$$E_m = \int_{-\pi}^{\pi} [f(x) - \sum_{n=1}^{m} b_n \sin nx]^2 dx$$

is the residual difference (or error) of the function over its primary interval

-π to π

We minimize this quantity with respet to a particular b_n in this case b_p

we take a derivative with respect to b_p (a particular p in sum of n = 1 to m <<< IMPORTANT)

The partial derivative is moved inside integral (assume integral converges) and applied!

$$\frac{\partial E_m}{\partial b_p} = \int_{-\pi}^{\pi} 2[f(x) - \sum_{n=1}^{m} b_n \sin nx][-\sin px]dx = 0$$

As a easy derivation that most students have seen previously lets look at a linear least square or fitting a simple linear funcion to a finite number of points.

Given a set of N points (x_i, y_i) where the y_is are taking the place of the function f(x) above

Lets fit the linear function $y = b_1 + b_2 x$ with coefficents (parameters) b_1, b_2 - replaces Fourier sum

For this simple case (finite number of points i = 1 to N) the integration is just replaced with a sum so that the error is represnted by

$$E_m = \sum_{n=1}^{N} [y_i - (b_1 + b_2 x_i)]^2$$

We minimize this quantity with respet to a particular b_1, b_2

$$\frac{\partial E_m}{\partial b_1} = \frac{\partial}{\partial b_1} \sum_{n=1}^{N} [y_i - (b_1 + b_2 x_i)]^2 = \sum_{n=1}^{N} \frac{\partial}{\partial b_1} [y_i - (b_1 + b_2 x_i)]^2 = \sum_{n=1}^{N} 2[y_i - (b_1 + b_2 x_i)](-1) = 0$$

$$\frac{\partial E_m}{\partial b_2} = \frac{\partial}{\partial b_2} \sum_{n=1}^{N} [y_i - (b_1 + b_2 x_i)]^2 = \sum_{n=1}^{N} \frac{\partial}{\partial b_2} [y_i - (b_1 + b_2 x_i)]^2 = \sum_{n=2}^{N} 2[y_i - (b_1 + b_2 x_i)](-x_i) = 0$$

Up to this point very similar steps to Fourier coefficient derivation, however we can finish this a little differently since we have a finite number of points to fit and the function being fit is different!

Factor out common terms and rewrite

$$\sum_{n=1}^{N} 2[y_i - (b_1 + b_2 x_i)](-1) = 0 \rightarrow \sum_{n=1}^{N} [y_i - (b_1 + b_2 x_i)] = 0 \rightarrow -\sum_{n=1}^{N} y_i + b_1 \sum_{n=1}^{N} 1 + b_2 \sum_{n=1}^{N} x_i = 0$$

$$\sum_{n=2}^{N} 2[y_i - (b_1 + b_2 x_i)](-x_i) = 0 \rightarrow \sum_{n=2}^{N} [y_i - (b_1 + b_2 x_i)](-x_i) = 0 \rightarrow -\sum_{n=1}^{N} y_i x_i + b_1 \sum_{n=1}^{N} x_i + b_2 \sum_{n=1}^{N} x_i^2 = 0$$

Rearrange

$$b_1 \sum_{n=1}^{N} 1 + b_2 \sum_{n=1}^{N} x_i = \sum_{n=1}^{N} y_i$$

$$b_1 \sum_{n=1}^{N} x_i + b_2 \sum_{n=1}^{N} x_i^2 = \sum_{n=1}^{N} y_i x_i$$

At this point we can solve for b_1, b_2 - two equations in two unknowns - I would do this using matrix notation and use Cramers rule but any easy algebraic technique will work!

Note the sums are known values (constants) since the points (x_i, y_i) are the given points to fit line to.