8.6

(a)

Follow technique example page 302

$$\text{Define}\langle g(x)|f(x)\rangle = \int_0^\infty f(x)g(x)\rho(x)dx = \int_0^\infty f(x)g(x)e^{-x}dx$$

$$\phi_0 = 1 \text{ and obviously } \hat{\phi}_0 = 1 \text{ since } |\phi_0| = 1$$

$$\phi_1 = x - \hat{\phi}_0\langle\hat{\phi}_0|x\rangle = x - 1\langle 1|x\rangle = x - \int_0^\infty 1\cdot xe^{-x}dx = x - \int_0^\infty xe^{-x}dx$$

$$\int_0^\infty xe^{-x}dx = (\text{by Table}) = \frac{e^{-x}}{(-1)^2}[-x-1]\Bigg|_0^\infty = -\lim_{x\to\infty} e^{-x}(x+1) - [-e^{-x}(x+1)]_{x=0} =$$

$$-\lim_{x\to\infty}\frac{x+1}{e^x} + [e^{-0}(0+1)] = -\lim_{x\to\infty}\frac{x+1}{e^x} + [1(0+1)] = -\lim_{x\to\infty}\frac{x+1}{e^x} + 1$$

$$\text{We are left with } \lim_{x\to\infty}\frac{x+1}{e^x} = (\text{use L'Hospital's rule}) = \lim_{x\to\infty}\frac{1}{e^x} = 0$$

$$\text{Alternatively } \int_0^\infty xe^{-x}dx = \Gamma(2) = 1! = 1$$

$$\text{Therefore either way } \int_0^\infty xe^{-x}dx = 1 \text{ and } \phi_1 = x - 1$$

$$|\phi_1| = \langle\phi_1|\phi_1\rangle^{1/2} = \int_0^\infty (x-1)(x-1)e^{-x}dx = \int_0^\infty x^2e^{-x} - 2xe^{-x} + e^{-x}dx =$$

$$\int_0^\infty x^2e^{-x}dx - 2\int_0^\infty xe^{-x}dx + \int_0^\infty e^{-x}dx$$

First integral is $\Gamma(3) = 2! = 2$; second integral is $\Gamma(2) = 1! = 1$

Last integral is easily done and =1

$$\text{Therefore } \int_0^\infty x^2e^{-x}dx - 2\int_0^\infty xe^{-x}dx + \int_0^\infty e^{-x}d = 2 - 2(1) + 1 = 1$$

$$\text{and } \hat{\phi}_1 = \frac{\phi_1}{|\phi_1|} = \frac{x-1}{1} = x - 1$$

$$\text{For(more of the same) } \phi_2 = x^2\text{-}4x\text{+}2,\ |\phi_2| = \langle\phi_2|\phi_2\rangle^{1/2} = 2 \text{ therefore } \hat{\phi}_2 = \frac{1}{2}(x^2\text{-}4x\text{+}2)$$

(b) Generate the Laguerre polynomials, see Page 362-365! They differ from our polynomials by an alternating negative sign

that is $(-1)^n$ and $\phi_n(x) = (-1)^n L_n(x)$.