 \cdot 2} a_1 = -\frac{1}{3 \cdot 2} = \frac{-1}{3!} \\
 & \cdot n = 4 \text{ then } a_4 = \frac{1}{4 \cdot 3} a_2 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{4!} \\
 & \cdot n = 5 \text{ then } a_5 = \frac{1}{5 \cdot 4} a_3 = \frac{-1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{-1}{5!} \\
 & \vdots \\
 & \cdot a_n = \frac{1}{n(n-1)} a_{n-2} = \dots = \frac{(-1)^n}{n!}
 \end{aligned}$$

We have the second solution:  $y_2(t) = \sum_{t=0}^{\infty} a_n t^n = \sum_{t=0}^{\infty} \frac{(-t)^n}{n!}$ , recognizing the last series as  $e^{-t}$ , we can write the general solution of the homogeneous equation as

$$y_H(t) = c_1 e^t + c_2 e^{-t}$$

which is the solution we found in question b.

d. The differential equation to solve is

$$\frac{d^2 y}{dt^2} - y = e^t$$

Next we use the variation of parameters method, we are looking for a solution  $y_p(t) = k_1(t)e^t + k_2(t)e^{-t}$ . We solve for derivatives of  $k$ 's a system of two equations:

$$\begin{cases} k_1' e^t + k_2' e^{-t} = 0 \\ k_1' e^t - k_2' e^{-t} = e^t \end{cases}$$

Multiplying through by  $e^t$  gives:

$$\begin{cases} k_1' e^{2t} + k_2' = 0 \\ k_1' e^{2t} - k_2' = e^{2t} \end{cases}$$

Adding first equation to second yields  $2k'_1 e^{2t} = e^{2t}$  or  $k'_1 = \frac{1}{2}$  and  $k_1 = \frac{t}{2}$ . Substitute

$$\begin{aligned} k'_2 &= -k'_1 e^{2t} \\ &= -\frac{1}{2} e^{2t} \end{aligned}$$

integrating

$$k_2 = -\frac{e^{2t}}{4}$$

Therefore:

$$\begin{aligned} y_p(t) &= k_1(t)e^t + k_2(t)e^{-t} \\ &= \frac{t}{2}e^t - \frac{e^{2t}}{4}e^{-t} \\ &= \frac{t}{2}e^t - \frac{e^t}{4} \\ &= \frac{e^t}{2}\left(t - \frac{1}{2}\right) \end{aligned}$$

- e. The general solution is:  $y(t) = y_H(t) + y_p(t) = c_1 e^t + c_2 e^{-t} + \frac{e^t}{2}(t - \frac{1}{2})$ , simplifying the constants, we can rewrite the general solution as  $y(t) = c_1 e^t + c_2 e^{-t} + \frac{t}{2}e^t$ . Plugging back  $x = e^t$  or  $t = \ln(x)$  gives

$$y(x) = c_1 x + \frac{c_2}{x} + \frac{x \ln x}{2}$$

- f. The total solution is

$$\begin{aligned} y(x) &= c_1 x + \frac{c_2}{x} + \frac{x \ln x}{2} \\ y'(x) &= c_1 - \frac{c_2}{x^2} + \frac{1}{2}(1 + \ln x) \end{aligned}$$

And the initial conditions are  $y(e) = 0, y'(e) = 2$ , plugging back these into the previous equations gives

$$\begin{aligned} &\begin{cases} y(e) = c_1 e + \frac{c_2}{e} + \frac{e \ln e}{2} = 0 \\ y'(e) = c_1 - \frac{c_2}{e^2} + \frac{1}{2}(1 + \ln e) = 2 \end{cases} \\ &\Rightarrow \begin{cases} c_1 e + c_2 e^{-1} = -\frac{e}{2} \\ c_1 - c_2 e^{-2} = 1 \end{cases} \\ &\Rightarrow \begin{cases} c_1 e^2 + c_2 = -\frac{e^2}{2} \\ c_1 e^2 - c_2 = e^2 \end{cases} \end{aligned}$$

Adding the first equation to the second equation leads to  $2e^2 c_1 = e^2 - \frac{e^2}{2} = \frac{e^2}{2}$ ,  $c_1 = \frac{1}{4}$ ,  $c_2 = e^2(c_1 - 1) = \frac{3}{4}e^2$ . Reporting these constants into the expression of the total solution gives:

$$\begin{aligned} y(x) &= \frac{1}{4}x - \frac{3}{4}e^2 \frac{1}{x} + \frac{x \ln x}{2} \\ y(x) &= \frac{x^2 + 2x^2 \ln(x) - 3e^2}{4x} \end{aligned}$$