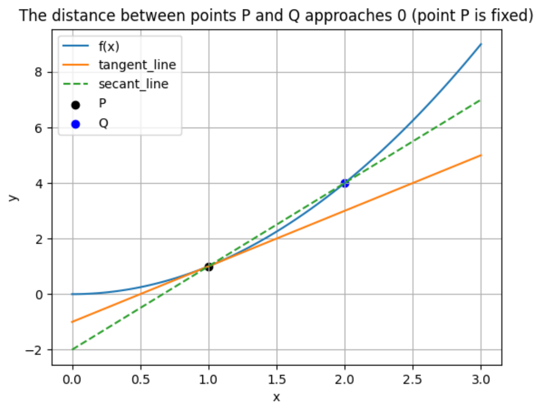
Unit 1 Derivatives

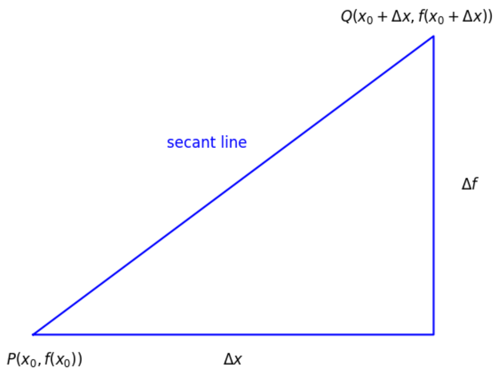
Lecture 1 What is a derivative?

1. The derivative is the limit of the secant line approaching the tangent line:

- Geometric interpretation:



- Algebraic Explanation:



$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$$

2. Example:

$$f(x) = \frac{1}{x}$$

$$\begin{aligned} proof: \frac{\Delta f}{\Delta x} &= \frac{\frac{1}{(x+\Delta x)} - \frac{1}{x}}{\Delta x} \\ &= \frac{-\Delta x}{\Delta x \cdot x \cdot (x+\Delta x)} \\ &= \frac{-1}{x \cdot (x+\Delta x)} \\ f'(x) &= \lim_{\Delta x \to 0} \frac{\Delta t}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-1}{x \cdot (x+\Delta x)} \\ &= -\frac{1}{x^2} \end{aligned}$$

$$f(x) = x^n$$

$$\begin{aligned} proof: \frac{\Delta f}{\Delta x} &= \frac{(x+\Delta x)^n - x^n}{\Delta x} \\ &= \frac{x^n + n \cdot \Delta x \cdot x^{n-1} + o[(\Delta x)^2] - x^n}{\Delta x} \\ &= n \cdot x^{n-1} + o(\Delta x) \\ f'(x) &= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \\ &= \lim_{\Delta x \to 0} [n \cdot x^{n-1} + o(\Delta x)] \\ &= n \cdot x^{n-1} \end{aligned}$$

3. Note:

Tangent Line Equation:

$$y - y_0 = f'(x_0)(x - x_0)$$

Lecture 2 Limits and Continuity

1. Limits:

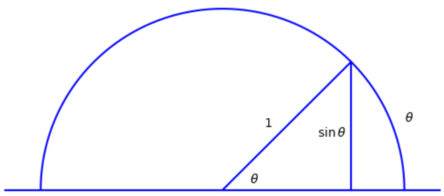
$$\lim_{x \to x_0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

2. Continuity:

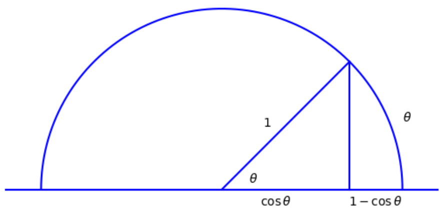
$$\lim_{x \to x_0} f(x) = f(x_0)$$

- Removable discontinuities of the first type:
  $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) \neq f(x_0)$ or $f(x_0)$ is not defined.
- Jump discontinuities of the first type:
  $$\lim_{x \to x_0^+} f(x) = a \neq b = \lim_{x \to x_0^-} f(x)$$
- Infinite discontinuities of the second type:
  $$\lim_{x \to x_0^+} f(x) / \lim_{x \to x_0^-} f(x) \to \infty$$
- Other discontinuities of the second type.

3. Two trigonometric limits:



$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$



$$\lim_{\theta \to 0} \frac{1 - \cos\,\theta}{\theta} = 0$$

4. Theorem:

A differentiable function must be continuous:

$$proof\colon \lim_{x\to x_0}(f(x)-f(x_0))$$
$$= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)$$
$$= f(x_0)\cdot 0 = 0$$

Lecture 3 Derivative formula

1. General formula:

- $(u+\varphi)' = u' + \varphi'$

Proof: $(u+\varphi)'(\delta) = \lim_{\Delta x\to 0}\frac{(u+\varphi)(\delta+\Delta x)-(u+\varphi)(\delta)}{\Delta x}$

$$= \lim_{\Delta x\to 0}\frac{u(\delta+\Delta x)+\varphi(\delta+\Delta x)-u(\delta)-\varphi(\delta)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\left(\frac{u(\delta+\Delta x)-u(\delta)}{\Delta x}+\frac{\varphi(\delta+\Delta x)-\varphi(\delta)}{\Delta x}\right)$$

$$= (\mathrm{u})'(\delta) + (\varphi)'(\delta)$$

- $(Cu)'(C\cdot u)' = C\cdot u'$

$$(C\cdot u)'(x)$$
$$= \lim_{\Delta x\to 0}\frac{C\cdot u(x+\Delta x)-C\cdot u(x)}{\Delta x}$$
$$= C\cdot \lim_{\Delta x\to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x}$$
$$= C\cdot u'(x)$$

- $(u\cdot v)' = u'\cdot v + u\cdot v'$

Proof: $(u\cdot v)' = \lim_{\Delta x\to 0}\frac{(u\cdot v)(x+\Delta x)-(u\cdot v)(x)}{\Delta x}$

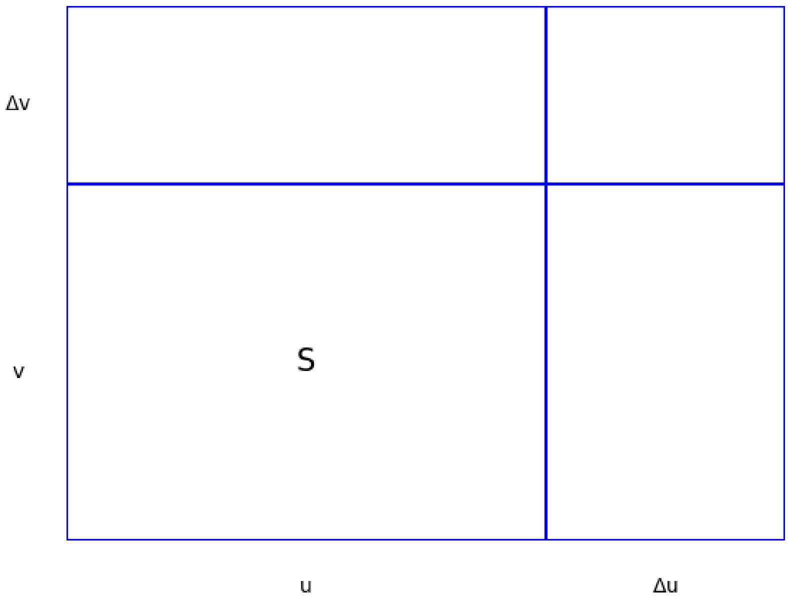
$$= \lim_{\Delta x\to 0}\frac{(u\cdot v)(x+\Delta x)+u(x+\Delta x)\cdot v(x)-u(x+\Delta x)\cdot v(x)-(u\cdot v)(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\left(u(x+\Delta x)\cdot\frac{v(x+\Delta x)-v(x)}{\Delta x}+v(x)\cdot\frac{u(x+\Delta x)-u(x)}{\Delta x}\right)$$

$$= u(x)\cdot v'(x) + v(x)\cdot u'(x)$$

Geometric interpretation:

$$\Delta S = \Delta v \cdot u + \Delta u \cdot v$$



- $(\frac{u}{v})' = \frac{u'v-uv'}{v'}$

$$Proof: \frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)}$$
$$= \frac{(u+\Delta u)\cdot v - (v+\Delta v)\cdot u}{(v+\Delta v)\cdot v}$$
$$= \frac{(\Delta u)\cdot v - (\Delta v)\cdot u}{(v+\Delta v)\cdot v}$$
$$\frac{1}{\Delta x}\left(\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)}\right)$$
$$= \frac{\frac{\Delta u}{\Delta x}\cdot v - \frac{\Delta v}{\Delta x}\cdot u}{(v+\Delta\theta)\cdot v}$$
$$\xrightarrow[\Delta x\to 0]{lim} \frac{\frac{du}{dx}\cdot v - \frac{dv}{dx}\cdot u}{v^2}$$
$$= \frac{u'v - v'u}{v^2}$$

- Note:

$$(u+v)(x) = u(x) + v(x)\text{、} uv(x) = u(x)\cdot v(x)$$

2. Special formula：

- $\frac{d}{dx}x^n = n\cdot x^{n-1}$
- $\frac{d}{dx}\sin x = \cos x$

Proof: $\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x)-\sin(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\sin x \cdot \frac{\cos \Delta x - 1}{\Delta x} + \cos x \cdot \frac{\sin \Delta x}{\Delta x} \right)$$

$$= \cos x$$

- $\frac{d}{dx}\cos\ x\& = -\sin\ x$

$$Proof: \frac{d}{dx}\cos x = \lim_{\Delta x \to 0} \frac{\cos(x + ax) - \cos(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\cos \mathrm{x} \cos x - \sin \mathrm{x} \sin x - \cos x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left(\cos x \cdot \frac{\cos x - 1}{\Delta x} - \sin x \cdot \frac{\sin x}{\Delta x} \right)$$
$$= -\sin x$$

Lecture 4 Chain Rule and Higher-Order Derivatives

1. Chain Rule:

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$$

$$\frac{d}{dt} f\big(g(t)\big) = f'\big(g(t)\big) \cdot g'(t)$$

$$(fog)(x) = f(g(x))、(gof)(x) = g(f(x))$$

2. Higher-Order Derivatives：

$$f^{(n)}(x) = D^n f = \frac{d^n f}{dx^n}$$

Lecture 5 Implicit functions and Inverse