Introduction

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Econometric Model

Model:

$$Y = f(X) + \epsilon \tag{1}$$

Parameterize:

$$Y = X\beta + \epsilon \tag{2}$$

Estimated:

$$Y = X\hat{\beta} + \hat{\epsilon} \tag{3}$$

Fitted:

$$\hat{Y} = X\hat{\beta} \tag{4}$$

Quality of an estimator

- Unbiasness

$$E[\hat{\beta}] = \beta$$

- Consistency

$$\text{plim}_{n\to\infty}\hat{\beta} = \beta$$

$$\lim_{n\to\infty} P(|\hat{\beta} - \beta| \geq \epsilon) = 0$$

- Efficiency

$$Var(\hat{\beta})$$

- Mean square error

$$MSE(\hat{\beta}) = E[(\hat{\beta} - \beta)^2]$$

- Computational Cost
- Robustness

Mean Square Error

$$
\begin{aligned}
MSE(\hat{\beta}) &= E[(\hat{\beta} - \beta)^2] \\
&= E[(\hat{\beta} - E[\hat{\beta}] + E[\hat{\beta}] - \beta)^2] \\
&= E[(\hat{\beta} - E[\hat{\beta}])^2 + 2(\hat{\beta} - E[\hat{\beta}])(E[\hat{\beta}] - \beta) + (E[\hat{\beta}] - \beta)^2] \\
&= E[(\hat{\beta} - E[\hat{\beta}])^2] + (E[\hat{\beta}] - \beta)^2 \\
&= Var(\hat{\beta}) + (Bias(\hat{\beta}))^2
\end{aligned}
$$

Unbiaseness of Sample Mean

$$
\begin{aligned}
\bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\
E[\bar{x}] &= E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] \\
&= \frac{1}{n}E\left[\sum_{i=1}^{n} x_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left[x_i\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{n}{n}\mu = \mu
\end{aligned}
$$

Consistency of Sample Mean

$$
\begin{aligned}
Var[\bar{x}] &= Var\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] \\
&= \sum_{i=1}^{n} Var\left[\frac{x_i}{n}\right] \\
&= \sum_{i=1}^{n} \frac{\sigma^2}{n^2} \\
&= \frac{\sigma^2}{n}
\end{aligned}
$$

- And it goes to 0 when $n \to \infty$

Consistency of Sample Mean

- By Chebyshev inequality,

$$\begin{aligned} P(|\bar{x} - \mu| \geq \epsilon) &\leq \frac{Var(\bar{x})}{\epsilon^2} \\ &= \frac{\sigma^2}{n\epsilon^2} \end{aligned}$$

- And therefore,

$$\lim_{n \to \infty} P(|\bar{x} - \mu| \geq \epsilon) = 0$$

Unbiaseness of Sample Variance

$$
\begin{aligned}
s^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i^2-2x_i\bar{x}+(\bar{x})^2) \\
&= \frac{1}{n-1}(\sum_{i=1}^{n}x_i^2-2\sum_{i=1}^{n}x_i\bar{x}+\sum_{i=1}^{n}(\bar{x})^2) \\
&= \frac{1}{n-1}(\sum_{i=1}^{n}x_i^2-2\bar{x}\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}(\bar{x})^2) \\
&= \frac{1}{n-1}(\sum_{i=1}^{n}x_i^2-2n(\bar{x})^2+n(\bar{x})^2) \\
&= \frac{1}{n-1}(\sum_{i=1}^{n}x_i^2-n(\bar{x})^2)
\end{aligned}
$$

Unbiaseness of Sample Variance

$$
\begin{aligned}
E[s^2] &= E\left[\frac{1}{n-1}(\sum_{i=1}^{n} x_i^2 - n(\bar{x})^2)\right] = \frac{1}{n-1}E\left[\sum_{i=1}^{n} x_i^2 - n(\bar{x})^2\right] \\
&= \frac{1}{n-1}\sum_{i=1}^{n} E[x_i^2] - nE[(\bar{x})^2] \\
&= \frac{1}{n-1} n\sigma^2 + n\mu^2 - n(\frac{\sigma^2}{n} + \mu^2) \\
&= \sigma^2
\end{aligned}
$$

Consistency of Sample Variance

$$
\begin{aligned}
s^2 &= \frac{1}{n-1}(\sum_{i=1}^{n} x_i^2 - n(\bar{x})^2) \\
&= \frac{n}{n-1}(\frac{1}{n}(\sum_{i=1}^{n} x_i^2) - (\bar{x})^2) \\
&\xrightarrow{p} E[x^2] - E[x]^2 \\
&= Var(x)
\end{aligned}
$$

How about Standard Deviation?

- Sample standard deviation is NOT an unbiased estimator of the true standard deviation
- Why? (Consider Jensen's Inequality)