Parametrized Surfaces (Solutions)

1. Determine the surface given by the parametric representation

$$r(u,v) = u\ \mathbf{i} + u\cos v\ \mathbf{j} + u\sin v\ \mathbf{k}.$$

Solution. $y^2 + z^2 = x^2$. It is a cone that opens along x-axis.

2. Show that

a) $G(u,v) = \langle 2u+1, u-v, 3u+v\rangle$ parametrizes the plane $2x - y - z = 2$. Then calculate $G_u,\ G_v$ and $N(u,v)$.

b) $T(r,\theta) = \langle r\cos\theta, r\sin\theta, 1-r^2\rangle$ parametrizes the paraboloid $z = 1 - x^2 - y^2$. Then, calculate $T_r,\ T_\theta$ and $N(r,\theta)$.

Solution. a)

$$2(2u+1) - (u-v) - (3u+v) = 4u + 2 - u + v - 3u - v = 2,$$

$$G_u = \langle 2,1,3\rangle, \quad G_v = \langle 0,-1,1\rangle,$$
$$N = G_u \times G_v = \langle 4,-2,-2\rangle.$$

3. Find a **parametrization** for:

a) the cylinder $x^2 + y^2 = R^2$, b) the sphere of radius 4,

c) the paraboloid given by $f(x,y) = x^2 + y^2$,

d) the single cone $z = \sqrt{x^2+y^2}$,

e) the portion S of the cone with equation $x^2 + y^2 = z^2$ lying above and below the disk $x^2 + y^2 = 9$.

Solution.
a. $G(\theta,z) = \langle R\cos\theta, R\sin\theta, z\rangle, \quad 0 \le \theta < 2\pi, \quad -\infty < z < +\infty.$
b. $r(\phi,\theta) = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi\rangle, \qquad 0 \le \theta < 2\pi, \quad 0 \le \phi \le \pi.$
c. $G(x,y) = \langle x, y, x^2+y^2\rangle, \qquad -\infty \le x < \infty, \qquad -\infty < y < +\infty,$ or
$G(u,\theta) = \langle u\cos\theta, u\sin\theta, u^2\rangle, \quad -\infty < u < \infty, \quad 0 < \theta < 2\pi,$
d. $G(u,t) = \langle u\cos t, u\sin t, u\rangle, \quad 0 < u < \infty, \quad 0 \le t < 2\pi.$
e. $r(z,\theta) = \langle z\cos\theta, z\sin\theta, z\rangle, \qquad 0 \le \theta < 2\pi, \quad -3 \le z \le 3.$

4. Determine the surface area of the portion of the surface given by the following parametric equation that lies inside the cylinder $u^2 + v^2 = 1$:

$$\mathbf{r}(u, v) = \langle uv, u + v, u - v\rangle.$$

Solution.

$$r_u = \langle v, 1, 1\rangle, \quad r_v = \langle u, 1, -1\rangle,$$
$$r_u \times r_v = \langle -2, u + v, v - u\rangle.$$

Then,

$$\begin{aligned} the\ surface\ area =& \int\int_D |r_u \times r_v|\ du\ dv \\ =& \int\int_D \sqrt{4 + (u + v)^2 + (v - u)^2}\ du\ dv \\ =& \int\int_D \sqrt{4 + 2(u^2 + v^2)}\ du\ dv, \end{aligned}$$

where D is inside the disc $u^2 + v^2 = 1$.
Change the variables to polar coordinates, i.e. let

$$u = r\cos\theta, \quad v = r\sin\theta, \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

Then, the double integral in above becomes,

$$\begin{aligned} the\ surface\ area =& \int_0^{2\pi} \int_0^1 \big(\sqrt{4 + 2r^2}\big) r\ dr\ d\theta \\ =& \frac{\pi}{3}\big(\sqrt{6^3} - \sqrt{4^3}\big). \end{aligned}$$

5. Use surface integrals to find the area of part of the cylinder $x^2 + y^2 = 9$ where $y \ge 0$, between the planes $z = -1$ and $z = 2$.
Solution.
Parametrization:

$$G(z, \theta) = \langle 3\cos\theta, 3\sin\theta, z\rangle, \quad -1 \le z \le 2, \quad 0 \le \theta \le \pi.$$

$$G_z = \langle 0, 0, 1\rangle, \quad G_\theta = \langle -3\sin\theta, 3\cos\theta, 0\rangle,$$
$$G_z \times G_\theta = \langle -3\cos\theta, -3\sin\theta, 0\rangle.$$

Then,

$$\begin{aligned} surface\ area =& \int_0^{\pi} \int_{-1}^2 |G_z \times G_\theta|\ dz\ d\theta \\ =& \int_0^{\pi} \int_{-1}^2 \sqrt{9\cos^2\theta + 9\sin^2\theta + 0}\ dz\ d\theta \\ =& \int_0^{\pi} \int_{-1}^2 3\ dz\ d\theta \\ =& 9\pi. \end{aligned}$$

6. Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder $x^2 + y^2 = 12$ and above $xy-$plane.

7. Find the area of the surface

$$S: \quad x = 2u, \quad y = uv, \quad z = 1 - 2v, \quad u^2 + v^2 \leq 4.$$