4.6

Especially for the joint whose axes is aligned with $\hat{y_s}$:

$$v = -\omega \times q$$

$$v_2 = -\begin{bmatrix}0\\1\\0\end{bmatrix} \times \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, v_4 = -\begin{bmatrix}0\\1\\0\end{bmatrix} \times \begin{bmatrix}W_1\\0\\L_1\end{bmatrix} = \begin{bmatrix}-L_1\\0\\-W_1\end{bmatrix}, v_6 = -\begin{bmatrix}0\\1\\0\end{bmatrix} \times \begin{bmatrix}0\\0\\L_1+L_2\end{bmatrix} = \begin{bmatrix}-(L_1+L_2)\\0\\0\end{bmatrix}$$

And for other joint:

$$v_1 = v_3 = v_5 = v_7 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\mathcal{S}_1 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} \mathcal{S}_2 = \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} \mathcal{S}_3 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}$$

$$\mathcal{S}_4 = \begin{bmatrix}0\\1\\0\\-L_1\\0\\-W_1\end{bmatrix} \mathcal{S}_5 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} \mathcal{S}_6 = \begin{bmatrix}0\\1\\0\\-(L_1+L_2)\\0\\0\end{bmatrix} \mathcal{S}_7 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}$$

4.9

$$M = \begin{bmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 3L\\0 & 0 & -1 & -2L\\0 & 0 & 0 & 1\end{bmatrix}$$

From left to right, from up to down, we define the joint 1-6.
In $\{0\}$:

$$v = -\omega \times q$$

$$v_1 = -\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}0\\0\\-L\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \mathcal{S}_1 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}$$

$$v_2 = -\begin{bmatrix}1\\0\\0\end{bmatrix} \times \begin{bmatrix}0\\0\\-2L\end{bmatrix} = \begin{bmatrix}0\\-2L\\0\end{bmatrix}, \mathcal{S}_2 = \begin{bmatrix}1\\0\\0\\0\\-2L\\0\end{bmatrix}$$

$$v_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathcal{S}_3 = \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}$$

$$v_4 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \mathcal{S}_4 = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}$$

$$v_5 = -\begin{bmatrix}0\\1\\0\end{bmatrix} \times \begin{bmatrix}0\\L\\-L\end{bmatrix} = \begin{bmatrix}L\\0\\0\end{bmatrix}, \mathcal{S}_5 = \begin{bmatrix}0\\1\\0\\L\\0\\0\end{bmatrix}$$

$$v_6 = -\begin{bmatrix}0\\0\\-1\end{bmatrix} \times \begin{bmatrix}0\\3L\\-L\end{bmatrix} = \begin{bmatrix}-3L\\0\\0\end{bmatrix}, \mathcal{S}_6 = \begin{bmatrix}0\\0\\-1\\-3L\\0\\0\end{bmatrix}$$

In $\{b\}$:

$$v_1 = -\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} -3L \\ 0 \\ -L \end{bmatrix} = \begin{bmatrix} 0 \\ -3L \\ 0 \end{bmatrix}, \mathcal{B}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ -3L \\ 0 \end{bmatrix}$$

$$v_2 = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -3L \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3L \end{bmatrix}, \mathcal{B}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -3L \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathcal{B}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \mathcal{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$v_5 = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -L \\ 0 \\ -L \end{bmatrix} = \begin{bmatrix} 0 \\ -L \\ 0 \end{bmatrix}, \mathcal{B}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -L \\ 0 \end{bmatrix}$$

$$v_6 = -\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}0\\0\\-L\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \mathcal{B}_6 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}$$

5.8

(a)

In $\{s\}$:

$$v = -\omega \times q$$

$$v_1 = -\begin{bmatrix}0\\1\\0\end{bmatrix} \times \begin{bmatrix}0\\L\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \mathcal{S}_1 = \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}$$

$$v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \ \mathcal{S}_2 = \begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}$$

$$v_3 = -\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}0\\2L\\0\end{bmatrix} = \begin{bmatrix}2L\\0\\0\end{bmatrix}, \ \mathcal{S}_3 = \begin{bmatrix}0\\0\\1\\2L\\0\\0\end{bmatrix}$$

$$J_s(\theta) = [\mathcal{S}_1, [\mathcal{A}d_{\hat{T}_1}]\mathcal{S}_2, [\mathcal{A}d_{\hat{T}_2}]\mathcal{S}_3]$$

$$\hat{T}_1 = e^{[\mathcal{S}_1]\theta_1}, \hat{T}_2 = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}$$

Calculate by MATLAB code:

```
function A = skew_symmetric(v)
    A = [0, -v(3), v(2);
         v(3), 0, -v(1);
         -v(2), v(1), 0];
end
syms theta1 theta2 L

w_1 = [0; 1; 0];
v_1 = [0; 0; 0];
S_1 = [w_1; v_1];

w_2 = [0; 0; 0];
v_2 = [0; 1; 0];
S_2 = [w_2; v_2];

w_3 = [0; 0; 1];
v_3 = [2*L; 0; 0];
S_3 = [w_3; v_3];

W_1 = skew_symmetric(w_1);
R_1 = eye(3,3) + W_1*sin(theta1) + W_1^2*(1-cos(theta1));
G_1 = eye(3,3)*theta1 + (1-cos(theta1))*W_1 + (theta1 - sin(theta1))*W_1^2;
p_1 = G_1*v_1;
[R_1, p_1]
E_1 = [R_1,p_1;zeros(1,3),1];
AdT_1 = [R_1,zeros(3,3);skew_symmetric(p_1)*R_1,R_1];

W_2 = skew_symmetric(w_2);
R_2 = eye(3,3) + W_2*sin(theta2) + W_2^2*(1-cos(theta2));
G_2 = eye(3,3)*theta2 + (1-cos(theta2))*W_2 + (theta2 - sin(theta2))*W_2^2;
p_2 = G_2*v_2;
E_2 = [R_2,p_2;zeros(1,3),1];
T2 = E_1*E_2;
AdT_2 = [T2(1:3,1:3),zeros(3,3);skew_symmetric(T2(1:3,4))*T2(1:3,1:3),T2(1:3,1:3)];

J = [S_1, AdT_1*S_2, AdT_2*S_3];
J
```

Result:

```
J =

[0, 0,                            sin(theta1)]
[1, 0,                                      0]
[0, 0,                            cos(theta1)]
[0, 0,   2*L*cos(theta1) + theta2*cos(theta1)]
[0, 1,                                      0]
[0, 0, - 2*L*sin(theta1) - theta2*sin(theta1)]
```

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & sin(\theta_1) \\ 1 & 0 & 0 \\ 0 & 0 & cos(\theta_1) \\ 0 & 0 & (2L+\theta_2)cos(\theta_1) \\ 0 & 1 & 0 \\ 0 & 0 & -(2L+\theta_2)sin(\theta_1) \end{bmatrix}$$

(b)

$$v_3 = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -L \\ 0 \end{bmatrix} = \begin{bmatrix} -L \\ 0 \\ 0 \end{bmatrix}, \mathcal{B}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -L \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathcal{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_1 = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -3L \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{B}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_b(\theta) = [[Ad_{\hat{T}_2}]\mathcal{B}_1, [Ad_{\hat{T}_3}]\mathcal{B}_2, \mathcal{B}_3]$$

$$\hat{T}_3 = e^{-[\mathcal{B}_3]\theta_3}, \hat{T}_2 = e^{-[\mathcal{B}_3]\theta_3}e^{-[\mathcal{B}_2]\theta_2}$$

Or

$$T_{bs} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J_b(\theta) = [Ad_{T_{bs}}]J_s(\theta)$$

$$J_b(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -L \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Suppose an external force

$$f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \in \mathbb{R}^3,$$

which is applied to the $\{b\}$ frame origin. The set of joint torques τ should satisfy:

$$\tau = J_b^T(\theta)F_b = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -L \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} 0 \\ f_y \\ -Lf_x \end{bmatrix}.$$

In this case, there are zero torques from the manipulator, which means

$$f_y = f_x = 0$$

The external force can be obtained as

$$f = \begin{bmatrix} 0 \\ 0 \\ f_z \end{bmatrix}$$