Lec 11

Bivariate transformation

Example 4.1.1 $f(x,y) = 3y$, $0<x<y<1$

Find the pdf of $U = XY$

[Recall. Ex 3.3.5 $f(x,y) = k e^{-x} e^{-y}$, $0<x<y$

(3) $P[X+Y \geq 1]$ i.e. $U = X+Y$, Find $P[U \geq 1]$]

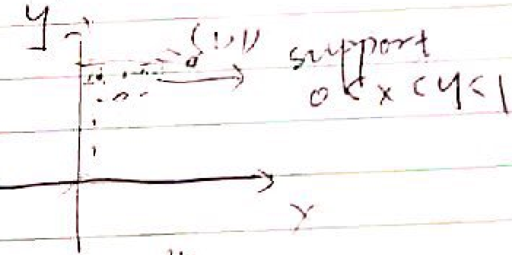
Notice $(X,Y) \to U$ Both in 4.1.1 and 3.3.5

this is NOT a 1-to-1 transformation

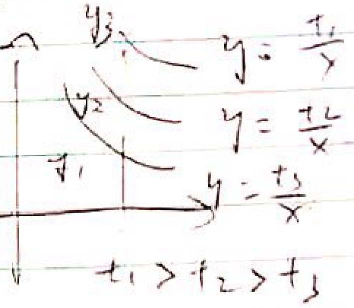
i.e. one pair of (X,Y) corresponds to one u

however one u does NOT correspond to one (X,Y)

$G(u) = P[U \leq u] = P[X \cdot Y \leq u]$



$$= \begin{cases} 0 & \text{if } u \leq 0 \\ (*) & \text{if } 0<u<1 \\ 1 & \text{if } u \geq 1 \end{cases}$$



$(*) = P[XY \leq u]$ for $0<u<1$

$= P[Y \leq \frac{u}{X}]$

$=$ 

$= 1 -$ 

$$= 1 - \int_{\sqrt{u}}^{1} \left[\int_{\frac{u}{y}}^{y} f(x,y) \right] dy = 1 - \int_{\sqrt{u}}^{1} 3y\left(y - \frac{u}{y}\right) dy$$

$$= 1 - \left(y^3 \Big|_{\sqrt{u}}^{1} - 3u(1-\sqrt{u}) \right) = 3u - 2u\sqrt{u}, \quad 0<u<1$$

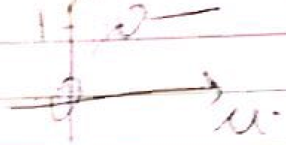
Exercise 4.1.2 Find the pdf of $V = \frac{Y}{X}$

Back 4.1.1

$$G(u) = \begin{cases} 0 & \text{if } u \le 0 \\ 3u - 2u\sqrt{u} & \text{if } 0 < u < 1 \\ 1 & \text{if } u \ge 1 \end{cases}$$

$g(u) = \frac{d}{du} G(u)$ ~~at~~ where $G(u)$ is differentiable.

$G(u)$ is NOT differentiable at $u = 0$ and $u = 1$.



$g(u) = 3 - 3\sqrt{u}, \quad 0 < u < 1.$

Although $G(u)$ is differentiable for $u < 0$, $\frac{d}{du} G(u) = \frac{d}{du} 0 = 0$ for $u < 0$

Also $G(u)$ is differentiable for $u > 1$,

$\frac{d}{du} G(u) = \frac{d}{du} 1 = 0$ for $u > 1$

Example 4.1.3 let $X_i \overset{IID}{\sim}$ with a common pdf $f(x)$ and cdf $F(x)$, $i = 1, 2, \dots n$

Find the pdf of $S = \underline{\max}(X_1, \dots X_n)$, $T = \underline{\min}(X_1, \dots X_n)$. Separately

$(X_1, X_2, \dots X_n) \to S$ is NOT 1-to-1.

$G(s) = P[S \le s] = P[\max(X_1, \dots X_n) \le s]$

$= P[X_1 \le s, X_2 \le s, \dots X_n \le s]$.

(Recall. $X_i \overset{IID}{\sim}$ with $f(x)$ and $F(x)$)

$= P[X_1 \le s] \cdot P[X_2 \le s] \dots P[X_n \le s]$

due to indep$\underline{t}$

| Recall. If X and Y are indpt

$P[X \in A, Y \in B] = P[X \in A] P[Y \in B]$, $A \subseteq R$, $B \subseteq R$

$= \underbrace{F(s) F(s) \dots F(s)}_{n \text{ of item}} = F^n(s)$

$g(s) = \frac{d}{ds} G(s) = \frac{d}{ds} F^n(s) = n F(s)^{n-1} \cdot f(s)$ where $f(s) = \frac{d}{ds}$

Exercise. Find the pdf of T

1-to-1 bivariate transformation

$(X, Y) \rightleftarrows (U, V)$ 1-to-1 transformation

Find the joint pdf of (U, V)

Recall: In univariate case $Y = h(X)$ is 1-to-1,

$X = h^{-1}(Y)$, $g(y) = f(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right|$

In the bivariate case, similarly, we need to find inverse transformation

i.e. $\begin{cases} U = h_1(X, Y) \\ V = h_2(X, Y) \end{cases}$ bivariate transformation

then $\begin{cases} X = w_1(U, V) \\ Y = w_2(U, V) \end{cases}$ inverse bivariate transformation

$$g(u, v) = f(w_1(u, v), w_2(u, v)) \left| \frac{\partial(w_1, w_2)}{\partial(u, v)} \right|$$

where $\left| \frac{\partial(w_1, w_2)}{\partial(u, v)} \right| = \det \begin{bmatrix} \frac{\partial w_1}{\partial u} & \frac{\partial w_1}{\partial v} \\ \frac{\partial w_2}{\partial u} & \frac{\partial w_2}{\partial v} \end{bmatrix} = \left(\frac{\partial w_1}{\partial u}\right)\left(\frac{\partial w_2}{\partial v}\right) - \left(\frac{\partial w_1}{\partial v}\right)\left(\frac{\partial w_2}{\partial u}\right)$

The support set for (U, V) is denoted as R_{UV}. and the support set for (X, Y) is denoted as R_{XY}

Find R_{UV} based on R_{XY}

Steps.

step 1: verify 1-to-1 transformation

step 2. Find inverse transformation and $\frac{\partial(w_1, w_2)}{\partial(u, v)}$

step 3. $g(u, v)$

Step 4. R_{UV} is re-describing R_{XY} in terms of (U, V).

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~~Met~~ last the: Bivariate transformation

Method 1: cdf technique (e.g. $(X,Y) \Rightarrow (U)$ is Not 1-to-1)

Method 2: 1-to-1 technique (e.g. $(X,Y) \rightleftarrows (U,V)$, 1-to-1)

$$g(u,v) = f(w_1(u,v), w_2(u,v)) \cdot \left| \frac{\partial(w_1,w_2)}{\partial(u,v)} \right| ; R_{u,v}$$

Today: Example(s) of 1-to-1 bivariate transformation

Correction: $\frac{\partial(w_1,w_2)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial w_1}{\partial u} & \frac{\partial w_1}{\partial v} \\ \frac{\partial w_2}{\partial u} & \frac{\partial w_2}{\partial v} \end{bmatrix}$

$\left| \frac{\partial(w_1,w_2)}{\partial(u,v)} \right| = \text{abs} \left| \frac{\partial(w_1,w_2)}{\partial(u,v)} \right|$ to make sure

Verifying 1-to-1 biverate transformation by "inverse mapping them" i.e. $\begin{cases} u = h_1(X,Y) \\ v = h_2(X,Y) \end{cases}$ is a 1-to-1

① $\frac{\partial h_1}{\partial x}, \frac{\partial h_1}{\partial y}, \frac{\partial h_2}{\partial x}, \frac{\partial h_2}{\partial y}$ are continuous functions.

② $\frac{\partial(h_1,h_2)}{\partial(x,y)} = \det \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix} \neq 0$ in R_{XY}

Example 4.2.4 $X \sim GAM(a,1)$ indpt of $Y \sim GAM(b,1)$

Let $\begin{cases} u = X+Y = h_1(X,Y) \\ v = \frac{X}{X+Y} = h_2(X,Y) \end{cases}$ Find the joint pdf of (u,v), $g(u,v)$

Answer: $f(x,y) = f_1(x) f_2(y)$ due to indpt

$$= \frac{1}{\Gamma(a)} x^{a-1} \exp(-x) \frac{1}{\Gamma(b)} y^{b-1} \exp(-y)$$

where $R_{XY} = R_X \times R_Y = (0,+\infty) \times (0,+\infty)$

$= \{(x,y): x>0, y>0\} \leftarrow$

Step 1 verify 1-to-1 bivarate transformation (through the inverse mapping them)

① $\frac{\partial h_1}{\partial x} = 1$, $\frac{\partial h_1}{\partial y} = 1$, $\frac{\partial h_2}{\partial x} = \frac{(x+y)-x}{(x+y)^2} = \frac{y}{(x+y)^2}$, $R_{XY}: \{(x,y): x>0, y>0\}$

$\frac{\partial h_2}{\partial y} = -\frac{x}{(x+y)^2}$ are cont functions of x and y in $\{(x,y): x>0, y>0\}$

(Notice that $\frac{\partial h_2}{\partial x}$ or $\frac{\partial h_2}{\partial y}$ are not cont if $x+y=0$)
but $x+y \neq 0$ in R_{XY}

② $\frac{\partial(h_1,h_2)}{\partial(x,y)} = \left(\frac{\partial h_1}{\partial x}\right)\left(\frac{\partial h_2}{\partial y}\right) - \left(\frac{\partial h_1}{\partial y}\right)\left(\frac{\partial h_2}{\partial x}\right)$

$= (1)\left(\frac{-x}{(x+y)^2}\right) - (1)\left(\frac{y}{(x+y)^2}\right) = -\frac{1}{x+y}$

Notice $\frac{\partial(h_1,h_2)}{\partial(x,y)} = -\frac{1}{x+y} \neq 0$ in $R_{XY} = \{(x,y): x>0, y>0\}$.

Therefore "1-to-1" due to ① + ②

Step 2: Inverse transformation (w_1, w_2) and $\frac{\partial(w_1,w_2)}{\partial(u,v)}$
Recall: $u = x+y \quad v = \frac{x}{x+y} \rightarrow x = uv$
$y = u - uv$

$\frac{\partial(w_1,w_2)}{\partial(u,v)} = \left(\frac{\partial w_1}{\partial u}\right)\left(\frac{\partial w_2}{\partial v}\right) - \left(\frac{\partial w_1}{\partial v}\right)\left(\frac{\partial w_2}{\partial u}\right)$

$= (v)(-u) - (u)(1-v) = -u$

Step 3: $g(u,v) = f(w_1, w_2)\left|\frac{\partial(w_1,w_2)}{\partial(u,v)}\right|$

Recall: $f(x,y) = \frac{1}{\Gamma(a)} \cdot \frac{1}{\Gamma(b)} x^{a-1} \exp(-x) y^{b-1} \exp(-y)$

Therefore
$g(u,v) = \frac{1}{\Gamma(a)} \cdot \frac{1}{\Gamma(b)} (u v)^{a-1} \exp(-uv) \cdot$
$(u(1-v))^{b-1} \exp(-u(1-u)) \left|-u\right|$

$= \underbrace{\frac{1}{\Gamma(a)} \frac{1}{\Gamma(b)} u^{a+b-1} \exp(-u)}_{\text{func of } u \text{ only}} \underbrace{v^{a-1} (1-v)^{b-1}}_{\text{func of } v \text{ only}}$

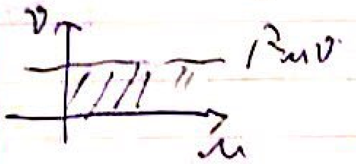
Step 4: Find R_{uv} (by redescribing R_{XY} in terms of w_1 and w_2)

$R(x,y) = \{(x,y): x>0, y>0\}$

Then $R_{uv} = \{(u,v): w_1(u,v) = uv > 0, w_2(u,v) = u - uv > 0\}$

First $uv > 0$ ($u>0, v>0$ or $u<0, v<0$)
Second $u - uv > 0$ ie ($u > uv > 0$ ie. $u>0, v>0$ and $v<1$)

In summary $R_{uv} = \{(u,v) : u>0, 0<v<1\}$



Therefore u and v are indept.

$$g_1(u) = \int_0^1 g(u,v)\,dv = \frac{1}{\Gamma(a+b)} u^{a+b-1} \exp(-u)$$

which is the pdf of GAM $(a+b, 1)$

$$g_2(v) = \int_0^\infty g(u,v)\,du = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} v^{a-1}(1-v)^{b-1}$$

which is the pdf of BETA (a,b)

Remark: Finding $\frac{\partial(w_1,w_2)}{\partial(u,v)}$ indirectly

Recall in step 1: verify 1-to-1
③ of the inverse mapping thm
$\frac{\partial(h_1,h_2)}{\partial(x,y)}$ to in R_{XY}

$$\begin{cases} u = h_1(x,y) \\ v = h_2(x,y) \end{cases} \xrightarrow[\text{transformation}]{\text{inverse}} \begin{cases} x = w_1(u,v) \\ y = w_2(u,v) \end{cases}$$

$$\frac{\partial(w_1,w_2)}{\partial(u,v)} = \left(\frac{\partial(h_1,h_2)}{\partial(x,y)}\right)^{-1}$$

and at the end, you replace (x,y) on the RHS by $(w_1(u,v), w_2(u,v))$ for notation consistency

Back to the example 1

$$\frac{\partial(w_1,w_2)}{\partial(u,v)} = |-u| \quad \text{and} \quad \frac{\partial(h_1,h_2)}{\partial(x,y)} = \left|-\frac{1}{x+y}\right|$$

$$\frac{\partial(w_1,w_2)}{\partial(u,v)} = \left(\frac{\partial(h_1,h_2)}{\partial(x,y)}\right)^{-1} = \left(-\frac{1}{x+y}\right)^{-1} = -(x+y) = -u$$

Example 4.1.1 $\boxed{\text{revisit}}$.

$f(x,y) = 3y, \quad 0 < x < y < 1.$

Find the pdf of $u = X \cdot Y$

$(X, Y) \to u$ is NOT 1-to-1

but $(X,Y) \rightleftarrows (u,v)$, 1-to-1 (make it one-to-one)

to find $g(u,v)$ then $g_1(u) = \int g(u,v)\, dv$

let $\begin{cases} u = X \cdot Y \\ v = X \end{cases}$ $\quad x=2 \quad y=8 \quad u = x \cdot y = 16$

$x = 4 \quad y = 4, \quad u = x \cdot y = 16$

$(X, Y) \to (u = x \cdot y,\ v = x) \leftarrow$ this in one-to-one.

if lock u, u is also locked

Lec 17 last time: 1-to-1 bivariate transformation

inverse mapping thm

Today: Examples of bivariate transformation and mgf method

Example 6.1.1 Revisit $f(x,y) = 3y$; $0<x<y<1$

Find the pdf of $U = X \cdot Y$ (through cdf technique)

Also: $\begin{cases} U = X \cdot Y \\ \end{cases}$

$\begin{cases} X = V = w_1(u,v) \\ Y = u/v = w_2(u,v) \end{cases}$

$\frac{\partial(w_1, w_2)}{\partial(u,v)} = -\frac{1}{v}$ (also $\frac{\partial(w_1,w_2)}{\partial(u,v)} = \left(\frac{\partial(h_1,h_2)}{\partial(x,y)}\right)^{-1}$ ← from step 1.

Step 3 $g(u,v) = \boxed{\frac{3u}{v^2}}$

Step 4. R_{UV}

Recall $R_{XY} = \{(x,y): 0<x<y<1\}$

$R_{UV} = \{(u,v): 0<v<u/v<1\}$

First of all $u>0$, $v>0$

$0 < v^2 < v \cdot \frac{u}{v} < 1 \cdot v$

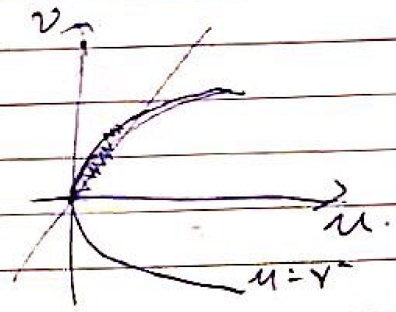
i.e. $0<v^2<u<v$

$R_{UV} = \{(u,v). 0<v^2<u<v\}$

$g_1(u) = \int_{u}^{\sqrt{u}} g(u,v)\,dv \quad 0<u<1$

$= \int_{u}^{\sqrt{u}} \frac{3u}{v^2}\,dv = 3 - 3\sqrt{u}, \quad 0<u<1$

Which is the same as we got using the cdf technique



Example 4.2.6 Box - Mueller transformation

$X \sim UNIF(0,1)$ indep't of $Y \sim UNIF(0,1)$

let $\begin{cases} u = (-2\log x)^{1/2} \cos(2\pi y) = h_1(x, y) \\ v = (-2\log x)^{1/2} \sin(2\pi y) = h_2(x, y) \end{cases}$

Find $g(u,v)$ and the dis'tr of u and v

Answer: Due to indpt

$f(x,y) = f_1(x) f_2(y) = 1 \cdot 1 = 1, \quad 0<x<1, 0<y<1$

i.e. $R_{XY} = \{(x,y) : 0<x<1, 0<y<1\}$

$$g(u,v) = f(w_1(u,v), w_2(u,v)) \left| \frac{\partial(w_1, w_2)}{\partial(u,v)} \right|$$

$$= \left| \left(\frac{\partial(h_1,h_2)}{\partial(x,y)} \right)^{-1} \right| \leftarrow$$

step 1: 1-to-1

① $\frac{\partial h_1}{\partial x} = (\frac{1}{2})(-\frac{2}{x})(-2\log x)^{-1/2} \cos(2\pi y)$

$\frac{\partial h_1}{\partial y} = (-2\pi)(-2\log x)^{1/2} \sin(2\pi y)$

$\frac{\partial h_2}{\partial x} = (\frac{1}{2})(-\frac{2}{x})(-2\log x)^{-1/2} \sin(2\pi y)$

$\frac{\partial h_2}{\partial y} = (2\pi)(-2\log x)^{1/2} \cos(2\pi y)$

are cont functions of x and y in $R_{xy} = \{(x,y) : 0<x<1, 0<y<1\}$

② $\frac{\partial(h_1,h_2)}{\partial(x,y)} = (\frac{\partial h_1}{\partial x})(\frac{\partial h_2}{\partial y}) - (\frac{\partial h_1}{\partial y})(\frac{\partial h_2}{\partial x})$ ← func of x

$= (\frac{1}{2})(-\frac{2}{x}) \cos(2\pi y)(2\pi) \cos(2\pi y)$

$\quad - (-2\pi) \sin(2\pi y) (\frac{1}{2})(-\frac{2}{x}) \sin(2\pi y)$

$= (-\frac{2\pi}{x}) \cos^2(2\pi y) - (+\frac{2\pi}{x}) \sin^2(2\pi y)$

$= (-\frac{2\pi}{x}) (\cos^2(2\pi y) + \sin^2(2\pi y)) = -\frac{2\pi}{x} \neq 0$ in

$R_{xy} = \{(x,y) ; 0<x<1, 0<y<1\}$

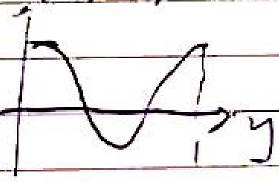
Step 2. Inverse transformations and $\frac{\partial(w_1, w_2)}{\partial(u,v)}$

Notice $\begin{cases} u = (-2\log x)^{1/2} \cos(2\pi y) \\ v = (-2\log x)^{1/2} \sin(2\pi y) \end{cases}$

$$u^2+v^2 = (-2\log X)\left[\cos^2(2\pi Y)+\sin^2(2\pi Y)\right]$$
$$= -2\log X$$
$$\to \quad X = \exp\left(-\tfrac{1}{2}(u^2+v^2)\right) = w_1$$

However, finding $Y = w_2(u,v)$ is not easy!

$\cos(2\pi Y)$



$\therefore y$ is not unique given $\cos(2\pi Y)$

similarly, given $\sin(2\pi y)$, y is also not unique

$$\frac{\partial(w_1,w_2)}{\partial(u,v)} = \left(\frac{\partial(h_1,h_2)}{\partial(x,y)}\right)^{-1} = \left(-\frac{2\pi}{x}\right)^{-1} = \boxed{-\frac{x}{2\pi}} = \boxed{-\frac{w_1(u,v)}{2\pi}}.$$

i.e. $\frac{\partial(w_1,w_2)}{\partial(u,v)} = \left(-\frac{1}{2\pi}\right)\exp\left(-\frac{1}{2}(u^2+v^2)\right)$

Step 3: $g(u,v) = f(w_1,w_2)\left|\frac{\partial(w_1,w_2)}{\partial(u,v)}\right| = 1\left|\left(-\frac{1}{2\pi}\right)\exp\left(-\frac{1}{2}(u^2+v^2)\right)\right|$

$$= \frac{1}{2\pi}\exp\left(-\tfrac{1}{2}u^2\right)\exp\left(-\tfrac{1}{2}v^2\right)$$

Step 4: R_{uv}

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)}_{\text{pdf of } N(0,1)}\underbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{v^2}{2}\right)}_{\text{pdf of } N(0,1)}$$

[Recall. $R_{XY} = \{(x,y): 0<x<1,\ 0<y<1\}$.]

I don't know $y = w_2(u,v)$, therefore it's hard to find R_{uv} by re-describing R_{xy} in terms of (w_1,w_2)

$u = (-2\log X)^{1/2}\cos(2\pi Y)$

$v = (-2\log X)^{1/2}\sin(2\pi Y)$

Since $x \in (0,1)$, $(-2\log X)^{1/2} > 0$

and $y \in (0,1)$, $-1 \le \cos(2\pi y) \le 1$, $-1 \le \sin(2\pi y) \le 1$

Therefore, $u \in R$ and $v \in R$.

i.e. $R_{uv} = \{(u,v): u \in R, v \in R\}$

Therefore, u and v are indpt and both are $N(0,1)$

[Remark: If you generate X indpt Y from $Uni(0,1)$ through Box-Mueller transformation, we can have (u,v) are indpt $N(0,1)$ r.v.'s.]