Instrumental Variables

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The Estimator

$$Y = X\beta + \epsilon \tag{1}$$

$$\hat{\beta} = (X'X)^{-1}X'Y \tag{2}$$

- OLS estimator consistency requires $plim\frac{1}{N}X'\epsilon = 0$
- However, with instrumental variables, we can use the IV estimator instead

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'Y \tag{3}$$

Z and X are correlated,

Consistency

$E[(Z^TX)^{-1}Z^T\varepsilon] \neq E[(Z^TX)^{-1}] \cdot E[Z^T\varepsilon]$

so, IV is biased

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'Y \tag{4}$$
$$= (Z'X)^{-1}Z'(X\beta + \epsilon)$$
$$= \beta + (Z'X)^{-1}Z'\epsilon)$$
$$= \beta + (\frac{1}{n}Z'X)^{-1}\frac{1}{n}Z'\epsilon) \tag{5}$$

- The $\hat{\beta}_{IV}$ is consistent if

$$plim\frac{1}{n}Z'X \neq 0 \tag{6}$$
$$plim\frac{1}{n}Z'\epsilon = 0 \tag{7}$$

Two Stage Least Squares

- However, the above method requires "exact identification". For situation when we have more instruments, we can apply the two-stage least squares.

$$\hat{\beta}_{2sls} = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'Y \tag{8}$$

- In this case the dimension of X is n by k and the dimension of Z is n by l, where $k \leq l$

Two Stage Least Squares

- A closer look to the estimator show how this is "two-stage" estimator

Since

$$\begin{aligned} X'Z(Z'Z)^{-1}Z' &= (Z(X'Z(Z'Z)^{-1})')' \\ &= (Z((Z'Z)^{-1})'(X'Z)')' \\ &= (Z(Z'Z)^{-1}Z'X)' \qquad (9) \end{aligned}$$

transpose transpose

how to get (3):
$$(\boxed{X'Z(Z'Z)^{-1}Z'}Z(Z'Z)^{-1}Z'X)^{-1}\boxed{X'Z(Z'Z)^{-1}Z'}Y$$

添加部分，运算结果不变

$$\begin{aligned} \hat{\beta}_{2sls} &= (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'Y \\ &= (X'Z(Z'Z)^{-1}Z'\underline{Z(Z'Z)^{-1}Z'}X)^{-1}X'Z(Z'Z)^{-1}Z'Y \\ &= (\boxed{(Z(Z'Z)^{-1}Z'X)'}\boxed{Z(Z'Z)^{-1}Z'X})^{-1}\boxed{(Z(Z'Z)^{-1}Z'X)'}Y \quad (3) \\ &= (\hat{X}'\hat{X})^{-1}\hat{X}'Y \qquad (10) \end{aligned}$$

Two Stage Least Squares

- Therefore to compute $\hat{\beta}_{2sls}$, it involves
  - Regress X on Z. Compute the fitted values.
  - Use the fitted values and instrument and compute $\hat{\beta}_{IV}$
- We can also use the "Fitted value maker":
- $\boxed{P_Z = Z(Z'Z)^{-1}Z'}$. $P_Z X$ is the "fitted value" from regressing X on Z
- And we can rewrite the IV estimator by

$$\begin{aligned}\hat{\beta}_{2sls} &= (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'Y \\ &= (X'P_Z X)^{-1}X'P_Z Y \qquad (11)\end{aligned}$$

Consistency and unbiasness?

$$
\begin{aligned}
\hat{\beta}_{2SLS} &= (X'P_Z X)^{-1} X'P_Z Y \\
&= (X'P_Z X)^{-1} X'P_Z (X\beta + \epsilon) \\
&= (X'P_Z X)^{-1} X'P_Z X\beta + (X'P_Z X)^{-1} X'P_Z \epsilon \\
&= \beta + (X'P_Z X)^{-1} X'P_Z \epsilon
\end{aligned}
$$

`not OLS, E(epsilon)=0?`

Standard Errors

- Recall for the OLS estimator, $Var(\hat{\beta}) = \sigma^2(X'X)^{-1}$, and the estimator of σ^2 is $s^2 = \frac{\hat{\epsilon}'\hat{\epsilon}}{n-k}$
- For the IV estimator, $Var(\hat{\beta}_{IV}) = \sigma^2(\hat{X}'\hat{X})^{-1}$
- To estimate σ^2, one may want to use the formula $s^2 = \frac{\hat{\epsilon}'\hat{\epsilon}}{n-k}$. However, since this is the residual from the second stage regression:

notice:this is x hat

$$\frac{\hat{\epsilon}'\hat{\epsilon}}{n-k} = \frac{(y - \hat{X}\hat{\beta}_{IV})'(y - \hat{X}\hat{\beta}_{IV})}{n-k} \tag{12}$$

- This is not the correct estimator of σ^2.

所以用IV回归的整体流程是？

如：earning\edu\exp\...如何IV

Standard Errors

- To get the correct estimator of σ^2, we need to use

$$\frac{(y - X\hat{\beta}_{IV})'(y - X\hat{\beta}_{IV})}{n-k} \tag{13}$$

- That is why we will get a wrong standard error estimate if we run the two-stage least square estimator in two steps manually and use the standard errors reported by the statistical softwares directly.

IV and FWL

Recall the OLS partitioned regression:

$$y = X_1\beta_1 + X_2\beta_2 + \epsilon \tag{14}$$

Let's say X_2 is endogenous and we have IV Z_2.
We have:

$$(Z'X)\hat{\beta}_{IV} = Z'y \tag{15}$$

where

$$X = \begin{pmatrix} X_1 & X_2 \end{pmatrix} \tag{16}$$

$$Z = \begin{pmatrix} X_1 & Z_2 \end{pmatrix} \tag{17}$$

$$\hat{\beta}_{IV} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} \tag{18}$$

IV and FWL

$$\begin{pmatrix} X_1'X_1 & X_1'X_2 \\ Z_2'X_1 & Z_2'X_2 \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} X_1'y \\ Z_2'y \end{pmatrix} \tag{19}$$

By solving the 1st equation,

$$(X_1'X_1)\hat{\beta}_1 + (X_1'X_2)\hat{\beta}_2 = X_1'y$$

we have

$$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'(y - X_2\hat{\beta}_2)$$

then we insert it into the 2nd equation, we'll have

$$\begin{aligned} Z_2'y = Z_2'X_1(X_1'X_1)^{-1}X_1'y &- Z_2'X_1(X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2 + Z_2'X_2\hat{\beta}_2 \\ Z_2'y - Z_2'X_1(X_1'X_1)^{-1}X_1'y &= Z_2'X_2\hat{\beta}_2 - Z_2'X_1(X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2 \\ (Z_2' - Z_2'X_1(X_1'X_1)^{-1}X_1')y &= [Z_2'X_2 - Z_2'X_1(X_1'X_1)^{-1}X_1'X_2]\hat{\beta}_2 \\ Z_2'(I - X_1(X_1'X_1)^{-1}X_1')y &= [Z_2'(I - X_1(X_1'X_1)^{-1}X_1')X_2]\hat{\beta}_2 \end{aligned} \tag{20}$$

IV and FWL

$$\hat{\beta}_2 = [Z_2'(I - X_1(X_1'X_1)^{-1}X_1')X_2]^{-1}Z_2'(I - X_1(X_1'X_1)^{-1}X_1')y \quad (21)$$
$$= (Z_2'M_1X_2)^{-1}(Z_2'M_1y) \quad (22)$$

Finally,

$$\begin{aligned} \hat{\beta}_2 &= (Z_2'M_1'M_1X_2)^{-1}(Z_2'M_1'M_1y) \\ &= \big((M_1Z_2)'M_1X_2\big)^{-1}\big((M_1Z_2)'M_1y\big) \\ &= (Z_2^{*\prime}X_2^*)^{-1}Z_2^{*\prime}y^{*\prime} \end{aligned} \quad (23)$$

2SLS and FWL

Similarly we can write the 2SLS as

$$\begin{pmatrix} \hat{X}_1{}'\hat{X}_1 & \hat{X}_1{}'\hat{X}_2 \\ \hat{X}_2{}'\hat{X}_1 & \hat{X}_2{}'\hat{X}_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \hat{X}_1{}'\hat{y} \\ \hat{X}_2{}'\hat{y} \end{pmatrix} \tag{24}$$

And we can use the same FWL trick for 2SLS.