Problem Set 1

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1 Question1

Formulate the strategic form games $G = (S_i, u_i)$

1.1 Cournot Duopoly

Given two firms, homogeneous goods, $MC = c > 0$, no fixed cost.

Demand Function:

$$P(Q) = a - bQ$$

where $a > c, b > 0, Q = q_1 + q_2$

So, the price function becomes this: $P = a - b(q_1 + q_2)$.

Profit for firm i is

$$\begin{aligned}\pi_i &= R_i - C_i \\ &= P \times q_i - c \times q_i\end{aligned}$$

For firm 1,

$$\begin{aligned}\pi_1 &= R_1 - C_1 \\ &= (a - bq_1 - bq_2)q_1 - cq_1 \\ &= aq_1 - bq_1^2 - bq_1q_2 - cq_1\end{aligned}$$

FOC wrt q_1:

$$\frac{\partial \pi_1}{\partial q_1} = a - 2bq_1 - bq_2 - c = 0 \tag{1}$$

$$q_1^* = \frac{a - bq_2 - c}{2b} \tag{2}$$

For firm 2,

$$\begin{aligned}
\pi_2 &= R_2 - C_2 \\
&= (a - bq_1 - bq_2)q_2 - cq_2 \\
&= aq_2 - bq_2^2 - bq_1q_2 - cq_2
\end{aligned}$$

FOC wrt q_2:

$$\frac{\partial \pi_2}{\partial q_2} = a - 2bq_2 - bq_1 - c = 0 \tag{3}$$

$$q_2^* = \frac{a - bq_1 - c}{2b} \tag{4}$$

We can see that the optimal output q_2 depends on q_2. Also, firm 2's output depends on firm 1's.

Now, let's define $\hat{q}_i$ as firm i's output in equilibrium. So, $\hat{q}_1 = q_1^*$, and $\hat{q}_2 = q_2^*$. We write down the equations below:

$$\hat{q}_1 = \frac{a - b\hat{q}_2 - c}{2b} \tag{5}$$

$$\hat{q}_2 = \frac{a - b\hat{q}_1 - c}{2b} \tag{6}$$

Substitute equation (6) into (5)

$$2b\frac{a-c-2b\hat{q}_1}{b} = a - b\hat{q}_1 - c \tag{7}$$

$$2a - 2c - 4b\hat{q}_1 = a - b\hat{q}_1 - c \tag{8}$$

$$a - c = 3b\hat{q}_1 \tag{9}$$

$$\hat{q}_1 = q_1^* = \frac{a-c}{3b} \tag{10}$$

$$\hat{q}_2 = q_2^* = \frac{a-c}{3b} \tag{11}$$

Conclusion:

At equilibrium, all firms produce the same amount of output q, where

$$q = \hat{q}_1 = \hat{q}_2 = q_1^* = q_2^*$$

If we think about this problem in a general form:

Define the profit for firm j

$$\pi_j = (a - b\sum_{i=1}^{N} q_i)q_j - cq_j \tag{12}$$

$$\pi_j = (a - b\sum_{i\neq j} q_i - bq_j)q_j - cq_j \tag{13}$$

FOCs wrt q_j:

$$\frac{\partial \pi_j}{\partial q_j} = a - b\sum_{i \neq j}^{N} q_i - 2bq_j - c = 0 \tag{14}$$

$$bq_j = a - b\sum_{i=1}^{N} q_i - c \tag{15}$$

$$bq = a - bNq - c \tag{16}$$

$$q^* = \frac{a-c}{b(N+1)} \tag{17}$$

$$Q^* = \sum_{i=1}^{N} q_i = Nq \tag{18}$$

$$= \frac{N(a-c)}{b(N+1)} \tag{19}$$

$$\tag{20}$$

We replace q_j with q in equation (16) because at equilibrium, $q_i = q_j = q$. It means each firm produce the same amount of output.

Now let's look at optimal price, P^*, and profit, π^*.

$$P^* = a - bQ \tag{21}$$

$$= a - b\frac{N(a-c)}{b(N+1)} \tag{22}$$

$$= \frac{a+Nc}{N+1} < a \tag{23}$$

$$\pi_i^* = (P^* - c)q \tag{24}$$

$$= \frac{(a-c)^2}{b(N+1)^2} \tag{25}$$

Now consider $P - c = \frac{a-c}{N+1}$,

$$\lim_{N\to\infty} \frac{a-c}{N+1} = 0$$

It means that when more and more firms get into the mkt, the equilibrium price and MC are getting closer and closer until $P^* = c$.

To be formal:

$$s_i = \{q_i | q_i \in [0, Q]\}$$

$$u_i = \{\pi_i\}$$

1.2 Bertrand Duopoly

Given two firms, each $MC = C_i$, price for two firms, P_1, P_2

The Game is designed in the form of:

$$S_i = \{p_1, p_2\}$$

$$u_i = \{\pi_1, \pi_2\}$$

Mechanism:

For each firm in the mkt, there are <u>three</u> possible outcomes:

1. If prices are same, then

$$
\begin{aligned}
p_{own} &= p_{mkt} = p_{rival} > c \\
q_{own} &= q_{rival} = \frac{Q}{2} \\
\pi_{own} &= \pi_{rival} = (p_{mkt} - c)\frac{1}{2}Q = \frac{1}{2}(p_{mkt} - c)(\alpha - \beta p_{own})
\end{aligned}
$$

2. If own price is lower than rival's:

$$
\begin{aligned}
p_{rival} > p_{mkt} = p_{own} > c \\
q_{own} &= Q = \alpha - \beta p_{own} \\
q_{rival} &= 0 \\
\pi_{own} &= (p_{own} - c)Q = (p_{own} - c)(\alpha - \beta p_{own}) \\
\pi_{rival} &= 0
\end{aligned}
$$

3. If own price is higher than rival's:

$$
\begin{aligned}
p_{own} > p_{mkt} = p_{rival} > c \\
q_{rival} &= Q = \alpha - \beta p_{rival} \\
q_{own} &= 0 \\
\pi_{rival} &= (p_{rival} - c)Q = (p_{rival} - c)(\alpha - \beta p_{rival}) \\
\pi_{own} &= 0
\end{aligned}
$$

Notice: firms will never set price lower than c, ($p \geq c$) The process would be like this:

One firm's price is lower than the other's, it serves the entire mkt, while rivals earn nothing. Then, in the next round, rivals will try to name a much lower price, which is even lower than the winner's in the previous round. So, all firms try to name a lower price, mkt price keep decreasing until it approach the marginal cost. No one would name an even lower price. Nash equilibrium achieved! Notice, the profit of Nash Equilibrium is not the maximized profit, but the mutual best response. For each firm, it can earn the maximum profit by name a price just slightly below it's rival's, so that it can serve the entire mkt. But this is not a stable solution, because rivals will name a even lower price in the next round at least it is not lower than c.

2 Question2

Consider a 3×3 game.

	C1	C2	C3
R1			
R2			
R3			

There should be at least one strategy being removed in each round. So, for player C, we need, at most 2 rounds to remove the other two strategies. So do player R. Then there should be at most 4 rounds in total to achieve the final decision. So, for a $n \times n$ game, with two players, there should be $n-1$ rounds for each player to drop all other strategies and remain only one strategy. Then, the upper bound for two-player iteration should be

$$upper\_bound = (n - 1) + (n - 1) = 2 \times (n - 1)$$

. If there are k players in total, then the bound becomes

$$upper\_bound = k \times (n - 1)$$

.

3 Question3

3.1 (a) The order of elimination does not matter.

Clearly, for row player, $U > D$, for column player, $M > R$.

1. If we remove D first:

Players	Column Player			
Row player		L	M	R
	U	2,1	1,1	0,0
	C	1,2	3,1	2,1

And for column player, $L > M > R$.

(1) If remove R first,

	L	M
U	2,1	1,1
C	1,2	3,1

Now, for column player, L is the dominant strategy. Then, if L is chose, row player chooses U. The outcome is (U, L).

(2) If remove M first,

	L	R
U	2,1	0,0
C	1,2	2,1

L is still the dominant strategy. Then end up with (U, L).

2. If remove R first:

Players	Column player		
Row player		L	M
	U	2,1	1,1
	C	1,2	3,1
	D	2,-2	1,-1

There is NO strictly dominated strategy for both players. Then, we should look at the best responses (BRs).

Players	Column player		
Row player		L	M
	U	2,1	1,1
	C	1,2	3,1
	D	2,-2	1,-1

Column Player

Row player

The game ends up with (U, L), because it is the mutual BR.
But if we still using iteratively eliminating dominated strategy, for player R, the payoff for U and D are exactly the same no matter what C does. then, if we remove D, then, we end up with (U, L). However, if we remove U, we can make final decision. That's why order matters. **Notice, weakly dominate only requires "$\geq$", it includes payoffs for R along a row is exactly the same as payoffs in another row. So do player C (columns)**

	Player C		
Player R		L	M
	C	1,2	3,1
	D	2,-2	1,-1

3.2 (b) Proof

In static and complete information games, players know everything in the game. So, they should know what are the strictly dominated strategies for each player. Then, they do not need to consider the response to those strictly dominated strategies, because rivals will never choose those ones. Use this game as an example, the game actually starts with only U, C, L, M, a 2×2 games rather 3×3

	L	M
U	2,1	1,1
C	1,2	3,1

Then we can find, for column player, M is the strictly dominated strategy. So, he will only choose L. Then, row player chooses U as response.

4 Question4

Define $\hat{s}$ as a weakly undominated strategy, and $\bar{s}$ as some other strategies. We call $\hat{s}$ as weakly undominated if

$$u_i(\hat{s}_i, s_{-i}) \geq u_i(\bar{s}_i, s_{-i}).$$

So, $\hat{s}_i$ survives the iterative weak dominance. It means that there is at least one cell for this strategy is greater or equal to some other strategy s_i. Otherwise,

$$u_i(\hat{s}_i, s_{-i}) < u_i(s_i, s_{-i}) \quad \forall s_i \in S_i$$

$\hat{s}_i$ is strictly dominated by s_i. Then, it is impossible for $\hat{s}_i$ to be a strictly dominated strategy.