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METHOD FOR THE CALCULATION OF VELOCITY, RATE OF FLOW AND VISCOUS DRAG IN ARTERIES WHEN THE PRESSURE GRADIENT IS KNOWN

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The experiments of McDonald and his co-workers (McDonald, 1952, 1955; Helps & McDonald, 1953) have shown that in the larger arteries of the rabbit and the dog there is a reversal of the flow. Measurements of the pressure gradient (Helps & McDonald, 1953) showed a phase-lag between pressure gradient and flow somewhat analogous with the phase-lag between voltage and current in a conductor carrying alternating current, and the simple mathematical treatment given below has strong similarities with the theory of the distribution of alternating current in a conductor of finite size.

Solution of the equation of motion

We consider a circular pipe of length l, radius R, filled with a viscous liquid of density ρ and viscosity μ. We shall need also the quantity $\nu=\mu/\rho$, the kinematic viscosity. To clarify what is to follow, the solution will be compared at each stage with the corresponding well-known Poiseuille solution for steady flow.

In steady flow, if p_1 and p_2 are the pressures at the ends of the pipe, the pressure-gradient is $(p_1-p_2)/l$.

If w is the longitudinal velocity of the liquid at points at a distance r from the axis of the pipe, the equation of motion of the liquid is

$$\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}+\frac{p_1-p_2}{\mu l}=0, \tag{1}$$

and its solution is
$$w=\frac{p_1-p_2}{4\mu l}(R^2-r^2),$$

which, if we write $y=r/R$, may be written

$$w=\frac{p_1-p_2}{4\mu l}R^2(1-y^2). \tag{2}$$

If now the pressure-gradient $(p_1-p_2)/l$ is not constant, equation (1) has a term $\frac{1}{\nu}\frac{\partial w}{\partial t}$ on the right-hand side. We consider a pressure gradient

$$\frac{p_1-p_2}{l}=A\,\mathrm{e}^{int}, \tag{3}$$

which is periodic in the time with a frequency

$$f=n/2\pi,$$

since the pulse is a periodic phenomenon, and any function which is periodic in the time can be expressed as the sum of a series of terms of this form. The equation of motion becomes

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{1}{\nu}\frac{\partial w}{\partial t}=-\frac{A}{\mu}\,\mathrm{e}^{int}, \tag{4}$$

and if we now write

$$w=u\,\mathrm{e}^{int}, \tag{5}$$

where u is a function of r alone, the equation for u is

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}-\frac{in}{\nu}u=-\frac{A}{\mu}. \tag{6}$$

If we write this equation in the form

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}+\frac{i^3n}{\nu}\,u=\frac{-A}{\mu} \tag{7}$$

(remembering that $-1=i^2$), its solution may be written

$$u=+\frac{A}{\rho}\frac{1}{in}\left\{1-\frac{J_0\left(r\sqrt{\frac{n}{\nu}}\,i^{\frac{3}{2}}\right)}{J_0\left(R\sqrt{\frac{n}{\nu}}\,i^{\frac{3}{2}}\right)}\right\}, \tag{8}$$

where $J_0(xi^{\frac{3}{2}})$ is a Bessel function of order zero and complex argument which is well known and arises in problems connected with the distribution of current in conductors of finite size. The quantity $R\sqrt{(n/\nu)}$ is a non-dimensional parameter. We shall write

$$R\sqrt{\frac{n}{\nu}}=\alpha.$$

If we also write $r/R=y$, then the velocity is given by

$$w=+\frac{A}{\rho}\frac{1}{in}\left\{1-\frac{J_0(\alpha y i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right\}\mathrm{e}^{int}. \tag{9}$$

This is still in complex form. If now we were to take as pressure gradient the real part of $A\,\mathrm{e}^{int}$, the corresponding flow would be the real part of (9).

A formula, essentially the same as the real part of (9) when A is real, was derived by Lambossy (1952) who also gave a form for the viscous drag, by separating $J_0(\alpha i^{\frac{3}{2}})$ into its real and imaginary parts. The conventional separation into real and imaginary parts leads to a very clumsy form for the results, and it is more convenient to express the results in terms of modulus and phase.

Tables of $J_0(xi^{\frac{3}{2}})$ are available in the form

$$J_0(xi^{\frac{3}{2}}) = M_0(x)\, e^{i\theta_0(x)},$$

where M_0 and θ_0 both vary with x, and by using these we are led at once to the amplitude and phase of the motion.

We write

$$\left.\begin{aligned} J_0(\alpha y i^{\frac{3}{2}}) &= M_0(y)\, e^{i\theta_0(y)}, \\ J_0(\alpha i^{\frac{3}{2}}) &= M_0\, e^{i\theta_0}. \end{aligned}\right\} \tag{10}$$

Then if the real part of $A\, e^{int}$ is $M \cos(nt+\phi)$ the corresponding velocity is

$$w = \frac{M}{\rho}\frac{1}{n}\left\{\sin(nt+\phi) - \frac{M_0(y)}{M_0}\sin(nt+\phi-\delta_0)\right\}, \tag{11}$$

where

$$\delta_0 = \theta_0 - \theta_0(y). \tag{12}$$

Tables of $M_0(y)$ and $\theta_0(y)$ are given by McLachlan (1941), and in a slightly different form by Jahnke & Emde (1938).

Equation (11) may be put in terms of a single phase-relationship, more suitable for calculation. If we write

$$h_0 = \frac{M_0(y)}{M_0}, \tag{13}$$

and define M_0' and ϵ_0 by the following

$$M_0' = \sqrt{(1 + h_0^2 - 2h_0 \cos\delta_0)}, \tag{14}$$

$$\tan\epsilon_0 = \frac{h_0 \sin\delta_0}{1 - h_0\cos\delta_0}, \tag{15}$$

then

$$w = \frac{M}{\rho}\frac{1}{n} M_0' \sin(nt+\phi+\epsilon_0). \tag{16}$$

To compare this with the steady-flow result, we use the relation $\alpha = R\sqrt{(n/\nu)}$, and in (16) write

$$\frac{1}{n} = \frac{1}{\nu}\frac{R^2}{\alpha^2} = \frac{\rho}{\mu}\frac{R^2}{\alpha^2}.$$

Then $\frac{1}{n}\frac{M}{\rho} = \frac{M}{\mu}\frac{R^2}{\alpha^2}$ and the expression for w becomes

$$w = \frac{M}{\mu}\frac{R^2}{\alpha^2} M_0' \sin(nt+\phi+\epsilon_0). \tag{17}$$

The quantity M_0'/α^2 takes the place of $\frac{1}{4}(1-y^2)$ in the formula for steady flow (equation (2)). It must also be noted that ϵ_0 varies with y, and therefore the phase varies across the pipe. The degree of departure from the normal parabolic form increases with α, i.e. with the frequency. The effect has a certain similarity with the 'skin effect' in electrical conductors, but the analogy cannot be pressed too closely because of the difference in surface conditions.

The rate of flow, i.e. the quantity of liquid passing through any cross-section per unit time, is given by

$$Q=2\pi\int_0^R wr\,\mathrm{d}r. \tag{18}$$

Writing $y=r/R$ this becomes

$$Q=2\pi R^2\int_0^1 wy\,\mathrm{d}y. \tag{19}$$

For steady flow
$$w=\frac{p_1-p_2}{\mu l}\frac{R^2}{4}(1-y^2),$$

and
$$Q=\frac{p_1-p_2}{4\mu l}\pi R^4\int_0^1(1-y^2)\,2y\,\mathrm{d}y=\frac{p_1-p_2}{8\mu l}\pi R^4, \tag{20}$$

which is Poiseuille's formula. If in (19) we substitute for w its value from (9) we have

$$Q=\frac{2\pi}{\rho}\frac{A}{in}\left\{\frac{R^2}{2}-\frac{R^2}{J_0(\alpha i^{\frac{3}{2}})}\int_0^1 J_0(\alpha y i^{\frac{3}{2}})\,y\,\mathrm{d}y\right\}\mathrm{e}^{int}. \tag{21}$$

Now $\int xJ_0(x)\,\mathrm{d}x=xJ_1(x)$ from known properties of Bessel functions and therefore

$$Q=\frac{\pi R^2}{\rho}\frac{A}{in}\left\{1-\frac{2\alpha i^{\frac{3}{2}}}{i^3\alpha^2}\frac{J_1(\alpha i^{\frac{3}{2}})}{J_0(\alpha i^{\frac{3}{2}})}\right\}\mathrm{e}^{int}. \tag{22}$$

Writing this again in terms of modulus and phase, if

$$J_1(\alpha i^{\frac{3}{2}})=M_1\mathrm{e}^{i\theta_1},$$

we may write (22) in the form

$$Q=\frac{\pi R^4}{\mu}\frac{M}{\alpha^2}\left\{\sin(nt+\phi)-\frac{2M_1}{\alpha M_0}\sin(nt+\phi-\delta_{10})\right\}, \tag{23}$$

where
$$\delta_{10}=135^\circ-\theta_1+\theta_0. \tag{24}$$

Tables of M_1 and θ_1 are given by McLachlan (1941), but a more convenient table is available in Jahnke & Emde (1938). Jahnke & Emde give a table of $\alpha M_0/2M_1$ and also of what they call

$$\beta_0-\beta_1=-\tfrac{1}{4}\pi-\delta_{10}.$$

It should be pointed out that this table (of $\beta_0-\beta_1$) is tabulated in decimals of a right-angle. To convert these values to degrees the tabular values should be multiplied by 90.

In the same way as for the velocity, the formula for the rate of flow may be reduced to a single-phase relationship. Since $\alpha M_0/2M_1$ is tabulated we write $\alpha M_0/2M_1 = k$. Then, in the same manner as before, defining

$$M'_{10} = \frac{1}{k}\sqrt{[\sin^2\delta_{10} + (k - \cos\delta_{10})^2]},$$

and
$$\tan\epsilon_{10} = \frac{\sin\delta_{10}}{k - \cos\delta_{10}},$$

we have
$$Q = \frac{\pi R^4}{\mu} M \frac{M'_{10}}{\alpha^2}\sin(nt + \phi + \epsilon_{10}). \tag{25}$$

The viscous drag on a cylinder of radius r is

$$F = -2\pi\mu r\frac{dw}{dr},$$

and since
$$\frac{d}{dx}[J_0(x)] = -J_1(x)$$

this can be expressed in terms of M_1, M_0, θ_1 and θ_0. Following exactly the same method as before, it reduces to

$$F = \pi M\frac{R^2}{\alpha^2}\alpha y\frac{M_1(y)}{M_0}\cos\{nt + \phi - \delta_1(y)\},$$

where
$$\delta_1(y) = 135^\circ - \theta_1(y) + \theta_0.$$

At $r = R$, where F is a maximum, the drag at the surface of the pipe is

$$F_{\max} = \pi MR^2\frac{2M_1}{\alpha M_0}\cos\{nt + \phi - \delta_{10}\}. \tag{26}$$

Numerical calculation of rate of flow from observed pressure gradient

McDonald (1955) obtained the pressure gradients corresponding to his average velocity measurements by direct difference between the readings of two manometers. One of his curves for the pressure gradient in the femoral artery of the dog is shown in Fig. 1.

The first step in the calculation of the corresponding rate of flow is to represent this pressure gradient as a Fourier series. If T is the pulse-time (in this case $\frac{1}{3}$ sec) we write $x = 2\pi\frac{t}{T}$ so that x, measured in degrees, runs from zero to 360° during one pulse period. The coefficients of the Fourier series were computed by direct summation, twenty-four ordinates being taken, 15° apart. The coefficients up to the sixth harmonic are shown in Table 1.

These were next converted to modulus and phase form, the results being given in the two right-hand columns.

The corresponding flow will be the sum of six terms of the form

$$Q=\frac{\pi MR^4}{\mu}\frac{M'_{10}}{\alpha^2}\sin(mx+\phi+\epsilon_{10}), \tag{27}$$

where m has the values 1, 2, ..., 6.

TABLE 1. Fourier components of pressure-gradient curve shown in Fig. 1

m	cosine term	sine term	M_m	ϕ_m
1	+0·8781	−0·7432	+1·1050	+40° 14′
2	+0·5415	+1·4327	+1·5316	−69° 17′
3	−0·7946	+0·5508	−0·9668	+34° 44′
4	−0·2375	−0·1588	−0·2857	−33° 47′
5	+0·0125	−0·2818	+0·2821	+87° 31′
6	−0·1917	−0·0167	−0·1924	− 4° 58′

The second step is the calculation of the value of α for each harmonic, and the corresponding values of M'_{10} and ϵ. The following figures have been used: diameter of artery, $2R=0\cdot3$ cm; viscosity, $\mu=0\cdot04$P; density, $\rho=1\cdot05$ g/ml.; pulse frequency, $f=3$ per sec. Hence $n=6\pi$ sec^{-1} for the fundamental, and

$$\alpha=0\cdot15\sqrt{\frac{6\pi\times1\cdot05}{0\cdot04}}=3\cdot337.$$

The values of α for the higher harmonics are obtained from this by simple multiplication. The corresponding values of M'_{10} and ϵ_{10} can be read from Table 3, which can be interpolated by proportional parts. It covers the values of α from $\alpha=0$ to $\alpha=10$. For values of $\alpha>10$ the author has derived the following asymptotic expressions:

$$\frac{M'_{10}}{\alpha^2}\doteqdot\frac{1}{\alpha^2}-\frac{\sqrt{2}}{\alpha^3}+\frac{1}{\alpha^4}, \tag{28}$$

$$\epsilon_{10}\,(\text{degrees})\doteqdot57\cdot296\left(\frac{\sqrt{2}}{\alpha}+\frac{1}{\alpha^2}+\frac{19}{24\sqrt{2}\,\alpha^3}\right). \tag{29}$$

Table 2 gives the values of α required for the pressure-gradient curve shown in Fig. 1 together with the corresponding values of M'_{10} and ϵ.

As pressures are normally measured in mm Hg, the conversion factor to dyne/cm^2 is included in the constant $\pi R^4/\mu$, which then becomes 53·05 for this example. The expression for Q is

$$\begin{aligned}Q=&\,3\cdot56\sin(x+71°\,13')\\&+2\cdot71\sin(2x-49°\,10')\\&-1\cdot20\sin(3x+50°\,33')\\&-0\cdot28\sin(4x-20°\,17')\\&+0\cdot22\sin(5x+99°\,28')\\&-0\cdot13\sin(6x+5°\,47').\end{aligned}$$

This has been plotted on Fig. 1 to show its phase-relationship to the pressure. The agreement with the rate of flow deduced from direct observations of the average velocity is not perfect, but, as will be seen from the results of similar calculations shown in McDonald's paper (1955), it is surprisingly good when

TABLE 2. Values of α, M'_{10} and ϵ_{10} for the Fourier components of the pressure-gradient curve shown in Fig. 1

m	α	α^2	M'_{10}	ϵ_{10}
1	3·34	11·13	0·6551	30° 59′
2	4·72	22·27	0·7436	19° 57′
3	5·78	33·40	0·7839	15° 49′
4	6·67	44·53	0·8096	13° 30′
5	7·46	55·67	0·8278	11° 57′
6	8·17	66·80	0·8416	10° 45′

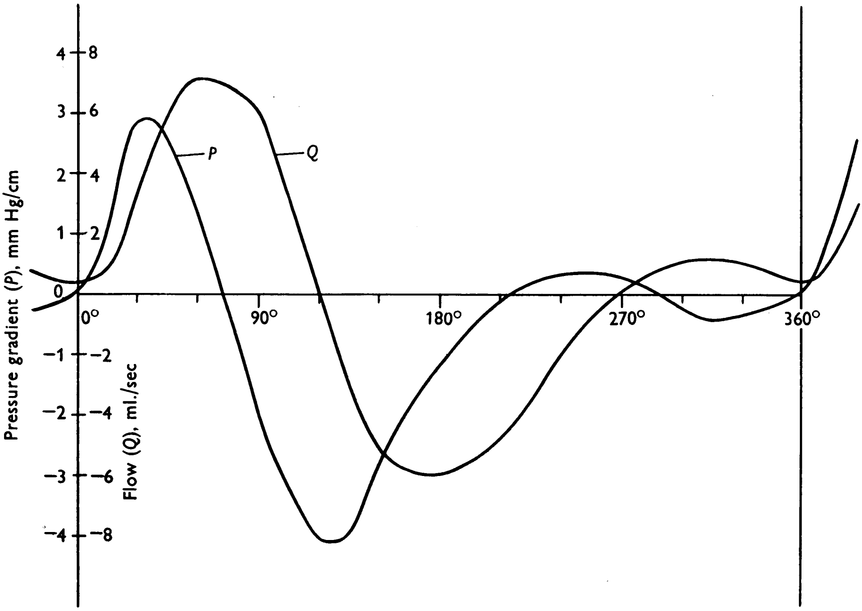

Fig. 1. Relation of flow (Q) to the pressure gradient (P) in the femoral artery of a dog. The equations for the curves are given in the text. The flow curve does not include the steady flow term.

it is recalled that the pressure determination was not made at the same time as the velocity determinations. Moreover, the plotting of the pressure-gradient curve by taking small differences between separate pressure determinations is subject to error.

The above method of setting out the calculation is the most suitable for demonstrating the way in which the components of the pressure gradient have to be modified, in amplitude and phase, to obtain the corresponding rate of flow. If the calculation is to be made as a routine, it is quicker to work in

TABLE 3. M'_{10}/α^2 and ϵ_{10} tabulated for values of α from 0 to 10.

α	M'_{10}/α^2	ϵ_{10}	α	M'_{10}/α^2	ϵ_{10}	α	M'_{10}/α^2	ϵ_{10}	α	M'_{10}/α^2	ϵ_{10}
0·00	0·1250	90·00	2·50	0·0855	44·93	5·00	0·0302	18·65	7·50	0·0147	11·87
·05	·1250	89·98	2·55	·0837	43·88	5·05	·0297	18·43	7·55	·0146	11·78
·10	·1250	89·90	2·60	·0819	42·86	5·10	·0292	18·23	7·60	·0144	11·70
·15	·1250	89·79	2·65	·0802	41·86	5·15	·0287	18·02	7·65	·0142	11·61
·20	·1250	89·62	2·70	·0784	40·90	5·20	·0282	17·83	7·70	·0140	11·53
0·25	0·1250	89·40	2·75	0·0767	39·96	5·25	0·0278	17·63	7·75	0·0139	11·45
·30	·1250	89·14	2·80	·0750	39·05	5·30	·0273	17·44	7·80	·0137	11·37
·35	·1250	88·83	2·85	·0734	38·17	5·35	·0269	17·26	7·85	·0136	11·29
·40	·1250	88·47	2·90	·0717	37·32	5·40	·0264	17·08	7·90	·0134	11·21
·45	·1249	88·07	2·95	·0701	36·50	5·45	·0260	16·90	7·95	·0133	11·14
0·50	0·1249	87·61	3·00	0·0685	35·70	5·50	0·0256	16·73	8·00	0·0131	11·06
·55	·1248	87·11	3·05	·0670	34·93	5·55	·0252	16·56	8·05	·0130	10·98
·60	·1248	86·57	3·10	·0655	34·18	5·60	·0248	16·39	8·10	·0128	10·91
·65	·1247	85·97	3·15	·0640	33·46	5·65	·0244	16·23	8·15	·0127	10·84
·70	·1246	85·33	3·20	·0626	32·77	5·70	·0240	16·07	8·20	·0125	10·77
0·75	0·1244	84·65	3·25	0·0612	32·09	5·75	0·0237	15·91	8·25	0·0124	10·70
·80	·1243	83·91	3·30	·0598	31·45	5·80	·0233	15·76	8·30	·0122	10·63
·85	·1240	83·14	3·35	·0585	30·82	5·85	·0230	15·61	8·35	·0121	10·56
·90	·1238	82·32	3·40	·0572	30·22	5·90	·0226	15·46	8·40	·0120	10·49
·95	·1235	81·45	3·45	·0559	29·64	5·95	·0223	15·32	8·45	·0119	10·42
1·00	0·1232	80·55	3·50	0·0547	29·08	6·00	0·0220	15·18	8·50	0·0117	10·36
1·05	·1228	79·60	3·55	·0535	28·53	6·05	·0216	15·04	8·55	·0116	10·29
1·10	·1224	78·61	3·60	·0523	28·01	6·10	·0213	14·90	8·60	·0115	10·22
1·15	·1219	77·59	3·65	·0512	27·51	6·15	·0210	14·77	8·65	·0114	10·16
1·20	·1213	76·53	3·70	·0501	27·02	6·20	·0207	14·63	8·70	·0112	10·10
1·25	0·1207	75·44	3·75	0·0490	26·55	6·25	0·0204	14·50	8·75	0·0111	10·04
1·30	·1200	74·31	3·80	·0480	26·10	6·30	·0201	14·38	8·80	·0110	9·97
1·35	·1193	73·16	3·85	·0470	25·66	6·35	·0199	14·25	8·85	·0109	9·91
1·40	·1185	71·98	3·90	·0460	25·24	6·40	·0196	14·13	8·90	·0108	9·85
1·45	·1176	70·77	3·95	·0451	24·83	6·45	·0193	14·01	8·95	·0107	9·79
1·50	0·1166	69·54	4·00	0·0441	24·43	6·50	0·0191	13·89	9·00	0·0106	9·73
1·55	·1156	68·30	4·05	·0432	24·05	6·55	·0188	13·77	9·05	·0104	9·68
1·60	·1144	67·03	4·10	·0424	23·68	6·60	·0185	13·66	9·10	·0103	9·62
1·65	·1133	65·76	4·15	·0415	23·32	6·65	·0183	13·54	9·15	·0102	9·56
1·70	·1120	64·47	4·20	·0407	22·98	6·70	·0181	13·43	9·20	·0101	9·51
1·75	0·1107	63·18	4·25	0·0399	22·64	6·75	0·0178	13·32	9·25	0·0100	9·45
1·80	·1093	61·89	4·30	·0391	22·32	6·80	·0176	13·21	9·30	·0099	9·40
1·85	·1078	60·59	4·35	·0384	22·00	6·85	·0173	13·11	9·35	·0098	9·34
1·90	·1063	59·30	4·40	·0376	21·70	6·90	·0171	13·00	9·40	·0097	9·29
1·95	·1047	58·02	4·45	·0369	21·40	6·95	·0169	12·90	9·45	·0096	9·24
2·00	0·1031	56·74	4·50	0·0362	21·11	7·00	0·0167	12·80	9·50	0·0096	9·18
2·05	·1015	55·47	4·55	·0355	20·84	7·05	·0165	12·70	9·55	·0095	9·13
2·10	·0998	54·22	4·60	·0349	20·56	7·10	·0163	12·60	9·60	·0094	9·08
2·15	·0980	52·98	4·65	·0342	20·30	7·15	·0161	12·50	9·65	·0093	9·03
2·20	·0963	51·77	4·70	·0336	20·05	7·20	·0159	12·41	9·70	·0092	8·98
2·25	0·0945	50·57	4·75	0·0330	19·80	7·25	0·0157	12·31	9·75	0·0091	8·93
2·30	·0927	49·39	4·80	·0324	19·55	7·30	·0155	12·22	9·80	·0090	8·88
2·35	·0909	48·24	4·85	·0319	19·32	7·35	·0153	12·13	9·85	·0089	8·84
2·40	·0891	47·11	4·90	·0313	19·09	7·40	·0151	12·04	9·90	·0088	8·79
2·45	·0873	46·01	4·95	·0308	18·86	7·45	·0149	11·95	9·95	·0088	8·74
2·50	0·0855	44·93	5·00	0·0302	18·65	7·50	0·0147	11·87	10·00	0·0087	8·69

terms of the usual Fourier coefficients. Thus, if we write for the mth component of the pressure gradient

$$A_m \cos mx + B_m \sin mx.$$

Then $$A_m = M_m \cos \phi_m, \quad -B_m = M_m \sin \phi_m,$$

and if we expand (27) it becomes

$$\frac{\pi R^4}{\mu} \frac{M'_{10}}{\alpha^2} (A_m \sin \epsilon_{10} - B_m \cos \epsilon_{10}) \cos mx$$
$$+ \frac{\pi R^4}{\mu} \frac{M'_{10}}{\alpha^2} (A_m \cos \epsilon_{10} + B_m \cos \epsilon_{10}) \sin mx, \qquad (30)$$

and this is the form recommended for use.

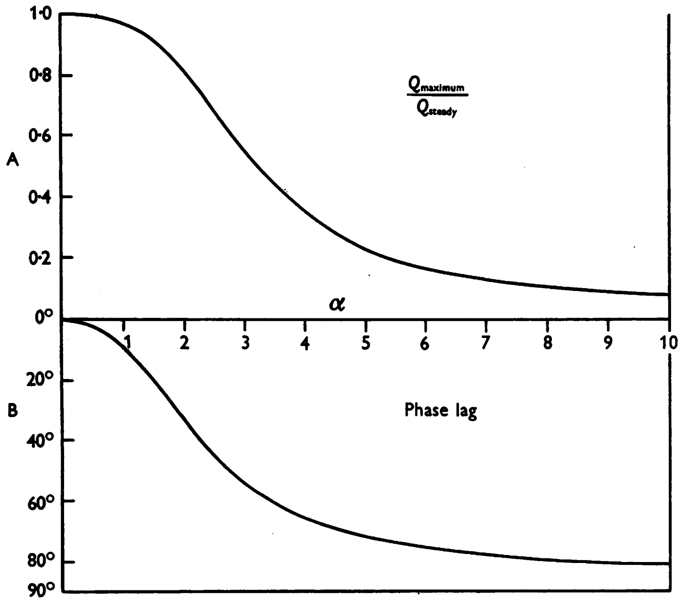

Fig. 2. The effect of changes in the non-dimensional constant α on: (A) the ratio of the maximum flow due to a given oscillating pressure to the corresponding steady, or Poiseuille, flow; and (B) the phase lag between the oscillating pressure and the flow generated.

It is appropriate at this point to consider how M'_{10} and ϵ_{10} vary as α changes. Fig. 2 gives a graphical indication of this. The lower curve (B) is a graph of the phase lag (i.e. of $90° - \epsilon_{10}$) against α. The phase lag tends to zero with frequency, as would be expected, but moves very slowly towards its asymptotic value of 90° at high values of α.

The upper curve shows the ratio of the maximum of the oscillatory flow to the Poiseuille flow for the same pressure gradient. This is obtained by dividing (27) by $\dfrac{M_m \pi R^4}{8\mu}$, and taking the maximum value, i.e. putting

$$\sin (mx + \phi_m + \epsilon_{10}) = 1.$$

This gives
$$\frac{Q_{\text{max}}}{Q_{\text{steady}}}=\frac{8M'_{10}}{\alpha^2}, \tag{31}$$

and it is this ratio which is plotted. As $\alpha \to 0$, $M'_{10} \to \frac{1}{8}\alpha^2$ so that $M'_{10}/\alpha^2 \to \frac{1}{8}$ and the flow at small values of α is the same as given by Poiseuille's formula. Above $\alpha=1$ it falls off sharply, until at $\alpha=10$ it is only about one-fifteenth of the corresponding Poiseuille flow. It will be seen also from this curve that the values of α which are used in the calculation of flow in the femoral artery fall in the range in which the variation with α is greatest.

This wide variation in the maximum rate of flow for different values of α raises the question: how much is α likely to vary in different animals? Taking the diameter of the human femoral artery as 0·5 cm, the pulse rate as 72 per min, and the same viscosity as for the dog,

$$\alpha=0{\cdot}25\sqrt{\frac{2\pi\times 72\times 1{\cdot}05}{60\times 0{\cdot}04}}=3{\cdot}52.$$

The corresponding values of α for the rabbit and the cat have also been calculated, and are of about the same magnitude. This indicates kinematical similarity in arterial flow in all these animals, and shows that the fluctuating flow in the great arteries in these experimental animals and in man has the same form, and differs only in scale.

The nature of the approximations and the possibility of measurement in living subjects

The simple theory outlined above contains two very drastic assumptions—the artery is regarded as a rigid tube, arterial expansion being neglected, and the pressure gradient is assumed to be a function of the time only, whereas it is generated by a pulse wave of finite velocity.

To a first approximation consider the pulse wave to be a wave travelling without distortion with velocity c. Then the pressure will have the form

$$p=p_0+f\left(t-\frac{z}{c}\right),$$

where z is distance along the artery. The first point to be observed is that in these circumstances
$$\frac{\partial p}{\partial z}=-\frac{1}{c}\frac{\partial p}{\partial t},$$

and therefore a good approximation to the pressure gradient over a short length of artery would be obtained by measuring the time derivative of the pressure at that point. An electrical manometer fitted with a time-differentiating input thus provides a direct record of the pressure gradient; and, if required, the flow at any point in the larger arteries of a human subject could be predicted, provided that the pulse could be observed at that point and the pulse-wave velocity were known. This procedure has been adopted by McDonald (1955).

If a complete analysis is attempted, taking into account the finite velocity of the pressure wave, the single equation (4) is no longer adequate, since there will be a radial component of velocity as well as a longitudinal one. It is true, however, that so long as the maximum velocity of the liquid is a small fraction of the wave velocity, the approximation is reasonably good.

A more detailed study of the more general equations is in preparation. It has been found that it is possible to consider this question, and that of arterial expansion, together, as parts of one general problem.

Note on the computation of Fourier components

If $f(x)$ is a periodic function defined at 24 equally spaced points (i.e. 15° apart if the full period is taken as 360°) with observed values f_r, $r=0, 1, 2, \ldots, 23$, then

$$f(x)=A_0+\sum_{m=1}^{m=12} A_m \cos mx+\sum_{m=1}^{m=12} B_m \sin mx,$$

where

$$A_0=\tfrac{1}{24}\sum_{r=0}^{r=23} f_r,$$

$$A_m=\tfrac{1}{12}\sum_{r=0}^{r=23} f_r \cos mr\times 15°,$$

$$B_m=\tfrac{1}{12}\sum_{r=0}^{r=23} f_r \sin mr\times 15°;$$

and the modulus and phase are

$$M_m=\sqrt{(A_m^2+B_m^2)},$$

$$\phi_m=-\tan^{-1} B_m/A_m.$$

SUMMARY

1. An exact solution of the equations of viscous fluid motion is given for the motion of a liquid in a circular tube under a pressure gradient which is a periodic function of the time. It is shown that there is a phase-lag between the motion of the liquid and the pressure gradient which causes it. Formulae are also given for the rate of flow and the viscous drag.

2. The calculation of the rate of flow from an observed pressure gradient is described, and tables are given to facilitate the calculation.

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