Distributing the Heat Equation

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1 Cellular automata

Question 1

Lemma 1. *N^2 applications of function δ are necessary to compute X^t from X^{t-1}.*

Proof. Each cell $X^t_{i,j}$ needs one application of δ to be computed from $X^{t-1}_{i,j}$. There are N^2 cells, so N^2 applications of δ are needed. □

Property 2. *tN^2 applications of function δ are necessary to compute X^t on $[\![0, N-1]\!]^2$.*

Proof. X^t is obtained after t applications of $\delta^\dagger$ on X^0. Each application needs N^2 calls to δ according to lemma 1. The whole computation needs tN^2 applications of δ. □

Question 2

Let p^2 be the number of processors.

For the sake of simplicity, we will assume throughout the homework that p divides N. Take $n = \frac{N}{p}$.

We divide the grid into square zones of size n. Each of this zones is given to one processor, which stores the data in its own memory and performs the computation of δ for all its cells. See figure 1 for an example.

At each step of computation, each processor updates its sub-matrix cells using a temporary sub-matrix that replaces the old one once the computation step is finished. Indeed, if we update the cells "in place", we overwrite values that are still necessary to compute other cells.

The computation of δ for the cells at the edges of the zones requires communication to retrieve the current states of their neighbours in other zones.

Question 3

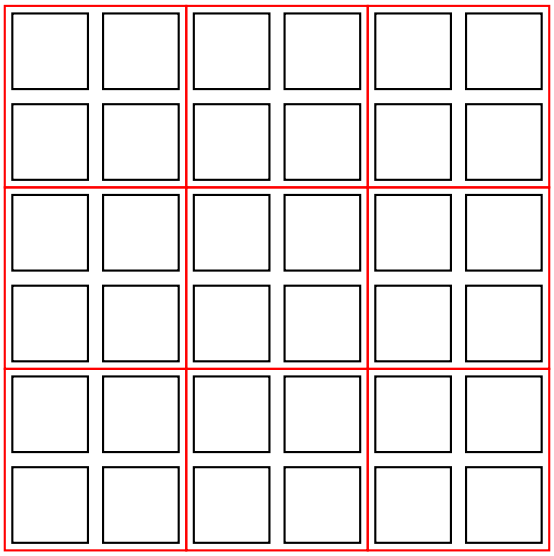
We assume that X^t is already distributed. Each processor is responsible for a submatrix of X^t of size $n \cdot n$. In practice, each processor also stores the 2 neighbours rows and the 2 neighbours colones of its submatrix. For each processor, we denote by $PREV$ the submatrix of size $(n+2) \times (n+2)$ it stores.

We also assume that the order of messages is preserved.

Let

$$\delta(PREV, i, j) = \delta\left(\begin{array}{|c|c|c|}\hline PREV[i-1,j-1] & PREV[i-1,j] & PREV[i-1,j+1] \\ \hline PREV[i,j-1] & PREV[i,j] & PREV[i+1,j+1] \\ \hline PREV[i+1,j-1] & PREV[i+1,j] & PREV[i+1,j+1] \\ \hline\end{array}\right)$$

Figure 1: Graphical representation of the topology for $N = 6$ and $p^2 = 9$.



We consider function `Send_X` (resp. `Receive_X`) for `X = Up, Down, Left, Right` which sends (resp. receives) to (resp. from) the corresponding processor. We suppose that this function has a time cost of 1 and a communication cost of $L + b$ where L is the latency and b the bandwidth.

We also consider functions to send an entire row (resp. column) as one single message, to decrease the overall latency. For instance, `Send_Down_Row(n,PREV)` will send to the down processor the n^{th} row, whereas `Send_Up_Row(0,PREV)` will receive from the up processor a row, which will be stored as the 0^{th} row. We suppose that this function has a time cost of n and a communication cost of $L + nb$ where L is

the latency, b the bandwidth, and n the size of the row/column.

Algorithm 1: Stencil algorithm on a toric 2D grid

Input: PREV: array[0..n+1,0..n+1] of real
Output: NEXT: array[0..n+1,0..n+1] of real

```
/* Columns and row                                          */
Send_Left_Column(1,PREV)
Send_Right_Column(n,PREV)
Send_Up_Row(1,PREV)
Send_Down_Row(n,PREV)
Receive_Left_Column(0,PREV)
Receive_Right_Column(n+1,PREV)
Receive_Up_Row(0,PREV)
Receive_Down_Row(n+1,PREV)
/* Corners                                                  */
Send_Up(PREV[1][0])
Send_Up(PREV[1][n+1])
Send_Down(PREV[n][0])
Send_Down(PREV[n][n+1])
Receive_Up(PREV[0][0])
Receive_Up(PREV[0][n+1])
Receive_Down(PREV[n+1][0])
Receive_Down(PREV[n+1][n+1])
/* Computation of δ                                         */
for i=1 to n do
    for j=1 to n do
        NEXT[i][j] = δ(PREV,i,j)
```

Time complexity

- *Each processor:* $8(n+1) + n^2\text{cost}(\delta) = 8(\frac{N}{p}+1) + \left(\frac{N}{p}\right)^2 \text{cost}(\delta) = \mathcal{O}\left(\left(\frac{N}{p}\right)^2\right)$ if $\text{cost}(\delta) = \mathcal{O}(1)$.
- *All processors:* $\mathcal{O}\left(N^2\right)$

Communication complexity

- *Each processor:* $8(L + nb + L + 1) = 8(2L + \frac{N}{p}b + 1)$.
- *All processors:* $8p^2(2L + \frac{N}{p}b + 1) = 8(2p^2L + Npb + p^2)$.

Non-toric grid

On a non-toric grid, processors placed on edges cannot send directly their information to their neighbours. They have to pass their messages throughout the whole grid. It needs p steps of communication instead of one.

Time complexity

- *Each processor on an edge and not on a corner of the grid:* $(6+2p)(8(n+1) + n^2\text{cost}(\delta))$.
- *Each processor on a corner:* $(4+4p)(8(n+1) + n^2\text{cost}(\delta))$.
- *Each processor not on an edge:* $8(n+1) + n^2\text{cost}(\delta)$.
- *All processors:* $4(p-2)(6+2p)(8(n+1) + n^2\text{cost}(\delta)) + 4(4+4p)(8(n+1) + n^2\text{cost}(\delta)) + (p-1)^2(8(n+1) + n^2\text{cost}(\delta)) = \mathcal{O}\left(N^2\right)$

Communication complexity

- *Each processor on an edge and not on a corner of the grid:* $6(L + nb + L + 1) + 2p(L + nb + L + 1) = (6 + 2p)(2L + \frac{N}{p}b + 1)$.
- *Each processor on a corner:* $4(L + nb + L + 1) + 4p(L + nb + L + 1) = (4 + 4p)(2L + \frac{N}{p}b + 1)$.
- *Each processor not on an edge:* $8(L + nb + L + 1) = 8(2L + \frac{N}{p}b + 1)$.
- *All processors:* $4(p-2)(6+2p)(2L + \frac{N}{p}b + 1) + 4(4+4p)(2L + \frac{N}{p}b + 1) + (p-1)^2 8(2L + \frac{N}{p}b + 1) = \mathcal{O}\left(p^2(L + \frac{N}{p}b)\right)$

Ring topology

On a ring topology we distribute the matrix in a different way. Each processor stores $\frac{N}{p^2}$ contiguous lines of the matrix. Thus, each processor only needs to get/send values to its 2 immediate neighbours in the ring.

Time complexity

- *Each processor:* $4n + N \cdot \frac{N}{p^2}\text{cost}(\delta)$ (4 rows are exchanged and $N \cdot \frac{N}{p^2}$ cells are updated).
- *All processors:* $p^2(4n + N \cdot \frac{N}{p^2}\text{cost}(\delta))$

Communication complexity

- *Each processor:* $4N(L + nb)$ (each processor sends 2 rows and receive 2 rows).
- *All processors:* $p^2 4N(L + nb)$

2 Average automata

Question 4

See the implementation in file `average.c`.

Question 5

Property 3. *In the case of a p-average automaton, $\delta^\dagger$ is linear.*

Proof. Let $\delta^\dagger$ be the global transition function of a *p-average automaton*. To prove that $\delta^\dagger$ is linear, it suffices to prove that the local transition function δ is linear:

$$\delta\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array}\right) = (1-p)\cdot e + p \cdot \frac{b+d+f+h}{4}$$

Let consider a real $k \in \mathbb{R}$ and two local configurations $\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array}$ and $\begin{array}{|c|c|c|}\hline a' & b' & c' \\ \hline d' & e' & f' \\ \hline g' & h' & i' \\ \hline\end{array}$. We have:

$$\delta\left(k\cdot\begin{array}{|c|c|c|}\hline a&b&c\\\hline d&e&f\\\hline g&h&i\\\hline\end{array}+\begin{array}{|c|c|c|}\hline a'&b'&c'\\\hline d'&e'&f'\\\hline g'&h'&i'\\\hline\end{array}\right)=(1-p)\cdot(k\cdot e+e')+p\cdot\frac{(k\cdot b+b')+(k\cdot d+d')+(k\cdot f+f')+(k\cdot h+h')}{4}$$

$$=k\cdot\left((1-p)\cdot e+p\cdot\frac{b+d+f+h}{4}\right)+(1-p)\cdot e'+p\cdot\frac{b'+d'+f'+h'}{4}$$

$$=k\cdot\delta\left(\begin{array}{|c|c|c|}\hline a&b&c\\\hline d&e&f\\\hline g&h&i\\\hline\end{array}\right)+\delta\left(\begin{array}{|c|c|c|}\hline a'&b'&c'\\\hline d'&e'&f'\\\hline g'&h'&i'\\\hline\end{array}\right)$$

Thus δ is linear, and $\delta^\dagger$ too. □

Definition 1. For $0\leqslant i,j\leqslant N-1$ we define the matrix $E^{i,j}$ such that $E^{i,j}_{i,j}=1$ and $E^{i,j}_{k,l}=0$ otherwise.

Lemma 4. *For all $0\leqslant i,j,k,l\leqslant N-1$, $\delta^{\dagger^t}(E^{i,j})_{k,l}=\delta^{\dagger^t}(E^{0,0})_{k-i,l-j}$ (indices are taken modulo N). Thus, knowing $\delta^{\dagger^t}(E^{0,0})$ we obtain $\delta^{\dagger^t}(E^{i,j})$ in constant time.*

Now let's consider a configuration X. We have : $X=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}X_{i,j}\cdot E^{i,j}$. Since $\delta^\dagger$ is linear, for all t :

$$\delta^{\dagger^t}(X)=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}X_{i,j}\cdot\delta^{\dagger^t}(E^{i,j})$$

Moreover :

$$\begin{aligned}\delta^{\dagger^{2t}}(X)&=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}X_{i,j}\cdot\delta^{\dagger^{2t}}(E^{i,j})\\&=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}X_{i,j}\cdot\delta^{\dagger^t}(\delta^{\dagger^t}(E^{i,j}))\\&=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}X_{i,j}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\delta^{\dagger^t}(E^{i,j})_{k,l}\cdot\delta^{\dagger^t}(E^{k,l})\\&=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}X_{i,j}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\delta^{\dagger^t}(E^{0,0})_{k-i,l-j}\cdot\delta^{\dagger^t}(E^{k,l})\end{aligned}\tag{1}$$

Thus, for all $0\leqslant m,n\leqslant N-1$:

$$\begin{aligned}\delta^{\dagger^{2t}}(X)_{m,n}&=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}X_{i,j}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\delta^{\dagger^t}(E^{0,0})_{k-i,l-j}\cdot\delta^{\dagger^t}(E^{k,l})_{m,n}\\&=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}X_{i,j}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\delta^{\dagger^t}(E^{0,0})_{k-i,l-j}\cdot\delta^{\dagger^t}(E^{0,0})_{m-k,n-l}\end{aligned}\tag{2}$$

Especially, for $X=E^{0,0}$:

$$\delta^{\dagger^{2t}}(E^{0,0})_{m,n}=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\delta^{\dagger^t}(E^{0,0})_{k,l}\cdot\delta^{\dagger^t}(E^{0,0})_{m-k,n-l}\tag{3}$$

Property 5. *Equations 2 and 3 enable us to compute $\delta^{\dagger^t}(X)$ in time $\mathcal{O}(\log(t))$ for a fixed N.*

Proof. For computing $\delta^{\dagger^{2t}}(X)$ we need to compute $\delta^{\dagger^t}(E^{0,0})$ using equation 3 and then apply equation 2 to get $\delta^{\dagger^{2t}}(X)$. The algorithm is:

Algorithm 2: Iteration on average automaton

Input: t, X
Output: $\delta^{\dagger^t}(X)$
$R \leftarrow E^{0,0}$;
for *i=1 to* $\log(t) - 1$ **do**
  Compute $R \leftarrow \delta^{\dagger^{2^i}}(E^{0,0})$ using $\delta^{\dagger^{2^{i-1}}}(E^{0,0})$ (previous value of R) and equation 3
Compute and return $\delta^{\dagger^t}(X)$ using $R = \delta^{\dagger^{t/2}}(E^{0,0})$ and equation 2

Let $T'(t)$ be the time needed to compute $\delta^{\dagger^t}(E^{0,0})$. According to equation 3 we have:

$$\begin{aligned} T'(t) &= T'(t/2) + \mathcal{O}(1) \\ &= \mathcal{O}(\log(t)) \end{aligned}$$

Now let $T(t)$ be the time needed to compute $\delta^{\dagger^t}(X)$. According to equation 2 and previous remarks, we have:

$$\begin{aligned} T(t) &= T'(t/2) + \mathcal{O}(1) \\ &= \mathcal{O}(\log(t)) \end{aligned}$$

Thus, applying the operations described in equations 2 and 3, we can compute $\delta^{\dagger^t}(X)$ in time $\mathcal{O}(\log(t))$. □

Property 6. *The time complexity $T_r(t, N)$ in terms of both t and N is $T_r(t, N) = \mathcal{O}\left(N^4 \cdot \log(t) + N^6\right)$.*

Proof. Let $T_e(t, N)$ be the time needed to compute $\delta^{\dagger^t}(E^{0,0})$. According to equation 3 we have:

$$\begin{aligned} T_e(t) &= T_e(t/2) + N^2 \cdot N^2 \text{ (there are } N^2 \text{ cells and each one needs to perform } N^2 \text{ sums)} \\ &= \mathcal{O}\left(N^4 \cdot \log(t)\right) \end{aligned}$$

According to equation 2 and the previous algorithm, knowing $\delta^{\dagger^{t/2}}(E^{0,0})$ it remains to perform for each cell the four sums described in equation 2 (it takes time $\mathcal{O}\left(N^4\right)$ per cell). Finally, we have:

$$T_r(t, N) = T_e(t/2, N) + N^2 \cdot N^4$$

Thus: $T_r(t, N) = \mathcal{O}\left(N^4 \cdot \log(t) + N^6\right)$.

According to question 3, the complexity in general case is $\mathcal{O}\left(N^2 \cdot t\right)$. The algorithm using the linearity of $\delta^\dagger$ is more interesting when the number of steps t increases. □

Property 7. *The space complexity $T_s(t, N)$ in terms of both t and N is $T_s(t, N) = \mathcal{O}\left(N^2\right)$.*

Proof. We need N^2 space to store the initial matrix.

When we compute $\delta^{\dagger^{2^i}}(E^{0,0})$ using $\delta^{\dagger^{2^{i-1}}}(E^{0,0})$, we need to use N^2 space to store $\delta^{\dagger^{2^i}}$. However, the space used by $\delta^{\dagger^{2^{i-1}}}$ can be re-use to compute $\delta^{\dagger^{2^{i+1}}}$. Thus, all the computation can be done using $\mathcal{O}\left(N^2\right)$ space.

The space complexity is the same than in the general case. □

Question 6

Matrices R and $\delta^{\dagger^t}(X)$ are distributed in the same way than in question 2 (each processor stores a submatrix of size $n \cdot n$).

Each step of the `for` loop of algorithm 2 is performed in the following way:

- each processor broadcasts its submatrix to all the other processors.
- using equation 3, each processor updates a temporary submatrix B initially null, as soon as it receives the submatrices it needs.
- each processor replaces the part of R it stores by the submatrix B.

At the end of the previous steps, $\delta^{\dagger^{t/2}}(E^{0,0})$ is computed and distributed. $\delta^{\dagger^t}(X)$ is computed in the same way than $\delta^{\dagger^{t/2}}(E^{0,0})$ (each processor broadcasts the parts of X^0 and $\delta^{\dagger^{t/2}}(E^{0,0})$ it stores, the computation is done using equation 2).

Property 8. *The time complexity is* $T_t(t, N) = \mathcal{O}\left(\log(t) \cdot N^4 + N^6\right)$ *(we do not count communication time).*

Proof. We recall that $n = \frac{N}{p}$.

At each step of the `for` loop, each processor performs $\mathcal{O}\left(n^2 \cdot N^2\right)$ operations according to equation 3. The final computation of $\delta^{\dagger^t}(X)$ using $\delta^{\dagger^{t/2}}(E^{0,0})$ needs $\mathcal{O}\left(N^2 \cdot N^4\right)$ operations. Thus $T_t(t, N) = \log(t) \cdot p^2 \cdot n^2 \cdot N^2 + N^6) = \mathcal{O}\left(\log(t) \cdot N^4 + N^6\right)$. □

Property 9. *The space complexity is* $T_s(t, N) = \mathcal{O}\left(N^2\right)$.

Proof. Each processor stores 3 arrays of size n^2 and need $\mathcal{O}\left(n^2\right)$ extra space during broadcast operations. Moreover, this space can be reused at each new step. Thus:

$$\begin{aligned} T_s(t, N) &= \mathcal{O}\left(p^2 \cdot n^2\right) \\ &= \mathcal{O}\left(N^2\right) \end{aligned}$$

□

Property 10. *The communication time complexity is* $T_c(t, N) = \mathcal{O}\left(c \cdot p^2 \cdot \log(t) \cdot N^2\right)$.

Proof. At each step of the `for` loop, each processor performs $\mathcal{O}\left(p^2\right)$ broadcasts of a matrix of size n^2. Thus:

$$\begin{aligned} T_c(t, N) &= \mathcal{O}\left(c \cdot p^2 \cdot p^2 \cdot \log(t) \cdot n^2\right) \\ &= \mathcal{O}\left(c \cdot p^2 \cdot \log(t) \cdot N^2\right) \end{aligned}$$

□

Question 7

Using the same reasoning than in question 5, for all $0 \leqslant m, n \leqslant N-1$ we have:

$$\begin{aligned} X^t_{m,n} &= \delta^{\dagger^t}(X^0)_{m,n} \\ &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} X^0_{i,j} \cdot \delta^{\dagger^t}(E^{i,j})_{m,n} \\ &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} X^0_{i,j} \cdot \delta^{\dagger^t}(E^{0,0})_{m-i,n-j} \\ &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} X^0_{i,j} \cdot Z^t_{m-i,n-j} \end{aligned} \tag{4}$$

Since X^0 is sparse, we assume that each processor p stores the list of *all* the triples $(i, j, X^0_{i,j})$ such that $X^0_{i,j} \neq 0$.

We use the following extra functions (`LocalizeCell` and `LocalizeProc` work on matrices M of size $N \cdot N$ distributed on p^2 processors according to question 2):

- `LocalizeCell(i,j)`: returns the triple (p, k, l) where p is the processor storing $M_{i,j}$ at position (k, l) in its submatrix.
- `LocalizeProc(p,k,l)`: returns the position (i, j) in M of the cell stored at position (k, l) in the submatrix of processor p.
- `Send(p,t,msg)`: send message `msg` at processor `p` with tag `t`.
- `Receive(p,t,dest)`: receive message of tag `t` sent by processor `p` and store it in `dest`.

For instance, the submatrix of M stored in processor p is:

$M_{\texttt{LocalizeProc(p,0,0)}}$	$\cdots$	$M_{\texttt{LocalizeProc(p,0,n-1)}}$
$\vdots$	$\vdots$	$\vdots$
$M_{\texttt{LocalizeProc(p,n-1,0)}}$	$\cdots$	$M_{\texttt{LocalizeProc(p,n-1,n-1)}}$

All processors execute the following algorithm in parallel:

Algorithm 3: Distributed iteration on average automaton with sparse initial condition

```
var L: list of non null cells of X^0, sorted in increasing order;
var A: array[0..n-1][0..n-1] part of Z^t handled by the processor (Z^t is distributed);
var B: array[0..n-1][0..n-1] part of X^t handled by the processor (X^t is distributed);
p ← My_Num();
for all (i, j, X^0_{i,j}) taken in L in increasing order do
    for l=0 to n-1 do
        for m=0 to n-1 do
            (p_s, q_s, r_s) ← LocalizeCell(LocalizeProc(p,l,m)+(i,j)) ;
            (p_r, _, _) ← LocalizeCell(LocalizeProc(p,l,m)-(i,j)) ;
            Send(p_s,(q_s,r_s),A[l][m]) || Receive(p_r,(l,m),temp);
            B[l][m] ← B[l][m] + X^0_{i,j}· temp
```

Initially, all the cells of B must be null.

Property 11. *The time complexity $T_t(N, |X_0|)$ of algorithm 3 is $T_t(N, |X_0|) = 3 \cdot |X_0| \cdot \frac{N^2}{p^2}$ (we do not count communication time).*

Proof. We recall that $n = \frac{N}{p}$.

Each processor performs 3 loops `for`, with three operations at each iteration. Thus, $T_t(N, |X_0|) = |X_0| \cdot n \cdot n \cdot 3 = 3 \cdot |X_0| \cdot \frac{N^2}{p^2}$. □

Property 12. *The space complexity $T_s(N, |X_0|)$ of algorithm 3 is $T_s(N, |X_0|) = 2N^2 + p^2 \cdot |X_0|$.*

Proof. Each processor stores two arrays of size n^2 and one list of size $|X_0|$. There are p^2 processors. We obtain:

$$\begin{aligned} T_s(N, |X_0|) &= p^2 \cdot (2n^2 + |X_0|) \\ &= 2N^2 + p^2 \cdot |X_0| \end{aligned}$$

□

Property 13. *The communication time complexity $T_c(N, |X_0|)$ of algorithm 3 is $T_c(N, |X_0|) = 2 \cdot c \cdot |X_0| \cdot \frac{N^2}{p^2}$.*

Proof. Each processor performs 3 loops `for`, with two communication operations at each iteration. Thus, $T_t(N, |X_0|) = 2 \cdot c \cdot |X_0| \cdot n \cdot n = 2 \cdot c \cdot |X_0| \cdot \frac{N^2}{p^2}$. □

Question 8

See the implementation in file `sparse.c`.

Question 9

Configurations with all their cells equal to a same constant c are example of fixed points in $\mathbb{R}^{\mathbb{Z}^d}$ and $\mathbb{R}^{[\![N-1]\!]^d}$.

3 Thermal reservoirs

Question 10

Example. The following configuration X^0 is a fixed point with one constant (in blue):

0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

Example. The following configuration X^0 is a fixed point with two constants (in blue):

1	2	0	−2	−1
2	5	0	−5	−2
4	16	0	−16	−4
2	5	0	−5	−2
1	2	0	−2	−1

Definition 2. We denote X^∞ the limit of (X^t) if it exists (i.e. (X^t) converges).

Lemma 14. *For each sequence (X^t), if X^∞ exists then it is a fixed point.*

Proof. Let us assume that X^∞ exists. By definition, for all position $X_{i,j}$ that is not a constant:

$$\begin{aligned}
\lim_{t\to\infty} \delta^{\dagger^t}(X)_{i,j} &= X^\infty_{i,j} \\
&= \lim_{t\to\infty} \left((1-p)\cdot \delta^{\dagger^{t-1}}(X)_{i,j} + p\cdot \frac{\delta^{\dagger^{t-1}}(X)_{i,j+1} + \delta^{\dagger^{t-1}}(X)_{i+1,j} + \delta^{\dagger^{t-1}}(X)_{i-1,j} + \delta^{\dagger^{t-1}}(X)_{i,j-1}}{4} \right) \\
&= (1-p)\cdot (\lim_{t\to\infty} \delta^{\dagger^{t-1}}(X)_{i,j}) + p\cdot \frac{(\lim_{t\to\infty} \delta^{\dagger^{t-1}}(X)_{i,j+1}) + (\lim_{t\to\infty} \delta^{\dagger^{t-1}}(X)_{i+1,j})}{4} \\
&\quad + p\cdot \frac{(\lim_{t\to\infty} \delta^{\dagger^{t-1}}(X)_{i-1,j}) + (\lim_{t\to\infty} \delta^{\dagger^{t-1}}(X)_{i,j-1})}{4} \\
&= (1-p)\cdot X^\infty_{i,j} + p\cdot \frac{X^\infty_{i+1,j} + X^\infty_{i,j+1} + X^\infty_{i-1,j} + X^\infty_{i,j-1}}{4}
\end{aligned}$$

Since $X^\infty_{i,j} = (1-p)\cdot X^\infty_{i,j} + p\cdot \frac{X^\infty_{i+1,j}+X^\infty_{i,j+1}+X^\infty_{i-1,j}+X^\infty_{i,j-1}}{4}$, X^∞ is a fixed point.

□

Lemma 15. *If X is a fixed point then for each position $X_{i,j}$ that is not a constant we have :*

$$X_{i,j} = \frac{X_{i,j+1} + X_{i+1,j} + X_{i-1,j} + X_{i,j-1}}{4}$$

Proof. If X is a fixed point then for each position $X_{i,j}$ that is not a constant:

$$X_{i,j} = (1-p)\cdot X_{i,j} + p\cdot \frac{X_{i,j+1} + X_{i+1,j} + X_{i-1,j} + X_{i,j-1}}{4}$$

If we remove $X_{i,j}$ from both side of this equality and then we simplify it by p ($p \neq 0$), we obtain the lemma. □

Lemma 16. *If X is configuration without any constant then for all t the sum of all values in $\delta^{\dagger^{t+1}}$ is equal to the sum of all values in $\delta^{\dagger^t}$.*

Proof. We directly obtain that:

$$\begin{aligned}
\sum_{i,j} \delta^{\dagger^{t+1}}(X)_{i,j} &= \sum_{i,j} (1-p)\cdot \delta^{\dagger^t}(X)_{i,j} + p\cdot \frac{\delta^{\dagger^t}(X)_{i+1,j} + \delta^{\dagger^t}(X)_{i,j+1} + \delta^{\dagger^t}(X)_{i-1,j} + \delta^{\dagger^t}(X)_{i,j-1}}{4} \\
&= \sum_{i,j} \delta^{\dagger^t}(X)_{i,j}
\end{aligned}$$

□

Property 17. *Let X^0 be a configuration without any constant. If it exists, the limit of (X^t) is $\overline{X^0}$ (the average value of all cells of X^0).*

Proof. We assume that the limit X^∞ of (X_t) exists.

First of all, we prove that all the values of X^∞ are equal.

Let us consider a value d contained into X^∞ and assume that not all the cells of X^∞ are equal to d.

First case There exists a cell in X^∞ that contains a value strictly lower than d. We consider a cell C of X^∞ containing the lowest value e of X^∞ and such that one of its 4 neighbours is different from e (at least one cell contains $d > e$, so C exists). According to lemmas 14 and 15, the value of C must be the average of its 4 neighbours. Since e is the lowest value of X^∞ and one of the neighbours of C contains a value different from e (so strictly greater than e), C cannot contains the average of its 4 neighbours. It is absurd.

Second case There exists a cell in X^∞ that contains a value strictly greater than d. As previously, we prove it is absurd.

Finally, it is absurd that one of the cells of X^∞ contains a value different from d. All the cells of X^∞ must be equal.

According to lemma 16 the sum of all values does not change at any step of the computation. Thus, all the cells of x^∞ must be equal to the average value of all the initial cells in X^0. □

Property 18. *Let X^0 be a configuration with only one constant c. If it exists, the limit of (X^t) is the configuration with all its cells equal to c.*

Proof. We assume that the limit X^∞ of (X_t) exists.

As in property 17, we prove by contradiction that all the cells of X^∞ must be equal to c, in the corresponding proof we choose c for the value d and the corresponding cell C (that is supposed to contain the lowest or greatest value of the configuration, different from $c = d$) cannot be a constant cell. □

Question 11

Property 19. *For a p-average automaton with constants, $\delta^\dagger$ can be non-linear.*

Proof. Let's consider the following local configuration with one constant (in blue):

0	0	0
0	1	0
0	0	0

We take $p = 0.5$ and we obtain:

$$\delta\left(\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 0&0&0\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 0&0&0\\\hline\end{array}\right) = \delta\left(\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&2&0\\\hline 0&0&0\\\hline\end{array}\right)$$
$$= 0.5 \cdot 2 = 1$$

However:

$$\delta\left(\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 0&0&0\\\hline\end{array}\right) + \delta\left(\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 0&0&0\\\hline\end{array}\right) = 1 + 1 = 2$$

δ is not linear, and so $\delta^\dagger$ too. This result proves that $\delta^\dagger$ can be non-linear for a *p-average automaton* with constants. □

Question 12

Question 13

See the implementation in file `average.c` (since the code is the same as in question 4).