14.6

(a)

$\tan z = \frac{\sin z}{\cos z}$

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}),\ \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

Zeros when tanz=0 $\rightarrow \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = 0 \rightarrow e^{iz} = e^{-iz}$

Substitute $z = x + iy$ gives $e^{i(x+iy)} = e^{-i(x+iy)} \rightarrow e^{ix}e^{-y} = e^{-ix}e^{y}$

Now $e^{ix}e^{-y} = (\cos x + i\sin x)e^{-y}$ and $e^{-ix}e^{y} = (\cos x - i\sin x)e^{y}$

Therefore $(\cos x + i\sin x)e^{-y} = (\cos x - i\sin x)e^{y}$

or $e^{-y}\cos x + ie^{-y}\sin x = e^{y}\cos x - ie^{y}\sin x$

Match real part to real part $e^{-y}\cos x = e^{y}\cos x \rightarrow e^{-y} = e^{y} \rightarrow y = 0$

Then match imaginary parts $e^{-y}\sin x = -e^{y}\sin x$ (now y = 0) $\rightarrow \sin x = -\sin x$

or $2\sin x = 0 \rightarrow \sin x = 0 \rightarrow x = n\pi$

Therefore $z = n\pi + i0 = n\pi$ are the zeros

Poles when denominator is zero $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = 0 \rightarrow e^{iz} = -e^{-iz} \equiv -1e^{-iz}$

Easier in this case to leave as function of z and do it similar

to that in lecture. that is $-1 = e^{i(2n+1)\pi}$ substitution gives $e^{iz} = e^{i(2n+1)\pi}e^{-iz}$

or $e^{iz} = e^{i[-z+(2n+1)\pi]} \rightarrow z = -z + (2n+1)\pi \rightarrow 2z = (2n+1)\pi \rightarrow z = (n + \frac{1}{2})\pi$ are the poles

What about $z \rightarrow \pm\infty$ $\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots$

Now as the book indicateds look at $z = \frac{1}{\xi}$ as $\xi = 0$

Therefore look at $\tan\frac{1}{\xi} = \frac{1}{\xi} + \frac{1}{3}\frac{1}{\xi^3} + \frac{2}{15}\frac{1}{\xi^5} + \cdots$ at $\xi = 0$

Now $\lim_{\xi_o \rightarrow 0}(\xi - \xi_o)^n\left[\frac{1}{\xi} + \frac{1}{3}\frac{1}{\xi^3} + \frac{2}{15}\frac{1}{\xi^5}\right]$ is not finite for some large n

and function $\tan\frac{1}{\xi}$ has essential singularity at $\xi = 0$

which is equivalent to $\tan z$ having essential singularity at $z \rightarrow \pm\infty$