# Generalized Least Squares

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# Basics

In the regression model

$$Y = X\beta + \epsilon \tag{1}$$

Suppose we drop the assumption of homoskedasticity. In other words, instead of assuming

variance comes from each sample

$$E[\epsilon\epsilon'] = \sigma^2 I \tag{2}$$

*omega is symmetric matrix*
*but may be heterskadasticity*

$$\begin{pmatrix} \sigma_1{}^2 & 0 & 0 \\ 0 & \sigma_2{}^2 & 0 \\ 0 & 0 & \sigma_3{}^3 \end{pmatrix}$$

we are assuming:

this is a more general matrix,
rather diagonal matrix like *I*
*generally,we do not know omega*

$$E[\epsilon\epsilon'] = \sigma^2\Omega \tag{3}$$

With this assumption we know $\Omega$ so we can decompose it:

*the sqrt(4) = 2, not (2,-2). sqrt is a fn. it can only yield one value*
*why we have(2,-2) in high school, because we are solving a equation y = x^2,*

$$\Omega^{-1} = P'_\Omega P_\Omega \tag{4}$$

*P(omega)^2*

# Unbiasesness

With $P_\Omega$ known in advance, we be premultiply the regression with it:

$$P_\Omega Y = P_\Omega X\beta + P_\Omega \epsilon \tag{5}$$

Instead of regressing $Y$ on $X$ , we are regressing $P_\Omega Y$ on $P_\Omega X$. The $\hat{\beta}$ would be unbiased (try to prove consistency yourself) because:

$$((X^*)^T X^*)^{-1} (X^*)^T Y^*$$

$$\begin{aligned}
E[\hat{\beta}|X] &= ((P_\Omega X)'(P_\Omega X))^{-1}(P_\Omega X)' P_\Omega Y &(6)\\
&= (X' P_\Omega' P_\Omega X)^{-1} X' P_\Omega' P_\Omega Y &(7)\\
&= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y &(8)\\
&= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}(X\beta+\epsilon) &(9)\\
&= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}X\beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\epsilon &(10)\\
&= \beta + (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}\epsilon = \beta &(11)
\end{aligned}$$

# Variance of $\hat{\beta}$

However, instead of $\epsilon$, we have $P_\Omega \epsilon$. The variance of $P_\Omega \epsilon$ is equal to:

we assume $E[\epsilon\epsilon'] = \sigma^2\Omega$

$$
\begin{aligned}
E[(P_\Omega \epsilon)(P_\Omega \epsilon)'] &= E[P_\Omega \epsilon\epsilon' P_\Omega'] && (12)\\
&= P_\Omega E[\epsilon\epsilon'] P_\Omega' && (13)\\
&= P_\Omega \sigma^2 \Omega P_\Omega' && (14)\\
&= \sigma^2 P_\Omega \Omega P_\Omega' && (15)\\
&= \sigma^2 P_\Omega \Omega \Omega^{-1} \Omega P_\Omega' && (16)\\
&= \sigma^2 P_\Omega \Omega P_\Omega' P_\Omega \Omega P_\Omega' && (17)
\end{aligned}
$$

$= \sigma^2 P_\Omega \Omega P_\Omega' = \sigma^2 P_\Omega \Omega P_\Omega' \boxed{P_\Omega \Omega P_\Omega'}$

*two equation are equal,*
*it mean, we multiply a term = 1*

*so* $P_\Omega \Omega P_\Omega' = I$.

$\Omega^{-1} = P_\Omega' P_\Omega$

For $\sigma^2 P_\Omega \Omega P_\Omega' = \sigma^2 P_\Omega \Omega P_\Omega' P_\Omega \Omega P_\Omega'$, $P_\Omega \Omega P_\Omega' = I$.
Therefore,

$$E[(P_\Omega e)(P_\Omega e)'] = \sigma^2 I \quad (18)$$

# Variance of $\hat{\beta}$

And we can calculate the variance of $\hat{\beta}$

$$\begin{aligned} Var(\hat{\beta}) &= \sigma^2((P_\Omega X)' P_\Omega X)^{-1} && (19) \\ &= \sigma^2(X' P'_\Omega P_\Omega X)^{-1} && (20) \\ &= \sigma^2(X' \Omega^{-1} X)^{-1} && (21) \end{aligned}$$

And hence we get the standard error of $\hat{\beta}$.

# Using FGLS

*Feasible Generalized Least Squares*

*this is used to estimate omega*

*if there's heteroskadasticity and don't know omega use FGLS*

How about if we do not know $\Omega$?

We can estimate it. One way to do it is to assume a structure of heteroskadasticity:

*this is the real residual, but in regression , we use epsilon hat*

$$\epsilon^2 = \sigma^2 exp(X\delta) \tag{22}$$

By using the residual $\hat{\epsilon}$ we can estimate the model:

FGLS is biased
*FGLS is not always consistent!!*
*It depends on your X*

$$log(\hat{\epsilon}^2) = X\delta + \eta \tag{23}$$

Recover the $\hat{\sigma}_i^2$ for individual obervations by taking exponential on the fitted value. *epsilon is the variance of each term* $\hat{\sigma}_i^2$

*regress* $log(\hat{\epsilon}^2)$ *on X, => E[* $log(\hat{\epsilon}^2)$ *] =* $X\hat{\delta}$

*you take exponential at both side E[* $\hat{\epsilon}^2$ *]*

# Use $\hat{\Omega}$

- Construct the matrix $\hat{\Omega}$ by $\hat{\sigma}_i^2$. We can use the $\hat{\Omega}$ to perform Generalized Least Squares.
- This is called Feasible Generalized Least Squares.
- What's the properties of this FGLS estimator?

*two problems of this method,*
*1.RHS of equation 22 can be wrong. if this eq is wrong, then our GLS can be wrong.*
*2.we use epsilon hat to run regression. but it is not real residual.*

*In your model, you have to adjust if* $\epsilon^2 = \sigma^2 exp(X\delta)$ *is good or not*