DS-GA 3001.001 Special Topics in Data Science: Probabilistic Time Series Analysis
Homework 5

Due date: Dec 13
YG390

Problem 4.6 in tsa4. (15pt)

(a) For $x_t = \phi x_{t-1} + w_t$ where $|\phi| < 1$ and $w_t \sim \mathcal{N}(0, \sigma_w^2)$, using the property 4.4 in tsa4; for this AR(1) process, $\theta(z) = 1$ and $\phi(z) = 1 - \phi z$, then:

$$\begin{aligned}|\phi(e^{-2\pi i w})|^2 &= |1 - \phi e^{-i2\pi w}|^2 \\ &= (1 - \phi e^{-i2\pi w})(1 - \phi e^{i2\pi w}) \\ &= 1 - \phi(e^{-i2\pi w} + e^{i2\pi w}) + \phi^2 \\ &= 1 + \phi^2 - 2\phi\cos(2\pi w)\end{aligned}$$

Substituting into (4.23) we have $f_x(w) = \frac{\sigma_w^2}{1+\phi^2-2\phi\cos(2\pi w)}$

(b) Assuming $\gamma_x(h) = \frac{\sigma_w^2 \phi^{|h|}}{1-\phi^2}$ for $h = 0, \pm 1, \pm 2, \ldots$

$$\begin{aligned}f_x(w) &= \sum_{h=-\infty}^{h=+\infty} \gamma(h) e^{-i2\pi wh} \\ &= \frac{\sigma_w^2}{1-\phi^2} \sum_{h=-\infty}^{h=+\infty} \phi^{|h|} e^{-i2\pi wh} \\ &= \frac{\sigma_w^2}{1-\phi^2}[1 + \sum_{h=1}^{h=+\infty} (\phi e^{-i2\pi w})^h + \sum_{h=1}^{h=+\infty} (\phi e^{i2\pi w})^h] \\ &= \frac{\sigma_w^2}{1-\phi^2}[1 + \frac{1}{1-\phi e^{-i2\pi w}} - 1 + \frac{1}{1-\phi e^{i2\pi w}} - 1] \\ &= \frac{\sigma_w^2}{1-\phi^2}[1 + \frac{\phi e^{-i2\pi w}}{1-\phi e^{-i2\pi w}} + \frac{\phi e^{i2\pi w}}{1-\phi e^{i2\pi w}}] \\ &= \frac{\sigma_w^2}{1-\phi^2} \frac{1-\phi^2}{1+\phi^2-2\phi\cos(2\pi w)} \\ &= \frac{\sigma_w^2}{1+\phi^2-2\phi\cos(2\pi w)}\end{aligned}$$

where we used in the fourth equality, the convergence of geometric series since $|\phi e^{-i2\pi w}| < 1$ and $|\phi e^{i2\pi w}| < 1$. And we find the expression of the power spectrum established in (a), confirming the equality for $\gamma_x(h)$.