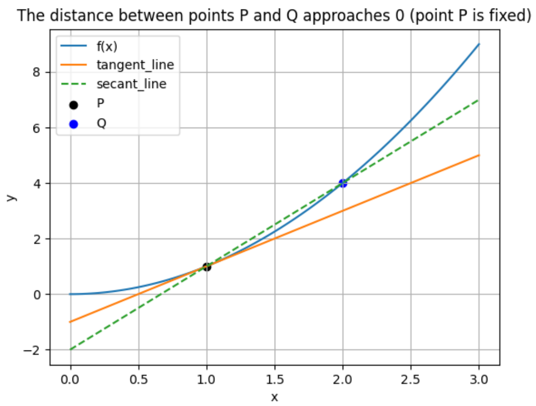
Unit 1 Derivatives

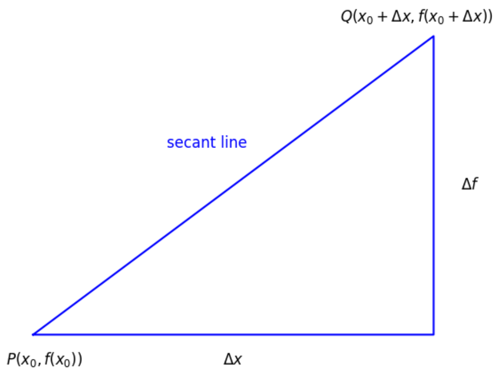
Lecture 1 What is a derivative?

1. The derivative is the limit of the secant line approaching the tangent line:

- Geometric interpretation:



- Algebraic Explanation:



$$\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0)$$

2. Example:

$$f(x)=\frac{1}{x}$$

$$\begin{aligned}
proof{:}\frac{\Delta f}{\Delta x}&=\frac{\frac{1}{(x+\Delta x)}-\frac{1}{x}}{\Delta x}\\
&=\frac{-\Delta x}{\Delta x\cdot x\cdot(x+\Delta x)}\\
&=\frac{-1}{x\cdot(x+\Delta x)}\\
f'(x)&=\lim_{\Delta x\to 0}\frac{\Delta t}{\Delta x}\\
&=\lim_{\Delta x\to 0}\frac{-1}{x\cdot(x+\Delta x)}\\
&=-\frac{1}{x^2}
\end{aligned}$$

$$f(x)=x^n$$

$$\begin{aligned}
proof{:}\frac{\Delta f}{\Delta x}&=\frac{(x+\Delta x)^n-x^n}{\Delta x}\\
&=\frac{x^n+n\cdot\Delta x\cdot x^{n-1}+o[(\Delta x)^2]-x^n}{\Delta x}\\
&=n\cdot x^{n-1}+o(\Delta x)\\
f'(x)&=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}\\
&=\lim_{\Delta x\to 0}[n\cdot x^{n-1}+o(\Delta x)]\\
&=n\cdot x^{n-1}
\end{aligned}$$

3. Note:

Tangent Line Equation:

$$y - y_0 = f'(x_0)(x - x_0)$$

Lecture 2 Limits and Continuity

1. Limits:

$$\lim_{x \to x_0} \frac{\Delta f}{\Delta x} = \lim_{x \to x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

2. Continuity:

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- Removable discontinuities of the first type:

  $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) \neq f(x_0)$ or $f(x_0)$ is not defined.

- Jump discontinuities of the first type:

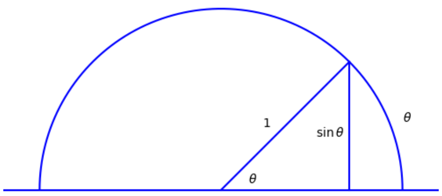
$$\lim_{x \to x_0^+} f(x) = a \neq b = \lim_{x \to x_0^-} f(x)$$

- Infinite discontinuities of the second type:

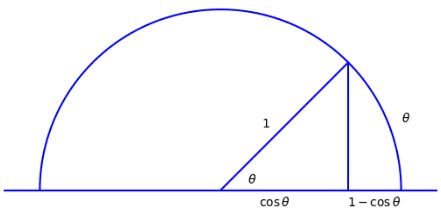
$$\lim_{x \to x_0^+} f(x) / \lim_{x \to x_0^-} f(x) \to \infty$$

- Other discontinuities of the second type.

3. Two trigonometric limits:



$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



$$\lim_{\theta \to 0} \frac{1 - \cos\ \theta}{\theta} = 0$$

4. Theorem:

A differentiable function must be continuous:

$$
\begin{aligned}
proof: & \lim_{x \to x_0} (f(x) - f(x_0)) \\
& = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \\
& = f(x_0) \cdot 0 = 0
\end{aligned}
$$

Lecture 3 Derivative formula

1. General formula:

- $(u + \varphi)' = u' + \varphi'$

Proof: $(u + \varphi)'(\delta) = \lim_{\Delta x \to 0} \frac{(u+\varphi)(\delta+\Delta x)-(u+\varphi)(\delta)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{u(\delta + \Delta x) + \varphi(\delta + \Delta x) - u(\delta) - \varphi(\delta)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{u(\delta + \Delta x) - u(\delta)}{\Delta x} + \frac{\varphi(\delta + \Delta x) - \varphi(\delta)}{\Delta x} \right)$$

$$= (\mathrm{u})'(\delta) \ + \ (\varphi)'(\delta)$$

- $(Cu)'(C \cdot u)' = C \cdot u'$

$$
\begin{aligned}
& (C \cdot u)'(x) \\
& = \lim_{\Delta x \to 0} \frac{C \cdot u(x + \Delta x) - C \cdot u(x)}{\Delta x} \\
& = C \cdot \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} \\
& = C \cdot u'(x)
\end{aligned}
$$

- $(u \cdot v)' = u' \cdot v + u \cdot v'$

Proof: $(u \cdot v)' = \lim_{\Delta x \to 0} \frac{(u \cdot v)(x+\Delta x)-(u \cdot v)(x)}{\Delta x}$

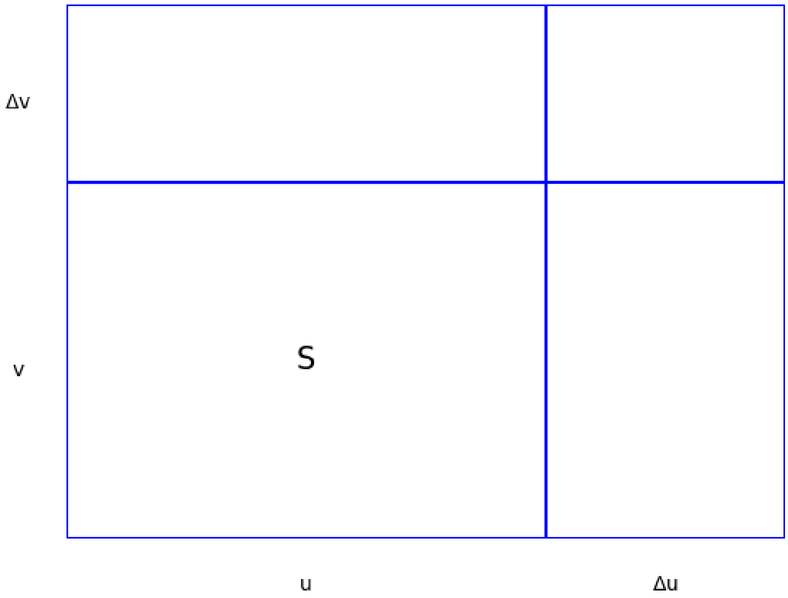
$$= \lim_{\Delta x \to 0} \frac{(u \cdot v)(x + \Delta x) + u(x + \Delta x) \cdot v(x) - u(x + \Delta x) \cdot v(x) - (u \cdot v)(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(u(x + \Delta x) \cdot \frac{v(x + \Delta x) - v(x)}{\Delta x} + v(x) \cdot \frac{u(x + \Delta x) - u(x)}{\Delta x} \right)$$

$$= u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

Geometric interpretation:

$$\Delta S = \Delta v \cdot u + \Delta u \cdot v$$



- $(\frac{u}{v})' = \frac{u'v-uv'}{v'}$

$$Proof: \frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)}$$
$$= \frac{(u+\Delta u)\cdot v - (v+\Delta v)\cdot u}{(v+\Delta v)\cdot v}$$
$$= \frac{(\Delta u)\cdot v - (\Delta v)\cdot u}{(v+\Delta v)\cdot v}$$
$$\frac{1}{\Delta x}\left(\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)}\right)$$
$$= \frac{\frac{\Delta u}{\Delta x}\cdot v - \frac{\Delta v}{\Delta x}\cdot u}{(v+\Delta\theta)\cdot v}$$
$$\xrightarrow[\Delta x\to 0]{lim} \frac{\frac{du}{dx}\cdot v - \frac{dv}{dx}\cdot u}{v^2}$$
$$= \frac{u'v - v'u}{v^2}$$

- Note:

$$(u+v)(x) = u(x) + v(x)\text{、}\ uv(x) = u(x)\cdot v(x)$$

2. Special formula：

- $\frac{d}{dx}x^n = n\cdot x^{n-1}$
- $\frac{d}{dx}\sin x = \cos x$

Proof: $\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x)-\sin(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos\Delta x + \sin\Delta x \cos x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\sin x \cdot \frac{\cos\Delta x - 1}{\Delta x} + \cos x \cdot \frac{\sin\Delta x}{\Delta x}\right)$$

$$= \cos x$$

- $\frac{d}{dx}\cos\ x\& = -\sin\ x$

$$Proof: \frac{d}{dx}\cos x = \lim_{\Delta x \to 0} \frac{\cos(x + ax) - \cos(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\cos \mathrm{x} \cos x - \sin \mathrm{x} \sin x - \cos x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left(\cos x \cdot \frac{\cos x - 1}{\Delta x} - \sin x \cdot \frac{\sin x}{\Delta x}\right)$$
$$= -\sin x$$

Lecture 4 Chain Rule and Higher-Order Derivatives

1. Chain Rule:

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$$

$$\frac{d}{dt} f\big(g(t)\big) = f'\big(g(t)\big) \cdot g'(t)$$

$$(fog)(x) = f(g(x))\text{、}(gof)(x) = g(f(x))$$

2. Higher-Order Derivatives：

$$f^{(n)}(x) = D^n f = \frac{d^n f}{dx^n}$$

Lecture 5 Implicit functions and Inverse

1. Implicit function calculation method：

Derivatives ($\frac{dy}{dx}$) are taken for all terms and finally separated.

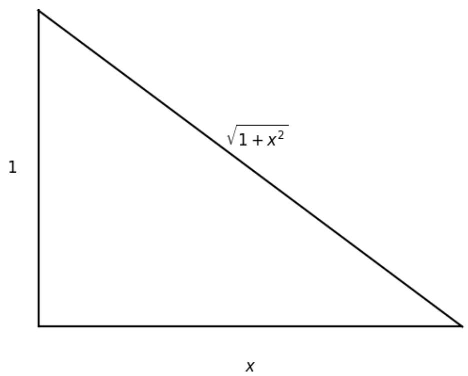
2. Inverse:

$$x = g(y) = f^{-1}(y)$$
$$\big(f^{-1}(y)\big)' \cdot y' = 1$$

$$
\begin{aligned}
&Proof:\ \because f^{-1}(y) = x \\
&\therefore \frac{d}{dx} f(y) = \frac{d}{dx} \cdot x = 1 \\
&\Rightarrow \frac{d}{dy} \cdot \frac{dy}{dx} f(y) = 1 \\
&\Rightarrow \frac{d}{dy} f(y) \cdot \frac{dy}{dx} = 1 \\
&\Rightarrow \left(f^{-1}(y)\right)' \cdot y' = 1
\end{aligned}
$$

- Among them, the nested expressions of trigonometric functions and inverse trigonometric functions can be expressed through geometric substitution:

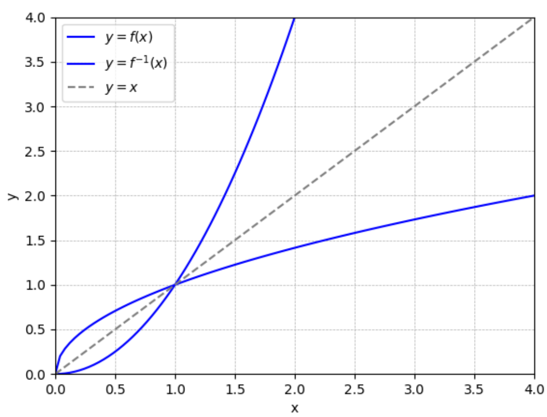
$$
\begin{aligned}
&y = \arctan x \\
&\Rightarrow \tan y = x \\
&\Rightarrow (\tan y)' \cdot y' = 1 \\
&\Rightarrow \frac{1}{\cos^2 y} \cdot y' = 1 \\
&\Rightarrow y' = \cos^2 y \\
&\Rightarrow y' = \cos^2(\arctan x) = \frac{1}{1+x^2}
\end{aligned}
$$



$$\Rightarrow \cos^2\ y = \frac{1}{1+x^2}$$

- Geometric relationship:

Symmetrical with the original function about line $y = x$.



- Notes:

$$f^{-1}(f(x)) = f(f^{-1}(x)) = f^{-1}of(x) = fof^{-1}(x) = x$$

Lecture 6 Exp and Log function derivatives

1. Exp function: $y = a^x$

$$\because \frac{d}{dx}a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

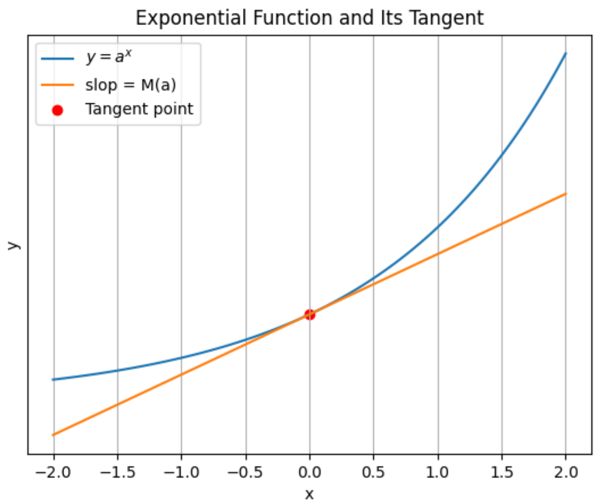
$$= \lim_{\Delta x \to 0} a^x \cdot \frac{a^{\Delta x} - 1}{\Delta x}$$

$$= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$\therefore \text{assuming } M(x) \equiv \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$\Rightarrow \frac{d}{dx}a^x = M(x) \cdot a^x$$

After analysis, $M(a) = \frac{d}{dx}a^x|_{x=0}$.



Consider a slope equal to 1

According to the geometric form of the derivative:

$$M(2) < 1, M(3) > 1$$
$$hence, M(e) \equiv 1, 2 < e < 3$$
$$\Rightarrow \lim_{\lambda \to 0} \frac{e^\lambda - 1}{\lambda} = 1 \Rightarrow \frac{d}{dx}e^x = e^x.$$

2. Log function: $y = lnx$ (Inverse $y = e^x$)

- Basic theory: $\ln(x_1 x_2) = lnx_1 + lnx_2$

$$\begin{aligned} &\because y = \ln x \Rightarrow e^y = x \\ &\Rightarrow (e^y)' \cdot y' = 1 \\ &\Rightarrow y' = \frac{1}{e^y} = \frac{1}{x} \end{aligned}$$

Then, for the exponential function $y = a^x$, we have:

$$y = \left(e^{\ln a}\right)^x = e^{x\ln a}$$

According to the chain rule:

$$\begin{aligned} y' &= \frac{d}{dx} e^{x\ln a} = (\ln a) \cdot e^{x\ln a} \\ &= (\ln a) \cdot a^x \\ \Rightarrow \frac{d}{dx} a^x &= (\ln a) \cdot a^x \end{aligned}$$

3. Derivative Applications:

$$\begin{aligned} &y = x^x \\ &\because y = e^{x\ln x} \Rightarrow \ln y = x\ln x \\ &\therefore y'/y = \ln x + 1 \\ &\Rightarrow y' = y(\ln x + 1) \\ &\Rightarrow y' = x^x \cdot (\ln x + 1) \end{aligned}$$

$$\lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k$$

$$\because \lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k = \lim_{k\to\infty} e^{k\cdot ln\left(1+\frac{1}{k}\right)}$$

Consider the exponential equation,

$$\begin{aligned} &y = \frac{1}{x}\ln(1+x), x = \frac{1}{k} \\ &\Rightarrow \lim_{x\to 0} y = \lim_{x\to 0}\frac{\ln(1+x) - \ln(1)}{x} \\ &= \frac{d}{dx}\ln(x)|_{x=1} = 1 \\ &\Rightarrow \lim_{k\to\infty} k\ln\left(1 + \frac{1}{k}\right) = 1 \\ &\Rightarrow \lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k = e \end{aligned}$$

Lecture 7 Hyperbolic function

1. Manifestation：

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}$$

2. Derivative：

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \cosh(x)$$

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

3. Theorem:

$$\cosh^2(x) - \sinh^2(x) = 1$$

Trigonometric functions (circular functions): $\sin^2 x + \cos^2 x = 1$

$$
\begin{aligned}
Proof:\ &\because \cosh^2(x) - \sinh^2(x) \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\
&= 1
\end{aligned}
$$