Distributing the Heat Equation

Tom Cornebize   Yassine Hamoudi

Sunday, December 7th 2014

1 Cellular automata

Question 1.

Lemma 1. *N^2 applications of function δ are necessary to compute X^t from X^{t-1}.*

Proof. Each cell $X_{i,j}^t$ needs one application of δ to be computed from $X_{i,j}^{t-1}$. There are N^2 cells, so N^2 applications of δ are needed. □

Property 2. *tN^2 applications of function δ are necessary to compute X^t on $[\![0, N-1]\!]^2$.*

Proof. X^t is obtained after t applications of $\delta^\dagger$ on X^0. Each application needs N^2 calls to δ according to lemma 1. The whole computation needs tN^2 applications of δ. □

Question 2. Let p^2 be the number of processors.

For the sake of simplicity, we will suppose that p divides N. Take $n = \frac{N}{p}$.

We divide the grid into square zones of size n. Each of this zones is given to one processor, which stores the data in its own memory and performs the computation of δ for all its cells. See figure 1 for an example.

At each step of computation, each processor updates its sub-matrix cells using a temporary sub-matrix that replaces the old one once the computation step is finished. Indeed, if we update the cells "in place", we overwrite values that are still necessary to compute other cells.

The computation of δ for the cells at the edges of the zones requires communication to retrieve the current states of their neighbours in other zones.

Question 3.

Algorithm 1: Stencil algorithm on a toric 2D grid

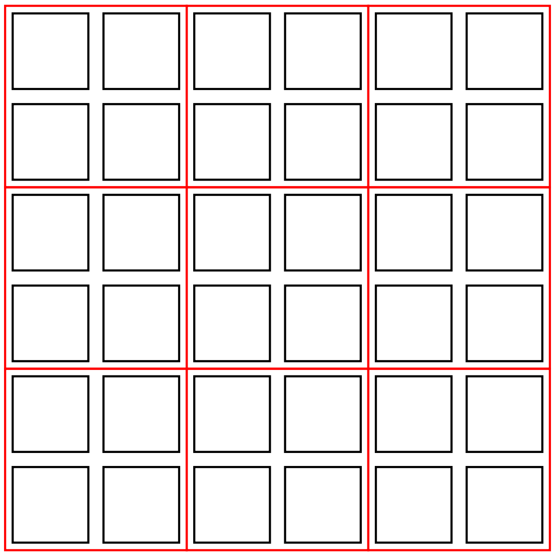
var A: array[0..n-1,0..n-1] of real
var NEXT: array[0..n+1,0..n+1] of real
$myrow \leftarrow$ `My_Proc_Row();` $mycol \leftarrow$ `My_Proc_Col();`

2 Average automata

Question 4. See the implementation in `average.c`.

Question 5.

Figure 1: Graphical representation of the topology for $N = 6$ and $p^2 = 9$.



Property 3. *In the case of a p-average automaton, $\delta^\dagger$ is linear.*

Proof. Let $\delta^\dagger$ be the global transformation function of a *p-average automaton*. To prove that $\delta^\dagger$ is linear, it suffices to prove that the local transformation function δ is linear:

$$\delta\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array}\right) = (1-p)\cdot e + p \cdot \frac{b+d+f+h}{4}$$

Let consider a real $k \in \mathbb{R}$, two local configurations $\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array}$ and $\begin{array}{|c|c|c|}\hline a' & b' & c' \\ \hline d' & e' & f' \\ \hline g' & h' & i' \\ \hline\end{array}$. We have:

$$\begin{aligned}
\delta\left(k\cdot\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array}+\begin{array}{|c|c|c|}\hline a' & b' & c' \\ \hline d' & e' & f' \\ \hline g' & h' & i' \\ \hline\end{array}\right) &= (1-p)\cdot(k\cdot e+e') + p\cdot\frac{(k\cdot b+b')+(k\cdot d+d')+(k\cdot f+f')+(k\cdot h+h')}{4} \\
&= k\cdot\left((1-p)\cdot e + p\cdot\frac{b+d+f+h}{4}\right) + (1-p)\cdot e' + p\cdot\frac{b'+d'+f'+h'}{4} \\
&= k\cdot\delta\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline\end{array}\right) + \delta\left(\begin{array}{|c|c|c|}\hline a' & b' & c' \\ \hline d' & e' & f' \\ \hline g' & h' & i' \\ \hline\end{array}\right)
\end{aligned}$$

Thus δ is linear, and $\delta^\dagger$ too. $\square$

Let's consider a configuration X. For $0 \leqslant i, j \leqslant N-1$ we define the matrix $E^{i,j}$ such that $E^{i,j}_{i,j} = 1$ and $E^{i,j}_{k,l} = 0$ otherwise. We obtain : $X = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} X_{i,j}\cdot E^{i,j}$

Since $\delta^\dagger$ is linear, for all t :

$$\delta^{\dagger^t}(X) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} X_{i,j}\cdot\delta^{\dagger^t}(E^{i,j})$$

Moreover :

$$
\begin{aligned}
\delta^{\dagger^{2t}}(X) &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} X_{i,j} \cdot \delta^{\dagger^{2t}}(E^{i,j}) \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} X_{i,j} \cdot \delta^{\dagger^{t}}(\delta^{\dagger^{t}}(E^{i,j})) \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} X_{i,j} \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \delta^{\dagger^{t}}(E^{i,j})_{k,l} \cdot \delta^{\dagger^{t}}(E^{i,j})
\end{aligned} \tag{1}
$$

Property 4. *Equation 1 enables us to compute $\delta^{\dagger^t}(X)$ in time $O(logt)$.*

Proof. First of all, if $\delta^{\dagger^t}(E^{0,0})$ is already computed, we directly obtained by translation $\delta^{\dagger^t}(E^{i,j})$, for all i, j.

Let $T(t)$ the time needed to compute $\delta^{\dagger^t}(X)$. According to equation 1 and previous remark, we have:

$$
\begin{aligned}
T(2t) &= T(t) + O(1) \\
&= O(\log t)
\end{aligned}
$$

Thus, using the algorithm described by equation 1, we can compute $\delta^{\dagger^t}(X)$ in time $O(\log t)$. □

3 Thermal reservoirs

Question 6.

Example.

Question 7.

Property 5. *For a p-average automaton with constants, $\delta^\dagger$ can be non-linear.*

Proof. Let's consider the following local configuration:

0	0	0
0	1	0
0	0	0

We assume that 1 is a constant, but 0 not. We take $p = 0.5$.

We have:

$$
\begin{aligned}
\delta\left(\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 0&0&0\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 0&0&0\\\hline\end{array}\right) &= \delta\left(\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&2&0\\\hline 0&0&0\\\hline\end{array}\right) \\
&= 0.5 \cdot 2 = 1
\end{aligned}
$$

However:

$$
\delta\left(\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 0&0&0\\\hline\end{array}\right) + \delta\left(\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&1&0\\\hline 0&0&0\\\hline\end{array}\right) = 1 + 1 = 2
$$

δ is not linear, and so $\delta^\dagger$ too. This result proves that $\delta^\dagger$ can be non-linear for a *p-average automaton* with constants. □