Interactive Assigment 4

21 pages

Problems

Yves GREATTI

Chapter 6 - Problem 6.10

Use the methods of Laplace transforms to solve

(a) $\frac{d^2 f}{dt^2} + 5\frac{df}{dt} + 6f = 0 \qquad f(0) = 1, \quad f'(0) = -4$

Taking the Laplace transform on both sides gives:

$$s^2 \bar{f}(s) - s f(0) - f'(0) + 5\left[s \bar{f}(s) - f(0) \right] + 6 \bar{f}(s) = 0$$

$$(s^2 + 5s + 6)\, \bar{f}(s) = s f(0) + f'(0) + 5 f(0)$$

$$\Rightarrow \bar{f}(s)\, (s^2 + 5s + 6) = (s+5) f(0) + f'(0)$$

$$= s + 5 - 4 = s + 1$$

$$\Rightarrow \bar{f}(s) = \frac{s+1}{s^2 + 5s + 6} = \frac{s+1}{(s+2)(s+3)}$$

Using the partial fraction method

$$\frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} = \frac{(A+B)s + (3A+2B)}{(s+2)(s+3)}$$

Next equating powers of s on LHS and RHS, gives:

s^1: $A + B = 1$

s^0: $3A + 2B = 1$

$\Rightarrow \quad A = -1$ and $B = 2$

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$$\bar{f}(s) = -\frac{1}{s+2} + \frac{2}{s+3}$$

Taking the inverse Laplace transform and using a table gives:

$$f(t) = -e^{-2t} + 2e^{-3t}$$

(b) $\frac{d^2f}{dt^2} + 2\frac{df}{dt} + 5f = 0$, $f(0) = 1$, $f'(0) = 0$

Taking the Laplace transform on both sides of this equation:

$$s^2\bar{f}(s) - sf(0) - f'(0) + 2[s\bar{f}(s) - f(0)] + 5\bar{f}(s) = 0$$

$$\Rightarrow (s^2 + 2s + 5)\bar{f}(s) = (s+2)f(0) = s+2$$

$$\Rightarrow \bar{f}(s) = \frac{s+2}{s^2+2s+5} = \frac{s+1+1}{(s+1)^2+4}$$

$$= \frac{s+1}{(s+1)^2+2^2} + \frac{1}{(s+1)^2+2^2}$$

Using a table of Laplace transform:

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)$$

$$= e^{-t}\left[\cos(2t) + \frac{\sin(2t)}{2}\right]$$

Chapter 6 - Problem 6.12

For $t \geq 0$ we have to solve the system of differential equations
$$\begin{cases} \ddot{x} + 2x + y = \cos t \\ \ddot{y} + 2x + 3y = 2\cos t \end{cases}$$

where $x(0) = \dot{x}(0) = 0$
$y(0) = \dot{y}(0) = 0$

Taking the Laplace transform on both sides of the previous equations gives:

$$\begin{cases} s^2 \bar{x}(s) - s\bar{x}(0) - \dot{x}(0) + 2\bar{x}(s) + \bar{y}(s) = \frac{s}{s^2+1} \\ s^2 \bar{y}(s) - s\,y(0) - \dot{y}(0) + 2\bar{x}(s) + 3\bar{y}(s) = \frac{2s}{s^2+1} \end{cases}$$

Grouping terms together and simplifying:

(1) $(s^2+2)\bar{x}(s) + \bar{y}(s) = \frac{s}{s^2+1}$

(2) $2\bar{x}(s) + (s^2+3)\bar{y}(s) = \frac{2s}{s^2+1}$

$2\times(1)-(2)$: $\quad 2(s^2+1)\bar{x}(s) - (s^2+1)\bar{y}(s) = 0$

$$\Rightarrow \bar{y}(s) = 2\bar{x}(s)$$

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Plugging back $\bar{y}(s)$ into (1) gives:

$$(s^2+2+2)\,\bar{x}(s) = \frac{s}{s^2+1} \Rightarrow \bar{x}(s) = \frac{s}{(s^2+1)(s^2+4)}$$

The partial fraction expansion is quite easy to determine:

$$\bar{x}(s) = 1/3 \left[\frac{s}{s^2+1} - \frac{s}{s^2+4} \right]$$

Taking the inverse laplace transform and using a table:

$$x(t) = \mathcal{L}^{-1}\{\bar{x}(s)\} = \frac{1}{3} \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= 1/3\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\} - 1/3\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}$$

$$= 1/3\,(\cos t - \cos 2t)$$

And $y(t) = \mathcal{L}^{-1}\{\bar{y}(s)\} = \mathcal{L}^{-1}\{2\bar{x}(s)\} = 2\mathcal{L}^{-1}\{\bar{x}(s)\}$

$$= 2\,x(t)$$

$$= \frac{2}{3}(\cos t - \cos 2t)$$

$y(t) = 2x(t)$: the motion takes place along a straight line in x-y plane.

Note the $\cos t$ term with frequency $\omega = 1$ is the same as that for the driving functions in the original D.E equations.

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Chapter 6 - Problem 6.12

We look carefully and recognize spring equation $\ddot{x}+x=0$ or $\ddot{x}=-x$. Since we expect oscillatory motion, we try:

$x(t)=A\,e^{i\omega t}$ and $y(t)=B\,e^{i\omega t}$

Substituting these into the equations, gives us:

$$\begin{cases} -A\omega^2 e^{i\omega t}+2Ae^{i\omega t}+B\,e^{i\omega t}=0 \\ -B\,\omega^2 e^{i\omega t}+2A\,e^{i\omega t}+3B\,e^{i\omega t}=0 \end{cases}$$

Since the exponential factor is common to all terms, we omit it and simplify:

$$\begin{cases} (2-\omega^2)\;A+B=0 \\ 2A+(3-\omega^2)B= \end{cases}$$

And in matrix representation: $\begin{bmatrix} 2-\omega^2 & 1 \\ 2 & 3-\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The non-trivial solutions are to be found for those values of ω whereby the matrix on the left is singular, i.e. not invertible. It follows that the determinant of the matrix must be equal to 0.

To find resonant behavior we are $\begin{vmatrix} 2-\omega^2 & 1 \\ 2 & 3-\omega^2 \end{vmatrix}=(2-\omega^2)(3-\omega^2)-2=0$

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$$\begin{vmatrix} 2-\omega^2 & 1 \\ 2 & 3-\omega^2 \end{vmatrix} = 0 \Rightarrow (\omega^2-4)(\omega^2-1)=0$$

which leads to $\omega = 1, 2$ resonant frequencies.

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Chapter 6 – Problem 6.13

We have for A: $A \to B, 3s^{-1}$ which we can write as $\frac{dA}{dt} = -3A - A$
$A \to C, s^{-1}$

$$\frac{dA}{dt} = -4A$$

For B: $A \to B, 3s^{-1}$: $\frac{dB}{dt} = 3A - 2B$
$B \to C, 2s^{-1}$

And for C: $A \to C, s^{-1}$: $\frac{dC}{dt} = A + 2B$
$B \to C, 2s^{-1}$

Taking the laplace transforms of each of these ODE's gives:

$$s\bar{A}(s) - x_0 = -4\bar{A}(s) \Rightarrow \bar{A}(s) = \frac{x_0}{s+4}$$

$$s\bar{B}(s) = 3\bar{A}(s) - 2\bar{B}(s) \Rightarrow \bar{B}(s) = \frac{3\bar{A}(s)}{s+2}$$

$$= \frac{3x_0}{(s+2)(s+4)}$$

$$s\,\bar{C}(s) = \bar{A}(s) + 2\bar{B}(s)$$

$$= \frac{x_0}{s+4} + \frac{6x_0}{(s+2)(s+4)}$$

$$= \frac{(s+6)x_0 + 2x_0}{(s+2)(s+4)}$$

$$= \frac{x_0 s + 8x_0}{(s+2)(s+4)} = x_0\frac{(s+8)}{(s+2)(s+4)} \Rightarrow \bar{C}(s) = x_0\frac{s+8}{s(s+2)(s+4)}$$

Chapter 6- Problem 6.13

Now we determine the partial fraction expansion of $\frac{s+8}{s(s+2)(s+4)}$

$$\frac{s+8}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$= \frac{(s+2)(s+4)A + s(s+4)B + s(s+2)C}{s(s+2)(s+4)}$$

We equate powers of s on each side of the previous equation:

(1) s^2: $A+B+C=0$

(2) s^1: $6A+4B+2C=1$

(3) s^0: $8A=8$

$\Rightarrow$ $A=1$

$C=-B-1$

And substituting in equation (2) gives:

$$6+4B-2(1+B)=1$$

$$\Rightarrow B=-3/2 \text{ and } C=1/2$$

Replacing these values onto the initial expression for $C(s)$:

$$C(s) = x_0 \left[\frac{1}{s} - \frac{3}{2(s+2)} + \frac{1}{2(s+4)} \right]$$

Taking its inverse Laplace transform yields:

$$c(t) = x_0 \left[1 - \frac{3}{2} e^{-2t} + \frac{1}{2} e^{-4t} \right]$$

Chapter 6 - Problem 6.21

Find the general solution of $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x$

given that $y(1) = 1$ and $y(e) = 2e$

We start by changing the variable $x = e^t$ or $t = \ln x$

Using the chain rule for differentiation

$$x\frac{dy}{dx} = e^t \frac{dy}{dx} = e^t \frac{dt}{dx}\frac{dy}{dt} = e^t \frac{d\ln x}{dx}\frac{dy}{dt} = e^t \frac{1}{x}\frac{dy}{dt}$$
$$= e^t \frac{1}{e^t} \times \frac{dy}{dt}$$

Thus we have $x\frac{dy}{dx} = \frac{dy}{dt}$

And since this is a legendre linear differential equation

we can use the expression for the second derivative term:

$$(\alpha x + \beta)^2 \frac{d^2y}{dx^2} = \alpha^2 \frac{d}{dt}\left[\frac{d}{dt} - 1\right] y$$

For this equation $\alpha = 1, \beta = 0$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{d}{dt} - 1\right] y = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

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By substitution into the differential equation, we have

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - \frac{dy}{dt} + y = e^t$$

or $$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t$$

Setting the RHS to zero, substituting $y = Ae^{\lambda t}$ and dividing through by $Ae^{\lambda t}$ gives the auxiliary equation

$$\lambda^2 - 2\lambda + 1 = 0$$

The root $\lambda = 1$ occurs twice and so, the full complementary solution is:

$$y_c(t) = (c_1 + c_2 t)\, e^t$$

We take as a particular solution initially $y_p(t) = be^t$

However we see that e^t is already contained in the complementary solution; and as te^t.

Multiplying the first guess by the lowest integer power of t such that the result does not appear in $y_c(t)$

we try $y_p(t) = bt^2e^t$. Substituting this into the ODE, we find $b = 1/2$ so $y_p(t) = t^2e^t/2$.

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The general solution of the differential equation is therefore

$$y(t) = (c_1 + c_2 t) e^t + t^2 e^t / 2$$

substituting $x = e^t$ back into this solution

$$y(x) = (c_1 + c_2 \ln x) . x + \frac{1}{2} \times x (\ln x)^2$$

with $\left.\begin{array}{l} y(1)=1 \\ y(e)=2e \end{array}\right\} \Rightarrow \begin{array}{l} (c_1 + c_2 \times \ln 1) \times 1 + \frac{1}{2} \times 1 \times (\ln 1)^2 = 1 \\ (c_1 + c_2 \times \ln e) \times e + \frac{1}{2} \times e \times (\ln e)^2 = 2e \end{array}$

$$\Rightarrow \quad c_1 = 1$$

$$(c_1 + c_2) \times e + \frac{e}{2} = 2e \Rightarrow c_1 + c_2 + \frac{1}{2} = 2$$

$$\Rightarrow c_2 = 2 - \frac{1}{2} - c_1 = 2 - \frac{1}{2} - 1$$

$$= 1/2$$

The general solution with conditions $y(1) = 1$ and $y(e) = 2e$

is:
$$y(x) = \left(1 + \frac{\ln x}{2}\right) . x + \frac{1}{2} \times x \times (\ln x)^2$$

$$= x + \frac{1}{2} \left[x \ln x \, (1 + \ln x) \right]$$

Chapter 6 - Problem 6.22

Find the general solution of

$$(x+1)^2 \frac{d^2y}{dx^2} + 3(x+1)\frac{dy}{dx} + y = x^2$$

This differential equation is a Legendre ODE with $\alpha = 1$, $\beta = 1$.
Thus by changing to $x+1 = e^t$ or $t = \ln(x+1)$, we have

$$(x+1)\frac{dy}{dx} = \frac{dy}{dt}$$

$$(x+1)^2 \frac{d^2y}{dx^2} = 1^2 \cdot \frac{d}{dt}\left[\frac{d}{dt} - 1\right] y$$

$$= \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

By substitution into the differential equation gives:

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 3\frac{dy}{dt} + y = e^{2t}$$

$$\Rightarrow \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{2t}$$

For the homogeneous ODE: $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$

take as the solution $y(t) = Ae^{\lambda t}$, $\frac{d^2y}{dt^2} = \lambda^2 A e^{\lambda t}$

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Chapter 6 – Problem 6.22

Substitution into the ODE gives the characteristic equation:

$$\lambda^2 + 2\lambda + 1 = 0$$

$\lambda = 1$ is a double root and so the complementary solution is:

$$y_c(t) = (c_1 + c_2 t)\, e^{-t}$$

Following the variation of parameters method, we are looking for a particular solution: $y_p(t) = k_1(t)\, e^{-t} + k_2(t)\, t e^{-t}$

And we find a system of two equations in unknown k'_1 and k'_2:

$$\begin{cases} k'_1(t)\, e^{-t} + k'_2(t)\, t\, e^{-t} = 0 \\ -k'_1(t) e^{-t} + k'_2(t)\, e^{-t}(1-t) = (e^t - 1)^2 \end{cases}$$

Multiplying both equations by e^t

$$\begin{cases} k'_1(t) + t\, k'_2(t) = 0 & (1) \\ -k'_1(t) + (1-t)\, k'_2(t) = e^t(e^t-1)^2 & (2) \end{cases}$$

The determinant of this system is:

$$D = \begin{vmatrix} 1 & t \\ -1 & 1-t \end{vmatrix} = 1 - t + t = 1 \neq 0$$

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Chapter 6 - Problem 6.22

From (1): $k'_1(t) = -t\, k'_2(t)$

and substituting $k'_1(t)$ into (2) gives:

$$t\, k'_2(t) + (1-t)\, k'_2(t) = e^t (e^t - 1)^2$$

$$\Rightarrow k'_2(t) = e^t (e^t - 1)^2$$

$$\Rightarrow k_2(t) = \int e^t (e^t - 1)^2 dt = \frac{1}{3} e^t (e^{2t} - 3e^t + 3)$$

$$\Rightarrow k'_1(t) = -t\, k'_2(t) = -t\, e^t (e^t - 1)^2$$

$$\Rightarrow k_1(t) = \frac{(1-3t)}{9} e^{3t} + (t - 1/2) e^{2t} + (1-t) e^t$$

hence

$$y_p(t) = k_1(t) e^{-t} + k_2(t)\, t e^{-t}$$

$$= e^{-t}\left[\frac{(1-3t)}{9} e^{3t} + (t - 1/2) e^{2t} + (1-t) e^t\right]$$

$$+ \frac{1}{3} e^t (e^{2t} - 3e^t + 3)\, t e^{-t}$$

$$= \frac{1}{9} e^{2t} - \frac{e^t}{2} + 1$$

Therefore $y(t) = y_c(t) + y_p(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{e^{2t}}{9} - \frac{e^t}{2} + 1$

plugging back: $x+1 = e^t$ or $t = \ln(x+1)$ gives

$$y(x) = \frac{c_1}{x+1} + c_2 \frac{\ln(x+1)}{x+1} + \frac{(x+1)^2}{9} - \frac{(x+1)}{2} + 1$$

Chapter 6. Problem 6.24

Use the method of variation of parameters to find the general solutions of

(a) $\frac{d^2y}{dx^2} - y = x^n$

Taking $y(x) = Ae^{\alpha x}$ we find the characteristic equation to be $\alpha^2 - 1 = 0$ with roots $\alpha = \pm 1$ so the complementary solution is $y_c(x) = c_1 e^x + c_2 e^{-x}$.

Next we consider the particular solution for the inhomogeneous equation: $y_p(x) = k_1(x) e^x + k_2(x) e^{-x}$

Following the variation of parameters method we find a system of two unknown functions:

$$\begin{cases} k_1'(x) e^x + k_2'(x) e^{-x} = 0 \\ k_1'(x) (e^x)' + k_2'(x) (e^{-x})' = x^n \end{cases}$$

or $$\begin{cases} e^x k_1'(x) + e^{-x} k_2'(x) = 0 \\ e^x k_1'(x) - e^{-x} k_2'(x) = x^n \end{cases}$$

Chapter 6 - Problem 6.24

Solving for $k'_1(x)$, $k'_2(x)$, Cramer's rule gives:

$$k'_1(x) = \frac{\begin{vmatrix} 0 & e^{-x} \\ x^n & -e^{-x} \end{vmatrix}}{D} \quad , \quad k'_2(x) = \frac{\begin{vmatrix} e^{x} & 0 \\ e^{x} & x^n \end{vmatrix}}{D}$$

where $D = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -e^{x}e^{-x} - e^{x}e^{-x} = -e^{0} - e^{0} = -2$

Next $k'_1(x) = -1/2 \; e^{-x} x^n$

Using a table of integrals with $\Gamma(n,x) = \int_0^x x^{n-1} e^{-x} \, dx$.

we have $k_1(x) = -\frac{\Gamma(n+1, x)}{2}$

Another expression of $k_1(x) = \frac{1}{2} e^{-x} (-1) \, n! \sum_{r=0}^{n} \frac{x^{n-r}}{(n-r)!}$

$k'_2(x) = \cdot 1/2 \; x^n e^{x}$

Note that $k'_1(-x) = -\frac{1}{2} e^{-(-x)} (-x)^n = (-1)^n \left(-\frac{1}{2} e^{x} x^n\right)$

$= k'_2(x) \, (-1)^n$

thus $k'_2(x) = (-1)^n k'_1(-x) \Rightarrow k_2(x) = -\frac{1}{2} (-1)^n \Gamma(n+1, -x)$

Finally $y_p(x) = -\frac{1}{2} e^{x} \Gamma(n+1, x) - \frac{1}{2} (-1)^n \Gamma(n+1, -x) \, e^{-x}$

And the total solution is

$$y(x) = y_c(x) + y_p(x) = C_1 e^{x} + C_2 e^{-x} - \frac{1}{2} e^{x} \Gamma(n+1, x) - \frac{1}{2} (-1)^n e^{-x} \Gamma(n+1, -x)$$

Chapter 6 - Problem 6.24

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2xe^x$

Taking $y(x) = Ae^{\lambda x}$ a solution of the homogeneous equation we find the characteristic equation: $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$ is a root of multiplicity 2 and so the complementary solution $y_c(x) = (C_1 + C_2 x)e^x$

Next we consider for the inhomogeneous equation the particular solution: $y_p(x) = k_1(x)e^x + k_2(x)\,x\,e^x$

Using the variation of parameter method we find a system of two unknown functions:

$$\begin{cases} e^x k_1'(x) + xe^x k_2'(x) = 0 \\ e^x k_1'(x) + (1+x)e^x k_2'(x) = 2xe^x \end{cases}$$

Using Gaussian elimination

$$k_1'(x) = \frac{\begin{vmatrix} 0 & xe^x \\ 2xe^x & (1+x)e^x \end{vmatrix}}{D}, \quad k_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & 2xe^x \end{vmatrix}}{D}$$

chapter 6 - Problem 6.24

$$D = \begin{vmatrix} e^x & x e^x \\ e^x & (1+x) e^x \end{vmatrix} = (1+x) e^{2x} - x e^{2x} = e^{2x}$$

Thus $k_1'(x) = \dfrac{\begin{vmatrix} 0 & x e^x \\ 2x e^x & (1+x) e^x \end{vmatrix}}{e^{2x}} = - \dfrac{2 x^2 e^{2x}}{e^{2x}} = -2x^2$

$$\Rightarrow k_1(x) = -2 \int x^2 dx = -2/3 \, x^3$$

$$k_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & 2x e^x \end{vmatrix}}{e^{2x}} = \frac{2x e^{2x}}{e^{2x}} = 2x \Rightarrow k_2(x) = x^2$$

Therefore $y_p(x) = -\frac{2}{3} x^3 e^x + x^3 e^x = 1/3 \, x^3 e^x$

And the general solution

$$y(x) = y_p(x) + y_c(x) = 1/3 \, x^3 e^x + (c_1 + c_2 x) e^x$$

Chapter 6- Problem 6.31

The characteristic equation for the homogeneous equation is easily determined (plug in $x(t) = Ae^{\lambda t}$):

$$\lambda^2 + \alpha\lambda = 0$$

$$\lambda(\lambda+\alpha) = 0 \Rightarrow \lambda = 0, -\alpha \text{ are the roots}$$

Therefore the complementary solution is: $x_c(t) = C_1 + C_2 e^{-\alpha t}$

Next we define our Green's function:

$$G(t,t_0) = \begin{cases} A(t_0) + B(t_0)\, e^{-\alpha t} & 0 \le t < t_0 \\ C(t_0) + D(t_0)\, e^{-\alpha t} & t > t_0 \end{cases}$$

Apply at $t=0$ the first condition, $x(0)=0$:

$$G(0,t_0) = 0 = A(t_0) + B(t_0)$$

$$\Rightarrow B(t_0) = -A(t_0)$$

Therefore $$G(t,t_0) = \begin{cases} A(t_0)\,(1 - e^{-\alpha t}) & 0 \le t < t_0 \\ C(t_0) + D(t_0)\, e^{-\alpha t} & t > t_0 \end{cases}$$

And $$\frac{\partial G(t,t_0)}{\partial t} = \begin{cases} \alpha A(t_0)\, e^{-\alpha t} & 0 \le t < t_0 \\ -\alpha D(t_0)\, e^{-\alpha t} & t > t_0 \end{cases}$$

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Apply at $t=0$, the second condition, $\dot{x}(0)=0$, gives:

$$\frac{\partial G(0,t_0)}{\partial t} = 0 = \alpha A(t_0)$$

Assume $\alpha \neq 0$: $A(t_0) = 0 = B(t_0)$

Now $G(t,t_0) = \begin{cases} 0 & 0 \leq t < t_0 \\ C(t_0) + D(t_0)\, e^{-\alpha t} & t > t_0 \end{cases}$

Next the step condition on $\frac{\partial G}{\partial t}$ at t_0 gives:

$$\Rightarrow \quad \lim_{\varepsilon \to 0} \frac{\partial G(t_0+\varepsilon, t_0)}{\partial t} - \frac{\partial G(t_0-\varepsilon, t_0)}{\partial t} = 1$$

$$\Rightarrow \quad \lim_{\varepsilon \to 0} \frac{\partial G(t_0+\varepsilon, t_0)}{\partial t} - 0 = 1$$

$$\Rightarrow \quad \lim_{\varepsilon \to 0} -\alpha D(t_0)\, e^{-\alpha(t_0+\varepsilon)} = 1$$

$$\Rightarrow \quad -\alpha D(t_0)\, e^{-\alpha t_0} = 1$$

$$\Rightarrow \quad D(t_0) = -1/\alpha \; e^{\alpha t_0}$$

Now $G(t,t_0) = \begin{cases} 0 & 0 \leq t < t_0 \\ C(t_0) - 1/\alpha \; e^{-\alpha(t-t_0)} & t > t_0 \end{cases}$

Chapter 6 Problem 6.31

$G(t,t_0)$ is continuous at $t=t_0$:

$$c(t_0) - 1/\alpha\, e^{-\alpha \cdot 0} = 0 \Rightarrow c(t_0) = 1/\alpha$$

Therefore $G(t,t_0) = \begin{cases} 0 & 0 \le t < t_0 \\ 1/\alpha\,(1-e^{-\alpha(t-t_0)}) & t > t_0 \end{cases}$ $\quad \alpha \neq 0$

or $x(t) = \int_0^t G(t,t_0)\, f(t_0)\, dt_0$

Plug in $f(t) = Ae^{-at}$ $(t > t_0,\ a \neq \alpha)$

$$x(t) = \int_0^t 1/\alpha\,(1-e^{-\alpha(t-t_0)})\, A\, e^{-at_0}\, dt_0$$

$$= A/\alpha \int_0^t (1-e^{-\alpha(t-t_0)})\, e^{-at_0}\, dt_0$$

$$\int_0^t (1-e^{-\alpha(t-t_0)})\, e^{-at_0} dt_0 = \int_0^t e^{-at_0} dt_0 - \int_0^t e^{-\alpha(t-t_0)} e^{-at_0} dt_0$$

$$\int_0^t e^{-at_0} dt_0 = -1/a\, [e^{-at_0}]_0^t = -1/a\,(e^{-at}-1) = 1/a\,(1-e^{-at})$$

$$\int_0^t e^{-\alpha(t-t_0)} e^{-at_0} dt = e^{-\alpha t}\int_0^t e^{(\alpha-a)t_0} dt_0$$

$$= \frac{e^{-\alpha t}}{\alpha-a}\,[e^{(\alpha-a)t_0}]_0^t$$

$$= \frac{e^{-\alpha t}}{\alpha-a}\,[e^{(\alpha-a)t}-1]$$

$$\Rightarrow x(t) = \frac{A}{\alpha a}(1-e^{-at}) - \frac{A}{\alpha(\alpha-a)}\, e^{-\alpha t}(e^{(\alpha-a)t}-1) \qquad \alpha \neq 0,\ a \neq 0,\ \alpha \neq a$$