Professor Rio
EN.585.615.81.SP21 Mathematical Methods
Final Exam
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Question 1

a. $f(x) = x$ is odd on $[-\pi, \pi]$ therefore its Fourier coefficients a_n are 0 and we need to find its b_n coefficients:

$$\begin{aligned}
b_n &= \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x) \sin(\frac{2\pi n x}{2\pi}) dx \\
&= \frac{4}{2\pi} \int_{0}^{\pi} x \sin(\frac{2\pi n x}{2\pi}) dx \\
&= \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) dx
\end{aligned}$$

Using integration by parts:

$$\begin{aligned}
\int_0^{\pi} x \sin(nx) dx &= [x(-\frac{\cos(nx)}{n})]_0^{\pi} + \int_0^{\pi} 1 \cdot \frac{\cos(nx)}{n} dx \\
&= (-\frac{\pi}{n}) \cos(n\pi) + \frac{1}{n} [\sin(nx)]_0^{\pi} \\
&= \frac{(-1)^{n+1}\pi}{n}
\end{aligned}$$

Thus $b_n = \frac{2}{\pi} \frac{(-1)^{n+1}\pi}{n} = \frac{(-1)^{n+1}2}{n}$ and the Fourier series of x, on $[-\pi, \pi]$, is:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nx)}{n}$$

b. If we integrate terms by terms the previous expression, the Fourier series of x over $[-\pi, \pi]$, we have:

$$\begin{aligned}
\frac{x^2}{2} &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-\frac{\cos(nx)}{n}) + c \quad c \text{constant of integration} \\
x^2 &= 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx) + c \text{ with } 2c \to c \\
&= c + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)
\end{aligned}$$

c. $f(x) = x^2$ is an even function, by Fourier Series for even function over symmetric range, we have:

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n x}{2\pi}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \ (1)$$

where

$$
\begin{aligned}
a_0 &= \frac{4}{2\pi}\int_0^{\pi} x^2 dx \\
&= \frac{2}{\pi}[\frac{x^3}{3}]_0^{\pi} \\
&= \frac{2}{3}\pi^2
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{4}{2\pi}\int_0^{\pi} x^2 \cos{(\frac{2\pi n x}{2\pi})} dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx \\
\int_0^{\pi} x^2\cos(nx)dx &= [x^2\ \frac{\sin(nx)}{n}]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin(nx)dx \\
&= 0 - \frac{2}{n}\frac{(-1)^{n+1}\pi}{n} \\
a_n &= \frac{2}{\pi}\frac{(-1)^n 2\ \pi}{n^2} \\
&= (-1)^n\frac{4}{n^2}
\end{aligned}
$$

Substituting for a_n in (1):

$$
x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)
$$

d. Fourier series of x^2 using integration terms by terms or calculating directly match, as required, by taking $c = \frac{\pi^2}{3}$ since x is a piecewise smooth function on the specified range.

Question 2

Consider the differential equation:

$$
z\frac{d^2y}{dy^2} + y = 0
$$

a. We put the equation in standard form:

$$
\frac{d^2y}{dy^2} + \frac{1}{z}y = 0
$$

$z\ p(z) = 0$ and $z^2q(z) = z$ therefore 0 is a regular singular point.

b. Take $y = z^{\sigma}\sum_{n=0}^{\infty} a_n z^n$ and the usual derivatives in the D.E. gives by substitution

$$
\begin{aligned}
z\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0 \\
\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-1} + \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0\ (1)
\end{aligned}
$$

Take the term with the lowest power of z, which is the first sum with $n = 0$, then since each power of z term must be equal to 0, we have

$$\sigma(\sigma - 1)a_0 z^{\sigma-1} = 0$$

Since $a_0 \neq 0$ and $z^{\sigma-1} \neq 0$, therefore $\sigma = 0, 1$.

c. We go back to equation (1) and take $\sigma = 1$ yields

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

Then reindex the second sum to get same power of z in both sums:

$$\sum_{n=0}^{\infty} n(n+1)a_n z^n + \sum_{n=1}^{\infty} a_{n-1} z^n = 0$$

Note, in first term $n = 0$ does not contribute so we can start index at $n = 1$ in the first sum, and combine both sums

$$\sum_{n=1}^{\infty} [n(n+1)a_n + a_{n-1}] z^n = 0$$

Since every power of z term must be 0 and $z^n \neq 0$, gives:

$$a_n = -\frac{1}{(n+1)n} a_{n-1}$$

Taking $a_0 = 1$, now

$$\begin{aligned}
n = 1\ a_1 &= -\frac{1}{2\ 1} a_0 = -\frac{1}{2\ 1} = \frac{(-1)^1}{2\ 1} \\
n = 2\ a_2 &= -\frac{1}{3\ 2} a_1 = \frac{1}{3\ 2\ 2\ 1} = \frac{(-1)^2}{(3\ 2\ 1)\ (2\ 1)} \\
n = 3\ a_3 &= -\frac{1}{4\ 3} a_2 = -\frac{1}{4\ 3\ 3\ 2\ 2\ 1} = \frac{(-1)^3}{(4\ 3\ 2\ 1)\ (3\ 2\ 1)} \\
&\vdots \\
a_n &= -\frac{1}{(n+1)n} a_{n-1} = \cdots = \frac{(-1)^n}{((n+1)\ n \cdots 1)\ (n\ (n-1) \cdots 1)} = \frac{(-1)^n}{(n+1)!n!}
\end{aligned}$$

Therefore one of the independent solution of the ODE is

$$y_1(z) = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!n!} z^n$$

Question 3

a. We have

$$
\begin{aligned}
&n=0,\ M=0,\ P_0(x)=\frac{(-1)^0(2\ 0-2\ 0)!}{2^0(0-0)!(0-2\ 0)!}x^{0-2\ 0}=1\\
&n=1,\ M=\frac{1-1}{2}=0,\ P_1(x)=\frac{(-1)^0(2\ 1-2\ 0)!}{2^1(1-0)!(1-2\ 0)!}x^{1-2\ 0}=\frac{1\ 2}{2\ 1!\ 1!}x^1=x\\
&n=2,\ M=\frac{2}{2}=1,\ P_2(x)=\frac{(-1)^0(2\ 2-2\ 0)!}{2^2(2-0)!(2-2\ 0)!}x^{2-2\ 0}+\frac{(-1)^1(2\ 2-2\ 1)!}{2^2(2-1)!(2-2\ 1)!}x^{2-2\ 1}\\
&\qquad P_2(x)=\frac{4!}{2^2\ 2!\ 2!}x^2-\frac{(2\ 2-2)!}{2^2\ 1!\ 0!}x^0\\
&\qquad P_2(x)=\frac{4\ 3\ 2\ 1}{4\ 2\ 2}x^2-\frac{2!}{4}\\
&\qquad P_2(x)=\frac{3}{2}x^2-\frac{1}{2}=\frac{1}{2}(3x^2-1)
\end{aligned}
$$

b. From

$$
a_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx=\frac{2n+1}{2}\int_{-1}^{1}xP_n(x)dx
$$

we have

$$
\begin{aligned}
n=0,\ a_0&=\frac{2\ 0+1}{2}\int_{-1}^{1}xP_0(x)dx\\
&=\frac{1}{2}\int_{-1}^{1}xdx=\frac{1}{2}[\frac{x^2}{2}]_{-1}^{1}=\frac{1}{4}[1^2-(-1)^2]=0\\
n=1,\ a_1&=\frac{2\ 1+1}{2}\int_{-1}^{1}xP_1(x)dx\\
&=\frac{3}{2}\int_{-1}^{1}x^2dx=\frac{3}{2}[\frac{x^3}{3}]_{-1}^{1}=\frac{1}{2}[1^3-(-1)^3]=\frac{1}{2}\ 2=1\\
n=2,\ a_2&=\frac{2\ 2+1}{2}\int_{-1}^{1}xP_2(x)dx\\
&=\frac{5}{2}\int_{-1}^{1}x[\frac{1}{2}(3x^2-1)]dx=\frac{5}{4}\int_{-1}^{1}(3\ x^3-x)dx\\
&=0\ \text{ since the powers of } x \text{ in the integrand are odd}
\end{aligned}
$$

Therefore the Fourier-Legendre series of x iss $x=1\cdot P_1(x)$ as required.

c. Using Rodrigues's formula

$$
P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]
$$

we have

$$
\begin{aligned}
n = 0,\ & \frac{d^0}{dx^0}[(x^2-1)^0] = (x^2-1)^0 = 1 \\
& P_0(x) = \frac{1}{2^0\ 0!}\ 1 = 1 \\
n = 1,\ & \frac{d}{dx}(x^2-1) = 2x \\
& P_1(x) = \frac{1}{2^1\ 1!}\ 2x = x \\
n = 2,\ & \frac{d^2}{dx^2}(x^2-1)^2 = \frac{d}{dx}\big[\frac{d}{dx}(x^2-1)^2\big] = \frac{d}{dx}\big[4x(x^2-1)\big] = \frac{d}{dx}\big[4x^3-4x\big] = 12\,x^2-4 \\
& P_2(x) = \frac{1}{2^2\ 2!}\ (12\,x^2-4) = \frac{4}{4\ 2}(3x^2-1) = \frac{1}{2}(3x^2-1)
\end{aligned}
$$

Question 4

Question 5

Question 6