Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Final Exam
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Question 1

a. $f(x) = x$ is odd on $[-\pi, \pi]$ therefore its Fourier coefficients a_n are 0 and we need to find its b_n coefficients:

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{4}{2\pi} \int_{0}^{\pi} x \sin(\frac{2\pi n x}{2\pi}) dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) dx \end{aligned}$$

Using integration by parts:

$$\begin{aligned} \int_0^{\pi} x \sin(nx) dx &= [x(-\frac{\cos(nx)}{n})]_0^{\pi} + \int_0^{\pi} 1 \cdot \frac{\cos(nx)}{n} dx \\ &= (-\frac{\pi}{n}) \cos(n\pi) + \frac{1}{n} [\sin(nx)]_0^{\pi} \\ &= \frac{(-1)^{n+1}\pi}{n} \end{aligned}$$

Thus $b_n = \frac{2}{\pi} \frac{(-1)^{n+1}\pi}{n} = \frac{(-1)^{n+1}2}{n}$ and the Fourier series of x, on $[-\pi, \pi]$, is:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

b. If we integrate terms by terms the previous expression, the Fourier series of x over $[-\pi, \pi]$, we have:

$$\begin{aligned} \frac{x^2}{2} &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-\frac{\cos(nx)}{n}) + c \quad c \text{constant of integration} \\ x^2 &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + c \text{ with } 2c \to c \\ &= c + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) \end{aligned}$$

c. $f(x) = x^2$ is an even function, by Fourier Series for even function over symmetric range, we have:

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n x}{2\pi}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \text{ (1)}$$

where

$$
\begin{aligned}
a_0 &= \frac{4}{2\pi}\int_0^{\pi} x^2 dx \\
&= \frac{2}{\pi}[\frac{x^3}{3}]_0^{\pi} \\
&= \frac{2}{3}\pi^2
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{4}{2\pi}\int_0^{\pi} x^2 \cos(\frac{2\pi n x}{2\pi})dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx \\
\int_0^{\pi} x^2\cos(nx)dx &= [x^2\ \frac{\sin(nx)}{n}]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin(nx)dx \\
&= 0 - \frac{2}{n}\frac{(-1)^{n+1}\pi}{n} \\
a_n &= \frac{2}{\pi}\frac{(-1)^n 2\,\pi}{n^2} \\
&= (-1)^n\frac{4}{n^2}
\end{aligned}
$$

Substituting for a_n in (1):

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

d. Fourier series of x^2 using integration terms by terms or calculating directly match, as required, by taking $c = \frac{\pi^2}{3}$ since x is a piecewise smooth function on the specified range.

Question 2

Consider the differential equation:

$$z\frac{d^2y}{dy^2} + y = 0$$

a. We put the equation in standard form:

$$\frac{d^2y}{dy^2} + \frac{1}{z}y = 0$$

$z\,p(z) = 0$ and $z^2q(z) = z$ therefore 0 is a regular singular point.

b. Take $y = z^{\sigma}\sum_{n=0}^{\infty} a_n z^n$ and the usual derivatives in the D.E. gives by substitution

$$
\begin{aligned}
z\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} &= 0 \\
\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} &= 0\ (1)
\end{aligned}
$$

Take the term with the lowest power of z, which is the first sum with $n = 0$, then since each power of z term must be equal to 0, we have

$$\sigma(\sigma - 1)a_0 z^{\sigma-1} = 0$$

Since $a_0 \neq 0$ and $z^{\sigma-1} \neq 0$, therefore $\sigma = 0, 1$.

c. We go back to equation (1) and take $\sigma = 1$ yields

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

Then reindex the second sum to get same power of z in both sums:

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=1}^{\infty} a_{n-1} z^n = 0$$

Note, in first term $n = 0$ does not contribute so we can start index at $n = 1$ in the first sum, and combine both sums

$$\sum_{n=1}^{\infty} [n(n+1)a_n + a_{n-1}] z^n = 0$$

Since every power of z term must be 0 and $z^n \neq 0$, gives:

$$a_n = -\frac{1}{(n+1)n} a_{n-1}$$

Taking $a_0 = 1$, now

$$\begin{aligned}
n = 1\ a_1 &= -\frac{1}{2\,1} a_0 = -\frac{1}{2\,1} = \frac{(-1)^1}{2\,1} \\
n = 2\ a_2 &= -\frac{1}{3\,2} a_1 = \frac{1}{3\,2\,2\,1} = \frac{(-1)^2}{(3\,2\,1)\,(2\,1)} \\
n = 3\ a_3 &= -\frac{1}{4\,3} a_2 = -\frac{1}{4\,3\,3\,2\,2\,1} = \frac{(-1)^3}{(4\,3\,2\,1)\,(3\,2\,1)} \\
&\vdots \\
a_n &= -\frac{1}{(n+1)n} a_{n-1} = \cdots = \frac{(-1)^n}{((n+1)\,n \cdots 1)\,(n\,(n-1) \cdots 1)} = \frac{(-1)^n}{(n+1)!n!}
\end{aligned}$$

Therefore one of the independent solution of the ODE is

$$y_1(z) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!n!} z^n$$

Question 3

a. We have

$$n=0,\ M=0,\ P_0(x)=\frac{(-1)^0(2\ 0-2\ 0)!}{2^0(0-0)!(0-2\ 0)!}x^{0-2\ 0}=1$$
$$n=1,\ M=\frac{1-1}{2}=0,\ P_1(x)=\frac{(-1)^0(2\ 1-2\ 0)!}{2^1(1-0)!(1-2\ 0)!}x^{1-2\ 0}=\frac{1\ 2}{2\ 1!\ 1!}x^1=x$$
$$n=2,\ M=\frac{2}{2}=1,\ P_2(x)=\frac{(-1)^0(2\ 2-2\ 0)!}{2^2(2-0)!(2-2\ 0)!}x^{2-2\ 0}+\frac{(-1)^1(2\ 2-2\ 1)!}{2^2(2-1)!(2-2\ 1)!}x^{2-2\ 1}$$
$$P_2(x)=\frac{4!}{2^2\ 2!\ 2!}x^2-\frac{(2\ 2-2)!}{2^2\ 1!\ 0!}x^0$$
$$P_2(x)=\frac{4\ 3\ 2\ 1}{4\ 2\ 2}x^2-\frac{2!}{4}$$
$$P_2(x)=\frac{3}{2}x^2-\frac{1}{2}=\frac{1}{2}(3x^2-1)$$

b. From

$$a_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx=\frac{2n+1}{2}\int_{-1}^{1}xP_n(x)dx$$

we have

$$n=0,\ a_0=\frac{2\ 0+1}{2}\int_{-1}^{1}xP_0(x)dx$$
$$=\frac{1}{2}\int_{-1}^{1}xdx=\frac{1}{2}[\frac{x^2}{2}]_{-1}^{1}=\frac{1}{4}[1^2-(-1)^2]=0$$
$$n=1,\ a_1=\frac{2\ 1+1}{2}\int_{-1}^{1}xP_1(x)dx$$
$$=\frac{3}{2}\int_{-1}^{1}x^2dx=\frac{3}{2}[\frac{x^3}{3}]_{-1}^{1}=\frac{1}{2}[1^3-(-1)^3]=\frac{1}{2}\ 2=1$$
$$n=2,\ a_2=\frac{2\ 2+1}{2}\int_{-1}^{1}xP_2(x)dx$$
$$=\frac{5}{2}\int_{-1}^{1}x[\frac{1}{2}(3x^2-1)]dx=\frac{5}{4}\int_{-1}^{1}(3\ x^3-x)dx$$
$$=0\ \text{ since the powers of } x \text{ in the integrand are odd}$$

Therefore the Fourier-Legendre series of x iss $x=1\cdot P_1(x)$ as required.

c. Using Rodrigues's formula

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

we have

$$
\begin{aligned}
n=0,\ & \frac{d^0}{dx^0}[(x^2-1)^0] = (x^2-1)^0 = 1 \\
& P_0(x) = \frac{1}{2^0\ 0!}\ 1 = 1 \\
n=1,\ & \frac{d}{dx}(x^2-1) = 2x \\
& P_1(x) = \frac{1}{2^1\ 1!}\ 2x = x \\
n=2,\ & \frac{d^2}{dx^2}(x^2-1)^2 = \frac{d}{dx}\Big[\frac{d}{dx}(x^2-1)^2\Big] = \frac{d}{dx}\Big[4x(x^2-1)\Big] = \frac{d}{dx}\Big[4x^3-4x\Big] = 12\,x^2-4 \\
& P_2(x) = \frac{1}{2^2\ 2!}\ (12\,x^2-4) = \frac{4}{4\ 2}(3x^2-1) = \frac{1}{2}(3x^2-1)
\end{aligned}
$$

Question 4

a.

$$\frac{\partial u}{\partial x} + 4xu = 0$$

Integration factor is

$$e^{\int 4xdx} = e^{4\int xdx} = e^{4\frac{x^2}{2}} = e^{2x^2}$$

Multiply the partial differential equation by the I.F.:

$$
\begin{aligned}
e^{2x^2}\frac{\partial u}{\partial x} + 4xe^{2x^2}u &= 0 \\
\frac{\partial}{\partial x}(e^{2x^2}u) &= 0
\end{aligned}
$$

Now integrate both sides with respect to x

$$
\begin{aligned}
e^{2x^2}u &= C\ C\text{:constant} \\
u(x) &= Ce^{-2x^2}
\end{aligned}
$$

b.

$$y^2u_x - x^2u_y = 0$$

Let $u(x,y) = X(x)Y(y)$, substitution into the D.E. gives

$$
\begin{aligned}
y^2X'Y - x^2XY' &= 0 \\
y^2\frac{X'Y}{XY} - x^2\frac{XY'}{XY} &= 0 \\
y^2\frac{X'}{X} - x^2\frac{Y'}{Y} &= 0 \\
\frac{1}{x^2}\frac{X'}{X} = \frac{1}{y^2}\frac{Y'}{Y} &= k
\end{aligned}
$$

Integrating $\ln X = \frac{1}{3}kx^3 + \ln(C)$ and $\ln Y = \frac{1}{3}ky^3 + \ln(D)$, so

$$X = Ce^{\frac{1}{3}kx^3}, Y = De^{\frac{1}{3}ky^3}$$

Therefore (with $CD = A$) $u(x,y) = A\ e^{\frac{1}{3}k(x^3+y^3)}$

Question 5

Question 6

a. Substituting $u(r,z) = R(r)Z(z)$ into the diffusion equation in cylindrical coordinates gives

$$R''Z + \frac{1}{r}R'Z + RZ'' = 0$$

Diving by RZ gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{Z''}{Z} = 0$$

Separation of variables gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = -k^2$$

or

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -k^2$$
$$\frac{Z''}{Z} = k^2$$

b. For $\frac{d^2}{dz^2}Z(z) = k^2 Z(z)$, we immediately see that $Z(z) = c_1 e^{kz} + c_2 e^{-kz}$ which we can reformulate as $Z(z) = A\sinh(kz) + B\cosh(kz)$.
For $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -k^2$, starting with $\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + k^2R(r) = 0$.

$$s = kr, r = \frac{s}{k}, \frac{ds}{dr} = k, R(r) \to R(s)$$
$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = k\frac{dR}{ds}$$
$$\frac{d^2R}{dr^2} = k\frac{d}{ds}\left(k\frac{dR}{ds}\right) = k^2\frac{d^2R}{ds^2}$$

Substitution into the ODE gives

$$k^2\frac{d^2R(s)}{ds^2} + \frac{1}{\frac{s}{k}}k\frac{dR(s)}{ds} + k^2R(s) = 0$$
$$k^2\frac{d^2R(s)}{ds^2} + \frac{1}{s}k^2\frac{dR(s)}{ds} + k^2R(s) = 0$$

Multiplying out by $(\frac{s}{k})^2$ gives

$$s^2\frac{d^2R(s)}{ds^2}+s\frac{dR(s)}{ds}+s^2R(s)=0$$
$$s^2\frac{d^2R(s)}{ds^2}+s\frac{dR(s)}{ds}+(s^2-0^2)R(s)=0$$

The last equation being a Bessel equation of order 0 therefore the solution is of the form

$$R(r)=C_1J_0(kr)+C_2Y_0(kr)$$

Since the temperature remains bounded at $r=0$ thus the term $Y_0(kr)$ has to be discarded, $C_2=0$, and $R(r)=CJ_0(kr), C=C_1$

c. Finally apply boundary conditions. First, $u(r,0)=R(r)Z(0)=0$, since in general $R(r)\neq 0$, thus $Z(0)=0$, which is $A\,0+B\,1=0\rightarrow B=0$ and $Z(z)=A\sinh(kz)$. Then $u(5,z)=R(5)Z(z)=0\rightarrow R(5)=0$, therefore $CJ_0(5k)=0\rightarrow J_0(5k)=0$. $5k$ represents the zero crossing for the Bessel function of order 0. We call them α_m and set $5k_m=\alpha_m\rightarrow k_m=\frac{\alpha_m}{5}$. Therefore the solutions are $R_m(r)=C_mJ_0(k_mr)=C_mJ_0(\frac{\alpha_m}{5}r)$. Note now that solutions in z are $Z_m(z)=A_m\sinh(k_mz)$. Finally applying the superposition principle, we have

$$u(r,z)=\sum_{m=1}^{\infty}A_m\sinh(k_mz)J_0(\frac{\alpha_m}{5}r)\ \text{ where } A_mC_m\rightarrow A_m$$

Applying the last boundary condition $u(r,20)=u_0, 0<r<5$, we get:

$$u(r,20)=\sum_{m=1}^{\infty}A_m\sinh(20k_m)J_0(\frac{\alpha_m}{5}r)=u_0$$

This is a Fourier Bessel series where the coefficients are given by

$$\sinh(20k_m)A_m=\frac{2}{5^2J_1^2(\alpha_m)}\int_0^5 ru_0J_0(\frac{\alpha_m}{5}r)dr=\frac{2u_0}{25J_1(\alpha_m)}\int_0^5 rJ_0(\frac{\alpha_m}{5}r)dr$$

Next, $\sinh(20k_m)=\sinh(20\frac{\alpha_m}{5})=\sinh(4\alpha_m)$, and we get

$$A_m=\frac{2u_0}{25J_1(\alpha_m)\sinh(4\alpha_m)}\int_0^5 J_0(\frac{\alpha_m}{5}r)rdr$$

Finally using $\frac{\partial}{\partial r}[rJ_1(r)]=rJ_0(r)$

$$\begin{aligned}A_m&=\frac{2u_0}{25J_1(\alpha_m)\sinh(4\alpha_m)}\left[rJ_1(\frac{\alpha_m}{5}r)\right]_0^5\\&=\frac{2u_0}{25J_1(\alpha_m)\sinh(4\alpha_m)}\left(5J_1(\frac{\alpha_m}{5}5)\right)\\&=\frac{2u_0}{5\sinh(4\alpha_m)}\end{aligned}$$