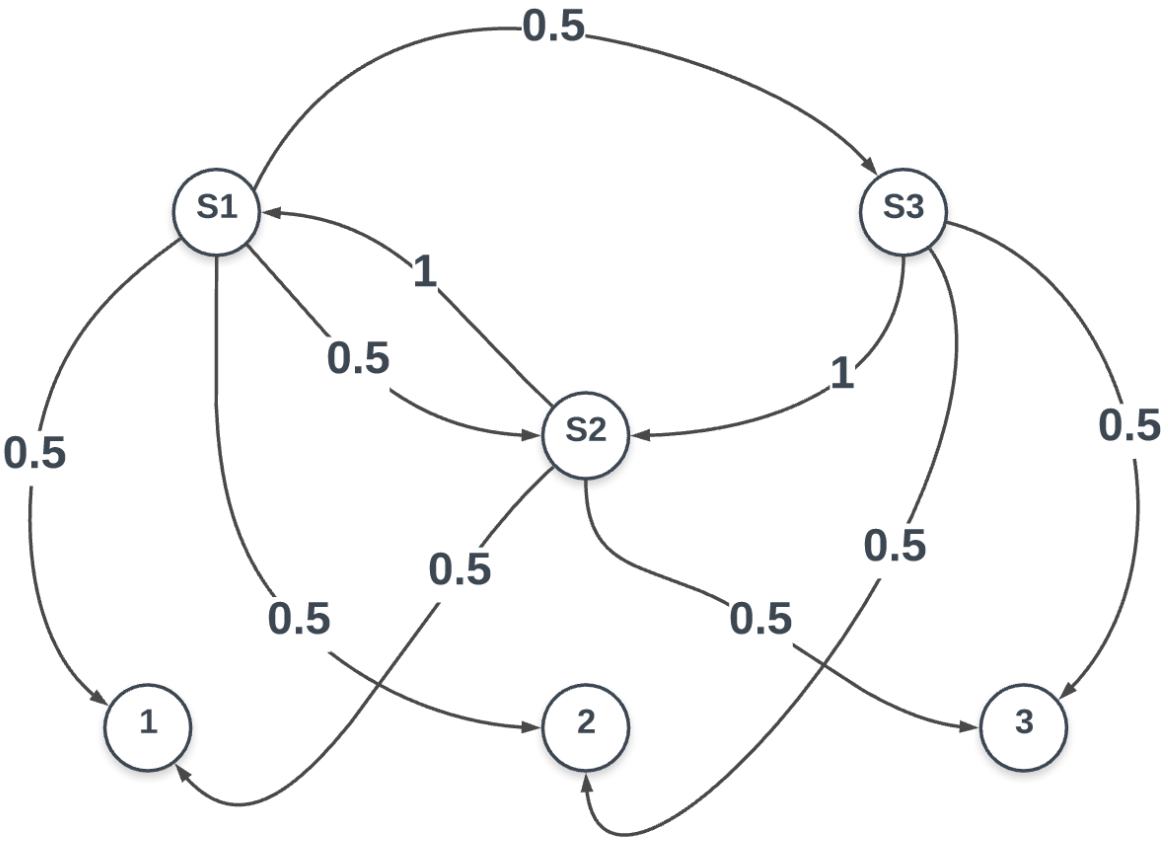
**DS-GA 3001.001 Special Topics in Data Science: Probabilistic Time Series Analysis**
**Homework 3**

**Due date: Oct 25, by 6pm**
YG390

**Problem 1.** (15p)
Consider the HMM with K=3 latent states and discrete observations $\{1, 2, 3\}$, with parameters specified by: initial distribution $\pi = [1,\ 0,\ 0]$, transition matrix $\mathbf{A} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, where $A_{ij} = \mathrm{P}(z_{t+1} = j | z_t = i)$ and likelihood $\mathrm{P}(x_t|z_t)$ described by matrix entries $B_{xz}$: $\mathbf{B} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$.
Write down all possible state sequences consistent with observations a) 1, 2, 3 and b) 1, 3, 1.
Let the three latent states be $\{S_1, S_2, S_3\}$. Given the HMM with parameters $\{A, B, \pi\}$, the model can be described as:




When observing the observations 1,2, and 3 with initial distribution $\pi$ which implies we start in state $S_1$ then observing 2 means we have 0.5 probability of being in state $S_1$ or $S_3$. However staying in state $S_1$ is not possible so the only state after seeing the sequence 1,2 is state $S_3$. Then the last observation 3 means we are in state $S_2$ or $S_3$. but from state $S_3$, the only existing transition is from $S_3$ to $S_2$. So the sequence of observations 1,2,3 corresponds to the state sequence $\{S_1, S_3, S_2\}$.

When observing 1,2,1 with initial distribution $\pi$, seeing 2 after 1 implies we can be in state $S_2$ 50% of the time or $S_3$ the rest of the time. The last observation 1 implies that the latent space is either $S_1$ or $S_2$. Based on the transition matrix $\boldsymbol{A}$ then the only possible state sequences for the sequence of observations $\{1, 2, 1\}$ are $\{S_1, S_2, S_1\}$ or $\{S_1, S_3, S_2\}$.

**Problem 2.** (15p)
Construct an HMM that generates the observation sequence $A^{k_1}C^{k_2}A^{k_3}C^{k_4}$ where $A^{k_1}$ denotes $k_1$ repeats of symbol $A$ and the number of repeats $k_i$ are drawn from the set $\{1, 2, 3\}$ with equal probability.

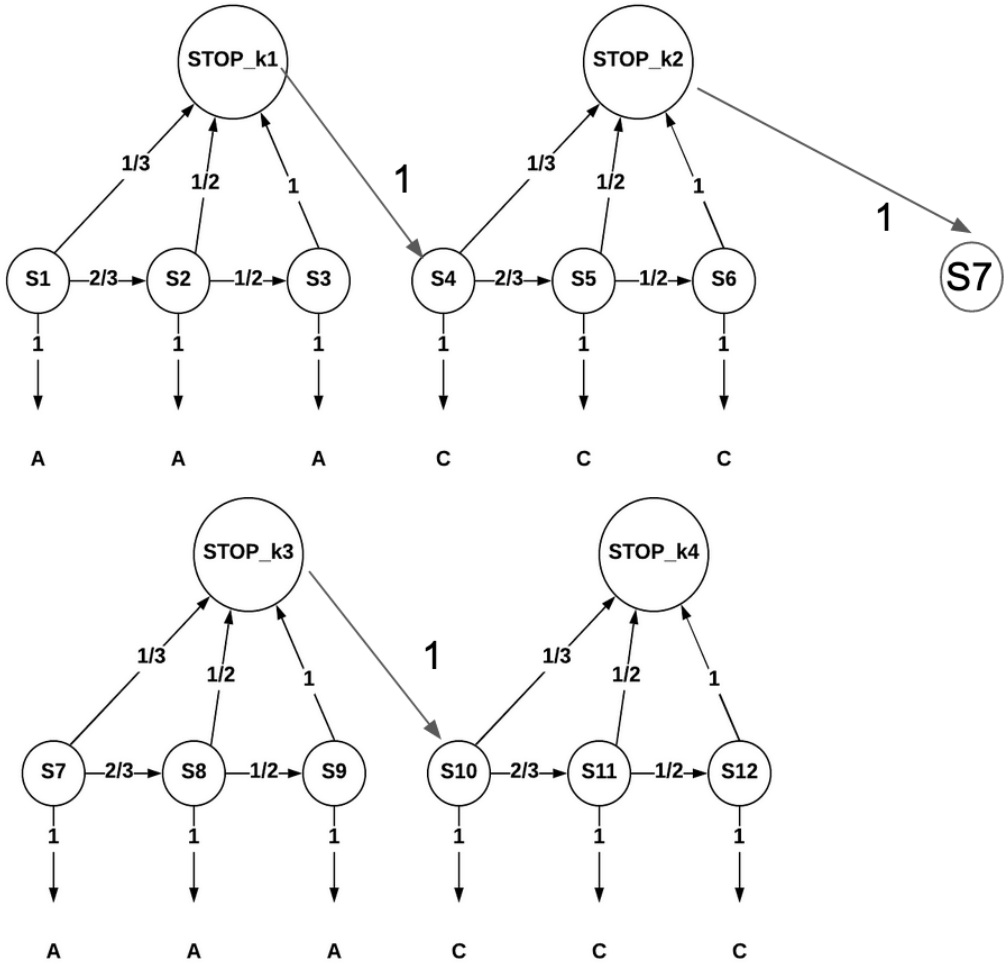
The HMM model is defined by three class of parameters $\{\boldsymbol{A}, \boldsymbol{B}, \pi\}$, and is described in the graph below. Note that there is a direct connection between the states STOP_k2 and S_7 which I broke down into two for convenience of representation. The latent states are $\{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{12}\}$. Sequences based on $k_i$ have equal probability $\frac{1}{3}$, meaning, that for example for $k_1$: A, AA, AAA have the same probability $\frac{1}{3}$. The transition matrix has for elements the probabilities indicated on the graph and $\sum_j A_{ij} = 1$.

$$B_{ih} = \frac{num\ of\ times\ state\ i\ emits\ h}{num\ state\ i}$$

The emission probabilities $\boldsymbol{B}$, is the identity matrix since there are as many observations repeated for A, then the number of state $S_i$, and the same for C and $S_j$.

$$\pi_i = \frac{num\ of\ chains\ start\ with\ i}{total\ num\ of\ chains}$$

All the chains start with a sequence of A in $S_1$, the initial distribution is $\pi = [100000000000]^T$.




**Problem 3.** (20p)
Implement EM for an HMM model with K states and gaussian observations (full derivations in handout). Use this code to fit the weekly S&P 500 returns data (data/sp500w.csv) for K = 2 vs. K = 3 and compare the two results.
Hint: Use Example 6.17 from tsa4 textbook as guideline for plots and interpretation.