Johns Hopkins
Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
EN. 585.409

JOHNS HOPKINS
WHITING SCHOOL
of ENGINEERING

Series solution of ODE at regular singular points

Like that for ordinary points We will again be looking at second order ODEs, however the techniques can be generalized to higher order equations. Furthermore we will look at the solutions at $z=z_0=0$. A shift of z can accommodate other non-zero point, that is we could make the variable substitution $\tilde{z}=z-z_0$ in our differential equation.

However to accommodate the solutions at regular singular points we will extend our series solution as follows

$$\boxed{y(z)=z^{\sigma}\sum_{n=0}^{\infty}a_n z^n}$$

where the a_n, σ are constants.

We also note that a_0 Will never be equal to 0 since

$$y(z)=z^{\sigma}\sum_{n=0}^{\infty}a_n z^n = z^{\sigma}(a_0+a_1z^1+a_2z^2+\cdots = z^{\sigma}(0+a_1z^1+a_2z^2+\cdots =$$

$$z^{\sigma}(a_1z^1+a_2z^2+\cdots = z^{\sigma}(0+a_1z^1+a_2z^2+\cdots = z^{\sigma+1}(a_1+a_2z^1+\cdots$$

$$\equiv z^{\sigma}(a_0+a_1z^1+\cdots$$

And we can redefine the a_n, σ in the last equality since they are unknown constants.

We can start as we did for ordinary points by finding the form of the first and second derivatives, that is

$$y(z)=\sum_{n=0}^{\infty}a_n z^{\sigma+n}, \quad y'(z)=\sum_{n=0}^{\infty}(\sigma+n)a_n z^{\sigma+n-1}, \quad y''(z)=\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_n z^{\sigma+n-2}$$

An example of a series solution of an ODE at an regular point

Take the differential equations $4zy''+2y'+y=0$

We are interested at the solution near or at z = 0.

The very first thing we do is to put the equation in standard form so that we can identify the functions p(z) and q(z).

$$\boxed{y''+\frac{1}{2z}y'+\frac{1}{4z}y=0}$$

Therefore $p(z)=\frac{1}{2z},\ q(z)=\frac{1}{4z}$

Clearly z = 0 is a singular point, so next we determine if it's a regular singular point.

$$(z-z_0)p(z)\Big|_{z=z_0}=(z)\frac{1}{2z}=\frac{1}{2}$$

$$(z-z_0)^2q(z)\Big|_{z=z_0}=(z)^2\frac{1}{4z}\Big|_{z=z_0}=\frac{z}{4}\Big|_{z=z_0}=0$$

Since both these functions are analytically defined z=0 is a regular singular point!

At this point we are ready to use our "extended" series solution method.
It will require a little more work that that for ordinary points as we have additional constants to find. But we start out by substitution into our standard form differential equation (by the way we could start with the original form of the ODE with some slight changes in the following steps).

$$y''+\frac{1}{2z}y'+\frac{1}{4z}y=\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_n z^{\sigma+n-2}+\frac{1}{2z}\sum_{n=0}^{\infty}(\sigma+n)a_n z^{\sigma+n-1}+\frac{1}{4z}\sum_{n=0}^{\infty}a_n z^{\sigma+n}=$$

$$\sum_{n=0}^{\infty}(\sigma+n-1)(\sigma+n)a_n z^{\sigma+n-2}+\sum_{n=0}^{\infty}\frac{1}{2}(\sigma+n)a_n z^{\sigma+n-2}+\frac{1}{4z}\sum_{n=0}^{\infty}a_n z^{\sigma+n}=$$

$$\sum_{n=0}^{\infty}[(\sigma+n-1)(\sigma+n)+\frac{1}{2}(\sigma+n)]a_n z^{\sigma+n-2}+\sum_{n=0}^{\infty}\frac{1}{4}a_n z^{\sigma+n-1}=0$$

KEY(as before): z or correspondingly z^n cannot equal 0 otherwise we have a degenerate solution, correct but of no interest. Furthermore since each power of z must not be zero that means that the other terms that include the coefficients must be zero.
We will start by looking at the lowest power of z that occurs in either sum.
This happens in the first sum when n = 0 and the lowest power of z is $\sigma-2$

We get

$$[(\sigma+0-1)(\sigma+0)+\frac{1}{2}(\sigma+0)]a_0=[(\sigma-1)\sigma+\frac{1}{2}\sigma]a_0=0$$

Indicial equation

$$\text{and since } a_0\neq 0 \text{ then } \boxed{(\sigma-1)\sigma+\frac{1}{2}\sigma=0} \text{ or } \sigma=0,\ \frac{1}{2}$$

When we solve the indicial equation for σ because we are working with a second order ODE we will get two solutions. We take up the one case here where their difference is not an integer (as seen for this example). In this case using this methodology will lead to two independent solutions to our differential equation as expected. We will look at the other cases next.

In order to continue we need to match up powers of z. There are a number of ways to do this. Here we start by factoring out the lowest power of z that occurs in either sum (as before this is the $\sigma-2$ power.

$$\sum_{n=0}^{\infty}[(\sigma+n-1)(\sigma+n)+\frac{1}{2}(\sigma+n)]a_n z^{\sigma+n-2}+\frac{1}{4z}\sum_{n=0}^{\infty}a_n z^{\sigma+n-1}=$$

$$z^{\sigma-2}\left\{\sum_{n=0}^{\infty}[(\sigma+n-1)(\sigma+n)+\frac{1}{2}(\sigma+n)]a_n z^n+\sum_{n=0}^{\infty}\frac{1}{4}a_n z^{n+1}\right\}=0$$

Since the factored out power of z cannot be zero we can focus on the bracket. In order to have the same powers of z in both sums we re-index the second sum by letting $n\to n-1$

$$\sum_{n=0}^{\infty}\frac{1}{4}a_n z^{n+1} \text{ with } n\to n-1 \text{ we get } \sum_{n-1=0}^{\infty}\frac{1}{4}a_{n-1}z^{(n-1)+1}\equiv\sum_{n=1}^{\infty}\frac{1}{4}a_{n-1}z^n$$

Substitution gives

$$\sum_{n=0}^{\infty}[(\sigma+n-1)(\sigma+n)+\frac{1}{2}(\sigma+n)]a_n z^n+\sum_{n=1}^{\infty}\frac{1}{4}a_{n-1}z^n=0$$

However having matched powers of z in both sums the indices don't match as one starts at n = 0 and the other at n =1. We will take see this is not a problem at all!

But first let's recalled from our indicial equation that $\sigma = 0,\ \frac{1}{2}$

Taking the larger value first (standard) let's rewrite our sums.

$$\sum_{n=0}^{\infty}[(\frac{1}{2}+n-1)(\frac{1}{2}+n)+\frac{1}{2}(\frac{1}{2}+n)]a_n z^n + \sum_{n=1}^{\infty}\frac{1}{4}a_{n-1}z^n =$$

$$\sum_{n=0}^{\infty}[n^2+\frac{n}{2}]a_n z^n + \sum_{n=1}^{\infty}\frac{1}{4}a_{n-1}z^n = 0$$

Note that for the first sum the n=0 term is 0! In fact we our guaranteed this since essentially it comes from the indicial equation (remember it was equal to 0 after we set n = 0 and solved for $\sigma = 1/2$). So just drop the first index term at n =0 In the first sum and combine the two sums!

$$\sum_{n=1}^{\infty}[(n^2+\frac{n}{2})a_n + \frac{1}{4}a_{n-1}]z^n = 0,\ \ n \ge 1$$

And again, since z^n is not equal to zero we have

$$(n^2+\frac{n}{2})a_n + \frac{1}{4}a_{n-1} = 0 \text{ or } \boxed{a_n} = \frac{-\frac{1}{4}a_{n-1}}{(n^2+\frac{n}{2})} = \frac{-a_{n-1}}{4(n^2+\frac{n}{2})} = \boxed{\frac{-a_{n-1}}{2n(2n+1)},\ \ n \ge 1}$$

A recursive definition for finding the constants a_n.

Let's work with this expression $a_n = \dfrac{-a_{n-1}}{2n(2n+1)}$ to determine the constants explicitly.

Remember a_0 is a constant not equal to zero so just pick something convenient like $a_0=1$, then start to increment n.

For $n = 1$ $\quad a_1 = \dfrac{-a_{1-1}}{2\cdot 1(2\cdot 1+1)} = \dfrac{-a_0}{2(3)} = \dfrac{-1}{3!}$

For $n = 2$ $\quad a_2 = \dfrac{-a_{2-1}}{2\cdot 2(2\cdot 2+1)} = \dfrac{-a_1}{4(5)} = \dfrac{-}{4(5)}\left(\dfrac{-1}{3!}\right) = \dfrac{1}{5!}$

$\vdots$

Therefore an explicit definition for the constants is $\boxed{a_n = \dfrac{(-1)^n}{(2n+1)!}}$

Therefore our first solution is $\boxed{y_1(z) = z^{1/2}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)!}z^n}$

For the other solution $\sigma = 0$ we get a similar result (try it yourself), that is

$a_n = \dfrac{-a_{n-1}}{2n(2n-1)}$ and an explicit form is $\boxed{a_n = \dfrac{(-1)^n}{(2n)!}}$

And our second solution is $\boxed{y_2(z) = z^0\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n)!}z^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n)!}z^n}$

Let's work with this expression $a_n = \frac{-a_{n-1}}{2n(2n+1)}$ to determine the constants explicitly.

Remember a_0 is a constant not equal to zero so just pick something convenient like $a_0=1$, then start to increment n.

For $n = 1$ $\quad a_1 = \frac{-a_{1-1}}{2\cdot 1(2\cdot 1+1)} = \frac{-a_0}{2(3)} = \frac{-1}{3!}$

For $n = 2$ $\quad a_2 = \frac{-a_{2-1}}{2\cdot 2(2\cdot 2+1)} = \frac{-a_1}{4(5)} = \frac{-}{4(5)}\left(\frac{-1}{3!}\right) = \frac{1}{5!}$

$\vdots$

Therefore an explicit definition for the constants is $\boxed{a_n = \frac{(-1)^n}{(2n+1)!}}$

Therefore our first solution is $\boxed{y_1(z) = z^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^n}$

For the other solution $\sigma = 0$ we get a similar result (try it yourself), that is

$a_n = \frac{-a_{n-1}}{2n(2n-1)}$ and an explicit form is $\boxed{a_n = \frac{(-1)^n}{(2n)!}}$

And our second solution is $\boxed{y_2(z) = z^0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^n}$

As before we can take this a little further to investigate the series solution

Let's look at the Taylor series for the sine function. The generating formula for a Taylor series near z_0=0 is (previously presented)

$$f(z)=\sum_{n=0}^{\infty} a_n z^n,\ a_n=\left.\frac{f^{(n)}(z)}{n!}\right|_{z=z_0=0}$$

Setting

$f(z)=\sin z$ then

$$a_0=\left.\frac{f^{(0)}(z)}{0!}\right|_{z=z_0=0}=\left.\frac{\sin z}{0!}\right|_{z=z_0=0}=\frac{\sin 0}{0!}=\frac{0}{1}=0,\ a_1=\left.\frac{f^{(1)}(z)}{1!}\right|_{z=z_0=0}=\left.\frac{\cos z}{1!}\right|_{z=z_0=0}=\frac{\cos 0}{1!}=\frac{1}{1}=1$$

$$a_2=\left.\frac{f^{(2)}(z)}{2!}\right|_{z=z_0=0}=\frac{-\sin z}{2!}=\frac{-\sin 0}{2!}=\frac{0}{2!},\ a_3=\left.\frac{f^{(2)}(z)}{3!}\right|_{z=z_0=0}=\frac{-\cos z}{3!}=\frac{-\cos 0}{3!}=\frac{-1}{3!}$$

$\vdots$

Therefore $f(z)=\sin z=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$ and making the substitution $z\to z^{1/2}$ gives

$$\sin z^{1/2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(z^{1/2})^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{n+1/2}=z^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^n=y_1(z)$$

The same as our previous series solution $y_1(z)$. $y_2(z)$ is the same as the Taylor expansion for $\cos z^{1/2}$. By the way the Wronskian is not equal to 0 in general and is only undefined at z = 0, our singular point!

$$W(z)=\begin{vmatrix} y_1(z) & y_2(z) \\ y_1'(z) & y_2'(z)\end{vmatrix}=\begin{vmatrix} \sin z^{1/2} & \cos z^{1/2} \\ \dfrac{\cos z^{1/2}}{2z^{1/2}} & \dfrac{-\sin z^{1/2}}{2z^{1/2}}\end{vmatrix}=\frac{-1}{2z^{1/2}}[(\sin z^{1/2})^2+(\cos z^{1/2})^2]=\frac{-1}{2z^{1/2}}$$