International Macroeconomics: Basic Models

Intro

1. We'll introduce some basic models of international macro basic.
2. We'll start with a basic 2-period SOE model without investment and uncertainty.
3. Next we'll add more time periods to be able to think more clearly about dynamics, shock persistence, etc.
4. We'll also add investment and uncertainty.

Two Period Model: The Basics

The basic 2 period consumption model looks like this:

1. Basic assumptions:
   a. World starts in year 1 and end after year 2.
   b. Output is given exogenously.
   c. There is an exogenously given interest rate $r \geqslant 0$.

2. Basic definitions:
   a. C_s consumption in year s.
   b. Y_s output (income) in year s.
   c. B_s assets (bonds) accumulated at the end of year *s - 1* ; interest on these assets is paid in year *s*.
   d. $\triangle B_{s+1} = B_{s+1} - B_s$

1. Life-time utility is called *U*. Starting in the first year (i.e. year 1) this life-time utility is given by:

$$U_1 = u(C_1) + \beta u(C_2)$$

where u'(C) > 0 and u''(C) < 0.

2. The representative agent maximizes U subject to the following “flow” budget constraints:

$r\, B_1 + Y_1 = C_1 + \triangle B_2$

$r\, B_2 + Y_2 = C_2 + \triangle B_3$

$B_{s+1} = (1 + r)\, B_s + Y_s - C_s$

which can be re-stated as

$B_2 = (1 + r)\, B_1 + Y_1 - C_1$

$B_3 = (1 + r)\, B_2 + Y_2 - C_2$

and combined to create the life-time budget constraint:

$$(1 + r)\, B_1 + Y_1 + \frac{1}{1 + r} Y_2 = C_1 + \frac{1}{1 + r} C_2 + \frac{1}{1 + r} B_3$$

We know that $B_3 = 0$. Why?

If we additionally assume that $B_1 = 0$, we get the familiar constraint:

$$Y_1 + \frac{Y_2}{1 + r} = C_1 + \frac{C_2}{1 + r}$$

The problem of the agent is therefore given by:

$$\max_{\{C_1,C_2\}} u(C_1) + \beta u(C_2) \text{ subject to}$$

$$Y_1 + \frac{Y_2}{1+r} = C_1 + \frac{C_2}{1+r}$$

Notice that the problem can be expressed as:

$$\max_{\{C_1,C_2\}} \sum_{s=1}^{2} \beta^{s-1} u(C_s) \text{ subject to}$$

$$\sum_{s=1}^{2} \left(\frac{1}{1+r}\right)^{s-1} Y_s = \sum_{s=1}^{2} \left(\frac{1}{1+r}\right)^{s-1} C_s$$

Model Quantities vs. Real-World Statisitcs

Let’s see how our model quantities align with real-world macro statistics. Let’s do this for year 1:

Y_1 = year 1 GDP

C_1 = year 1 GNE (no I or G in this model)

$TB_1 = Y_1 - C_1$= year 1 trade balance (recall that GDP = GNE + TB)

$r B_1$ = year 1 NIFA

$r B_1 + Y_1$= year 1 GNI (equal to GNDI since we don’t have NUT in this model)

$CA_1 = r B_1 + Y_1 - C_1 = NFIA + TB$ = year 1 current account balance

- $\triangle B_2$ = year 1 financial account

B_2= external (and total) wealth at the end of year 1

There are several ways to solve this problem. The most straight-forward one is to substitute for C_1 using the budget constraint and solve the univariate maximization problem. The First order condition is the well-known Euler equation:

$$\boxed{u'(C_1) = (1+r)\beta u'(C_2)}$$

To get closed form solutions need further assumptions on u(C) or β and r. For example:

1. If $(1+r)\beta = 1$, then $C_1 = C_2$ and

$$C_1 = \frac{1+r}{2+r}\left[Y_1 + \frac{Y_2}{1+r}\right]$$

2. If u(C) = ln(C), then

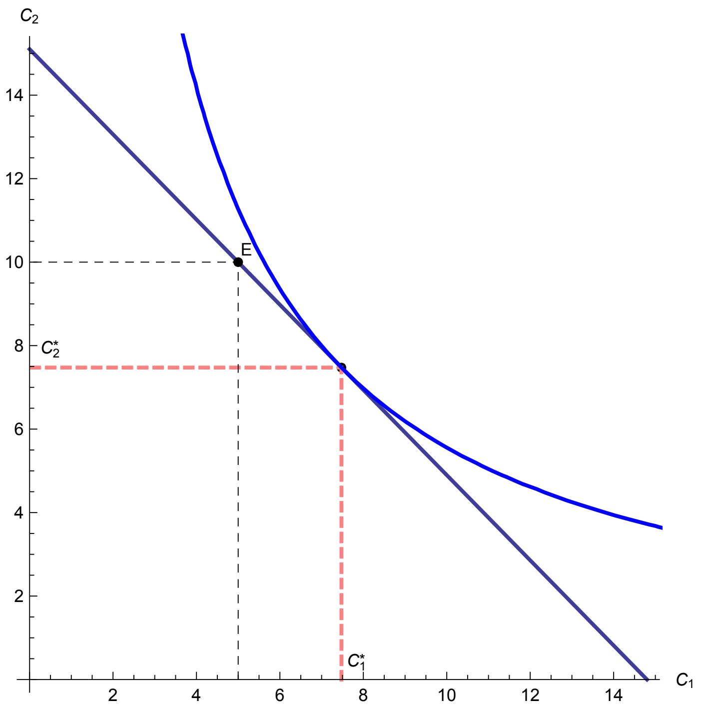
$$C_1 = \frac{1}{1+\beta}\left[Y_1 + \frac{Y_2}{1+r}\right]$$

2. If $u(C) = \frac{C^{1-1/\sigma}}{1-1/\sigma}$.

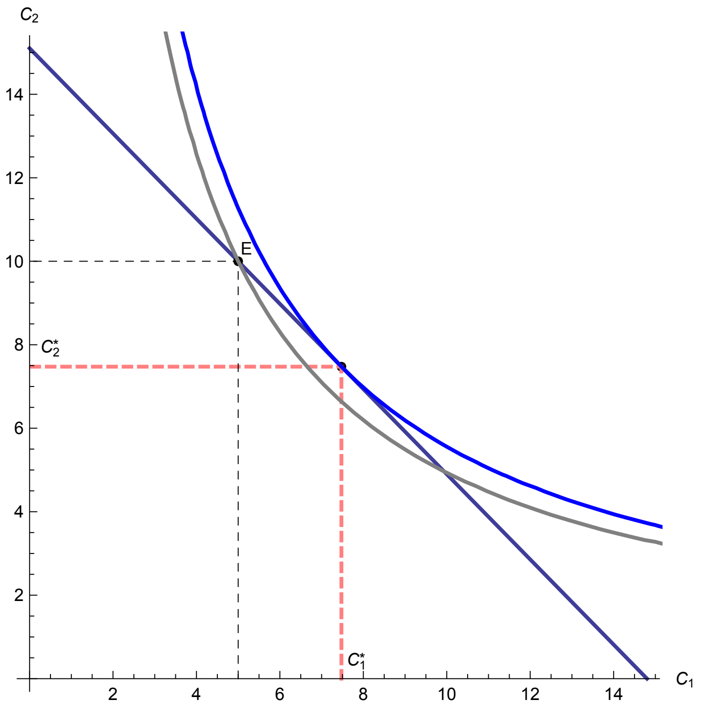
$$C_1 = \frac{1}{1+(1+r)^{\sigma-1}\beta^{\sigma}}\left[Y_1 + \frac{Y_2}{1+r}\right]$$

Graphical Solution

Let's draw some graphs.
Here is an example of a solution when $Y_2 > Y_1$and $(1+r)\beta = 1$.
This could be China?

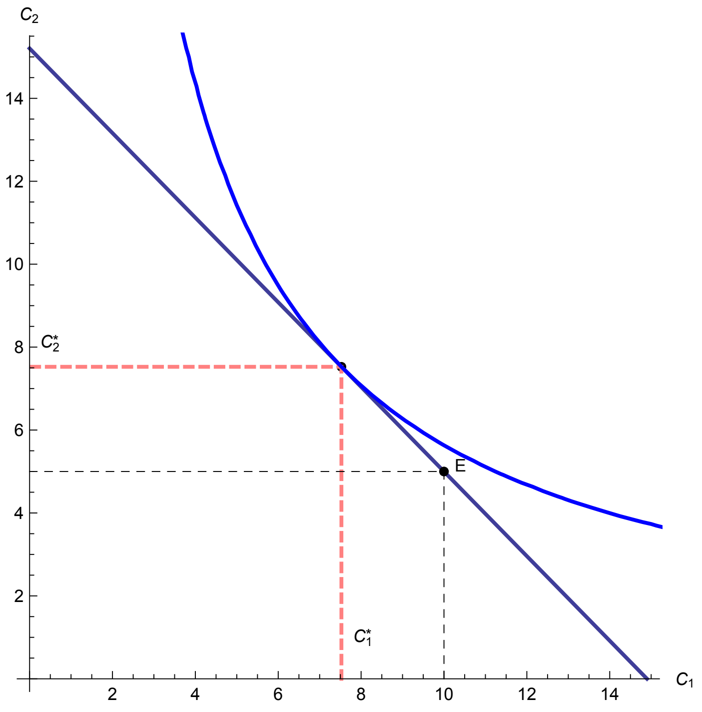


Notice that China is better off (on a higher indifference cureve) when it is open to capital flows than when it is closed.



Graphical Solution

Here is an example of a solution when $Y_2 < Y_1$ and $(1+r)\beta = 1$.
Norway?



Interest Rate in Autarky

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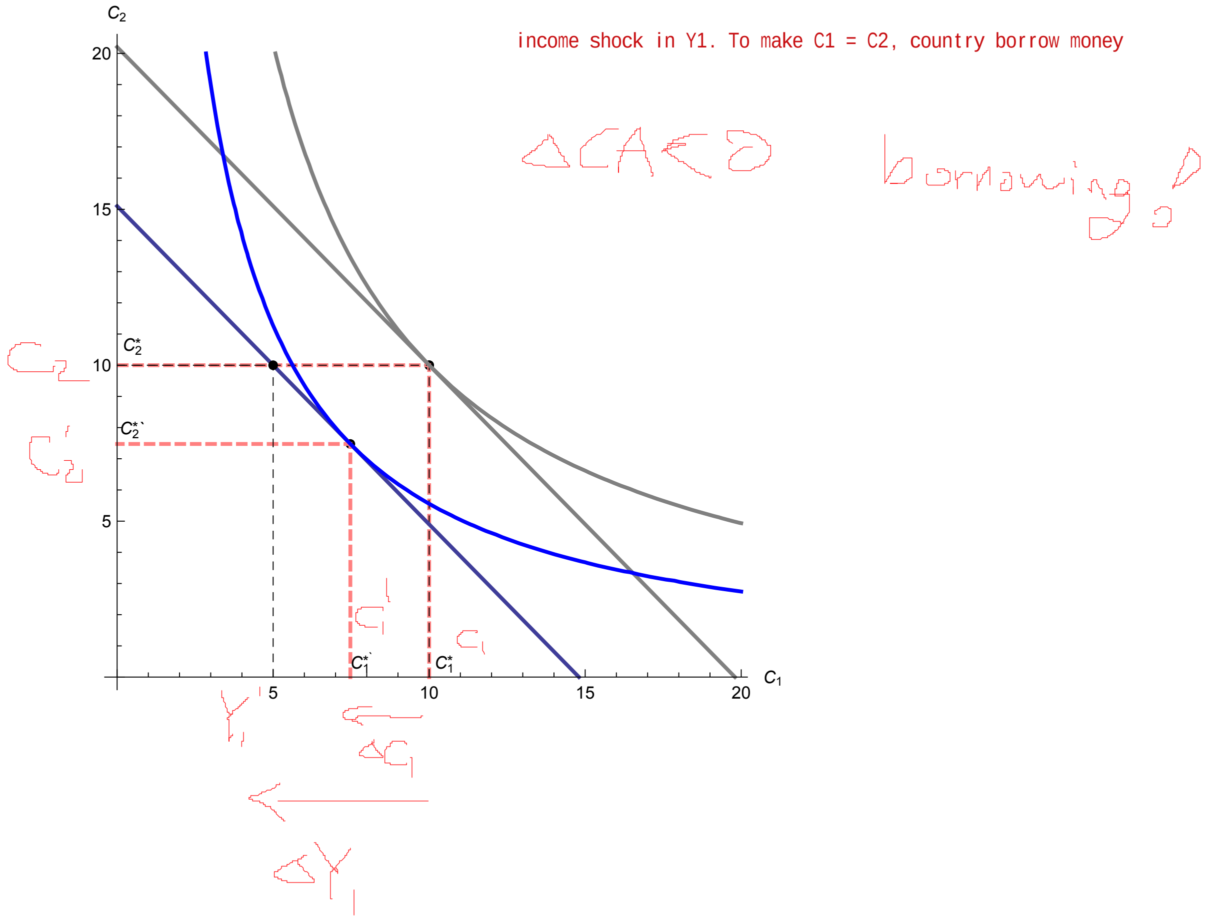
Case 1.

Income Shocks: a transitory shock income

With just two periods a transitory shock is a fall in Y_1 without a change in Y_2.

For simplicty, let's assume we start with $Y_2 = Y_1$and $(1+r)\beta = 1$ so that initially (before the shock) the optimal consumption would have been $C_1 = Y_1$and $C_2 = Y_2$.

What will happend to consumption? It falls but by less than the income shock. Why? Consumption smoothing. How? International borrowing! Current account goes into deficit.



Does the math check out? For example, when $(1+r)\beta = 1$ we got

$$C_1 = \frac{1+r}{2+r}\left[Y_1 + \frac{Y_2}{1+r}\right]$$

Which implies

$$\triangle C_1 = \frac{1+r}{2+r}\triangle Y_1 < \triangle Y_1$$

Check other cases on your own.

Case 2

Income Shocks: a permanent shock income

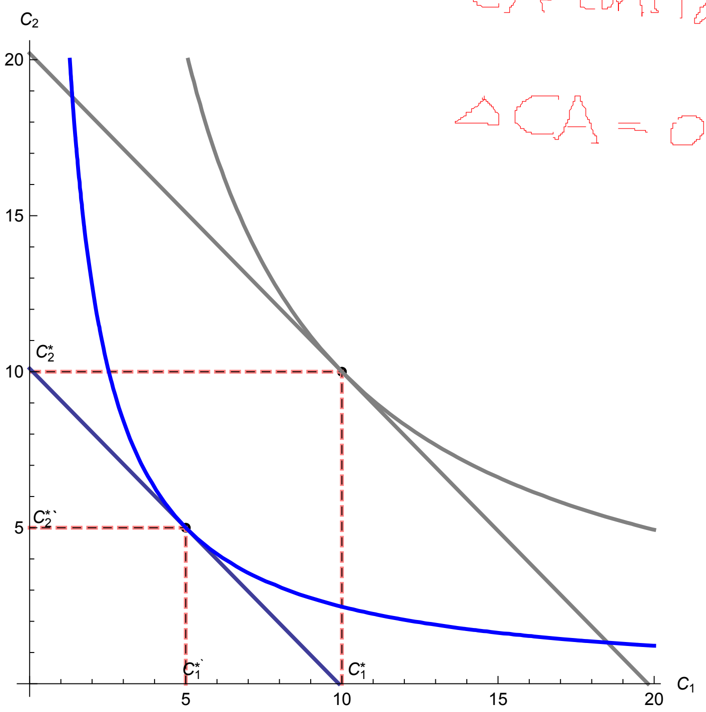
With just two periods a permanent shock is a fall in both Y_1 and Y_2 (of eqiual size).

For simplicity, let’s assume we start with $Y_2 = Y_1$and $(1+r)\beta = 1$ so that initially (before the shock) the optimal consumption would have been $C_1 = Y_1$and $C_2 = Y_2$.

What will happend to consumption? It falls by exactly the same amount as the income shock. Why? Consumption smoothing is not possible; this is a permanet shock. No change in international borrowing! Current account is unchanged.

CA constant

$\Delta CA = 0$



$Y_2 = C_2$

$Y_2' = C_2'$

$Y_1' = C_1'$ $Y_1 = C_1$

Does the math check out? For example, when $(1+r)\beta = 1$ we got

$$C_1 = \frac{1+r}{2+r}\left[Y_1 + \frac{Y_2}{1+r}\right]$$

Which implies (with $\triangle Y_1 = \triangle Y_2 = \triangle Y$)

$$\triangle C_1 = \frac{1+r}{2+r}\left[\triangle Y_1 + \frac{\triangle Y_2}{1+r}\right] = \frac{1+r}{2+r}\left[\triangle Y + \frac{\triangle Y}{1+r}\right] = \frac{1+r}{2+r}\left[\frac{2+r}{1+r}\triangle Y\right] = \triangle Y$$

Check other cases on your own.

Case 3

Income Shocks: a shock to future income

A fall in Y_2 without a change in Y_1.

For simplicty, let's assume we start with $Y_2 = Y_1$ and $(1+r)\beta = 1$ so that initially (before the shock) the optimal consumption would have been $C_1 = Y_1$ and $C_2 = Y_2$.

What will happend to consumption? It falls. Why? Consumption smoothing. How? International lending! Current account goes into surplus.

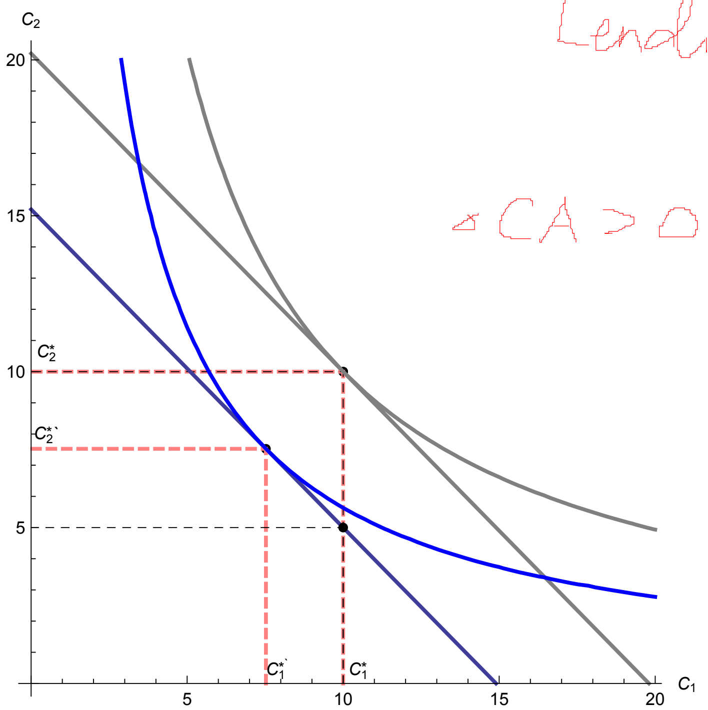
Lending ;)

$\Delta CA > 0$

$Y_2 = C_2$

C_2'

Y_2'



C_1' $Y_1 = Y_1' = C_1$

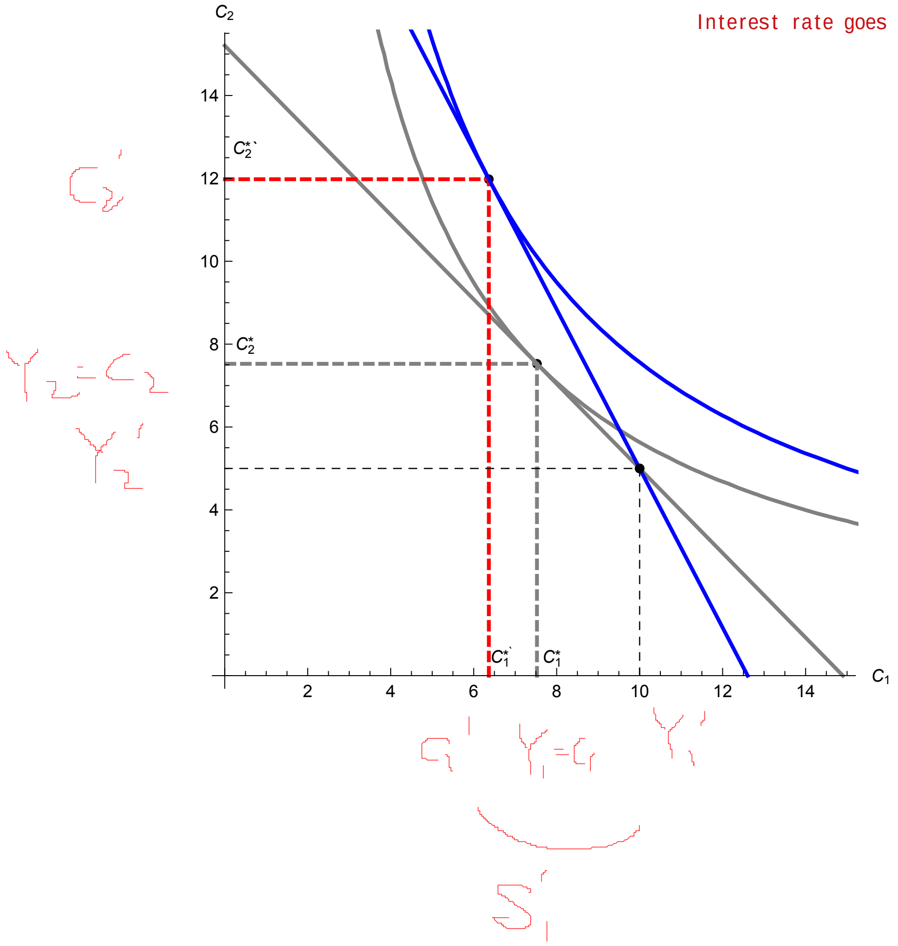
Case 4

Change in World Interest Rate

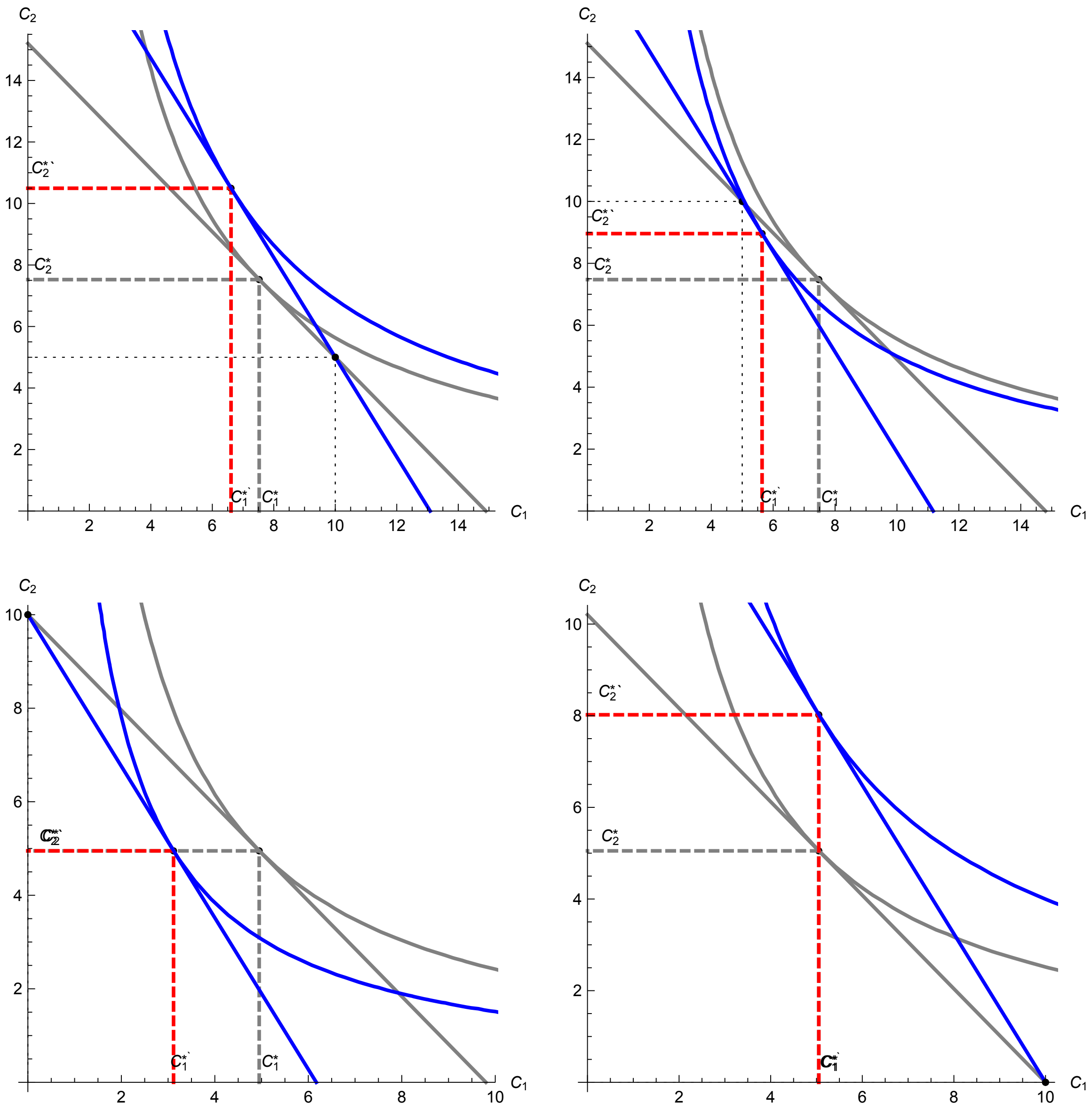
r ↑

What happens to C1 and CA when the interest rate goes up?
In the graph below, C1 falls and CA1 improves (thus saving increase). But is this necessarily always true?

Interest rate goes up --> MC goes up for C1, steeper blue line (price)



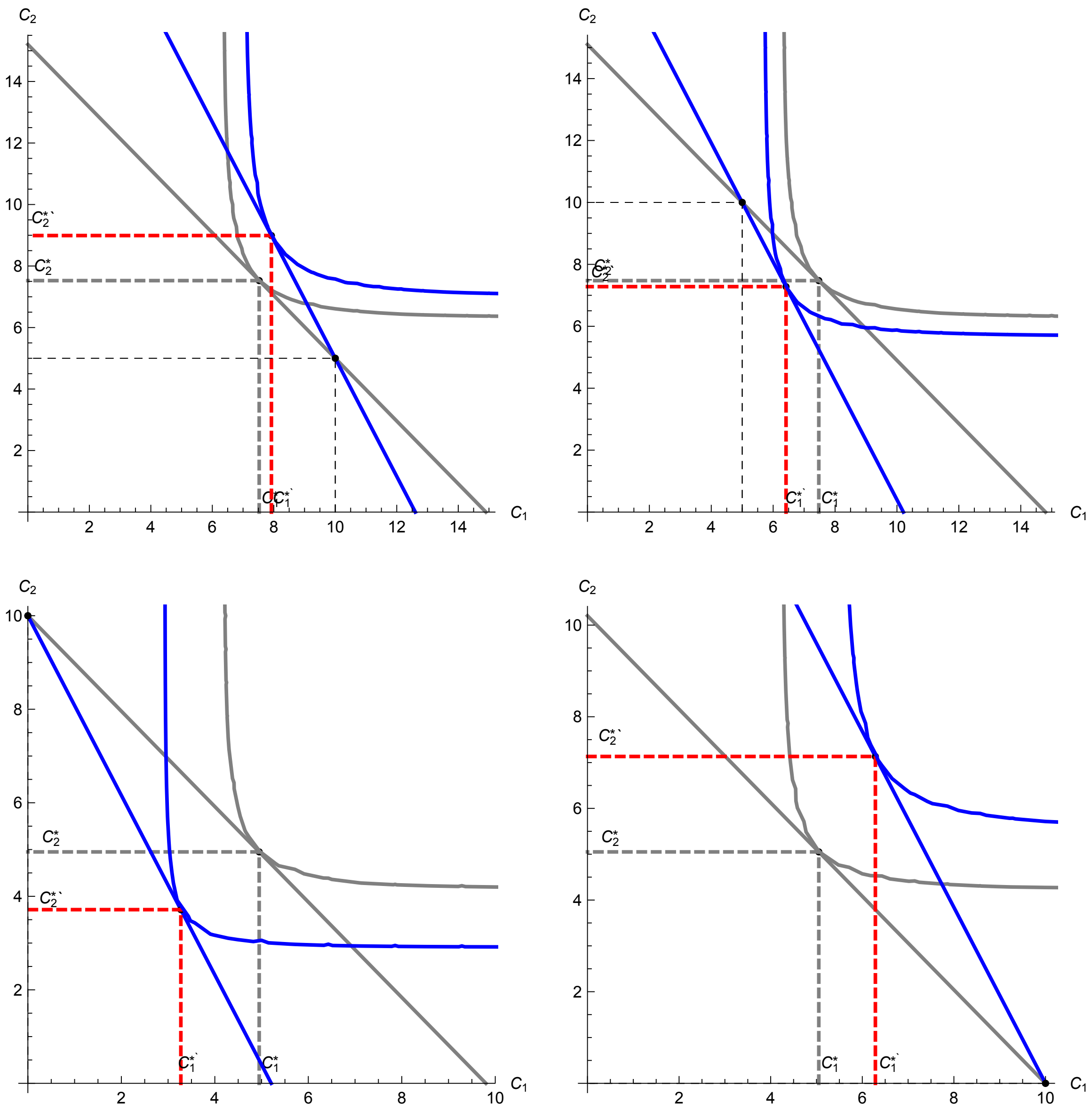
Change in World Interest Rate (σ = 1; log-utility)



Change in World Interest Rate (σ < 1)

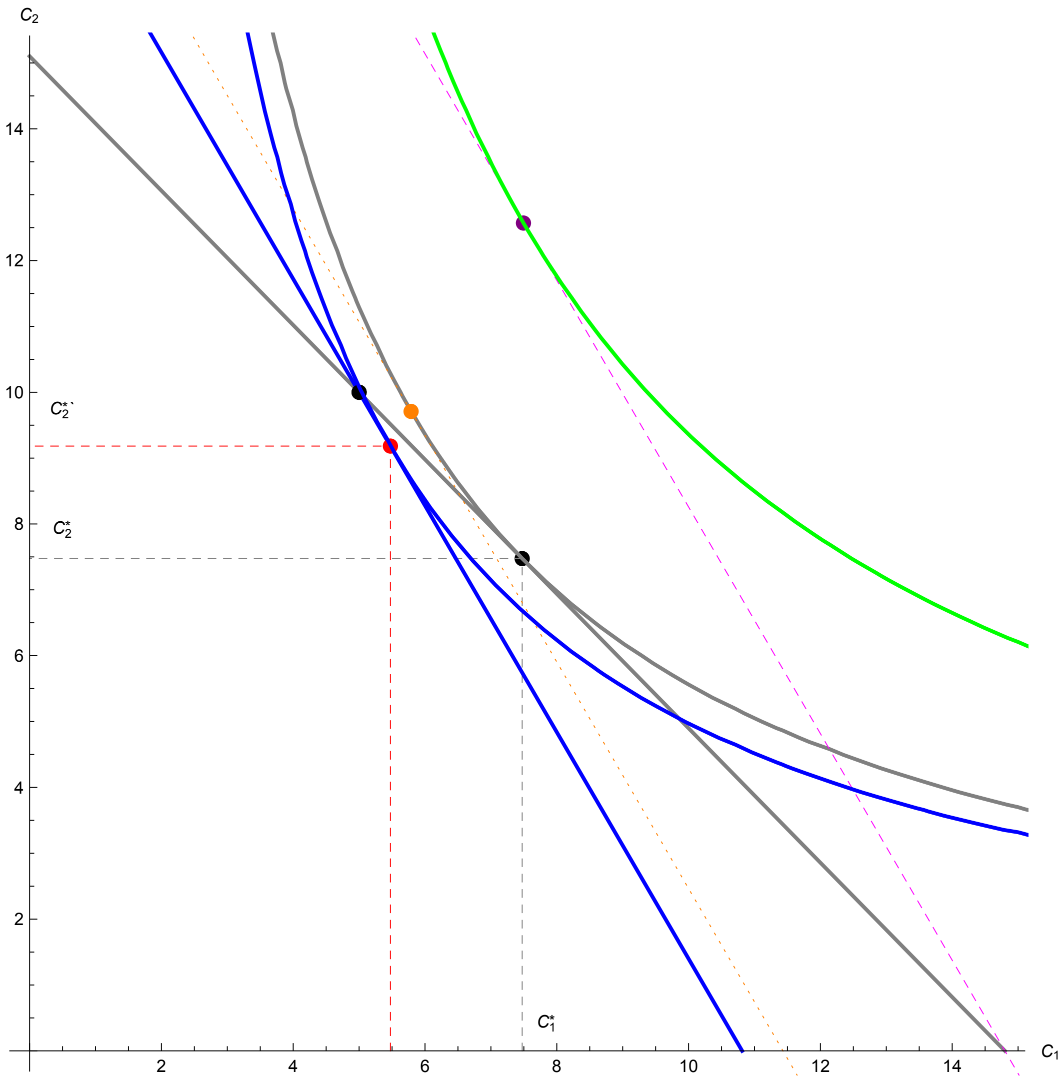
sigma < 1, Consume more in this period, C1

What happens to C1 and CA1 when the interest rate goes up? Consider the following cases:

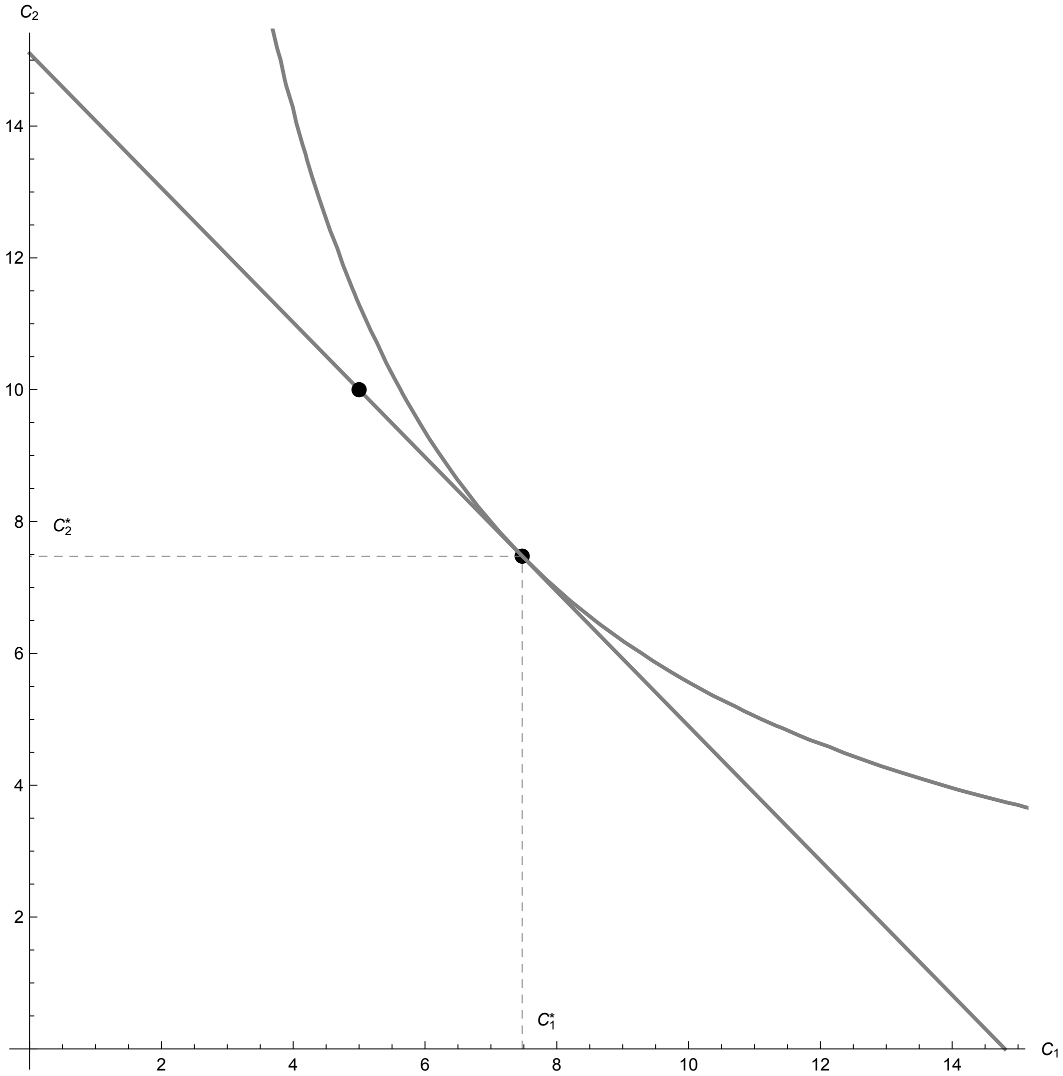


Substitution, Income and Wealth Effects: Graphical Analysis

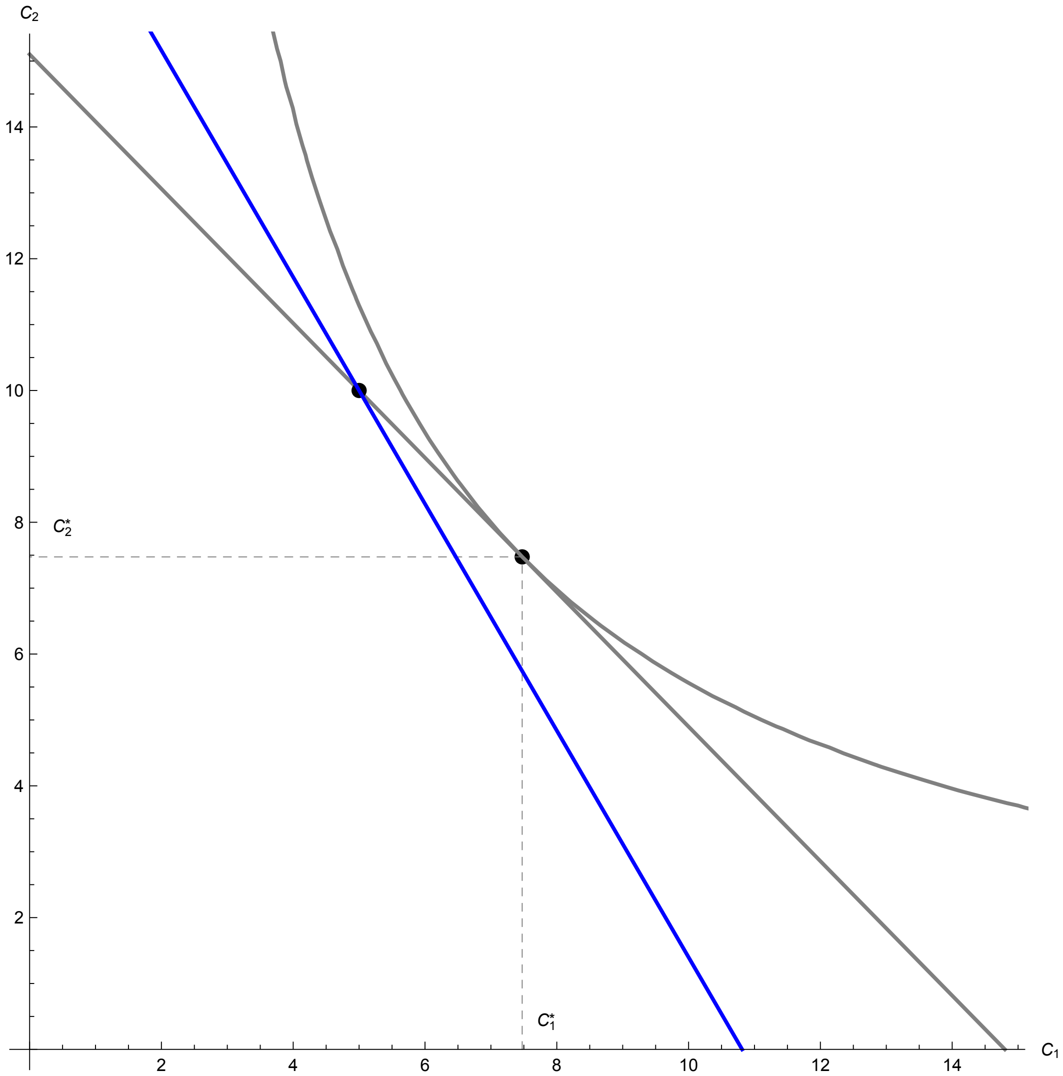
Let’s decompose the change in C1 into the substitution, income and wealth effects. The Original optimal choice is the black dot where the gray curves are tangent. The new optimum is the red point.



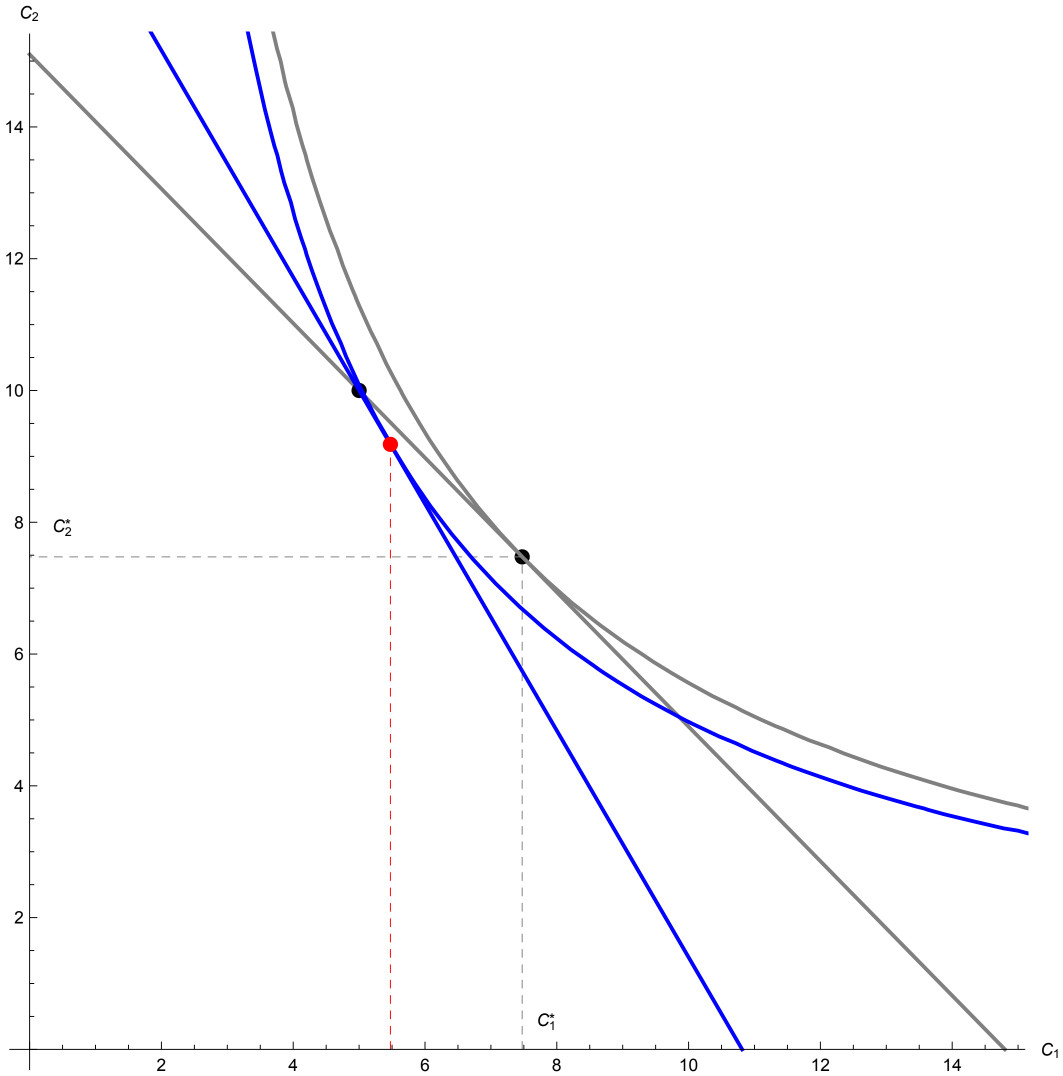
Original optimal choice



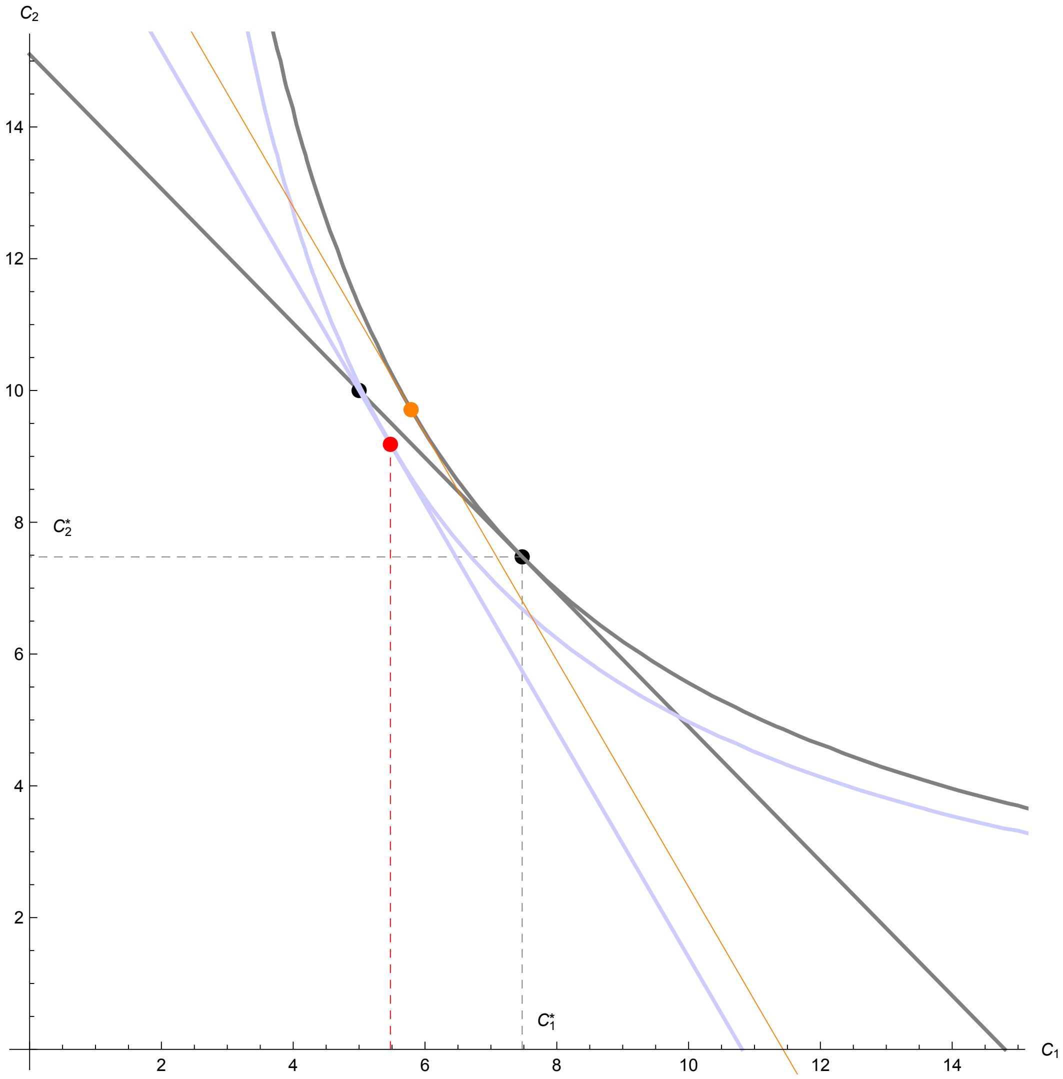
Original optimal choice + new budget line with higher *r*.



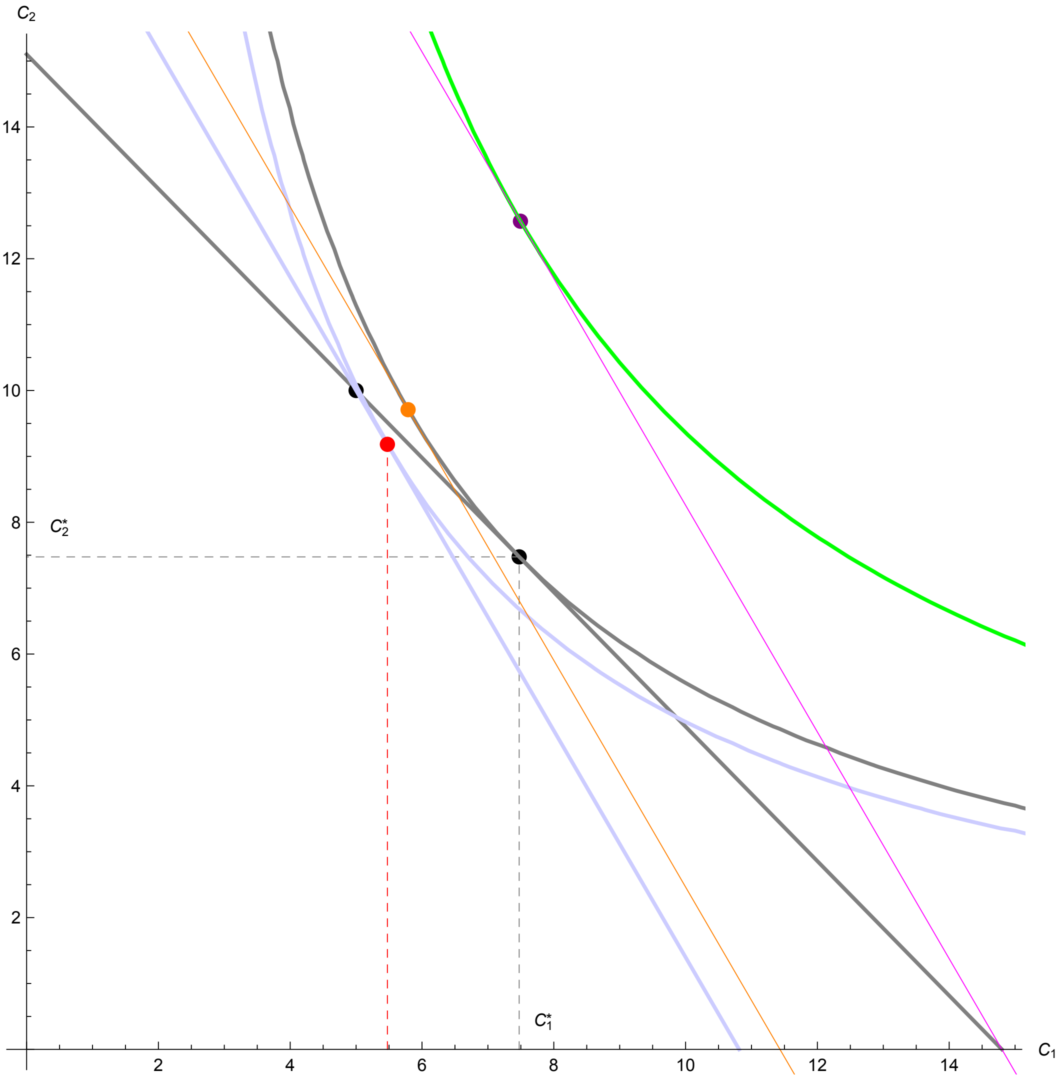
Original optimum choice + new optimal choice (red)



Original optimal choice + new optimal choice (red) + substitution effect (oranage)



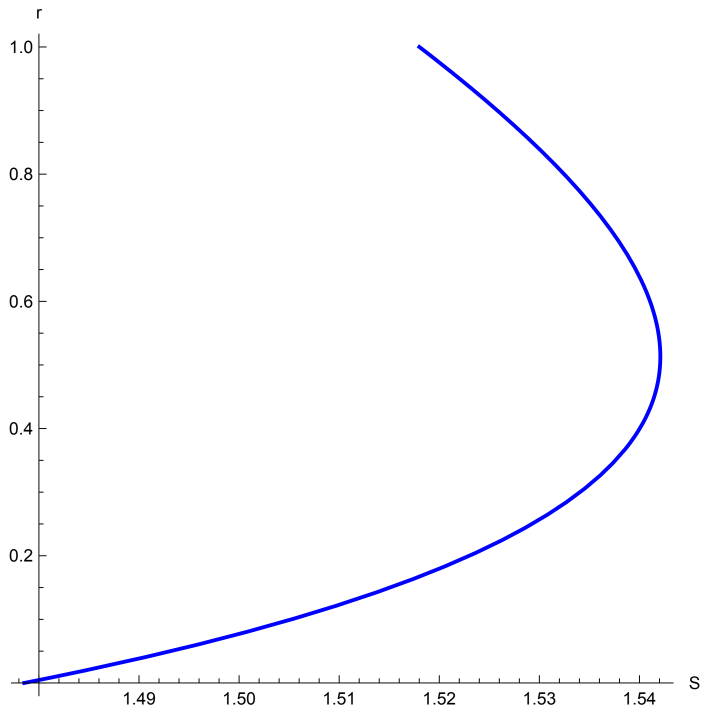
Original optimal choice + new optimal choice (red) + substitution effect (orange) + substitution & income effects (purple).
We can find the income effect by subtracting the substitution effect (purple - orange).
We can find the wealth effect by subtracting the substitution & income effects from total change (red - purple).



C_2
14
12
10
8
C_2^*
6
4
2
C_1^*
2
4
6
8
10
12
14
C_1

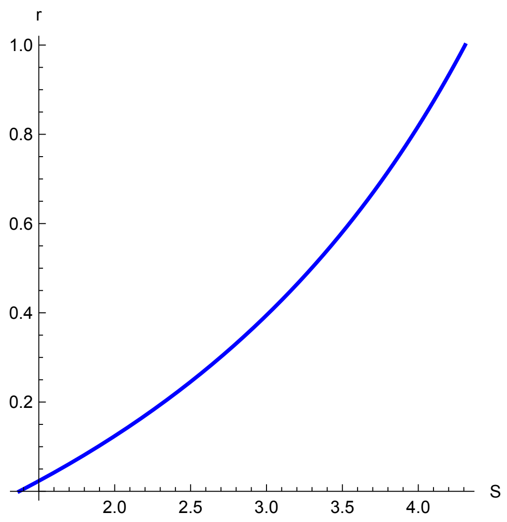
Example of a Backward-bending Saving Curve

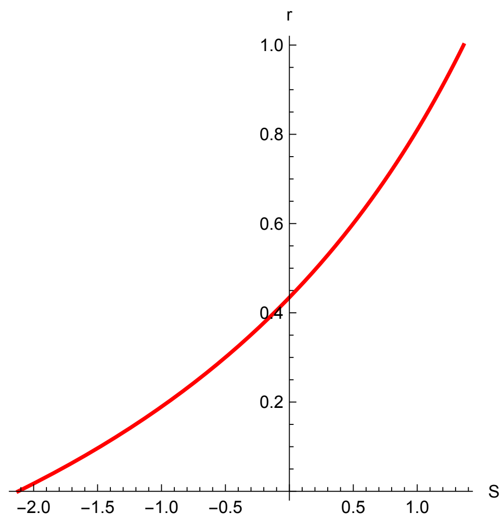
With sufficiently low elasticity of intertemporal substitution (σ), saving increase with r for sufficiently high r.

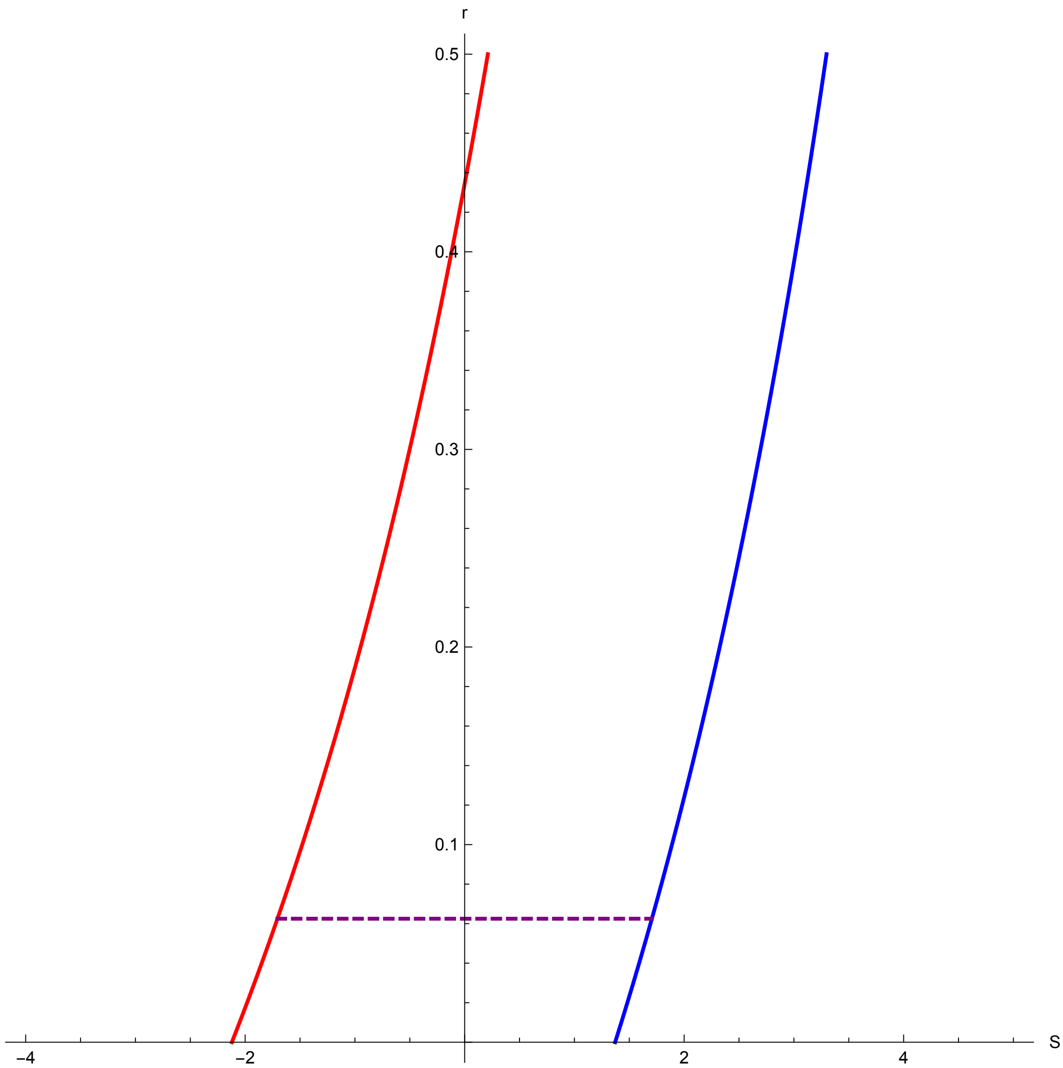


A Global Model

Let’s pause for a second and think about how the world interest rate is determined. For this let us drop the small open economy assumption. Let’s think of there being two large economies (or one large and another is just the rest of the world).
Important point: neither economy realizes (i.e. internalizes) its effect on the equilibrium interest rate.
Let’s also assume that saving is an non-decreasing function of the interest rate (for example log utility will do it).

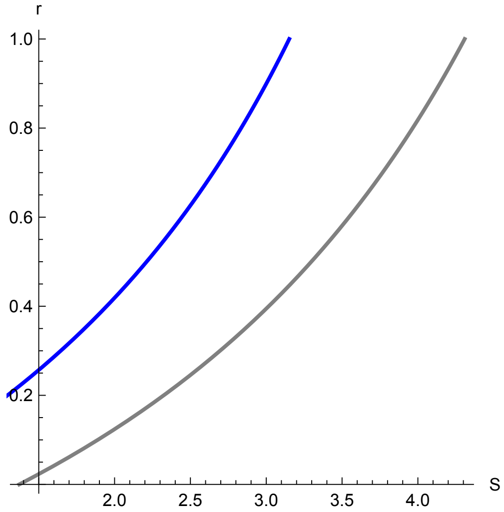


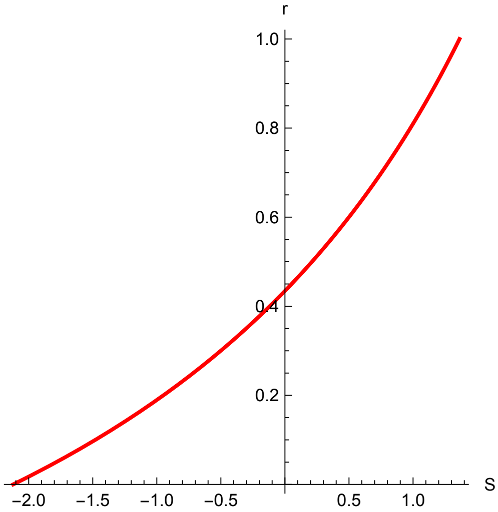


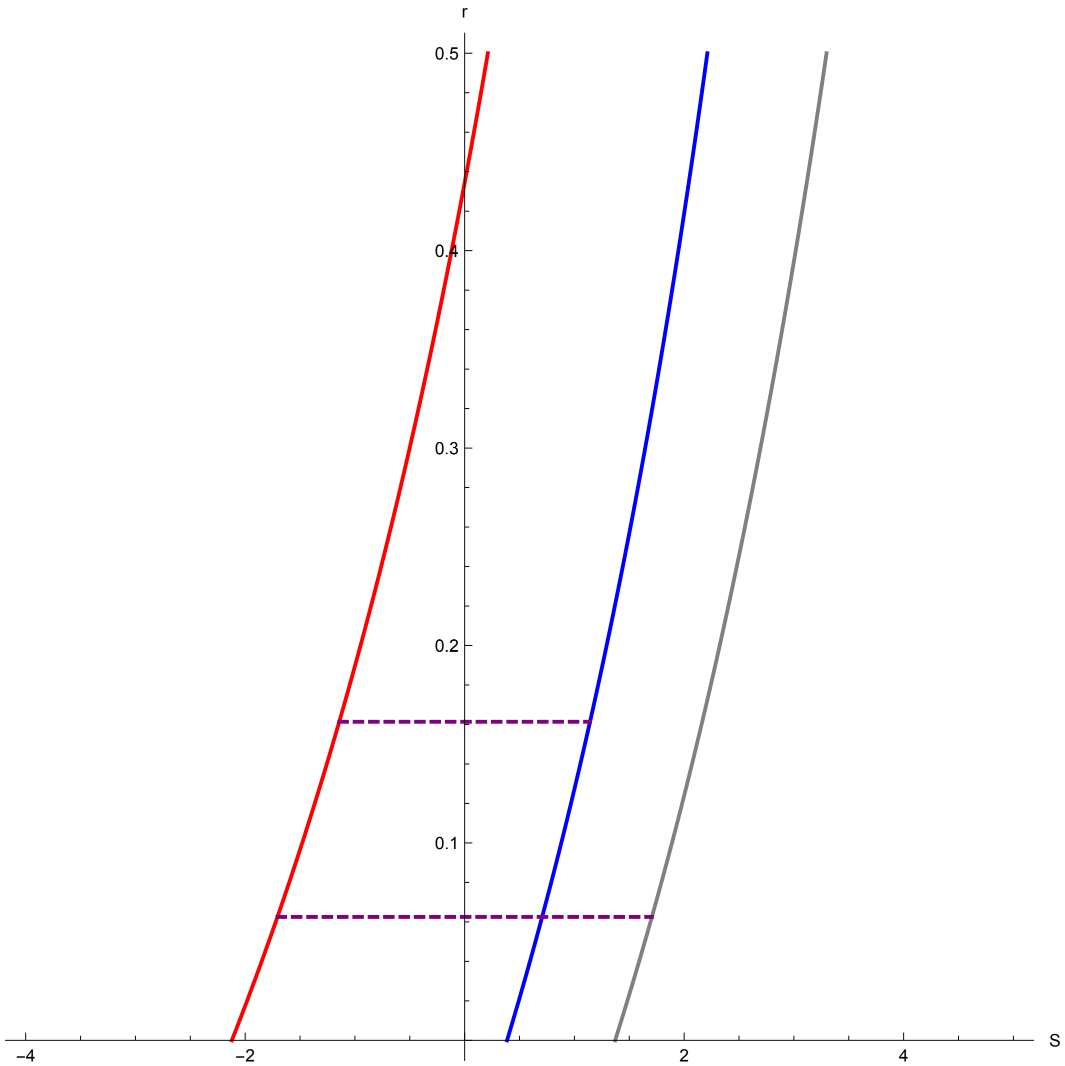

r
0.5
0.4
0.3
0.2
0.1
S
−4
−2
2
4

Changes in World Interest Rate

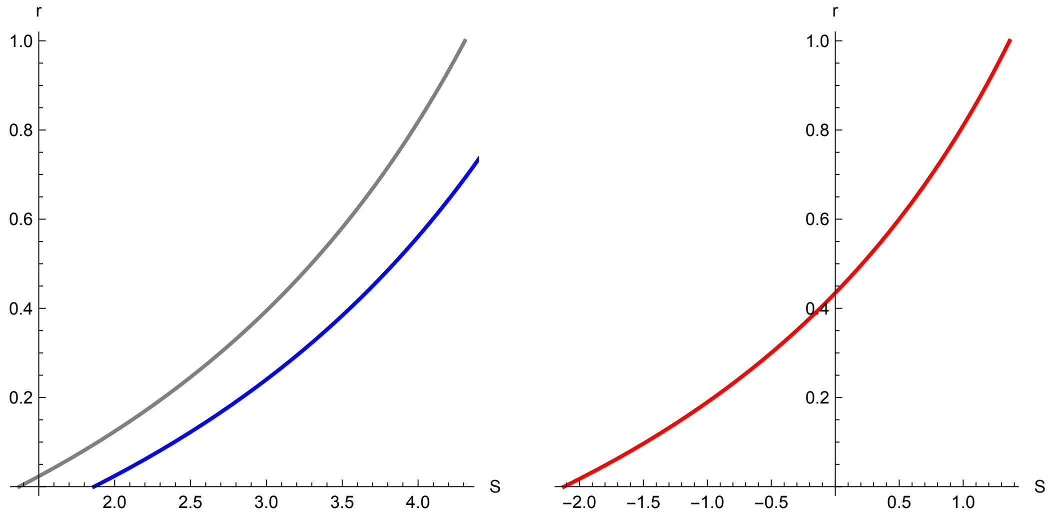
Suppose Home income in year 1 falls.







Suppose Home income in year 1 rises. Can this Home worse off?



World Equilibrium Interest Rate: Algebra

Let's solve for the equilibrium interest rate. Recall that the equilibrium condition is $CA_1 + CA_1^* = 0$ but since there is no investment CA = S so we have

$$S_1(r) + S_1^*(r) = 0$$

where $S_1(r) = Y_1 - C_1$, is the Home saving in year 1.

Let's assume log utility, so that

$$C_1 = \frac{1}{1+\beta}\left[Y_1 + \frac{Y_2}{1+r}\right]$$

$$C_1^* = \frac{1}{1+\beta^*}\left[Y_1^* + \frac{Y_2^*}{1+r}\right]$$

And the equilibrium condition becomes

$$Y_1 - \frac{1}{1+\beta}\left[Y_1 + \frac{Y_2}{1+r}\right] + Y_1^* - \frac{1}{1+\beta^*}\left[Y_1^* + \frac{Y_2^*}{1+r}\right] = 0$$

Multiplying by *(1+r)* and re-arranging

$$1 + r = \frac{Y_2 + Y_2^*}{\beta Y_1 + \beta^* Y_1^*} \qquad ??$$

Investment & Government

So far we have been assuming that output is exogenous (i.e. given, not determined inside the model). We'll relax this assumption by assuming that output is produced using capital, labor and technology according to

$$Y_t = A_t F(K_t, L_t)$$
$$F_K > 0, F_L > 0$$
$$F_{KK} < 0, F_{LL} < 0$$

Also, we will soften make the following assumption

$$Y_t = A_t F(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha}, \quad 0 \leq \alpha \leq 1.$$

Initial capital K_1 will be given just like initial assets B_1 were (and will be here as well).
Of course, K_1 cannot be assumed to equal 0 as we often do with B_1. (Why?)
K_2 will be determined by investment decisions.

We won't have much to say about labor for now so let's assume: $L_t = 1$.

Nature of Capital

We will make the so-called putty-putty assumption: namely we will assume that capital and output can be costlessly turned into one another. That is you can take a unit of Y and at no cost make it a unit of K and then you can costlessly turn it back into output to be consumed.
Q: what does this imply about the price of capital?
What are the alternatives?

Closed Economy: PPF

First we'll consider a closed economy. We'll also add a government which taxes (T) and spends (G). Let the taxes be lump sum.
Let's start by thinking about the budget constraints.

Recall that K_1 and B_1 are given.

$$Y_1 - T_1 = C_1 + I_1$$
$$Y_2 - T_2 = C_2 + I_2$$

In addition capital evolves according to

$$K_2 = (1 - \delta) K_1 + I_1$$
$$K_3 = (1 - \delta) K_2 + I_2$$

where δ is the depreciation rate. If we assume that it is 0 (for simplicity), and recalling that capital can be costlessly converted to output we have to conclude that $K_3 = 0$ and so

$$K_2 = K_1 + I_1$$
$$I_2 = -K_2$$

Together this gives us:

$$\begin{aligned} A_1 F(K_1) - T_1 &= C_1 + I_1 \\ A_2 F(K_2) - T_2 &= C_2 + I_2 \\ &= C_2 - K_2 \\ &= C_2 - (K_1 + I_1) \end{aligned}$$

From the top equation we have $I_1 = A_1 F(K_1) - T_1 - C_1$, which we can substitute for I_1 in the second equation to get:

$$A_2 F(K_2) - T_2 = C_2 - K_1 - (A_1 F(K_1) - T_1) + C_1$$
$$A_2 F(K_1 + I_1) - T_2 = C_2 - K_1 - (A_1 F(K_1) - T_1) + C_1$$
$$A_2 F(K_1 + A_1 F(K_1) - T_1 - C_1) - T_2 = C_2 - K_1 - (A_1 F(K_1) - T_1) + C_1$$

or re-arranging terms

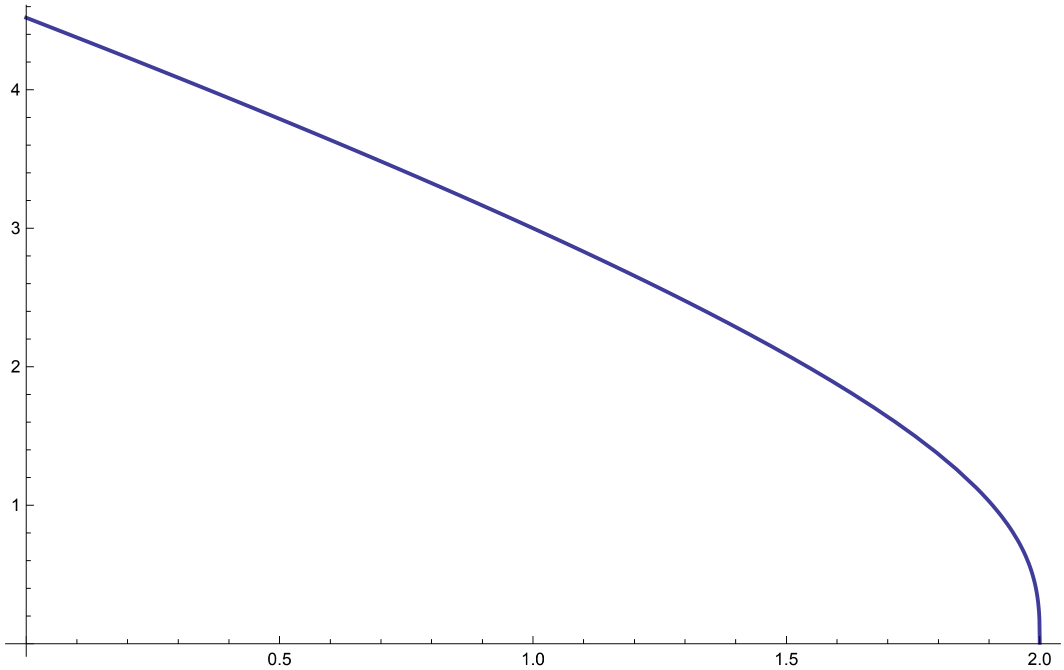
$$C_2 = A_2 F(K_1 + A_1 F(K_1) - T_1 - C_1) - T_2 + K_1 + (A_1 F(K_1) - T_1) - C_1$$

This gives us all the feasible C_2 as a function of C_1; this is the so-called Production Possibilities Fronteir (PPF).

Closed Economy: PPF

Here is what the PPF looks like

$$C_2 = A_2 F(K_1 + A_1 F(K_1) - T_1 - C_1) - T_2 + K_1 + (A_1 F(K_1) - T_1) - C_1$$



Closed Economy: Solution

The problem is therefore reduced to

$$\max_{\{C_1, C_2, K_2\}} u(C_1) + \beta u(C_2)$$

$$\text{subject to}$$

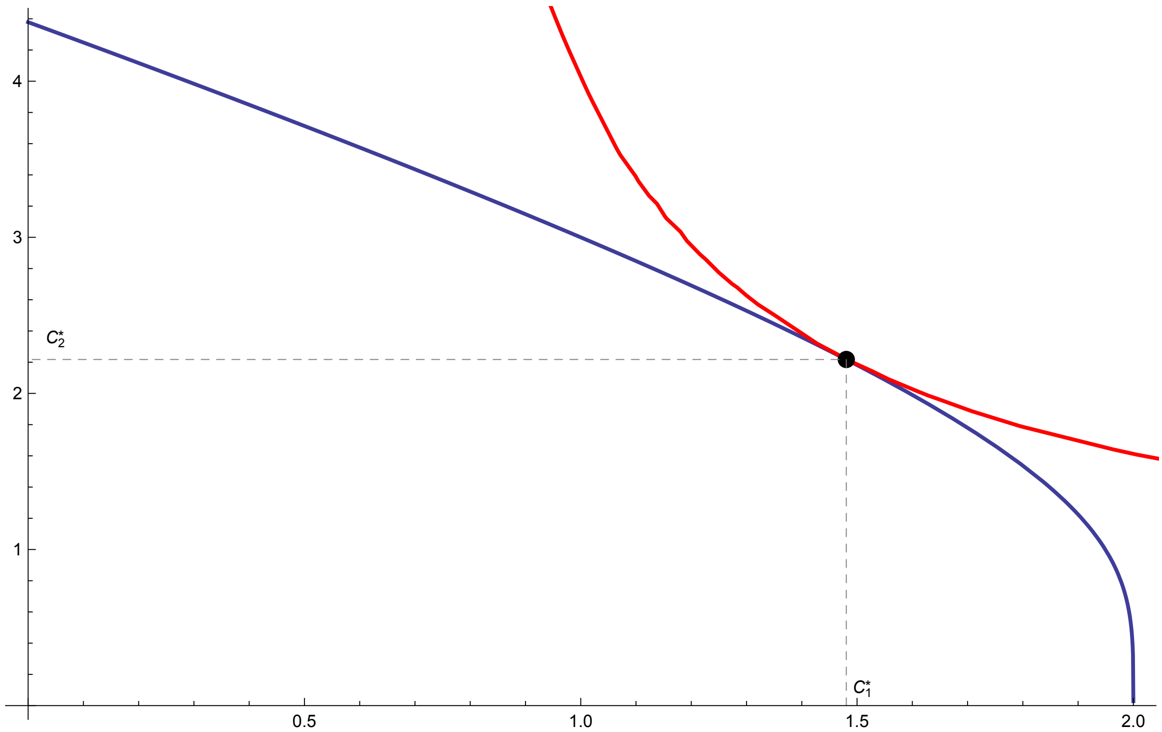
$$C_2 = A_2 F(K_1 + A_1 F(K_1) - T_1 - C_1) - T_2 + K_1 + (A_1 F(K_1) - T_1) - C_1$$

By substituting the budgeth constraint into teh objective function and taking the first-order condition we get: w.r.t. C1

$$u'(C_1) = \beta u'(C_2)(1 + A_2 F'(K_2))$$

(If we didn't assume depreciation away, the condition would be:
$u'(C_1) = \beta u'(C_2)(1 + A_2 F'(K_2) - \delta)$. Can you derive it?)

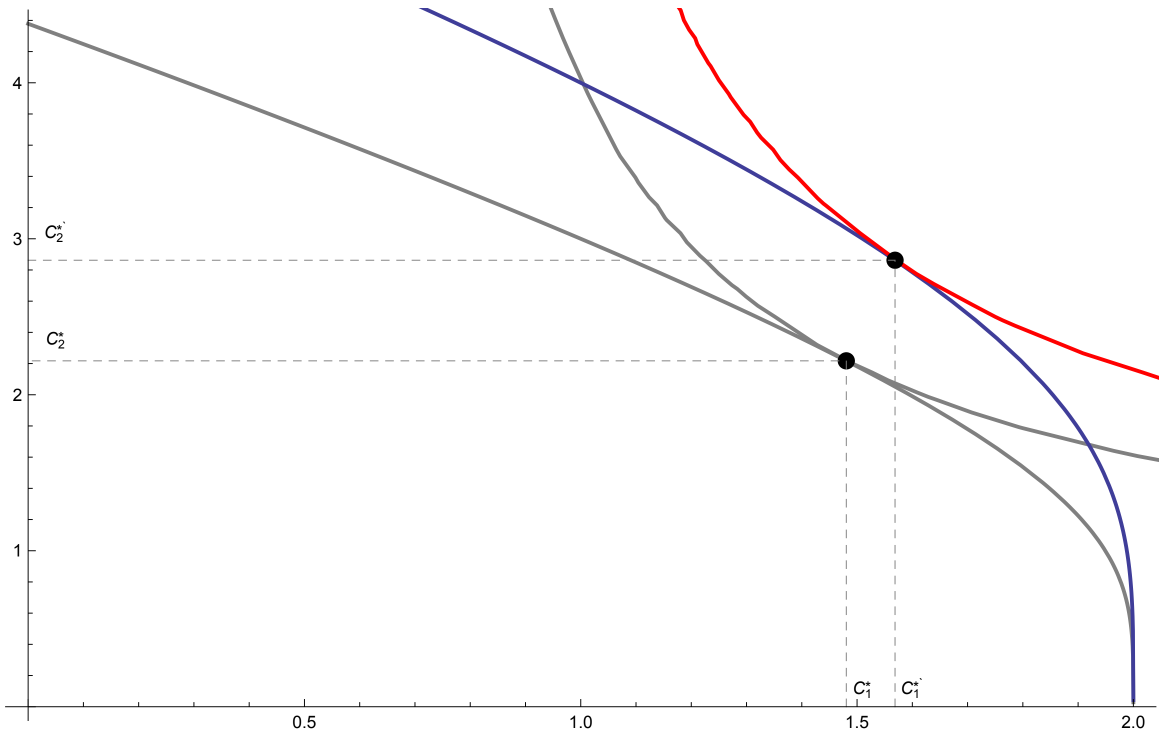
This is another version of the Euler equation. In the perhaps more familiar form
$u'(C_1) / \beta u'(C_2) = 1 + A_2 F'(K_2)$ it says that the MRS = MRT, or the indifference curve is tangent to the PPF.



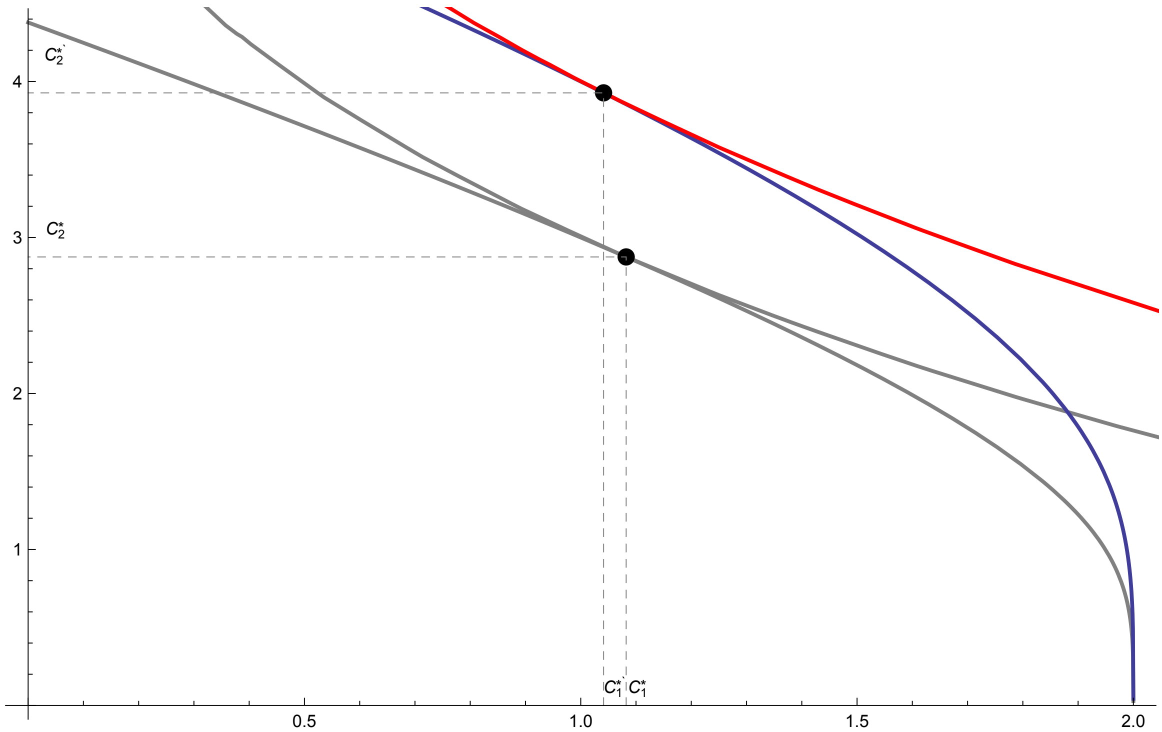
Closed Economy: Productivity Growth

An interesting question is the following: Suppose future productivity (A_2) increases (what about current productivity?), what effect does this have on consumption and investment?
Turns out we can have two cases. Because C and I are tied in year 1 via $Y_1 - T_1 = C_1 + I_1$, if one goes up, the other must fall.
This is really too bad since you'd like both to occur. Why? think about it: higher A_2 means that marginal product of capital will be high and this is a good time to invest, i.e. you'd like to increase investment I_1 but at the same time higher A_2 means future output will be higher (even more so if you do respond and boost investment!) so consumption smoothing tells you, to increase C_1. Unfortunately, in closed economy cannot do both!

CASE 1: Consumption increases but investment falls.



CASE 2: Consumption falls but investment increase.


$C_2^{*'}$
4
C_2^*
3
2
1
$C_1^{*'}$
C_1^*
0.5
1.0
1.5
2.0

Open Economy: Budget Constraints with Investment & Government

First let's assume that two assets are available: riskless bonds (B) and physical capital (K). Riskless bonds pay a fixed real interest rate r; this is again our small open economy assumption.
We'll also add a government which taxes (T) and spends (G). Let the taxes be lump sum.

Let's start by thinking about the budget constraints.

Recall that K_1 and B_1 are given.

$$B_2 = (1+r)\, B_1 + Y_1 - T_1 - C_1 - I_1$$

$$B_3 = (1+r)\, B_3 + Y_2 - T_2 - C_2 - I_2$$

Now let's combine these assuming B_1= 0 and recalling that B_3 must be 0. We get:

$$Y_1 - T_1 + \frac{Y_2 - T_2}{1+r} = C_1 + I_1 + \frac{C_2 + I_2}{1+r}$$

In addition capital evolves according to

$$K_2 = (1-\delta)\, K_1 + I_1$$

$$K_3 = (1-\delta)\, K_2 + I_2$$

where δ is the depreciation rate. If we assume that it is 0 (for simplicity), and recalling that capital can be costlessly converted to output we have to conclude that $K_3 = 0$ and so

$$K_2 = K_1 + I_1$$

$$I_2 = -K_2$$

Together this gives (assuming, as before $B_1 = 0$).

$$(1+r)\,(Y_1 - T_1 - (C_1 + I_1)) + (Y_2 - T_2) = C_2 + I_2$$

$$(1+r)\,(Y_1 - T_1 - (C_1 + I_1)) + (Y_2 - T_2) = C_2 - K_2$$

$$C_2 = (1+r)\,(Y_1 - T_1 - (C_1 + I_1)) + (Y_2 - T_2) + K_1 + I_1$$

$$C_2 = (1+r)\,(Y_1 - T_1 - (C_1 + I_1)) + A_2\, F\,(K_1 + I_1) - T_2 + K_1 + I_1$$

The general statement of the problem is:

$$\max_{\{C_1, C_2, I_1\}} u(C_1) + \beta u(C_2)$$

$$\text{subject to}$$

$$B_{t+1} = (1+r) B_t + Y_t - C_t - I_t$$

$$K_{t+1} = (1-\delta) K_t + I_t$$

But with the above transformations we can express it as:

$$\max_{\{C_1, C_2, I_1\}} u(C_1) + \beta u(C_2)$$

$$\text{subject to}$$

$$C_2 = (1+r)(Y_1 - T_1 - (C_1 + I_1)) + A_2 F(K_1 + I_1) - T_2 + K_1 + I_1$$

Which can be solved by substituting the constraint into the objective function to get

$$\max_{\{C_1, I_1\}} u(C_1) + \beta u[(1+r)(Y_1 - T_1 - (C_1 + I_1)) + A_2 F(K_1 + I_1) - T_2 + K_1 + I_1]$$

The first order conditions for this problem are:

$$u'(C_1) = \beta (1+r) u'(C_2) \quad \text{wrt C}$$

$$\beta u'(C_2)(A_2 F'(K_2) - r) = 0 \quad \text{wrt I}$$

Or simply:

$$u'(C_1) = \beta (1+r) u'(C_2)$$

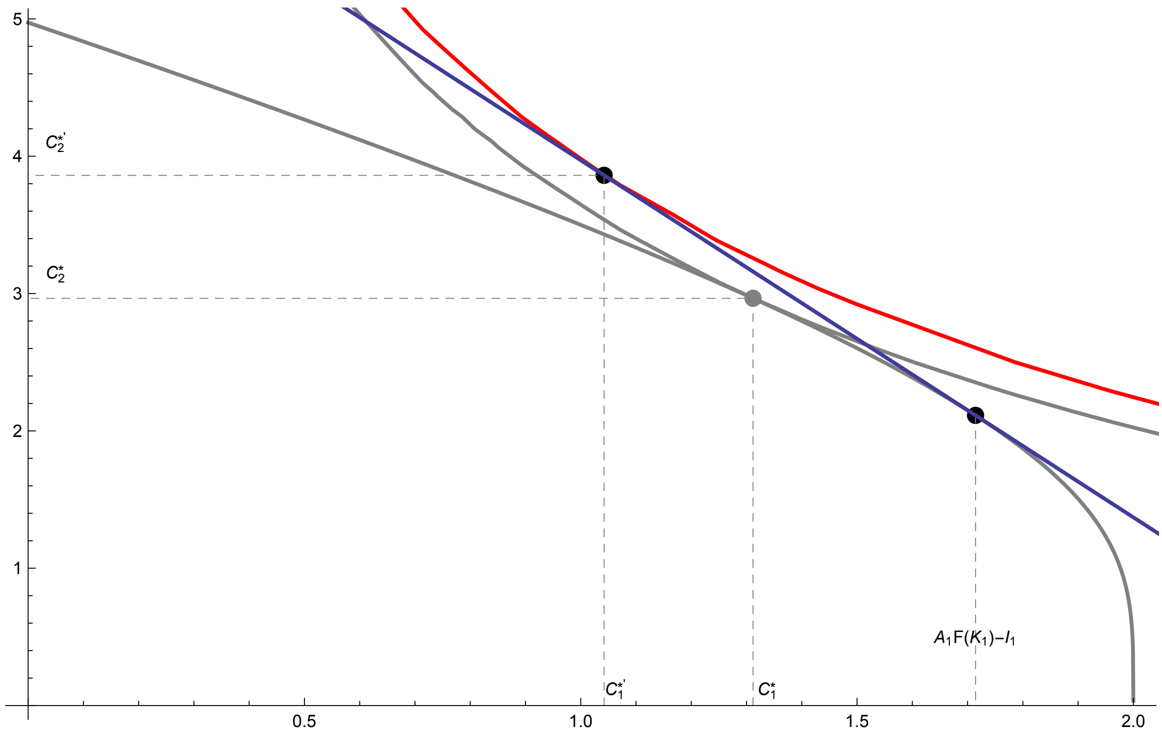
$$A_2 F'(K_2) = r$$

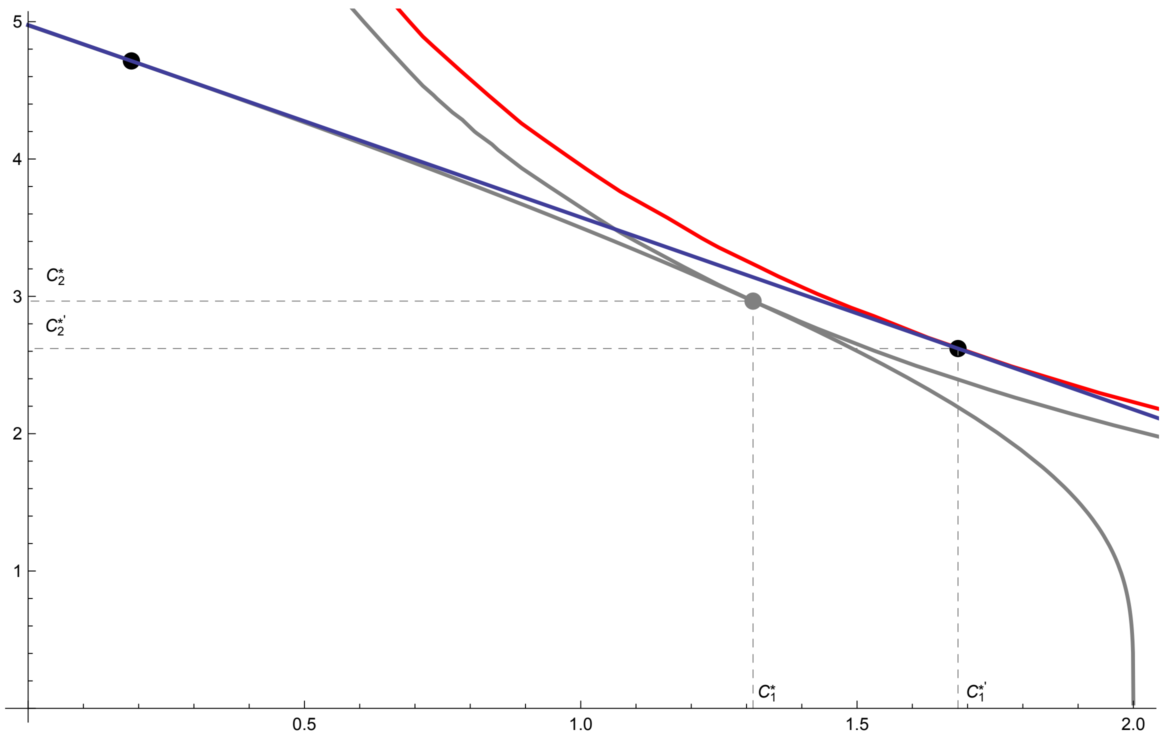
Investment and consumption decisions are separated. The FONC's say that to get to the optimum you need

1. Pick I_1 according to $A_2 F'(K_2) = r$. Your consumption choice is irrelevant here.
2. Pick consumption optimally just like in the economy without investment; That is max $u(C_1) + \beta u(C_2)$ subject to $Y_1 - T_1 + \frac{Y_2 - T_2}{1+r} = C_1 + I_1 + \frac{C_2 + I_2}{1+r}$, where Y_2, I_1 and I_2 follow from your choice in (1) so are given. Thus, it's just like exogenous income path we studied before!

Open Economy: Solution

Two cases relative to a closed economy. Either r is greater or less than the autarky interest rate at home (i.e. the marginal product of capital).

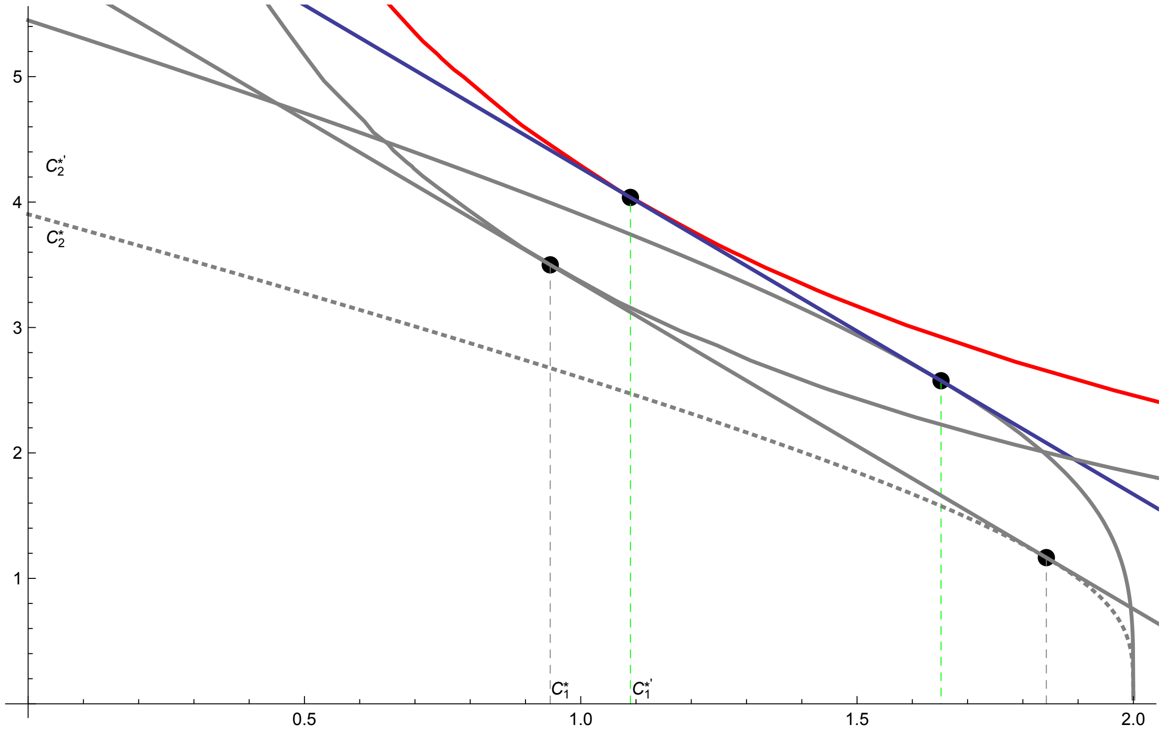




Open Economy: Productivity Growth

Let’s revisit the question of productivity growth: Suppose future productivity increases, what effect does this have on consumption and investment?
Recall that in the closed economy we had two cases. Because C and I were tied in year 1 via $Y_1 - T_1 = C_1 + I_1$, if one went up, the other had to fall.

Here things are different! Investment and consumption are separated! In response to an increase in A_2 , we have both consumption and investment go up.
Of course, that means current account decreases, since CA = S - I = Y - C - I.



We can borrow money in an open economy (at r), both C and I can go up.

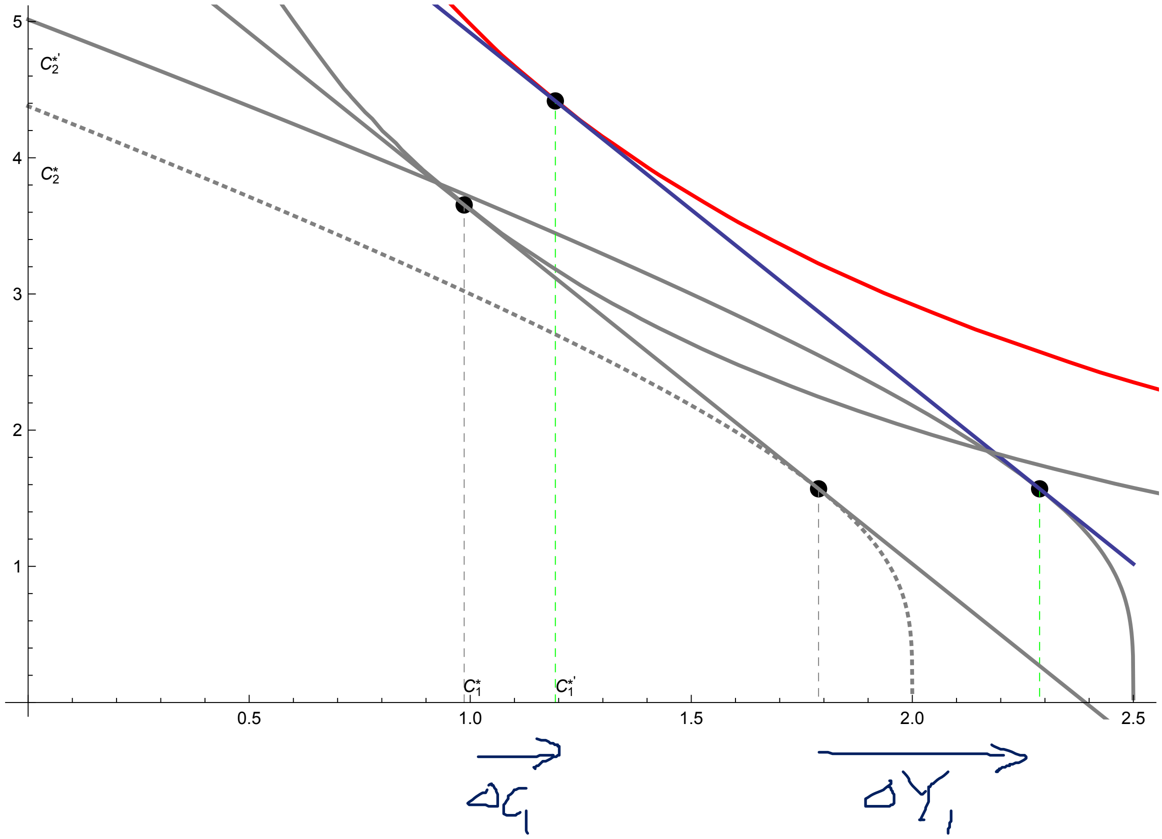
An example: Norway

Norway discovers oil under the North Sea and this follows:

Case 1

Increase in Current Productivity Level

What would happen if current productivity (A_1) went up instead?



World Interest Rate Equilibrium with Investment

In the model with investment, we can again ask what will determine the equilibrium world interest rate.
Here the equilibrium condition will be

$$CA_1(r) + CA_1^*(r) = 0$$

$$S_1(r) - I_1(r) + S_1^*(r) - I_1^*(r) = 0$$

Q: what will dS/dr look like now?
A: the same as before!

Start with the Euler equation $u'(C_1) = (1+r)\beta u'(C_2)$ and substitute for C_2 using the budget constraint (ignore taxes) to get

$$u'(C_1) = (1+r)\beta u[(1+r)(Y_1 - (C_1 + I_1)) + A_2 F(K_1 + I_1) + K_1 + I_1]$$

And totally differentiate wrt r and C_1.

$$u''(C_1)\,dC_1 =$$
$$\beta u'(C_2)\,dr - (1+r)^2 \beta u''(C_2)\,dC_1 + (1+r)\beta u''(C_2)(Y_1 - T_1 - (C_1 + I_1))\,dr +$$
$$(1+r)\beta u''(C_2)\left(A_2 F'(K_2)\frac{\partial I_1}{\partial r} - r\frac{\partial I_1}{\partial r}\right) + \frac{\partial I_1}{\partial r} \quad \beta$$

The last part comes from the fact that now change sin r affect investment, which in turn affects the budget constraint. However, note that this term is zero at the opium!

$$\left(A_2 F'(K_2)\frac{\partial I_1}{\partial r} - r\frac{\partial I_1}{\partial r}\right) = (A_2 F'(K_2) - r)\frac{\partial I_1}{\partial r} = 0$$

So the equation becomes

$$\left(u''(C_1) + 1 + r)^2 \beta u''(C_2)\right) dC_1 =$$
$$(\beta u'(C_2) + (1+r)\beta u''(C_2)(Y_1 - (C_1 + I_1)))\,dr$$

Collecting terms and re-arranging

$$\frac{dC_1}{dr} = \frac{\beta u'(C_2) + (1+r)\beta u''(C_2)(Y_1 - (C_1 + I_1))}{u''(C_1) + (1+r)^2 \beta u''(C_2)}$$
$$= (\beta u'(C_2)/u'(C_1) + (1+r)\beta u''(C_2)/u'(C_1)(Y_1 - (C_1 + I_1)))\Big/$$
$$\left(u''(C_1)/u'(C_1) + (1+r)^2 \beta u''(C_2)/u'(C_1)\right)$$
$$= \frac{1/(1+r) + (1+r)\frac{C_2 u''(C_2)\beta u'(C_2)}{C_2 u'(C_1) u'(C_2)}(Y_1 - (C_1 + I_1))}{\frac{1}{C_1}\frac{C_1 u''(C_1)}{u'(C_1)} + -(1+r)^2 \frac{C_2 u''(C_2)\beta u'(C_2)}{C_2 u'(C_1) u'(C_2)}}$$

Defining, as before, σ to be the intertemporal elasticity of substitution (or inverse of the coefficient of relative risk aversion)

$$\sigma(C) = -\frac{u'(C)}{C\,u''(C)}$$

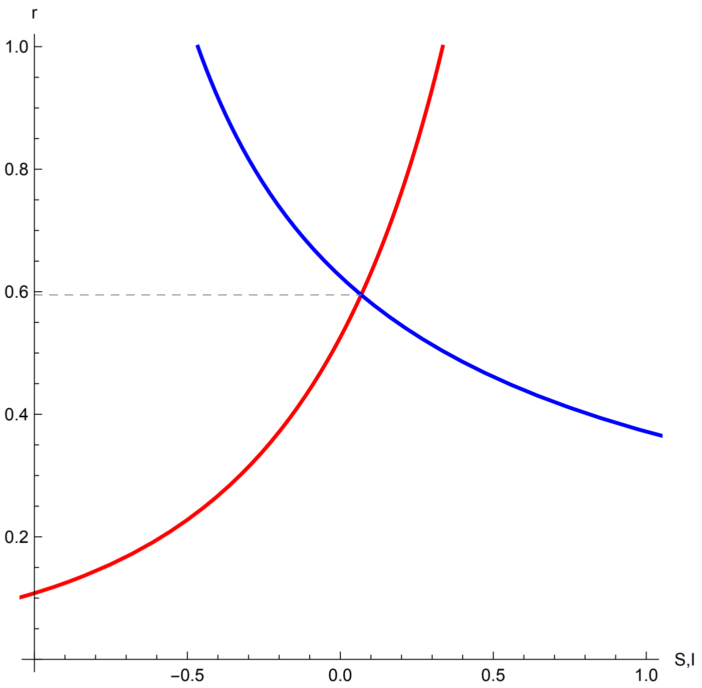
$$\frac{dC_1}{dr} = \frac{1/(1+r) - \frac{1}{\sigma(C_2)\,C_2}\,(Y_1 - (C_1 + I_1))}{\frac{1}{C_1}\,\frac{1}{\sigma(C_1)} - (1+r)\,\frac{1}{\sigma(C_2)\,C_2}}$$

Assuming σ is constant and multiplying by $C_2\,\sigma$

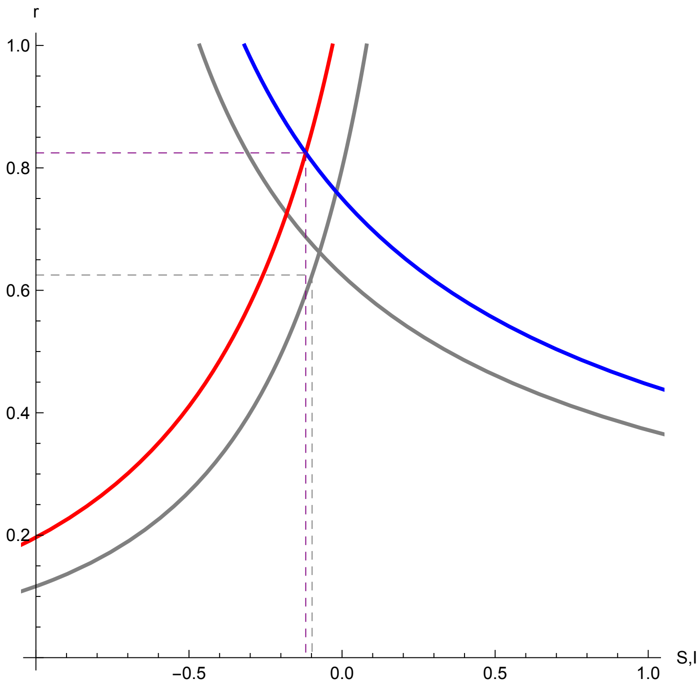
$$\frac{dC_1}{dr} = \frac{(Y_1 - C_1 - I_1) - \sigma\,C_2/(1+r)}{1 + r + \frac{C_2}{C_1}}$$

Slide

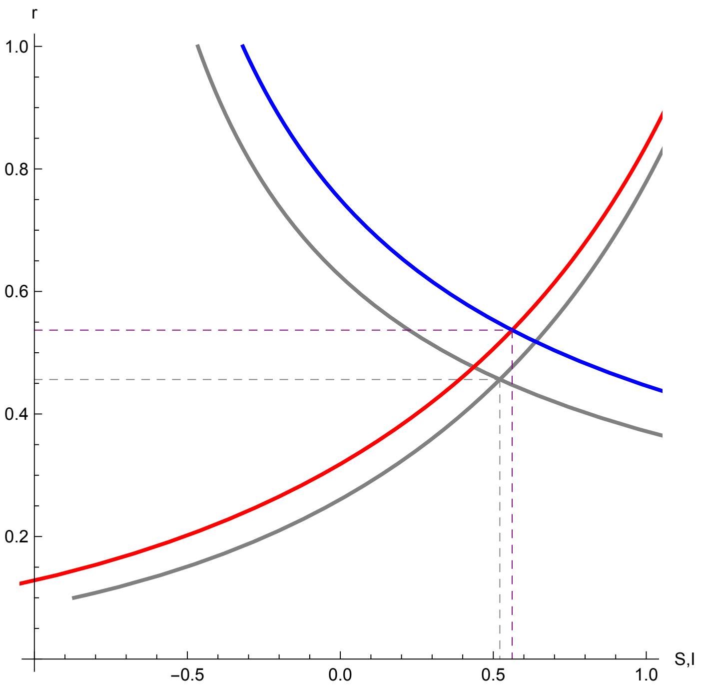
This is a text cell.



So here is an interesting question related to Barro Sala-i-martin paper. Suppose future productivity A2 is expected to be higher. What will happen to r and investment?
Well, we know that the investment demand curve will shift up (look back at the FONC if you need convincing). That by itself would push r and I up. But at the same time the saving schedule will shift to the left as well (consumption smoothing). This will work to push r further up and it will also work to push equilibrium investment down.
The result is that r unambiguously goes up but I may go up or down. Here is a case for log utility where I actually falls.



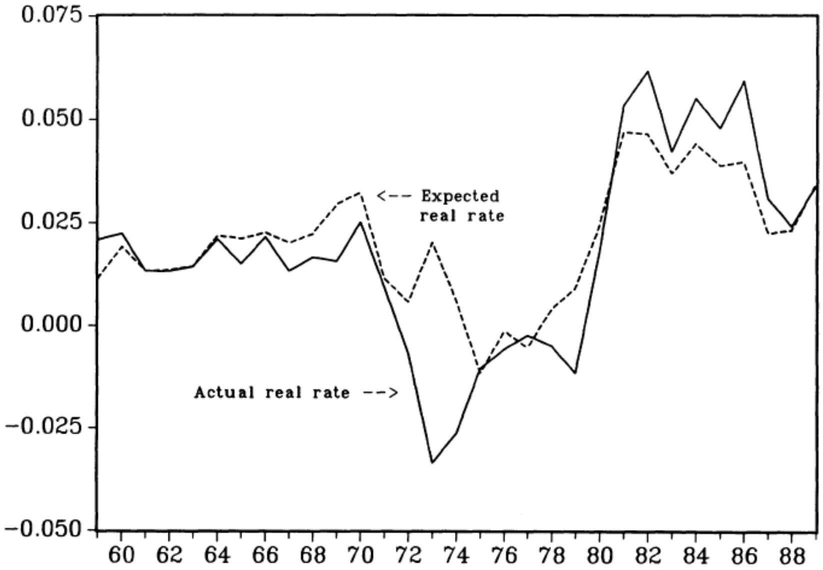
Now Barro and Sala-i-Martin thought r went up because of high anticipated A2 (measured by stock market returns). They also pointed to higher levels of investment as evidence that demand for I went up. So, when would that happen?
The answer is: it will occur as long as the shift in the saving supply is smaller relative to that in I demand. But that takes a large value of intertemporal elasticity of substitution. Here is an example with $\sigma = 3$.
But most studies find σ <0.5!


r
1.0
0.8
0.6
0.4
0.2
-0.5
0.0
0.5
1.0
S,I

Barro and Sala-i-Martin (1990)

During the 1980’s the world real interest (r) was unusually high. BSM want to explain this.

Figure 3 WORLD ACTUAL AND EXPECTED REAL INTEREST RATES



In order to do that they will run regressions of the form:

$$r_t^e = X_t \beta + \epsilon_t$$

Where r_t^e is the expected real interest rate.
The first question they must confront is how to measure the world real interest rate. They choose a weighted avarage of 10 largest OECD economies’ short term interst rates. Is short term rate what we mean by *r*? Not really but they claim that ratesof different maturties move together.

The second problem is that that they only have data on the nominal interest rates (i_t), where (using “e” to denote “expected”) and what they really want is the expected real interest rate. The Fisher identity tells us that

$$r_t^e = i_t - \pi_t^e$$

where π_t^e is the expected inflation rate.
They could just use realzied (“ex-post”) real rates given by:

$$r_t = i_t - \pi_t$$

where π_t is the actula (realzied) infaltion.
Combining the last two equations:

$$r_t^e = r_t - (\pi_t^e - \pi_t)$$

where the last term $(\pi_t^e - \pi_t)$ is the “inflation forecast error”.
So if the use ex-post rates in place of expected they end up with

$$r_t^e = X_t \beta + \epsilon_t$$

$$r_t - (\pi_t^e - \pi_t) = X_t \beta + \epsilon_t$$

$$r_t = X_t \beta + \epsilon_t + (\pi_t^e - \pi_t) = X_t \beta + \mu_t$$

The problem with this regression is that even if ϵ_t is uncorrelated with X_t, μ_t may be. The key question is whether:

$$E_t[X_t (\pi_t^e - \pi_t)] = 0$$

If this is true, then using ex-post rates is OK.
Turns out this condition is true under Rational Expectations. But they are not willing to just use the ex-post real interest rates. assume Instead they use a different way of getting r_t^e. They run same autoregressive (ARMA(1,1)) regressions to come up with a good forecast of inflation π_t^e and use $r_t^e = i_t - \pi_t^e$, to compute expected rates. This is alos consistent with rational expectations if ipast nflation is the only variable you need to know to forecats future inflation...
[ADD: notes on distributed lag models?]

The next problem is that the interest rate is determined by supply and demand. When we run the regression of price P on quantity Q, are you estimating the demand or the supply curve?''

[More to be added
here...]

We used the structural model to determine a reduced form for the "world" expected real interest rate and ratio of investment to GDP. The main predictions are that more favorable stock returns raise the real interest rate and investment, higher oil prices increase the real interest rate but decrease investment, higher monetary growth lowers the real interest rate and stimulates investment, and greater fiscal expansion raises the real interest rate and reduces investment.

More Time Periods

We'll think of adding more periods to the model. This will allows us to think more clearly about dynamics, shock persistence, etc.
Before we do that, we'll relax the arbitrary convention we have adopted in the two period model, namely that the first period is labeled 1. Instead the first period will be labeled *t*.

We'll start by thinking of an economy that goes on for T periods ("cap-t"). That is the final year is t + T.
In our two period model, we had t =1 and T = 1. The big change now will be that we'll make T > 1.

Basic Setup

All the basics are just like our 2 period endowment economy except we'll have more periods

$$U_t = u(C_t) + \beta u(C_{t+1}) + \beta^2 u(C_{t+2}) + \beta^3 u(C_{t+3}) + ... = \sum_{s=t}^{t+T} \beta^{s-t} u(C_s)$$

The flow budget constraint is

$$B_{s+1} = (1+r) B_s + Y_s - C_s$$

cont.

To add the flow budget constraints start with this:

$$(1+r)\, B_s = B_{s+1} - Y_s + C_s \tag{1}$$

Forwarding equation (1) by one period

$$(1+r)\, B_{s+1} = B_{s+2} - Y_{s+1} + C_{s+1} \Longrightarrow B_{s+1} = \frac{B_{s+2}}{1+r} - \frac{Y_{s+1}}{1+r} + \frac{C_{s+1}}{1+r}$$

And substituting above

$$(1+r)\, B_s = \frac{B_{s+2}}{1+r} - \frac{Y_{s+1}}{1+r} + \frac{C_{s+1}}{1+r} - Y_s + C_s$$

Forwarding (1) by two period

$$B_{s+2} = \frac{B_{s+3}}{1+r} - \frac{Y_{s+2}}{1+r} + \frac{C_{s+2}}{1+r}$$

And substituting again

$$(1+r)\, B_s = \frac{1}{1+r}\left(\frac{B_{s+3}}{1+r} - \frac{Y_{s+2}}{1+r} + \frac{C_{s+2}}{1+r}\right) - \frac{Y_{s+1}}{1+r} + \frac{C_{s+1}}{1+r} - Y_s + C_s =$$
$$\frac{B_{s+3}}{(1+r)^2} + C_s + \frac{C_{s+1}}{1+r} + \frac{C_{s+2}}{(1+r)^2} - Y_s - \frac{Y_{s+1}}{1+r} - \frac{Y_{s+2}}{(1+r)^2}$$

If we keep doing this long enough, we get:

$$(1+r)\, B_t + \sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t} Y_s = \sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t} C_s + \left(\frac{1}{1+r}\right)^{T} B_{t+T+1}$$

where as before we'll often assume $B_t = 0$.
Also, as long as T is finite we can conclude that $\left(\frac{1}{1+r}\right)^{T} B_{t+T+1} = 0$.

Solving the consumer problem

We have the following problem:
Choose a sequence of $\{C_s\}_{s=t}^{t+T}$ and $\{B_s\}_{s=t+1}^{t+T}$ to solve

$$\max_{\{C\}_t^{t+T}} \sum_{s=t}^{t+T} \beta^{s-t} u(C_s) \text{ subject to}$$

$$\sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} Y_s = \sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} C_s$$

We'll go through two simple ways of solving this problem and then we'll introduce a third, slightly different way.

Method I: Substitution

We could proceed as in the two period model: use the flow budget constraint to substitute for C_s

$$\max \sum_{s=t}^{t+T} \beta^{s-t} u((1+r) B_s + Y_s - B_{s+1})$$

and find the FONC's with respect to $\{B_s\}_{s=t+1}^{t+T}$ remembering that any $B_{\tau+1}$ appears in two terms of the above summation

$$U = \ldots + \beta^{\tau-1-t} u((1+r) B_{\tau-1} + Y_{\tau-1} - B_\tau) + \beta^{\tau-t} u((1+r) B_\tau + Y_\tau - B_{\tau+1}) + \beta^{\tau+1-t} u((1+r) B_{\tau+1} + Y_\tau - B_{\tau+2}) + \beta^{\tau+2-t} u((1+r) B_{\tau+2} + Y_\tau - B_{\tau+3}) + \ldots$$

For example, starting with t = 1 we'd have to maximize:

$$U = u((1+r) B_1 + Y_1 - B_2) + \beta u((1+r) B_2 + Y_2 - B_3) + \beta^2 u((1+r) B_3 + Y_3 - B_4) + \beta^3 u((1+r) B_4 + Y_4 - B_5) + \ldots$$

with respect to $\{B_s\}_{s=1}^{T+1} = \{B_1, B_2, B_3, \ldots, B_{T+1}\}$.

So the FONC w.r.t. a generic $B_{\tau+1}$ is:

$$dU/dB_{\tau+1} = -\beta^{\tau-t} u'\left((1+r) B_{\tau} + Y_{\tau} - B_{\tau+1}\right) + (1+r) \beta^{\tau+1-t} u'\left((1+r) B_{\tau+1} + Y_{\tau+1} - B_{\tau+2}\right) = 0$$

or

$$\beta^{\tau-t} u'(C_{\tau}) = (1+r) \beta^{\tau+1-t} u'(C_{\tau+1})$$

or simplifying and noting that an equation like this must hold for any $\tau = \{t, \ldots, t+T\}$

$$u'(C_s) = (1+r) \beta u'(C_{s+1})$$

Which is, of course, just our good friend the Euler equation.

Or, using the t=1 example, the FONC w.r.t. a B_3 is:

$$dU/dB_3 = -\beta u'\left((1+r) B_2 + Y_2 - B_3\right) + (1+r) \beta^2 u'\left((1+r) B_3 + Y_3 - B_4\right) = 0$$

or

$$\beta u'(C_2) = (1+r) \beta^2 u'(C_3)$$

$$u'(C_2) = (1+r) \beta u'(C_3)$$

Method 2: Lagrangian

Alternatively, we can set up a Lagrangian

$$\mathcal{L}_t = \sum_{s=t}^{t+T} \beta^{s-t} u\,(C_s) - \lambda\left[\sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} C_s - \sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} Y_s\right]$$

With the resulting Khun-Tucker necessary conditions being (ignoring slackness conditions)

$$\frac{d\mathcal{L}_t}{dC_s} = 0 \text{ for } s = t, \ldots, t+T$$

$$\frac{d\mathcal{L}_t}{d\lambda} = 0$$

Which turn out to be:

$$\beta^{s-t} u'\,(C_s) = \lambda \left(\frac{1}{1+r}\right)^{s-t} \text{ for } s = t, \ldots, t+T$$

$$\sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} Y_s = \sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} C_s$$

Combining the first equation for *s* and *s+1, we get*

$$u'\,(C_s) = (1+r)\,\beta\, u'\,(C_{s+1})$$

Which is, of course, again just our good friend the Euler equation.

An example

Suppose that $(1+r)\beta = 1$.
The Euler equation then implies that consumption is constant, i.e.

$C_t = C_{t+1} = C_{t+2} = \ldots = C_{t+T} = C$.

Plugging this into the budget constraint we get

$$\sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} C_s = \sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t$$

$$C \sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} = \sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t$$

Now, we'll need the following fact:

$$1 + q + q^2 + q^3 + \ldots + q^P = \sum_{j=0}^{P} q^P = \frac{1 - q^{P+1}}{1-q}$$

As an aside:

$$1 + q + q^2 + q^3 + \ldots = \sum_{j=0}^{\infty} q^P = \lim_{P\to\infty} \frac{1 - q^{P+1}}{1-q} = \frac{1}{1-q}$$

$$\sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} = 1 + \frac{1}{1+r} + \left(\frac{1}{1+r}\right)^2 + \ldots + \left(\frac{1}{1+r}\right)^T =$$

$$\frac{1 - \left(\frac{1}{1+r}\right)^{T+1}}{1 - \frac{1}{1+r}} = \frac{1+r}{r}\left(1 - (1+r)^{-(T+1)}\right) = \left(\frac{1}{1-(1+r)^{-(T+1)}}\, \frac{r}{1+r}\right)^{-1}$$

$$C = \left(\frac{1}{1-(1+r)^{-(T+1)}}\, \frac{r}{1+r}\right)\left(\sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t\right)$$

Let's look at some examples.

Constant Income

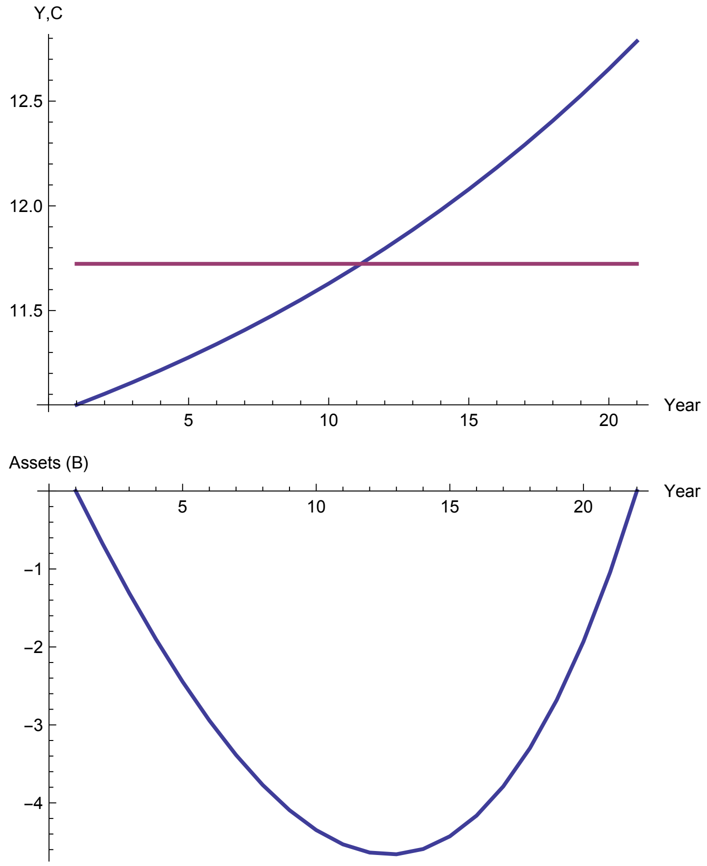
Suppose that initial assets $B_t = 0$ and income is expected to be constant and equal to $\overline{Y}$. Then

$$C = \left(\frac{1}{1-(1+r)^{-(T+1)}}\ \frac{r}{1+r}\right)\left(\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\,B_t\right) =$$

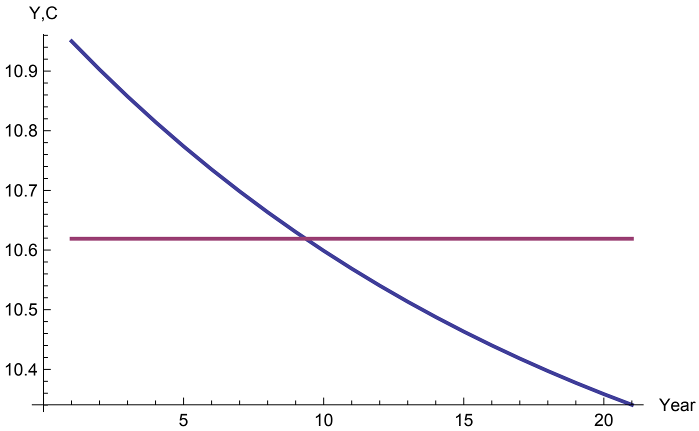
$$C = \left(\frac{1}{1-(1+r)^{-(T+1)}}\ \frac{r}{1+r}\right)\left(\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t} \overline{Y}\right) =$$

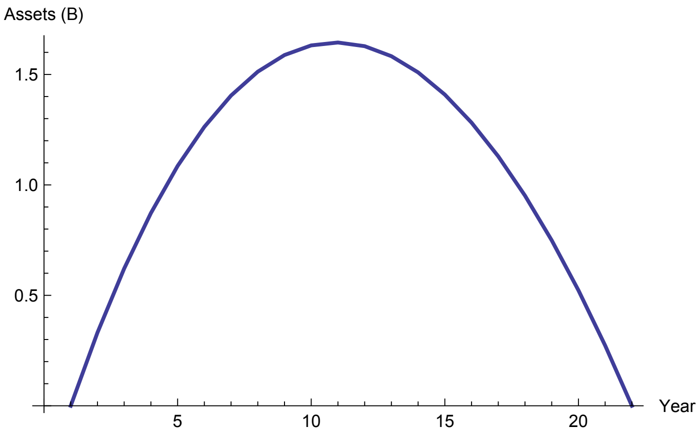
$$\overline{Y}\left(\frac{1}{1-(1+r)^{-(T+1)}}\ \frac{r}{1+r}\right)\left(\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t}\right) = \overline{Y}$$

Rising Income

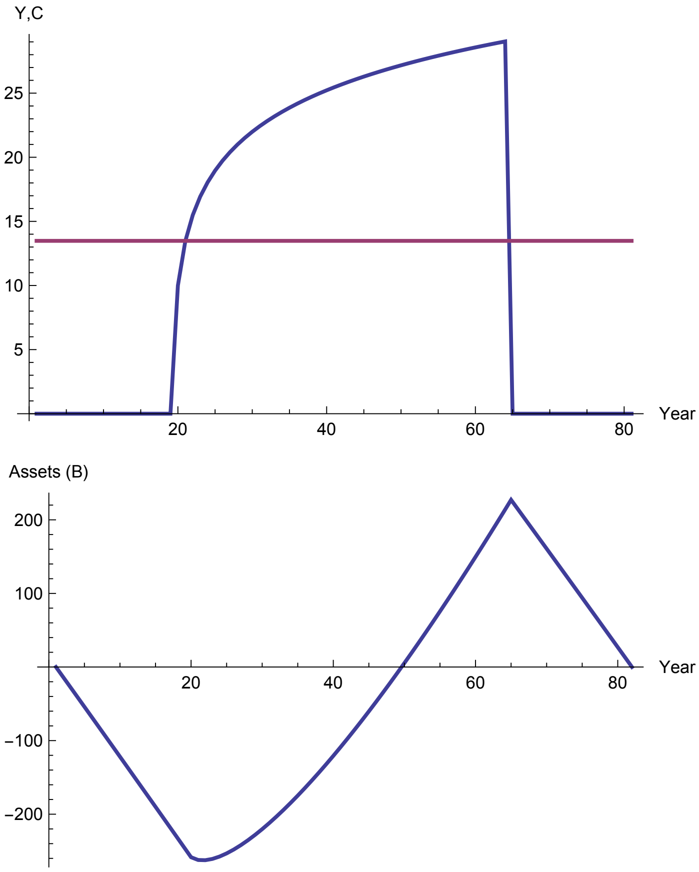


Falling Income

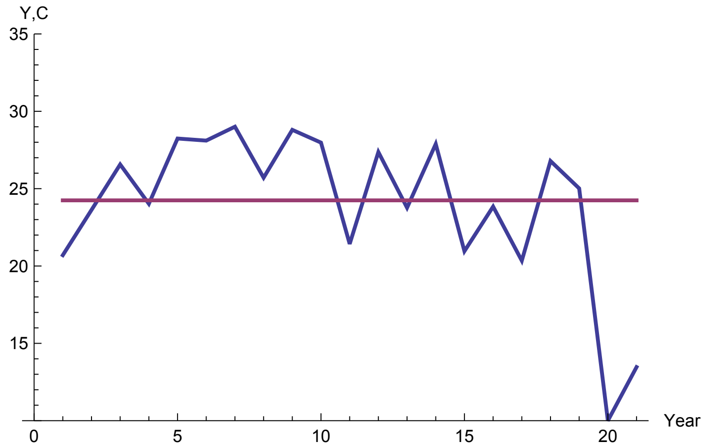


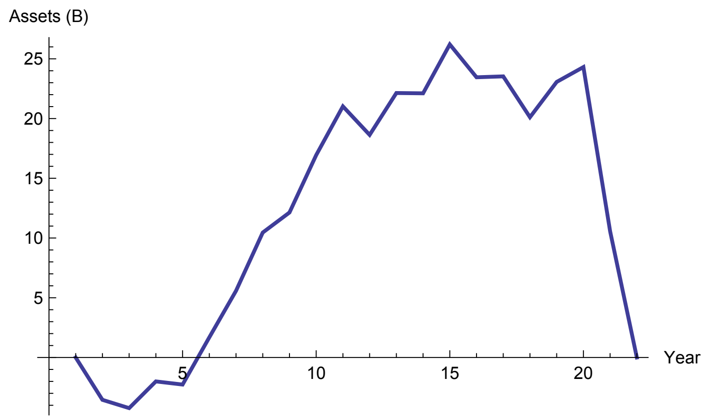


Life Cycle (Modigliani)



Variable Income





An example: 2

Let (1+r)$\beta \neq 1$.
Suppose we assume an isoelastic utility function (Q: what is σ)?

$$u(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma}$$

An aside:
This is macroeconomists' favorite utility function. You should know it well. What is u(c) when $\sigma = 1$, how about $\sigma \to \infty$ or $\sigma \to 0$?

The Euler equation is $C_s{}^{-1/\sigma} = (1+r)\,\beta\, C_{s+1}{}^{-1/\sigma}$ or

$$C_{s-1}{}^{-1/\sigma} = (1+r)\,\beta\, C_s{}^{-1/\sigma}$$

$$C_s = [(1+r)\,\beta]^{\sigma} C_{s-1} = \ldots$$
$$= [(1+r)\,\beta]^{2\sigma}$$

$$C_{s-2} = [(1+r)\,\beta]^{3\sigma} C_{s-3} = \ldots$$

$$= [(1+r)\,\beta]^{\sigma(s-t)} C_t$$

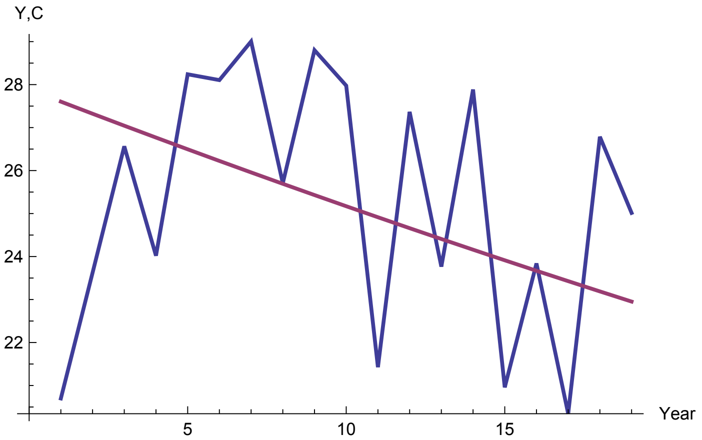
Plugging this into the budget constraint we get

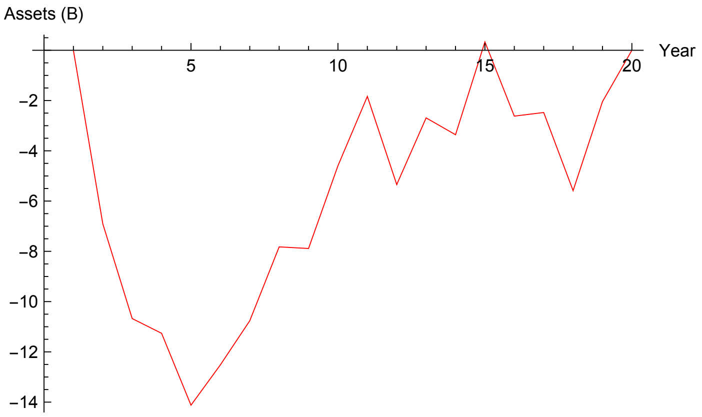
$$\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t} C_s = \sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t$$

$$C_t \sum_{s=t}^{t+T}\left(\frac{(1+r)\,\beta]^{\sigma}}{1+r}\right)^{s-t} = C_t \sum_{s=t}^{t+T}\left((1+r)^{\sigma-1}\beta^{\sigma}\right)^{s-t} = \sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t$$

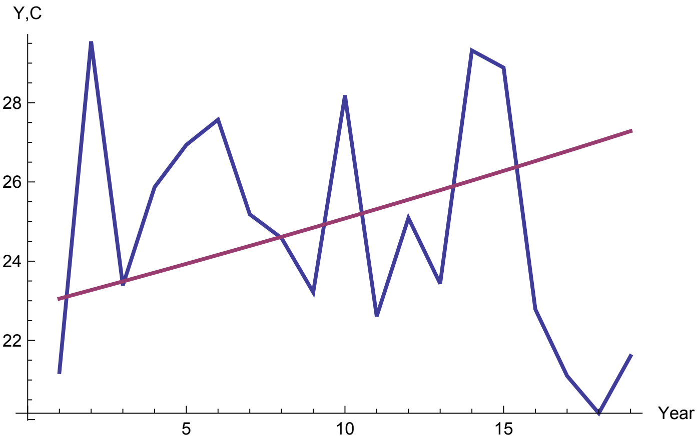
$$C_t = \left(\frac{1}{\sum_{s=t}^{t+T}\left((1+r)^{\sigma-1}\beta^{\sigma}\right)^{s-t}}\right)\left(\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t\right)$$

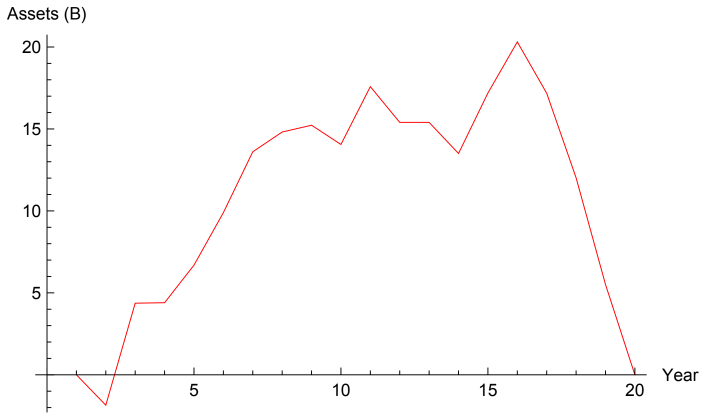
$(1 + r)\,\beta < 1$





$(1 + r)\,\beta > 1$





Infinite Horizon

Things actually get easier

$$V_t = \sum_{s=t}^{\infty} \beta^{s-t}\, u\,(C_s) \text{ subject to}$$

$$\lim_{t\to\infty} \left\{ (1+r)\, B_t + \sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} Y_s = \sum_{s=t}^{t+T} \left(\frac{1}{1+r}\right)^{s-t} C_s + \left(\frac{1}{1+r}\right)^{T} B_{t+T+1} \right\}$$

which becomes

$$(1+r)\, B_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s = \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} C_s$$

with the assumption

$$\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^{T} B_{t+T+1} = 0$$

This is a No-Ponzi game condition; it says debt cannot grow faster than interest rate!

An example I with ∞ horizon

Suppose that $(1+r)\beta=1$.
The Euler equation then implies $C_t = C_{t+1} = C_{t+2} = \ldots = C_{t+T} = C$.
Plugging this into the budget constraint we get

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} C_s = \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t$$

$$C\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} = \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t$$

$$C = \left(\frac{r}{1+r}\right)\left(\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t\right)$$

Important question: What is the MPC?

Permanent vs. Transitory Income Shocks

Imagine that income is subject to shocks. This is cheating a bit since we didn't introduce any uncertainty when we were solving the model. So we have to pretend that the shock is a one-time, unanticipated shocks and it won't ever happen again. It works. We'll see that under certain assumptions, we get the same result form a proper model with uncertainty.

Suppose that initial assets $B_t = 0$ and income is expected to be constant and equal to $\overline{Y}$. Consumption is constant in every period and given by our consumption function

$$C = \left(\frac{r}{1+r}\right)\left(\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t\right) = \left(\frac{r}{1+r}\right)\left(\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} \overline{Y}\right) = \overline{Y}$$

Since in every period C = $\overline{Y}$, assets remain at 0.

Now imagine a unexpected shock to income occurs in some year j and raises it to $\overline{Y}' > \overline{Y}$. Since income will be equal $\overline{Y}'$ to in all future years, this is a permanent shocks. Starting in year j however, the situation is exactly analogous to the original problem, except with higher income level $\overline{Y}'$. The optimal consumption path is again constant, lets denote it by C', and given by

$$C' = \left(\frac{r}{1+r}\right)\left(\sum_{s=j}^{\infty}\left(\frac{1}{1+r}\right)^{s-j} Y_s + (1+r)\, B_j\right) = \left(\frac{r}{1+r}\right)\left(\sum_{s=j}^{\infty}\left(\frac{1}{1+r}\right)^{s-j} \overline{Y}'\right) = \overline{Y}'$$

Again since in every period C = $\overline{Y}'$, assets remain at 0.

So the change in consumption, relative to change in income (ΔC/ΔY), or the marginal propensity to consume, is one

$$\frac{\Delta C}{\Delta Y} = \left(\frac{C' - C}{\overline{\overline{Y}}' - \overline{Y}}\right) = 1$$

Now imagine a unexpected shock to income occurs in some year j and raises by ϵ but only for year j and in all the future years income is again equal to $\overline{Y}$. Since income will be equal unchanged to in all future years, this is a temporary (transitory) shocks. Starting in year j , the optimal consumption path is again constant and given by

$$C' = \left(\frac{r}{1+r}\right)\left(\sum_{s=j}^{\infty}\left(\frac{1}{1+r}\right)^{s-j} Y_s\right) =$$

$$\left(\frac{r}{1+r}\right)\left(\overline{Y} + \in + \sum_{s=j+1}^{\infty}\left(\frac{1}{1+r}\right)^{s-j}\overline{Y}\right) = \left(\frac{r}{1+r}\right)\left(\in + \sum_{s=j}^{\infty}\left(\frac{1}{1+r}\right)^{s-j}\overline{Y}\right) = \overline{Y} + \frac{r}{1+r}\in$$

So the change in consumption is

$$\frac{\triangle C}{\triangle Y} = \left(\frac{C' - C}{\overline{\overline{Y}} + \in - \overline{\overline{Y}}}\right) = \frac{r}{1+r} \approx r$$

Answer: MPC out of transitory change sin income = r/(1+r) ≈ r. What is a reasonable value for r? 3-7%

That’s waaaay lower than 1/3 we think MPC out of temporary changes in income is (M. Friedman said it is about 1/3).

Example 2 with Infinite Horizon

Let (1+r)β ≠1.

Suppose we assume an isoelastic utility function (Q: what is σ)?

$$u(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma}$$

Recall that the Euler equation $C_s = [(1+r)\,\beta]^{\sigma\,(s-t)}\,C_t$ implies.

$$C_t = \left(\frac{1}{\sum_{s=t}^{t+T}\left((1+r)^{\sigma-1}\,\beta^{\sigma}\right)^{s-t}}\right)\left(\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\,B_t\right)$$

With T → ∞ we need to assume $(1+r)^{\sigma-1}\,\beta^{\sigma} < 1$, then this becomes

$$C_t = \left(1-(1+r)^{\sigma-1}\,\beta^{\sigma}\right)\left(\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\,B_t\right)$$

Note: what does $(1+r)^{\sigma-1}\,\beta^{\sigma} < 1$ mean? It means that $(1+r)^{\sigma}\,\beta^{\sigma} < 1+r$ which means consumption grows at the rate less than *r*. What if it did not? The LHS of the budget constraint is infinite! The problem does not have a solution. This is one reason to like (1+r)β =1.
Note: you can also get weird results with (1+r)β =1. Think of a case of growing Y. If C is constant then C/Y → 0.

An Alternative Solution Method: The Dynamic Programming Approach

Now the dynamic programming approach follows a different path. It, of course, gives the same solution. But it has several important advantages: it helps highlight some economic intuition, it is well suited to deal with stochastic problems and discrete choice models.

Let $V(B_t)$ be the maximized -- subject to the constraint -- value of lifetime utility as of period *t*

$$V(B_t) = \max_{\{C_s\}_t^\infty} \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) \quad \text{subject to } B_{s+1} = (1+r) B_s + Y_s - C_s$$

V is called the “value function” of this problem. It should be easy to see that

$$V(B_t) = \max_{\{C_s\}_t^\infty} \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) =$$

$$\max_{C_t} \left\{ u(c_t) + \beta \max_{\{C_s\}_{t+1}^\infty} \sum_{s=t+1}^{\infty} \beta^{s-t} u(C_s) \right\} = \max_{C_t} \{ u(c_t) + \beta V(B_{t+1}(C_t, B_t)) \}$$

where, of course, B_{t+1} depends on C_t and through $B_{t+1} = (1+r) B_t + Y_t - C_t$ so we can write $B_{t+1}(C_t, B_t)$.

This is called the Optimality Principle. It is a very important result. It is the fundamental equation of Dynamic Programming; it’s called the Bellman Equation and is a recursive representation of the optimization problem

$$V(B_t) = \max_{\{C_t\}} \{ u(c_t) + \beta V(B_{t+1}) \}$$

$$\text{subject to } B_{t+1} = (1+r) B_t + Y_t - C_t$$

In words this says: If you know you will maximize from tomorrow onwards, the best thing to do today is to optimize taking future optimal plans as give.

How do we proceed next? Note that we have two variables: a control variable C and a state variable B.
We are looking for a solution to the above (functional) equation; this solution is an equation, the so-called policy function:

$$C_t = h(B_t)$$

This function gives us the optimal consumption in period t as a function of the state variable (B) in that period (and possibly other exogenous variables like Y, but I have suppressed that in the notation above).

So we have transformed a problem of finding an infinite sequence $\{C_s\}_t^\infty$ into one of finding a function $h(B_t)$ (We may also want to know the value function itself $V(B_t)$).

Here is how we go about finding the solution.
First, since the policy function solves the problem, we have

$$V(B_t) = u(h(B_t)) + \beta V(B_{t+1}(h(B_t), B_t))$$

where again we have used the following notation $B_{t+1} = (1+r) B_t + Y_t - C_t \equiv B_{t+1}(h(B_t), B_t)$

What are the FONC's of this problem? Clearly on of them is:

$$u'(c_t) + \beta V'(B_{t+1}) \frac{\partial B_{t+1}}{\partial C_t} = 0$$

and, since $\frac{\partial B_{t+1}}{\partial C_t} = -1$, we get

$$\mathbf{u'(c_t) = \beta V'(B_{t+1})}$$

Suppose we have the policy function (really we don't have it YET) then
$V(B_t) = u(h(B_t)) + \beta V((1+r) B_t + Y_t - h(B_t))\}$ But notice that it is also true (by the envelope theorem) that

$$V'(B_t) = \frac{\partial u(h(B_t))}{\partial B_t} + \beta V'(B_{t+1}) \frac{\partial B_{t+1}}{\partial B_t}$$

$$= u'(c_t) \frac{\partial h(B_t)}{\partial B_t} + \beta V'(B_{t+1}) \left((1+r) - \frac{\partial h(B_t)}{\partial B_t}\right) =$$

$$= (u'(c_t) - \beta V'(B_{t+1})) \frac{\partial h(B_t)}{\partial B_t} + (1+r) \beta V'(B_{t+1})$$

But using the FONC $u'(c_t) = \beta V'(B_{t+1})$ we get that

$$\mathbf{V'(B_t) = \beta V'(B_{t+1}) (1+r)}$$

In words, the envelope coditions says this: Along an optimal path the value of consumption and saving must be the same on the margin (otherwise I coudl make myself better off by shifting resources from one to another). So what is the effect on my lifetime utility $V(B_t)$ of having a little extra assets today, i.e. what is $\mathbf{V'(B_t)}$? It must be equal to the value of saving that extra bit of assets, which is $\mathbf{\beta V'(B_{t+1}) (1+r)}$.

These two are our key equations

$$V'(B_t) = \beta (1+r) V'(B_{t+1})$$
$$u'(c_t) = \beta V'(B_{t+1})$$

We can “forward” both of them by one period

$$V'(B_{t+1}) = \beta (1+r) V'(B_{t+2})$$
$$u'(c_{t+1}) = \beta V'(B_{t+2})$$

and substitute

$$V'(B_{t+1}) = \beta V'(B_{t+2}) (1+r)$$
$$u'(c_{t+1}) = \frac{V'(B_{t+1})}{(1+r)}$$

which imply:

$$(1+r) u'(c_{t+1}) = V'(B_{t+1})$$

and together with the FONC $u'(c_t) = \beta V'(B_{t+1})$ we have again

$$(1+r) u'(c_{t+1}) = \frac{u'(c_t)}{\beta} \quad \text{or}$$

$$u'(c_t) = (1+r) \beta u'(c_{t+1})$$

The Euler equation; same solution as using the Lagrangian!

Example of Dynamic Programing

Let's go back to our simple model of consumption with exogenous income stream.

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$$

The flow budget constraint is

$$B_{s+1} = (1+r) B_s + Y_s - C_s$$

And the No-Ponzi game condition

$$\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^T B_{t+T+1} >= 0$$

As long as there is no satiation of the utility function, we know that $\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^T B_{t+T+1} = 0$ and together these imply the familiar life-time budget constraint:

$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} C_s = \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r) B_t$$

It turns out to be useful to summarize the exogenous income stream (Y) and the stocks of assets (B) together as one state variable. Let’s call it wealth (W) and define it as:

$$W_{t+1} = (1+r)\, B_{t+1} + \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-(t+1)} Y_s$$

Now using $B_{t+1} = (1+r)\, B_t + Y_t - C_t$ we can write

$$W_{t+1} = (1+r)\, \left((1+r)\, B_t + Y_t - C_t\right) + (1+r) \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s$$

$$= (1+r) \left[(1+r)\, B_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s - C_t\right]$$

$$= (1+r)\, (W_t - C_t)$$

So our problem can be restated in terms of W

$$V(W_t) = \max_{\{C_t\}} \{u(c_t) + \beta V(W_{t+1})\}$$

$$\text{subject to}$$

$$W_{t+1} = (1+r)(W_t - C_t)$$

or simply

$$V(W_t) = \max_{\{C_t\}} \{u(c_t) + \beta V((1+r)(W_t - C_t))\}$$

The FONC is

$$u'(c_t) = \beta(1+r)V'(W_{t+1}) \tag{2}$$

Recall that we used the envelope theorem to show

$$V'(W_t) = \beta V'(W_{t+1}) \frac{\partial W_{t+1}}{\partial W_t}$$

Here,

$$V'(W_t) = (1+r)\beta V'(W_{t+1})$$

Together the two blue equations imply:

$$u'(c_t) = V'(W_t) \tag{3}$$

Combining (1) and (2), give us

$$u'(c_t) = (1+r)\,\beta\,u'(c_{t+1}) \tag{4}$$

OK, so far so good. We got the Euler equation. Now however, we'll see how DP in action! Suppose we assume an isoelastic utility function

$$u(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma} \tag{5}$$

To solve the DP we need to find the policy function C(W) and possibly the value function V(W). Let's start by guessing that the value function will also be isoelastic:

$$V(W) = \frac{A}{1-1/\sigma} W^{1-1/\sigma} \tag{6}$$

We have to find an A that will make our guess work. Use $u'(C_t) = (1+r)\,\beta\,V'(W_{t+1})$ to write

$$C_t^{-1/\sigma} = (1+r)\,\beta\,A\,W_{t+1}^{-1/\sigma} = (1+r)\,\beta\,A\,((1+r)\,(W_t - C_t))^{-1/\sigma} \tag{7}$$

which can be solved (raise both sides to power σ and rearrange) to get:

$$C_t(W_t) = \frac{1}{1+(1+r)^{\sigma-1}\beta^\sigma A^\sigma} W_t \tag{8}$$

This is the policy function (here consumption function). But it still has the unknown A in it...
However, recall that the Bellman equation says:
$V(W_t) = u(c(W_t)) + \beta V((1+r)(W_t - c(W_t)))$ so we can write:

$$\begin{aligned} A\frac{W_t^{1-1/\sigma}}{1-1/\sigma} = {} & \left[\frac{1}{1+(1+r)^{\sigma-1}\beta^\sigma A^\sigma}\right]^{1-1/\sigma} \frac{W_t^{1-1/\sigma}}{1-1/\sigma} + \\ & \beta A\left[\frac{(1+r)^\sigma \beta^\sigma A^\sigma}{1+(1+r)^{\sigma-1}\beta^\sigma A^\sigma}\right]^{1-1/\sigma} \frac{W_t^{1-1/\sigma}}{1-1/\sigma} \end{aligned} \tag{9}$$

This looks like a mess but is actually easy to solve for A since it implies
$A = \frac{1}{\left[1+(1+r)^{\sigma-1}\beta^\sigma A^\sigma\right]^{1-1/\sigma}} + \frac{(1+r)^{\sigma-1}\beta^\sigma A^\sigma}{\left[1+(1+r)^{\sigma-1}\beta^\sigma A^\sigma\right]^{1-1/\sigma}}$ which leads to

$$A = \left[\frac{1}{1-(1+r)^{\sigma-1}\beta^\sigma}\right]^{1/\sigma} \tag{10}$$

Which gives us the solution we were looking for

$$\begin{aligned} c_t(W_t) &= \left[1+(1+r)^{\sigma-1}\beta^\sigma\right] W_t \\ V(W) &= \left[\frac{1}{1-(1+r)^{\sigma-1}\beta^\sigma}\right]^{1/\sigma} \frac{W^{1-1/\sigma}}{1-1/\sigma} \end{aligned} \tag{11}$$

Which, of course, is the same solution we got in Example 2 above.

$$C_t = \left(1 - (1+r)^{\sigma-1}\beta^{\sigma}\right)\left(\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)\, B_t\right)$$

Dynamics of CA

If we included government and investment the consumption functions would be

$$C_t = \left(\frac{r}{1+r}\right)\left(\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - G_s - I_s) + (1+r)\,B_t\right)$$

Let us define a **permanent level** of some variable X by $\tilde{X}$

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}\tilde{X}_t = \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}X_s$$

That is: $\tilde{X}$ is the counterfactual value that X would have to take on if it were to (a) be constant from time t on and (b) maintain the same present discounted value as the actual path of X. Note that, since $\tilde{X}$ is constant, the above implies:

$$\tilde{X}_t = \left(\frac{r}{1+r}\right)\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}X_s$$

Dynamics of CA (cont.)

Let's assume (1+r)β = 1.
The consumption function and our new definitions imply:

$$C_t = \left(\frac{r}{1+r}\right)\left(\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - G_s - I_s) + (1+r)\,B_t\right) = \tilde{Y}_t - \tilde{G}_t - \tilde{I}_t + rB_t$$

i.e consumption is equal to permanent income minus permanent level of government spending minus permanent level if investment (plus asset return). Recall that the current account is given by $CA_t = Y_t + rB_t - G_t - I_t - C_t$; it follows that

$$CA_t = \left(Y_t - \tilde{Y}_t\right) - \left(G_t - \tilde{G}_t\right) - \left(I_t - \tilde{I}_t\right)$$

Current account reflects differences between current and permanent levels of Y, G and I. Note: I and Y are related: more I today means higher Y in the future.

Bankruptcy

When is a country bankrupt?
Re-write the inter-temporal budget constraint as:

$$(1+r)\,B_t = \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - G_s - I_s - C_s) = \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} TB_s$$

What does this say? Same thing we saw in the 2 period model. Negative external wealth ($B_t < 0$) means the PDV of trade balances must be positive. A country must eventually pay off its debts by sending more stuff abroad than it imports for the rest of the world.

However, note that it is possible to run perpetual CA deficits. Suppose that output grows at a constant rate *g,* so that $Y_{s+1} = (1+g)\,Y_s$.
Assume that the country -- with debt (i.e. B < 0) -- wants to stabilize its external debt/GDP ratio B_s / Y_s; clearly that means that debt must also grow at the rate *g*, that is: $B_{s+1} = (1+g)\,B_s$.
Recall the asset accumulation equation (the flow budget constraint)

$$B_{s+1} - B_s = rB_s + Y_s - C_s - G_s - I_s = rB_s + TB_s$$

Substitution $B_{s+1} - B_s = gB_s$ and rearranging we get

$$\frac{TB_s}{Y_s} = \frac{-(r-g)\,B_s}{Y_s}$$

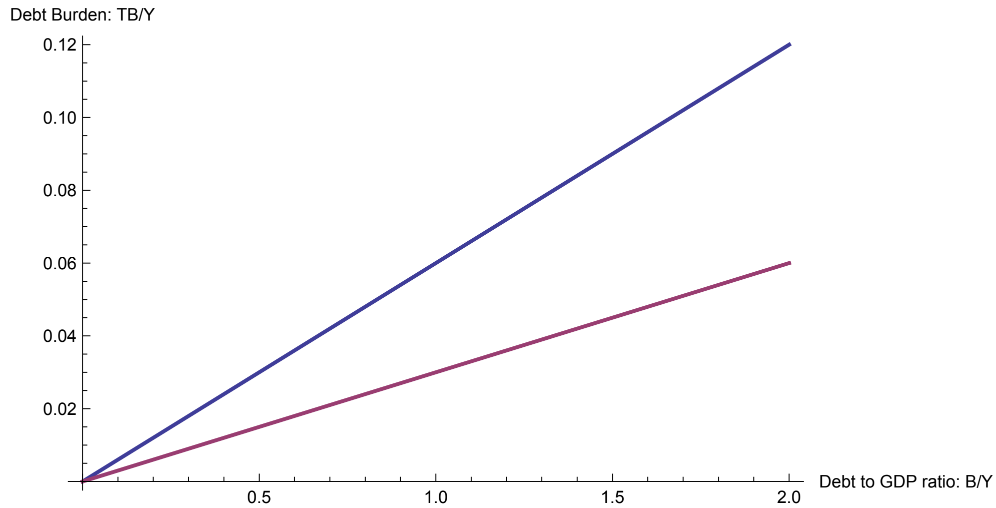
$$\frac{TB_s}{Y_s} = \frac{-B_s}{Y_s/(r-g)} > 0$$

Two lessons from this equation:

(1) To stabilize B/Y need to run TB surpluses equal to $\frac{-B_s}{Y_s/(r-g)}$

(2) The the debt burden (how much trade surplus does the debt require?) is determined not by the debt to GDP ratio (B/Y) but by the ratio of debt to $Y_s/(r-g)$, which is the present discounted value of GDP from now until the end of the world.

Suppose r = 8% (return on equities) and g = 2% or 5%.



Clearly, even a B/Y ratio of 100% corresponds to a relatively low debt burden, especially when you are growing fast. So why do investors worry when B/Y in Greece or Ireland goes up past 100%? Countries can and do default: more on this later...

A Note on Italy' s Debt

Here is an article form the Financial Times (11/2011) by the economist Martin Feldstein (the one from Feldsetin-Horioka Puzzle).

Italy can save itself and the euro

The euro currency may soon collapse even though there is no fundamental reason for it to fail.Everything depends on Italy, because financial markets now fear that it may be insolvent.If the Italian government has to continue paying a seven or even eight per cent interest rate to finance its debt, the country' s total debt will grow faster than its annual output and therefore faster than its ability to service that debt.If investors expect that to persist, they will stop lending to Italy.At that point, it will be forced to leave the euro.And if it does, the value of the "new lira" will reduce the price of Italian goods in general and Italian exports in particular.The resulting competitive pressure could then force France to leave the euro as well, bringing the monetary union to an end.But this need not happen.Italy can save both its own economic sovereignty and the euro if it acts decisively and quickly to convince the financial markets that it will balance its budget and increase its rate of economic growth so that the ratio of its public debt to its gross domestic product will decline in a steady and predictable way.If markets have confidence in that, Italy' s interest rate could decline to the four per cent that it paid before the crisis began.Italy is in a good position to achieve this.It already has a "primary budget surplus", with tax revenues exceeding total non - interest government outlays.It can eliminate its small overall budget deficit if it cuts spending and raises revenue by a total of just three per cent of its GDP–an amount not impossible to find in a public budget that now equals 50 per cent of GDP.The country also has a positive growth rate of about one per cent per year.If reforms to strengthen incentives and reduce regulatory impediments raise that growth rate to two per cent, that together with a long - term balanced budget would cause Italy' s public debt to decline from today' s 120 per cent of GDP to about 65 per cent over the next 15 years.That is similar to what happened in the US after the second world war when a combination of a balanced budgets, 2.3 per cent growth and 3.3 per cent inflation brought the debt to GDP ratio from 109 per cent in 1946 to 46 per cent in 1960.)

Italy' s situation is totally different from Greece's. The latter has a budget deficit of nine per cent of GDP and its real GDP is declining at seven per cent, driving its debt from 150 per cent of GDP today to 170 per cent after just one year.The over - valued exchange rate results in a current account deficit of ten per cent of its GDP.Greece would be better off if it abandons the euro, devalues its new currency, and defaults on its debt.A decision by Athens to leave the euro and default could cause a run on the euro and on Italian debt in particular.That' s why it is so important for Italy to stress that its conditions are totally different from those in Greece, and that its new policies will soon produce budget balance and a declining ratio of debt to GDP.Italy can do all of this itself.It does not need assistance from Frankfurt, Brussels, or Washington.The proposed policies for help from the European Central Bank, the European Commission, and the International Monetary Fund would ultimately weaken Italy and undermine its economic independence.There are strong voices, including the French government, calling for the ECB to buy the sovereign debt of Italy and other countries in order to keep the level of their interest rates within about 200 basis points of the rate on

German bonds and therefore low enough to avoid an automatic rise in their debt to GDP ratios.But this would violate the "no bail - out" provision of the Maastricht treaty, put Germany at risk if any countries are eventually forced to default, cause an explosive inflationary supply of euros, and remove any market feedback about whether Italy and other governments have done enough to control future deficits.In exchange for supporting the debt of these countries, the ECB or the EC would have to be able to veto national budget decisions.Italy, like Greece today, would become an economic vassal of Germany.After the clear failure to expand the European Financial Stability Facility from €400bn euros to the thousands of billions needed to backstop borrowing by Italy and Spain, the EC recently proposed an alternative policy of creating 'stability bonds'.Every EMU country would be able to issue these 'eurobonds' that would be guaranteed by all 17 eurozone members.This would only be feasible if the national budgets were subject to control by the EC, which would be dominated by Germany as the primary guarantor of the new bonds.The IMF in turn has suggested that it create a fund that would lend to troubled eurozone members and perhaps be used to put a cap on their interest rates.This fund would be financed by loans from the ECB, thus deftly circumventing the Maastricht treaty' s rules against bail - outs and against buying new bonds issued by member governments.Under this plan, some combination of the IMF and the EC would have to control the budgets of the borrowing nations.Any of these proposed programmes would create new conflicts within Europe as borrower governments are forced to relinquish their ability to set their own national tax and spending policies.Moreover, what would start as EC limits on fiscal irresponsibility could evolve into limits designed to prevent trade advantages.Ireland' s low corporate tax rate would be an obvious target for its eurozone competitors.The riots and political upheavals in Greece are a symptom of what would happen more generally if the Brussels bureaucracy and the German Chancellor came to dominate national economic policies.Fortunately, none of this is necessary if Italy now acts forcefully to create budget and growth conditions that imply sustainable debt outcomes.But impatience and scepticism in financial markets may cause a deeper financial crisis before Italy has time to prove itself.

The writer is professor of economics at Harvard University and former chairman of the Council of Economic Advisers and president Ronald Reagan' s chief economic adviser.

Feldstein is applying the same method to thinking of Italy's debt as we have when thinking of the measure of external debt burden. Using the following notation: D = debt, Y = GDP, r = interest rate that Italy is paying on its debt, G = gov't spending, T = taxes, and T - G = B (primary budget balance, i.e. budget balance before paying interest on debt). We have

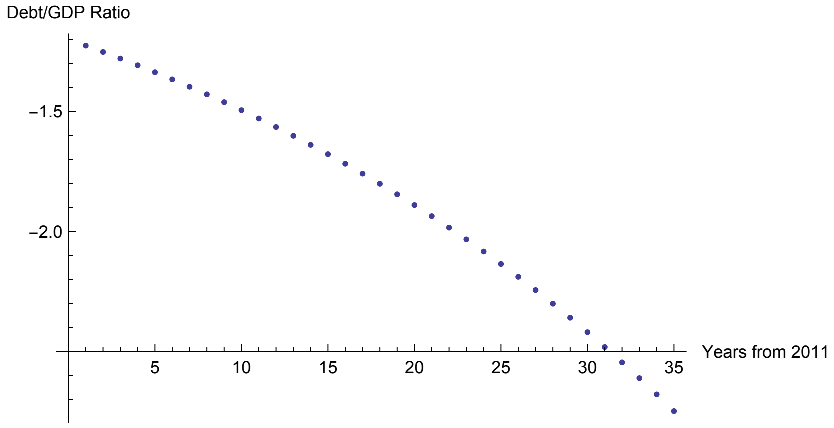
$$\triangle D_{t+1} = T_t - G_t + r D_t$$

we'll also assume that Italy's GDP is growing at a constant rate *g*

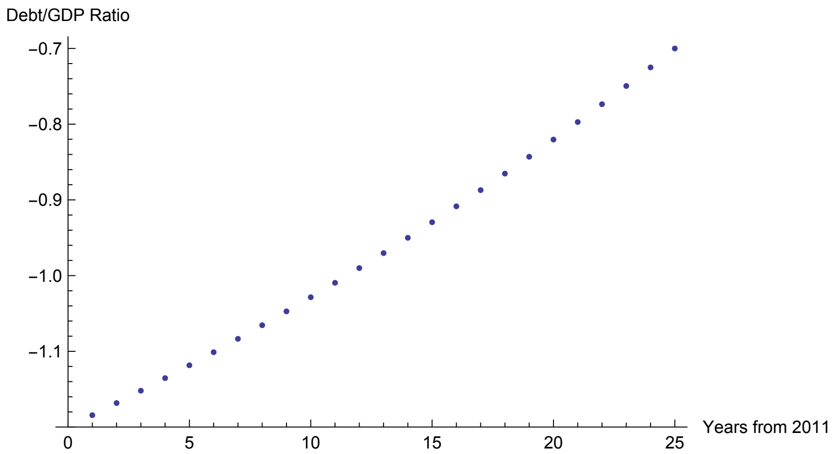
$$Y_{t+1} = (1 + g) Y_t$$

Italy's today government debt to GDP is currently 120%. Its budget deficit (T - G +rD) is -4% of GDP and its primary balance (T - G) is about 1%. The growth rate of GDP has been very slow in Italy lately, about 1% per year. The interest rate Italy has been paying on its debt is about 4% (it has gone up in last two years to as much as 8% but since the ECB's announcement in August have fallen and the spread over 10-year Germanbasis bonds is now only 300 basis points but let's ignore all of that for now, in any case the a higher rate would apply to new debt and since Italy isn't borrowing very much (they don't have a huge deficit) most of their debt is still serviced at about 4%). Based on our earlier calculations, the primary surplus necessary to stabilized D/Y ratio would be $\frac{-D_s}{Y_s/(r-g)} = 1.7 \times (.04 - .01) = 5\,\%$. That's a large surplus!

Alternatively, we can ask what will be the steady state D/Y ratio i.e. if *r = .04* , *(T - G)/Y* is stable at 1%



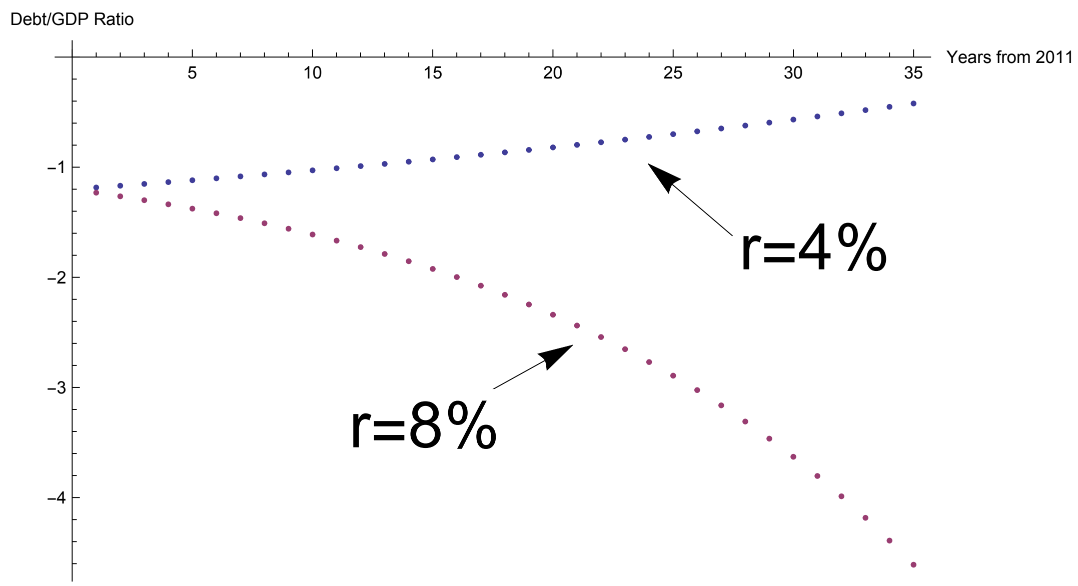
So this situation is clearly unsustainable as debt/GDP increases exponentially. What if instead Italy is able to increase its primary balance by 3%, boost growth to 2%, and is able to borrow at 4% as Feldstein hopes? Well, the debt is clearly sustainable



I'm not sure where Feldstein gets the 65% in 15 number from as the above calculation suggest something like a 95% debt/GDP ratio in 15 years, but the bottom line is the debt is clearly sustainable (debt/GDP ratio is falling, i.e. becoming less negative).
However, consider the point made by Brad DeLong:

Yes, if Italy cuts spending by 1% of GDP, improves tax compliance by 2% of GDP, and manages to boost its economic growth rate from 1% per year to 2% per year, then if it can borrow at 4% per year then its debt is sustainable--and if you bet in financial markets that Italy will default, you will lose your money.

But even if Italy cuts spending by 1% of GDP, improves tax compliance by 2% of GDP, and manages to boost its economic growth rate from 1% per year to 2% per year, if it nevertheless finds that it can only borrow at 8% per year then its debt is unsustainable--and if you bet in financial markets that Italy will not default, you will lose your money.



Here is more from DeLong:

There are two equilibria out there if Italy reforms its policies: There is one in which investors lose money if they bet Italy will default even if Italy undertakes minor policy reforms--and if investors believe that we are in this equilibrium we will indeed be in this equilibrium and investors who bet Italy will default will lose their money. There is another equilibrium in which investors lose money if they bet Italy will not default if Italy undertakes minor policy reforms--and if investors believe that we are in that equilibrium we will indeed be in that equilibrium, and investors who bet Italy will not default will lose their money, and all the policy adjustments Italy can undertake by itself will not help.

What can the world do to make financial markets--and Italy, and the eurozone--settle at the first equilibrium?

Martin Feldstein appears to argue that the world should let Italy undertake its economic policy reforms by itself, and otherwise it should stand back and pray for the appearance of the Confidence Fairy.

I don’t think that the Confidence Fairy is guaranteed to show up.

I think that if the assembled credit-worthy sovereigns of the globe--the money-printers: the BOJ, the ECB, the FRB, the BoE, and the IMF--show up and say that if you bet on the bad equilibrium we will ruin you, because we will buy up as many Italian government bonds as needed to cut Italy to the good equilibrium, and if they then start buying, then we do not need to pray for the Confidence Fairy because we are in the good equilibrium whether the Confidence Fairy shows up or not.

And by one of the arcana imperii of monetary and financial policy, the best way to guarantee that the Confidence Fairy will show up is not to need to pray for it--and so if the money-printers show up, they probably won’t have to buy Italian government bonds because everybody will see that holding Italian government bonds is a way to profit and selling Italian government bonds short is a way to bankruptcy.

The problem, of course, is what if the money-printers show up but Italy does not undertake its structural adjustment...

Stochastic Model

Let's introduce uncertainty by assuming that output is random.
Note: the only asset we'll consider is a riskless bond. There are no Arrow-Debreu securities for now.

Important notation: $E_t[X_s]$ is the expected value of some variable X at time s conditional on the information available at time t, i.e. the "year-t information set". Clearly, $E_t[X_s] = X_s$ for any $s \leqslant t$, since everything up until year t is in the "year-t information set".

Here is the Bellman Equation

$$V(B_t) = \max_{\{C_t\}} \{u(C_t) + \beta E_t[V(B_{t+1})]\}$$
$$\text{subject to } B_{t+1} = (1+r) B_t + Y_t - C_t$$

The FONC is

$$u'(C_t) = \beta E_t[V'(B_{t+1})]$$

As before, using the envelope theorem we have

$$V'(B_t) = (1+r) \beta E_t[V'(B_{t+1})]$$

Now we can forward these equations as before

$$u'(C_{t+1}) = \beta E_{t+1}[V'(B_{t+2})]$$
$$V'(B_{t+1}) = (1+r) \beta E_{t+1}[V'(B_{t+2})]$$

Then take the expectations of both side conditional on info available at time t, i.e. E_t

$$E_t\{u'(C_{t+1})\} = E_t\{\beta E_{t+1}[V'(B_{t+2})]\}$$
$$E_t\{V'(B_{t+1})\} = E_t\{(1+r) \beta E_{t+1}[V'(B_{t+2})]\}$$

Now, here is a very important part: $E_t\{E_{t+1}[X_{t+s}]\} = E_t[X_{t+s}]$ for $s \geqslant 1$. This called the Law of Iterated Expectations.
The last two equations become

$$E_t\{u'(C_{t+1})\} = \beta E_t[V'(B_{t+2})]$$
$$E_t\{V'(B_{t+1})\} = (1+r) \beta E_t[V'(B_{t+2})]$$

and they imply

$$E_t\{u'(C_{t+1})\} = E_t\{V'(B_{t+1})\} / (1+r)$$

Plugging this back into the first order condition we get the stochastic Euler equation for bonds

$$u'(C_t) = (1+r)\,\beta\,E_t[u'(C_{t+1})]$$

If we assume quadratic utility and (1+r)β=1, we get Hall’s random walk result

$$E_t[C_{t+1}] = C_t$$

cont.

In a model with uncertainty the intertemporal budget constraint is

$$(1+r)\ B_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s = \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} C_s$$

and it has to hold for every possible realization of the path of Y. Which means, it also must hold “in expectations”

$$(1+r)\ B_t + E_t\left[\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s\right] = E_t\left[\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} C_s\right]$$

But from the FONC we know that $E_t[C_s] = E_t[C_{s-1}] = \ldots = E_t[C_{t+1}] = C_t$ so we can substitute $C_t$into the budget constraint to get

$$(1+r)\ B_t + E_t\left[\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s\right] = C_t \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t}$$

And the consumption function is:

$$C_t = \left(\frac{r}{1+r}\right)\left[(1+r)\ B_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} E_t\, Y_s\right] \tag{12}$$

Compare with the one from the model without uncertainty: people with quadratic preferences follow the *certainty equivalence.*

Persistence of Output Shocks

Suppose

$$Y_{t+1} - \overline{Y} = \rho \left(Y_t - \overline{Y}\right) + \epsilon_{t+1}$$

where $E_t \epsilon_{t+1} = 0$, and $0 \leqslant \rho \leqslant 1$. As long as ρ <1 this process is mean-reverting. For ρ = 0 this process is called white noise. Higher ρ means shocks are more persistent.

```
Ȳ := 100;
Y[1] := Ȳ;
ρ =.
T =.
ϵ =.
T = 150;
ϵ = RandomVariate[NormalDistribution[0, 5], T];
Manipulate[ListLinePlot[
  Table[Y[t + 1] = Ȳ + ρ (Y[t] - Ȳ) + ϵ[[t]], {t, 1, T}], AxesLabel → {"Year", "Y"},
  PlotMarkers → {"●", 4}], {ρ, 0, 1}, TrackedSymbols → {ρ}]
```

```
ρ

ListLinePlot[Table[Y[t + 1] = Ȳ + FE`ρ$$1 (Y[t] - Ȳ) + ϵ[[t]], {t, 1, T}],
  AxesLabel → {Year, Y}, PlotMarkers → {●, 4}]
```

Table: Iterator {t, 1, T} does not have appropriate bounds.

Table: Iterator {t, 1.000000000000000, T} does not have appropriate bounds.

Table: Iterator {t, 1., T} does not have appropriate bounds.

General: Further output of Table::iterb will be suppressed during this calculation.

ListLinePlot: Table[Y[t + 1.] = Ȳ + FE`ρ$$1 (Y[t] − 1. Ȳ) + ϵ[[t]], {t, 1., T}] is not a list of numbers or pairs of numbers.

ListLinePlot: Table[Y[t + 1.] = Ȳ + FE`ρ$$1 (Y[t] − 1. Ȳ) + ϵ[[t]], {t, 1., T}] is not a list of numbers or pairs of numbers.

ListLinePlot: Table[Y[t + 1.] = Ȳ + FE`ρ$$1 (Y[t] − 1. Ȳ) + ϵ[[t]], {t, 1., T}] is not a list of numbers or pairs of numbers.

General: Further output of ListLinePlot::lpn will be suppressed during this calculation.

An Aside on lag Operators

Let’s define an operator L, called ‘the lag operator’, in the following way:

$LX_t = X_{t-1}$

$LLX_t = L^2 X_t = X_{t-2}$

$L^{-1} X_t = X_{t+1}$

With this definition -- and also defining $\tilde{Y}_t = Y_t - \overline{Y}$ we can re-write the equation for output as

$$Y_{t+1} - \overline{Y} = \rho \left(Y_t - \overline{Y}\right) + \epsilon_{t+1}$$

$$\tilde{Y}_{t+1} = \rho \tilde{Y}_t + \epsilon_{t+1}$$

$$\tilde{Y}_{t+1} = \rho L \tilde{Y}_{t+1} + \epsilon_{t+1}$$

$$\tilde{Y}_{t+1} (1 - \rho L) = \epsilon_{t+1}$$

$$\tilde{Y}_{t+1} = \frac{\epsilon_{t+1}}{1 - \rho L}$$

Now, recall that for x < 1 we have

$$\sum_{s=t}^{\infty} x^{s-t} = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

So as long as ρ < 1 can write

$$\sum_{s=t}^{\infty} (\rho L)^{s-t} = 1 + \rho L + (\rho L)^2 + (\rho L)^3 + \ldots = \frac{1}{1-\rho L}$$

so it follows that

$$\tilde{Y}_{t+1} = \epsilon_{t+1} \sum_{s=t}^{\infty} (\rho L)^{s-t}$$

$$Y_{t+1} = \overline{Y} + \sum_{s=-\infty}^{t+1} \rho^{t-s} \epsilon_s$$

A higher value of ρ means that shocks are more persistent (take more time to go away). In the extreme cases of ρ=0 shocks last only for one period and ρ=1 shocks are permanent (lasts forever).

```
Ȳ := 100;
Y[1] := Ȳ;
ρ =.
T =.
ϵ =.
T = 100;
```

```
Manipulate[
 ListLinePlot[Table[Y[t + 1] = Ȳ + ρ (Y[t] - Ȳ) + If[t == 10, 15, 0], {t, 1, T}],
  AxesLabel → {"Year", "Y"}, PlotRange → {{0, T}, {90, 120}},
  PlotMarkers → {"●", 6}], {ρ, 0, 1}, TrackedSymbols → {ρ}]
```

```
ρ

ListLinePlot[
Table[Y[t + 1] = Ȳ + FE`ρ$$2 (Y[t] - Ȳ) + If[t == 10, 15, 0], {t, 1, T}],
AxesLabel → {Year, Y},
PlotRange → {{0, T}, {90, 120}}, PlotMarkers → {●, 6}]
```

Table::iterb : Iterator {t, 1, T} does not have appropriate bounds. ≫

ListLinePlot::lpn : Table[Y[t + 1] = Ȳ + FE`ρ$$21 (Y[t] – Ȳ) + If[t == 10, 15, 0], {t, 1, T}] is not a list of numbers or pairs of numbers. ≫

... Table: Iterator {t, 1, T} does not have appropriate bounds.

... Table: Iterator {t, 1.000000000000000, T} does not have appropriate bounds.

... Table: Iterator {t, 1., T} does not have appropriate bounds.

... General: Further output of Table::iterb will be suppressed during this calculation.

... ListLinePlot: Value of option PlotRange –> {{0, T}, {90, 120}} is not All, Full, Automatic, a positive machine number, or an appropriate list of range specifications.

... ListLinePlot: Table[Y[t + 1.] = Ȳ + FE`ρ$$2 (Y[t] – 1. Ȳ) + If[t == 10., 15., 0.], {t, 1., T}] is not a list of numbers or pairs of numbers.

... ListLinePlot: Value of option PlotRange –> {{0, T}, {90, 120}} is not All, Full, Automatic, a positive machine number, or an appropriate list of range specifications.

... ListLinePlot: Table[Y[t + 1.] = Ȳ + FE`ρ$$2 (Y[t] – 1. Ȳ) + If[t == 10., 15., 0.], {t, 1., T}] is not a list of numbers or pairs of numbers.

... ListLinePlot: Value of option PlotRange –> {{0, T}, {90, 120}} is not All, Full, Automatic, a positive machine number, or an appropriate list of range specifications.

... General: Further output of ListLinePlot::prng will be suppressed during this calculation.

... ListLinePlot: Table[Y[t + 1.] = Ȳ + FE`ρ$$2 (Y[t] – 1. Ȳ) + If[t == 10., 15., 0.], {t, 1., T}] is not a list of numbers or pairs of numbers.

... General: Further output of ListLinePlot::lpn will be suppressed during this calculation.

Consumption Function with a Specific Process for Y

$$Y_{t+1} - \overline{Y} = \rho \left(Y_t - \overline{Y}\right) + \epsilon_{t+1}$$

Implies that

$$Y_{t+2} - \overline{Y} = \rho \left(Y_{t+1} - \overline{Y}\right) + \epsilon_{t+2} = \rho^2 \left(Y_t - \overline{Y}\right) + \rho\epsilon_{t+1} + \epsilon_{t+2}$$
$$Y_{t+3} - \overline{Y} = \rho \left(Y_{t+2} - \overline{Y}\right) + \epsilon_{t+3} = \rho^3 \left(Y_t - \overline{Y}\right) + \rho^2 \epsilon_{t+1} + \rho\epsilon_{t+2} + \epsilon_{t+3}$$

or for any s≥t

$$E_t\left[Y_s - \overline{Y}\right] = \rho^{s-t} \left(Y_t - \overline{Y}\right)$$

which we can substitute into the consumption function

$$C_t = \left(\frac{r}{1+r}\right)\left[(1+r)\ B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t\, Y_s\right]$$

to get

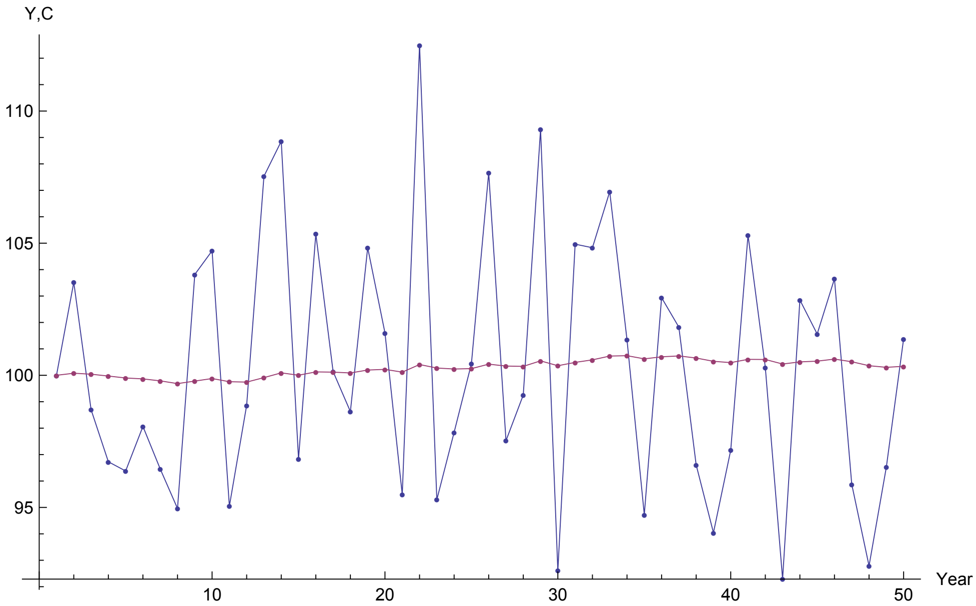
$$C_t = \overline{Y} + \left(\frac{r}{1+r}\right)\left[(1+r)\ B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} \rho^{s-t} \left(Y_t - \overline{Y}\right)\right]$$

$$\boxed{C_t = \overline{Y} + rB_t + \frac{r\left(Y_t - \overline{Y}\right)}{1+r-\rho}}$$

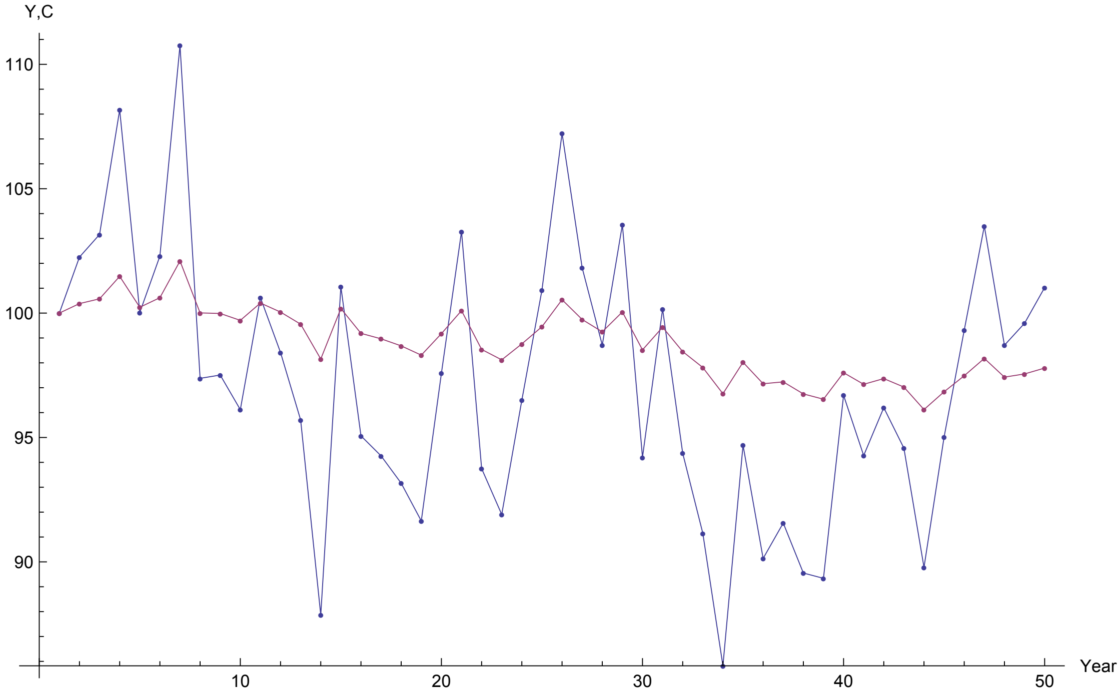
Consumption Variability

Notice that as long as ρ<1 we know that $\frac{r}{1+r-\rho} < 1$ so Var($C_t \mid Y_{t-1}$) < Var(ϵ_t).

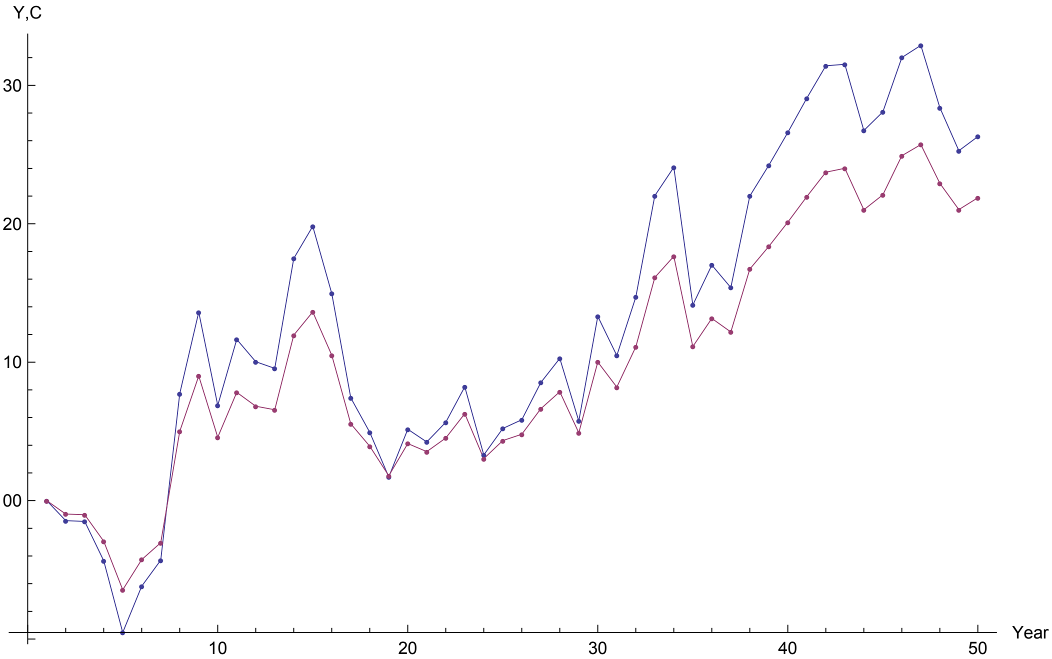
Notice that the higher ρ is, the more consumption tracks income.
Here is an example for ρ =.1 (Consumption is purple, income is blue)



here is one for ρ = 0.9.



And here is one for ρ = 0.99.


Y,C
130
120
110
100
10
20
30
40
50
Year

Can ρ be greater than 1?

Sure. Consider the process that is not mean-reverting in levels (like the one above) but mean-reverting in growth rates:

$$Y_{t+1} - Y_t = \eta\ (Y_t - Y_{t-1}) + \epsilon_{t+1}$$

Note: η = 0 corresponds to ρ=1.

```
Ȳ := 100;
Y[1] := Ȳ;
Y[2] := Ȳ;
ρ =.
T =.
ϵ3 =.
T = 50;
ϵ3 = RandomVariate[NormalDistribution[0, 5], T];
Manipulate[
 ListLinePlot[Table[Y[t + 1] = Y[t] + η (Y[t] - Y[t - 1]) + ϵ3[[t]], {t, 2, T}],
  AxesLabel → {"Year", "Y"}, PlotMarkers → {"●", 4}],
 {η, 0, 1}, TrackedSymbols → {η}]
```

```
η

ListLinePlot[
Table[Y[t + 1] = Y[t] + FE`η$$3 (Y[t] - Y[t - 1]) + ϵ3[[t]], {t, 2, T}],
AxesLabel → {Year, Y}, PlotMarkers → {●, 4}]
```

Table: Iterator {t, 2, T} does not have appropriate bounds.

Table: Iterator {t, 2.000000000000000, T} does not have appropriate bounds.

Table: Iterator {t, 2., T} does not have appropriate bounds.

General: Further output of Table::iterb will be suppressed during this calculation.

ListLinePlot: Table[Y[t + 1.] = Y[t] + FE`η$$3 (Y[t] – 1. Y[Plus[≪2≫]]) + ϵ3[[t]], {t, 2., T}] is not a list of numbers or pairs of numbers.

ListLinePlot: Table[Y[t + 1.] = Y[t] + FE`η$$3 (Y[t] – 1. Y[Plus[≪2≫]]) + ϵ3[[t]], {t, 2., T}] is not a list of numbers or pairs of numbers.

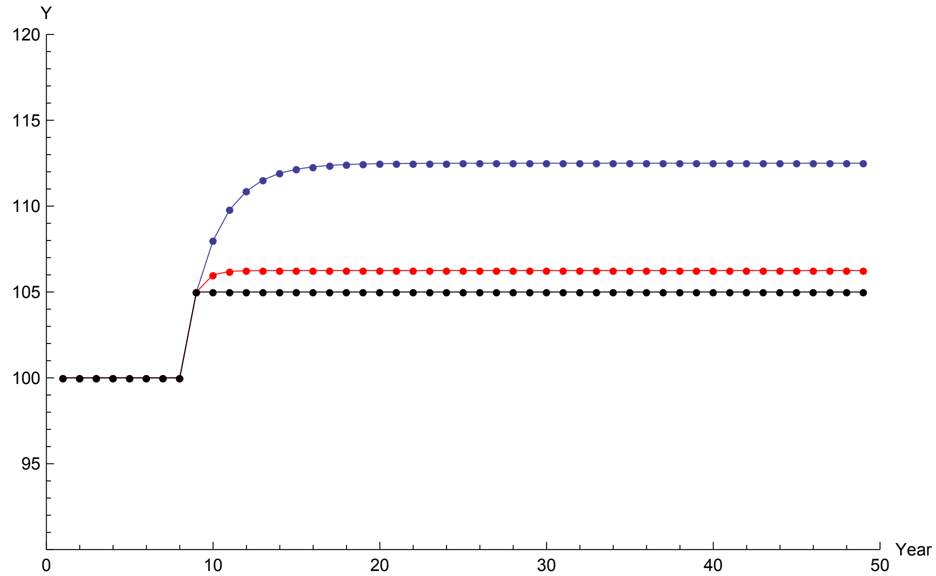
ListLinePlot: Table[Y[t + 1.] = Y[t] + FE`η$$3 (Y[t] – 1. Y[Plus[≪2≫]]) + ϵ3[[t]], {t, 2., T}] is not a list of numbers or pairs of numbers.

General: Further output of ListLinePlot::lpn will be suppressed during this calculation.

cont.

Notice that using lag operators (as we did above) we can show

$$Y_{t+1} = Y_t + \sum_{s=-\infty}^{t+1} \eta^{t-s} \epsilon_s$$



Closed Form Solution

Start by observing that with $Y_{t+1} - Y_t = \eta\ (Y_t - Y_{t-1}) + \epsilon_{t+1}$

$$Y_{t+2} - Y_{t+1} = \eta\ (Y_{t+1} - Y_t) + \epsilon_{t+2} = \eta^2\ (Y_t - Y_{t-1}) + \eta\epsilon_{t+1} + \epsilon_{t+2}$$
$$Y_{t+3} - Y_{t+2} = \eta\ (Y_{t+2} - Y_{t+1}) + \epsilon_{t+3} = \eta^3\ (Y_t - Y_{t-1}) + \eta^2\ \epsilon_{t+1} + \eta\epsilon_{t+2} + \epsilon_{t+3}$$

or for any s⩾t

$$E_t[Y_s - Y_{s-1}] = \eta^{s-t}\ (Y_t - Y_{t-1})$$

We can use this to establish that

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s - Y_{s-1}] = \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} \eta^{s-t}\ (Y_t - Y_{t-1}) = \frac{(1+r)}{1+r-\eta}\ (Y_t - Y_{t-1}) \tag{13}$$

Notice that we can also do the following neat transformation.
Start by writing

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s - Y_{s-1}] = \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s] - \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_{s-1}]$$

Then observe that

$$\begin{aligned}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_{s-1}] &= Y_{t-1} + \left(\frac{1}{1+r}\right) Y_t + \left(\frac{1}{1+r}\right)^2 E_t[Y_{t+1}] + \left(\frac{1}{1+r}\right)^3 E_t[Y_{t+2}] + \ldots \\ &= Y_{t-1} + \left(\frac{1}{1+r}\right)\left(Y_t + \left(\frac{1}{1+r}\right) E_t[Y_{t+1}] + \left(\frac{1}{1+r}\right)^2 E_t[Y_{t+2}] + \ldots\right) \\ &= Y_{t-1} + \left(\frac{1}{1+r}\right)\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s]\end{aligned}$$

Which implies

$$\begin{aligned}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s - Y_{s-1}] &= \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s] - Y_{t-1} - \left(\frac{1}{1+r}\right)\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s] \\ &= \frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s] - Y_{t-1}\end{aligned}$$

From equation (2) above we have

$$\frac{(1+r)}{1+r-\eta}(Y_t - Y_{t-1}) = \frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s] - Y_{t-1}$$

or

$$\frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t[Y_s] = Y_{t-1} + \frac{(1+r)}{1+r-\eta}(Y_t - Y_{t-1})$$

Plugging this into equation (2) (the consumption function:
$C_t = \left(\frac{r}{1+r}\right)\left[(1+r) B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} E_t Y_s\right]$) we get

$$\boxed{C_t = rB_t + Y_{t-1} + \frac{1+r}{1+r-\eta}(Y_t - Y_{t-1})}$$

Consumption Variability

Notice that unless $\eta=0$, we know that $\frac{1+r}{1+r-\eta} > 1$ so $\mathrm{Var}(C_t \mid Y_{t-1}) > \mathrm{Var}(\epsilon_t)$.

When η>0 consumption becomes more volatile than output!

```
Ȳ := 100;
Y[1] := Ȳ;
Y[2] := Ȳ;
B[1] := 0;
ρ =.
T =.
ϵ4 =.
T = 30;


ϵ4 = RandomVariate[NormalDistribution[0, 5], T];

Manipulate[Table[{Y[t + 1] = Y[t] + η (Y[t] - Y[t - 1]) + ϵ4[[t]],
    Cx[t] = Y[t - 1] + r B[t] + ((1 + r) (Y[t] - Y[t - 1]))/(1 + r - η),
    B[t + 1] = (1 + r) B[t] + Y[t] - Cx[t]}, {t, 2, T}];
 ListLinePlot[{Table[Y[t], {t, 2, T}], Table[Cx[t], {t, 2, T}]},
  AxesLabel → {"Year", "Y,C"}, PlotMarkers → {"●", 4}], {η, 0, 1, .02}]
```

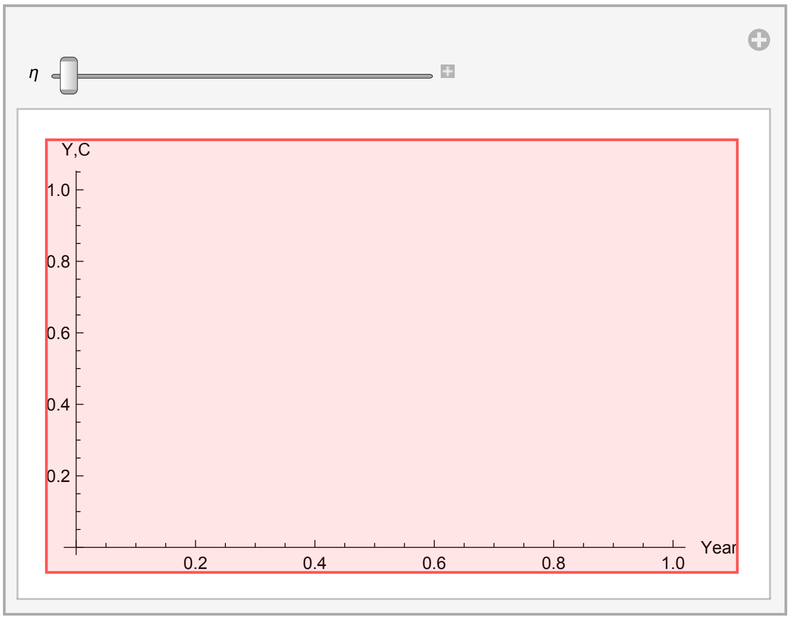


Table: Iterator {t, 2, T} does not have appropriate bounds.

Table: Iterator {t, 2, T} does not have appropriate bounds.

Table: Iterator {t, 2, T} does not have appropriate bounds.

General: Further output of Table::iterb will be suppressed during this calculation.

Here are the relative volatilities of consumption and output (from Aguiar and Gopinath, JPE 2009)

Table 2A: Volatility and Autocorrelation of Filtered Income and Growth Rates

	$\sigma(Y)$		$\sigma(\Delta Y)$		$\rho(Y_t, Y_{t-1})$		$\rho(\Delta Y_t, \Delta Y_{t-1})$	
Emerging Markets								
Argentina	3.68	(0.42)	2.28	(0.37)	0.85	(0.02)	0.61	(0.08)
Brazil	1.98	(0.20)	1.69	(0.33)	0.65	(0.04)	0.35	(0.15)
Ecuador	2.44	(0.52)	1.52	(0.38)	0.82	(0.05)	0.15	(0.14)
Israel	1.95	(0.14)	1.99	(0.17)	0.50	(0.10)	0.27	(0.05)
Korea	2.51	(0.46)	1.71	(0.27)	0.78	(0.08)	0.17	(0.19)
Malaysia	3.10	(0.65)	1.84	(0.37)	0.85	(0.02)	0.56	(0.16)
Mexico	2.48	(0.33)	1.53	(0.25)	0.82	(0.01)	0.27	(0.11)
Peru	3.68	(0.70)	2.97	(.50)	0.64	(0.11)	0.12	(0.10)
Philippines	3.00	(0.43)	1.66	(0.27)	0.87	(0.07)	0.17	(0.15)
Slovak Republic	1.24	(0.20)	1.06	(0.24)	0.66	(0.18)	0.20	(0.13)
South Africa	1.62	(0.16)	0.82	(0.11)	0.89	(0.06)	0.58	(0.06)
Thailand	4.35	(0.65)	2.25	(0.40)	0.89	(0.02)	0.42	(0.20)
Turkey	3.57	(0.41)	2.92	(0.36)	0.67	(0.06)	0.05	(0.13)
MEAN	**2.74**		**1.87**		**0.76**		**0.23**	

Table 2A: Volatility and Autocorrelation of Filtered Income and Growth Rates

	$\sigma(Y)$		$\sigma(\Delta Y)$		$\rho(Y_t, Y_{t-1})$		$\rho(\Delta Y_t, \Delta Y_{t-1})$	
Emerging Markets								
Argentina	3.68	(0.42)	2.28	(0.37)	0.85	(0.02)	0.61	(0.08)
Brazil	1.98	(0.20)	1.69	(0.33)	0.65	(0.04)	0.35	(0.15)
Ecuador	2.44	(0.52)	1.52	(0.38)	0.82	(0.05)	0.15	(0.14)
Israel	1.95	(0.14)	1.99	(0.17)	0.50	(0.10)	0.27	(0.05)
Korea	2.51	(0.46)	1.71	(0.27)	0.78	(0.08)	0.17	(0.19)
Malaysia	3.10	(0.65)	1.84	(0.37)	0.85	(0.02)	0.56	(0.16)
Mexico	2.48	(0.33)	1.53	(0.25)	0.82	(0.01)	0.27	(0.11)
Peru	3.68	(0.70)	2.97	(.50)	0.64	(0.11)	0.12	(0.10)
Philippines	3.00	(0.43)	1.66	(0.27)	0.87	(0.07)	0.17	(0.15)
Slovak Republic	1.24	(0.20)	1.06	(0.24)	0.66	(0.18)	0.20	(0.13)
South Africa	1.62	(0.16)	0.82	(0.11)	0.89	(0.06)	0.58	(0.06)
Thailand	4.35	(0.65)	2.25	(0.40)	0.89	(0.02)	0.42	(0.20)
Turkey	3.57	(0.41)	2.92	(0.36)	0.67	(0.06)	0.05	(0.13)
MEAN	**2.74**		**1.87**		**0.76**		**0.23**	

Developed Markets								
Australia	1.39	(0.21)	0.84	(0.10)	0.84	(0.04)	0.36	(0.10)
Austria	0.89	(0.09)	0.47	(0.00)	0.90	(0.08)	0.52	(0.09)
Belgium	1.02	(0.09)	0.71	(0.05)	0.79	(0.05)	0.18	(0.09)
Canada	1.64	(0.21)	0.81	(0.09)	0.91	(0.04)	0.55	(0.11)
Denmark	1.02	(0.16)	1.04	(0.09)	0.49	(0.14)	0.15	(0.11)
Finland	2.18	(0.39)	1.32	(0.11)	0.85	(0.09)	0.01	(0.20)
Netherlands	1.20	(0.13)	0.88	(0.09)	0.77	(0.07)	0.03	(0.08)
New Zealand	1.56	(0.20)	1.13	(0.14)	0.77	(0.10)	0.02	(0.13)
Norway	1.40	(0.10)	1.46	(0.13)	0.48	(0.11)	0.46	(0.10)
Portugal	1.34	(0.14)	1.03	(0.13)	0.72	(0.11)	0.28	(0.17)
Spain	1.11	(0.12)	0.75	(0.09)	0.82	(0.03)	0.08	(0.18)
Sweden	1.52	(0.20)	1.45	(0.32)	0.53	(0.21)	0.35	(0.11)
Switzerland	1.11	(0.13)	0.50	(0.04)	0.92	(0.05)	0.81	(0.04)
MEAN	**1.34**		**0.95**		**0.75**		**0.09**	

Note: The series for each country was deseasonalized if a significant seasonal component was identified. The income series were then logged and filtered using an HP filter with a smoothing parameter of 1600. For growth rates the unfiltered series was used. GMM estimated standard errors are reported in parenthesis. The standard deviations are reported in percentage terms.

Developed Markets								
Australia	1.39	(0.21)	0.84	(0.10)	0.84	(0.04)	0.36	(0.10)
Austria	0.89	(0.09)	0.47	(0.00)	0.90	(0.08)	0.52	(0.09)
Belgium	1.02	(0.09)	0.71	(0.05)	0.79	(0.05)	0.18	(0.09)
Canada	1.64	(0.21)	0.81	(0.09)	0.91	(0.04)	0.55	(0.11)
Denmark	1.02	(0.16)	1.04	(0.09)	0.49	(0.14)	0.15	(0.11)
Finland	2.18	(0.39)	1.32	(0.11)	0.85	(0.09)	0.01	(0.20)
Netherlands	1.20	(0.13)	0.88	(0.09)	0.77	(0.07)	0.03	(0.08)
New Zealand	1.56	(0.20)	1.13	(0.14)	0.77	(0.10)	0.02	(0.13)
Norway	1.40	(0.10)	1.46	(0.13)	0.48	(0.11)	0.46	(0.10)
Portugal	1.34	(0.14)	1.03	(0.13)	0.72	(0.11)	0.28	(0.17)
Spain	1.11	(0.12)	0.75	(0.09)	0.82	(0.03)	0.08	(0.18)
Sweden	1.52	(0.20)	1.45	(0.32)	0.53	(0.21)	0.35	(0.11)
Switzerland	1.11	(0.13)	0.50	(0.04)	0.92	(0.05)	0.81	(0.04)
MEAN	**1.34**		**0.95**		**0.75**		**0.09**	

Note: The series for each country was deseasonalized if a significant seasonal component was identified. The income series were then logged and filtered using an HP filter with a smoothing parameter of 1600. For growth rates the unfiltered series was used. GMM estimated standard errors are reported in parenthesis. The standard deviations are reported in percentage terms.

Table 2B: Relative Volatility of Consumption, Investment, and Net Exports

	σ(C)/σ(Y)		σ(I)/σ(Y)		σ(NX/Y)	
Emerging Markets						
Argentina	1.38	(0.07)	2.53	(0.01)	2.56	(0.67)
Brazil	2.01	(0.07)	3.08	(0.03)	2.61	(0.92)
Ecuador	2.39	(0.01)	5.56	(0.01)	5.68	(1.07)
Israel	1.60	(0.00)	3.42	(0.04)	2.12	(0.18)
Korea	1.23	(0.06)	2.50	(0.04)	2.32	(0.51)
Malaysia	1.70	(0.03)	4.82	(0.02)	5.30	(0.77)
Mexico	1.24	(0.05)	4.05	(0.02)	2.19	(0.32)
Peru	0.92	(0.08)	2.37	(0.01)	1.25	(0.15)
Philippines	0.62	(0.12)	4.66	(0.02)	3.21	(0.34)
Slovak Republic	2.04	(0.08)	7.77	(0.02)	4.29	(0.56)
South Africa	1.61	(0.08)	3.94	(0.03)	2.57	(0.50)
Thailand	1.09	(0.07)	3.49	(0.01)	4.58	(0.85)
Turkey	1.09	(0.06)	2.71	(0.03)	3.23	(0.40)
MEAN	**1.45**		**3.91**		**3.22**	

Table 2B: Relative Volatility of Consumption, Investment, and Net Exports

	σ(C)/σ(Y)		σ(I)/σ(Y)		σ(NX/Y)	
Emerging Markets						
Argentina	1.38	(0.07)	2.53	(0.01)	2.56	(0.67)
Brazil	2.01	(0.07)	3.08	(0.03)	2.61	(0.92)
Ecuador	2.39	(0.01)	5.56	(0.01)	5.68	(1.07)
Israel	1.60	(0.00)	3.42	(0.04)	2.12	(0.18)
Korea	1.23	(0.06)	2.50	(0.04)	2.32	(0.51)
Malaysia	1.70	(0.03)	4.82	(0.02)	5.30	(0.77)
Mexico	1.24	(0.05)	4.05	(0.02)	2.19	(0.32)
Peru	0.92	(0.08)	2.37	(0.01)	1.25	(0.15)
Philippines	0.62	(0.12)	4.66	(0.02)	3.21	(0.34)
Slovak Republic	2.04	(0.08)	7.77	(0.02)	4.29	(0.56)
South Africa	1.61	(0.08)	3.94	(0.03)	2.57	(0.50)
Thailand	1.09	(0.07)	3.49	(0.01)	4.58	(0.85)
Turkey	1.09	(0.06)	2.71	(0.03)	3.23	(0.40)
MEAN	**1.45**		**3.91**		**3.22**	

Developed Markets						
Australia	0.69	(0.00)	3.69	(0.03)	1.08	(0.12)
Austria	0.87	(0.14)	2.75	(0.04)	0.65	(0.04)
Belgium	0.81	(0.13)	3.72	(0.04)	0.91	(0.07)
Canada	0.77	(0.09)	2.63	(0.03)	0.91	(0.08)
Denmark	1.19	(0.10)	3.90	(0.02)	0.88	(0.14)
Finland	0.94	(0.07)	3.26	(0.02)	1.11	(0.10)
Netherlands	1.07	(0.09)	2.92	(0.03)	0.71	(0.09)
New Zealand	0.90	(0.10)	4.38	(0.02)	1.37	(0.18)
Norway	1.32	(0.12)	4.33	(0.03)	1.73	(0.19)
Portugal	1.02	(0.11)	2.88	(0.05)	1.16	(0.12)
Spain	1.11	(0.07)	3.70	(0.03)	0.86	(0.07)
Sweden	0.97	(0.14)	3.66	(0.04)	0.94	(0.09)
Switzerland	0.51	(0.31)	2.56	(0.05)	0.96	(0.09)
MEAN	**0.94**		**3.41**		**1.02**	

Note: The series for each country was deseasonalized if a significant seasonal component was identified. The series were then logged (except for TB/Y) and filtered using an HP filter with a smoothing parameter of 1600. GMM estimated standard errors are reported in parenthesis. The standard deviation of the ratio of net export to GDP are reported in percentage terms.

Developed Markets						
Australia	0.69	(0.00)	3.69	(0.03)	1.08	(0.12)
Austria	0.87	(0.14)	2.75	(0.04)	0.65	(0.04)
Belgium	0.81	(0.13)	3.72	(0.04)	0.91	(0.07)
Canada	0.77	(0.09)	2.63	(0.03)	0.91	(0.08)
Denmark	1.19	(0.10)	3.90	(0.02)	0.88	(0.14)
Finland	0.94	(0.07)	3.26	(0.02)	1.11	(0.10)
Netherlands	1.07	(0.09)	2.92	(0.03)	0.71	(0.09)
New Zealand	0.90	(0.10)	4.38	(0.02)	1.37	(0.18)
Norway	1.32	(0.12)	4.33	(0.03)	1.73	(0.19)
Portugal	1.02	(0.11)	2.88	(0.05)	1.16	(0.12)
Spain	1.11	(0.07)	3.70	(0.03)	0.86	(0.07)
Sweden	0.97	(0.14)	3.66	(0.04)	0.94	(0.09)
Switzerland	0.51	(0.31)	2.56	(0.05)	0.96	(0.09)
MEAN	**0.94**		**3.41**		**1.02**	

Note: The series for each country was deseasonalized if a significant seasonal component was identified. The series were then logged (except for TB/Y) and filtered using an HP filter with a smoothing parameter of 1600. GMM estimated standard errors are reported in parenthesis. The standard deviation of the ratio of net export to GDP are reported in percentage terms.

$$Y_t = e^{z_t} K_t^{1-\alpha} (\Gamma_t L_t)^\alpha, \tag{1}$$

where $\alpha \in (0,1)$ represents labor's share of output. The parameters z_t and Γ_t represent productivity processes. The two productivity processes are characterized by different stochastic properties. Specifically, z_t follows an AR(1) process

$$z_t = \rho_z z_{t-1} + \varepsilon_t^z \tag{2}$$

with $|\rho_z| < 1$, and ε_t^z represents *iid* draws from a normal distribution with zero mean and standard deviation σ_z.

The parameter Γ_t represents the cumulative product of "growth" shocks. In particular,

$$\begin{aligned} \Gamma_t &= g_t \Gamma_{t-1} = \prod_{s=0}^{t} g_s \\ \ln(g_t) &= (1-\rho_g)\log(\mu_g) + \rho_g \ln(g_{t-1}) + \varepsilon_t^g, \end{aligned}$$

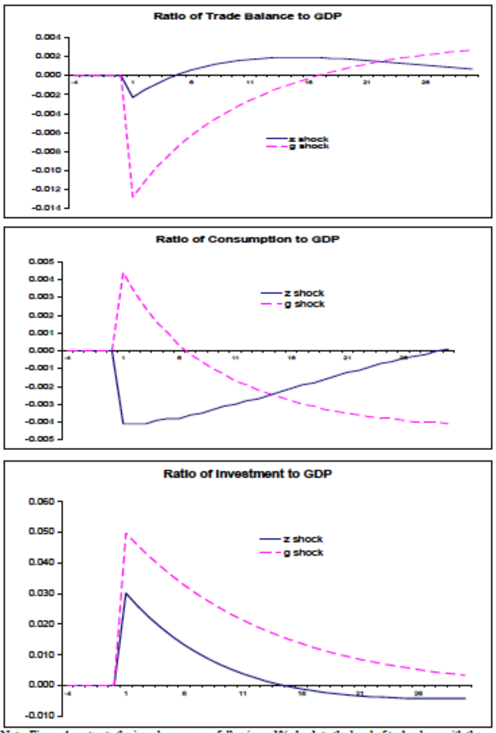
where $|\rho_g| < 1$ and ε_t^g represents *iid* draws from a normal distribution with zero mean and standard deviation σ_g. The term μ_g represents productivity's long run mean growth rate. We loosely refer to the realizations of g as the "growth" shocks as they constitute the stochastic trend of productivity. We use separate notation for shocks to the "level" of productivity (z_t) and the "growth" of productivity (g_t) to simplify exposition and calibration.[9]

Table 4: Estimated Parameters

		"Developed" (Canada)		"Emerging Market" (Mexico)	
		GHH	**Cobb Douglas**	**GHH**	**Cobb Douglas**
Mean Growth Rate	μ_g	1.007	1.007	1.006	1.005
		(0.001)	(0.001)	(0.001)	(0.001)
Volatility of z	σ_z	0.57	0.72	0.41	0.46
		(0.04)	(0.09)	(0.42)	(0.37)
Autocorrelation of z	ρ_z	0.88	0.96	0.94	0.94
		(0.08)	(0.02)	(0.29)	(0.13)
Volatility of g	σ_g	0.14	0.44	1.09	2.50
		(0.06)	(0.32)	(0.37)	(0.27)
Autocorrelation of g	ρ_g	0.94	0.50	0.72	0.06
		(0.04)	(0.26)	(0.08)	(0.04)
Adjustment Cost Parameter	ϕ	2.63	3.76	3.79	2.82
		(1.25)	(0.52)	(0.96)	(0.48)
Test of Model Fit (P-Value)		0.12	0.16	0.13	0.44

Note: GMM estimates with standard errors in parentheses. See text for details of estimation. Standard deviations are reported in percentage terms.

Figure 4: Impulse Responses from the Model



Note: Figure 4 contrasts the impulse response following a 1% shock to the level of technology with the impulse response to a 1% growth shock. The values plotted are deviations from steady state. The parameterization corresponds to the Emerging Market Parameterization using GHH preferences.

Cuberes and Jerzmanowski (EJ 2010)

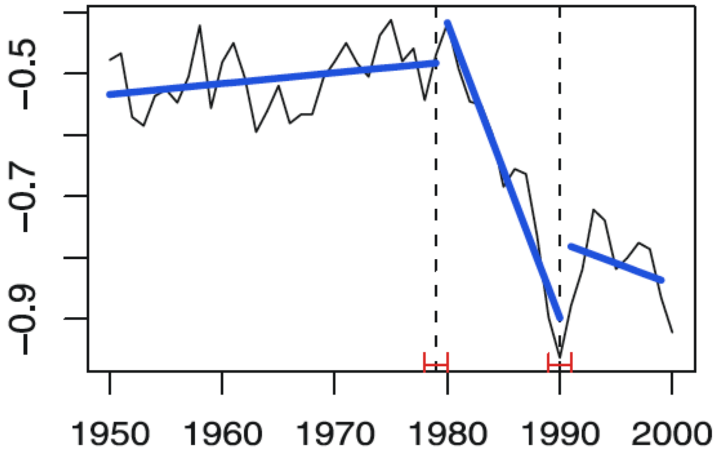


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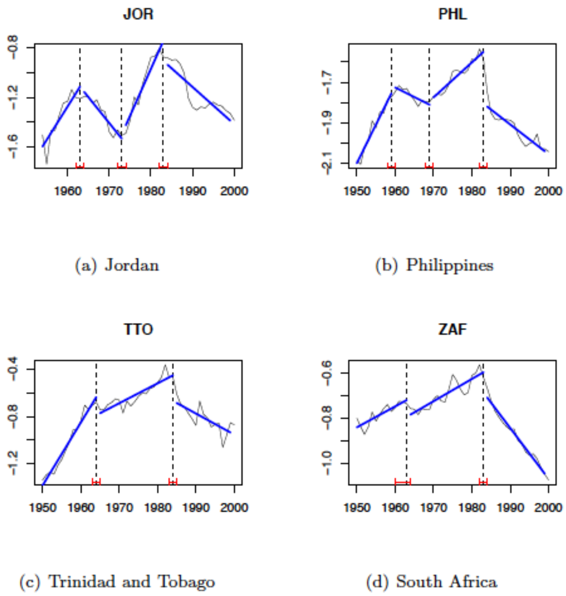
Figure 1: Path of Argentina's log real output per worker relative to the United States. Bai-Perron break dates are indicated by the vertical dashed lines. The solid lines show the estimate of the trend part of $y_t = \alpha_s + g_s t$. The figure also shows the confidence intervals around the estimated break dates.

Consider the following simple model

$$y_t = \alpha_s + g_s t + \varepsilon_t \text{ for } t_{s-1} < t \leq t_s, \forall t = 1, ...T \tag{1}$$

where y_t represents the logarithm of real output per worker relative to the United States, which is taken to be the technological leader. The variable t indexes time, and it is multiplied by the constant trend growth g_s. Finally, ε_t is a white noise error term.[9] That is, between two break dates t_{s-1} and t_s output per worker grows at a constant rate g_s relative to the technological frontier – which we assume is given by the U.S. Each time a break occurs there is a change in one or both of the parameters – the trend-growth rate and the intercept. We focus our attention of the former.



(a) Jordan

(b) Philippines

(c) Trinidad and Tobago

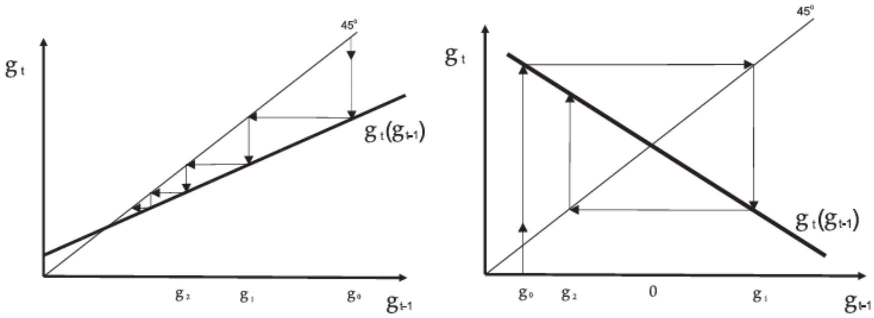
(d) South Africa

Figure 3: Examples of growth reversals.

Our first approach is to estimate the following regression

$$g_{is+1} = \beta_0 + \beta_1 g_{is} + \varepsilon_{is+1} \tag{2}$$

where g_{is} represents the growth rate in regime s for country i estimated from (1). We are interested in the coefficient β_1, i.e. the existence and direction of a relationship between pre-break and post-break growth rates. The basic idea of this approach is as follows. Depending on the value of the β_1 parameter we can have three interesting cases. First, if $\hat{\beta}_1 = 0$ then, on average, the growth rate before a break does not help predicting the growth rate after it; there is no memory across breaks. If $\hat{\beta}_1 \in (0, 1)$ then there is monotonic convergence in growth rates. This is a reversion-to-the-mean dynamics, i.e. exceptionally fast growers before the break still grow fast after the break, just slightly less so; in the long run there is convergence to the steady state. Figure 4(a) illustrates the dynamics of this system for the case where initial growth is above the long run equilibrium value. When interpreting the figure recall that "periods" here are not calendar years but break dates. Thus growth



(a) Monotonic Convergence

(b) Cycling

Figure 4: Panel (a): Monotonic convergence in growth rates ($\hat{\beta}_1 \in (0,1)$). Panel (b): Growth reversals ($\hat{\beta}_1 \in (-1,0)$).

$$g_{is+1} = \beta_0 + \beta_{11} g_{is} + \beta_{12}\, D_{is+1} g_{is} + \beta_{13} y_{is+1} g_{is} + \beta_2 y_{is+1} + \beta_3 D_{is+1} + \varepsilon_{is+1} \tag{3}$$

where y_{is+1} is the average of (log of) real per worker output, relative to the U.S. over the 5 year period prior to the break "into" growth regime $s+1$. We take these averages to smooth out any abnormal change in GDP in the year of the break. D_{is+1} is the log of our measure of democracy. This variable, obtained from the Polity IV database from Marshall and Jaggers (2002), records several regime characteristics for every independent state above half million total population. The measure we use in the analysis is *polity2* which is an average of the autocracy and the democracy score. It ranges from -10 to 10 (-10 = high autocracy; 10 = high democracy) and it includes specific indexes meant to capture constraints on the executive, the degree of political competition, effectiveness of legislature, etc. Here too, we take the average over five years prior to the break. Finally, ε_{is+1} is a white noise error term.

	Fixed Effects	Pooled OLS	GMM	System GMM Growth before Break
Growth before Break	-1.103***	-0.453	-1.157***	-0.992***
	(0.332)	(0.320)	(0.361)	(0.323)
Democracy × Growth Before Break	0.485*	0.573**	0.764*	1.014***
	(0.265)	(0.258)	(0.398)	(0.329)
Democracy × Income	-0.184	-0.050	-0.182	-0.154
	(0.194)	(0.080)	(0.177)	(0.138)
Initial Income	-0.014	0.002	-0.011	0.003
	(0.017)	(0.003)	(0.034)	(0.007)
Democracy	-0.028	0.000	-0.014	-0.005
	(0.019)	(0.009)	(0.016)	(0.014)
Hansen p-value			0.20	0.26
R^2	0.49	0.15		
N	197	197	95	197

Table 1: Magnitude of growth breaks; estimates of equation (3). Robust standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Adding Investment: Small Open Economy

So far-- in the infinite horizon setting -- we have been assuming that output is exogenous (i.e. given, not determined inside the model). We'll relax this assumption by assuming, as in the 2-period model, that output is produced using capital, labor and technology according to

$$Y_t = A_t F(K_t, L_t)$$
$$F_K > 0,\ F_L > 0$$
$$F_{KK} < 0,\ F_{LL} < 0$$

Again, we will soften make the following assumption

$$Y_t = A_t F(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha},\quad 0 \leqslant \alpha \leqslant 1.$$

First let's assume that two assets are available: riskless bonds (B) and physical capital (K). Riskless bonds pay a fixed real interest rate r; this is a small open economy assumption.

$$V(B_t, K_t) = \max_{\{C_t\}} \{u(C_t) + \beta V(B_{t+1}, K_{t+1})\}$$

$$\text{subject to}$$

$$B_{t+1} = (1+r) B_t + Y_t - C_t - I_t$$
$$K_{t+1} = (1-\delta) K_t + I_t$$

This problem has two control variables C and I and two state variables B and K. The first order conditions are

$$\text{FONC wrt C}: u'(C_t) + \beta V_B(B_{t+1}, K_{t+1}) \frac{dB_{t+1}}{dC_t} + \beta V_K(B_{t+1}, K_{t+1}) \frac{dK_{t+1}}{dC_t} = 0$$

$$\text{FONC wrt I}: 0 + \beta V_B(B_{t+1}, K_{t+1}) \frac{dB_{t+1}}{dI_t} + \beta V_K(B_{t+1}, K_{t+1}) \frac{dK_{t+1}}{dI_t}$$

Where we can use the facts that

$$\frac{dB_{t+1}}{dC_t} = -1$$

$$\frac{dK_{t+1}}{dC_t} = 0$$

$$\frac{dB_{t+1}}{dI_t} = -1$$

$$\frac{dK_{t+1}}{dI_t} = 1$$

to write the FONC's as:

$$u'(C_t) = \beta V_B(B_{t+1}, K_{t+1})$$

$$V_B(B_{t+1}, K_{t+1}) = V_K(B_{t+1}, K_{t+1})$$

Think about what these say! The first one should be familiar. What does the second one mean? Bonds and capital are perfect substitutes!

We also have the Envelope Theorem conditions

$$V_B(B_t, K_t) = \beta V_B(B_{t+1}, K_{t+1}) \frac{dB_{t+1}}{dB_t} + \beta V_K(B_{t+1}, K_{t+1}) \frac{dK_{t+1}}{dB_t}$$
$$V_K(B_t, K_t) = \beta V_B(B_{t+1}, K_{t+1}) \frac{dB_{t+1}}{dK_t} + \beta V_K(B_{t+1}, K_{t+1}) \frac{dK_{t+1}}{dK_t}$$

Where we can use the facts that

$$\frac{dB_{t+1}}{dB_t} = (1 + r)$$
$$\frac{dK_{t+1}}{dB_t} = 0$$
$$\frac{dB_{t+1}}{dK_t} = A_t F_K(K_t, L_t)$$
$$\frac{dK_{t+1}}{dK_t} = 1 - \delta$$

To write these as

$$V_B(B_t, K_t) = \beta V_B(B_{t+1}, K_{t+1})(1 + r)$$
$$V_K(B_t, K_t) = \beta V_B(B_{t+1}, K_{t+1}) A_t F_K(K_t, L_t) + \beta V_K(B_{t+1}, K_{t+1})(1 - \delta)$$

Let's collect all the equations we have so far (two FONC's and two Envelope Theorem conditions)

$$u'(C_t) = \beta V_B(B_{t+1}, K_{t+1}) \tag{14}$$

$$V_B(B_{t+1}, K_{t+1}) = V_K(B_{t+1}, K_{t+1}) \tag{15}$$

$$V_B(B_t, K_t) = \beta V_B(B_{t+1}, K_{t+1})(1+r) \tag{16}$$

$$V_K(B_t, K_t) = \beta V_B(B_{t+1}, K_{t+1}) A_t F_K(K_t, L_t) + \beta V_K(B_{t+1}, K_{t+1})(1-\delta) \tag{17}$$

First noticeEliminate that equations (13) and (15) are the exact same equations and in the model without investment, and the same transformation (forward them by one period, rearrange, etc.) can be used to get

$$u'(C_t) = (1+r)\beta u'(C_{t+1})$$

Next let us forward (16) by one period

$$V_K(B_{t+1}, K_{t+1}) = \beta V_B(B_{t+2}, K_{t+2}) A_{t+1} F_K(K_{t+1}, L_{t+1}) + \beta V_K(B_{t+2}, K_{t+2})(1-\delta)$$

Next observe that this together with (14) now implies

$$V_B(B_{t+1}, K_{t+1}) = \beta V_B(B_{t+2}, K_{t+2})(F_K(K_t, L_t) + 1 - \delta)$$

Now forward (15) one year and use it to eliminate $V_B(B_{t+1}, K_{t+1})$ to get

$$\beta V_B(B_{t+2}, K_{t+2})(1+r) = \beta V_B(B_{t+2}, K_{t+2})(F_K(K_t, L_t) + 1 - \delta)$$

$$1 + r = F_K(K_{t+1}, L_{t+1}) + 1 - \delta$$

Or simply

$$r = F_K(K_{t+1}, L_{t+1}) - \delta$$

Thus, the optimal solution involves two, independent conditions for I and C.

$$u'(C_t) = (1+r)\,\beta\, u'(C_{t+1})$$
$$r = F_K(K_{t+1}, L_{t+1}) - \delta$$

and the two budget constraints (or equations of motion) for B and K

$$B_{t+1} = (1+r)\,B_t + Y_t - C_t - I_t$$
$$K_{t+1} = (1-\delta)\,K_t + I_t$$

The key is that these conditions determine C and I independently of each other. This follows from some assumptions that went into our model but is a key result.

Example: let's assume

$$Y_t = A_t\,F(K_t, L_t) = A_t\,K_t^{\alpha}\,L_t^{1-\alpha}, \quad 0 \leqslant \alpha \leqslant 1.$$

In this case $F_K(K_{t+1}, L_{t+1}) = \alpha\,A_{t+1}\,K_{t+1}^{\alpha-1}\,L_{t+1}^{1-\alpha}$, or -- if we normalize L_t=1 for all t to simplify -- $F_K(K_{t+1}, L_{t+1}) = \alpha\,A_{t+1}\,K_{t+1}^{\alpha-1}$. With this, and $K_{t+1} = (1-\delta)\,K_t + I_t$, we can solve for I.

$$K_{t+1} = \left[\frac{\alpha\,A_{t+1}}{r+\delta}\right]^{\frac{1}{1-\alpha}}$$

$$I_t = \left[\frac{\alpha\,A_{t+1}}{r+\delta}\right]^{\frac{1}{1-\alpha}} - (1-\delta)\,K_t$$

Adding Investment: Closed Economy

As above we are assuming that output is produced using capital, labor and technology according to

$$Y_t = A_t F(K_t, L_t)$$
$$F_K > 0,\ F_L > 0$$
$$F_{KK} < 0,\ F_{LL} < 0$$

However, now let's assume that there is only one asset are available: physical capital (K). Technically, we are assuming that riskless borrowing does exist but the interest at which it happens is equal to the rental rate of capital. Since the economy is closed the aggregate supply of bonds must equal 0 all the time. This is a closed open economy assumption. Another way to state it, is simply to say $Y_t = C_t + I_t$. The problem facing a representative consumer is

$$V(K_t) = \max_{\{C_t\}} \{u(C_t) + \beta V(K_{t+1})\}$$
$$\text{subject to}$$
$$K_{t+1} = (1-\delta) K_t + Y_t - C_t$$

The FONC is

$$u'(C_t) + \beta V'(K_{t+1}) \frac{dK_{t+1}}{dC_t} = 0$$

or, since $\frac{dK_{t+1}}{dC_t} = -1$,

$$u'(C_t) = \beta V'(K_{t+1})$$

The Envelope Theorem condition is

$$V'(K_t) = \beta V'(K_{t+1}) \frac{dK_{t+1}}{dK_t}$$

And using the capital accumulation equation we know

$$\frac{dK_{t+1}}{dK_t} = F_K(K_t, L_t) + 1 - \delta$$

we have

$$V'(K_t) = \beta V'(K_{t+1}) (F_K(K_t, L_t) + 1 - \delta)$$

So we're down to these two equations:

$$u'(C_t) = \beta V'(K_{t+1}) \tag{18}$$

$$V'(K_t) = \beta V'(K_{t+1}) (F_K(K_t, L_t) + 1 - \delta) \tag{19}$$

Proceeding as before, forward equation (17) and (18) to get

$$u'(C_{t+1}) = \beta V'(K_{t+2})$$

$$V'(K_{t+1}) = \beta V'(K_{t+2}) (F_K(K_{t+1}, L_{t+1}) + 1 - \delta)$$

And the use (17) and (18) to substitute and arrive at

$$u'(C_t) = \beta u'(C_{t+1}) [F_K(K_{t+1}, L_{t+1}) + 1 - \delta]$$

The solution is therefore characterized by this first order condition and the equation of motion for K

$$u'(C_t) = \beta u'(C_{t+1})\,[F_K(K_{t+1}, L_{t+1}) + 1 - \delta]$$
$$K_{t+1} = (1-\delta) K_t + Y_t - C_t$$

OK, now what...

We'll conjecture that there is a steady state (sometimes called the Balanced Growth Path; BGP), where variables grow at constant rate. For this model, specifically, we'll conjecture that this rate is 0. This turns out to be right, since someone else did this before. But if we were doing it for the first time, it could turn out that our conjecture was wrong; such is life....

Here if our conjecture is true, along the BGP

$$C_t = C_{t+1} = \overline{C}$$
$$K_t = K_{t+1} = \overline{K}$$

If we plug this into the above equations

$$u'(\overline{C}) = \beta u'(\overline{C})\left[F_K(\overline{K}) + 1 - \delta\right]$$
$$\overline{K} = (1-\delta)\overline{K} + F(\overline{K}) - \overline{C}$$

Which re-arranges to

$$1 = \beta \left[F_K(\overline{K}) + 1 - \delta\right]$$
$$\overline{C} = F(\overline{K}) - \delta \overline{K}$$

Notice that since the interest rate must equal the MPK minus the rate of depreciation, i.e. $r = F_K(\overline{K}) - \delta$, we get $\beta(1+r) = 1$. We have been making this assumption previously.

So the questions is how does the economy get to this BGP. Or whether it does at all?
Let’s think about the dynamics of this problem. For this it will be helpful to assume a particular form of the utility function and the production function. Let’s go with u(C) = log(C) and $F(K) = K^{\alpha}$.
With this assumption, the Euler equation becomes

$$\frac{C_{t+1}}{C_t} = \beta\left[\alpha K_{t+1}^{\alpha-1} + 1 - \delta\right]$$

which is an equation relating the growth rate of C to K.
We can convert the other equation into a growth rate equation as in

$$K_{t+1} - K_t = K_t^{\alpha} - C_t - \delta K_t$$

These two equations tell us how C and K evolve jointly, in other words this is a dynamical system with two variables:

$$\frac{C_{t+1}}{C_t} = \beta\left[\alpha K_{t+1}^{\alpha-1} + 1 - \delta\right]$$
$$K_{t+1} - K_t = K_t^{\alpha} - C_t - \delta K_t$$

To study it, take the first equation and ask, when will C be constant (recall that is our conjectured long run equilibrium). The answer is that C will be constant (or $\frac{C_{t+1}}{C_t}$=1), when

$$1 = \beta\left[\alpha K^{\alpha-1} + 1 - \delta\right]$$

Which is just the definition of the BGP K from above

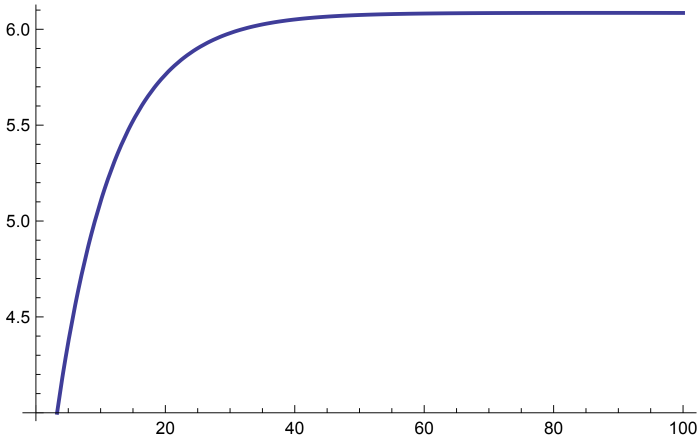
$$\frac{1}{\beta} - 1 + \delta = \alpha K^{\alpha-1}$$

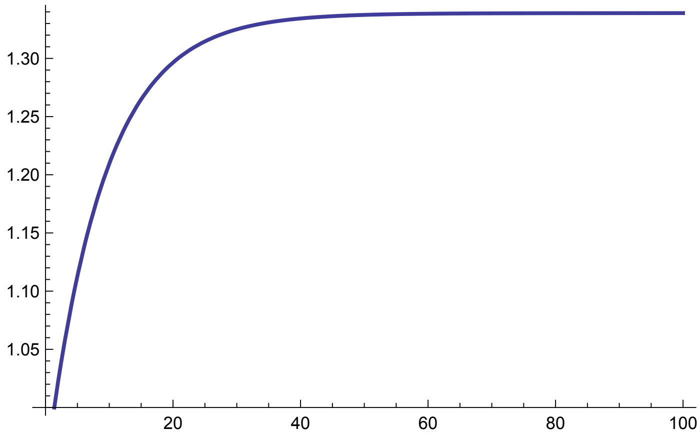
$$\overline{K} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta}\right)^{\frac{1}{1-\alpha}}$$

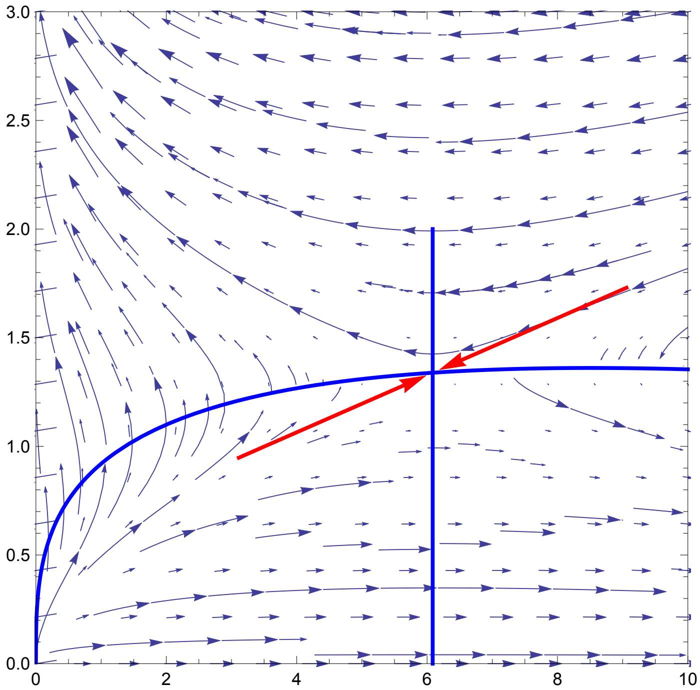
Now take the second equation and ask when will k be constant (or $K_{t+1} - K_t$ = 0).

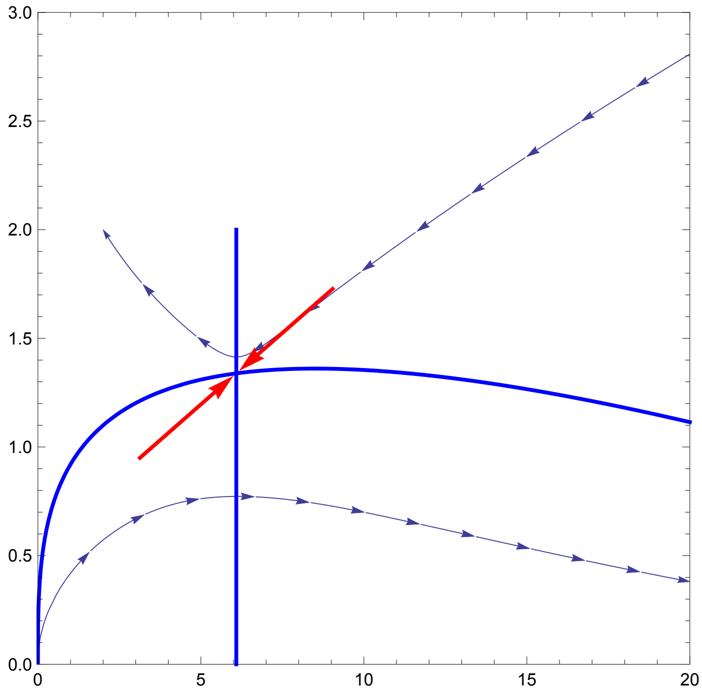
$$C = K^{\alpha} - \delta K$$

Numerical Examples





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20

Stochastic Model with Investment

We'll now add investment to the stochastic model. Let's start with the incomplete markets stochastic version of the problem: the only asset we'll consider is a riskless bond. There are no Arrow-Debreu securities for now.

This is a fundamental equation of Dynamic Programming; it's called the Bellman Equation

$$V(B_t, K_t) = \max_{\{C_t, I_t\}} \{u(C_t) + \beta E_t[V(B_{t+1}, K_{t+1})]\}$$

$$\text{subject to}$$

$$B_{t+1} = (1+r) B_t + Y_t - C_t - I_t$$

$$K_{t+1} = (1-\delta) K_t + I_t$$

$$Y_t = F(K_t) = A_t K_t^{\alpha}$$

The FONC's with respect to C and I respectively are:

$$u'(C_t) = (1+r)\beta E_t[V_B(B_{t+1}, K_{t+1})] \tag{20}$$

$$E_t[V_B(B_{t+1}, K_{t+1})] = E_t[V_K(B_{t+1}, K_{t+1})] \tag{21}$$

We've seen equation (3) before. Think about what equation (4) says: K and B are perfect substitutes in terms of transferring resources across periods. If I increase I_t by one unit I will decrease B_{t+1} by one unit but increase K_{t+1} by one unit. It must be the case that the benefits of increasing B and K are the same, which is what (4) states.

The envelope theorem further gives us:

$$V_B(B_t, K_t) = (1+r)\,\beta\,E_t[V_B(B_{t+1}, K_{t+1})] \tag{22}$$

$$V_K(B_t, K_t) = (1-\delta)\,\beta\,E_t[V_K(B_{t+1}, K_{t+1})] + \beta\,E_t[V_B(B_{t+1}, K_{t+1})]\,F'(K_t) \tag{23}$$

Using equations (3) and (5) we get our good old stochastic Euler equation:

$$\boxed{u'(C_t) = (1+r)\,\beta\,E_t[u'(C_{t+1})]}$$

Using (4) and (5) we can re-write (6) a s

$$\begin{aligned} V_K(B_t, K_t) &= (1-\delta)\,\beta\,E_t[V_B(B_{t+1}, K_{t+1})] + \beta\,E_t[V_B(B_{t+1}, K_{t+1})]\,F'(K_t) \\ &= \beta\,(1-\delta+F'(K_t))\,E_t[V_B(B_{t+1}, K_{t+1})] \end{aligned} \tag{24}$$

Use equation (3) to write

$$V_K(B_t, K_t) = (1-\delta+F'(K_t))\,\frac{u'(C_t)}{1+r}$$

Forward the last equation by one period and take expectations as of period t

$$E_t[V_K(B_{t+1}, K_{t+1})] = E_t\left[(1-\delta+F'(K_{t+1}))\frac{u'(C_{t+1})}{1+r}\right]$$

And make use of (4) to get

$$E_t[V_B(B_{t+1}, K_{t+1})] = E_t\left[(1-\delta+F'(K_{t+1}))\frac{u'(C_{t+1})}{1+r}\right]$$

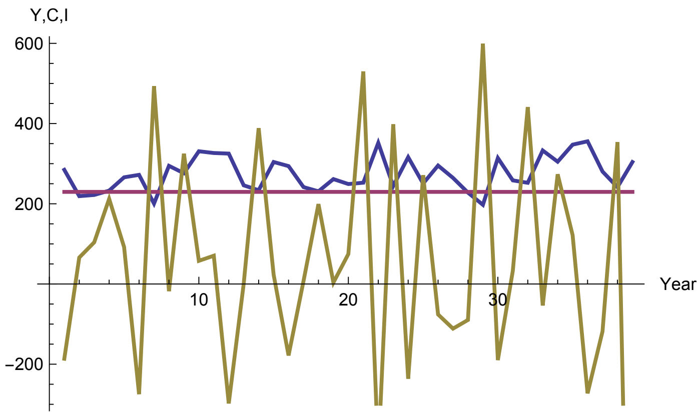
$$\boxed{1 = E_t\left[(1-\delta+F'(K_{t+1}))\frac{\beta u'(C_{t+1})}{u'(C_t)}\right]} \qquad (25)$$

Note that in the deterministic case (without uncertainty) this implies

$$\boxed{\begin{aligned} u'(C_t) &= (1+r)\beta u'(C_{t+1}) \\ F'(K_{t+1})-\delta &= r \end{aligned}}$$

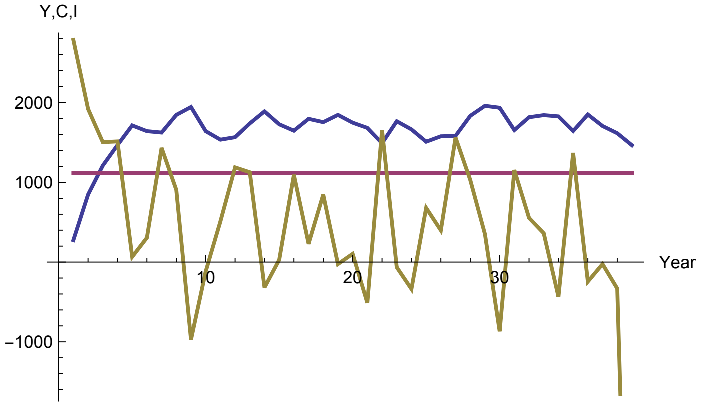
Which are the familiar FONC's from our 2 period model without uncertainty.

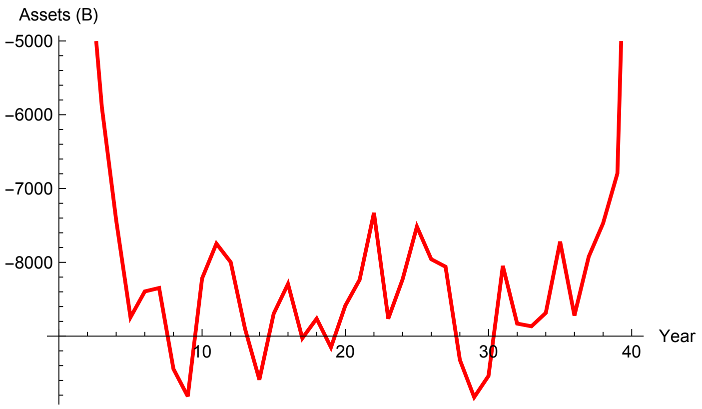
Here is an example of the deterministic case. Assuming $\beta(1+r)=1$ and letting A follow a random path (ranmom in the sense that it varies over time but it is know in advance by the agent; this si teh determinsitic case) we get: (blue line: Y, purple: C, yellow: I)

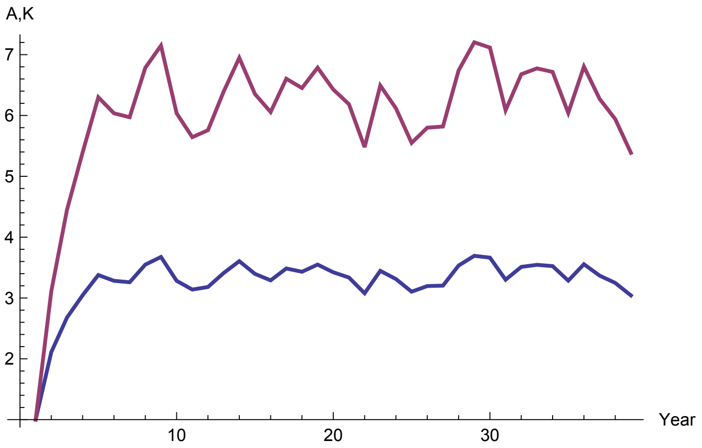


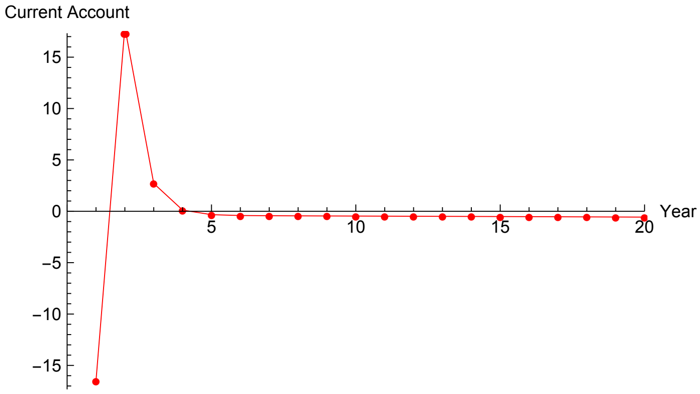
Here is another example, this time assuming

$$A_{t+1} - \overline{A} = \rho \left(A_t - \overline{A}\right) + \epsilon_{t+1}$$





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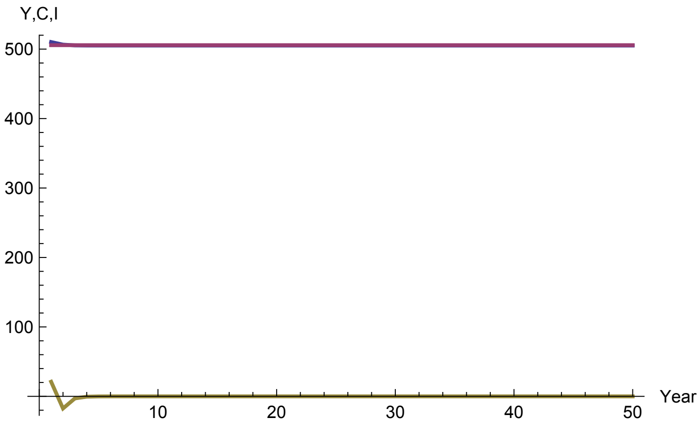
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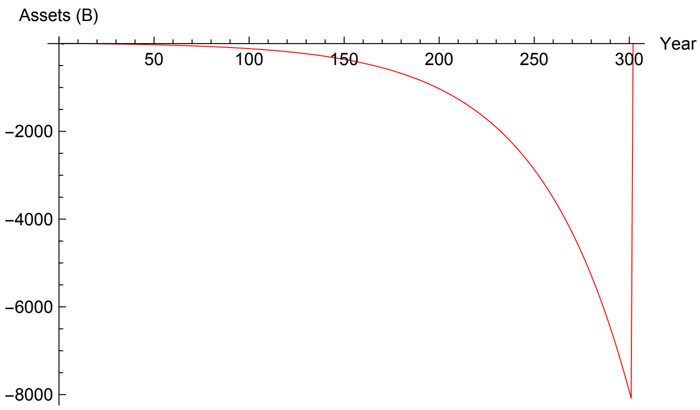
Assets (B)
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200
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300
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−4000
−6000
−8000

Small Open Economy Model with Uncertainty

So far we’ve considered certain environment -- for example future (period 2) income and the interest rate were both know with certainty. But the real world is not like that. We don’t know what our future income will be (people who purchased Lehman Bros. in summer of 2008 year didn’t know what rate of return they would be getting (but they were probably hoping it would be slightly more than the realized -100%).

Aside from being more realistic, a model with uncertainty will allow us to talk about assets and their prices, insurance, risk sharing and pooling...all the stuff that makes modern finance (and modern economies) go...or go bust. Also the consequences of changes in the level of risk (due to say...a major meltdown in credit markets), etc.

We have seen so far that the concavity of the utility function (intertemporal elasticity of substitution) lead to a preferences for consumption smoothing. Concavity of u(.) will turn out to play a role for choice under uncertainty. But let’s start with some basics

Introduction to Choice Under Uncertainty

Suppose the Bernoulli Fairy comes down from the sky and offers you a choice: she can flip a coin and with heads your lifetime annual income will be \$50,000 while with tails your lifetime annual income will be \$200,000, or she can eliminate the risk and give you a lifetime annual income of \$X. What, then, is the highest annual income at which you would still gamble?

Bernoulli Fairy Encounter

For most people the tipping point for $X = $100,000 or so: each doubling seeming equally worthwhile, and so a 50% chance of loosing half your lifetime income being worth risking only if it comes with a 50% chance of a double. Some people appear to be more risk-averse: you can find people for whom $X is as low as $68,000--although they are rare.

But if you take a look at financial market returns--stocks, junk bonds, corporate bonds, and Treasuries--over the past century, and you ask what is the value of $X implicit in the risk premia that financial markets have yielded, you get an answer more like $55,000: the market acts as though prices are set by people--as if the market is ruled by “representative agents” of the kind who appear in economists’ models and theories--who are indifferent between (a) a $55,000 annual lifetime income with certainty, and (b) a 50-50 gamble between a lifetime annual income of $50,000 and one of $200,000 (see Mehra and Prescott (1985), “The Equity-Premium: A Puzzle,” Journal of Monetary Economics 15, 145-62; Rabin and Thaler (2001), “Anomalies: Risk Aversion” http://www.behaviouralfinance.net/risk/RaTh01.pdf; DeLong and Magin (2008), “The Equity Premium: Past, Present, and Future” http://delong.typepad.com/berkeley_pe_notes/2008/09/2008-9-1-delong.html; and millions of others).

St. Petersburg Paradox

Imagine your offered the following game:
You toss a fair coin and you win \$1 if it comes up tails on the first try, if not you toss again and the reward is doubled, that is you win \$2 if you toss HT; this continues so if you toss HHT you win \$4, if you toss HHHT you win \$8 and so on...

How much would you pay top play?

What is the expected reward?

$$E(R) = \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 4 + \frac{1}{16} \times 8 + \ldots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \sum_{k=0}^{\infty} \frac{1}{2} = \infty$$

That’s a lot of money!
Would you be willing to pay any amount to play?

Choice under Uncertainty (review)

How should we model behavior under uncertainty? What are people's preferences over random outcomes? The umbrella seller above, what is his utility in this situation?

The following make sense -- (a) we probably don't like uncertainty for its own sake and (b) we're willing to put up with some risk to get a nice reward (Arrow).

Suppose, for example, that I know that there is a 50% chance that my consumption will be $100 and a 50% chance that my consumption will be $200. How do we calculate my expected utility?
There are two ways that you might consider doing it: could take the expected value of my utilities, or the utility of my expected consumption.

$$V = 1/2\ U(100) + 1/2\ U(200)$$

or

$$V = U\ (1/2*100 + 1/2*200)$$

The first of these methods of calculating utility from an probabilistic situation is called <u>Von Neumann - Mortgenstern (VNM) utility</u>. This is the approach that we always use. The second method is called wrong.

Expected Utility

More generally, EUT proved to be a very tractable, although with many shortcomings, theory of utility under uncertainty. It says

$V = p_1\, u(X_1) + p_2(X_2) + p_3\, u(X_3) + \ldots + p_N\, u(X_N)$,

where U is the v.N-M utility function and u(.) is a Bernoulli utility.

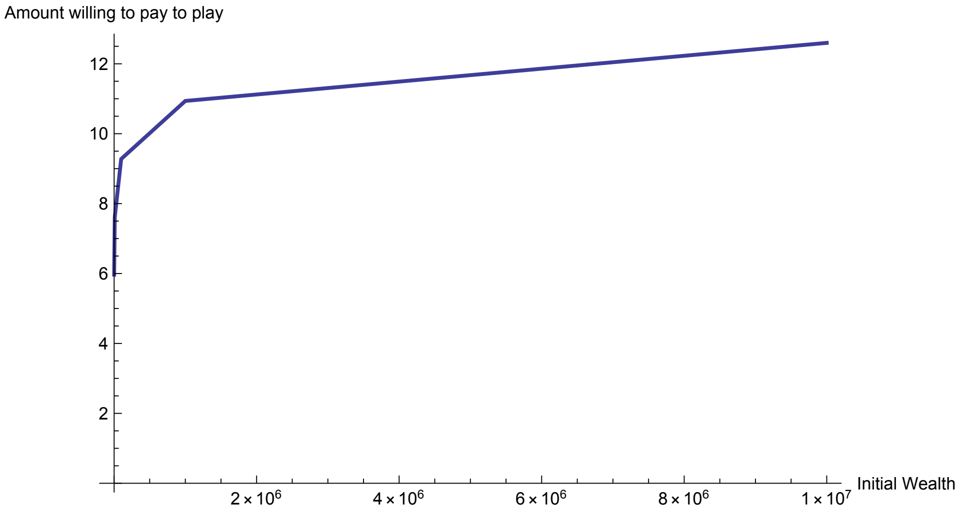
How do we know the VNM utility is the right way to think about utility when there are different possible states of the world? Here is a simple demonstration: Suppose that you can have either $150 with certainty, or a lottery where you have a chance of getting either $100 or $200, each with a probability of .5. Which would you prefer? Almost everyone would say they prefer the certain allocation. This is a simple example of risk aversion. But notice that if we chose the second technique for adding up utility across states of the world, we would say that you should be indifferent.

The fact that uncertainty lowers your utility is called risk aversion. Notice that risk aversion is a direct implication of the utility function being curved. (The mathematical rule that shows this is called Jensen's inequality: if U is concave, then **U(E(C)) > E(U(C)),** where E is the expectation operator.) If the utility function were a straight line then the utility of $150 with certainty would be the same as the utility of a lottery with equal chances of getting $100 and $200. A person who indeed gets equal utility from these two situations is called risk neutral.

What are the consequences of risk aversion? Clearly this is the motivation for things like insurance, etc. Similarly, this is why in financial theory we say that people trade off risk and return: to accept more risk, an investor has to be promised a higher expiated return.

Back to St. Petersburg

What is the answer to St. Petersburg paradox under Expected Utility?



CRRA vs. CARA

Am example of CARA utility function is u(x) = $-\frac{1}{\alpha} e^{-\alpha x}$. The Arrow-Pratt coefficient of ARA is -u''(x)/u'(x) (it measures the curvature of u(.)). Here it is equal to α. Decreasing absolute risk aversion means willing to take on more risk with higher wealth -- this is plausible so we like to assume that absolute risk aversion is not constant.

Relative risk aversion is useful because it allows us to talk about proportional gains or losses as opposed to absolute. The coefficient of RRA is - x u''(x)/u(x). CRAR (sometimes?) implies decreasing absolute risk aversion.

The difference between relative risk aversion and absolute risk aversion can be thought of this way.

Suppose that I am willing to pay 10 to avoid the uncertainty of a lottery that gives me either 150 or 50 each with probability 50% that is, I consider certain consumption of 90 to have utility equal to the lottery.

If utility is CRRA, then I will also be willing to pay 100 to avoid a lottery of 1500 or 500. If utility is CARA, then I will be willing to pay 10 to avoid a lottery of 1050 or 950.

CRRA vs. CARA

Note that with CARA I will be willing to more than 100 to avoid a lottery of 1500 or 500 this should be obvious for the following reason: the larger is uncertainty, the more (at the margin) you are willing to pay to avoid it. So if a CARA consumer with expected income of 1000 will pay 10 to avoid 50 worth of uncertainty, he will pay more than 100 to avoid 500 worth of uncertainty.

To see this consider:

$-.5\ \mathrm{Exp}[-\alpha\ (X+z)] - .5\ \mathrm{Ex}[-\alpha(X - z)] = -\ \mathrm{Exp}[-\alpha(X - \mu)]$

implies

$-.5\ \mathrm{Exp}[-\alpha\ z] - .5\ \mathrm{Ex}[\alpha z] = -\ \mathrm{Exp}[\ \alpha\ \mu]$ (**)

so if I increase the risk 10 times is $\mu' = 10\ \mu$?

$-.5\ \mathrm{Exp}[-\alpha\ 10\ z] - .5\ \mathrm{Ex}[\alpha\ 10z] = -\ \mathrm{Exp}[\ \alpha\ \mu']$

from (**) above we have

$-.5\ \mathrm{Exp}[-\alpha'\ 10\ z] - .5\ \mathrm{Ex}[\alpha'\ 10z] = -\ \mathrm{Exp}[\ \alpha'\ 10\ \mu]$

where $\alpha' = \alpha/10$

So a person with risk aversion of one-tenth of α would be willing to pay 10μ to avoid the pet of 10z and -10z....it follows that a person as risk averse as α would be willing to pay much more than 10μ!

Expected Utility

Often it is enough to think of a world with just two states to get the essential insights. this is what we will do for the most part. With two states of the world we have

$$V = p_1 u(X_1) + p_2 u(X_2) = p_1 u(X_1) + (1 - p_1) u(X_2)$$

Our choice of the Bernoulli utility function will be

$$u(C) = \frac{C^{1-\rho}}{1-\rho}$$

This function is called a CRRA utility function. What does that mean?

ρ is the coefficient of relative risk aversion -- it measures the curvature of the utility function (it also measures how willing a person is to substitute intertemporally in response to changes in the interest rate. These two -- risk aversion and IES -- are intimately tied with this utility function. This is probably not so good.

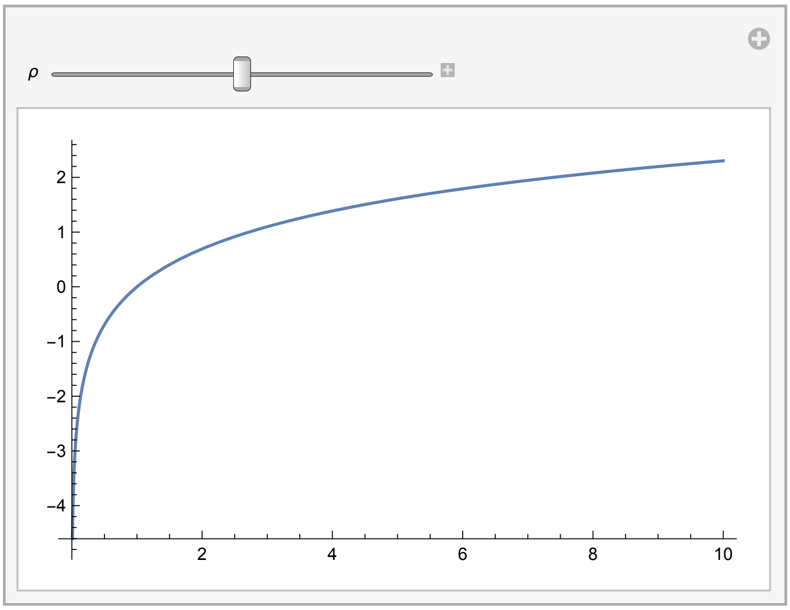
The Relation Between Risk Aversion and Consumption Smoothing

Now we get to the really big idea: risk aversion and consumption smoothing are really two sides of the same coin: they are both results of the curvature of the utility function. If the utility function were linear (and so the marginal utility of consumption constant) then people would not care about smoothing consumption, and their expected utility would not be lowered by risk. This will be important for many reasons: among them is that even when we are talking about a world with no uncertainty, we will often use the idea of risk aversion to measure the curvature of the
One can come up with many instances of risk neutrality or even risk-loving (i.e. more uncertainty raises utility) behavior, such as participating in lotteries, flipping a coin with your friend for who will buy coffee, etc. However, it is unlikely that these exceptions tell us much about the vast majority of consumption decisions. utility function.

The greater the ρ, the more curved the utility function. For $\rho = 0$ the utility is linear, for $\rho = 1$ it is log-utility and for $\rho \to \infty$ we get a Leontieff utility functions (can you show this?)

Example: $u(C) = \frac{C^{1-\rho}}{1-\rho}$

```
Manipulate[Plot[If[ρ == 1, Log[c], c^(1 - ρ) / (1 - ρ)], {c, 0.01, 10},
  AxesOrigin → {0, If[ρ == 1, Log[c], c^(1 - ρ) / (1 - ρ)] /. c → 0.01}],
 {ρ, 0, 2, .1}, TrackedSymbols → {ρ}]
```



Back to the Bernoulli fairy Encounter

The thick black segment shows the amount a person is willing to pay, in order to get the expected value ($125,000) of the Fairy's bet instead of the gamble itself.

```
π1 := .5;
x1 := 50;
x2 := 200;
xmin = 10;
u[c_, ρ_] := If[ρ == 1, Log[c], c^(1 - ρ) / (1 - ρ)];
Manipulate[Show[Plot[u[c, ρ], {c, xmin, 250}, AxesOrigin → {xmin, u[xmin, ρ]}],
  Graphics[Line[{{x1, u[x1, ρ]}, {x2, u[x2, ρ]}}]],
  Graphics[{Dashed, Red, Line[{{xmin, π1 u[x1, ρ] + (1 - π1) u[x2, ρ]},
      {π1 x1 + (1 - π1) x2, π1 u[x1, ρ] + (1 - π1) u[x2, ρ]}}]}],
  Graphics[{Dashed, Blue, Line[{{xmin, u[π1 x1 + (1 - π1) x2, ρ]},
      {π1 x1 + (1 - π1) x2, u[π1 x1 + (1 - π1) x2, ρ]}}]}],
  Graphics[{Dashed, Green, Line[{{π1 x1 + (1 - π1) x2, u[xmin, ρ]},
      {π1 x1 + (1 - π1) x2, u[π1 x1 + (1 - π1) x2, ρ]}}]}],
  Graphics[{Thick, Line[{{π1 x1 + (1 - π1) x2, π1 u[x1, ρ] + (1 - π1) u[x2, ρ]},
      {FindRoot[0 == π1 u[x1, ρ] + (1 - π1) u[x2, ρ] - u[x, ρ] /. ρ → 1, {x, 5}][[1,
        2]], u[FindRoot[0 == π1 u[x1, ρ] + (1 - π1) u[x2, ρ] - u[x, ρ] /. ρ → 1,
          {x, 5}][[1, 2]], ρ]}}]}],
  Graphics[{Dashed, Green, Line[{{FindRoot[0 == π1 u[x1, ρ] +
             (1 - π1) u[x2, ρ] - u[x, ρ] /. ρ → 1, {x, 5}][[1, 2]], u[FindRoot[
          0 == π1 u[x1, ρ] + (1 - π1) u[x2, ρ] - u[x, ρ] /. ρ → 1, {x, 5}][[1, 2]], ρ]},
      {FindRoot[0 == π1 u[x1, ρ] + (1 - π1) u[x2, ρ] - u[x, ρ] /. ρ → 1, {x, 5}][[
        1, 2]], u[xmin, ρ]}}]}]
 ],
 {ρ,
  0,
  3,
  .2}]
```

ρ

```
Show[Plot[u[c, FE`ρ$$6], {c, xmin, 250},
  AxesOrigin → {xmin, u[xmin, FE`ρ$$6]}],  ,
  ,  ,  ,  ,  ]
```

Plot::plln : Limiting value xmin in {c, xmin, 250} is not a machine-sized real number. ≫

FindRoot::nlnum : The function value {u[5., 1.] − 1. π1 u[x1, 1.] − 1. (1. − 1. π1) u[x2, 1.]} is not a list of numbers with dimensions {1} at {x} = {5.}. ≫

FindRoot::nlnum : The function value {u[5., 1.] − 1. π1 u[x1, 1.] − 1. (1. − 1. π1) u[x2, 1.]} is not a list of numbers with dimensions {1} at {x} = {5.}. ≫

FindRoot::nlnum : The function value {u[5., 1.] − 1. π1 u[x1, 1.] − 1. (1. − 1. π1) u[x2, 1.]} is not a list of numbers with dimensions {1} at {x} = {5.}. ≫

General::stop : Further output of FindRoot::nlnum will be suppressed during this calculation. ≫

Show::gcomb : Could not combine the graphics objects in

Show[Plot[u[c, FE`ρ$$105], {c, xmin, 250}, AxesOrigin → {xmin, u[xmin, FE`ρ$$105]}], ,

Plot::plln : Limiting value xmin in {c, xmin, 250} is not a machine-sized real number. ≫

FindRoot::nlnum : The function value {u[5., 1.] − 1. π1 u[x1, 1.] − 1. (1. − 1. π1) u[x2, 1.]} is not a list of numbers with dimensions {1} at {x} = {5.}. ≫

FindRoot::nlnum : The function value {u[5., 1.] − 1. π1 u[x1, 1.] − 1. (1. − 1. π1) u[x2, 1.]} is not a list of numbers with dimensions {1} at {x} = {5.}. ≫

FindRoot::nlnum : The function value {u[5., 1.] − 1. π1 u[x1, 1.] − 1. (1. − 1. π1) u[x2, 1.]} is not a list of numbers with dimensions {1} at {x} = {5.}. ≫

General::stop : Further output of FindRoot::nlnum will be suppressed during this calculation. ≫

Show::gcomb : Could not combine the graphics objects in

Show[Plot[u[c, FE`ρ$$78], {c, xmin, 250}, AxesOrigin → {xmin, u[xmin, FE`ρ$$78]}], ,

... Plot: Limiting value xmin in {c, xmin, 250} is not a machine-sized real number.

... FindRoot: The function value {u[5., 1.] − 1. π1 u[x1, 1.] − 1. (1. − 1. π1) u[x2, 1.]} is not a list of numbers with dimensions {1} at {x} = {5.}.

FindRoot: The function value {u[5., 1.] − 1. π1 u[x1, 1.] − 1. (1. − 1. π1) u[x2, 1.]} is not a list of numbers with dimensions {1} at {x} = {5.}.

FindRoot: The function value {u[5., 1.] − 1. π1 u[x1, 1.] − 1. (1. − 1. π1) u[x2, 1.]} is not a list of numbers with dimensions {1} at {x} = {5.}.

General: Further output of FindRoot::nlnum will be suppressed during this calculation.

Show: Could not combine the graphics objects in

Show[Plot[u[c, FE`ρ$$6], {c, xmin, 250}, AxesOrigin → {xmin, u[xmin, FE`ρ$$6]}], , ,

Diversification: An Illustration

This is a text cell.

a

	A	A	A	A	B	B
	Labor	Capital	GNI	GDP	Labor	Capital
% Home K owned by Foreign	1.25	1.25	1.25	1.25	1.25	1.25
State 1	54.	46.	100.	90	66.	34.
State 2	66.	34.	100.	110	54.	46.
Variabilty	6.	−6.	0.	10.	−6.	6.

The Model: Assumptions

1. Small Open Economy
2. Two Periods
3. Two States of the world
4. Output in year 2 is uncertain
5. Expected Utility:

$$U = \pi(1)\, u(x(1)) + \pi(2)\, u(x(2)) + \ldots = \sum_{s=1}^{S} \pi(s)\, u(x(s))$$

Assumptions cont.

6. Output is exogenous: in year one it is given by Y_1, and in period 2 it is given by $Y_2(s)$, where s represents the state fo the world (for now $s = 1, 2$).
7. The states of the world occur with known probabilities $\pi(s)$

$$\pi(1) + \pi(2) = 1$$

Lifetime utility is given by $U = u(C_1) + \beta[\pi(1)u(C_2(1)) + \pi(2)u(C_2(2))]$

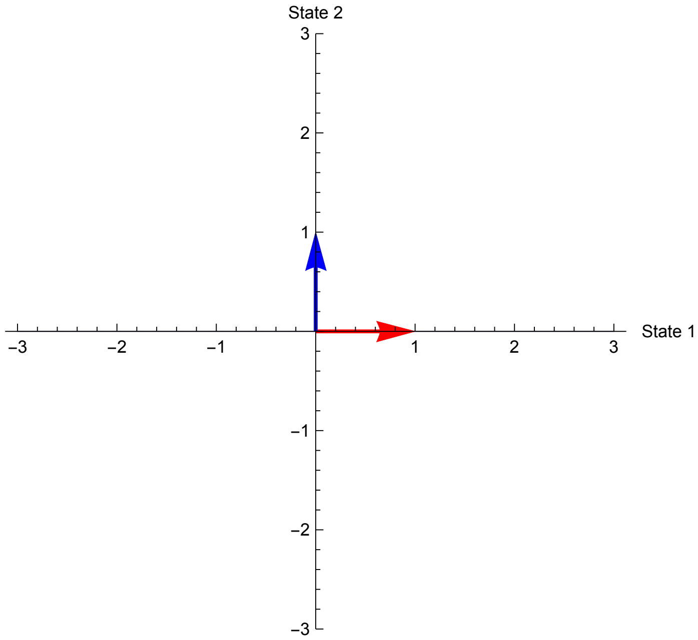
8. Markets are complete: i.e. exists the following securities for <u>each state s</u>:
    1. The security pays 1 unit of output if state s occurs
    2. Security pays nothing otherwise

These are called Arrow-Debreu securities.

The price of a Arrow-Debreu security (in terms of period 1 consumption) is $\frac{p(s)}{1+r}$

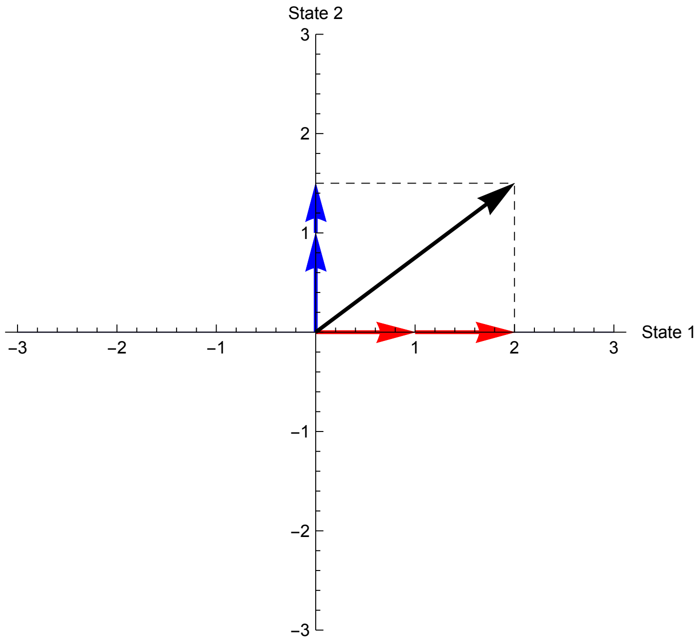
Arrow-Debreu Securities

If we have an Arrow-Debreu security for each state of the world then we say that “markets are complete”. What does that mean?



Arrow-Debreu Securities

Suppose you want to create a synthetic security that pays off 2 units of output in state 1 and 1.5 units in state 2. With complete markets you can create any such security. In this case you have to combine two state 1 Arrow-Debreu securities and 1.5 state 2 Arrow-Debreu securities. This would cost 2 p(1)/(1+r) + 1.5 p(2)/(1+r)



Riskless bonds

In fact you can create a riskless bond using A-D securities: if you buy (1+r) of state 1 A-D securities and (1+r) of state 2 A-D securities your payoff in year 2 will be (1+r) for certain! This means bonds are redundant when thinking of asset allocation.
So suppose that riskless bonds exist; what does that imply for the prices of A-D securities? Well, since the bond costs 1 in terms of period 1 consumption, no-arbitrage requires that:

$$(1+r)\frac{p(1)}{1+r} + (1+r)\frac{p(2)}{1+r} = 1$$

$$p(1) + p(2) = 1$$

Numerical Example of No Arbitrage

Suppose the above condition does not hold, i.e.. $p(1) + p(2) \neq 1$.
Lets start by assuming $p(1) + p(2) < 1$. In this case it is cheaper to move resources risklessly between periods using Arrow-Debreu securities. Suppose you buy (1+r) of each state 1 and 2 AD securities. This will cost you p(1) + p(2) and you will receive a certain payoff of (1+r). To buy them you simply borrow (issue a riskless bond) in the amount p(1) + p(2). In year 2 your net return is your payoff (1+r) minus payment to your creditors (i.e. interest and principal on the bond you sold): (1+r)(p(1) +p(2)).

$$\text{Return} = (1+r) - (1+r)\,(p(1) + p(2)) = (1+r)\,(1-(p(1) + p(2))) > 0$$

where the last inequality follows from $p(1) + p(2) < 1$. Now imagine that you scale this up, but (1+r)X of each AD security and borrow X(p(1) +p(2)). Your return will go up by a factor of X! You can make it as big as possible by making X as big as possible. This is called arbitrage. If everyone tries to do this however, they will simply bid up the prices of the AD securities until p(1) + p(2)=1.

For example if p(1) = 0.4 and p(2) =.2, and r =.05, then the return would be

```
(1 + .05) (1 - (.4 + .2))
0.42
```

How about the case when $p(1) + p(2) > 1$. How would you arbitrage in this case and how would that affect p(1) and p(2)?

Budget Constraints

Let $B_2(s)$ be the net purchase of state s A-D securities (with price of $\frac{p(s)}{1+r}$).

Given the SOE assumption, the period 1 budget constraint is

$$\frac{p(1)}{1+r} B_2(1) + \frac{p(2)}{1+r} B_2(2) = Y_1 - C_1$$

The period 2 budget constraints are

$$C_2(1) = Y_2(1) + B_2(1)$$
$$C_2(2) = Y_2(2) + B_2(2)$$

Combining them we get the intertemporal budget constraint

$$C_1 + \frac{p(1)\, C_2(1) + p(2)\, C_2(2)}{1+r} = Y_1 + \frac{p(1)\, Y_2(1) + p(2)\, Y_2(2)}{1+r}$$

Smoothing Across States

Notice that there is a symmetry between this set-up and the two-period deterministic economy. There trading in assets (bonds) allowed a country to smooth consumption across time. Here, in addition to smoothing across time, the country can smooth across states.
In particular, a country can fully insure by:

1. selling $Y_2(1)$ for $\frac{p(1)}{1+r} Y_2(1)$
2. selling $Y_2(2)$ for $\frac{p(2)}{1+r} Y_2(2)$

and investing the proceeds in riskless bonds. Their second period consumption would then be

$$C_2(1) = p(1)\,Y_2(1) + p(2)\,Y_2(2)$$
$$C_2(2) = p(1)\,Y_2(1) + p(2)\,Y_2(2)$$

Solving the Model

Using the budget constraint to substitute for C_1 in the utility function we get

$$u(C_1) + \beta[\pi(1)\ u(C_2(1)) + \pi(1)\ u(C_2(2))] =$$

$$u\left(Y_1 + \frac{p(1)\ Y_2(1) + p(2)\ Y_2(2)}{1+r} - \frac{p(1)\ C_2(1) + p(2)\ C_2(2)}{1+r}\right) +$$

$$\beta[\pi(1)\ u(C_2(1)) + \pi(1)\ u(C_2(2))]$$

Taking the first order conditions wrt $C_2(1)$ and $C_2(2)$ we get

First Order Conditions

$$\frac{p(1)}{1+r} u'(C_1) = \pi(1)\,\beta u'(C_2(1))$$
$$\frac{p(2)}{1+r} u'(C_1) = \pi(2)\,\beta u'(C_2(2))$$

which have the familiar interpretations. Alternatively, they can be re-written as

$$\frac{\pi(1)\,\beta u'(C_2(1))}{u'(C_1)} = \frac{p(1)}{1+r}$$
$$\frac{\pi(2)\,\beta u'(C_2(2))}{u'(C_1)} = \frac{p(2)}{1+r}$$

Bond Euler Equation

Recall that (from the no-arbitrage condition above)

$$p(1) + p(2) = 1$$

Adding the two first order conditions above (and multiplying by (1+r)) we get

$$\begin{aligned}(p(1) + p(2))\, u'(C_1) &= (1 + r)\, \beta\, (\pi(1)\, u'(C_2(1)) + \pi(2)\, u'(C_2(2))) \\ &= (1 + r)\, \beta\, E_1[u'(C_2)]\end{aligned}$$

where $E_1[\,]$ is the expectations operator (conditional on period 1 info set). This yields the Stochastic Euler equations:

$$u'(C_1) = (1 + r)\, \beta\, E_1[u'(C_2)]$$

Implications

First, we have identified third source of gains from trade in assets; along consumption smoothing across time and separation of investment from saving we now also have smoothing across states of nature or risk diversification.
Open economy is at least as well off as an autarky economy, and in most cases better off.

Question: does this risk diversification take place in the real world?
Answer: Not as much as we think it should.

Let's look at the Home Bias paper by French and Poterba (AER 1991) before we move on to the other implications of the model.

French and Poterba (1991)

Do investors hold a lot of foreign assets? Not really:

TABLE 1—EQUITY PORTFOLIO WEIGHTS: BRITISH, JAPANESE, U.S. INVESTORS

	Portfolio Weight			Adj. Market Value
	U.S.	Japan	U.K.	
U.S.	.938	.0131	.059	$2941.3
Japan	.031	.9811	.048	1632.9
U.K.	.011	.0019	.820	849.8
France	.005	.0013	.032	265.4
Germany	.005	.0013	.035	235.8
Canada	.010	.0012	.006	233.5

Note: Estimates correspond to portfolio holdings in December, 1989. They are based on the authors' tabulations using data from the *U.S. Treasury Bulletin* and Michael Howell and Angela Cozzini (1990). Adjusted market values exclude intercorporate cross-holdings from total market value, and correspond to June 1990 values.

French and Poterba quantify it thsi way: Suppose utility over wealth is given by

$$U(W) = -\exp(-\lambda W / W0)$$

and returns are normally distributed N(μ,Σ).
Recall that if X is log normally distributed N(μ,σ^2) then, E(X^a) = $e^{a\mu+\lambda^2\sigma^2/2}$
So expected utilty is

$$E(U(W)) = -e^{-\lambda(\mu' w - \lambda w' \Sigma w/2)}$$

where *w* are portfolio weights.
Frist order coditions are:

$$\mu = \lambda \Sigma w$$

French and Poterba look at six biggest economies. They measure Σ (covariance matrix of returns) and assume λ=3. They have data on w (see table 1 above). They don't have data on μ. So they use the first order condition to calculate what μ would have to be if the observed weights w were optimal; call it μ1. They find that expected returns vary greatly in roder satisfy the above equation: people have to expect much higher returns at home than anywhere else.

TABLE 2—EXPECTED REAL RETURNS IMPLIED BY ACTUAL PORTFOLIO HOLDINGS

	U.S.	Japan	U.K.
A. Expected Returns Needed to Justify Observed Portfolio Weights			
U.S.	5.5	3.1	4.4
Japan	3.2	6.6	3.8
U.K.	4.5	3.8	9.6
France	4.3	3.4	5.3
Germany	3.6	3.0	4.8
Canada	4.7	3.0	4.0

Then they calculate what μ would have to be it the weights were equal to the value-weight of each market (last column in Table 1); call it $\mu 2$. As we'll se later the our model will predict something like that. They find that $\mu 1$ and $\mu 2$ are very different!

TABLE 2—EXPECTED REAL RETURNS
IMPLIED BY ACTUAL PORTFOLIO HOLDINGS

	U.S.	Japan	U.K.
A. Expected Returns Needed to Justify Observed Portfolio Weights			
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Japan	3.2	6.6	3.8
U.K.	4.5	3.8	9.6
France	4.3	3.4	5.3
Germany	3.6	3.0	4.8
Canada	4.7	3.0	4.0

This is the home bias result (or international diversification puzzle). It has diminished somewhat in recent years but is still there. See the paper for French and Poterba's suggested explanations.

Hall's Random Walk

The stochastic Euler equation.

$$u'(C_1) = \beta(1+r)\, E_1\{u'[C_2(s)]\}$$

(Compare with the case under certainty.)

If we assume, quadratic utility and β (1+r) = 1, then this says consumption follows a random walk process (Hall 1979).

$$E_1\{C_2\} = C_1$$

More on this later...

Will countries fully insure?

In the deterministic model (just like here) there was preference for smoothing across time; but unless β(1+r) = 1 this smoothing was never complete. Same applies to smoothing across states. Dividing the two FONC's by each other we get

$$\frac{\pi(1)\, u'(C_2(1))}{\pi(2)\, u'(C_2(2))} = \frac{p(1)}{p(2)}$$

which clearly implies that the country will completely insure against risk, i.e. set $C_2(1) = C_2(2)$, if and only if

$$\frac{\pi(1)}{\pi(2)} = \frac{p(1)}{p(2)}$$

If this is the case, we call the prices of A-D securities actuarially fair.

Role of Risk Aversion

What is the role of risk aversion? Start with

$$\frac{\pi(1)\, u'(C_2(1))}{\pi(2)\, u'(C_2(2))} = \frac{p(1)}{p(2)}$$

Totally differentiate wrt $\ln(C_2(1))$, $\ln(C_2(2))$ and $\ln\left(\frac{p(1)}{p(2)}\right)$ and assume *-C u''(C)/u'(C)* is constant and equal to ρ to get $\frac{\pi(1)}{\pi(2)}\left(\frac{C_2(2)}{C_2(2)}\right)^{\rho} = \frac{p(1)}{P(2)}$ or

$$d \ln\left(\frac{C_2(2)}{C_2(1)}\right) = \frac{1}{\rho}\, d \ln\left(\frac{p(1)}{p(2)}\right)$$

Recall that the Coefficient of relative risk aversion is given by ρ is just the inverse of the intertemporal elasticity of substitution σ. The latter controls how responsive consumption growth is to the real interest rate. Something similar is true of the CRRA, it control how responsive consumption allocation across states is to the relative prices of A-D securities. High degree of risk aversion means you want certain outcomes and will not change your allocation much when prices of A-D change a little.

Global Model

Just like in the deterministic model we wanted to know how the world real interest rate was determined. We can ask the same question about the A-D prices.
Recall, in teh detrminsitic model we said: suppose there are two countries Home and Foreign (*). In equailibrium we must have

$$C_1 + C_1^* \;=\; Y_1 + Y_1^* = Y_1^W$$
$$C_2 + C_2^* \;=\; Y_2 + Y_2^* = Y_2^W$$

and since for each country we had

$$u'(C_1) = \beta(1+r)\,u'(C_2)$$

if we assume CRRA utility we get

$$C_1 = (\beta(1+r))^{-1/\rho}\,C_2$$
$$C_1^* = (\beta(1+r))^{-1/\rho}\,C_2^*$$

By substituting into the budget balance conditions

$$(\beta\,(1+r))^{-1/\rho}\,Y_2^W = Y_1^W$$
$$1+r \;=\; \frac{1}{\beta}\left(\frac{Y_2^W}{Y_1^W}\right)^{1/\rho}$$

Global model cont.

We'll follow the same approach here. The equilibrium conditions are

$$C_1 + C_1^* = Y_1 + Y_1^* = Y_1^W$$

$$C_2(s) + C_2^*(s) = Y_2(s) + Y_2^*(s) = Y_2^W(s) \quad \text{for } s = 1, 2$$

and teh fonc's are

$$C_2(s) = \left[\frac{\pi(s)\,\beta\,(1+r)}{p(s)}\right]^{1/\rho} C_1$$

$$C_2^*(s) = \left[\frac{\pi(s)\,\beta\,(1+r)}{p(s)}\right]^{1/\rho} C_1^*$$

And we get

$$\frac{p(s)}{1+r} = \pi(s)\,\beta\left(\frac{Y_2^W(s)}{Y_1^W}\right)^{-1/\rho} \quad \text{for } s = 1, 2$$

also

$$\frac{p(s)}{p(s')} = \frac{\pi(s)}{\pi(s')}\left(\frac{Y_2^W(s)}{Y_2^W(s')}\right)^{-1/\rho} \quad \text{for } s = 1, 2$$

So prices will be actuarially fair iff $Y_2(s) = Y_2^*(s)$, which means no aggregate uncertaity.

Example

Global Model cont.

Since *p(1)* + *p(2)=1* it also follows that

$$p(s') = \frac{\pi(s')\, Y_2^W(s')^{-\rho}}{\sum_{s=1}^{S} \pi(s)\, Y_2^W(s)^{-\rho}}$$

and

$$1 + r = \frac{Y_1^W(s')^{-\rho}}{\beta \sum_{s=1}^{S} \pi(s)\, Y_2^W(s)^{-\rho}}$$

So the state-s AD security will be highe if world output in state s is relatively more scarce (lower $Y_2^W(s)$) and if the probability of state s occuring is higher (greater π(s)).

Implications for International Consumption Levels

Let’s explore some more implications of the complete markets model.
Recall the FONC’s must hold for both countires

$$\frac{\pi(s)\beta u'(C_2(s))}{u'(C_1)} = \frac{p(s)}{1+r} = \frac{\pi(s)\beta u'(C_2^*(s))}{u'(C_1^*)}$$

for every state

$$\frac{\pi(s')\beta u'(C_2(s'))}{u'(C_1)} = \frac{p(s')}{1+r} = \frac{\pi(s')\beta u'(C_2^*(s'))}{u'(C_1^*)}$$

, which implies the euqlization of marginal rates of substitution between states across countries

$$\frac{\pi(s)\beta u'(C_2(s))}{\pi(s')\beta u'(C_2(s'))} = \frac{p(s)}{p(s')} = \frac{\pi(s)\beta u'(C_2^*(s))}{\pi(s')\beta u'(C_2^*(s'))}$$

Which, if you recall from your micro class, is the condition for efficiency (i.e. there are no more gains from trade to be achieved). Thus our market equilibrium with complete Arrow-Debreu asset markets is effiicient.

Furthermore, assuming CRRA utility we have *u'(c)* = $c^{-\rho}$, from the above equation (multiplying by π(s')/π(s) and raising both sides to the power -1/ρ) we get

$$\frac{C_2(s)}{C_2(s')} = \frac{C_2^*(s)}{C_2^*(s')} = \left[\frac{p(s)}{p(s')}\frac{\pi(s')}{\pi(s)}\right]^{-1/\rho}$$

and from our discussion of teh AD prices we know that

$$p(s') = \frac{\pi(s')\,Y_2^W(s')^{-\rho}}{\sum_{s=1}^{S}\pi(s)\,Y_2^W(s)^{-\rho}} \text{ so that } \frac{p(s)}{p(s')}\frac{\pi(s')}{\pi(s)} = \left[\frac{Y_2^W(s')}{Y_2^W(s)}\right]^{-\rho}$$

which means that

$$\frac{C_2(s)}{C_2(s')} = \frac{C_2^*(s)}{C_2^*(s')} = \frac{Y_2^W(s')}{Y_2^W(s)}$$

Similarly we could show that

$$\frac{C_2(s)}{C_1} = \frac{C_2^*(s)}{C_1^*} = \frac{Y_2^W(s)}{Y_1^W}$$

What do these equations mean? Note that the right hand sides depend only on world values so are not country-specific. For example, the second equation says that growth rates of consumption are the same across countries. This is a pretty start implication.

The first equation implies that a country always consumes a fixed fraction of the world output:

$$\frac{C_2(s)}{Y_2^W(s)} = \frac{C_2(s')}{Y_2^W(s')} \equiv \mu \quad \text{and} \quad \frac{C_2^*(s)}{Y_2^W(s)} = \frac{C_2^*(s')}{Y_2^W(s')} \equiv \mu^* = 1 - \mu$$

as well as

$$\frac{C_1}{Y_1^W} = \mu \quad \text{and} \quad \frac{C_1^*}{Y_1^W} = \mu^* = 1 - \mu$$

Using the budget constraint $C_1 + \frac{p(1)\, C_2(1) + p(2)\, C_2(2)}{1+r} = Y_1 + \frac{p(1)\, Y_2(1) + p(2)\, Y_2(2)}{1+r}$ and the expressions for AD prices we can show that

$$\mu = \frac{Y_1\, (Y_1^W)^{-\rho} + \beta \sum_{s=1}^{S} \pi(s)\, Y_2(s)\, Y_2^W(s)^{-\rho}}{(Y_1^W)^{1-\rho} + \beta \sum_{s=1}^{S} \pi(s)\, Y_2^W(s)^{1-\rho}}$$

So a country’s share in world consumption is equal to the present discounted value of its expected lifetime income evaluated using the equailibrium AD prices.

Take-away

So in this mdel consumption moves with world output NOT domestic output. The way we have presented it above consumption of any two countries co-move perfectly (correlation of 1). If we allow for rate of risk aversion and discount factors to differ we won't get as strong a result but we'll still get that correlation of consumption growth rates must be much higher than correlation of country output growth rates. Is this true in the data? Unfortunately, not really. This is sometimes referred to as the Backus, Kehoe, and Kydland Puzzle (BKK, 1992).

BKK (1992)

International Real Business Cycles

David K. Backus
New York University

Patrick J. Kehoe
University of Minnesota and Federal Reserve Bank of Minneapolis

Finn E. Kydland
Carnegie Mellon University

We ask whether a two-country real business cycle model can account simultaneously for domestic and international aspects of business cycles. With this question in mind, we document a number of discrepancies between theory and data. The most striking discrepancy concerns the correlations of consumption and output across countries. In the data, outputs are generally more highly correlated across countries than consumptions. In the model we see the opposite.

BKK Puzzle cont.

Here are the numbers for the correlation of growths o consumption ($\hat{C}$) and output ($\hat{Y}$) with the world couterparts ($\hat{C}^W$,$\hat{Y}^W$) for a group of rich countries calcualted by Obstfeld and Rogoff (1996).

Table 5.1
Consumption and Output: Correlations between Domestic and World Growth Rates, 1973–92

Country	Corr $(\hat{c}, \hat{c}^w)$	Corr $(\hat{y}, \hat{y}^w)$
Canada	0.56	0.70
France	0.45	0.60
Germany	0.63	0.70
Italy	0.27	0.51
Japan	0.38	0.46
United Kingdom	0.63	0.62
United States	0.52	0.68
OECD average	0.43	0.52
Developing country average	−0.10	0.05

Note: The numbers Corr$(\hat{c}, \hat{c}^w)$ and Corr$(\hat{y}, \hat{y}^w)$ are the simple correlation coefficients between the annual change in the natural logarithm of a country's real per capita consumption (or output) and the annual change in the natural logarithm of the rest of the world's real per capita consumption (or output), with the "world" defined as the 35 benchmark countries in the Penn World Table (version 5.6). Average correlations are population-weighted averages of individual country correlations. The OECD average excludes Mexico.

Precautionary Saving

We now ask a different question about uncertainty: how does uncertainty affect behavior? It is easy to show that uncertainty makes you worse off. In the single period case, we know that, if c is some random variable:

$$E(u(c)) < u(E(c))$$

So, if you could pay to reduce uncertainty, you would do so (for example, buying insurance). But now we want to ask a different question: does uncertainty raise saving?

Let's examine the question in a simple two period model. For simplicity, we will assume that there is no discounting and no interest rate. Also, there are no Arrow-Debreu securities; there is only a riskless bond. In period 1 you get income Y and consume C_1 . In period 2, you participate in some lottery: with probability .5 you get some amount L given to you. With probability .5 you have the same amount taken away from you. So your consumption in period 2 is

$C_2 = Y - C_1 + L$ with probability .5 and
$C_2 = Y - C_1 - L$ with probability .5

Precautionary Saving cont.

Your only choice variable is C1. We will want to answer the question: how does saving (or first period consumption) change when the size of the lottery changes (that is, as uncertainty is increased).

Expected utility is :

$$E(U) = U(C_1) + E(U(C_2))$$

(As before we don't need to put an expected value symbol in from of C1,since it is known).

$$E(U) = U(C_1) + .5 \times U(Y - C_1 + L) ++ .5 \times U(Y - C_1 - L)$$

We maximize this by taking the derivative with respect to C1 and setting it equal to zero:

$$0 = U'(C_1) - .5 \times U'(Y - C_1 + L) - .5 \times U'(Y - C_1 - L)$$

The optimal value of C_1 is the value that solves this equation. That is, the equation implicitly tells us C_1 as a function of L. What we care about is how C_1 changes as L changes. To find this derivative, we use the implicit function theorem (See Chiang for example). If we write this as *0 = F(*C_1*,L)*, the implicit function
theorem tells us that :

$$\frac{dC_1}{dL} =$$

$$-\frac{F_L}{F_{C_1}} = -\frac{-.5[U''(Y - C_1 + L) - U''(Y - C_1 - L)]}{U''(C_1) - .5[-U''(Y - C_1 + L) - U''(Y - C_1 - L)]}$$

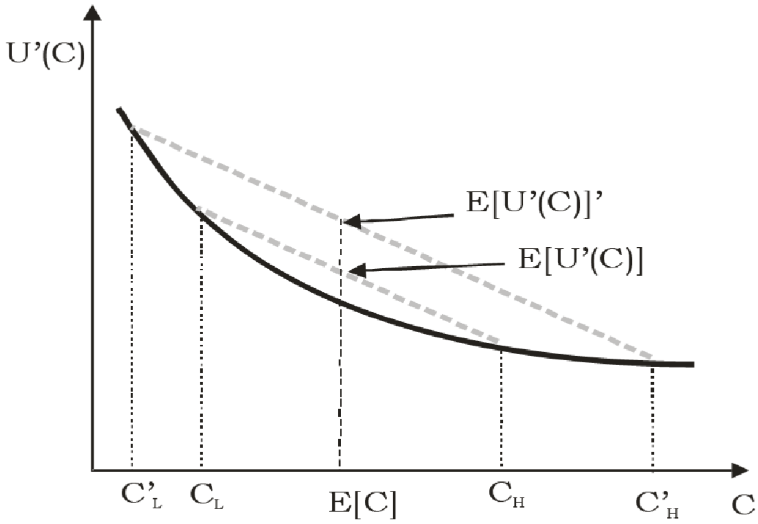
Notice that since U’’(C)<0, the term in the denominator that is in square brackets is positive, and so the whole denominator is negative. So the key term is:

$$U''(Y - C1 + L) - U''(Y - C_1 - L)$$

When will this term be positive? When U'''(C) > 0. In this case, the whole expression is negative, and so in this case there is precautionary saving.

$$\frac{dC_1}{dL} < 0$$

Assuming U'''>0 is what means the marginal utility curve looks like this



The intuition is that spreading out consumption in the second period raises the expected value of the marginal utility of consumption in the second period. This means that in taking part of consumption away from the first period and moving it to the second period (ie by saving more), you can, in expectation, get higher marginal utility.

In other words

$$u'(C_1) = (1+r)\beta E_1\{u'[C_2(s)]\}$$

says that a mean-preserving spread increases RHS so need to set LHS higher (which means consume less or save more).

This result about U'''>0 being necessary for precautionary saving is not unique to the two period model -- it holds generally. Many of our favorite utility functions, such as log and CRRA, have positive third derivatives, and thus imply precautionary savings.

A utility function that doesn't imply precautionary savings is quadratic utility:

$$U(C) = a\,C - \frac{b}{2}\,C^{2}$$

Here the first derivative is positive (for low enough C), the second negative, and the third is zero.

(We know that quadratic utility cannot be globally correct, since it implies that marginal utility becomes negative at some point. But it can still serve as a useful approximation as long as we restrict our attention to a limited range of values of C) Also it implies increasing absolute risk aversion (check it as an exercise).

We often use quadratic utility for mathematical simplicity, and also sometimes precisely because it gets rid of precautionary savings.

If there is no precautionary savings (and also no precautionary dis-savings), then the economy or consumer is said to display "certainty equivalence." That is, they act as if future income at every period t were certain to be equal to E(Yt).

====> The general way to deal with these sort of uncertainty problems is via dynamic programming, which we saw eariler...

Back to the SOE 2-period stochastic model: Closed Form Solutions with Log Utility

Now back to our SOE two period stochastic model.
We can get closed form solutions if we assume utility is log.
Lets define lifetime wealth as

$$W_1 = Y_1 + \frac{p(1)\,Y_2(1) + p(2)\,Y_2(2)}{1+r}$$

Using the first order conditions it is easy to show that:

$$C_1 = \frac{1}{1+\beta} W_1$$

$$C_2(1) = \frac{\pi(1)\,\frac{\beta}{1+\beta}\,W_1}{\frac{p(1)}{1+r}}$$

$$C_2(2) = \frac{\pi(2)\,\frac{\beta}{1+\beta}\,W_1}{\frac{p(2)}{1+r}}$$

Numerical Example (coming soon)

Asset Pricing

Now let's use what we have learned to think about asset pricing.
Start with present discounted values:
What is the value (in today's $) of an asset that pays Y_2 on date 2?
Answer: present discounted value of Y_2 which is

$$V_1 = \frac{Y_2}{1 + r}$$

where r is the riskless interest rate?

Now what if Y is actually uncertain?

$$V_1 = E_1 \frac{Y_t}{(1 + \tilde{r})}$$

where $\tilde{r}$ is some appropriate discount rate (risky discount rate).

What about more periods?

$$V_1 = \sum_{t=2}^{\infty} \frac{Y_t}{(1+r)^t}$$

Again if Y is uncertain we would use

$$V_1 = E_1 \sum_{t=2}^{\infty} \frac{Y_t}{(1+\tilde{r}_t)^t}$$

where $\tilde{r}$ is some appropriate discount rate (risky discount rate). If $\tilde{r}$ is constant then we get:

$$V_1 = E_1 \sum_{t=2}^{\infty} \frac{Y_t}{(1+\tilde{r})^t}$$

e.g. the Gordon model.
But what if $\tilde{r}$ is not constant. Also what should it really be? Simple finance models treat these as exogenous. But our current model sheds light on this.

Take an asset that pays of $Y_2(1)$ and $Y_2(2)$. Our models says that

$$V_1 = \frac{p(1)}{(1+r)} Y_2(1) + \frac{p(2)}{(1+r)} Y_2(2)$$

Now if both Arrow-Debreu securities are traded (or can be constructed from other actually traded securities) the we say that <u>markets are complete</u>. If this is the case we have a price p(s)/(1+r) for each state and can price the asset as above.

<u>From exogenous prices to (sort of) exogenous consumption.</u>

Now the question is what will the Arrow-Debreu asset prices be? If you're a SOE we already figured this out.

Demand for assets will be given (implicitly -- we don't have closed form solutions) by these two equations :

$$\frac{\pi_1 \, \beta \, u'[C_2(1)]}{u'(C_1)} = \frac{p(1)}{1+r}$$

$$\frac{\pi_2 \, \beta \, u'[C_2(2)]}{u'(C_1)} = \frac{p(2)}{1+r}$$

Now the prices will depend on demand and supply. But whatever the supply of assets is we know that in equilibrium prices must be equal to the marginal rates of substitutions given above! That is, with our model we can now ask: given a time (and state) path of consumption -- here this means C_1, $C_2(1)$, and $C_2(2)$ -- what does the model predict about prices of various assets (or their returns).

Using the fact that prices must be equal to the marginal rate of substitution we can write:

$$V_1 = \frac{\pi_1\,\beta\, u'[C_2(1)]}{u'(C_1)} Y_2(1) + \frac{\pi_2\,\beta\, u'[C_2(2)]}{u'(C_1)} Y_2(2) = E_1\left\{\frac{\beta\, u'[C_2(s)]}{u'(C_1)} Y_2(s)\right\}$$

so the right discount factor $1/(1+\tilde{r}_t)$ is the <u>intertemporal marginal rate of substitution</u>!

We can write this as

$$V_1 = E_1\left\{\frac{\beta\, u'[C_2(s)]}{u'(C_1)} Y_2(s)\right\} = E_1\left\{\frac{\beta\, u'[C_2(s)]}{u'(C_1)}\right\} E_1\{Y_2(s)\} + \mathrm{Cov}_1\left\{\frac{\beta\, u'[C_2(s)]}{u'(C_1)},\ Y_2(s)\right\}$$

and further using the stochastic Euler equation for the riskless bond ($u'(C_1) = \beta(1+r)\,E_1\{u'[C_2(s)]\}$)

$$V_1 = \frac{E_1\{Y_2(s)\}}{1+r} + \mathrm{Cov}_1\left\{\frac{\beta\, u'[C_2(s)]}{u'(C_1)},\ Y_2(s)\right\} = \frac{E_1\{Y_2(s)\}}{1+r} + \mathrm{Cov}_1\left\{\beta\left(\frac{C_2(s)}{C_1}\right)^{-\rho},\ Y_2(s)\right\}$$

Where the second equality assumes CRRA.

The price of the asset is thus made up of two components

1. The discounted expected payoff: $\frac{E_1\{Y_2(s)\}}{1+r}$
2. Conditional covariance between the relative marginal utility at date 2 and the payoff (or consumption growth C_2/C_1): $\text{Cov}_1\left\{\beta\left(\frac{C_2(s)}{C_1}\right)^{-\rho},\ Y_2(s)\right\}$

This covariance reflects risk (or alternatively the value of the asset as insurance). This is called the Consumption CAPM (capital asset pricing model).

Note: Are we treating consumption as exogenous? There are two answers we can give:

1. No we are not. Keep in mind of course that the covariance of consumption growth with asset reruns is endogenous -- that is by purchasing more of a given asset you change increase the co-movement of your consumption with that asset. That's OK since it doesn't affect the validity of the above equations, which says that after all asset allocation is done those assets whose returns co-move (have a positive correlation) with consumption growth will command a lower price).

2. In some sense consumption at the aggregate level is exogenous. In a closed economy I can borrow from you and you can buy some of state 1 Arrow - Debreu security from me but at the end of the day there is a fixed supply of all these assets in the economy and so our total payoff -- or aggregate GDP or consumption -- is given exogenously. All that Euler equations do is tell us how prices should align so all asset demands = asset supplies. In an open economy of course we can buy or sell more of a given asset from another country so we don't have to hold the same amount of it regardless of our preferences. But then we can think of the whole world and we're back to asset supplies are fixed.

Now let's think in terms of rates of return.
Define $\tilde{r} = \frac{Y_2 - V_1}{V_1}$ as the risky rate of return, then multiplying both sides of the above equation by $(1+r)/V_1$, we get

$$E_1\{\tilde{r}\} - r = -(1+r)\,\mathrm{Cov}_1\left\{\frac{\beta\, u'[C_2(s)]}{u'(C_1)},\ 1+\tilde{r}\right\}$$

$$= -(1+r)\,\mathrm{Cov}_1\left\{\frac{\beta\, u'[C_2(s)]}{u'(C_1)},\ \tilde{r} - r\right\}$$

Expected excess return depends negatively on the covariance of growth of marginal utility and excess return or positively related to the covariance of consumption growth and excess return. Assets with negative correlation are good insurance -- i.e. are not risky --

Using CRRA we have

$$E_1\{\tilde{r}\} - r = -(1+r)\,\mathrm{Cov}_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho},\ \tilde{r} - r\right\}$$

Now the correlation between consumption growth and asset returns is positive (implying a risk premium) but it is way too small. To see this we use again the formula *Cov(X,Y)* = *E(XY)* - *E(X)E(Y)* to get

$$\mathrm{Cov}_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho},\ \tilde{r}\ -r\right\} = E_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho}(\tilde{r}\ -r)\right\} - E_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho}\right\}\times E_1\{(\tilde{r}\ -r)\} =$$

$$E_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho}\tilde{r}\right\} - r\,E_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho}\right\} - E_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho}\right\}E_1\,\tilde{r} + r\,E_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho}\right\} =$$

$$E_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho}(\tilde{r}\ -E_1\,\tilde{r})\right\}$$

Now we can take the first-order Taylor approximation around $C_2/C_1 = 1$ to

$$E_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho}(\tilde{r}\ -E_1\,\tilde{r})\right\} \approx E_1\left\{\left[-\rho\beta\left(\frac{C_2}{C_1}\right)^{-\rho-1}(\tilde{r}\ -E_1\,\tilde{r})\right]_{\frac{C_2}{C_1}=1}\left(\frac{C_2}{C_1}-1\right)\right\} =$$

$$-\rho\beta\,E_1\left\{(\tilde{r}\ -E_1\,\tilde{r})\left(\frac{C_2}{C_1}-1\right)\right\}$$

once again using the fact that *Cov(X,Y) = E(XY) - E(X)E(Y)* get the following expression:

$$E_1\left\{\beta\left(\frac{C_2}{C_1}\right)^{-\rho}(\tilde{r} - E_1\,\tilde{r})\right\} \approx -\rho\beta\,\mathrm{Cov}_1\left\{(\tilde{r} - E_1\,\tilde{r})\left(\frac{C_2}{C_1}-1\right)\right\}$$

Which, after using the definition of correlation coefficient, becomes

$$\mathrm{E}\left\{\tilde{r}\right\} - r = (1+r)\,\beta\,\rho\,\kappa\,\mathrm{Std}\left\{C_2 / C_1 - 1\right\}\mathrm{Std}\left\{\tilde{r} - r\right\}$$

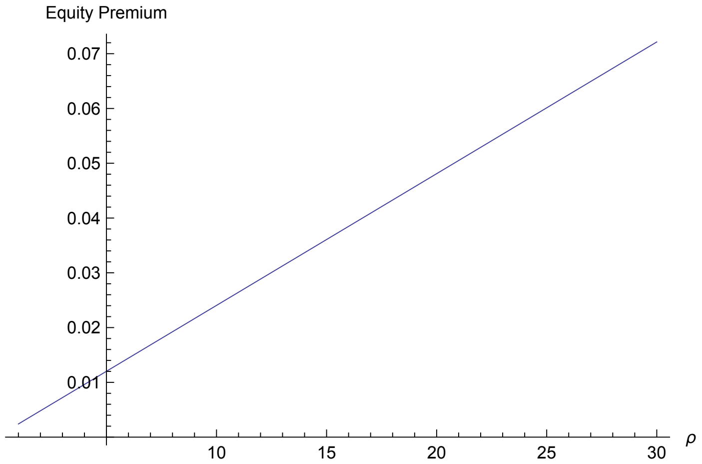
where κ = correlation coefficient between $C_2/C_1 - 1$ and $\tilde{r}$ - r.

We can use data to compute these quantities. $\kappa = 0.4$, Std of consumption growth = 0.036, std of excess returns is 0.167. Under a plausible value of ρ, say 3, what do these imply for the average equity premium E{$\tilde{r}$} – r? Assume (1+r)β = 1

```
3 × .4 × .036 × .167
```

```
0.0072144
```

or 0.7 %. What was the true average equity premium in the data? About 6%!
What ρ would it take to get 6%? Answer: about 26!



This is called the Equity Premium Puzzle (Mehar and Prescott 1985).

This is an implausibly high number (we would see a lot more insurance if people were this risk averse). And even if you believe this level of risk aversion you still don't save the model because now it implies it predicts way too high a risk-free rate (why? -- also maybe this is not really a problem?). This last problem follows form the link between intertemporal and risk preferences in our approach.

Some of the proposed solutions:

Disasters
Uninsurable labor income
Transaction costs/low stock market participation
Habits
The puzzle went (or is going) away

None are 100% convincing.

Show::gcomb : Could not combine the graphics objects in

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