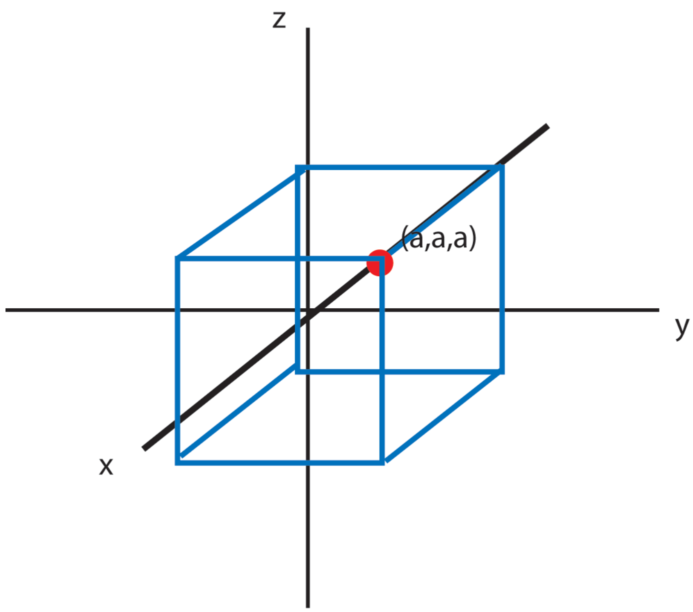
11.2 First, Verify solution



Taking $u(x,y,z,t) = X(x)Z(z)T(t) = A\cos\frac{\pi}{a}x\sin\frac{\pi}{a}ze^{-\left(\frac{2k\pi^2 t}{a^2}\right)}$

for solution to $k\nabla^2 u = \frac{\partial u}{\partial t}$

Verify u is a solution (note no y dependence)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = ZT\frac{\partial^2 X}{\partial x^2} + 0 + XT\frac{\partial^2 Z}{\partial z^2}$$

and $\frac{\partial^2 X}{\partial x^2} = \left(-\frac{\pi^2}{a^2}\right)X,\ \frac{\partial^2 Z}{\partial z^2} = \left(-\frac{\pi^2}{a^2}\right)Z$

Therefore

$$k\nabla^2 u = k(-\frac{\pi^2}{a^2}XZT - \frac{\pi^2}{a^2}XZT) = -2k\frac{\pi^2}{a^2}XZT$$

Now

$$\frac{\partial u}{\partial t} = \frac{\partial XZT}{\partial t} = XZ\frac{\partial T}{\partial t}$$

and $\frac{\partial T}{\partial t} = -\frac{2k\pi^2}{a^2}T$

Therefore $\frac{\partial u}{\partial t} = -\frac{2k\pi^2}{a^2}XZT$

Done

Next, look for heat flow across faces

Taking $u(x,y,z,t) = X(x)Z(z)T(t) = A\cos\frac{\pi}{a}x\sin\frac{\pi}{a}ze^{-\left(\frac{2k\pi^2 t}{a^2}\right)}$

KEY: We define heat flow via Fourier law for heat flow

$Q = -k\nabla u$, k is thermal conductivity

Then lets look at heat flow in x, y and z direction at $x = \pm a$, $y = \pm a$, $z = \pm a$, respectively

$$Q_x = -k\frac{\partial u}{\partial x} = -kZT\frac{\partial X}{\partial x} = kZT\left(-\frac{\pi}{a}\right)\sin\frac{\pi}{a}x = 0 \text{ for } x = \pm a$$

$$Q_y = -k\frac{\partial u}{\partial y} = 0 \text{ (no y dependence)}$$

$$Q_z = -k\frac{\partial u}{\partial z} = -kXT\frac{\partial Z}{\partial z} = -kXT\left(\frac{\pi}{a}\right)\cos\frac{\pi}{a}z \neq 0 \text{ for } z = \pm a$$

Therefore there is heat flow across faces in z!!

Finally, calculate the heat flow at $(x,y,z) = (\frac{3a}{4},\frac{a}{4},a)$ and $t = \frac{a^2}{k\pi^2}$

Taking $u(x,y,z,t) = X(x)Z(z)T(t) = A\cos\frac{\pi}{a}x\sin\frac{\pi}{a}ze^{-\left(\frac{2k\pi^2 t}{a^2}\right)}$

$$Q_z = -k\frac{\partial u}{\partial z} = -kA\left(\frac{\pi}{a}\right)\cos\frac{\pi}{a}z\cos\frac{\pi}{a}xe^{-\left(\frac{2k\pi^2 t}{a^2}\right)}$$

At $(x,y,z) = (\frac{3a}{4},\frac{a}{4},a)$ and $t = \frac{a^2}{k\pi^2}$

Then

$$Q_z = -kA\left(\frac{\pi}{a}\right)\cos(\frac{\pi}{a}a)\cos(\frac{\pi}{a}\frac{3a}{4})e^{-\left(\frac{2k\pi^2}{a^2}\frac{a^2}{k\pi^2}\right)} = -kA\left(\frac{\pi}{a}\right)\cos\pi\cos\frac{3\pi}{4}e^{-2}$$

$$= kA\left(\frac{\pi}{a}\right)\frac{\sqrt{2}}{2}e^{-2}$$