Notes for Game Theory

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August 29, 2020

1 Lecture 1

Date: August 24, 2020

1.1 Strategic Interdependent (S.I.)

DEF:

If you know your rival's choices, you could find your best choice. And your best choice is vary with your rival's choices.

Example:

Coke's best choices often depends on Pepsi's choice of pricing, ads, new product information, etc.

S.I. does not fit with 2 standard approaches:

- Perfect competition: Because Coke affects mkt outcome.
- Monopoly: Pepsi also affect mkt outcomes.

Some History:

In 1970s, economists got interested in strategic Interdependent. Rediscovered work on game theory from 1950s.

S.I. Application:

- I.O: pricing, capacity choice, cartel.
- Labor: negotiation, strikes, compensation, schemes.

- International trade: strategic trade policy.
- Macro: money supply/inflation.
- Law and Econ: Jury rules, settling law suit, allocation court costs.
- Political: International relations.
- Biology: competition for food and shelter.
- Industrial Engineering: autonomous vehicles.

Important Point:

Game theory is just an extension of basic 801, i.e. individual decision making. Payoff-maximizer interacting with other payoff maximizer, in setting strategic interdependence.

1.2 Chuck's Approach for This Course.

Part I:

Introduce concepts/tools, which is very abstract and almost no economics. Then, we goes to four types of games and their associated equilibrium concepts.

	Complete Information	Incomplete Infromation
Static	(1)	(3)
Dynamic	(2)	(4)

Part II: Standard game-theoretic models

- Models that all economists should know.

- Tips/tricks/techniques for solving broad classes of model.

Part III: Information Economics

Part IV: Auctions and Mechanism Design

2 Lecture 2

Date: August 26, 2020

2.1 Game Theory Basics

When formalize a strategic scenario, think about relevant structural features:

- Who are the players?
- What can they do?
- What are payoffs for different combinations of choices by all players?
- What is timing of moves?
- What information is available to players at every point in time?

Let's start with static games of complete Information:

- All players make simultaneous choices, once (one-time decision making).
- All players have common knowledge (C.K.) about game's structure: players, payoffs, timing, information, etc.

NB: C.K. refers to every player know everything about the game.

Now, let's be formal:

- **Players**: Normally indexed by $i, i = 1, 2, 3, ..., N$.

- **Strategies**:S_i is a set of strategies available to player i. it could be infinite/finite, and encompass dynamics (NOT in static games).

- **Payoffs**: $u_i : \times_{j=1}^{N} S_j \to \mathbf{R}$, player i's payoff is a function of all player's strategies.

2.1.1 Strategic-form game

Define a strategic form game as

$$G = (S_i, u_i)_{i=1}^{N}$$

Notice: $\times_{j=1}^{N} S_j$ is the cross product of all s_j

Consider

$$S = \times_{j=1}^{N} S_j \quad s \in S,$$

then we can write $s = (s_1, s_2, ..., s_N)$.

Each S_j is a vector (each element of S), where a particular vector is the strategies for particular player.

Example:

Now, let's try to formalize the half-average game we have played on Monday. The game requires each player say a number from 1 to 100. The closest number to the half-average ($HA = \frac{Ave}{2}$) get \$5 reward.

(1)Strategy space:

$$S_i = \{1, 2, ..., 100\}$$

Or

$$S_i = \{x | x \in \{1, 2, ..., 100\}\}$$

(2)Payoffs:

$$u_i = \begin{cases} \emptyset & if |s_i - HA| > \min_j |s_j - HA| \\ \frac{5}{|w|} & if |s_i - HA| = \min_j |s_j - HA| \end{cases}$$

$$HA = \frac{\sum_{j=1}^{n} s_j}{2N}$$

$$w = \{k : |s_k - HA| = \min_j |s_j - HA|\}$$

- $|s_i - HA|$ refers to the distance of your answer and HA
- $\min_j |s_j - HA|$ is the nearest distance to HA, which is the distance can win the game.
- k refers to how many players win the game. k can be greater than 1 when multiply player have same distance.
- w is the set of winners. $|w|$ is not the **absolute value**. It is the number of winners.

2.2 Dominant Strategy

2.2.1 Intuition

Consider players R and C in a one-shot, simultaneous-move game. Each player has 2 choices: $\{R_1, R_2\}, \{C_1, C_2\}$. Payoffs are $\{\pi_R, \pi_C\}$ for different combination of actions as following:

	R1,C1	R1,C2	R2,C1	R2,C2
π_R	10	2	15	5
π_C	10	15	2	5

How do you think the player will choose? Let's write down the payoff matrix in this way.

	Player C		
Player R		C1	C2
	R1	10,10	2,15
	R2	15,2	5,5

If you are player R, think about payoffs from different combinations of strategies.

- If C chooses C_1, then R prefers R_2, because $15 > 10$.
- If C chooses C_2, then R prefers R_2, because $5 > 2$.

So, whatever C choose, R will choose R_2. Then, we call R_2 is a **dominant strategy** that maximizes player's payoff, regardless of rival's choice.

2.2.2 Formalization

Now, let's be formal:

(1) For all players' strategies:

$$S = \times_(j = 1)^N S_j$$

$$s \in S, \quad s_i = (s_1, s_2, ..., s_N)$$

(2) For all players' strategies without player i's:

$$S_{-i} = \times_{j \neq i}^N S_j$$

$$s_{-i} \in S_{-i}, \quad s_{-i} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_N)$$

NB: we define $-i$ as all players except player i. So, S_{-i} is the joint strategy space except player i.

Definition: A strategy $\hat{s}_i$ for player i is **strictly dominat** If

$$u_i(\hat{s}_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}, s_i \neq \hat{s}_i$$

Notice, $u_i(\hat{s}_i, s_{-i})$ means the payoff you will receive if you choose $\hat{s}_i$, and other player choose s_{-i}.
So, $\hat{s}_i$ is dominant strategy if the payoff choosing it is greater than all other strategies you will choose, no matter what strategy is chosen by others.

We can have **weakly dominant** if replace ">" with "≥", i.e.

$$u_i(\hat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$$

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2.3 Dominate and Dominated

Now let's view **dominant strategy** from another perspective:

R_1 is worse than R_2, no matter what C does. Then, we say that R_1 is a **dominated strategy**. It is dominated by R_2.

Definition:

Player i's strategy $\bar{s}_i$ is **strictly dominated** if $\exists \tilde{s}_i$

$$u_i(\tilde{s}_i, s_{-i}) > u_i(\bar{s}_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

.

Alternative:

Player i's strategy $\tilde{s}_i$ **strictly dominates** another strategy $\bar{s}_i$ if:

$$u_i(\tilde{s}_i, s_{-i}) > u_i(\bar{s}_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

we can also define "**weakly dominate**" by replacing ">" with "≥".

Games with this structure is referred to as Prisoner's Dilemma.

- US vs Russia (arms races)

- Advertisement
- Pricing
- Hiring lawyer
- ...

3 Lecture 3

Date: August 28, 2020

3.1 Iterated Elimination

How to play this game?

	C1	C2	C3
R1	4,8	7,20	11,10
R2	8,4	10,8	12,12
R3	10,10	3,4	15,6

Obviously, there is NO dominant strategy for each player. But for player R, R_2 is always better than R_1, whatever C does.

So, R will never choose R_1. R_1 is ruled out! The game becomes this:

	C1	C2	C3
R2	8,4	10,8	12,12
R3	10,10	3,4	15,6

We say R_2 is strictly better than R_1, or R_2 strictly dominates R_1.

Then, in the revised game, we see, for player C, C_3 strictly dominates C_2 $(12 > 8, 6 > 4)$.

So, C will never choose C_2.

Then, the game looks like this:

	C1	C2
R2	8,4	12,12
R3	10,10	15,6

For player R, R_3 strictly dominates $R_2 (10 > 8, 15 > 12)$. Then, if C know R will pick up R_3, C will choose $C_1, (10 > 6)$.
So, game ends up with $(R_3, C_1) \rightarrow (10, 10)$.

The main idea of this game is to find the strategies can strictly dominates others. The process of iterated elimination of strictly dominated strategies, is called IESDS.
Or called:
(1) IEWDS: Iterated elimination of weakly dominated strategies.
(2) IEDS: Iterated elimination of dominated strategies.

So, IEDS can eliminate some strategies from consideration as equilibrium. And it might get to unique prediction.

Now, let's reconsider the half-average game. A rational player will never choose number that is above 50. Why? Even if everyone choose 100, the $HA = \frac{\sum 100}{n} \times \frac{1}{2} = 50$, so $HA_0 \in [0, 50]$. With this consideration, if everyone know $HA_0 \in [0, 50]$, rational players will not choose number above 25, as $HA_1 \in [0, 25]$. Then, players will not choose number above 12.5, ..., and so forth. In the end, everyone will choose 1.

3.2 Best Response (BR)

Now consider a new game.

	C1	C2	C3
R1	8,2	3,3	10,5
R2	10,15	12,9	8,10
R3	15,8	4,12	2,3

In this game, IEDS does nothing because there is NO strategy can be strictly dominated by the others. Let's think about the Best Responses (BRs). **DEF:** BR is payoff maximizing choice for a particular choice chose by rival.

(1) For player R:

When C chooses C_1, R_3 is the best. $\{R_3, C_1\}$.

When C chooses C_2, R_2 is the best. $\{R_2, C_2\}$.

When C chooses C_3, R_1 is the best. $\{R_1, C_3\}$.

(2) For player C:

When R chooses R_1, C_3 is the best. $\{R_1, C_3\}$.

When R chooses R_2, C_1 is the best. $\{R_2, C_1\}$.

When R chooses R_3, C_2 is the best. $\{R_3, C_2\}$.

	C1	C2	C3
R1	8,2	3,3	10,5
R2	10,15	12,9	8,10
R3	15,8	4,12	2,3

R's best choice

C's best choice

Clearly $\{R_1, C_3\}$ is the only one choice that is selected by both players. $\{R_1, C_3\}$ is the only combination of strategies for which each player is best-responding to rival's choice.

3.3 Nash Equilibrium (N.E.)

Informal Definition:

A set of strategies, one for each player, that are mutual best-responses.

Formal Definition:

For a strategic-form game $G = (S_i, u_i)_{i=1}^{N}$ that joint strategy $\hat{s} \in S$ is a pure strategy nash equilibrium if for all i:

$$u_i(\hat{s}_i, \hat{s_{-i}}) \geq u_i(s_i, \hat{s_{-i}}) \quad \forall s_i \in S_i$$

$u_i(\hat{s}_i, \hat{s_{-i}})$ is the payoff that all players choose $\hat{s}$ type of strategy.

$u_i(s_i, \hat{s_{-i}})$ is the payoff that all other players choose $\hat{s}$ but you DO NOT.

Important:

$\hat{s}_i$ is a best-response to $\hat{s_{-i}}$. But it is not necessarily a best-response to any (or every) s_{-i}.

N.E. is the strategies, not the payoffs. So, in our game, N.E. is $\{R_1, C_3\}$. BR illustrates link between game theory and decision theory (or 'monopoly' theory). BR is 'monopoly choice: holding rival's choice constant,

> It means we hold C_1 and find best R_i, then hold C_2 and find best R_i...

N.E. is the intersection of BR function.

Then questions come up:

1. Does games tend to have N.E. ?
2. Can a game have multiple N.E. ?

For the first question, the answer is YES, if we modify one thing.
For the second question, the answer is YES. we can even have a game with infinite numbers of N.E. But if so, we cannot predict how players are going to make decision.

Good exercise for constant 3×3 games such that:

1. No dominant strategy but IEDS gives unique solution.
2. IEDS does nothing but intersecting BR function does.
3. No mutual BRs(BR function do not intersect)

To ensure existence of N.E., we need to enrich the strategy space. Consider a game like this:

	C1	C2
R1	5,5	3,10
R2	2,12	10,8

R's best choice

C's best choice

There is no intersecting BRs. Now what? How to enrich the strategy space? Answer: we need to let the strategies include a random component(Prob.). Strategies considered so far, in our class, are pure strategies(P.E.). P.E. means the particular choice is made with $Prob. = 1$. Now, we are going to design a mix strategy, puts Prob. mass on one or more P.E.

For example:

Play R_1 with $Prob. = .3$

Play R_2 with $Prob. = .7$

We will discuss this next week!