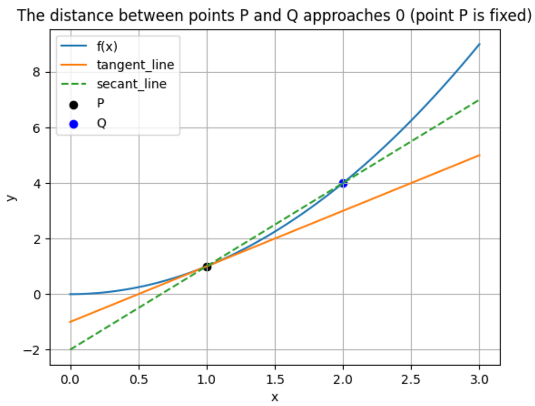
Unit 1 Derivatives

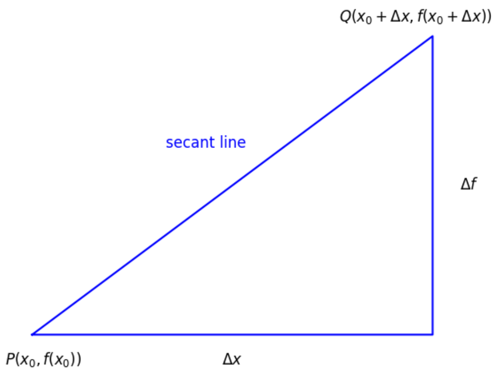
Lecture 1 What is a derivative?

1. The derivative is the limit of the secant line approaching the tangent line:

- Geometric interpretation:



- Algebraic Explanation:



$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$$

2. Example:

$$f(x) = \frac{1}{x}$$

$$\begin{aligned} proof: \frac{\Delta f}{\Delta x} &= \frac{\frac{1}{(x+\Delta x)} - \frac{1}{x}}{\Delta x} \\ &= \frac{-\Delta x}{\Delta x \cdot x \cdot (x+\Delta x)} \\ &= \frac{-1}{x \cdot (x+\Delta x)} \\ f'(x) &= \lim_{\Delta x \to 0} \frac{\Delta t}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-1}{x \cdot (x+\Delta x)} \\ &= -\frac{1}{x^2} \end{aligned}$$

$$f(x) = x^n$$

$$\begin{aligned} proof: \frac{\Delta f}{\Delta x} &= \frac{(x+\Delta x)^n - x^n}{\Delta x} \\ &= \frac{x^n + n \cdot \Delta x \cdot x^{n-1} + o[(\Delta x)^2] - x^n}{\Delta x} \\ &= n \cdot x^{n-1} + o(\Delta x) \\ f'(x) &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \\ &= \lim_{\Delta x \to 0} [n \cdot x^{n-1} + o(\Delta x)] \\ &= n \cdot x^{n-1} \end{aligned}$$

3. Note:

Tangent Line Equation:

$$y - y_0 = f'(x_0)(x - x_0)$$

Lecture 2 Limits and Continuity

1. Limits:

$$\lim_{x \to x_0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

2. Continuity:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- Removable discontinuities of the first type:

  $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) \neq f(x_0)$ or $f(x_0)$ is not defined.

- Jump discontinuities of the first type:

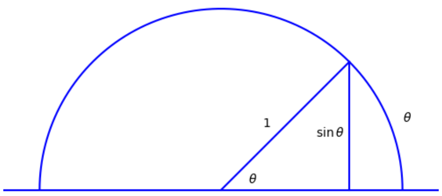
  $$\lim_{x \to x_0^+} f(x) = a \neq b = \lim_{x \to x_0^-} f(x)$$

- Infinite discontinuities of the second type:

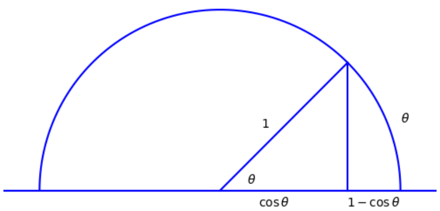
  $$\lim_{x \to x_0^+} f(x) / \lim_{x \to x_0^-} f(x) \to \infty$$

- Other discontinuities of the second type.

3. Two trigonometric limits:



$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



$$\lim_{\theta \to 0} \frac{1 - \cos\ \theta}{\theta} = 0$$

4. Theorem:

A differentiable function must be continuous:

$$
\begin{aligned}
proof\colon & \lim_{x \to x_0} (f(x) - f(x_0)) \\
&= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \\
&= f(x_0) \cdot 0 = 0
\end{aligned}
$$