(1) $S(\theta)=l'(\theta)$, $I(\theta)=-l''(\theta)=-S'(\theta)$, $I(\hat\theta)$ and $J(\theta)=E[I(\theta;x)]$

(2) Find $\hat\theta$ when the support set depends on the unknown parameter θ

Invariance property of MLE and Newton's Method

Example:

Suppose $X_i \overset{iid}{\sim} \text{Poi}(\theta)$, $i=1,\dots,n$. $\theta>0$

Find the MLE of $\tau=P[X_i=0]$

Answer: to find the MLE of θ, we find $L(\theta), l(\theta), S(\theta)$ and solve $S(\theta)=0$ to find $\hat\theta$

$$\tau=P[X_1=0]=\frac{e^{-\theta}\theta^0}{0!}=e^{-\theta}$$

To find the MLE of τ we find $L(\tau), l(\tau), S(\tau)$ and solve $S(\tau)=0$ to find $\hat\tau$

$$L(\tau)=\prod_{i=1}^{n} f(x_i;\tau)$$

[Recall: $f(x;\theta)=\frac{e^{-\theta}\theta^x}{x!}$, and $\tau=e^{-\theta}$, $\theta=(-\log\tau)$ then $f(x;\tau)=\frac{e^{-\tau}\tau^x}{x!}$]

$$\therefore L(\tau)=\prod_{i=1}^{n}\frac{\tau(-\log\tau)^{x_i}}{x_i!}=\frac{\tau^n(-\log\tau)^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$$

and $l(\tau)=n\log(\tau)+(\sum_{i=1}^n x_i)\log(-\log\tau)-\log\prod_{i=1}^n x_i!$

$$S(\tau)=\frac{\partial l(\tau)}{\partial\tau}=\frac{n}{\tau}+\frac{\sum_{i=1}^n x_i}{-\log\tau}\left(-\frac{1}{\tau}\right)$$

$$=\frac{n\log\tau+\sum_{i=1}^n x_i}{\tau(\log\tau)}$$

Solve $S(\tau)=0$ to find $n\log\tau+\sum_{i=1}^n x_i=0$

i.e $\hat\tau=e^{-\bar x}$

Verify $S(\tau)=0$ if $\tau=\hat\tau$ and $S(\tau)<0$ of $\tau=\hat\tau$?

As $S(\tau)=\frac{n\log\tau+\sum_{i=1}^n x_i}{\tau(\log\tau)}$, $\tau\log\tau<0$, as

$\tau=P[X_i=0]\in(0,1)$

$S(\tau)>0$ if $n\log\tau+\sum_{i=1}^n x_i<0$ (i.e $\tau<e^{-\bar x}=\hat\tau$)

and $S(\tau)<0$ if $n\log(\tau)+\sum_{i=1}^n x_i>0$ (i.e $\tau>e^{-\bar x}=\hat\tau$)

In Summary, the MLE of τ is

$\hat\tau=e^{-\bar x}$ (recall $\tau=P[X_i=0]=e^{-\theta}$)

Thm 6.2.17 Invariance property of MLE

If $\hat\theta$ is the MLE of θ and $\tau=\tau(\theta)$ is a 1-to-1 function of θ then the MLE of τ.

$$\hat\tau=\tau(\hat\theta)$$

Back to the example. $\hat\theta=\bar x$ and

$\tau=e^{-\theta}=\tau(\theta)$ is 1-to-1 i.e $\tau(\theta)$ is strictly montone decreasing in θ), then according to 6.2.17, the MLE of τ,

$$\hat\tau=\tau(\hat\theta)=e^{-\hat\theta}=e^{-\bar x}$$

Example 6.3.18

Suppose $X_i\overset{iid}{\sim}f(x;\theta)=\theta\cdot x$, $0<x<1$, $\theta>0$. Find the MLE of the median

Answer:

In 6.2.12 we found

$$\hat\theta=-\frac{n}{\sum_{i=1}^n \log x_i}$$

let τ be the median of $f(x;\theta)$ and find

$\tau=\tau(\theta)$, a 1-to-1 function of θ, then due to 6.2.7

$\hat\tau=\tau(\hat\theta)$

let $F(\cdot)$ be the c.d.f of .

$$X_i\overset{iid}{\sim}f(x;\theta)=\theta x^{\theta-1}, \quad 0<x<1$$

$$F(x)=\begin{cases}0 & \text{if } x\le 0\\ 1 & \text{if } x\ge 1\\ (*) & \text{if } 0<x<1\end{cases}$$

$$(*)=\int_0^x f(x;\theta)\,dx=\int_0^x \theta\cdot x^{\theta-1}\,dx=x^\theta$$

Here,

$F(\tau)=\tau^\theta=\frac{1}{2}$, $0<\tau<1$

i.e $\tau=(\frac{1}{2})^{\frac{1}{\theta}}=\tau(\theta)$

and $\tau(\theta)$ is strictly monotone increasing in θ

then $\hat\tau=\tau(\hat\theta)=(\frac{1}{2})$

Exercise 6.2.19

Example 6.2.16 Suppose $X_i \overset{iid}{\sim} WEI(1,\theta), \theta>0, i=1,\dots,n$

Find the MLE of θ.

Answer: $L(\theta)=\prod_{i=1}^{n} f(x_i;\theta)=\prod_{i=1}^{n}\theta x_i^{\theta-1}e^{-x_i^{\theta}}$

$=\theta^n\left(\prod_{i=1}^{n} x_i\right)^{\theta-1} e^{-\sum_{i=1}^{n} x_i^{\theta}}$

$l(\theta)=n\log(\theta)+(\theta-1)\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n} x_i^{\theta}$

$S(\theta)=\frac{\partial l(\theta)}{\partial(\theta)}=\frac{n}{\theta}+\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}(\log(x_i))x_i^{\theta}$

[Notice that $\frac{\partial(x^\theta)}{\partial(\theta)}=\log(x)x^\theta$ but $\frac{\partial(x^\theta)}{\partial x}=\theta x^{\theta-1}$

Solve $S(\theta)=0$.]

i.e. $\frac{n}{\theta}=\sum_{i=1}^{n}(\log x_i)x_i^{\theta}-\sum_{i=1}^{n}\log x_i$

$\therefore$ There is no explicit solution(s) to $S(\theta)=0$

Next, we find the solution to $S(\theta)=0$, $\hat\theta$ normally through iteration, such as Newton's Method.

Step1: Find the initial value $\theta^{(0)}$

Step2: $\theta^{(r+1)}=\theta^{(r)}+\frac{S(\theta^{(r)})}{I(\theta^{(r)})}$

Therefore,

$\theta^{(1)}=\theta^{(0)}+\frac{S(\theta^{(0)})}{I(\theta^{(0)})}$

$\theta^{(2)}=\theta^{(1)}+\frac{S(\theta^{(1)})}{I(\theta^{(1)})}$

Step3: let $\theta^{(k)}$ be the numerical solution to

$S(\theta)=0$, if $|\theta^{(k+1)}-\theta^{(k)}|<\varepsilon$, where ε is a prespecified small value. $\varepsilon>0$, e.g. $\varepsilon=10^{-5}$

Remark:

Verify $\theta^{(k)}$ is a maximizer at the end of step 3.

Back to 6.2.16.

$S(\theta)=\frac{n}{\theta}+\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}(\log x_i)x_i^{\theta}$

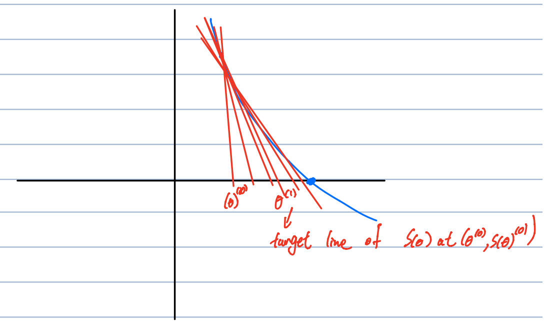
$I(\theta)=-\frac{\partial(S(\theta))}{\partial(\theta)}=-\left[-\frac{n}{\theta^2}-\sum_{i=1}^{n}(\log x_i)^2x_i^{\theta}\right]=\frac{n}{\theta^2}+\sum_{i=1}^{n}(\log x_i)^2x_i^{\theta}$

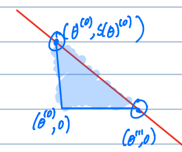
$\therefore$

$$\theta^{(r+1)}=\theta^{(r)}+\frac{\frac{n}{\theta^{(r)}}+\sum_{i=1}^{n}\log x_i-\sum_{i=1}^{n}(\log x_i)x_i^{\theta^{(r)}}}{\frac{n}{(\theta^{(r)})^2}+\sum_{i=1}^{n}(\log x_i)^2x_i^{\theta^{(r)}}}$$

Remark:

Newton's Method does not always converge unless the "Quadratic Converge Conditions" are satisfied (will not be required for this course).





Slope of the tangent line is $S'(\theta)^{(0)}$

also the slope can be calculated.

$$\frac{(S(\theta)^{(0)}-0)}{\theta^{(0)}-\theta^{(1)}}=S'(\theta^{(0)})$$

That is $\theta^{(1)}=\theta^{(0)}+\frac{S(\theta^{(0)})}{-S'(\theta^{(0)})}$

and $I(\theta)=-\frac{\partial(S(\theta))}{\partial(\theta)}$, i.e. $-S'(\theta)^{(0)}=I(\theta^{(0)})$