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Q1a 3

Question 1 [12 points]

Suppose that $(X_1, X_2) \sim \text{MULT}(n, p_1, p_2)$, $p_i \in (0,1)$, $i = 1, 2$, and $0 < p_1 + p_2 < 1$.

(a) (4pts) Prove that the joint mgf of (X_1, X_2) is

$$M(t_1, t_2) = (e^{t_1}p_1 + e^{t_2}p_2 + 1 - p_1 - p_2)^n, \ t_i \in \Re, \ i = 1, 2.$$

and derive the marginal distribution of X_1 using the **mgf method**.

$$f(x_1,x_2) = \frac{n!}{x_1!x_2!(n-\sum_{i=1}^{2} x_i)!} p_1^{x_1} p_2^{x_2} (1-\sum_{i=1}^{2} p_i)^{n-\sum_{i=1}^{2} x_i}$$

$$M(t_1,t_2) = E[e^{t_1X_1+t_2X_2}]$$

$$= \sum_{x_1=0}^{n} \sum_{x_2=0}^{n-x_1} e^{t_1x_1} e^{t_2x_2} \frac{n!}{x_1!x_2!(n-\sum_{i=1}^{2}x_i)!} p_1^{x_1} p_2^{x_2} (1-\sum_{i=1}^{2}p_i)^{n-\sum_{i=1}^{2}x_i}$$

$$= \sum_{x_1=0}^{n} \sum_{x_2=0}^{n-x_1} \frac{n!}{x_1!x_2!(n-\sum_{i=1}^{2}x_i)!} (e^{t_1}p_1)^{x_1} (e^{t_2}p_2)^{x_2} (1-\sum_{i=1}^{2}p_i)^{n-\sum_{i=1}^{2}x_i}$$

by Multinomial theorem

$$= \left[e^{t_1}p_1 + e^{t_2}p_2 + (1-\sum_{i=1}^{2}p_i)\right]^n$$

$$= \left[e^{t_1}p_1 + e^{t_2}p_2 + 1 - p_1 - p_2\right]^n$$

Thus $M_X(t) = M(t_1 = t, t_2 = 0)$

$$= (e^{t}p_1 + 1 - p_1 - p_2)^n$$

marginal distribution is wrong

Fact 1: The joint pf of $(Y_1, Y_2, \cdots, Y_k) \sim \text{MULT}(n, p_1, p_2, \cdots, p_k)$ is

$$f(y_1, y_2, \cdots, y_k) = \frac{n!}{y_1!y_2!\cdots y_k!\left(n - \sum_{i=1}^k y_i\right)!} p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k} \left(1 - \sum_{i=1}^k p_i\right)^{n - \sum_{i=1}^k y_i}$$

for $0 \le y_i \le n$, $i = 1, 2, \cdots, k$ and $0 \le \sum_{i=1}^k y_i \le n$.

Fact 2: Multinomial theorem: If n is a positive integer and $a_1, \cdots, a_\ell$ are real numbers, then

$$(a_1 + a_2 + \cdots + a_\ell)^n = \sum_{z_1} \sum_{z_2} \cdots \sum_{z_\ell} \frac{n!}{z_1!z_2!\cdots z_\ell!} a_1^{z_1} a_2^{z_2} \cdots a_\ell^{z_\ell}$$

where the summation extends over all non-negative integers $z_1, z_2, \cdots, z_\ell$ with $z_1 + z_2 + \cdots + z_\ell = n$.

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Question 1

Q1bc 4

(b) (...s) Derive $Cov(X_1, X_2)$ based on (a). Show your work.

$$Cov(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2]$$

$$E[X_1X_2] = \frac{\partial}{\partial t_1 \partial t_2} M(t_1,t_2)\Big|_{t_1=0, t_2=0} = \frac{\partial}{\partial t_1 \partial t_2}[e^{t_1}p_1 + e^{t_2}p_2 + 1 - p_1 - p_2]^n$$

$$= \frac{\partial}{\partial t_2} e^{t_1}p_1 n [e^{t_1}p_1 + e^{t_2}p_2 + 1 - p_1 - p_2]^{n-1}$$

$$= e^{t_1}p_1 e^{t_2}p_2 n(n-1)[e^{t_1}p_1 + e^{t_2}p_2 + 1 - p_1 - p_2]^{n-2}$$

$$= n(n-1)p_1p_2$$

$$E[X_1] = \frac{\partial}{\partial t} M_{X_1}(t_1)\Big|_{t=0} = \frac{\partial}{\partial t}(e^t p_1 + 1 - p_1 - p_2)^n$$

$$= ne^{t_1}(e^t p_1 + 1 - p_1 - p_2)^{n-1} = np_1$$

$$E[X_2] = np_2$$

$$\Rightarrow Cov(X_1, X_2) = n(n-1)p_1p_2 - n^2p_1p_2 = -np_1p_2$$

(c) (1pts) Derive the distribution of $T = X_1 + X_2$ based on (a). Show your work.

$$M_T(t) = E[e^{tT}] = E[e^{tX_1 + tX_2}]$$

$$= M(t_1 = t, t_2 = t) \quad (a)$$

$$= (e^t p_1 + e^t p_2 + 1 - p_1 - p_2)^n$$

$$= (e^t(p_1 + p_2) + 1 - (p_1 + p_2))^n$$

Thus according to the uniqueness of mgf,

$T \sim BIN(n, p_1 + p_2)$, $0 \le t \le n, 0 < p_1 + p_2 < 1$

Fact 3: For $Y \sim \text{BIN}(n,p)$, the pf is

$$f(y) = \frac{n!}{y!(n-y)!}p^y(1-p)^{n-y}, 0 \le y \le n, \quad 0 < p < 1;$$

the mgf is $M_Y(t) = (e^t p + 1 - p)^n$, $t \in \Re$, and $E(Y) = np$ and $Var(Y) = np(1-p)$.

Q1d 4

Question 1

(4pts) Derive the conditional distribution of X_1 given $T = t$ where T is given in (c).

$$X_1 \mid X_1 + X_2 = t$$

$$P(X_1 = x_1, X_1 + X_2 = t) = P(X_1 = x_1, X_2 = t - x_1)$$

$$= \frac{n!}{x_1!(t-x_1)!(n-(x_1+t-x_1))!} p_1^{x_1} p_2^{t-x_1} (1-p_1-p_2)^{n-(x_1+t-x_1)}$$

$$= \frac{n!}{x_1!(t-x_1)!(n-t)!} p_1^{x_1} p_2^{t-x_1} (1-p_1-p_2)^{n-t}$$

$$P(T=t) = \frac{n!}{t!(n-t)!} p^t (1-p)^{n-t} \quad (p = p_1 + p_2)$$

$$\Rightarrow f(x_1|t) = \frac{f(x_1,t)}{f(t)} = \frac{\frac{\cancel{n!}}{x_1!(t-x_1)!\cancel{(n-t)!}} p_1^{x_1} p_2^{t-x_1} \cancel{(1-p_1-p_2)^{n-t}}}{\frac{\cancel{n!}}{t!\cancel{(n-t)!}} p^t \cancel{(1-p)^{n-t}}}$$

$$= \frac{t!}{x_1!(t-x_1)!} \left(\frac{p_1}{p_1+p_2}\right)^{x_1} \left(1 - \frac{p_1}{p_1+p_2}\right)^{t-x_1} \quad p = p_1 + p_2)$$

Thus from pdf of $f(x_1|t)$, we can tell

then $X_1 | X_1 + X_2 = t \sim BIN\left(t, \frac{p_1}{p_1+p_2}\right)$ ☐

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Question 2 **[16 points]**

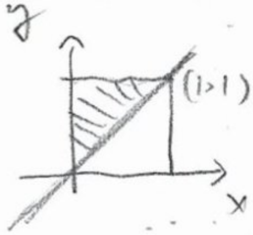
Q2a 4.5

(a) (6pts) Suppose X and Y are continuous r.v.'s with joint pdf

$$f(x,y) = 2(x+y),\ 0 < x < y < 1.$$

Find the pdf of $U = XY$ using the **cdf technique**. Be sure to specify its support set.

$f(x,y) = 2(x+y),\quad 0<x<y<1$



$U = XY$

$\Rightarrow Y = \frac{U}{X}$

$G(U) = P(U<u)$

$= P(XY<u)$

$= P(Y<\frac{u}{x})$

$\Rightarrow \begin{cases} 0 & u<0 \\ (*) & 0\le u<1 \\ 1 & u\ge 1 \end{cases}$

when $0\le u<1$, $G(u) = (*) = 1 -$ ▽

▽ $= \int_{\sqrt{u}}^{1}\int_{\frac{u}{y}}^{y} 2(x+y)\,dx\,dy$

$= \int_{\sqrt{u}}^{1} 2\left[\frac{1}{2}x^2+yx\right]_{\frac{u}{y}}^{y} dy$

$= \int_{\sqrt{u}}^{1} 2\left(\frac{1}{2}y^2+y^2-\left(\frac{1}{2}\left(\frac{u}{y}\right)^2+u\right)\right) dy$

$= \int_{\sqrt{u}}^{1} 3y^2 - \left(\frac{u}{y}\right)^2 + 2u\, dy$

$= \left[y^3+\frac{u^2}{y}+2uy\right]_{\sqrt{u}}^{1}$

$= (1+u^2+2u) - \left(u\sqrt{u}+\frac{u^2}{\sqrt{u}}+2u\sqrt{u}\right)$

$= 1+u^2+2u-3u\sqrt{u}-u\sqrt{u}$

$= 1+u^2+2u-4u\sqrt{u}$

Thus $G(u) = 1-1-u^2-2u+4u\sqrt{u}$

$\frac{\partial}{\partial u}G(u) = 6\sqrt{u}-2u-2$

Thus $g(u) = \begin{cases} 0 & u<0 \\ \cancel{6\sqrt{u}}-2u-2 & 0\le u<1 \\ 1 & u\ge 1 \end{cases}$

-0.5

$= 4u\sqrt{u} - u^2 - 2u$

-1

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Q2b 9

Question 2

(10pts) Let $Z \sim N(0,1)$ independently of $X \sim \chi^2(n)$. Find the pdf of $U = \frac{Z}{\sqrt{X/n}}$ using **1-to-1 bivariate transformation** by including $V = X$.

$U = \frac{z}{\sqrt{x/n}}$ (h_1) ① verify 1-to-1

$V = X$ (h_2) $\frac{\partial h_1}{\partial z} = \frac{1}{\sqrt{\frac{x}{n}}}$ $\frac{\partial h_1}{\partial x} = -\frac{1}{2}\sqrt{n}\, z\, x^{-\frac{3}{2}}$

$\frac{\partial h_2}{\partial z} = 0$ $\frac{\partial h_2}{\partial x} = 1$

$R_{zx} = \{(z,x) \mid z \in R, x > 0\}$

$\frac{\partial h_1}{\partial z}, \frac{\partial h_1}{\partial x}, \frac{\partial h_2}{\partial z}, \frac{\partial h_2}{\partial x}$ are obvious continuous in R_{zx}

$\begin{vmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial x} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial x} \end{vmatrix} = \sqrt{\frac{n}{x}} - 0 = \sqrt{\frac{n}{x}} \neq 0$ in R_{zx} $(x > 0)$

② $X = V$ w_2 by inverse mapping theorem

$Z = U\sqrt{\frac{V}{n}}$ w_1 $\begin{vmatrix} \frac{\partial w_1}{\partial u} & \frac{\partial w_1}{\partial v} \\ \frac{\partial w_2}{\partial u} & \frac{\partial w_2}{\partial v} \end{vmatrix} = \sqrt{\frac{v}{n}}$

Since Z and X are independent,

$f(z,x) = f(z) \cdot f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}$

$\Rightarrow g(u,v) = f(w_1, w_2) \begin{vmatrix} \frac{\partial w_1}{\partial u} & \frac{\partial w_1}{\partial v} \\ \frac{\partial w_2}{\partial u} & \frac{\partial w_2}{\partial v} \end{vmatrix} = \frac{1}{\sqrt{2\pi}} e^{-\frac{w_1^2}{2}} \frac{1}{2^{n/2}\Gamma(\frac{n}{2})} w_2^{n/2-1} e^{-\frac{w_2}{2}} \left|\sqrt{\frac{v}{n}}\right|$

$= \frac{1}{\sqrt{2\pi}\, 2^{n/2}\Gamma(\frac{n}{2})} v^{n/2-1} e^{-\left(\frac{u^2 v}{n} + v\right)} \left|\sqrt{\frac{v}{n}}\right|$ $\sqrt{\frac{v}{n}} > 0$

$g(u) = \int_0^\infty g(u,v)\, dv$ $R_{u,v} = \{(u,v) \mid u\sqrt{\frac{v}{n}} \in R, v > 0\}$

$= \{(u,v) \mid u \in R, v > 0\}$

$=$

final answer? -1

Fact 4: The pdf of $X \sim \chi^2(n)$ is $f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}$, $x > 0$, $n = 1, 2, \cdots$.

Fact 5: The pdf of $X \sim \text{GAM}(\alpha, \beta)$ is $f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$, $x > 0$, $\alpha > 0$, $\beta > 0$ and $\int_0^{+\infty} f(x)dx = 1$.

Fact 6: The pdf of $Z \sim N(\mu, \sigma^2)$ is $f(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$, $z \in \Re$, $\mu \in \Re$ and $\sigma > 0$.

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Question 3 **[12 points]**

Q3a 4

Suppose that $\underset{\sim}{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \text{BVN}(\underset{\sim}{\mu}, \Sigma)$ and $Y \sim \text{N}(\mu, \sigma^2)$.

(a) (4pts) Prove that the mgf of Y is $M_Y(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$, $t \in \Re$. Hint: You can use the fact that $\int_{\Re} f(x)dx = 1$, where $f(x)$ is the pdf of any normally distributed random variable.

$$M_Y(t) = E[e^{tY}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{(y-\mu)^2}{2\sigma^2}\right)} e^{ty} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2-2\mu y+\mu^2}{2\sigma^2} + ty} dy$$

$$= e^{t\mu+\frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2-2\mu y+\mu^2}{2\sigma^2} + ty - t\mu - \frac{1}{2}t^2\sigma^2} dy.$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2-2\mu y+\mu^2 + 2\sigma^2 ty - 2\sigma^2 t\mu - t^2\sigma^4}{2\sigma^2}} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2-(2\mu-2\sigma^2 t)y + \mu^2-2\sigma^2 t\mu + t^2\sigma^4}{2\sigma^2}} dy \quad (\mu-\sigma^2 t)^2.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-(\mu+\sigma^2 t))^2}{2\sigma^2}} dy = 1$$

$$\text{Thus } M_Y(t) = e^{t\mu+\frac{1}{2}t^2\sigma^2} \times 1 = e^{t\mu+\frac{1}{2}t^2\sigma^2}$$

Fact 6: The pdf of $Y \sim N(\mu, \sigma^2)$ is $f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$, $y \in \Re$, $\mu \in \Re$ and $\sigma > 0$.

Q3b 4

Question 3

(b) [4pts] Given the fact that the mgf of $\underset{\sim}{X}$ is

$$M(t_1, t_2) = \exp\left(\underset{\sim}{t}^T \underset{\sim}{\mu} + \frac{1}{2}\underset{\sim}{t}^T \Sigma \underset{\sim}{t}\right), \ \underset{\sim}{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, t_1 \in \Re, t_2 \in \Re.$$

Prove that X_1 and X_2 are independent if and only if $\rho = 0$.

Prove this by showing $M(t_1, t_2) = M_{X_1}(t_1) M_{X_2}(t_2)$.

Since $M(t_1,t_2) = \exp(\underset{\sim}{t}^T\underset{\sim}{\mu} + \frac{1}{2}\underset{\sim}{t}^T\Sigma\underset{\sim}{t})$

$① = \exp(t_1\mu_1 + t_2\mu_2 + \frac{1}{2}t_1^2\sigma_1^2 + \frac{1}{2}t_2^2\sigma_2^2 + \rho\sigma_1\sigma_2 t_1 t_2)$

$M_{X_1}(t) = M(t_1, 0) = \exp(t_1\mu_1 + \frac{1}{2}t_1^2\sigma_1^2)$

$M_{X_2}(t) = M(0, t_2) = \exp(t_2\mu_2 + \frac{1}{2}t_2^2\sigma_2^2)$

$\Rightarrow M_{X_1}(t_1) M_{X_2}(t_2) = \exp(t_1\mu_1 + t_2\mu_2 + \frac{1}{2}t_1^2\sigma_1^2 + \frac{1}{2}t_2^2\sigma_2^2)$. ②

① = ② iff $\rho\sigma_1\sigma_2 t_1 t_2 = 0$ for $\forall t_1, t_2$, $\sigma_1 > 0$, $\sigma_2 > 0$

Thus $\rho = 0$ iff X_1 and X_2 are independent.

✓

Fact 7: The joint pdf of $\underset{\sim}{X} \sim \text{BVN}(\underset{\sim}{\mu}, \Sigma)$ is:

$$f(x_1, x_2) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\underset{\sim}{x} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu})\right\}, \ x_1 \in \Re, x_2 \in \Re,$$

where $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$ is a positive definite 2×2 covariance matrix with $\mu_i \in \Re$, $-1 < \rho < 1$, and $\sigma_i > 0$ for $i = 1, 2$.

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Question 3

Q3c 4

(… ts) Use the **mgf method** to prove that if A is a 2×2 nonsingular matrix and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is a constant vector, then

$$\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = A\underset{\sim}{X} + \underset{\sim}{b} \sim \text{BVN}(A\underset{\sim}{\mu} + \underset{\sim}{b}, A\Sigma A^T).$$

$$M_{\underset{\sim}{Y}}(\underset{\sim}{t}) = E[e^{\underset{\sim}{t}^T \underset{\sim}{Y}}]$$

$$= E[e^{\underset{\sim}{t}^T(A\underset{\sim}{X}+\underset{\sim}{b})}]$$

$$= E[e^{\underset{\sim}{t}^T A\underset{\sim}{X} + \underset{\sim}{t}^T \underset{\sim}{b}}]$$

$$= e^{\underset{\sim}{t}^T \underset{\sim}{b}} E[e^{\underset{\sim}{t}^T A\underset{\sim}{X}}] \quad \text{let } \underset{\sim}{t}^{*T} = \underset{\sim}{t}^T A$$

$$\Rightarrow e^{\underset{\sim}{t}^T \underset{\sim}{b}} E[e^{\underset{\sim}{t}^{*T}\underset{\sim}{X}}]$$

$$= e^{\underset{\sim}{t}^T \underset{\sim}{b}} M_{\underset{\sim}{X}}(\underset{\sim}{t}^*) = e^{\underset{\sim}{t}^T \underset{\sim}{b}} e^{\underset{\sim}{t}^{*T}\underset{\sim}{\mu} + \frac{1}{2}\underset{\sim}{t}^{*T}\Sigma \underset{\sim}{t}^*}$$

$$= e^{\underset{\sim}{t}^T A\underset{\sim}{\mu} + \underset{\sim}{t}^T\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{t}^T A\Sigma A^T t}$$

$$= e^{t^T\underset{\sim}{(A\underset{\sim}{\mu}+\underset{\sim}{b})} + \frac{1}{2}\underset{\sim}{t}^T \underset{\sim}{A\Sigma A^T} t}$$

Since the uniqueness of mgf,

$$\underset{\sim}{Y} = A\underset{\sim}{X} + \underset{\sim}{b} \sim BVN(A\underset{\sim}{\mu}+\underset{\sim}{b}, A\Sigma A^T) \quad \square$$

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