14.3

(a)

$$a_n = \frac{1}{\ln n}$$

Use equation 14.13

$$\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$$

Substitution gives

$$\lim_{n\to\infty} \left|\frac{1}{\ln n}\right|^{1/n} = \lim_{n\to\infty} \frac{1}{|\ln n|^{1/n}}$$

Now for large n $\lim\limits_{n\to\infty} |\ln n|^{1/n} = \lim\limits_{n\to\infty} (\ln n)^{1/n} = \lim\limits_{n\to\infty} \left[e^{\ln(\ln n)}\right]^{1/n}$

$$= \lim_{n\to\infty} e^{\frac{1}{n}\ln(\ln n)} = e^0 = 1$$

Therefore R = 1 and therefore converges for $-1 < z < 1$

Now at z = -1 obviously still converges

However at z = 1 we have

$$\sum_{n=2}^{\infty} \frac{1^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

Comparison with $\sum\limits_{n=2}^{\infty} \frac{1}{n}$ which diverges (see class notes) show that $\sum\limits_{n=2}^{\infty} \frac{1}{\ln n}$ diverges

Therefore covergence for $-1 \le z < 1$

(c)

$a_n = n^{\ln n}$

Use equation 14.13

$$\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$$

Substitution gives

$$\lim_{n\to\infty} \left|n^{\ln n}\right|^{1/n}$$

Now for large n $\lim_{n\to\infty} \left|n^{\ln n}\right|^{1/n} = \lim_{n\to\infty} n^{\frac{1}{n}\ln n} = \lim_{n\to\infty} \left[e^{\ln n}\right]^{\frac{1}{n}(\ln n)} = \lim_{n\to\infty} e^{\frac{1}{n}(\ln n)^2}$

Look at $\lim_{n\to\infty} \frac{1}{n}(\ln n)^2$ using L'H rule

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{2(\ln n)\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2(\ln n)}{n} = \lim_{n\to\infty} \frac{2\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{2}{n} = 0$$

Therefore

$\lim_{n\to\infty} e^{\frac{1}{n}(\ln n)^2} = \lim_{n\to\infty} e^0 = 1$ and R = 1 and therefore converges for $-1 < z < 1$

However at z = 1 we have

$\sum_{n=2}^{\infty} n^{\ln n}$ which obviously diverges