

Making Bipartite Graphs DM-irreducible

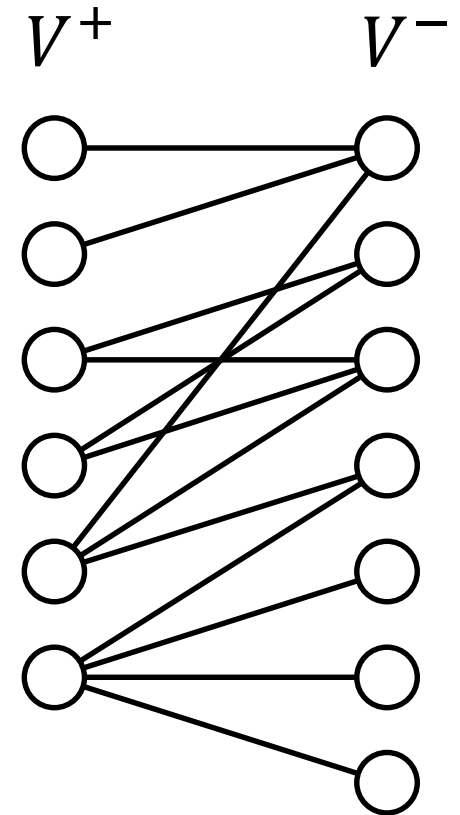
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1. Eötvös Lorand University, Hungary.
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4. Osaka University, Japan.

Dulmage–Mendelshon Decomposition

[Dulmage–Mendelsohn 1958,59]

$G = (V^+, V^-; E)$: Bipartite Graph

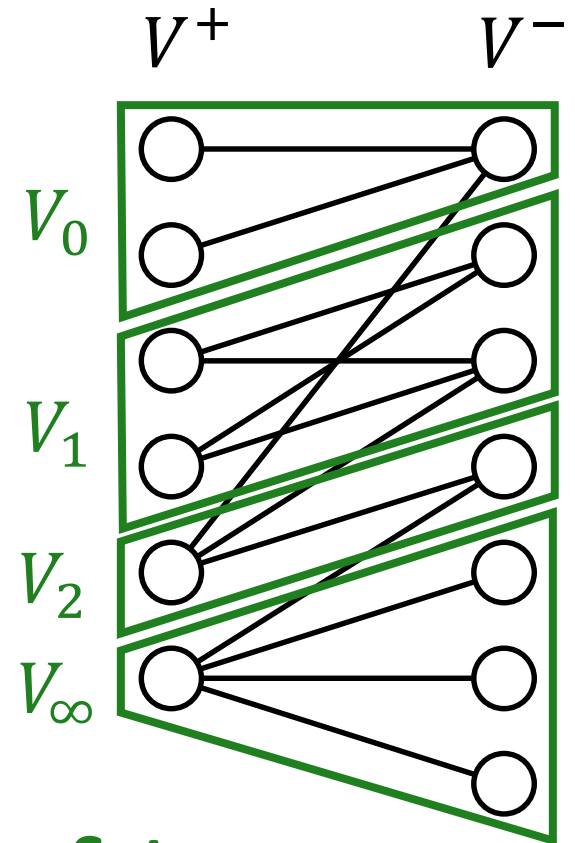


Dulmage–Mendelshon Decomposition

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- $|V_0^+| > |V_0^-|$ or $V_0 = \emptyset$
- $|V_i^+| = |V_i^-|$ ($i \neq 0, \infty$)
- $|V_\infty^+| < |V_\infty^-|$ or $V_\infty = \emptyset$
- \forall **Max. Matching** in G is a union of **Perfect Matchings** in $G[V_i]$



Unique Partition of Vertex Set

reflecting Structure of **Maximum Matchings**

(Definition will be given later)

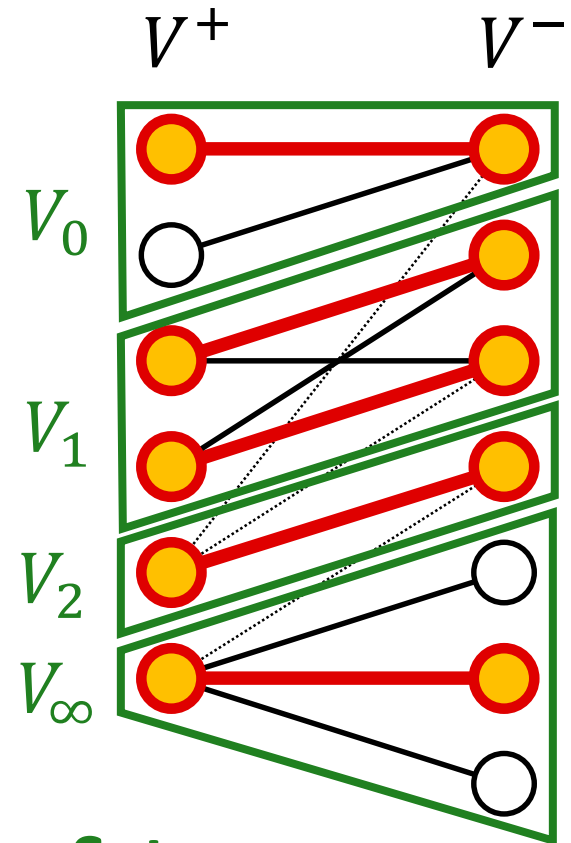
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Covering ALL vertices in one side



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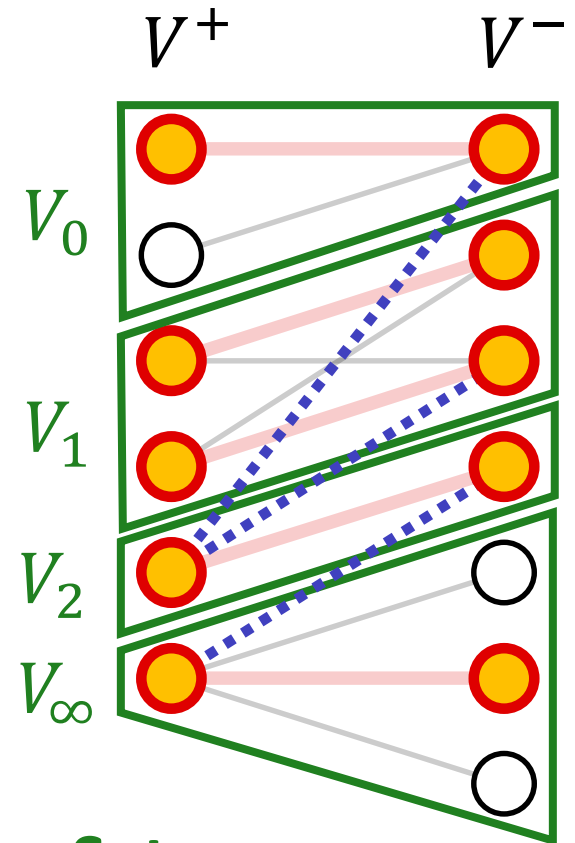
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- $\forall e$: Edge in $G[V_i]$,
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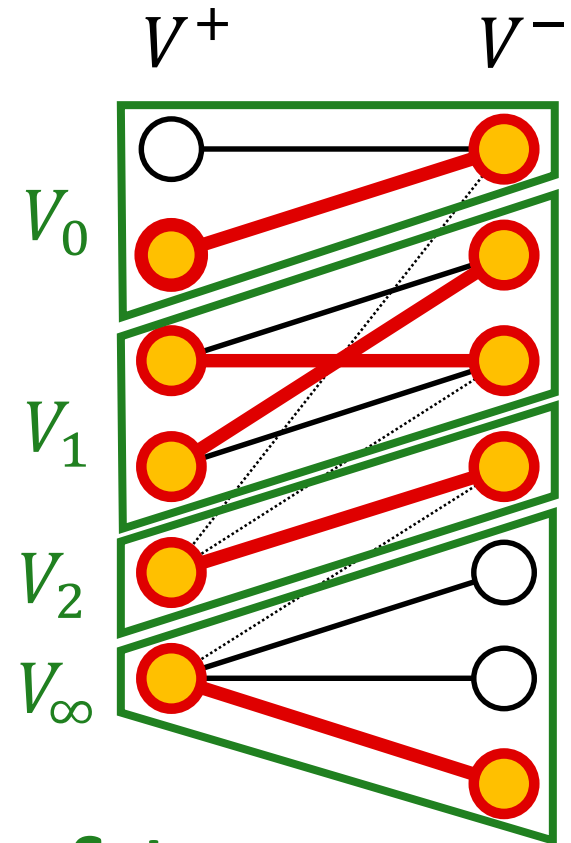
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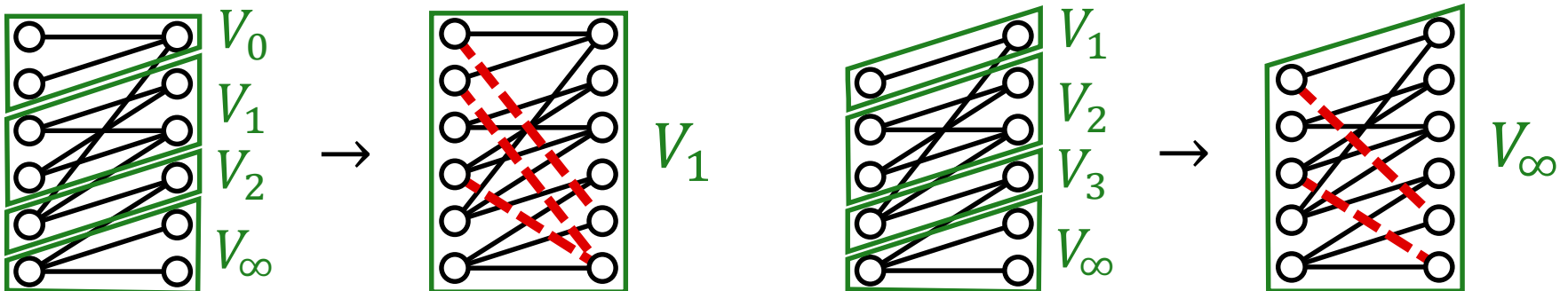
Our Problem

Input $G = (V^+, V^-; E)$: Bipartite Graph

Goal Find a **Minimum Number of Additional Edges** to Make G **DM-irreducible**

DM-decomposition consists of a **Single Component**

$= [\forall e, \exists M: \text{Perfect Matching s.t. } e \in M] + \alpha \text{ (Some Connectivity)}$



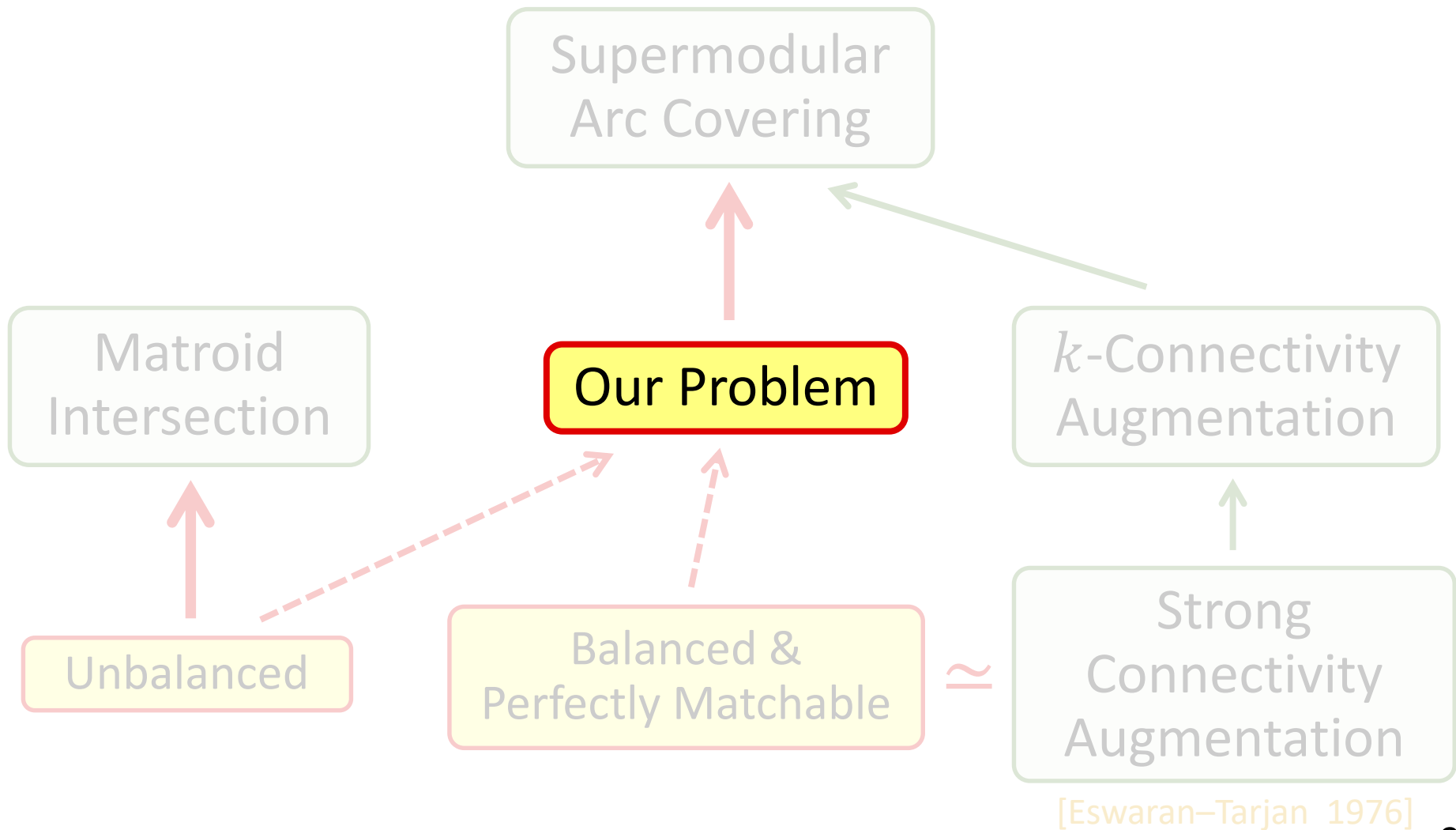
Our Results (Summary)

Input $G = (V^+, V^-; E)$: Bipartite Graph

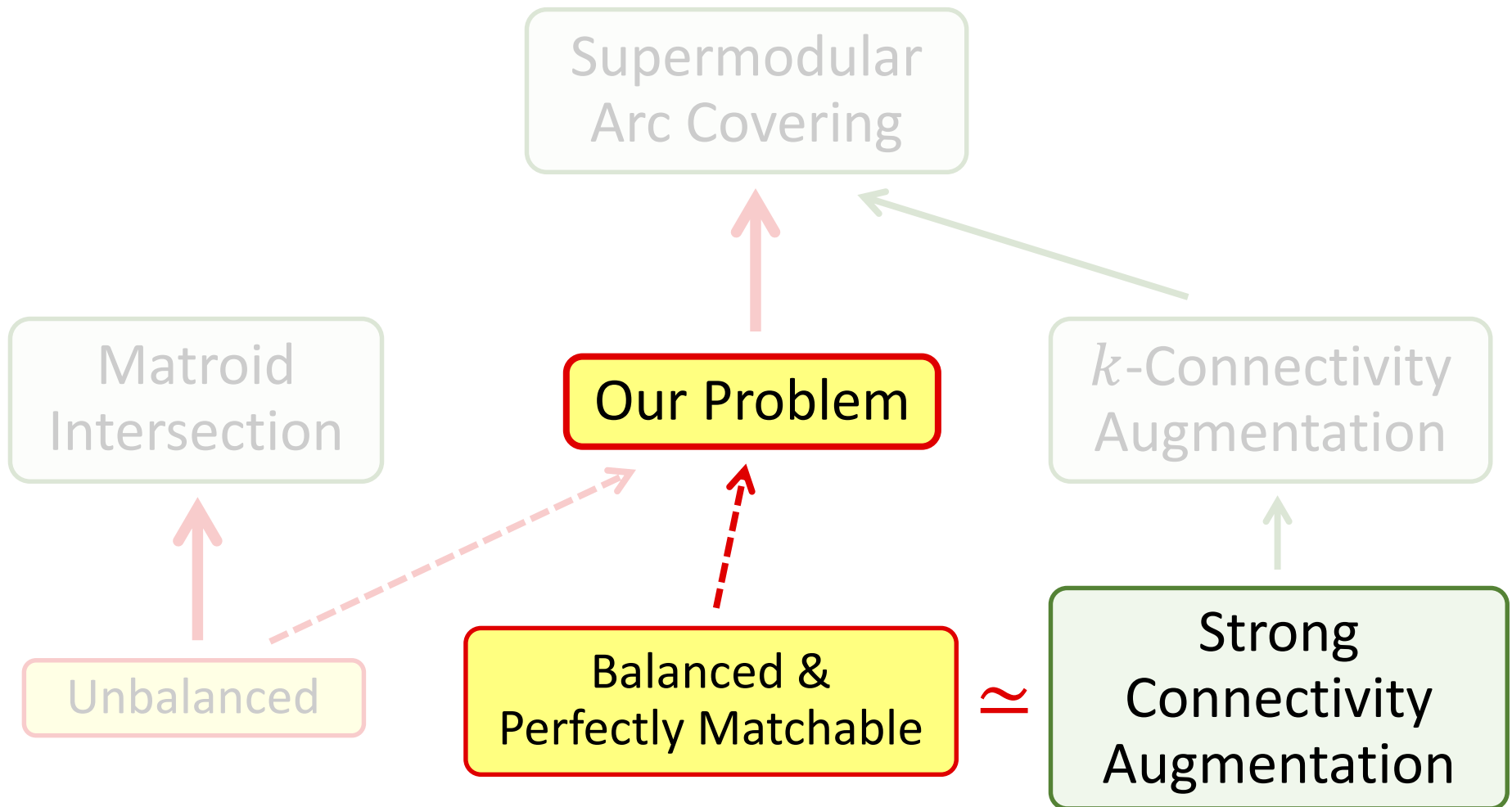
Goal Find a **Minimum Number of Additional Edges**
to Make G **DM-irreducible**

- **Min-Max Duality** via **Supermodular Arc Covering**
[Frank–Jordán 1995]
- Unbalanced ($|V^+| \neq |V^-|$) \subseteq **Matroid Intersection**
- Balanced ($|V^+| = |V^-|$) & Perfectly Matchable
 \simeq **Strong Connectivity Augmentation** [Eswaran–Tarjan 1976]
- Balanced & NOT P.M. \rightarrow **Direct $O(nm)$ -time Algorithm**
(Moreover, **General Case**)

Overview



Overview



Overview

- Min-Max Duality
- Polytime by Ellipsoid

[Frank–Jordán 1995]

Supermodular
Arc Covering

Combinatorial
Pseudopolytime

[Végh–Benczúr 2008]

Min-Max

Polytime

Polytime

Matroid
Intersection

Our Problem

k -Connectivity
Augmentation

Unbalanced

Balanced &
Perfectly Matchable

\approx

Strong
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[Eswaran–Tarjan 1976]

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Weighted ver. OK

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$O(nm)$ -time

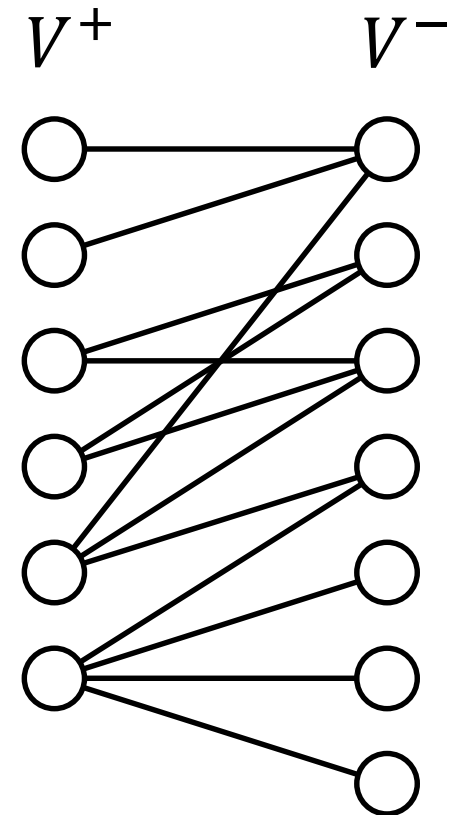
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How to Compute DM-decomposition

$G = (V^+, V^-; E)$: Bipartite Graph

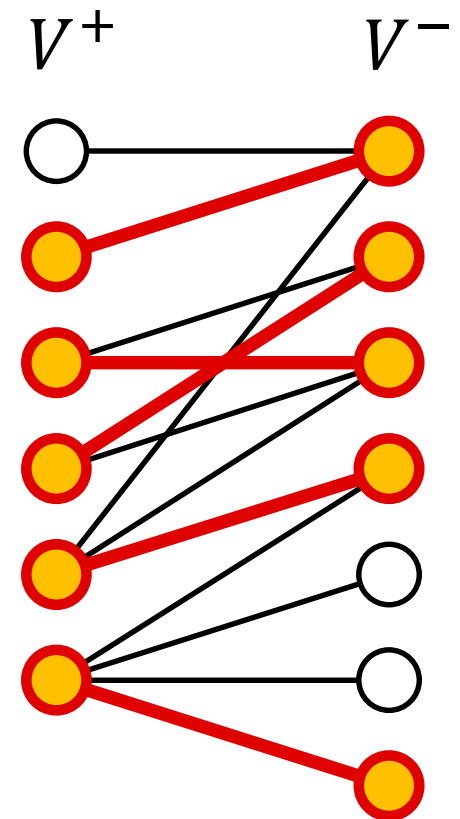
- Find a **Maximum Matching** M in G
- Orient Edges so that
$$\begin{array}{ll} M \Rightarrow \text{Both Directions} & \leftrightarrow \\ E \setminus M \Rightarrow \text{Left to Right} & \rightarrow \end{array}$$
- V_0 : Reachable from $V^+ \setminus \partial^+ M$
- V_∞ : Reachable to $V^- \setminus \partial^- M$
- V_i : Strongly Connected Component of $G - V_0 - V_\infty$



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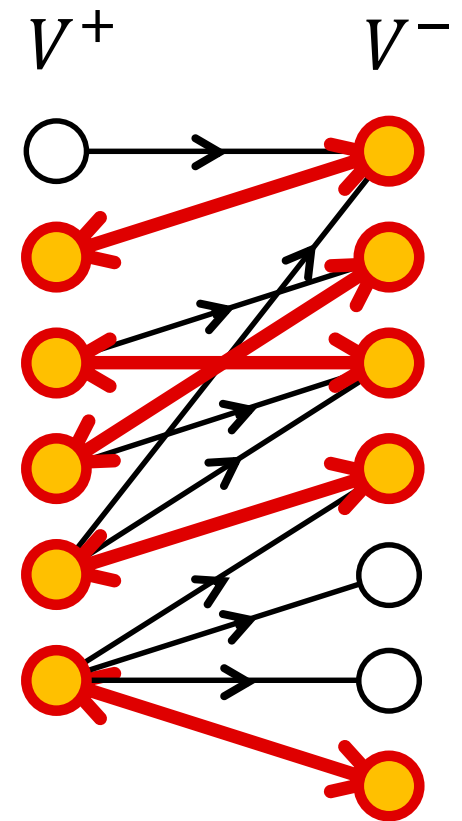
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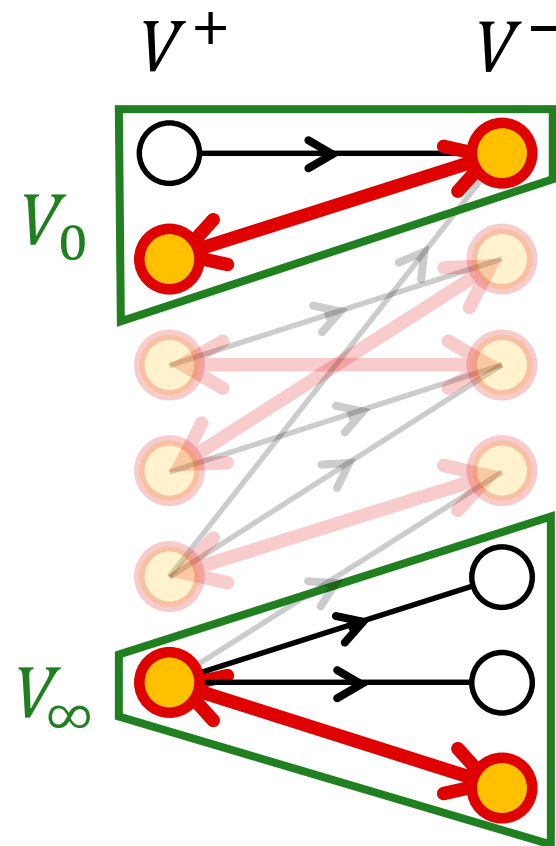
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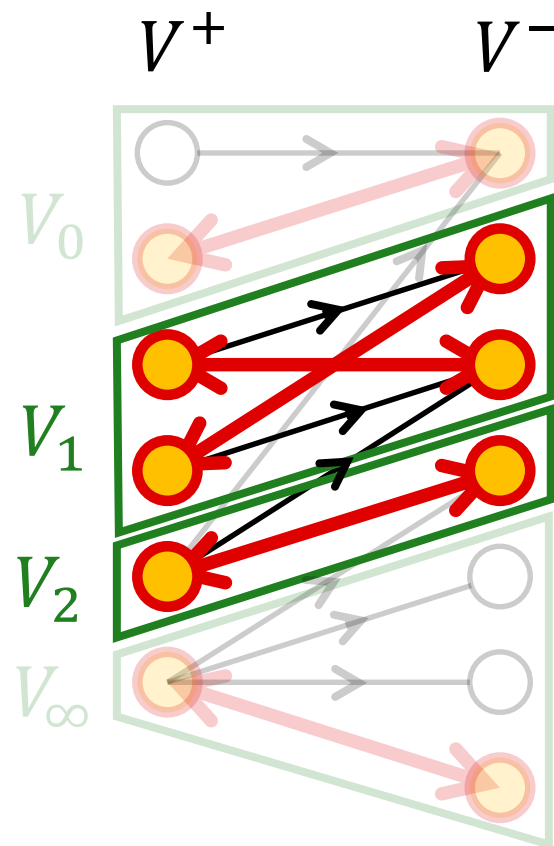
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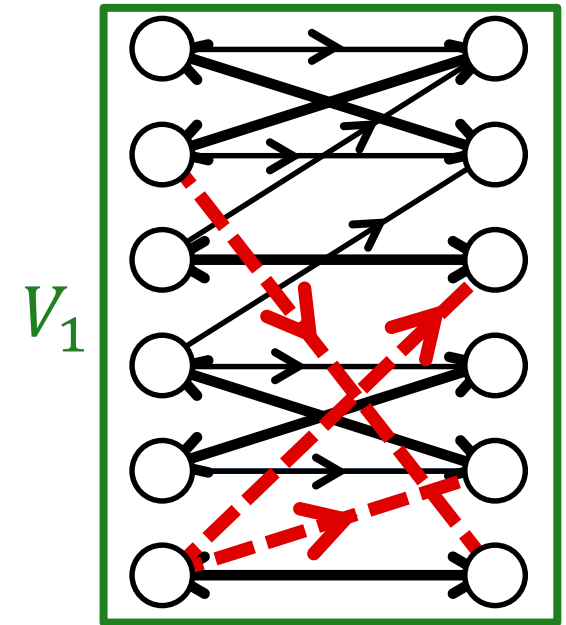
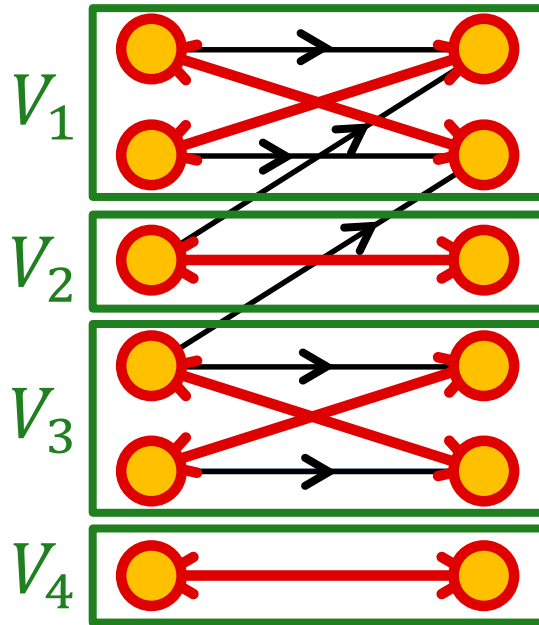
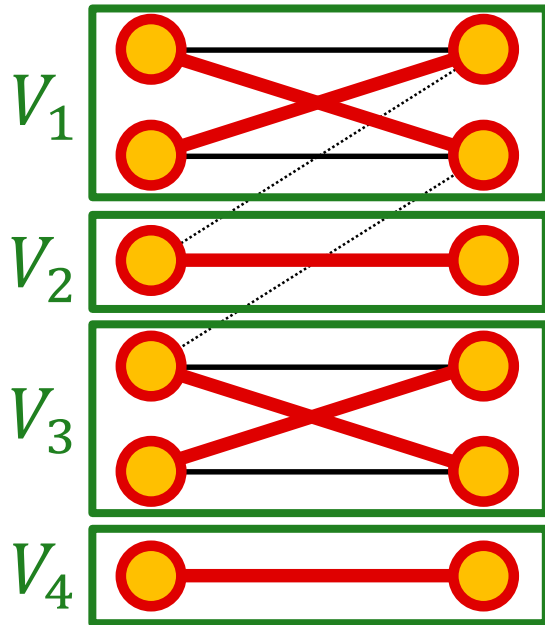
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When Balanced & Perfectly Matchable



DM-decomposition = Decomposition into Strg. Conn. Comps. → Make it Strg. Conn. by Adding Edges

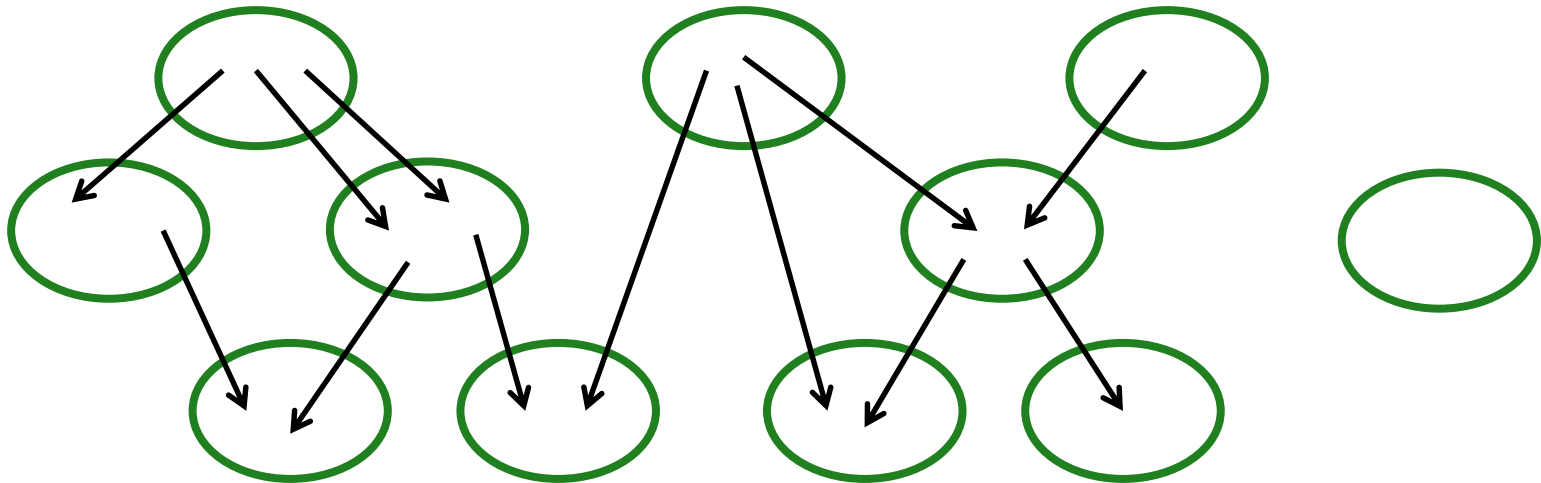
Obs. DM-irreducibility is Equivalent to Strong Connectivity of the Oriented Graph

Strong Connectivity Augmentation

Input $G = (V, E)$: Directed Graph (NOT Strg. Conn.)

Goal Find a **Minimum Number of Additional Edges** to Make G **Strongly Connected**

○: Strg. Conn. Comp.



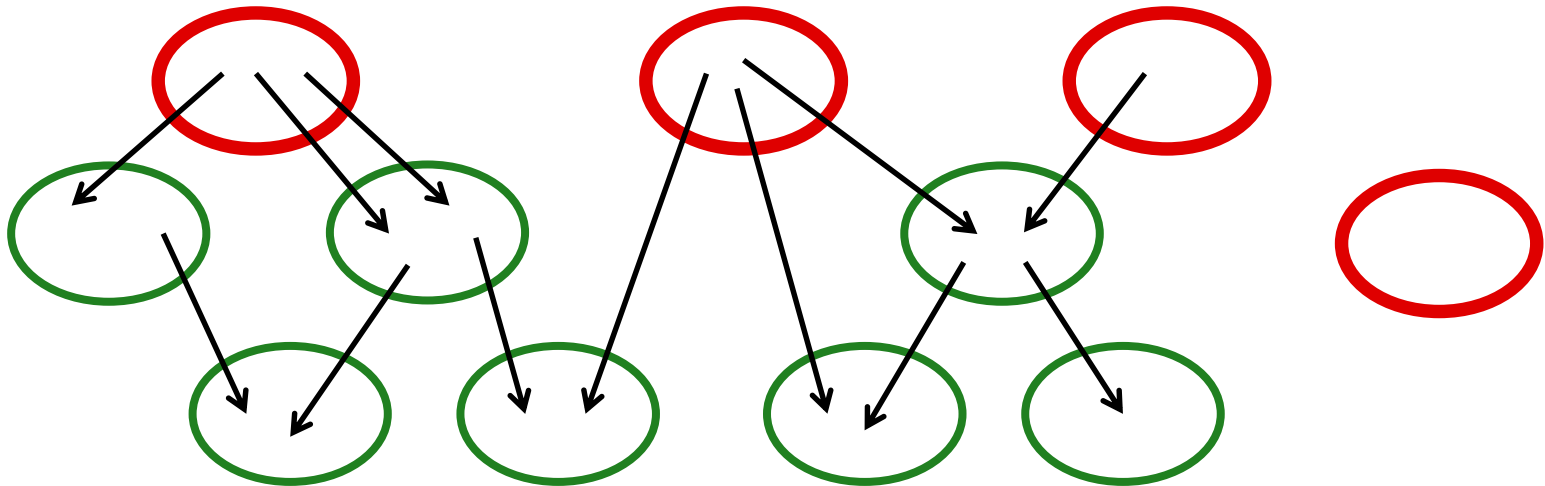
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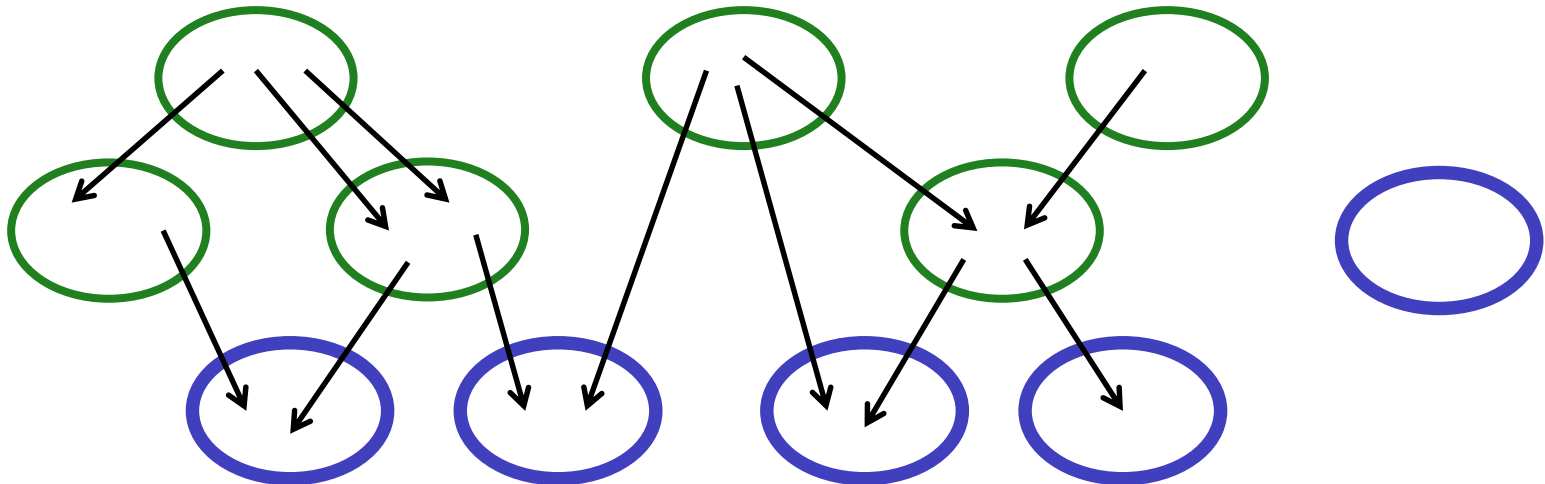
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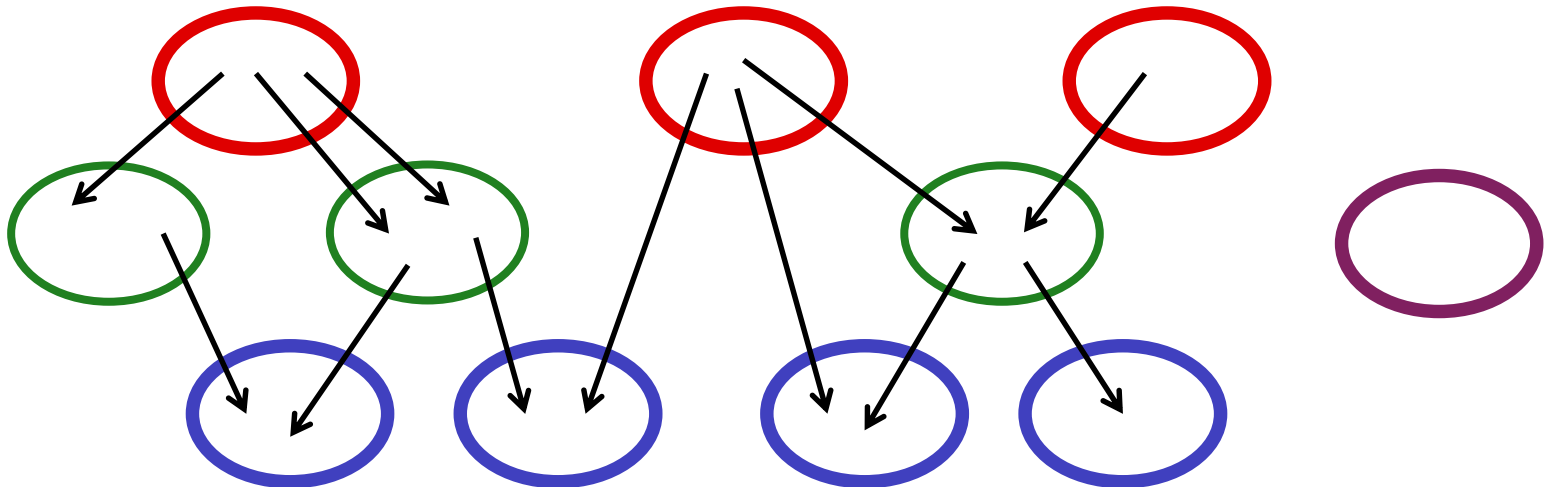
Each **Sink** needs a **Leaving Edge**

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$\max \{ \# \text{ of } \textcolor{red}{\text{Sources}}, \# \text{ of } \textcolor{blue}{\text{Sinks}} \}$ edges are **Necessary**.



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Thm. It is also **Sufficient**.

One can find such an edge set in **Linear Time**.

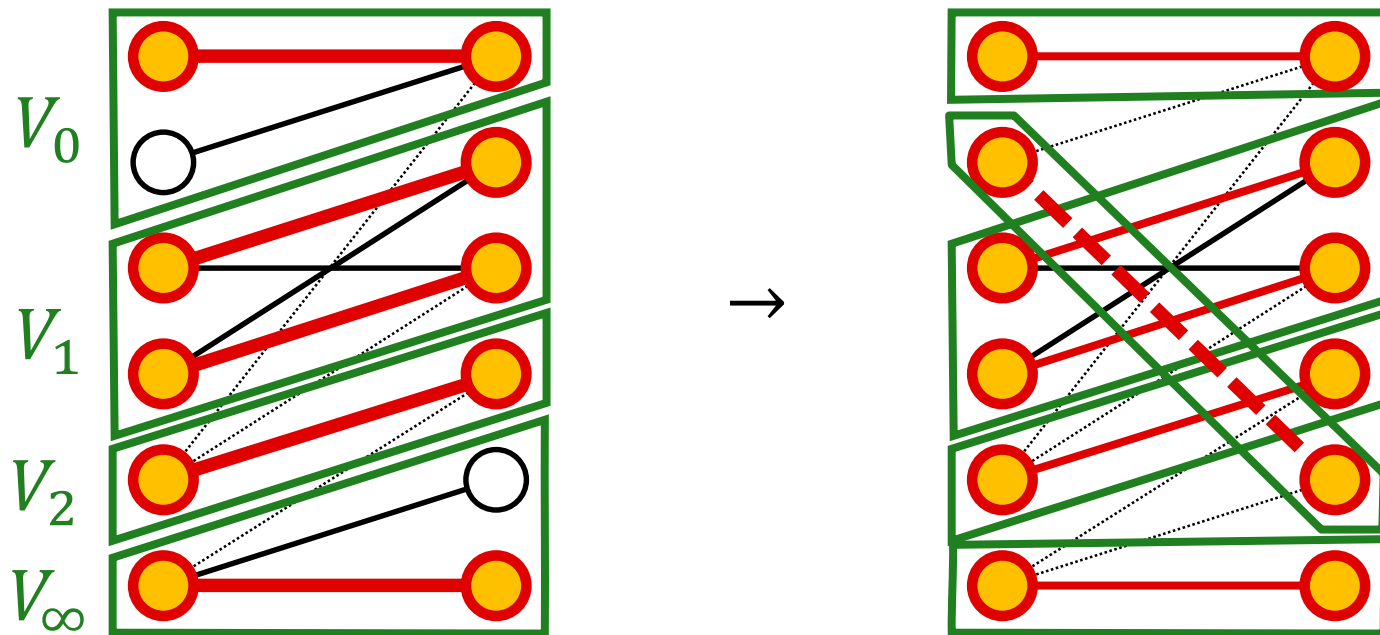
[Eswaran–Tarjan 1976]

Cor. If the input is **Balanced with Perfect Matching**,
Our Problem can be solved in **Linear Time**.

When Balanced & NOT Perfectly Matchable

Idea Reduce to P.M. Case by **Connecting Exposed Vertices**

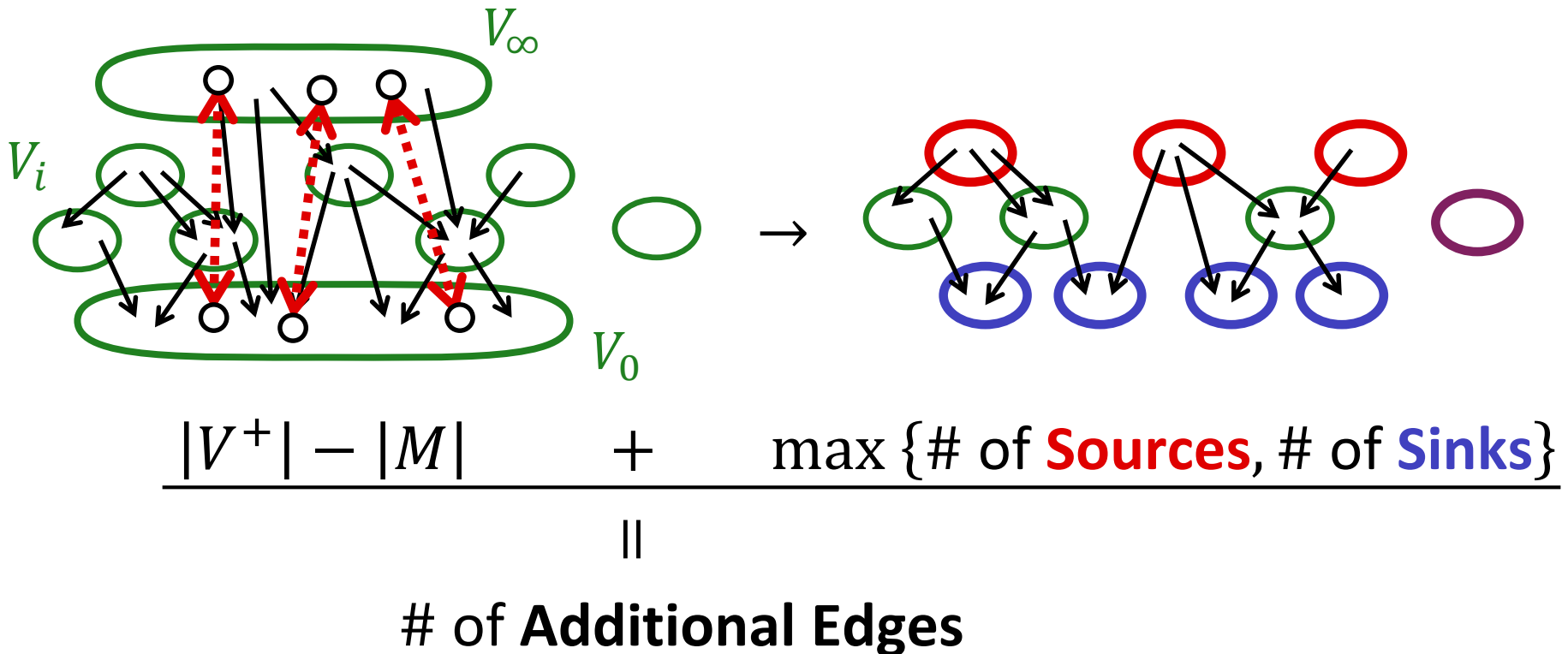
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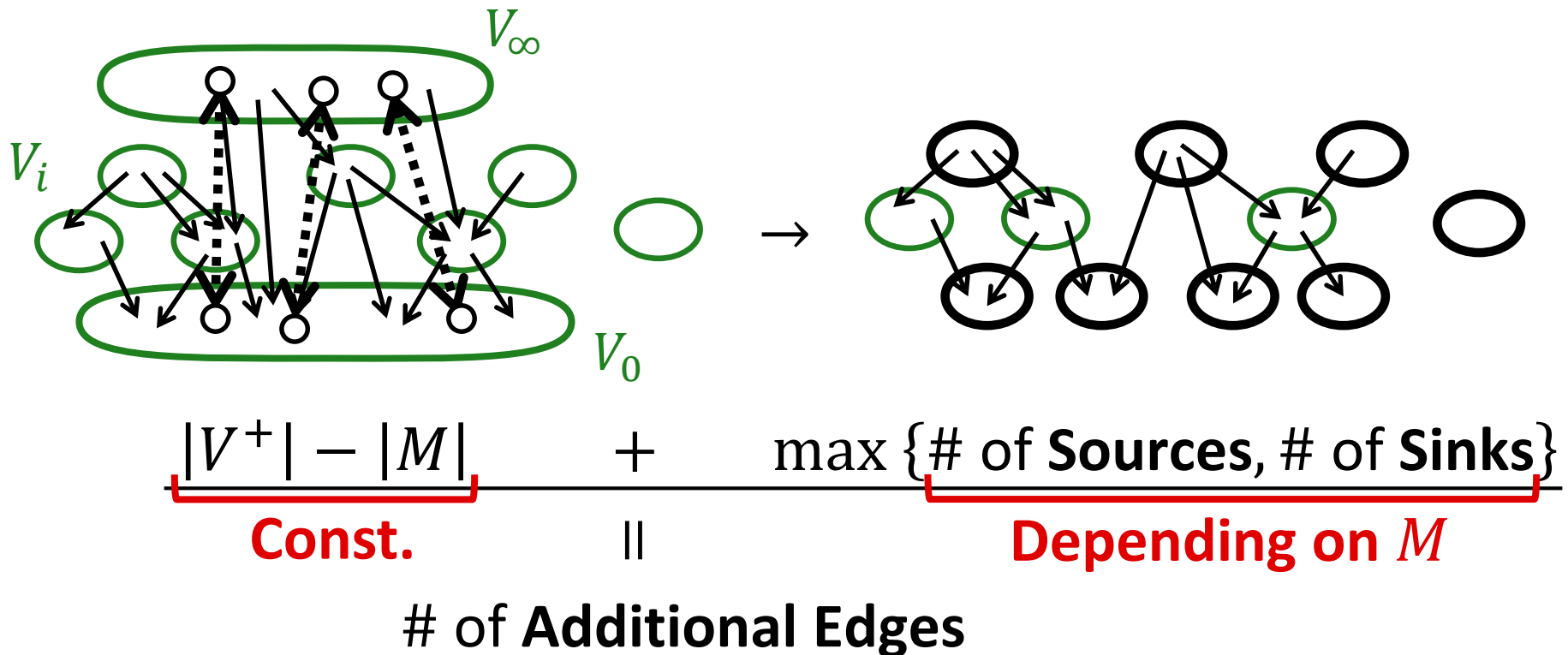
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- (# of **Sources** or of **Sinks**) depends on Max. Matching M
- Find Eligible Perfect Matchings in $G[V_\infty]$ and in $G[V_0]$
 - Minimizing (# of **Sources in V_∞**) and (# of **Sinks in V_0**)
 - Just by finding two edge-disjoint $s-t$ paths $O(n)$ times
- Optimality is guaranteed by **Min-Max Duality**

Thm. If the input is Balanced (in fact, NOT necessary),
Our Problem can be solved in $O(nm)$ time.

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$G = (V^+, V^-; E)$: Bipartite Graph

$$f_G(X^+) := |\Gamma_G(X^+)| - |X^+| \quad (X^+ \subseteq V^+)$$

(Surplus for Hall's Condition)

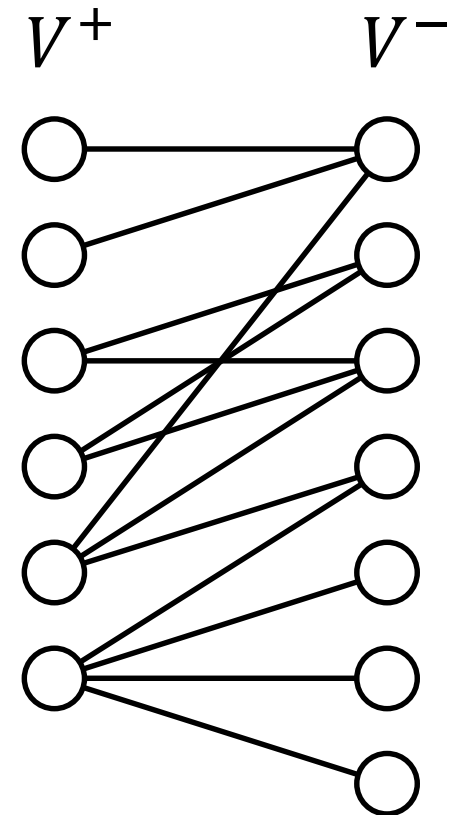
f_G is Submodular

- Minimizers form Distributive Lattice
- $X_0^+ \subsetneq X_1^+ \subsetneq \dots \subsetneq X_k^+$: Maximal Chain

$$V_0^+ := X_0^+, \quad V_0^- := \Gamma_G(X_0^+)$$

$$V_i^+ := X_i^+ \setminus X_{i-1}^+, \quad V_i^- := \Gamma_G(X_i^+) \setminus \Gamma_G(X_{i-1}^+)$$

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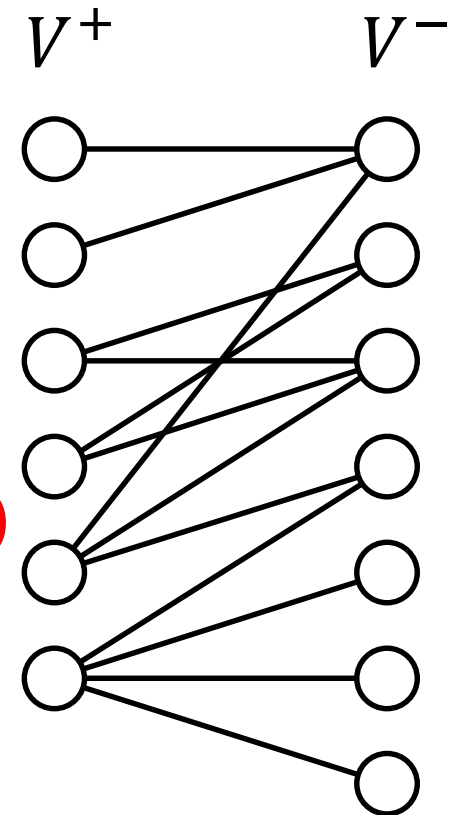
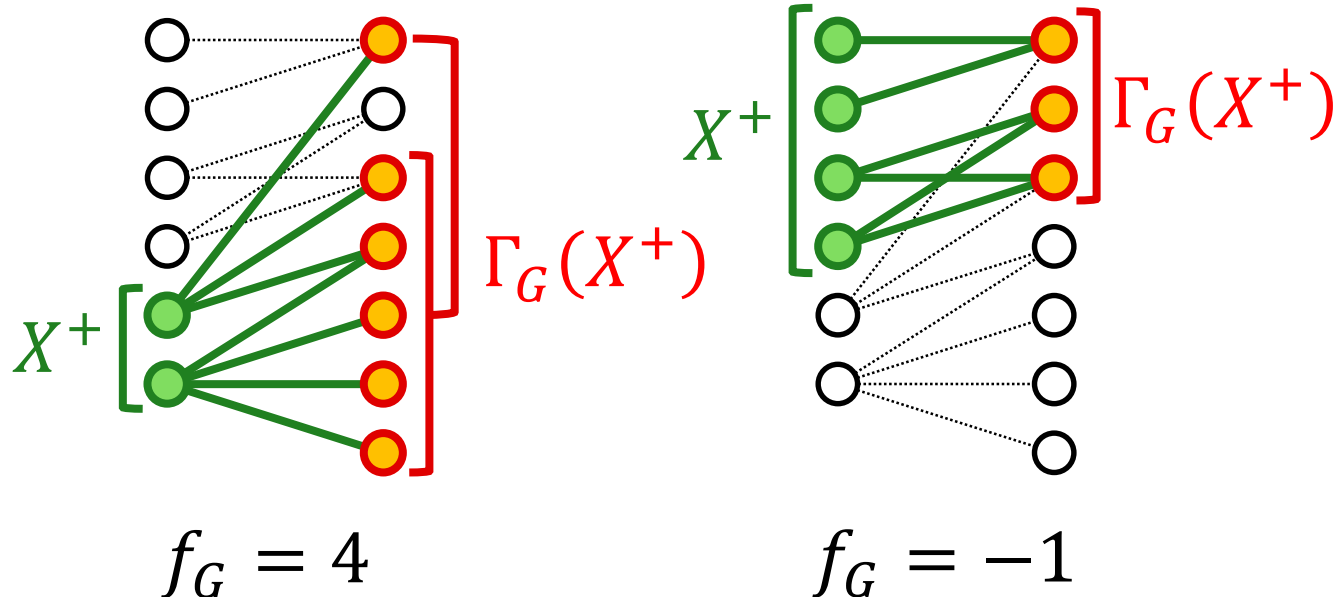


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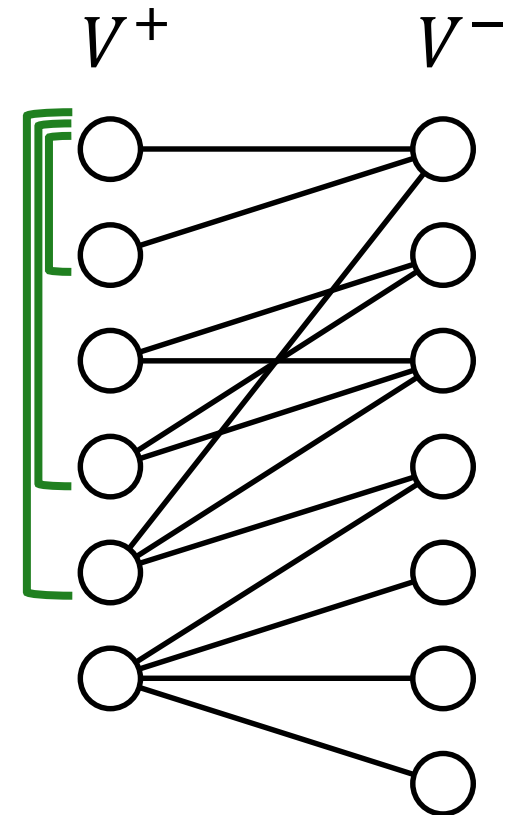
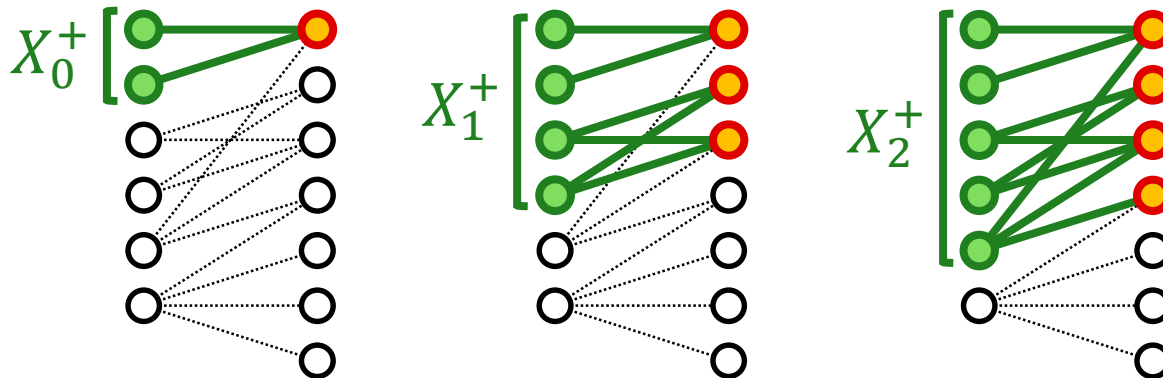
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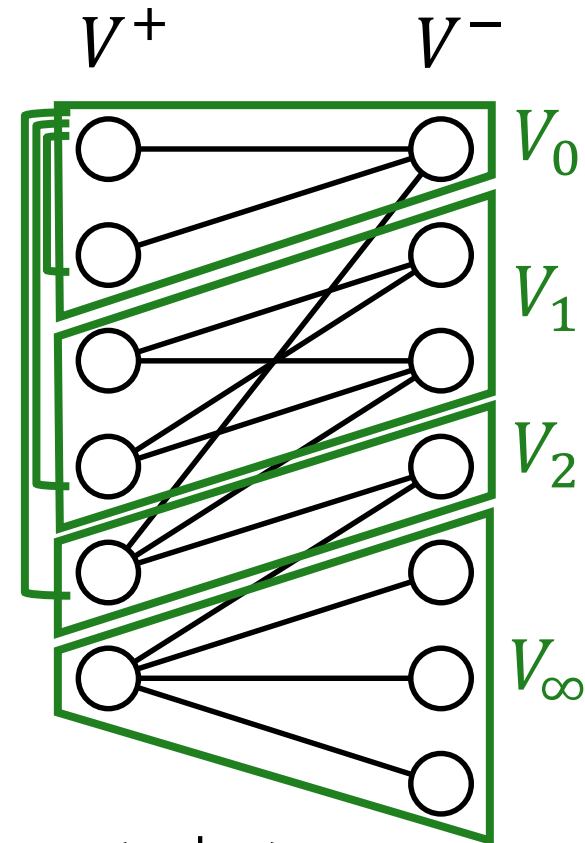
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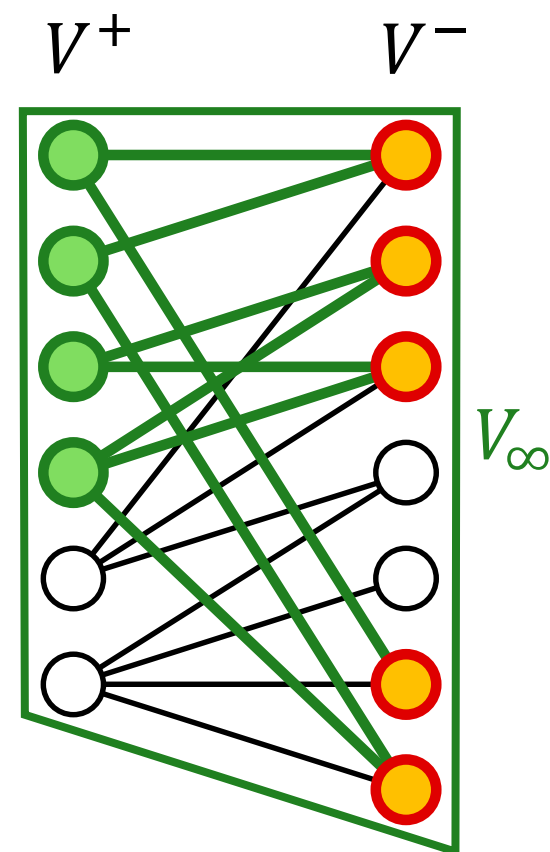
Rephrasing of DM-irreducibility

$G = (V^+, V^-; E)$: Bipartite Graph

$$f_G(X^+) := |\Gamma_G(X^+)| - |X^+| \quad (X^+ \subseteq V^+)$$

(Surplus for Hall's Condition)

- When $|V^+| < |V^-|$ (Unbalanced)
 - $\emptyset \subseteq V^+$ is a unique minimizer
 - $\Leftrightarrow |\Gamma_G(X^+)| > |X^+| \quad (\emptyset \neq \forall X^+ \subseteq V^+)$
- When $|V^+| = |V^-|$ (Balanced)
 - Only \emptyset and V^+ are minimizers
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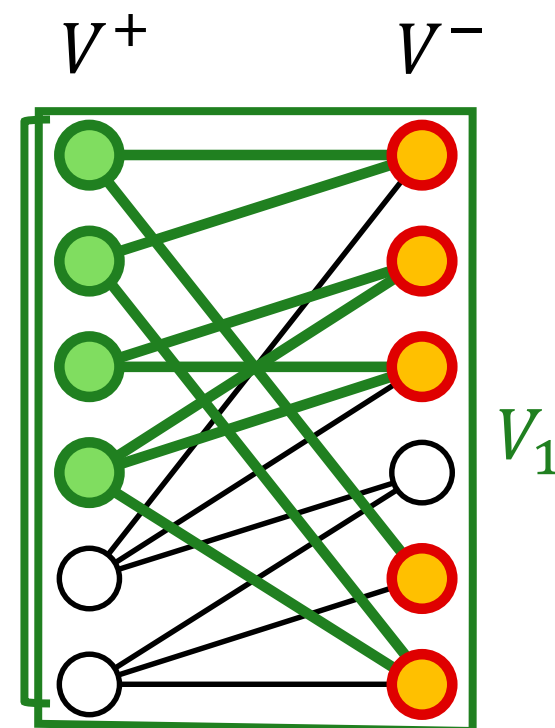
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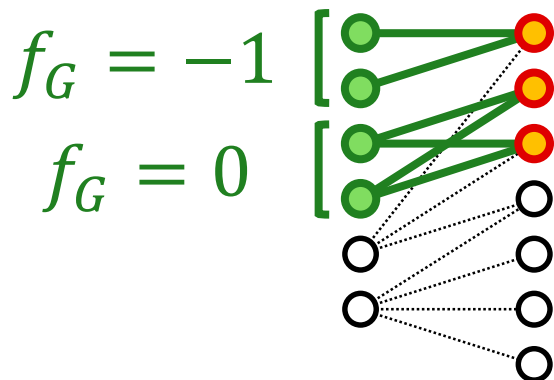
Symmetrically
for V^-

Min-Max Duality (Unbalanced Case)

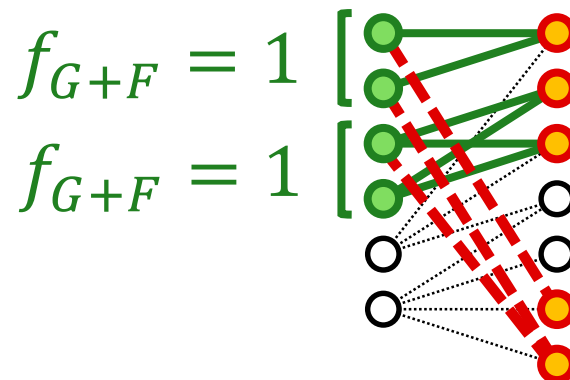
Input $G = (V^+, V^-; E)$: Bipartite Graph ($|V^+| < |V^-|$)

Goal Find a **Smallest Set F of Additional Edges**
 s.t. $|\Gamma_{G+F}(X^+)| > |X^+|$ ($\emptyset \neq \forall X^+ \subseteq V^+$)

$$|F| \geq \sum_{X^+ \in \mathcal{X}^+} \left(1 - \underbrace{f_G(X^+)}_{|\Gamma_G(X^+)| - |X^+|} \right) \quad (\forall X^+: \text{Subpartition of } V^+)$$



→



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Thm. $\min \{ |F| \mid G + F \text{ is DM-irreducible} \}$

\parallel

$\max \{ \sum_{X^+ \in \mathcal{X}^+} (1 - f_G(X^+)) \mid \mathcal{X}^+: \text{Subpartition of } V^+ \}$

Min-Max Duality (Balanced Case)

Input $G = (V^+, V^-; E)$: Bipartite Graph ($|V^+| = |V^-|$)

Goal Find a **Smallest Set F of Additional Edges**
 s.t. $|\Gamma_{G+F}(X^\pm)| > |X^\pm|$ ($\emptyset \neq \forall X^\pm \subsetneq V^\pm$)

Thm. $\min \{ |F| \mid G + F \text{ is DM-irreducible} \}$
 $\parallel \boxed{\mathcal{X}^\pm \neq \{V^\pm\}}$
 $\max \left\{ \begin{array}{l} \max \{ \tau_G(\mathcal{X}^+) \mid \mathcal{X}^+ : \text{Proper Subpartition of } V^+ \}, \\ \max \{ \tau_{\bar{G}}(\mathcal{X}^-) \mid \mathcal{X}^- : \text{Proper Subpartition of } V^- \} \end{array} \right\}$

$$\tau_G(\mathcal{X}^+) := \sum_{X^+ \in \mathcal{X}^+} (1 - f_G(X^+))$$

[BIKY 2018]

$$\bar{G} := (V^-, V^+; \bar{E}): \text{Interchanging } V^+ \text{ and } V^-$$

Supermodular Arc Covering

Thm. V^+, V^- : Finite Sets (possibly intersecting)
 $\mathcal{F} \subseteq 2^{V^+} \times 2^{V^-}$: Crossing Family (Constraint Set)
 $g: \mathcal{F} \rightarrow \mathbf{Z}_{\geq 0}$ Supermodular (Demand on \mathcal{F})

The minimum cardinality of a multiset $A: V^+ \times V^- \rightarrow \mathbf{Z}_{\geq 0}$ of directed edges in $V^+ \times V^-$ that **covers** g is equal to

$$\max_{\mathcal{S} \subseteq \mathcal{F}} \left\{ \sum_{(X^+, X^-) \in \mathcal{S}} g(X^+, X^-) \mid \mathcal{S}: \text{pairwise independent} \right\}$$

[Frank–Jordán 1995]

- Packing (Max) vs. Covering (Min) type Strong Duality
- Polytime Solvability by Ellipsoid Method
- Including Directed k -Connectivity Augmentation etc.

Supermodular Arc Covering

Thm. V^+, V^- : Finite Sets (possibly intersecting)
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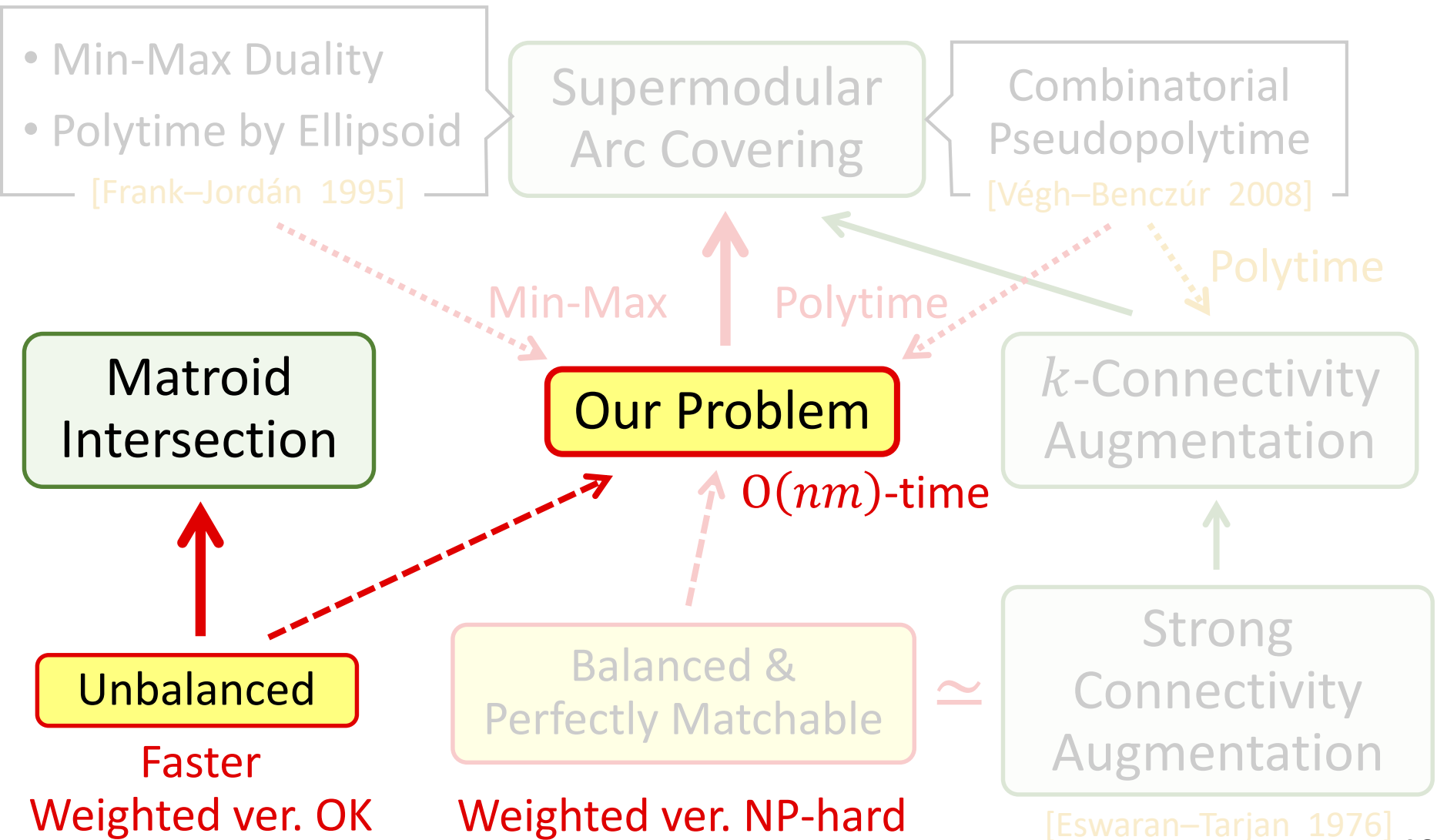
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[Frank–Jordán 1995]

By defining \mathcal{F} and g appropriately for Our Problem, we obtain Min-Max Duality Theorems for Our Problem (via some nontrivial Uncrossing arguments)

Overview

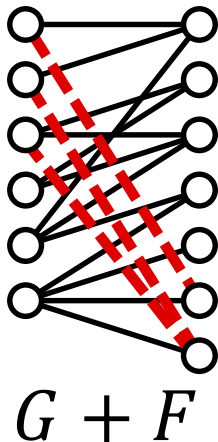


Weighted Problem (Unbalanced Case)

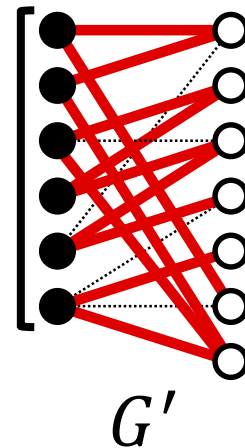
Input $G = (V^+, V^-; E)$: Bipartite Graph ($|V^+| < |V^-|$)
 $c: (V^+ \times V^-) \setminus E \rightarrow \mathbf{R}_{>0}$ (Cost on Addition)

Goal Find a **Minimum-Cost Set F of Additional Edges**
s.t. $G + F$ is **DM-irreducible**

Lem. $\exists G' \subseteq G + F$: **Forest** s.t. $|\Gamma_{G'}(v)| = 2 \ (\forall v \in V^+)$



$$|\Gamma_{G'}(v)| = 2$$



[BIKY 2018]

**Minimally
DM-irreducible**

Weighted Problem (Unbalanced Case)

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Reduce to find a **Min-Weight Common Base** in [BIKY 2018]

\mathbf{M}_1 : Graphic Matroid (with $2|V^+|$ -Truncation)

\mathbf{M}_2 : Partition Matroid (Degree Constraint on V^+)

$\gamma: V^+ \times V^- \rightarrow \mathbf{R}_{\geq 0}$ (Weight); $\gamma(e) := \begin{cases} 0 & (e \in E) \\ c(e) & (e \notin E) \end{cases}$

Our Results (Summary)

Input $G = (V^+, V^-; E)$: Bipartite Graph

Goal Find a **Minimum Number of Additional Edges**
to Make G **DM-irreducible**

- **Min-Max Duality** via **Supermodular Arc Covering**
[Frank–Jordán 1995]
- Unbalanced ($|V^+| \neq |V^-|$) \subseteq **Matroid Intersection**
- Balanced ($|V^+| = |V^-|$) & Perfectly Matchable
 \simeq **Strong Connectivity Augmentation** [Eswaran–Tarjan 1976]
- Balanced & NOT P.M. \rightarrow **Direct $O(nm)$ -time Algorithm**
(Moreover, **General Case**)