Derivation of the Universal Formula for the Time Required for Releasing Water ΔV

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Abstract

In Interdisciplinary Discovery for Educational Advancement (IDEA) Project, we tried to create a fully-automated, self-sustaining, botanical garden with data-monitoring capabilities, for residential use. In order to always provide the plants appropriate amount of water, there was a necessity of deriving a general formula for the water released, and this paper shows the detail approaches to the formula.

1 Introduction

In order to water the plants properly, we need to find the amount of water running out of the tank. However, as the amount of the remaining water in the tank decreases, the pressure on the valves changes, meaning the amount of water released in a certain time changes as well. Here, taking the changes in the water pressure into consideration, we are trying to calculate the time required to release a certain amount of water through a valve.

2 Derivation

Here are the abbreviations for the terms we use in this paper.

$$\begin{aligned} R &= \text{Radius of the tank} \\ r &= \text{Radius of the valve} \\ y &= \text{Height of the water level in the tank} \\ C_d &= \text{the discharge coefficient} \\ C_c &= \text{the contraction coefficient (sharp edge aperture: } C_c\text{= 0.62, well-rounded aperture: } C_c = 0.97\text{)} \end{aligned}$$

$$\frac{dy}{dt} = C_v\sqrt{2gy} \quad \text{where} \quad C_v = 0.97$$

Using the information above, the equation is derived through the following steps.

$$\begin{aligned} \frac{dV}{dt} &= C_d A\sqrt{2gy} \quad \text{where} \quad C_d = C_c * C_v = 0.62 * C_v = 0.6 \qquad (1) \\ A &= \pi r^2 \end{aligned}$$

Put C_d and A in the equation (1)

$$\begin{aligned} \frac{dV}{dt} &= C_d A\sqrt{2gy} \\ &= 0.6\pi r^2\sqrt{2gy} \end{aligned}$$

$$\begin{aligned} V &= \pi R^2 y \quad \text{(volume of a cylinder)} \\ \frac{dV}{dt} &= \pi R^2 \frac{dy}{dt} \qquad (2) \end{aligned}$$

Then, we combine the equation (1) with (2).

$$
\begin{aligned}
0.6\pi r^2\sqrt{2gy} &= \pi R^2\frac{dy}{dt} \\
0.6r^2\sqrt{2gy^2} &= R^2\frac{dy}{dt} \\
\frac{0.6r^2}{R^2}\sqrt{2g}dt &= y^{-1/2}dy \quad \text{(separation of variables)} \\
\frac{0.6r^2}{R^2}\sqrt{2g}\int dt &= \int y^{-1/2}dy \\
\frac{0.6r^2}{R^2}\sqrt{2g}t + C_1 &= 2y^{-1/2} + C_2 \\
y &= (\frac{0.3r^2}{R^2}\sqrt{2g}t + C)^2
\end{aligned}
$$

Because t=0, y=h, C=$\sqrt{h}$.

$$
y = (\frac{0.3r^2}{R^2}\sqrt{2g}t + \sqrt{h})^2 \tag{3}
$$

Plug in the equation (3) to the equation (1).

$$
\begin{aligned}
\frac{dV}{dt} &= 0.6\pi r^2\sqrt{2g}[\frac{0.3r^2}{R^2}\sqrt{2g}t + \sqrt{h}] \\
\frac{dV}{dt} &= C_0\left[\frac{C_0}{2R^2}t + \sqrt{h}\right] \quad \text{where} \quad C_0 = 0.6\pi r^2\sqrt{2g} \\
dV &= C_0\left[\frac{C_0}{2R^2}t + \sqrt{h}\right]dt \\
\int_{V_i}^{V_f} dV &= C_0\left[\frac{C_0}{2R^2}\int_{t_i}^{t_f} tdt + \sqrt{h}\int_{t_i}^{t_f} dt\right] \\
\Delta V &= \frac{{C_0}^2}{4R}({t_f}^2 - {t_i}^2) + C_0\sqrt{h}(t_f - t_i) \\
0 &= \frac{{C_0}^2}{4R}{t_f}^2 + C_0\sqrt{h}t_f - \left[\frac{{C_0}^2}{4R}{t_i}^2 + C_0\sqrt{h}t_i + \Delta V\right] \\
t_f &= \frac{-C_0\sqrt{h} + \sqrt{{C_0}^2 h + \frac{{C_0}^2}{R}\left[\frac{{C_0}^2}{4R}{t_i}^2 + C_0\sqrt{h}t_i + \Delta V\right]}}{\frac{{C_0}^2}{2R}} \\
t_f &= \frac{-\sqrt{h} \pm \sqrt{h + \frac{1}{R}\left[\frac{{C_0}^2}{4R}{t_i}^2 + C_0\sqrt{h}t_i + \Delta V\right]}}{\frac{C_0}{2R}}
\end{aligned}
$$

As a result of the calculation above, the time required to release ΔV amount of water can be derived using this universal formula. ΔV is the constant amount of water we want to release from the tank each time a plant is watered.

3 Example

Now, using the above formula, we will use some examples to show how it actually works.

Example 1: Find t_f when we want to release 0.00050 m^3, and the water tank is full. Use the following values:

h=0.20 m
R=0.20 m
r=0.00025 m
ΔV_i=0.00050 m^3
t_i=0

Firstly, we need to calculate the C_0.

$$\begin{aligned} C_0 &= 0.6 * \pi * r^2 \sqrt{2g} \\ &= 0.6 * \pi * 0.00025^2 \sqrt{2 * 9.80} \\ &= 0.000052 \\ &= 5.2 * 10^{-5} \end{aligned}$$

Then, plug in the C_0 into the universal formula we got in the last section.

$$\begin{aligned} t_f &= \frac{-\sqrt{h} \pm \sqrt{h + \frac{1}{R}\left[\frac{C_0{}^2}{4R} t_i{}^2 + C_0\sqrt{h} t_i + \Delta V\right]}}{\frac{C_0}{2R}} \\ t_f &= \frac{-\sqrt{0.20} \pm \sqrt{0.20 + \frac{1}{0.20}\left[\frac{5.2*10^{-5}{}^2}{4*0.20} 0^2 + 5.2 * 10^{-5}\sqrt{0.20}0 + 0.00050\right]}}{\frac{5.2*10^{-5}}{2*0.20}} \\ &= \frac{-\sqrt{0.20} + \sqrt{0.20 + \frac{0.00050}{0.20}}}{\frac{5.2*10^{-5}}{0.20}} \\ &= 10.7169... \\ &\approx 10.7 \text{ seconds} \end{aligned}$$

This value means that it takes 10.7 seconds to release 0.00050 m^3 from one valve when the water tank is full.

Example 2: Calculate the time required to release 0.00050 m^3 when the height of the water left in the tank is 0.10 m. Use the following values:

h=0.20 m
R=0.20 m
r=0.00025 m
ΔV_i=0.00050 m^3
t_i=0

This example is much more complex than the first one, and the solution will be composed of two parts. The first part will calculate the time required to release water until the height of the water left in the tank reaches 0.1 m. This time will be the t_i of the next part. Here, we call the amount of water that is released until the height reaches 0.1 m ΔV_i.

As ΔV_i is the amount of water already released, the amount corresponds to the volume of the empty part of the tank, which can be derived from the formula: $V = R^2 * h * \pi$.

$$\begin{aligned} \Delta V_i &= 0.2^2 * 0.2 * \pi \\ &= 0.012566... \\ &\approx 1.3 * 10^{-2} \end{aligned}$$

Using this number and other given values, we now calculate the t_i.

$$
\begin{aligned}
t_i &= \frac{-\sqrt{h} + \sqrt{h + \frac{\Delta V_i}{R}}}{\frac{C_0}{R}} \\
&= \frac{-\sqrt{0.20} + \sqrt{0.20 + \frac{1.3*10^{-2}}{0.20}}}{\frac{5.2*10^{-5}}{0.20}} \\
&= 259.877... \\
&\approx 260 \text{ seconds}
\end{aligned}
$$

This means it takes 260 seconds to release 0.13 m^3, which corresponds to the half volume of the tank.

In the second part, we will calculate the time required to release 0.00050 m^3 in the situation described above using the values gained in the first part.

$$
\begin{aligned}
t_f &= \frac{-\sqrt{h} + \sqrt{h + \frac{1}{R}\left[\frac{{C_0}^2}{4R}{t_i}^2 + C_0\sqrt{h}t_i + \Delta V\right]}}{\frac{C_0}{2R}} \\
&= \frac{-\sqrt{0.20} + \sqrt{0.20 + \frac{1}{0.20}\left[\frac{{5.2*10^{-5}}^2}{4*0.20}260^2 + 5.2*10^{-5}\sqrt{0.20}*260 + 0.00050\right]}}{\frac{5.2*10^{-5}}{2*0.20}} \\
&= 44.86 \text{ seconds}
\end{aligned}
$$

After the calculation, we set up the same conditions as the example and checked if our universal formula and the answer were correct. In the actual measurement, it took 45 seconds to release 0.00050 m^3, which is so close to our answer derived from the calculation!!