9.8 (a)

For example to get $H_2(x)$ we need to take the

second order partial derivative of G(x,h) and then set h = 0

$$G(x,h) = e^{2hx-h^2} = \sum_{n=0}^{\infty} H_n(x)h^n$$

Take partial derivative of both sides

$$\frac{\partial}{\partial h} e^{2hx-h^2} = \frac{\partial}{\partial h} \sum_{n=0}^{\infty} H_n(x)h^n$$

For the LHS $\frac{\partial}{\partial h} e^{2hx-h^2} = (2x-2h)e^{2hx-h^2}$

On the RHS $\frac{\partial}{\partial h} \sum_{n=0}^{\infty} H_n(x)h^n = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)nh^{n-1}$

For the second sum you can start the index at n=1

since for n=0 we have zero contribution to the sum!

Then also replace $\frac{n}{n!} = \frac{1}{(n-1)!}$ and we get

$$(2x-2h)e^{2hx-h^2} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_n(x)h^{n-1}$$

Now take another partial derivative

$$(-2)e^{2hx-h^2} + (2x-2h)^2 e^{2hx-h^2} = \frac{\partial}{\partial h} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_n(x)h^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_n(x)(n-1)h^{n-2}$$

As before the index can start at n = 2 and simplify the factorial

$$(-2)e^{2hx-h^2} + (2x-2h)^2 e^{2hx-h^2} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} H_n(x)h^{n-2} = \frac{1}{(2-2)!} H_2(x)h^{2-2} + \frac{1}{(3-2)!} H_3(x)h^{3-2} + \text{higher order}$$

Now set h = 0 [IMPORTANT note $0^0 = 1$, $0! = 1$]

$$(-2)e^{2(0)x-(0)^2} + (2x-2(0))^2 e^{2(0)x-(0)^2} = \frac{1}{(2-2)!} H_2(x)(0)^0 + \frac{1}{(3-2)!} H_3(x)(0)^1 + \text{higher order } = H_2(x)$$

Therefore $H_2(x) = (-2)e^{2(0)x-(0)^2} + (2x-2(0))^2 e^{2(0)x-(0)^2} = 4x^2 - 2$

(b) Before we can evaluate the integrals given we need the following

Aside: The book gives the integral relationship

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = \frac{(2n)!\sqrt{\pi}}{2^{2n} n!}$$

Unfortunately we also need the integral for odd powers of x. So lets proceed

$\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2} dx$ by parts

Let dv = $xe^{-x^2}dx$, therefore $v = -\frac{1}{2}e^{-x^2}$ and $u = x^{2n}$ therefore $du = 2nx^{2n-1}dx$

Therefore $\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2} dx = x^{2n}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(-\frac{1}{2}e^{-x^2}\right)2nx^{2n-1}dx$

The first term on the RHS is zero since for the upper or lowe bound

$\lim_{x\to\pm\infty} x^{2n}e^{-x^2} = \lim_{x\to\pm\infty} \frac{x^{2n}}{e^{x^2}}$(taking derivative top and bottom, L'Hospitals rule, 2n times gives)

$\lim_{x\to\pm\infty} \frac{1}{(2x)^{2n}e^{x^2}} = 0$

Therefore $\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2} dx = n\int_{-\infty}^{\infty} x^{2n-1} e^{-x^2} dx$

We will come back to this in a moment, first

$\int_{-\infty}^{\infty} x^1 e^{-x^2} = -\frac{1}{2}e^{-x^2}\Big|_{-\infty}^{\infty} = 0 - 0 = 0$

Then from above take n = 1 $\int_{-\infty}^{\infty} x^{2(1)+1} e^{-x^2} dx = (1)\int_{-\infty}^{\infty} x^{2(1)-1} e^{-x^2} dx$

Therefore $\int_{-\infty}^{\infty} x^3 e^{-x^2} dx = (1)\int_{-\infty}^{\infty} x^1 e^{-x^2} dx = 0$ and so on for all odd powers of x

That is $\int_{-\infty}^{\infty} x^{2n+1} e^{-x^2} dx = 0$

Back:

Back:

Now we can evaluate, eg.

(i) Set p=2, q=3 we get odd powers of x, therefore use the integral we derived

$$\int_{-\infty}^{\infty} e^{-x^2} H_2(x) H_3(x)dx = \int_{-\infty}^{\infty} e^{-x^2}(4x^2-2)(8x^3-12x)dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2}(32x^5-48x^3-16x^3+24x)dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} 32x^5-48x^3-16x^3+24x)dx$$

$$= 32\int_{-\infty}^{\infty} x^5 e^{-x^2}dx - 64\int_{-\infty}^{\infty} x^3 e^{-x^2}dx + 24\int_{-\infty}^{\infty} xe^{-x^2}dx = 0$$

All these have odd powers of x therefore all 0!

(ii) For the case p=2 and q =4 you get even powers of x therefore you have to use the formula in the book Note this leads to a zero answer also , but it is because the various integrals cancel out! Try it!