Johns Hopkins
Engineering for Professionals

Mathematical Methods for Applied Biomedical Engineering
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WHITING SCHOOL
of ENGINEERING

Continuing our study of advanced differential equations methods we start by looking at some generalizing concepts:

- Eigenfunctions
- Adjoint and Hermitian operators

Eigenfunctions as applied to differential equations

As we have previously seen an inhomogeneous differential equation can be written in terms of a linear differential operator $\mathcal{L}$, a function f(x) and the solution y(x) with appropriate boundary conditions (e.g. at x = a and b).

$$\mathcal{L}y(x) = f(x)$$

KEY: Often the function f(x) is not simple enough so that we can find a particular integral solution. However since the operator is linear we can solve the differential equation for functions, $y_i(x)$ that also satisfy the same boundary conditions as the original equation and are eigenfunctions of the differential operator.

$$\boxed{\mathcal{L}y_i(x) = \lambda_i y_i(x)}$$

This allows us to express the solution y(x) for the original differential equations as a superposition of these eigenfunctions.

Here is an example of a system and solution which should already be familiar to you.

$$\text{Let } \mathcal{L} = -\frac{d^2}{dt^2},\ \lambda_i = \omega_i^2 \ \rightarrow -\frac{d^2 y_i}{dt^2} = \omega_i^2 y_i$$

Some possible solutions are $y_i(t) = \sin\omega_i t$ or $y_i(t) = \cos\omega_i t$

Of course we have already seen that these functions are the basis functions for Fourier series and a superposition of these can be used to represents any function on a particular interval.

We can address more general and interesting differential equations by extending our eigenfunction centered differential equations with what is referred to as a weight function. That is,

$$\mathcal{L} y_i(x) = \lambda_i \boxed{\rho(x)} y_i(x)$$

Note that $\rho(x)$ is required to be real and non-negative on the interval [a,b] and often has the value 1 (e.g. for the Fourier basis functions).

Aside: Infinite-dimensional vector spaces

Properties of any vector space as expressed for functional space, for functions f(x), g(x) and h(x) and constants λ, μ and defined on the interval $a \le x \le b$

Closed under addition and

commutative $f(x)+g(x)=g(x)+f(x)$

associative $[f(x)+g(x)]+h(x)=g(x)+[f(x)+h(x)]$

Closed under scalar multiplication and

distributive $\lambda[f(x)+g(x)]=\lambda g(x)+\lambda f(x)$

distributive $(\lambda+\mu)f(x)=\lambda f(x)+\mu f(x)$

associative $\lambda[\mu f(x)]=(\lambda\mu)f(x)$

There exist an additive identity, 0 such that $f(x)+0=f(x)$

There exist an scalar multiplcative identity, 1 such that $1\cdot f(x)=f(x)$

Additive inverse $f(x)+[-f(x)]=0$

Define linear combinations and linear (in)dependence

Define a linear combination $\sum_{n=0}^{\infty} c_n y_n(x)$ where c_n are constants

Then $\sum_{n=0}^{\infty} c_n y_n(x) = 0 \Rightarrow \forall n \; c_n = 0$ we then define the functions $y_n(x)$ as linearly independent, otherwise they are dependent

Define basis functions – a linearly independent spanning set for functions that satisfy the Dirichlet conditions (the Fourier series lectures). Thus a function f(x) can be expressed as a combination of the basis functions on the interval of interest [a,b]. That is

$$f(x) = \sum_{i=0}^{\infty} a_i y_i(x)$$

Define inner product (in terms of bra-ket notation) $\langle f | g \rangle = \int_a^b f^*(x) g(x) \rho(x) dx$

By the way an infinite dimensional vector space with a defined inner product is called a Hilbert space.

Finally **define standard norm** (magnitude) $\| f(x) \| = \left[\langle f(x) | f(x) \rangle \right]^{1/2}$

And **define orthogonally** between any two functions if $\langle f(x) | g(x) \rangle = 0$

How to construct from functions that form basis (but not orthogonal) a basis that is also orthogonal.

It is often the case that while the basis functions are linearly independent they are not orthogonal. However any set of basis functions can be made orthogonal using the **Gram-Schmidt** procedure. Succinctly given a set of basis functions $y_i(x)$, I = 1, 2,...

$$\text{Let } \phi_0 = y_0 \rightarrow \hat{\phi}_0 = \frac{\phi_0}{\|\phi_0\|}$$

$$\text{Next construct } \phi_1 = y_1 - \hat{\phi}_0 \langle \hat{\phi}_0 | y_1 \rangle \rightarrow \hat{\phi}_1 = \frac{\phi_1}{\|\phi_1\|}$$

$$\vdots$$

$$\text{In general } \boxed{\phi_n = y_n - \sum_{i=0}^{n-1} \hat{\phi}_i \langle \hat{\phi}_i | y_n \rangle}$$

Back:

Hermitians operators

First **define the adjoint operator** – given an operator T we require

$$\left\langle f(x) \middle| T[g(x)] \right\rangle = \left\langle T^*[f(x)] \middle| g(x) \right\rangle + \text{boundary terms}$$

where T* is the adjoint operator

An operator is **self-adjoint** if T*=T

Finally an operator is Hermitian given

$$\left\langle f(x) \middle| T[g(x)] \right\rangle = \left\langle T[f(x)] \middle| g(x) \right\rangle$$

You can prove that an operator is Hermitian if and only if it is diagonalizable in an orthonormal basis with **real eigenvalues**.

In the next lecture we will show that a very important and far reaching set of differential equations (**describing many physical systems**) have as there differential operator a Hermitian operator.

Example of finding an adjoint operator

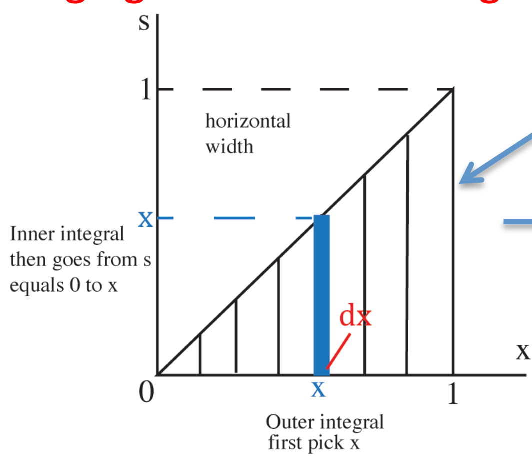
Define the inner product for the real valued functions f(x), g(x) on interval [0,1], that is

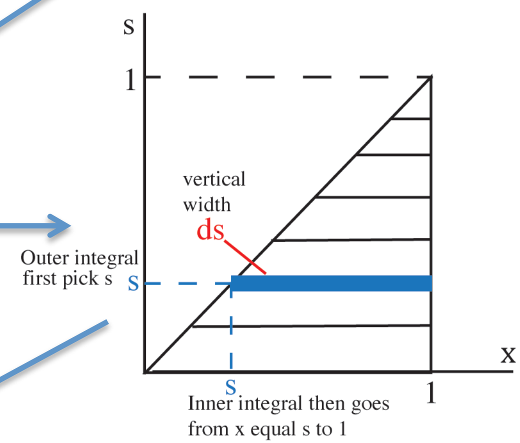
$\langle f | g \rangle = \int_0^1 f(x) g(x) dx$ and operator defined by $\boxed{T[f(x)] = \int_0^x f(s) ds}$

The defining relation for the adjoint is $\langle Tf | g \rangle = \langle f | T^* g \rangle$ and we will use this to find the adjoint operator for T.

Start with $\langle Tf | g \rangle = \int_0^1 Tf(x) g(x) dx = \int_0^1 \left[\int_0^x f(s) ds \right] g(x) dx =$

KEY: Changing the order of integration





Gives $= \int_0^1 f(s) \left[\int_s^1 g(x) dx \right] ds = \int_0^1 f(x) T^* g(x) dx = \langle f | T^* g \rangle$

Therefore the adjoint operator (written as a function of s) is $\boxed{T^*[g(x)] = \int_x^1 g(s) ds}$

Example of finding an adjoint operator (cont.)

Let's use specific functions for $f(x)=x$ and $g(x)=x^2$ as a check for our example.

First calculate the inner product with the operator T

$$\langle Tf|g\rangle = \int_0^1 \left[Tf(x)\right] g(x)\,dx = \int_0^1 \left[Tx\right] x^2\,dx = \int_0^1 \left[\int_0^x s\,ds\right] x^2\,dx = \int_0^1 \left[\frac{x^2}{2}\right] x^2\,dx = \left[\frac{x^5}{2\cdot 5}\right]_0^1 = \boxed{\frac{1}{10}}$$

Then calculate the inner product with the adjoint operator T^*

$$\langle f|T^*g\rangle = \int_0^1 f(x)\left[T^*g(x)\right]dx = \int_0^1 x\left[\int_x^1 s^2\,ds\right]dx = \int_0^1 x\left[\frac{s^3}{3}\right]_x^1 dx = \int_0^1 x\left[\frac{1^3}{3} - \frac{x^3}{3}\right]dx =$$

$$\frac{1}{3}\int_0^1 x - x^4\,dx = \frac{1}{3}\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{1}{3}\left[\frac{1^2}{2} - \frac{1^5}{5}\right] = \frac{1}{3}\left[\frac{3}{10}\right] = \boxed{\frac{1}{10}}$$

Therefore $\langle Tf|g\rangle = \langle f|T^*g\rangle$ as expected!