6.21

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=x$$

Euler therefore use change of variable $x=e^t$ or $t=\ln x$

Here is how to do it or use use formulas directly from book!!

$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{dy}{dt}\frac{d\ln x}{dx}=\frac{dy}{dt}\frac{1}{x}=\frac{1}{x}\frac{dy}{dt}$ That is $\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dt}=e^{-t}\frac{dy}{dt}$

That is $\frac{dy}{dx}=e^{-t}\frac{dy}{dt}$ or $\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dt}$ which gives $x\frac{dy}{dx}=\frac{dy}{dt}$

and for the operator itself $\frac{d}{dx}=e^{-t}\frac{d}{dt}$

Use this to calculate second derivative

$$\frac{d^2y}{dx^2}=\frac{d}{dt}(\frac{dy}{dt})=e^{-t}\frac{d}{dt}(e^{-t}\frac{dy}{dt})=e^{-t}(-e^{-t}\frac{dy}{dt}+e^{-t}\frac{d^2y}{dt^2})=(e^{-t})^2(-\frac{dy}{dt}+\frac{d^2y}{dt^2})$$

That is $\frac{d^2y}{dx^2}=(e^t)^{-2}(-\frac{dy}{dt}+\frac{d^2y}{dt^2})=(\frac{1}{x})^2(-\frac{dy}{dt}+\frac{d^2y}{dt^2})$ or $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$

Substitute into above equation

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=x$$

$(\frac{d^2y}{dt^2}-\frac{dy}{dt})-\frac{dy}{dt}+y=e^t$ or

$$\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=e^t$$

The homogenous equation is

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$$

Assume solution $y(t) = Ae^{mt}$ gives solution is $y(t) = e^t$ where characteristic equation has a double root m = 1

so referring to page 234 in the book (or your undergradte DE course) another solution (independent!) is $y(t) = te^t$

Therefore the homogenous solution is $y(t) = c_1e^t + c_2te^t$

Next use variation of parameters on DE equation as a function of t.

Take

$$y_p(t) = k_1e^t + k_2te^t$$

Proceed to solve for derivatives of k's using using equations 6.57 (see example pg. 252)

$$k_1'e^t + k_2'te^t = 0$$

$$k_1'e^t + k_2'(e^t + te^t) = e^t$$

Note it helps if you multiple both equations by $e^{-t}$

Once you solve for $k_1', k_2'$ integrate to find $k_1$ and $k_2$

$$y_p(t) = k_1e^t + k_2te^t = \left(-\frac{t^2}{2}\right)e^t + (t)te^t = \frac{t^2}{2}e^t$$

Substitute for t=lnx and x=$e^t$ gives

$$y(t) = y_h(t) + y_p(t) = c_1e^t + c_2te^t + \frac{t^2}{2}e^t \to y(x) = c_1x + c_2x\ln x + \frac{(\ln x)^2}{2}x$$

You can solve for constants $c_1, c_2$ by using initial conditions y(1)=1, y(e) = 2e to finish

I will let you finish!