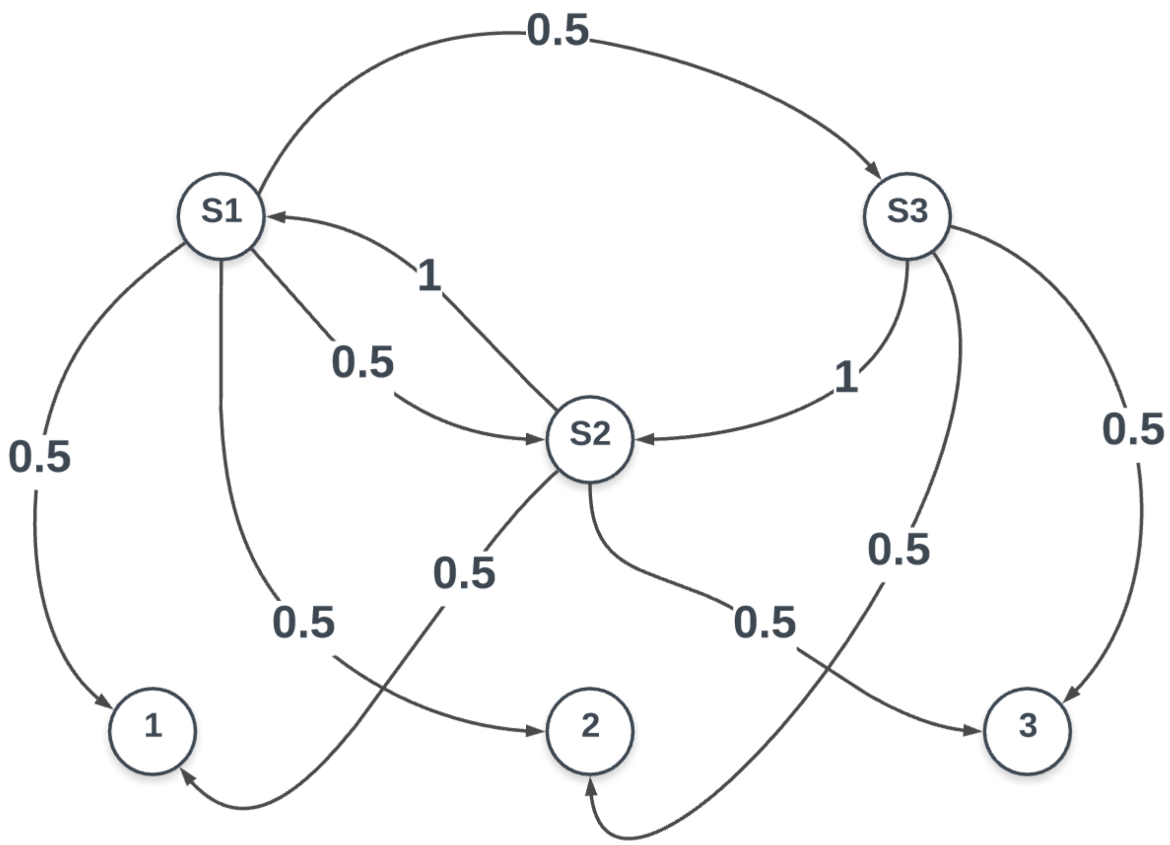
DS-GA 3001.001 Special Topics in Data Science: Probabilistic Time Series Analysis
Homework 3

Due date: Oct 25, by 6pm
YG390

Problem 1. (15p)
Consider the HMM with K=3 latent states and discrete observations $\{1, 2, 3\}$, with parameters specified by: initial distribution $\pi = [1,\ 0,\ 0]$, transition matrix $\mathbf{A} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, where $A_{ij} = \mathrm{P}(z_{t+1} = j | z_t = i)$ and likelihood $\mathrm{P}(x_t|z_t)$ described by matrix entries B_{xz}: $\mathbf{B} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$.
Write down all possible state sequences consistent with observations a) 1, 2, 3 and b) 1, 3, 1.
Let the three latent states be $\{S_1, S_2, S_3\}$. Given the HMM with parameters $\{A, B, \pi\}$, the model can be described as:



When observing the observations 1,2, and 3 with initial distribution π which implies we start in state S_1 then observing 2 means we have 0.5 probability of being in state S_1 or S_3. However staying in state S_1 is not possible so the only state after seeing the sequence 1,2 is state S_3. Then the last observation 3 means we are in state S_2 or S_3. but from state S_3, the only existing transition is from S_3 to S_2. So the sequence of observations 1,2,3 corresponds to the state sequence $\{S_1, S_3, S_2\}$.

When observing 1,2,1 with initial distribution π, seeing 2 after 1 implies we can be in state S_2 50% of the time or S_3 the rest of the time. The last observation 1 implies that the latent space is either S_1 or S_2. Based on the transition matrix $\boldsymbol{A}$ then the only possible state sequences for the sequence of observations $\{1, 2, 1\}$ are $\{S_1, S_2, S_1\}$ or $\{S_1, S_3, S_2\}$.

Problem 2. (15p)
Construct an HMM that generates the observation sequence $A^{k_1}C^{k_2}A^{k_3}C^{k_4}$ where A^{k_1} denotes k_1 repeats of symbol A and the number of repeats k_i are drawn from the set $\{1, 2, 3\}$ with equal probability.

The HMM model is defined by three class of parameters $\{\boldsymbol{A}, \boldsymbol{B}, \pi\}$. If we call $\{S_1, S_2\}$ the latent states. For a observation sequence $A^{k_1}C^{k_2}A^{k_3}C^{k_4}$ the transition matrix $\boldsymbol{A}$ is computed so that

$$a_{ij} = \frac{num\ state\ transitions\ from\ i\ to\ j}{num\ state\ transitions\ from\ i}$$

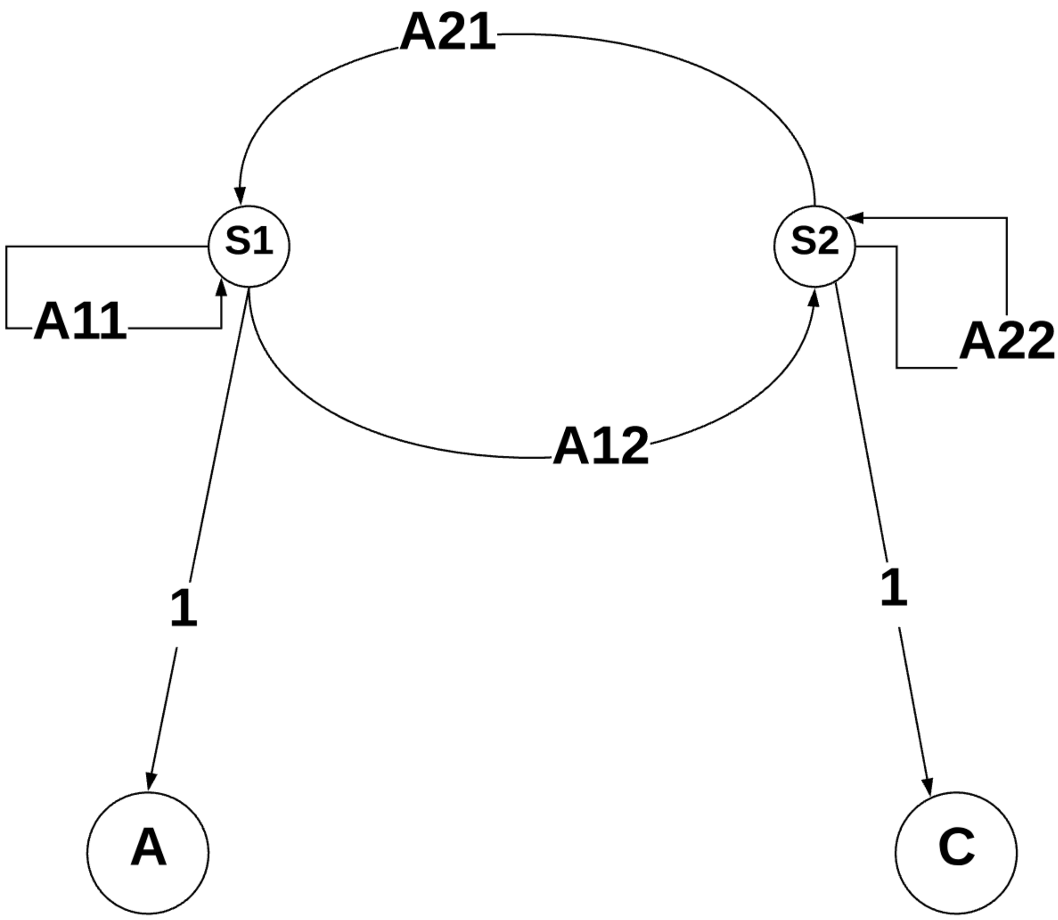
$$A_{11} = \frac{k_1 - 1 + k_3 - 1}{k_1 - 1 + k_3 - 1 + 2} = \frac{k_1 + k_3 - 2}{k_1 + k_3}$$
$$A_{12} = \frac{2}{k_1 + k_3}$$
$$A_{21} = \frac{1}{k_2 - 1 + 1 + k_4 - 1} = \frac{1}{k_2 + k_4 - 1}$$
$$A_{22} = \frac{k_2 - 1 + k_4 - 1}{k_2 + k_4 - 1} = \frac{k_2 + k_4 - 2}{k_2 + k_4 - 1}$$

$$B_{ih} = \frac{num\ of\ times\ state\ i\ emits\ h}{num\ state\ i}$$

Since the number of repeats are drawn with equal probabilities, the emission probabilities $\boldsymbol{B}$, is the identity matrix since there are as many observations repeated for A, then the number of state S_1, and the same for C and S_2.

$$\pi_i = \frac{num\ of\ chains\ start\ with\ i}{total\ num\ of\ chains}$$

We have only one chain starting with a sequence of A, which implies that we start in S_1 and the initial distribution is $\pi = [1,\ 0]$.



Problem 3. (20p)
Implement EM for an HMM model with K states and gaussian observations (full derivations in handout). Use this code to fit the weekly S&P 500 returns data (data/sp500w.csv) for $K = 2$ vs. $K = 3$ and compare the two results.
Hint: Use Example 6.17 from tsa4 textbook as guideline for plots and interpretation.