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The Inversion of Correlation Matrix for MA(1) Process

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Abstract—The exact expression for the inverse of the correlation matrix for the moving average order one, MA(1) process, is obtained. Its application in the context of longitudinal data analysis is discussed.

Keywords—Inversion of the correlation matrix of AR(1) process, Partitioning, Inversion of a submatrix.

1. INTRODUCTION

Consider a time-series realization $\{y_t: t = 1, \dots, n\}$ of length n from a stationary Gaussian process. Suppose that $\rho_{tt'}$ refers to the correlation between y_t and $y_{t'}$ for $t, t' = 1, \dots, n$. Obviously, $\rho_{tt} = 1$. If $\rho_{tt'} = \rho$ (say), for all $t \neq t'$, $t, t' = 1, \dots, n$, then the process is known to be exchangeable or equicorrelation (EQC) process. Further, suppose that $C_E = (1-\rho)I_n + \rho J_n$ denotes the $n \times n$ correlation matrix of this EQC process, where J_n is an $n \times n$ matrix of 1s. Then its inverse C_E^{-1} from [1] is given by $C_E^{-1} = (a-b)I_n + bJ_n$, where $a = \{1 + (n-2)\rho\}/[(1-\rho)\{1+(n-1)\rho\}]$ and $b = -\rho/[(1-\rho)\{1+(n-1)\rho\}]$.

If $\{y_t: t = 1, \dots, n\}$ follow the AR(1) process $y_t = \phi y_{t-1} + a_t$, where $-1 < \phi < 1$ is the parameter of the process, and $a_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma_a^2)$, then the correlation matrix $C_A = (\phi^{|t-t'|})$ of this

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process has its inverse C_A^{-1} [2]

$$C_A^{-1} = \frac{1}{1-\phi^2}\begin{bmatrix} 1 & -\phi & 0 & 0 & \cdot & 0 & 0 \\ -\phi & 1+\phi^2 & -\phi & 0 & \cdots & 0 & 0 \\ 0 & -\phi & 1+\phi^2 & -\phi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1+\phi^2 & -\phi \\ 0 & 0 & 0 & 0 & \cdots & -\phi & 1 \end{bmatrix}. \tag{1.1}$$

If $\{y_t : t = 1, \dots, n\}$, however, follow the MA(1) process $y_t = a_t - \theta a_{t-1}$, where $-1 < \theta < 1$ is the parameter of the process, and $a_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma_a^2)$, then it is not so easy to invert the correlation matrix C_M of this process [2], where the $n \times n$ C_M matrix is given by

$$C_M = \begin{bmatrix} 1 & \theta_1 & 0 & 0 & \cdots & 0 & 0 \\ \theta_1 & 1 & \theta_1 & 0 & \cdots & 0 & 0 \\ 0 & \theta_1 & 1 & \theta_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \theta_1 \\ 0 & 0 & 0 & 0 & \cdots & \theta_1 & 1 \end{bmatrix}, \tag{1.2}$$

where $\theta_1 = -\theta/(1+\theta^2)$. As a remedy, some approximations were suggested to obtain the inverse of the C_M matrix [3,4]. More specifically, let $c^{tt'}$ denote the $(t,t')^{\text{th}}$ element of the approximate inverse matrix of C_M. Following [3,4], these elements are given by

$$\frac{(-1)^{|t-t'|}}{(1+\theta_1^2)\sqrt{1-4\theta_1^2}}\left\{\frac{1-\sqrt{1-4\theta_1^2}}{2\theta_1}\right\}^{|t-t'|}. \tag{1.3}$$

Later on, Shaman [5] considered three new approximations to derive the inverse of the C_M matrix. Under certain modifications, these three approaches appear to agree with the approximation given in (1.3).

In the next section, we provide the exact expression for the inverse C_M^{-1} of the correlation matrix C_M. A numerical illustration is given in Section 3 to compare the exact inverse C_M^{-1} with the approximate inverse of C_M. In Section 4, we discuss an immediate application of this inversion process to the longitudinal data analysis.

2. DERIVATION OF C_M^{-1}

THEOREM 2.1. *For $t, t' = 1, \dots, n$, the $(t,t')^{th}$ element of the inverse matrix of C_M (1.2) is given by*

$$\frac{1+\theta^2}{1-\theta^2}\left\{\theta^{|t-t'|} - \theta^{2(n+2)-t-t'-2}\right\} - \frac{\theta^{t+t'}}{1-\theta^{2(n+2)-2}}\left\{\left(1-\theta^{2(n+2)-2t-2}\right)\left(1-\theta^{2(n+2)-2t'-2}\right)\right\}.$$

PROOF. The technique of the derivation depends on the fact that the inverse of the correlation matrix of the AR(1) process, i.e., C_A^{-1}, contains an $(n-2)\times(n-2)$ symmetric matrix which has the same structure as the correlation matrix of the MA(1) process. More specifically, partition the C_A^{-1} matrix in (1.1) as

$$C_A^{-1} = \frac{1+\phi^2}{1-\phi^2}\begin{bmatrix} P & G \\ G^\top & Q \end{bmatrix}, \tag{2.1}$$

where $G = [0, 0, \dots, -\phi/(1+\phi^2)]^\top$ is the $(n-1)\times 1$ vector, $Q = 1/(1+\phi^2)$ is a scalar quantity, $G^\top$ is the transpose of G, and P is the leading $(n-1)\times(n-1)$ symmetric matrix which may further be partitioned as

$$P = \begin{bmatrix} A & B \\ B^\top & D_M \end{bmatrix}, \tag{2.2}$$

where $A = 1/(1+\phi^2)$ is a scalar quantity, $B = [-\phi/(1+\phi^2), 0, \ldots, 0]$ is the $1 \times (n-2)$ row vector, and D is the $(n-2) \times (n-2)$ symmetric matrix given by

$$D_M = \begin{bmatrix} 1 & \phi_1 & 0 & 0 & \cdots & 0 & 0 \\ \phi_1 & 1 & \phi_1 & 0 & \cdots & 0 & 0 \\ 0 & \phi_1 & 1 & \phi_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \phi_1 \\ 0 & 0 & 0 & 0 & \cdots & \phi_1 & 1 \end{bmatrix} \tag{2.3}$$

with $\phi_1 = -\phi/(1+\phi^2)$. Notice that D_M has the exact same form as C_M in (1.2). The difference between these two matrices is: C_M is an $n \times n$ symmetric matrix, whereas D_M is an $(n-2) \times (n-2)$ symmetric matrix. For convenience, we denote the C_M matrix by $C_M(\theta_1)$ and similarly the D_M matrix by $D_M(\phi_1)$. Here our objective is to obtain $D_M^{-1}(\phi_1)$ and extend its dimension to obtain $C_M^{-1}(\phi_1)$ or $C_M^{-1}(\theta_1)$.

To obtain $D_M^{-1}(\phi_1)$, we first obtain P^{-1} from (2.1) and then use (2.2). Since $C_A = (\phi^{|t-t'|})$, and from [1]

$$\begin{pmatrix} P & G \\ G^\top & Q \end{pmatrix}^{-1} = \begin{pmatrix} P^{-1} + FE^{-1}F^\top & -FE^{-1} \\ -E^{-1}F & E^{-1} \end{pmatrix} \tag{2.4}$$

with $E = Q - G^\top P^{-1} G$ and $F = P^{-1}G$, it follows that

$$E^{-1} = \frac{1+\phi^2}{1-\phi^2}, \qquad -FE^{-1} = \left\{\frac{1+\phi^2}{1-\phi^2}\right\} \left[\phi^{n-1}, \phi^{n-2}, \ldots, \phi\right]^\top,$$

implying that $F = -[\phi^{n-1}, \phi^{n-2}, \ldots, \phi]$ and

$$FE^{-1}F^\top = \left\{\frac{1+\phi^2}{1-\phi^2}\right\} \left(\phi^{2n-t-t'}\right) : (n-1) \times (n-1), \tag{2.5}$$

for $t, t' = 1, \ldots, n-1$. This yields the $(n-1) \times (n-1)$ P^{-1} matrix as

$$P^{-1} = \left\{\frac{1+\phi^2}{1-\phi^2}\right\} \left(\phi^{|t-t'|} - \phi^{2n-t-t'}\right). \tag{2.6}$$

Next, we partition the P^{-1} matrix as

$$P^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{2.7}$$

which is

$$\begin{pmatrix} A & B \\ B^\top & D_M \end{pmatrix}^{-1}, \tag{2.8}$$

the same as (2.2). Now by equating (2.8) to (2.7), we obtain D_M^{-1} (i.e., $D_M^{-1}(\phi_1)$). From [6]

$$B_{22} = D_M^{-1} - B_{21} B D_M^{-1}, \tag{2.9}$$

where $B_{21} = -D_M^{-1} B^\top B_{11}$, yielding $-D_M^{-1} B^\top = B_{21}/B_{11}$ and

$$B_{22} = D_M^{-1} + \frac{B_{21} B_{21}^\top}{B_{11}}, \tag{2.10}$$

B_{11} being a scalar quantity. Thus, we obtain

$$D_M^{-1} = B_{22} - \frac{1}{B_{11}} B_{21} B_{12}, \tag{2.11}$$

where, by (2.6),

$$
\begin{aligned}
B_{11} &= \left\{\frac{1+\phi^2}{1-\phi^2}\right\}\left[1-\phi^{2n-2}\right],\\
B_{12} &= \left\{\frac{1+\phi^2}{1-\phi^2}\right\}\left[\phi\left\{1-\phi^{2n-2-2}\right\},\phi^2\left\{1-\phi^{2n-2-4}\right\},\dots,\phi^{n-2}\left\{1-\phi^{2n-2-2(n-2)}\right\}\right],\\
B_{21} &= B_{12}^{\mathrm{T}},\\
B_{22} &= \left\{\frac{1+\phi^2}{1-\phi^2}\right\}\left(\phi^{|t-t'|}-\phi^{2n-t-t'-2}\right):(n-2)\times(n-2).
\end{aligned}
$$

Consequently, the $(t,t')^{\text{th}}$ element of the $(n-2)\times(n-2)$ D_M^{-1} matrix is given by

$$
\left\{\frac{1+\phi^2}{1-\phi^2}\right\}\left[\left\{\phi^{|t-t'|}-\phi^{2n-t-t'-2}\right\}-\frac{\phi^{t+t'}}{1-\phi^{2n-2}}\left\{\left(1-\phi^{2n-2t-2}\right)\left(1-\phi^{2n-2t'-2}\right)\right\}\right]. \tag{2.12}
$$

Now by replacing n with $n+2$ in (2.12), we obtain the inverse of the $n\times n$ D_M matrix, that is, the inverse of $D_M(\phi_1): n\times n$ with $\phi_1=-\phi/(1+\phi^2)$. Next, by replacing ϕ with θ, i.e., ϕ_1 with $\theta_1=-\theta/(1+\theta^2)$, we obtain the $C_M^{-1}(\theta_1)$ as in the theorem.

3. A NUMERICAL VERIFICATION

In this section, we verify the exactness of our results given in Theorem 2.1 and also we check the performance of the approximation (1.3) for the inversion of the correlation matrix C_M. We consider a time-series realization $\{y_t: t=1,\dots,4\}$ from an MA(1) process $y_t=a_t+0.5a_{t-1}$, so that in the notation of (1.2) and (1.3), $\theta=-0.5$ and the 4×4 correlation matrix C_M is given by

$$
C_M=\begin{bmatrix}1 & 0.4 & 0 & 0\\ 0.4 & 1 & 0.4 & 0\\ 0 & 0.4 & 1 & 0.4\\ 0 & 0 & 0.4 & 1\end{bmatrix}. \tag{3.1}
$$

Now using our results from Theorem 2.1, the inverse of the C_M matrix is computed as

$$
C_M^{-1}=\begin{bmatrix}1.2463 & -0.6158 & 0.2933 & -0.1173\\ -0.6158 & 1.5396 & -0.7331 & 0.2933\\ 0.2933 & -0.7331 & 1.5396 & -0.6158\\ -0.1173 & 0.2933 & -0.6158 & 1.2463\end{bmatrix}, \tag{3.2}
$$

which is the inverse of the C_M matrix as it is readily verified that $C_M C_M^{-1}=I_4$, where I_4 is the 4×4 identity matrix. Next, by using equation (1.3), the approximate inverse of C_M is computed as

$$
C_M^{-1}=\begin{bmatrix}1.3333 & -0.6667 & 0.3333 & -0.1667\\ -0.6667 & 1.3333 & -0.6667 & 0.3333\\ 0.3333 & -0.6667 & 1.3333 & -0.6667\\ -0.6667 & 0.3333 & -0.6667 & 1.3333\end{bmatrix}, \tag{3.3}
$$

which, after multiplying by the C_M matrix does not yield the I_4 matrix. Note that it is not only that the elements of the approximate inverse matrix in (3.3) are different from those in (3.2), the first and last leading diagonal elements in the approximate inverse matrix also appear to be the same as other diagonal elements of this matrix, whereas in the exact inverse matrix, the first and the last diagonal elements are the same but they are different than the other diagonal elements.

4. APPLICATION

The inverse of the correlation matrix of the MA(1) process is used in estimation and forecasting in various ways in time series analysis. One of the main advantages of having the exact expression of $C_M^{-1}(\theta)$ available is that one can avoid the computer-based calculations for this which may not be feasible sometimes especially for large n, size of the series.

Recently, there have been studies [7] in other areas, such as in cluster longitudinal set-up, where the inversion of the correlation matrix of the moving average process of order 1 is used to compute the standard errors of the regression coefficient obtained by applying the so-called generalized estimating equations (GEE) approach. But the inversion itself is done based on a computer program. To be specific, in the longitudinal set-up discussed by Liang and Zeger [7], suppose that the so-called 'working' and 'true' correlation structures in their notations are considered to be the structures of the MA(1) and EQC processes, respectively. Then, for associated design matrix X_i of order $n_i \times p$, and diagonal variance matrix A_i of order $n_i \times n_i$ under the i^{th} $(i = 1, \dots, K)$ cluster, the asymptotic (as $K \to \infty$) covariance matrix of the GEE-based regression estimator is given by

$$\begin{aligned} V\left(\hat{\beta}_G\right) = \lim_{K\to\infty} & \left(\sum_{i=1}^{K} X_i^\top A_i^{1/2} C_{iM}^{-1}(\theta) A_i^{1/2} X_i\right)^{-1} \\ & \times \left\{\sum_{i=1}^{K} X_i^\top A_i^{1/2} C_{iM}^{-1}(\theta) C_{iA}(\phi) C_{iM}^{-1}(\theta) A_i^{1/2} X_i\right\} \\ & \times \left(\sum_{i=1}^{K} X_i^\top A_i^{1/2} C_{iM}^{-1}(\theta) A_i^{1/2} X_i\right)^{-1}, \end{aligned} \tag{4.1}$$

where $C_{iM}(\theta)$ and $C_{iA}(\phi)$ are the correlation matrices of the 'working' MA(1) and true AR(1) processes. It is clear that one may now use the exact expression given in Theorem 2.1 for the $n_i \times n_i$ (for any small or large n_i) inverse matrix $C_{iM}^{-1}(\theta)$ in (3.1) to compute such covariance matrices.

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