Johns Hopkins Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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JOHNS HOPKINS
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Residue Theorem

Let's prove the residue theorem. Given the tools we already have it is fairly straight forward.

Suppose we want to evaluate the following integral $\oint_{\gamma} f(z)dz$

Take the general Laurent expansion $f(z)=\sum_{n=-m}^{\infty} a_n(z-z_o)^n$ with pole of order m and substitute into our integral.

$$\oint_{\gamma} \sum_{n=-m}^{\infty} a_n(z-z_o)^n dz = \sum_{n=-m}^{\infty} a_n \oint_{\gamma} (z-z_o)^n dz$$

Take as our closed patha circle centered at z_o with radius ρ

parameterized in terms of θ (as we have done previously)

$z = z_o + \rho e^{i\theta}$, $dz = i\rho e^{i\theta} d\theta$

Substitution gives

$$\sum_{n=-m}^{\infty} a_n \int_0^{2\pi} (z_o + \rho e^{i\theta} - z_o)^n i\rho e^{i\theta} d\theta = i \sum_{n=-m}^{\infty} a_n \int_0^{2\pi} (\rho e^{i\theta})^n \rho e^{i\theta} d\theta =$$

$$i \sum_{n=-m}^{\infty} a_n \rho^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$

Next evaluate the integral $\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta$ This can be done with two cases.

Case 1: $n \neq -1$

$$\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = \left.\frac{e^{i(n+1)\theta}}{i(n+1)}\right|_0^{2\pi} = \frac{e^{i(n+1)2\pi}}{i(n+1)} - \frac{e^{i(n+1)0}}{i(n+1)} = \frac{e^{i(n+1)2\pi}}{i(n+1)} - \frac{e^{0}}{i(n+1)} =$$

$$\frac{e^{i(n+1)2\pi}}{i(n+1)} - \frac{1}{i(n+1)} = \frac{1}{i(n+1)}\left[e^{i(n+1)2\pi} - 1\right] = \frac{1}{i(n+1)}\left[\cos i(n+1)2\pi + i\sin i(n+1)2\pi - 1\right] =$$

$$\frac{1}{i(n+1)}\left[1 + i(0) - 1\right] = 0 \text{ and our integral is } \oint_\gamma f(z)dz \equiv i\sum_{n=-m}^{\infty} a_n \rho^{n+1} 0 = 0$$

Case 2: n = -1

$$\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta \to \int_0^{2\pi} e^{i(-1+1)\theta}\,d\theta = \int_0^{2\pi} e^{i(0)\theta}\,d\theta = \int_0^{2\pi} d\theta = 2\pi$$

Therefore the only contribution to the sum is for n = -1 and our integral

$$\oint_\gamma f(z)dz \equiv i a_{-1}\rho^{-1+1}\int_0^{2\pi} e^{i(-1+1)\theta}\,d\theta = i a_{-1}\rho^{0}2\pi = 2\pi i a_{-1}$$

Key: Notice which coefficient is used to evaluate the integral a_{-1} the residue!

Residue theorem

For multiple poles this result is easily generalized where the R_js are the residues for each pole

$$\oint_\gamma f(z)dz = 2\pi i a_{-1} \to \boxed{\oint_\gamma f(z)dz = 2\pi i \sum_j R_j}$$

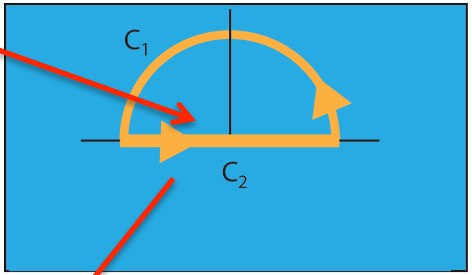
Examples of the use of Residue theorem

Let's use the residue theorem to evaluate the following general real integral

$$\int_{-\infty}^{\infty} f(x)dx$$

This can be done via the Residue theorem

Consider the integral $\oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 2\pi i \sum_j R_j$



where $C = C_1 + C_2$ and the semicircle path has radius **R**

Now for path C_2 the function is restricted to the REAL axis

therefore $\int_{C_2} f(z)dz \to \int_{-R}^{R} f(x)dx$

Key: C_2 on REAL axis therefore f(z)-> f(x) real

Substitution gives $\oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{-\mathbf{R}}^{\mathbf{R}} f(x)dx = 2\pi i \sum_j R_j$

or $\int_{-\mathbf{R}}^{\mathbf{R}} f(x)dx = 2\pi i \sum_j R_j - \int_{C_1} f(z)dz$

Now we will make an assumption for the form of f(z)so the can

evaluate the integral $\int_{C_1} f(z)dz$

Take $f(z) = \frac{p(z)}{q(z)}$ where the deg[q(z)] >deg[p(z)] +1

Key: Otherwise the Integral would diverge

Therefore $f(z) < \frac{k}{|z|^2}$ for $|z| = \mathbf{R}$ and

$$\left| \int_{C_1} f(z)dz \right| < \frac{k}{\mathbf{R}^2} \int_{C_1} dz = \frac{k}{\mathbf{R}^2}(2\pi\mathbf{R}) = \frac{2\pi k}{\mathbf{R}}$$

Substitution gives $\int_{-\mathbf{R}}^{\mathbf{R}} f(x)dx = 2\pi i \sum_j R_j - \frac{2\pi k}{\mathbf{R}}$

Finally let $\mathbf{R}$ go to ∞, that is

$$\lim_{\mathbf{R}\to\infty} \int_{-\mathbf{R}}^{\mathbf{R}} f(x)dx = 2\pi i \sum_j R_j - \lim_{\mathbf{R}\to\infty} \frac{2\pi k}{\mathbf{R}}$$

or

$\boxed{\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_j R_j - 0 = 2\pi i \sum_j R_j}$ where R_j are the residues of f(z)

Examples of the use of Residue theorem

Let's use the residue theorem to evaluate the following real integral

$$\int_0^{\infty} \frac{dx}{1+x^4}$$

Start by finding poles of $f(z)=\dfrac{1}{1+z^4}$ that is $1+z^4=0 \rightarrow z^4=-1=1(\cos\pi+i\sin\pi)$

$|z|=1(\text{real}),\ \arg z=\pi$ therefore using our process for finding roots of a complex value

$z=(-1)^{1/4}=1(\text{real})^{1/4}\cos(\dfrac{\pi+2k\pi}{4})+i\sin(\dfrac{\pi+2k\pi}{4})$ and k = 0, 1, 2, 3

This gives values for z = $e^{\frac{\pi}{4}i}$, $e^{\frac{3\pi}{4}i}$ and $e^{\frac{-3\pi}{4}i}$, $e^{\frac{-\pi}{4}i}$

We only need the first two poles as our integral only involve the half-plane!

If a function $f(z)=\dfrac{g(z)}{h(z)}$ has a simple pole at z_o then the residue is $\mathrm{Res}(z_o)=\dfrac{g(z_o)}{h'(z_o)}$

Therefore the residues for the two poles for our function $f(z)=\dfrac{g(z)}{h(z)}=\dfrac{1}{1+z^4}$ where $h'(z)=4z^3$

$$\mathrm{Res}(e^{\frac{\pi}{4}i})=\frac{1}{4\left(e^{\frac{\pi}{4}i}\right)^3}=\frac{1}{4e^{3\frac{\pi}{4}i}}=\frac{1}{4}e^{-3\frac{\pi}{4}i}=-\frac{1}{4}e^{\frac{\pi}{4}i}$$ (key: graph the angles to see the relation)

$$\mathrm{Res}(e^{\frac{3\pi}{4}i})=\frac{1}{4\left(e^{\frac{3\pi}{4}i}\right)^3}=\frac{1}{4e^{9\frac{\pi}{4}i}}=\frac{1}{4}e^{-9\frac{\pi}{4}i}=\frac{1}{4}e^{-\frac{\pi}{4}i}$$ (key: graph the angles to see the relation)

Finally $$\int_0^\infty \frac{dz}{1+z^4}=2\pi i\sum_j R_j \rightarrow 2\pi i(R_1+R_2)=2\pi i\left(-\frac{1}{4}e^{\frac{\pi}{4}i}+\frac{1}{4}e^{-\frac{\pi}{4}i}\right)=\frac{-2\pi i}{4}[e^{\frac{\pi}{4}i}-e^{-\frac{\pi}{4}i}]=\frac{-\pi i}{2}[2i\sin\frac{\pi}{4}]=$$
$$\frac{-\pi i}{2}[2i\sin\frac{\pi}{4}]=-\pi i^2\sin\frac{\pi}{4}=-\pi(-1)\frac{\sqrt{2}}{2}=\pi\frac{\sqrt{2}}{2}$$

Therefore $$\boxed{\int_0^\infty \frac{dx}{1+x^4}=\pi\frac{\sqrt{2}}{2}}$$

Example of the use of Residue theorem to perform the inverse Laplace transform

You may have noticed (or not) earlier in the course we had a Fourier transform and its inverse. One might ask what does the inverse Laplace transform look like. Its called The Bromwich integral and has the following form

$$\text{Given } L\{f(x)]=\tilde{f}(s)=\int_0^\infty e^{-sx} f(x)dx$$

then

$$f(x)=L^{-1}\{\tilde{f}(s)]=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{sx}\tilde{f}(s)ds,\ \lambda>0$$

Bromwich Integral

where s is considered a complex variable and

by the residue theorem

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{-sx}\tilde{f}(s)ds=\sum_{\text{all poles}} \mathrm{Res}[e^{-sx}f(s)]$$

Take $\tilde{f}(s) = \dfrac{s}{s^2 - k^2} = \dfrac{s}{(s+k)(s-k)}$

Now $e^{sx}\tilde{f}(s) = \dfrac{se^{sx}}{(s+k)(s-k)} = \dfrac{g(s)}{h(s)}$

where $g(s) = se^{sx}$, $h(s) = s^2 - k^2$ and $h'(s) = 2s$

The poles are at $s_o = k, -k$ and the residues are

$$R_1(k) = \frac{g(k)}{h'(k)} = \frac{ke^{kx}}{2k} = \frac{e^{kx}}{2} \text{ and } R_2(k) = \frac{g(-k)}{h'(-k)} = \frac{-ke^{-kx}}{2(-k)} = \frac{e^{-kx}}{2}$$

Therefore $\boxed{f(x) = L^{-1}\{\tilde{f}(s)] = \sum_{\text{all poles}} \mathrm{Res}[e^{-sx}f(s)] = \dfrac{e^{kx}}{2} + \dfrac{e^{-kx}}{2} \equiv \cosh kx}$