JOHNS HOPKINS
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Modeling Approaches to Cell and Tissue Engineering

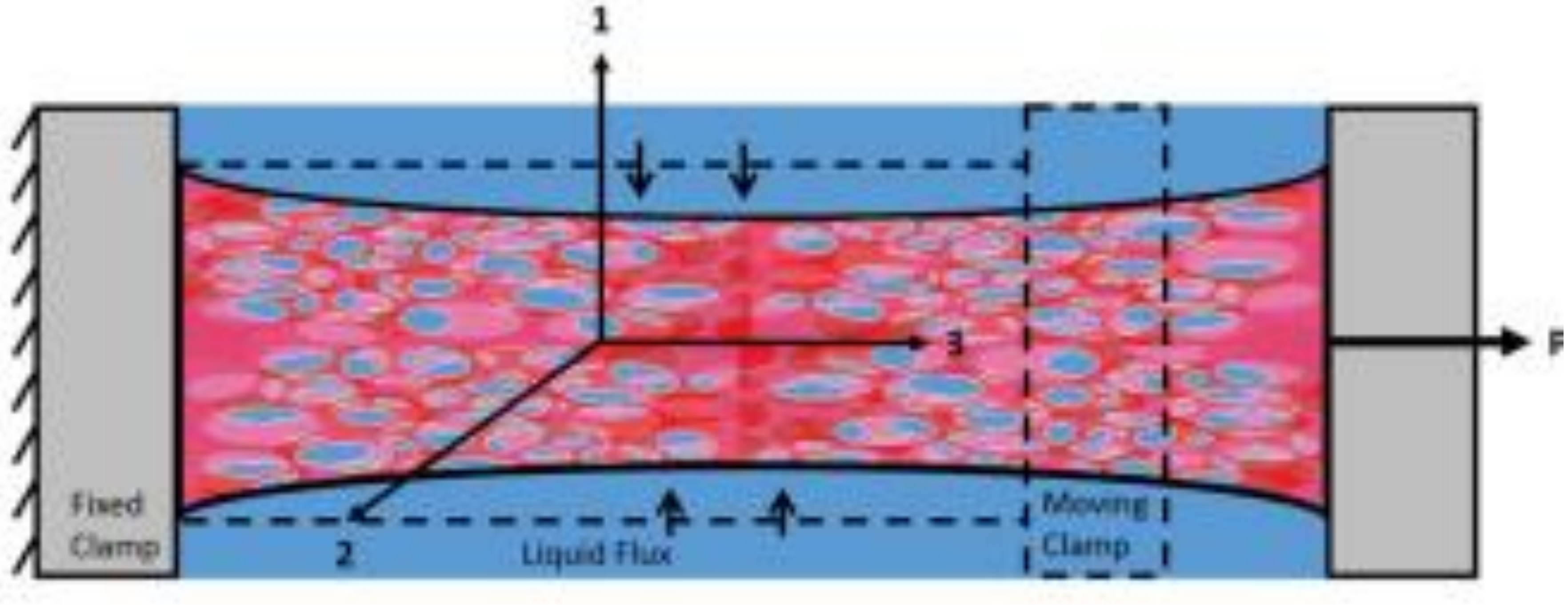
Poroelastic (Biphasic) Tissues

Overview of the Lecture

(From Yuan et al., 2018)

We consider an extension of a poroelastic transversely isotropic incompressible cylindrical scaffold via the application of the ramp strain. The relaxation process of the corresponding axial force is analyzed theoretically and experimentally. In our theoretical analysis, an analytical expression of the force as a function of time is derived. Then we do optimization of the model parameters by fitting the experimental data. As a result, we estimate the poroelastic parameters of the scaffold that include two Young's moduli, two Poisson's ratios, and gel diffusion time. We also consider the dependence of the scaffold parameters on its porosity. The obtained results are important for tissue (skeletal muscle) regeneration.

Cylindrical Poroelastic Scaffold



Stress Tensors for Solid (Including Elastic) and Fluid Components; Total Stress Tensor

$$\sigma^s = (1 - \varnothing)p\boldsymbol{I} + \sigma^E$$

$$\sigma^f = \varnothing p\boldsymbol{I}$$

$$\sigma^t = \sigma^s + \sigma^f$$

Fundamental Equations for Two-Phase Material

Continuity Equation

$$div\left[(1-\varnothing)\bar{v}^s + \varnothing\bar{v}^f\right] = 0$$

Equilibrium equations for the solid and fluid phases, total stress

$$-(1-\varnothing)\nabla p + \operatorname{div}\sigma^E + \varnothing^2/k\left(\bar{v}^f - \bar{v}^s\right) = 0$$

$$-(1-\varnothing)\nabla p - \varnothing^2/k\left(\bar{v}^f - \bar{v}^s\right) = 0$$

$$\operatorname{div}\boldsymbol{\sigma}^t = 0$$

Stresses, Strains, and Velocities

We treat the scaffold as a transversely isotropic poroelastic cylinder under the action of an axial load resulting in axisymmetric stresses, strains, and velocities. In the cylindrical system (r, Ø, z) with the (r, Ø) coordinates within x_1x_2-plane, the stresses have $\sigma_{rr}, \sigma_{\varphi\varphi}$ and σ_{zz} components, the displacements have u_r and u_z components, and the strains have $\epsilon_{rr} = \partial u_r/\partial r, \epsilon_{\phi\phi} = u_r/r,$ and $\epsilon_{zz} = \partial u_z/\partial z$ components. The scaffold fibers are long with the actual length-to-radius ratios of about 10. Because of this, we assume that $\epsilon_{zz} = \varepsilon(t)$ (where $\varepsilon(t)$ is the externally applied tensile strain) and other components of the strains, stresses, and displacements do not depend on the z-variable and are functions of the radius, r and time, t. For infinitesimal strains, the elastic stresses are related to strains by the equations.

Elastic Stresses and Strains in Transversely Isotropic Cylindrical Scaffold

$$\begin{pmatrix} \sigma_{rr}^{E} \\ \sigma_{\varphi\varphi}^{E} \\ \sigma_{zz}^{E} \end{pmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{11} & C_{13} \\ C_{13} & C_{13} & C_{33} \end{bmatrix} \begin{pmatrix} \varepsilon_{rr} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{zz} \end{pmatrix}$$

$$C_{11} = E_1\left(1 - v_{31}^2 E_1/E_3\right)/\left[(1+\nu_{21})\Delta_1\right] C_{12} = E_1\left(\nu_{21} - v_{31}^2 E_1/E_3\right)/\left[(1+\nu_{21})\Delta_1\right]$$

$$C_{13} = E_1\nu_{31}/\Delta_1 C_{33} = E_3\left[\left(1 + v_{31}^2 E_1/E_3\right)/\Delta_1\right]$$

Time Relaxation of the Load Intensity

Where $\Delta_1 = 1 - v_{21} - 2v_{31}^2 E_1/E_3$.

Here, E_3 and E_1 are Young's moduli along the fiber direction (in tension) and in the plane of isotropy (in compression), respectively.

Also, v_{21} and v_{31} are Poisson's ratios given by the following ratios of the strain components $v_{21} = -\varepsilon_{\phi\phi}/\varepsilon_{rr}$ and $v_{31} = -\varepsilon_{rr}/\varepsilon_{z3}$.

We will analyze the scaffold stress relaxation by considering the time course of the change in intensity of the tensile load applied to the cylinder, which is given by the equation:

$$P(t) = \frac{2}{a^2} \int_0^a \sigma_{zz}^t r dr$$

Where a is the radius of the cylindrical scaffold.

Derivation of the Load Intensity

The main steps of the derivation of the load intensity, P(t), are as follows.

First, the continuity equation is used to obtain the fluid velocity, in terms of the radial displacement, u.

Then, the obtained fluid velocity is used in the equilibrium equation to derive pressure, p, in terms of the radial displacement, u. Thus, the problem reduces to a second-order ODE in terms of u-displacement as a function of r and t.

Using Laplace transform over time, t, the problem reduces to a Bessel equation in terms of the Laplace transform of u. The inverse Laplace transform is obtained by calculating the residues.

Finally, the longitudinal stress is obtained in terms of the displacement, u, and integrated over the scaffold cross-section resulting in an expression for the load intensity.

Time Relaxation of the Load Intensity in Response to the Application of the Ramp Strain

$$\varepsilon(t) = \begin{bmatrix} \dot{\varepsilon}_0 t \text{ for} & 0 \leq t \leq t_0 \\ \varepsilon_0 = t_0 \dot{\varepsilon}_0 & \text{for } t > t_0 \end{bmatrix}$$

$$P(t) = E_3 \dot{\varepsilon_0} t + E_1 \dot{\varepsilon_0} t_g F_1 \left(t_g, E_1, E_3, v_{21}, v_{31}\right) \text{ for } 0 \leq t \leq t_0$$

$$P(t) = E_3 \dot{\varepsilon_0} t_g - E_1 \dot{\varepsilon_0} t_g F_2 \left(t_g, E_1, E_3, v_{21}, v_{31}\right) \text{ for } t \geq t_0$$

Where:

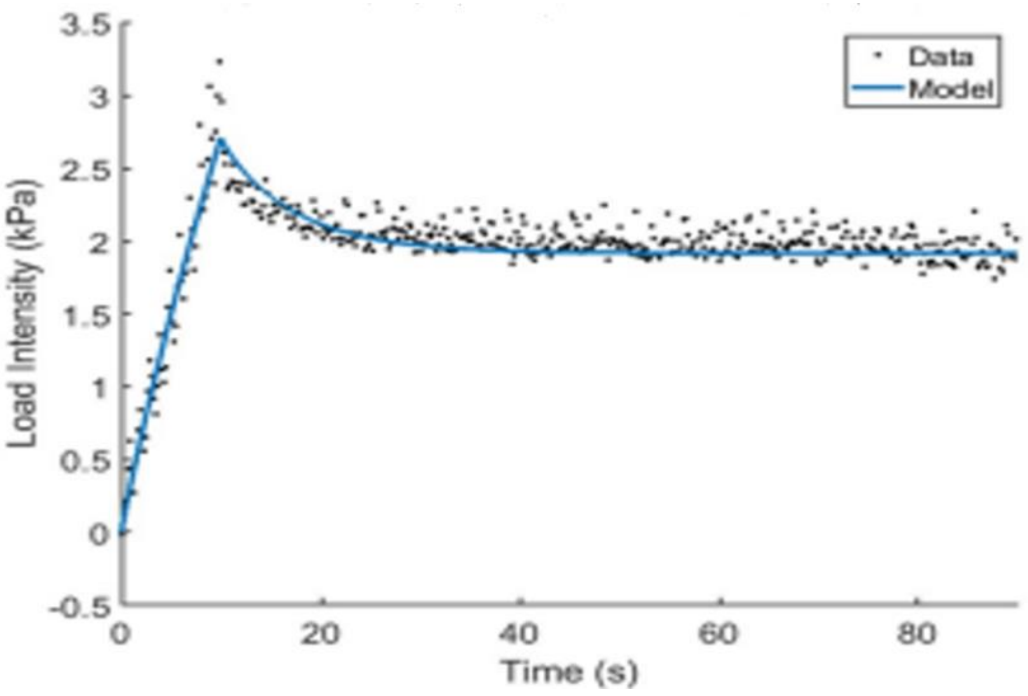
$$F_1 = \Delta_3 \left\{ \frac{1}{8} - \sum_n \exp\left(-\alpha_n^2 \frac{t}{t_g}\right) \left(\propto_n^2 \left[\Delta_2^2 \propto_n^2 - \Delta_1 \left(1 + v_{21}\right)\right]\right)^{-1} \right\}$$

$$F_2 = \Delta_3 \sum_n \left[\exp\left(-\alpha_n^2 \frac{t}{t_g}\right) - \exp\left(-\alpha_n^2 \frac{t - t_0}{t_g}\right)\right] \left(\alpha_n^2 \left[\Delta_2^2 \alpha_n^2 - \Delta_1 \left(1 + v_{21}\right)\right]\right)^{-1}$$

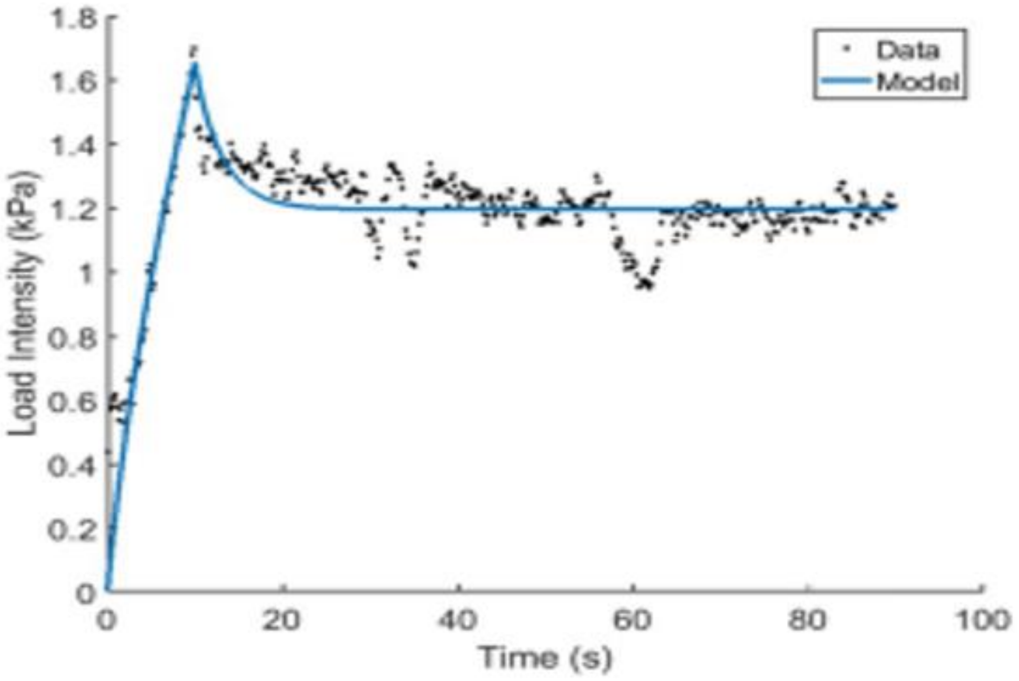
$$\Delta_2 = \left(1 - v_{31}^2 \frac{E_1}{E_3}\right) \text{ and } \Delta_3 = \left(1 - \frac{2 v_{31}^2 \Delta_2}{\Delta_1}\right)$$

Fitting the Experimental Data

50% Scaffold Porosity, 1%/sec Strain Rate



70% Scaffold Porosity, 1%/sec Strain Rate



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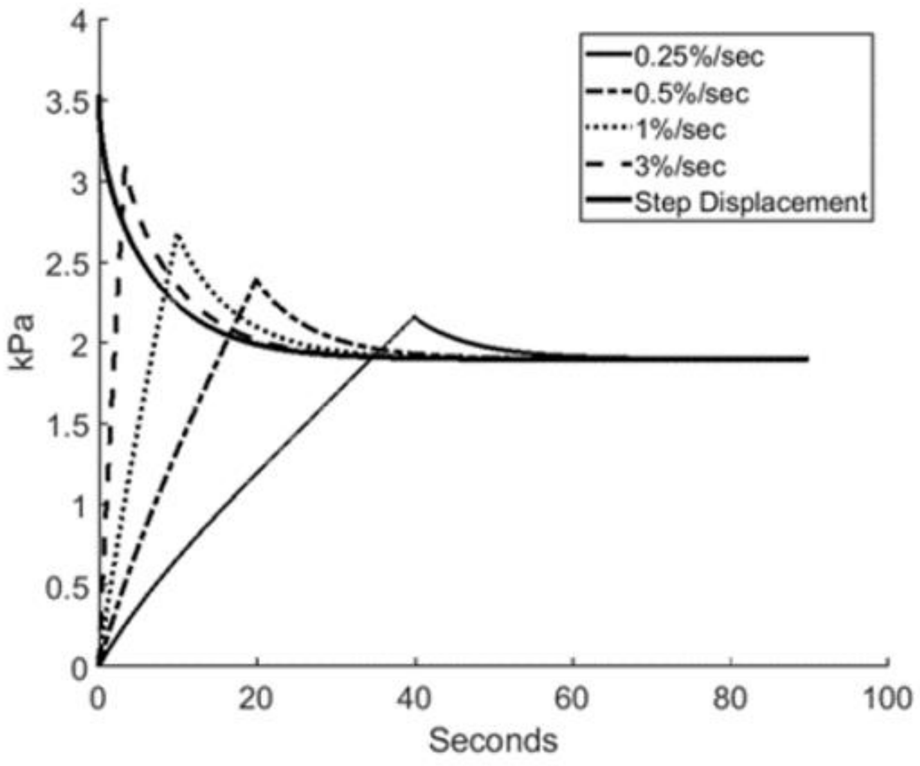
Optimization of the Model Parameters

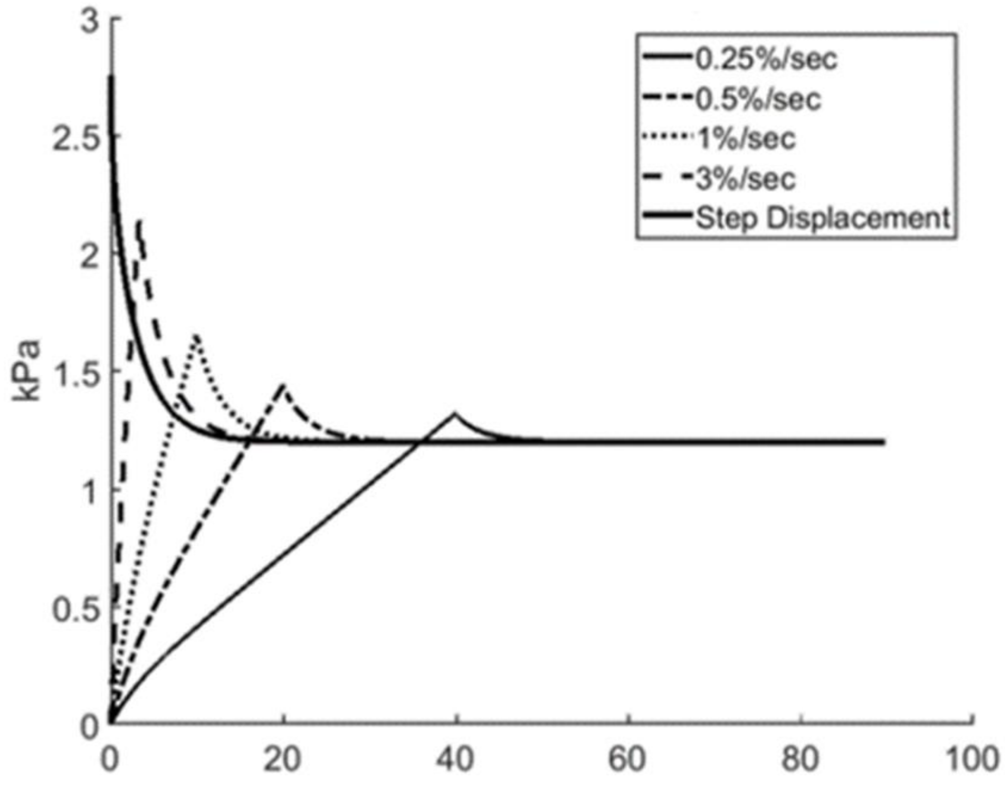
Model Parameters	50% Porosity	70% Porosity
E_1, kPa	8.49	5.61
t_g, s	40.62	17.58
v_{21}	0.75	0.82
E_3, kPa	19.19	11.97
ν_{31}	0.24	0.24

Load Intensity Relaxation for Different Rates of the Applied Strain

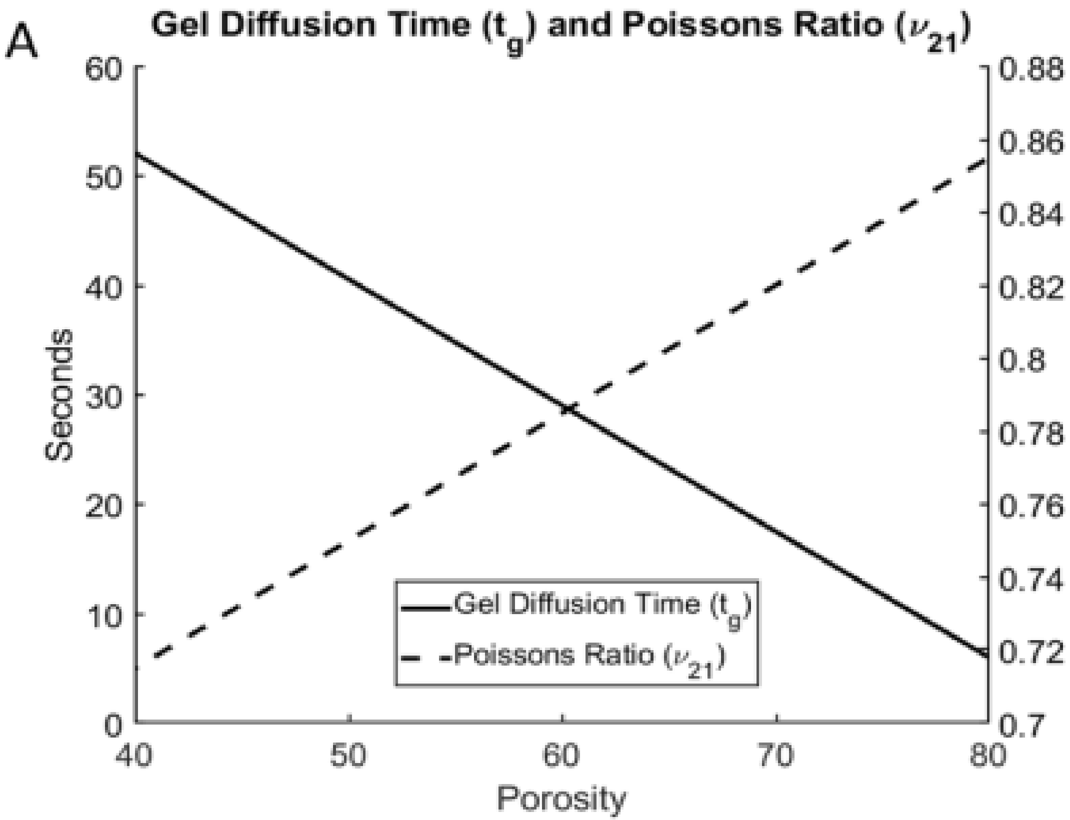
50% Scaffold Porosity

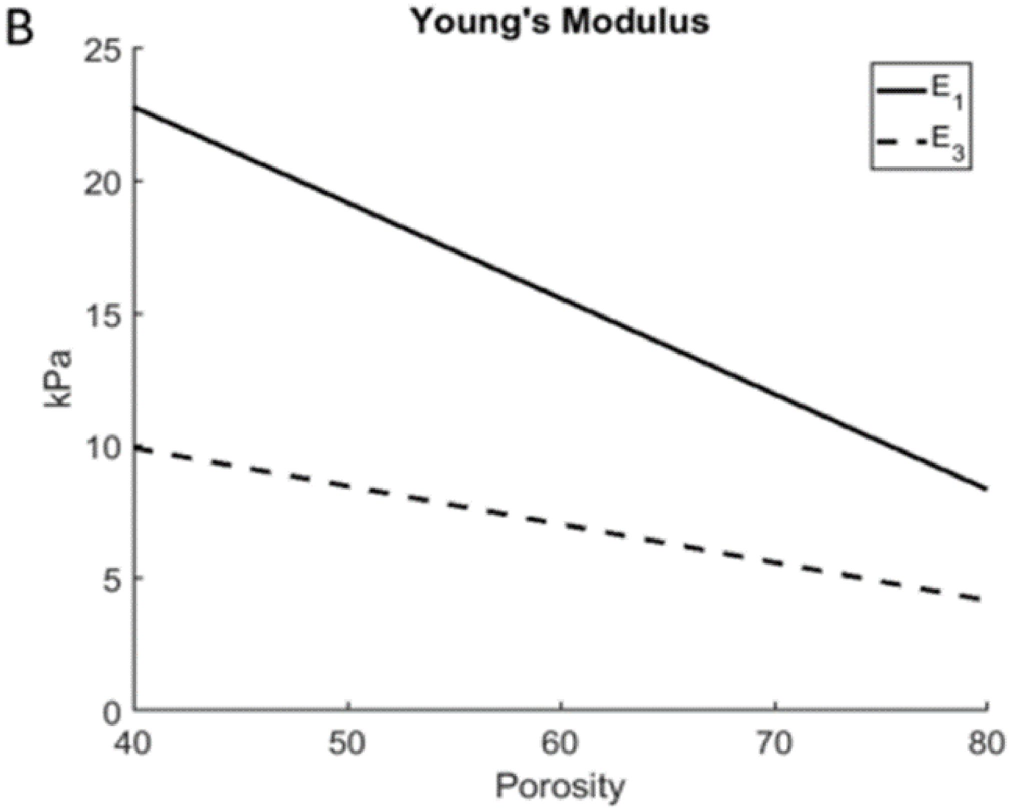


70% Scaffold Porosity



The Model Parameters as Functions of the Scaffold Porosity





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