Lecture 18

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Last time: 1-to-1 bivariate transformation

Find $\frac{\partial(w_1,w_2)}{\partial(u,v)}$ indirectly via $\frac{\partial(h_1,h_2)}{\partial(x,y)}$

Find $g(u,v)$ without the complete inverse transformation (w_1, w_2)

Today: Univarite, bivariate, multivarite transformation via

Method 3: MGF method

Basic: Let $M_X(t)$ be the mgf of X, for $|t|<h$, some $h>0$

Let $Y = a\cdot X + b$, $a \neq 0$, $b \in \mathbb{R}$. Find $M_Y(t)$

$M_Y(t) = E[e^{t\cdot Y}] = E[e^{t(ax+b)}] = e^{tb}E[e^{(ta)x}]$

let $t^* = ta$, then $M_Y(t) = e^{tb}\cdot E[e^{t^*\cdot x}]$

$= e^{tb}\cdot M_X(t^*)$

$= e^{tb} M_X(ta)$

Notice $M_Y(t) = e^{tb}\cdot M_X(t^*)$ therefore, $M_Y(t)$ exists only if $M_X(t^*)$ exists, i.e. $|t^*|<h$

i.e. $|ta|<h$ i.e. $|t|<\frac{h}{|a|}$

4.3.2 Special Results

(1) If $X \sim GAM(\alpha, \beta)$ and α is positive integer

Let $Y = \frac{2X}{\beta}$, then $Y \sim \chi^2(2\alpha)$

Proof: $Y = \frac{2X}{\beta}$ i.e. $a = \frac{2}{\beta}$, $b=0$

$= \frac{2}{\beta}x + 0$

$M_Y(t) = e^{t\cdot 0} M_X(t\cdot\frac{2}{\beta}) = M_X(\frac{2t}{\beta}) \xlongequal{t^*=\frac{2t}{\beta}} M_X(t^*)$

$\boxed{\text{Recall: If } X \sim GAM(\alpha,\beta),\ M_X(t) = \frac{1}{(1-\beta t)^\alpha},\ t < \frac{1}{\beta}}$

$M_Y(t) = M_X(t^*) = \frac{1}{\quad}$ $t^* < \frac{1}{\quad}$

$$M_Y(t) = M_X(t^*) = \frac{1}{(1-\beta t^*)^\alpha}, \quad t^* < \frac{1}{\beta}$$

$$t^* = \frac{2t}{\beta}$$

$$= \frac{1}{(1-\beta\frac{2t}{\beta})^\alpha}, \quad \frac{2t}{\beta} < \frac{1}{\beta}$$

i.e. $M_Y(t) = \frac{1}{(1-2t)^{\frac{2\alpha}{2}}}, \quad t < \frac{1}{2}$

[Notice If $X \sim \chi^2(n)$, $M_X(t) = \frac{1}{(1-2t)^{\frac{n}{2}}}$, $t < \frac{1}{2}$]

Therefore, $M_Y(t)$ is identical to the mgf of $\chi^2(2\alpha)$
i.e. $Y \sim \chi^2(2\alpha)$ due to the uniqueness thm of mgf

(2) If $X_i \sim GAM(\alpha_i, \beta)$, $i = 1, \dots, n$, indep.
then $\sum_{i=1}^n X_i \sim GAM(\sum_{i=1}^n \alpha_i, \beta)$

Proof: $M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2+\dots+X_n)}]$

$$= E[e^{tX_1} e^{tX_2} \cdots e^{tX_n}]$$

$$= \underbrace{E[e^{tX_1}]}_{M_{X_1}(t)} \cdot \underbrace{E[e^{tX_2}]}_{M_{X_2}(t)} \cdots \underbrace{E[e^{tX_n}]}_{M_{X_n}(t)} \quad \text{due to indep.}$$

$$= \frac{1}{(1-\beta t)^{\alpha_1}} \cdot \frac{1}{(1-\beta t)^{\alpha_2}} \cdots \frac{1}{(1-\beta t)^{\alpha_n}}, \quad t < \frac{1}{\beta}$$

[Recall: $X_i \sim GAM(\alpha_i, \beta)$ $M_{X_i}(t) = \frac{1}{(1-\beta t)^{\alpha_i}}$, $t < \frac{1}{\beta}$]

$$= \frac{1}{(1-\beta t)^{\sum_{i=1}^n \alpha_i}}, \quad t < \frac{1}{\beta}$$

which is identical to the mgf of $GAM(\sum_{i=1}^n \alpha_i, \beta)$

i.e. $Y \sim GAM(\sum_{i=1}^n \alpha_i, \beta)$ due to the uniqueness thm

Exercise:
(3) If $X_i \overset{IID}{\sim} EXP(\beta)$, then $\sum_{i=1}^n X_i \sim GAM(n, \beta)$

Hint: $Exp(\beta) = GAM(1, \beta)$

(4) If $X_i \sim \chi^2(k_i)$ $i=1,\dots,n$ indep.
then $\sum_{i=1}^{n} X_i \sim \chi^2(\sum_{i=1}^{n} k_i)$

Hint: similar to the proof (2)

(5) If $X_i \overset{IID}{\sim} N(\mu, \sigma^2)$, then $\sum_{i=1}^{n} \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(n)$

Hint: $\frac{X_i-\mu}{\sigma} \sim N(0,1)$ due to 2.5.1
$\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2(1)$ due to 2.5.1

(6) If $X_i \sim POI(\mu_i)$ $i=1,\dots,n$ indep.
Then $\sum_{i=1}^{n} X_i \sim POI(\sum_{i=1}^{n} \mu_i)$

(7) If $X_i \sim BIN(n_i, p)$, $i=1,\dots,n$ indep.
$\sum_{i=1}^{n} X_i \sim BIN(\sum_{i=1}^{n} n_i, p)$

Skip (8) which is related to NB

Thm 4.3.5: If $X_i \sim N(\mu_i, \sigma_i^2)$ $i=1,\dots,n$ indep.
then $\sum_{i=1}^{n} a_i X_i \sim N(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2)$

Proof: Exercise.

Corollary: 4.3.6 If $X_i \overset{IID}{\sim} N(\mu, \sigma^2)$, then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

Proof: Let $a_i = \frac{1}{n}$

$$\sum_{i=1}^{n} a_i X_i = \sum_{i=1}^{n} \frac{1}{n} X_i = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

then $\bar{X} \sim N(\sum_{i=1}^{n} \frac{1}{n}\cdot\mu, \sum_{i=1}^{n} (\frac{1}{n})^2 \sigma^2)$

$= N(\mu, \frac{\sigma^2}{n})$

Thm 4.3.8: If $X_i \overset{IID}{\sim} N(\mu, \sigma^2)$ then $\underbrace{\bar{X} \sim N(\mu, \frac{\sigma^2}{n})}_{(4.3.6)}$
is indep. of $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

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where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is the sample variance.

Proof: Steps to complete the proof:

1: $\bar{X}$ and $(n-1)s^2$ are indep. (by the indep thm of mgf's)

2. $$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} = \frac{n(\bar{X}-\mu)^2}{\sigma^2} + \frac{(n-1)s^2}{\sigma^2}$$
$$(\chi^2(n)) \qquad (\chi^2(1)) + (\chi^2(n-1))$$

3. $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$ (due to special result 4)

Step 1:
$$(n-1)s^2 = \sum_{i=1}^{n}(X_i-\bar{X})^2 = (X_1-\bar{X})^2+(X_2-\bar{X})^2+\cdots+(X_n-\bar{X})^2$$

$(n-1)s^2$ is a function of $\{(X_1-\bar{X}),(X_2-\bar{X}_2),\ldots,(X_n-\bar{X})\}$

To show $\bar{X}$ is indep. of $(n-1)s^2$, it suffices to show $\bar{x}$ is indep of $\{(X_1-\bar{X}),\ldots,(X_n-\bar{X})\}$

Let $U_i = X_i - \bar{X}$, find the joint mgf of $(\underbrace{U_1,U_2,\ldots,U_n,\bar{X}}_{n+1 \text{ entries}})$ and therefore the marginal mgf's of $(U_1,\ldots,U_n)$ and $\bar{x}$, respectively.