Introduction to the Fundamental Theorem of Asse pricing

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Part I

Discrete time, finite assets, and finite terminal models

1 Settings

(1) $\mathbb{N} = \{1, 2, \ldots\},\ T \in \mathbb{N},\ d \in \mathbb{N}$. T denotes the terminal time and d does the number of asset prices in a market.

(2) $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\{0,1,\ldots,T\}})$: a filtered probability space.

(3) An $\mathbb{R}^d$-dimensional adapted process $S = (S_t)_{t\in\{0,\ldots,T\}}$ is called a price process. . This denotes the price process of d kinds of risky assets. In addition, we suppose that cash is also traded in the market.

(4) An $\mathbb{R}^d$-valued predictable process $H = (H_t)_{t\in\{0,\ldots,T\}}$ is called a (trading) strategy or a portfolio process . For fixed t, the $\mathbb{R}^d$-valued random variable H_t is called the portfolio at time t.

(5) For a price process S and a strategy H, we define a new process $H \bullet S$ as follows.

$$
\begin{aligned}
&(H \bullet S)_0 = 0, \\
&(H \bullet S)_t = \sum_{u=1}^{t} H_u \cdot \Delta S_u = \sum_{u=1}^{t} H_u \cdot (S_u - S_{u-1}) \quad t \in \{1, \ldots, T\},
\end{aligned}
$$

where $\cdot$ is the usual inner product of $\mathbb{R}^d$. $H \bullet S$ is often called the discrete stochastic integral or martingale transform of H by S. This can be interpreted as the accumulated wealth process of the portfolio trading strategy H with initial valued 0.

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(6) An *arbitrage* is a strategy H which satisfies

$$(H \bullet S)_T \geq 0 \ a.s. \quad \text{and} \quad \mathbb{P}\left[(H \bullet S)_T > 0\right] > 0.$$

If there is no arbitrage, we say that S satisfies the *No Arbitrage (NA)* condition.

(7) A probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ is called a *equivalent martingale measure (EMM)* of S if S is a $\mathbb{Q}$-martingale.

(8) $\mathbb{R}_+ = [0, \infty[$, $\mathbb{R}_{++} =]0, \infty[$.

(9) Given a class of real random variables $\mathscr{A}$, we define subsets of $\mathscr{A}$ as follows.

$$\begin{aligned} \mathscr{A}_+ &= \{X \in \mathscr{A} \mid X \geq 0 \text{ a.s.}\} \\ \mathscr{A}_{++} &= \{X \in \mathscr{A} \mid X > 0 \text{ a.s.}\} \\ \mathscr{A}_- &= \{X \in \mathscr{A} \mid X \leq 0 \text{ a.s.}\} \end{aligned}$$

(10) A $\{0, \ldots, T\} \cup \infty$-valued random variable τ is called a stopping time if

$$\forall t \in \{0, \ldots, T\} \quad \{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathscr{F}_t.$$

(11) $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ for $a, b \in \mathbb{R}$.

(12) Let $\mathscr{A}_1$, $\mathscr{A}_2$ be a subset of an algebra. We define the Minkowski addition (subtraction, product) $\mathscr{A}_1 + \mathscr{A}_2$ ($\mathscr{A}_1 - \mathscr{A}_2$, $\mathscr{A}_1 \overset{m}{\times} \mathscr{A}_2$, respectively) by the following formulae.

$$\begin{aligned} \mathscr{A}_1 + \mathscr{A}_2 &= \{a + b \mid a \in \mathscr{A}_1,\ b \in \mathscr{A}_2\} \\ \mathscr{A}_1 - \mathscr{A}_2 &= \{a - b \mid a \in \mathscr{A}_1,\ b \in \mathscr{A}_2\} \\ \mathscr{A}_1 \overset{m}{\times} \mathscr{A}_2 &= \{ab \mid a \in \mathscr{A}_1,\ b \in \mathscr{A}_2\} \end{aligned}$$

2 The fundamental theorem of asset pricing on finite probability spaces (The Harrison-Kreps theorem)

Theorem 2.1 (Harrison and Kreps [10])**.** On a finite probability space, the following conditions are equivalent.

(1) S satisfies (NA).

(2) S has an EMM.

Proof. **Step 1: (ii) $\Longrightarrow$ (i).** Let $\mathbb{Q}$ be a EMM for S. Then $K \bullet S$ is again a $\mathbb{Q}$-martingale for any strategy K. [1]. Therefore

$$\mathbb{E}^{\mathbb{Q}}[(K \bullet S)_T] = \mathbb{E}^{\mathbb{Q}}[(K \bullet S)_0] = 0. \tag{1}$$

holds for every strategy K. Choose a strategy H such that $(H \bullet S)_T \geq 0$ $\mathbb{Q}$-a.s.. By (1) and nonnegativeness of $(H \bullet S)_T$, we get $(H \bullet S)_T = 0$ $\mathbb{P}$-a.s.. Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, we see that

$$(H \bullet S)_T \geq 0,\ \mathbb{P}\text{-a.s.} \implies (H \bullet S)_T = 0,\ \mathbb{P}\text{-a.s.} \tag{2}$$

[1] All predicable process is bounded because Ω is finite.

holds for all the strategies.

Step 2: (i) $\Longrightarrow$ (ii). We can assume, without loss of generality, that $\mathscr{F} = \mathscr{F}_T = 2^\Omega$ and $\mathbb{P}(\{\omega\}) > 0$ holds for all $\omega \in \Omega$. Let $\#\Omega = n$ and let us identify a random variable: $\Omega \to \mathbb{R}$ with an element of $\mathbb{R}^n$. Moreover, a probability measure $\mathbb{P}$ is identified with a vector $(\mathbb{P}(\omega))_{\omega\in\Omega} \in \mathbb{R}^n$. From now on we use $p = (p_1, \dots, p_n)$ to denote his probability vector. Then the expectation of a random variable $x \in \mathbb{R}^n$ by the probability p is equal to the inner product $x \cdot p$.

We define a subset $\mathscr{A}$ of $\mathbb{R}^n$ as

$$\mathscr{A} = \{(H \bullet S)_T \mid H \text{ is predictable}\}.$$

$\mathscr{A}$ is a closed[2] vector subspace [3] of $\mathbb{R}^n$, and, in addition, convex [4]. Let $(e_k)_{k\in\{1,\dots,n\}}$ be a standard basis of $\mathbb{R}^n$ and $\mathscr{B}$ be the convex convex hull of $\{e_1, \dots, e_n\}$[5]. $\mathscr{B}$ is clearly bounded and closed, and hence, compact. Since $x \in \mathscr{B}$ are nonnegative and satisfies $x \cdot p > 0$, we have $P(x > 0) > 0$. Then by the condition (NA), we know that $x \notin \mathscr{A}$. Therefore $\mathscr{A} \cap \mathscr{B} = \emptyset$. Here we use the following Hahn-Banch separation theorem.

Hahn-Banach Separation Theorem

Let X be a locally convex TVS. We consider two nonempty disjoint subsets $\mathscr{A}$ and $\mathscr{B}$ of X. Suppose that $\mathscr{A}$ is nonempty, closed, and convex, and $\mathscr{B}$ is nonempty, compact, convex. Then there exists a $g \in X^*$ such that $\inf_{x\in\mathscr{B}} g(x) > \sup_{x\in\mathscr{A}} g(x)$.

The linear form g is represented by a vector $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ as $g(x) = q \cdot x$. Since $\mathscr{A}$ is a vector subspace, we have for any $x \in \mathscr{A}$

$$g(\mathscr{A}) = \{q \cdot y \mid y \in \mathscr{A}\} \supset \{\lambda(q \cdot x) \mid \lambda \in \mathbb{R}\} = \mathbb{R}(q \cdot x).$$

This inclusion and the upper boundedness of $g(\mathscr{A})$[6] implies that $g(x) = q \cdot x = 0$ holds for all $x \in \mathscr{A}$. Hence, $\sup_{x\in\mathscr{A}} g(x) = 0$ and $q \in \mathscr{A}^\perp$ follows. We can deduce $q \in \mathbb{R}_{++}$ from the following inequality.

$$q_k = q \cdot e_k \geq \inf_{x\in\mathscr{B}} q \cdot x > 0 = \sup_{x\in\mathscr{A}} q \cdot x \quad k \in \{1, \dots, n\}$$

Thus q can be identified with a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ [7].

It remains to prove that S is a $\mathbb{Q}$-martingale. Since q belongs to $\mathscr{A}^+$, we see that $\mathbb{E}^{\mathbb{Q}}[(H \bullet S)_T] = 0$ holds for any strategy H. Choose $i \in \{1, \dots, d\}$, $t \in \{0, \dots, T\}$, and $B \in \mathscr{F}_t$ arbitrary. We define a strategy $H^{(j)}$ by the equation

$$H^{(j)} = \begin{cases} \mathbb{1}_{B\times\{t+1,\dots,T\}} & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

This strategy obviously satisfies

$$(H \bullet S)_T = \mathbb{1}_B(S_T^{(i)} - S_t^{(i)}).$$

[2]Every finite dimensional topological vector subspace is closed.

[3]This follows from the linearity of martingale transformation.

[4]Every vector subspace is convex.

[5]That is

$$\mathscr{B} = \left\{ \sum_{1\leq k\leq n} \alpha_k e_k \,\middle|\, \alpha_k \geq 0, \sum_{1\leq k\leq n} \alpha_k = 1 \right\}.$$

[6]$\inf_{x\in\mathscr{B}} g(x)$ is an upper bound.

[7]$\mathbb{P}' \sim \mathbb{P}$ is equivalent to the condition "$\forall\omega \in \Omega,\ \mathbb{P}'(\omega) > 0$" because $\mathbb{P}(\omega) > 0$ holds for all ω.

Therefore, we have

$$\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_B(S_T^{(i)} - S_t^{(i)})] = \mathbb{E}^{\mathbb{Q}}[(H \bullet S)_T] = 0.$$

This proves that S satisfies $S_t^{(i)} = \mathbb{E}[S_T^{(i)}|\mathscr{F}_t]$ for ann i, t, and consequently S is a $\mathbb{Q}$-martingale. □

Remark 2.2. Theorem 2.1 is equivalent to the following "Gordan's theorem (or Stiemke's lemma), which is known as a theorem of linear algebra.

Gordan's Theorem

Let $\mathscr{A}$ be a vector subspace of $\mathbb{R}^n$. Then the following two conditions are equivalent.

(1) $\mathscr{A} \cap \mathbb{R}_+^n = \{0\}$.

(2) $\mathscr{A}^\perp \cap \mathbb{R}_{++}^n \neq \emptyset$.

Remark 2.3. The discussion in the proof of Theorem 2.1 does not work if Ω is infinite. We need to apply the Hahn-Banach theorem to a different object. Let

$$\mathscr{C} := \mathscr{A} - \mathbb{R}_+^n = \mathscr{A} + \mathbb{R}_-^n.$$

Under the situation of Theorem 2.1 $\mathscr{C}$ is a finitely generated convex cone [8] [9]. and hence, closed [10].

Theorem 2.4 (Kreps-Yan)**.** Let $(\Omega, \mathscr{F}, \mathbb{P})$ be an arbitrary probability space, and E be a locally convex TVS that includes $L^\infty(\mathbb{P})$. Suppose that every element of the dual E^* is identified with a continuous linear form defined by a element of some Banach space $F \subset L^1(\mathbb{P})$ [11]; i.e. any bounded linear form on E has the following representation.

$$E \ni X \longmapsto E^{\mathbb{P}}[ZX] \in \mathbb{R}$$

where Z is a proper random variable of F. We consider a closed convex cone $\mathscr{C} \subset E$ that satisfies $\mathscr{C} \supset E_-$ and $\mathscr{C} \cap E_+ = \{0\}$. There there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ such that $\mathbb{P} \sim \mathbb{Q}$, $\frac{d\mathbb{Q}}{d\mathbb{P}} \in F$ and for any $X \in \mathscr{C}$, $\mathbb{E}^{\mathbb{Q}}[X] \leq 0$.

Proof. Let us define a nonempty convex cone $\mathscr{D}$ of F as the following.

$$\mathscr{D} = \left\{ Y \in F_+ \mid \forall X \in \mathscr{C}\ \mathbb{E}^{\mathbb{P}}[YX] \leq 0 \right\}.$$

Zero element obviously belongs to $\mathscr{D}$. We will show that $\mathscr{D}$ has non-zero elements. Since the constant function $1 \in E_+$ does not belongs to $\mathscr{C}$, we have $\{1\} \cap \mathscr{C} = \emptyset$. The set $\{1\}$ is clearly compact and $\mathscr{C}$ is, by assumption, closed and convex. Therefore by the Hahn-Banach separation theorem, there is a $g_0 \in E^*$ such that

$$g_0(1) > \sup_{X \in \mathscr{C}} g_0(X).$$

[8] A cone is a subset of a vector space which is closed under nonnegative scalar multiplication.

[9] $\mathscr{C}$ is a finitely generated convex cone if there are elements $x_1, \ldots, x_k$ such that

$$\mathscr{C} = \left\{ \sum_{j=1}^k \alpha_j x_j \,\middle|\, \alpha_1, \ldots, \alpha_k \geq 0 \right\},$$

[10] Every finitely generated convex cone is closed.

[11] F does not need to be the "topological" subspace of $L^1(\mathbb{P})$.

Take a unique element $Z_0 \in F$ that satisfies

$$g_0(X) = \mathbb{E}^{\mathbb{P}}[Z_0 X] \quad \forall X \in \mathscr{C}.$$

Then

$$\mathbb{E}^{\mathbb{P}}[Z_0] > \sup_{X \in \mathscr{C}} \mathbb{E}^{\mathbb{P}}[Z_0 X]. \tag{3}$$

We see that

$$\forall X \in \mathscr{C} \quad \mathbb{E}^{\mathbb{P}}[Z_0 X] \leq 0 \tag{4}$$

because the image $g_0(\mathscr{C})$ of the cone $\mathscr{C}$ is upper bounded. For any $A \in \mathscr{F}$ we have $-\mathbb{1}_A \in L^\infty_- \subset E_- \subset \mathscr{C}$, and hence, $\mathbb{E}^{\mathbb{P}}[Z_0 \mathbb{1}_A] \geq 0$ follows. This implies that Z_0 is nonngetive, and $Z_0 \in \mathscr{D}$. Combinig (3) and (4), we get

$$\mathbb{E}^{\mathbb{P}}[Z_0] > \sup_{X \in \mathscr{C}} \mathbb{E}^{\mathbb{P}}[Z_0 X] = 0.$$

Consequently $Z_0 \in \mathscr{D}$ is not 0.

The next step of this proof is to construct a r.v. $Z \in \mathscr{D}$ such that

$$\mathbb{P}[Z > 0] = \sup_{Y \in \mathscr{D}} \mathbb{P}[Y > 0] > 0.$$

For $k \in \mathbb{N}$, choose an element Y_k of $\mathscr{D}$ that satisfies

$$\mathbb{P}[Y_k > 0] > \left(-\frac{1}{k} + \sup_{Y \in \mathscr{D}} \mathbb{P}[Y > 0] \right) \vee 0.$$

Note that these Y_k's are not the zero element of F [12]. Let

$$Z = \sum_{k \in \mathbb{N}} \frac{1}{2^k} \frac{Y_k}{\|Y_k\|_F}.$$

This series converges absolutely, and therefore converges with respect to the norm of F. We want to prove that Z is in $\mathscr{D}$. A partial sum

$$Z^k = \sum_{1 \leq l \leq k} \frac{1}{2^k} \frac{Y_k}{\|Y_k\|_F}$$

of Z satisfies

$$\mathbb{E}^{\mathbb{P}}[Z^k X] = \sum_{1 \leq l \leq k} \frac{1}{2^k \|Y_k\|_F} \mathbb{E}^{\mathbb{P}}[Y_k X] \leq 0 \quad \forall X \in \mathscr{C}.$$

Then $Z^k \in \mathscr{D}$. Since the sequence (Z^k) converges to Z with respect to the norm of F, it converges in weak*-topology. In particular we have

$$\mathbb{E}^{\mathbb{P}}[Z^k X] \xrightarrow[k \to \infty]{} \mathbb{E}^{\mathbb{P}}[Z X] \quad \forall X \in \mathscr{C}.$$

This prove that $\mathbb{E}^{\mathbb{P}}[ZX] \leq 0$ holds for all $X \in \mathscr{C}$. Consequently, $Z \in \mathscr{D}$. By the definition of Z and Y_k, we see that $\{Y_k > 0\} \subset \{Z > 0\}$ and

$$\mathbb{P}[Z > 0] \geq \mathbb{P}[Y_k > 0] \geq \left(-\frac{1}{k} + \sup_{Y \in \mathscr{D}} \mathbb{P}[Y > 0] \right) \vee 0.$$

[12] By assumption the zero element of F is also zero of $L^1(\mathbb{P})$. These Y_k's are not, however, zero of $L^1(\mathbb{P})$.

Letting $k \to \infty$, we obtain

$$\mathbb{P}[Z > 0] \geq \sup_{Y \in \mathscr{D}} \mathbb{P}[Y > 0].$$

The inverse of this inequality follow from the fact that $Z \in \mathscr{D}$. Thus Z satisfies

$$\mathbb{P}[Z > 0] = \sup_{Y \in \mathscr{D}} \mathbb{P}[Y > 0] > 0.$$

We will next show that Z is almost surely positive. Assume $\mathbb{P}[Z > 0] < 1$. Since Z is nonnegative, $\mathbb{P}[Z > 0] < 1$ implies $\mathbb{1}_{\{Z=0\}} \neq 0$. Then, as in the discussion of the first paragraph of this proof, we can pick a $\widetilde{Z} \in \mathscr{D}$ such that

$$\mathbb{E}^{\mathbb{P}}[\widetilde{Z}\mathbb{1}_{\{Z=0\}}] > \sup_{X \in \mathscr{C}} \mathbb{E}^{\mathbb{P}}[\widetilde{Z}X] = 0.$$

Now we have $\mathbb{P}[Z = 0,\ \widetilde{Z} > 0] > 0$, and hence

$$\mathbb{P}[Z + \widetilde{Z} > 0] = \mathbb{P}[Z = 0,\ \widetilde{Z} > 0] + \mathbb{P}[Z > 0] > \mathbb{P}[Z > 0].$$

On the other hand, from $\widetilde{Z} + Z \in \mathscr{D}$, we can deduce

$$\mathbb{P}[Z > 0] = \sup_{Y \in \mathscr{D}} P[Y > 0] \geq \mathbb{P}[Z + \widetilde{Z} > 0].$$

In consequence, we see that the following inequality holds.

$$\mathbb{P}[Z + \widetilde{Z} > 0] > \mathbb{P}[Z > 0] \geq \mathbb{P}[Z + \widetilde{Z} > 0]$$

This is a contradiction, which proves $\mathbb{P}[Z > 0] = 1$.

Let $c = 1/E[Z] > 0$ and

$$\mathbb{Q}(A) = \int_{\Omega} cZ(\omega)\mathbb{P}(d\omega).$$

Then $\mathbb{Q}$ is a probability measure absolutely continuous w.r.t. $\mathbb{P}$. Since $Z \in F$ is almost surely positive valued, $\mathbb{Q}$ is, in fact, equivalent to $\mathbb{P}$. Recall that Z is a element of $\mathscr{D}$, by definition. cZ also belongs to $\mathscr{D}$ because $\mathscr{D}$ is a cone. Therefore $\mathbb{Q}$ satisfies

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[(cZ)X] \leq 0 \quad \forall X \in \mathscr{C}.$$

This completes the proof. □

Remark 2.5. (1) The Minkowski addition $\mathscr{A}_1 + \mathscr{A}_2$ of two convex cones $\mathscr{A}_1, \mathscr{A}_2 \in \mathbb{R}^n$ is again a convex cone, but not necessarily closed. Here is a counter example. Let

$$\mathscr{A}_1 = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2} \leq z\},$$
$$\mathscr{A}_2 = \{-t(1, 0, 1) \in \mathbb{R}^3 \mid t \in \mathbb{R}^3_+\}.$$

Though they are closed convex cones, $\mathscr{A}_1 + \mathscr{A}_2$ is not closed.

(2) Assume that two closed cones[13] $\mathscr{A}_1$ and $\mathscr{A}_2$ of $\mathbb{R}^n$ satisfy [14]

$$\forall x \in \mathscr{A}_1\ \forall y \in \mathscr{A}_2\ (x + y = 0 \implies x = y = 0).$$

Then $\mathscr{A}_1 + \mathscr{A}_2$ is a closed cone.

[13] It does not need to be convex.

[14] This condition is sometimes called "positively semi-independent".

$\because$ Suppose that a sequence $(z_k) = (x_k + y_k)_{k\in\mathbb{N}}$ of $\mathscr{A}_1 + \mathscr{A}_2$ $(x_k \in \mathscr{A}_1, y_k \in \mathscr{A}_2)$ converges to $z_\infty \in \mathbb{R}^n$.

If $(x_k)_{k\in\mathbb{N}}$ is bounded, we get $x_\infty := \lim_l x_{k_l} \in \mathscr{A}_1$ by taking a convergent subsequence $(x_{k_l})_{l\in\mathbb{N}}$. Then $(y_{k_l}) = (z_{k_l} - x_{k_l})$ also converges, and its limit $y_\infty := \lim_l y_{k_l}$ belongs to $\in \mathscr{A}_2$. Therefore we have $z_\infty = x_\infty + y_\infty \in \mathscr{A}_1 + \mathscr{A}_2$. Similarly we see that $z_\infty \in \mathscr{A}_1 + \mathscr{A}_2$ if (y_k) is bounded.

We consider the case where (x_k) and (y_k) are unbounded. Let us choose a subsequence such that $x_{k_l}, y_{k_l} \neq 0$ and $\|x_{k_l}\|, \|y_{k_l}\| \to \infty$. By the assumption $x_k + y_k$ are bounded[15], and hence

$$\frac{x_{k_l} + y_{k_l}}{\|x_{k_l}\|} \xrightarrow[l\to\infty]{} 0. \tag{5}$$

[16] Now we have $x_{k_l}/\|x_{k_l}\| \in \mathscr{A}_1$ and $y_{k_l}/\|x_{k_l}\| \in \mathscr{A}_2$ because $\mathscr{A}_1$ are $\mathscr{A}_2$ cones. A bounded sequence $x_{k_l}/\|x_{k_l}\|$ has a convergent subsequence and its limit belongs to $\mathscr{A}_1$[17]. Since the norm of each $x_{k_l}/\|x_{k_l}\|$ is equal to 1, the limit of a convergent subsequence has the same norm. However, by (5) and positively semi-independence, it must holds that

$$\lim_{k\to\infty} \frac{x_k}{\|x_k\|} = \lim_{k\to\infty} \frac{y_k}{\|x_k\|} = 0.$$

This is a contradiction and hence, one of (x_k) and (y_k) is bounded.

From the discussion above, we see that the limit of a convergent sequence of $\mathscr{A}_1 + \mathscr{A}_2$ still belongs to $\mathscr{A}_1 + \mathscr{A}_2$. Consequently $\mathscr{A}_1 + \mathscr{A}_2$ is closed.

3 Remarks on martingales and local martingales

In this section we review some important notions in the theory of martingales with a discrete time parameter. We assume that a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\{0,\dots,T\}}, \mathbb{P})$ is given throughout this section.

An adapted process $(M_t)_{t\in\{0,\dots,T\}}$ is called a *martingale* if

(1) for all $t \in \{0, \dots, T\}$, M_t is integrable,

(2) $E[M_t|\mathscr{F}_s] = M_s$ (a.s.) holds for all $0 \le s \le t \le T$.

We say that M is a *local martingale* if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that

(1) $\tau_1 \le \tau_2 \le \cdots \to \infty$ a.s.,

(2) $M^{\tau_n} - M_0$ is a martingale.

Proposition 3.1. Every martingale M is a H^1 martingale. That is, $\mathbb{E}\left[\max_{t\in\{0,\dots,T\}}|M_t|\right] < \infty$ is satisfied.

Proof. We have the inequality

$$\max_{t\in\{0,\dots,T\}}|M_t(\omega)| \le \sum_{t=0}^{T}|M_t(\omega)|$$

[15] It is supposed to be a convergent sequence.

[16] Recall that $\|x_{k_l}\| \to \infty$.

[17] Because $\mathscr{A}_1$ is closed.

for all ω, and therefore,

$$\mathbb{E}\left[\max_{t\in\{0,\dots,T\}}|M_t|\right] \leq \sum_{t=0}^{T}\mathbb{E}\left[|M_t|\right] < \infty.$$

$\square$

Proposition 3.2. Let X be a real local martingale. If X_t is integrable for all t, X is indeed a martingale.

Proof. Pick a localizing sequence (τ_n) of X such that

$$\mathbb{E}\left[X_t^{\tau_n}|\mathscr{F}_s\right] = X_s^{\tau_n} \quad s \leq t \tag{6}$$

for all n. We can easily prove that

$$\mathbb{E}\left[\max_{t\in\{0,\dots,T\}}|X_t|\right] < \infty$$

as same as in the proof of Proposition 3.1. Note that $|X_t^{\tau_n}| \leq \max_{t\in\{0,\dots,T\}}|X_t|$ holds for all n. Applying the dominated convergence theorem for conditional expectation to (6), we obtain

$$\mathbb{E}\left[X_t|\mathscr{F}_s\right] = X_s.$$

$\square$

Definition 3.3. A real adapted process X is called a *σ-martingale* if it is represented as $X = X_0 + H \bullet X$ by a predictable process H and a martingale M.

Theorem 3.4. Let $X = (X_t)_{t\in\{0,\dots,T\}}$ be a adapted process. There is a equivalence between:

(1) X is a local martingale;

(2) X is a σ-martingale;

(3) there exist a predictable process H and a local martingale M such that $X = X_0 + H \bullet M$.

Proof. It is sufficient to prove when $X_0 = 0$.

(i) $\implies$ (ii). Let (τ_n) be a localizing sequence for X and $E_{t,n} := \{t \leq \tau_n\}$ for $t \in \{1,\dots,T\}$. Then $E_{t,n}$ is $\mathscr{F}_{t-1}$-measurable and it satisfies

$$\mathbb{E}\left[|X_t|1_{E_{t,n}}\right] = \mathbb{E}\left[|X_t^{\tau_n}|1_{\{t\leq\tau_n\}}\right] < \infty.$$

Therefore X_t is $\mathscr{F}_{t-1}$-σ-integrable [18], and so is $\Delta X_t = X_t - X_{t-1}$. This shows that the conditional expectation $\mathbb{E}[|\Delta X_t||\mathscr{F}_{t-1}]$ is almost surely finite [19].

Let us define a strictly positive predicable process K by the following formula.

$$K_t = \frac{1}{1+\mathbb{E}[|\Delta X_t||\mathscr{F}_{t-1}]}.$$

[18] A random variable Y is *$\mathscr{G}$-σ-integrable* if there exists a sequence (E_n) of $\mathscr{G}$ such that $\Omega = \bigcup_n E_n$ and $\mathbb{E}[|Y|1_{E_n}] < \infty$.

[19] $\mathbb{E}[Y|\mathscr{G}]$ is a.s. finite if and only if Y is $\mathscr{G}$-integrable. See He, Wang, and Yan [12, 1.17 Theorem]

It satisfies

$$
\begin{aligned}
\mathbb{E}\left[|K_t \Delta X_t|\right] &= \mathbb{E}\left[\left|\frac{\Delta X_t}{1+\mathbb{E}[|\Delta X_t||\mathscr{F}_{t-1}]}\right|\right] \\
&= \mathbb{E}\left[\frac{|\Delta X_t|}{1+\mathbb{E}[|\Delta X_t||\mathscr{F}_{t-1}]}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{|\Delta X_t|}{1+\mathbb{E}[|\Delta X_t||\mathscr{F}_{t-1}]}\middle|\mathscr{F}_{t-1}\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}\left[|\Delta X_t||\mathscr{F}_{t-1}\right]}{1+\mathbb{E}[|\Delta X_t||\mathscr{F}_{t-1}]}\right] \\
&\leq 1.
\end{aligned} \tag{7}
$$

Set $M = K \bullet X$. We will now show that M is a martingale. M is clearly adapted to $(\mathscr{F}_t)$. The integrability of M follows from (7). For $t \in \{1, \cdots T\}$ and $n \in \mathbb{N}$ we have

$$
\begin{aligned}
\mathbb{E}\left[M_t^{\tau_n} - M_{t-1}^{\tau_n}|\mathscr{F}_{t-1}\right] &= \mathbb{E}\left[K_t \Delta X_t^{\tau_n}|\mathscr{F}_{t-1}\right] \\
&= \mathbb{E}\left[\frac{\Delta X_t^{\tau_n}}{1+\mathbb{E}[|\Delta X_t||\mathscr{F}_{t-1}]}\middle|\mathscr{F}_{t-1}\right] \\
&= \frac{\mathbb{E}\left[\Delta X_t^{\tau_n}|\mathscr{F}_{t-1}\right]}{1+\mathbb{E}[|\Delta X_t||\mathscr{F}_{t-1}]} \\
&= 0.
\end{aligned}
$$

The process M is, hence, a local martingale. By Proposition 3.2, it is in fact a martingale. The associativity of discrete stochastic integration implies

$$
\frac{1}{K} \bullet M = \frac{1}{K} \bullet (K \bullet X) = \left(\frac{1}{K} K\right) \bullet X = X,
$$

which shows X is a σ-martingale.

(ii) $\Longrightarrow$ (iii). Trivial.

(iii) $\Longrightarrow$ (ii). Suppose X is represented as $X = H \bullet M$ where H is predictable and M is a local martingale. We can assume, without loss of generality, $M_0 = 0$. Take a localizing sequence (τ_n) for M. We define another sequence of stopping times by

$$
\tau_n = \inf\{t \in \{0, \ldots, T-1\} \mid |H_{t+1}| > k\}.
$$

Let $\sigma_n = \tau_n \wedge \tau_n'$. Then (σ_n) is again a localizing sequence for the local martingale M. Using a property of discrete stochastic integration, we have

$$
X^{\sigma_n} = (H \bullet M)^{\sigma_n} = H^{\sigma_n} \bullet M^{\sigma_n}.
$$

Since H^{σ_n} is bounded and M^{σ_n} is a martingale, X^{σ_n} is also a martingale. In consequence, X is a local martingale with a localizing sequence (σ_n). $\square$

4 L^0 Spaces

In this section, we study some fundamental (topological) properties of L^0 spaces, the space of all random variables.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space. The space $L^0(\Omega, \mathscr{F}, \mathbb{P})$ (or simply, L^0) denotes the set of all random variables. As usual, an element X of L^0 is identified with another element Y if they are $\mathbb{P}$-almost surely equal.

Proposition 4.1. Suppose that a function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing, bounded, continuous, concave and satisfies the following condition.

$$f(x) = 0 \iff x = 0.$$

Let us define a function $d\colon L^0 \times L^0 \to \mathbb{R}_+$ by the following formula.

$$d_f(X,Y) = \mathbb{E}\left[f(|X-Y|)\right] \quad \text{for } X, Y \in L^0.$$

Then (L^0, d) is a complete metric space.

Proof. **Step 1: "d is a metric".** Only the triangular inequality is not clear. We first prove the sub-additivity of f. Let $0 \le a \le b$. If $a = 0$, the inequality $f(a+b) \le f(a) + f(b)$ is obvious. If not, we have

$$f(b) = f\left(\frac{b-a}{b}(a+b) - \frac{a}{b}a\right) \ge \frac{b-a}{b} f(a+b) + \frac{a}{b} f(a) = f(a+b) - \frac{a}{b}\left\{f(a+b) - f(a)\right\}$$

$$f(a) \ge f\left(\frac{a}{a+b}(a+b) + \frac{b}{a+b}0\right) \ge \frac{a}{a+b} f(a+b) + \frac{b}{a+b} f(0) \ge \frac{a}{a+b} f(a+b)$$

by the concavity of f. Hence,

$$\begin{aligned} f(a+b) &\le f(b) + \frac{a}{b}\left\{f(a+b) - f(a)\right\} \\ &\le f(b) + \frac{a}{b}\left\{\frac{a+b}{a} f(a) - f(a)\right\} \\ &= f(a) + f(b). \end{aligned}$$

We return to the proof of the triangular inequality. For any $x, y, z \in \mathbb{R}$, we see that

$$\begin{aligned} f(|x-z|) &\le f(|x-y| + |y-z|) && (\because\ f \text{ is non-decreasing}) \\ &\le f(|x-y|) + f(|y-z|) && (\because\ \text{sub-additivity of } f). \end{aligned}$$

Combining this and the linearity of expectations, we obtain

$$\mathbb{E}\left[f(|X-Z|)\right] \le \mathbb{E}\left[f(|X-Y|)\right] + \mathbb{E}\left[f(|Y-Z|)\right].$$

Thus the function d_f is a metric.

Step 2: completeness. Let $(X_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of (L^0, d_f). Choose a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N}, \quad d_f\left(X_{n_k}, X_{n_{k+1}}\right) < \frac{1}{2^k}.$$

By the monotone convergence theorem, we have

$$E\left[\sum_{k\in\mathbb{N}} f\left(|X_{n_{k+1}} - X_{n_k}|\right)\right] = \sum_{k\in\mathbb{N}} d_f\left(X_{n_k}, X_{n_{k+1}}\right) < \infty.$$

Therefore,

$$\sum_{k\in\mathbb{N}} f\left(|X_{n_{k+1}}(\omega) - X_{n_k}(\omega)|\right) < \infty \quad \mathbb{P}\text{-a.e.}\omega. \tag{8}$$

Note that there exists a $M\colon \Omega \to \mathbb{R}_{++}$[20] such that

$$|X_{n_{k+1}}(\omega) - X_{n_k}(\omega)| \le M(\omega) f\left(|X_{n_{k+1}}(\omega) - X_{n_k}(\omega)|\right) \quad \text{for a.a. } \omega. \tag{9}$$

Indeed,

$$M(\omega) = \begin{cases} \dfrac{1}{f'_+\left(\sup_k |X_{n_{k+1}}(\omega) - X_{n_k}(\omega)|\right)} & \text{if } 0 < \sup\limits_k |X_{n_{k+1}}(\omega) - X_{n_k}(\omega)| < \infty, \\ 0 & \text{otherwise} \end{cases}$$

satisfies (9)[21]. We can deduce from (8) and (9) that

$$\sum_{k\in\mathbb{N}} \left|X_{n_{k+1}}(\omega) - X_{n_k}(\omega)\right| < \infty \quad \mathbb{P}\text{-a.s.}.$$

Hence, $(X_{n_k})_{k\in\mathbb{N}}$ converges almost surely to a random variable, denoted by X_∞. Since f is a bounded continuous function,

$$f(|X_{n_k} - X_\infty|) \to 0 \ \text{ a.s.,} \quad \text{and} \quad f(|X_{n_k} - X_\infty|) \le \sup_x f(x) < \infty.$$

Applying the dominated convergence theorem, we obtain

$$\lim_{k\to\infty} d_f(X_{n_k} - X_\infty) = \mathbb{E}\left[\lim_{k\to\infty} f(|X_{n_k} - X_\infty|)\right] = 0.$$

Consequently the subsequence (X_{n_k}) converges almost surely, and so the original Cauchy sequence (X_n). □

Proposition 4.2. Under the same assumptions as in Proposition 4.1, "convergence in (L^0, d_f)" is equivalent to "convergence in probability".

Proof. **Step 1: the proof of "in $(L^0, d_f) \implies$ in prob.".** Suppose that (X_n) converges to X in (L^0, d_f). Then by Chebyshev's inequality we have

$$\begin{aligned} \mathbb{P}\left[|X_n - X| \ge c\right] &= \mathbb{P}\left[f\left(|X_n - X|\right) \ge f(c)\right] \\ &\le \frac{\mathbb{E}[f(|X_n - X|)]}{f(c)} \end{aligned}$$

for any $c > 0$. Let $n \to \infty$, and we obtain

$$\lim_{n\to\infty} \mathbb{P}\left[|X_n - X| \ge c\right] \le \frac{1}{f(c)} \lim_{n\to\infty} \mathbb{E}[f(|X_n - X|)] = 0.$$

Hence (X_n) converges to X in probability.

Step 2: the proof of "in prob. $\implies$ in (L^0, d_f).". Assume that (X_n) converges to X in probability. Take $c > 0$ arbitrary. Then we have

$$\begin{aligned} \mathbb{E}\left[f(|X_n - X|)\right] &= \mathbb{E}\left[f(|X_n - X|)\mathbb{1}_{\{|X_n - X| \le c\}}\right] + \mathbb{E}\left[f(|X_n - X|)\mathbb{1}_{\{|X_n - X| > c\}}\right] \\ &\le \mathbb{E}\left[f(c)\mathbb{1}_{\{|X_n - X| \le c\}}\right] + \mathbb{E}\left[\left(\sup_{x\in\mathbb{R}_+} f(x)\right)\mathbb{1}_{\{|X_n - X| > c\}}\right] \\ &\le f(c) + \left(\sup_{x\in\mathbb{R}_+} f(x)\right)\mathbb{E}\left[|X_n - X| > c\right]. \end{aligned}$$

[20] It need not be measurable

[21] f'_+ denotes the right derivative of f.

Therefore

$$\overline{\lim_{n\to\infty}}\, \mathbb{E}\left[f(|X_n - X|)\right] \le f(c)$$

for all $c > 0$. This shows

$$\lim_{n\to\infty} \mathbb{E}\left[f(|X_n - X|)\right] = 0.$$

□

As a consequence of Proposition 4.2, any metric d_f defined by a function f satisfying the assumptions of Proposition 4.1 generates the same topology. From now on, we use the function defined by

$$f(x) = x \wedge 1 \quad x \in \mathbb{R}_+.$$

Remark 4.3. If Ω is countable, L^0 space is locally convex TVS. In general case, however, the space L^0 is not necessarily locally convex. See Schaefer [16, Chapter 1, Section 4].

Proposition 4.4. The topology of a L^0 space is invariant under equivalent change of probability measures.

Proof. Let $(\Omega, \mathscr{F})$ be a measurable space and $\mathbb{P}$, $\mathbb{Q}$ be two equivalent probability measures on it. Let Z denotes a version of the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$. We will show that the identity map $L^0(\mathbb{P}) \to L^0(\mathbb{Q})$ is a homeomorphism. Fix $\varepsilon > 0$ arbitrary. Then we can choose a positive number C such that $\mathbb{E}^{\mathbb{P}}\left[Z \mathbb{1}_{\{Z>C\}}\right] < \varepsilon/2$ [22]. If $X, Y \in L^0$ satisfy $\mathbb{E}^{\mathbb{P}}[1 \wedge |X - Y|] < \varepsilon/(2C)$, then

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[|X - Y| \wedge 1\right] &= \mathbb{E}^{\mathbb{P}}\left[(|X - Y| \wedge 1) Z\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[(|X - Y| \wedge 1) Z \mathbb{1}_{\{Z>C\}}\right] + \mathbb{E}^{\mathbb{P}}\left[(|X - Y| \wedge 1) Z \mathbb{1}_{\{Z\le C\}}\right] \\
&\le \mathbb{E}^{\mathbb{P}}\left[Z \mathbb{1}_{\{Z>C\}}\right] + C\mathbb{E}^{\mathbb{P}}\left[|X - Y| \wedge 1\right] \\
&\le \frac{\varepsilon}{2} + C\frac{\varepsilon}{2C} = \varepsilon.
\end{aligned}$$

Hence the identity map $L^0(\mathbb{P}) \to L^0(\mathbb{Q})$ is continuous. The continuity of the inverse is proved similarly. □

Definition 4.5. A subset $\mathscr{A}$ of $L^0(\Omega, \mathscr{F}, \mathbb{P})$ is *bounded* if for any neighborhood U of 0 there exists a $c > 0$ such that $c^{-1}\mathscr{A} \subset U$.

Proposition 4.6. $\mathscr{A} \subset L^0(\mathbb{P})$ is bounded if and only if the following condition is satisfied.

$$\lim_{c\to\infty} \sup_{X\in\mathscr{A}} \mathbb{P}\left[|X| > c\right] = 0.$$

Proposition 4.7. Let $\mathscr{A} \subset L^0$.

(1) If $\mathscr{A}$ is almost surely bounded, then it is bounded in L^0.

(2) If $\mathscr{A}$ is L^p bounded for some $p \in (0, \infty)$, then it is L^0-bounded.

Proposition 4.8. Assume that a sequence $(B_i)_{i\in\mathbb{N}}$ of $\mathscr{F}$-measurable set satisfies the following conditions.

$$\mathbb{P}\left(\bigcup_{i\in\mathbb{N}} B_i\right) = 1, \quad \mathbb{P}\left(B_i \cap B_j\right) = 0 \text{ (if } i \neq j).$$

Then,

[22] By the integrability of Z.

(1) a sequence (X_n) converges to Y in L^0 if and only if $(X_n \mathbb{1}_{B_i})$ converges to $Y\mathbb{1}_{B_i}$ for all i.

(2) $\mathscr{A} \subset L^0$ is bounded in L^0 if and only if $\{X\mathbb{1}_{B_i} \mid X \in \mathscr{A}\}$ is L^0-bounded for all i.

Proposition 4.9. Let $g \colon \mathbb{R}^k \to \mathbb{R}$ be a continuous function and $\mathscr{A}_1, \dots, \mathscr{A}_k$ be L^0-bounded subsets. Then,

$$g_*(\mathscr{A}_1 \times \cdots \times \mathscr{A}_k) = \{g(X_1, \dots, X_k) \mid X_i \in \mathscr{A}_i \ \forall i \in \{1, \dots, k\}\}$$

is L^0-bounded. In particular any finite Minkowski sums and Mikowski products of L^0 bounded sets are L^0-bounded.

Proposition 4.10. Let $(\mathscr{A}_i)_{i\in\mathbb{N}}$ be a sequence of L^0-bounded and $\tau \colon \Omega \to \mathbb{N}$ be a random variable. Then,

$$\left\{ \sum_{i=1}^{\tau} X_i \,\middle|\, X_i \in \mathscr{A}_i \ \forall i \in \mathbb{N} \right\} \quad \text{and} \quad \left\{ \prod_{i=1}^{\tau} X_i \,\middle|\, X_i \in \mathscr{A}_i \ \forall i \in \mathbb{N} \right\}$$

are bounded in L^0.

Proposition 4.11 (Delbaen and Schachermayer [4, Lemma A 1.1], D-S [8, Lemma 9.8])**.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}_+$-valued random variables. Then there exists $[0, \infty]$-valued random variables (Y_n) such that $Y_n \in \mathrm{Conv}(X_n, X_{n+1}, \dots)$ and (Y_n) converges almost surely. If $\mathrm{Conv}(\{X_n; n \in \mathbb{N}\})$ is L^0-bounded, the limit of (Y_n) is almost surely finite. Moreover if (X_n) satisfies the following condition, the limit of (Y_n) is not zero.

$$\exists \alpha > 0 \quad \exists \delta > 0 \quad \forall n \in \mathbb{N} \quad \mathbb{P}[X_n > 0] > \delta.$$

5 The Fundamental Theorem on Arbitrary Probability Spaces (The Dalang-Morton-Willinger Theorem)

In this section, our goal is to prove the following two important theorems.

Theorem 5.1. For any price process S, the set

$$\mathscr{A} = \{(H \bullet S)_T \mid H \colon \text{predictable}\}$$

is closed in L^0.

Theorem 5.2 (Dalang-Morton-Willinger [3])**.** For any price process S, the following four conditions are equivalent.

(1) (NA)

(2) (NA) and $\mathscr{C} := \mathscr{A} - L^0_+$ is L^0-closed.

(3) There exists an EMM for S.

(4) There is an EMM $\mathbb{Q}$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$.

Remark 5.3. Without the (NA) condition, there is an example where $\mathscr{C}$ is not L^0-closed. This example comes from Delbaen-Schachermayer [8, p. 94]. Let $d = 1$, $T = 1$. We consider a measurable space $(\Omega, \mathscr{F}) = ((0, 1], \mathscr{B}((0, 1])$ and the Lebesgue measure $\mathbb{P}$ on it. A filtration $(\mathscr{F}_t)_{t\in\{0,1\}}$ is defined by

$$\mathscr{F}_0 = \{\emptyset, \Omega\}, \quad \mathscr{F}_1 = \mathscr{F}.$$

We define a price process S as follows.

$$S_0 = 0, \quad S_1(\omega) = \omega.$$

Then the constant function 1 does not belong to $\mathscr{C}$. Let $f_n = n\mathbb{1}_{]0,\frac{1}{n}]} S_1 + 1_{[\frac{1}{n},1]}$. Each f_n satisfies

$$f_n \le g_n := n\Delta S \in \mathscr{A}$$

and therefore $f_n = g_n - (g_n - f_n)$ belongs to $\mathscr{C}$. Clearly (f_n) converges pointwise (and hence in probability!) to the constant function 1. Therefore $1 \in \overline{\mathscr{C}}$. This proves that $\mathscr{C}$ is not closed in L^0.

Lemma 5.4 (Delbaen and Schachermayer [8, Proposition 6.3.3 and 6.3.4])**.** Let $(\mathscr{K}, d)$ be a compact metric space and (X_n) a sequence of $\mathscr{K}$-valued random variables.

(1) There exists a strictly increasing sequence (ν_k) of $\mathbb{N}$-valued random variables such that (X_{ν_k}) converges a.s..

(2) Let $x \in \mathscr{K}$ and

$$B_x = \{\omega \in \Omega \mid x \text{ is an accumulation point of } (X_n(\omega))_{n\in\mathbb{N}}\}.$$

Then the sequence in (1) may be chosen such that

$$\lim_{k\to\infty} X_{\nu_k}(\omega) = x \quad \text{for all } \omega \in B_x.$$

Proof. (1) Take a finite open covering, denoted by $A_1^n, \dots, A_{N_n}^n$, of $\mathscr{K}$ whose diameter are less than $1/n$. We define a sequence (ν_k) as follows.

$$\begin{aligned}
I^0 &= \mathbb{N},\\
j_k(\omega) &= \min\left\{ j \in \{1,\dots,N_k\} \mid A_j^k \text{ contains infinitely many points of } (X_n(\omega))_{n\in I^{k-1}(\omega)} \right\}\\
I^k(\omega) &= \left\{ n \in I^{k-1}(\omega) \mid X_n(\omega) \in A_{j_k}^k(\omega) \right\}\\
\nu_k(\omega) &= \text{the } k\text{-th element of } I^k(\omega)
\end{aligned}$$

It is easy to see that ν_k is measurable and (X_{n_k}) converges.

(2) It suffices to choose the coverings (A_j^n) in the proof of (1) such that $x \in A_1^n$ for all n. ☐

Proof of Theorem 5.1. **Step 1.** We first assume the following.

$$(H \bullet S)_T = 0 \text{ a.s.} \implies H = 0 \text{ a.s.}$$

Let (H^n) be a sequence of strategies such that

$$(H^n \bullet S)_T \xrightarrow[n\to\infty]{\text{in prob.}} {}^\exists Z.$$

Replacing it, if necessary, with proper subsequence, we can assume that it converges almost surely. Our purpose is to find a strategy K satisfying $Z = (K \bullet S)_T$.

Case 1: the case in which (H^n) is a.s. bounded. Suppose that

$$\forall i \in \{1,\dots,d\} \quad \forall t \in \{0,\dots,T\} \quad \text{for a.e. } \omega \quad \sup_{n\in\mathbb{N}} |H_t^{n,(i)}(\omega)| < \infty$$

By Lemma 5.4, we can choose a sequence of $\mathbb{N}$-valued $\mathscr{F}_0$ measurable functions $(\nu_{1,k})$ such that

$$\lim_{k\to\infty} H_1^{\nu_{1,k}} = {}^\exists K_1 \quad \text{a.s..}$$

The sequence $(H^{\nu_{1,k}})$ satisfies the following properties.

- For each $t \in \{2, \dots, T\}$, $H_t^{\nu_{1,k}}$ is $\mathscr{F}_{t-1}$-measurable.
- $(H^{\nu_{1,k}} \bullet S)_T \to Z$ a.s.

Let $\widehat{H}^n = K_1 \mathbb{1}_{\{1\}} + H^{\nu_{1,n}} \mathbb{1}_{\{2,\dots,T\}}$. Then we see that

$$(\widehat{H}^n \bullet S)_T = (H^{\nu_{1,n}} \bullet S)_T + (K_1 - H_1^{\nu_{1,n}})(S_1 - S_0) \xrightarrow[n\to\infty]{\text{a.s.}} Z.$$

Next, take a sequence $(\nu_{2,k})$ such that

- each $\nu_{2,k}$ is $\mathscr{F}_1$-measurable $\mathbb{N}$-valued r.v.,
- $\widehat{H}_2^{\nu_{2,k}} \to {}^{\exists}K_2$ a.s.
- For each $t \in \{3, \dots, T\}$, $\widehat{H}_t^{\nu_{2,k}}$ is $\mathscr{F}_{t-1}$-measurable.
- $(\widehat{H}^{\nu_{2,k}} \bullet S)_T \to Z$ a.s.

Inductively we define a process $K = (K_t)_{t\in\{1,\dots,T\}}$. This is what we need to construct.

Case 2: the case (H^n) is not bounded. Let

$$t^* = \min\{t \in \{0, \dots, T\} \mid (H_t^n)_{n\in\mathbb{N}} \text{ not a.s. bounded}\}.$$

We can define, in the same way as Case 1, a predictable process $(\widehat{H}_t)_{t\in\{1,\dots,T\}}$ such that

- $\lim_{n\to\infty}(\widehat{H}^n \bullet S)_T = Z$ a.s.
- $\widehat{H}_u^n = K_u$ for $u < t^*$,
- $(\widehat{H}_{t^*}^n)$ is not a.s. bounded.

By assumption, the event

$$B = \left\{\omega \in \Omega \,\middle|\, \sup_{n\in\mathbb{N}} |\widehat{H}_t^n(\omega)| = \infty \right\} \in \mathscr{F}_{t-1}$$

has positive probability,. Then, by Lemma 5.4, there exists a sequence of $\mathscr{F}_{t-1}$-measurable random variables (τ_k) such that

$$\lim_{k\to\infty} |\widehat{H}_{t^*}^{\tau_k}| = \infty \quad \text{a.s. on } B.$$

Define a process $\widetilde{H}$ by

$$\widetilde{H}_u^k = \begin{cases} 1_B \dfrac{\widehat{H}_u^{\tau_k}}{|\widehat{H}_t^{\tau_k}|} & \text{on } \{\widehat{H}^{\tau_k} \neq 0\} \times \{t^*, \dots, T\}, \\ 0 & \text{otherwise.} \end{cases}$$

The process $\widetilde{H}$ is predictable and it satisfies

$$(\widetilde{H}^k \bullet S)_T = \frac{1_B}{|\widehat{H}_{t^*}^{\tau_k}|} \left\{ (\widehat{H}^{\tau_k} \bullet S)_T - (\widehat{H}^{\tau_k} \bullet S)_{t^*-1} \right\}.$$

Since $(\widehat{H}^{\tau_k} \bullet S)_T$ converges a.s. to Z, we have

$$\frac{1_B}{|\widehat{H}_{t^*}^{\tau_k}|} (\widehat{H}^{\tau_k} \bullet S)_T \xrightarrow[k\to\infty]{\text{a.s.}} 0.$$

By definition $\widetilde{H}^{\tau_k}_u$ $(u < t^*)$ does not depend on k, and hence

$$\frac{1_B}{|\widehat{H}^{\tau_k}_{t^*}|}(\widehat{H}^{\tau_k} \bullet S)_{t^*-1} \xrightarrow[k\to\infty]{\text{a.s.}} 0.$$

Consequently,

$$(\widetilde{H}^k \bullet S)_T \xrightarrow[k\to\infty]{a.s.} 0.$$

Now we got a sequence $(\widetilde{H}^k)$ which satisfies

(1) $(\widetilde{H}^k \bullet S)_T \to 0$,

(2) For any $t \in \{0, \dots, t^*\}$ and for almost all ω, the sequence $(\widetilde{H}^k_t(\omega))_{k\in\mathbb{N}}$ is bounded .

By a similar discussion as above, we can obtain a predictable process $\widetilde{K}$ such that

(1) $\widetilde{K}_{t^*} \neq 0$ with positive probability,

(2) $(\widetilde{K} \bullet S)_T = 0$ a.s.

This contradicts the assumption in Step 1.

Step 2: general cases. □

6 The NUPBR condition

Definition 6.1. Let $a \in \mathbb{R}_{++}$. A strategy H is *a-admissible* if

$$(H \bullet S)_t \geq -a \quad \forall t \in \{0, \dots, T\} \quad P\text{-a.s.}.$$

We say that H is *admissible for S* if it is a-admissible for some a.

Definition 6.2. We say that S satisfies the (NUPBR) condition if

$$\mathscr{K}_1 = \{(H \bullet S)_T \mid H \text{ is 1-admissible for } S\}$$

is bounded in L^0.

Remark 6.3. There is a case where $\mathscr{K}_1 = \{0\}$.

Proposition 6.4. In a discrete time model, the condition (NUPBR) is equivalent to the four conditions in Theorem 5.2.

7 Numeraire

Part II
Continuous time, finite assets, finite terminal models

8 Difficulties in continuous time settings and some examples

There are some important difference between continuous time models and discrete time modes such as

- traders can choose the "doubling strategy" even in a finite time interval—"admissibility",
- local martingales and martingales are essentially different.

Example 8.1. Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a 1-dimensional BM $W = (W_{t\in[0,T]})$. The filtration $(\mathscr{F}_t)_{t\in[0,T]}$ is supposed to be generated by W. We define a process S by

$$S_t = W_t + 2\sqrt{T-t}.$$

Then there is no EMM for S in this model. Let

$$X_t = -1 + \exp\left(-\int_0^t \frac{dW_u}{\sqrt{T-u}} + \frac{1}{2}\int_0^t \frac{du}{T-u} \right) \quad t < T.$$

9 Semimartingales, stochastic integrals, local martingales and σ-martingales

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ a filtered probability space with the usual hypothesis.

Stopping times

Definition 9.1. A random variable $\tau\colon \Omega \to [0,T] \cup \{\infty\}$ is called a stopping time if it satisfies

$$\forall t \in [0,T] \quad \{\omega \in \Omega \mid \tau(\omega) \le t\} \in \mathscr{F}_t$$

Local martingales

Definition 9.2. A càdlàg adapted process $M = (M_t)_{t\in[0,T]}$ is a local martingale if there exists a sequence of stopping times that satisfies the following conditions.

(1) $\tau_1 \le \tau_2 \le \cdots \to \infty$ a.s..

(2) $M^{\tau_n} - M_0$ is a local martingale.

Proposition 9.3. Suppose that a local martingale X is bounded from below and satisfies $X_0 \in L_1(\mathbb{P})$. Then X is a super martingale.

Proof. We can assume that X is nonnegative without loss of generality [23]. Pick a localizing sequence (τ_n) of X. We have for any n and $s \le t$

$$\mathbb{E}\left[X_t^{\tau_n} | \mathscr{F}_s\right] = M_s^{\tau_n} \quad \text{a.s..} \tag{10}$$

Let $s = 0$ in (9.7) and we obtain, by Fatou's lemma,

$$\mathbb{E}\left[|X_t|\right] = \mathbb{E}\left[X_t\right] \le \varliminf_{n\to\infty} \mathbb{E}\left[X_t^{\tau_n}\right] \le \varliminf_{n\to\infty} \mathbb{E}\left[X_{\tau_n\wedge 0}\right] = \mathbb{E}\left[X_0\right] < \infty.$$

This implies the integrability of each X_t. Applying Fatou's lemma for conditional expectations to (9.7), we can verify the following inequality.

$$\mathbb{E}\left[X_t | \mathscr{F}_s\right] \le \varliminf_{n\to\infty} \mathbb{E}\left[X^{\tau_n} | \mathscr{F}_s\right] = M_s \quad \text{a.s.}$$

Therefore (X_t) is a supermartingale. □

[23] If not, it is enough to consider the process $X - a$ where a is a lower bound of X.

Definition 9.4. Two local martingales M and N is said to be *orthogonal* if MN is a local martingale and $M_0N_0 = 0$. A *purely discontinuous martingale* is a local martingale which is orthogonal to all continuous local martingales.

Proposition 9.5. A local martingale M is uniquely decomposed as follow.

$$M = M_0 + M^c + M^d$$

where M^c is a continuous local martingale, M^d is a purely discontinuous local martingale, and $M_0^c = M_0^d = 0$.

For any locally square integrable martingale M, there exists a unique predictable increasing process, denoted by $\langle M, M\rangle$, such that $M^2 - M_0^2 - \langle M, M\rangle$ is a locally integrable martingale that starts from 0. If M is square integrable, this is a direct consequence of the Doob-Meyer decomposition theorem. The general case is proved by localization.

Definition 9.6. Let M be a local martingale. We define the *quadratic variation* of M by

$$[M, M] = \langle M^c, M^c\rangle + \sum_{0<s\leq \cdot} (\Delta M_s)^2$$

where M^c denotes the continuous part of M in the sense of Proposition 9.5.

The process $[M, M]$ is a unique adapted increasing process such that $M^2 - M_0^2 - [M, N]$ is a local martingale with initial value 0, and $\Delta[M, M] = (\Delta M)^2$. If M is locally square integrable, $\langle M, M\rangle$ is the predictable compensator of $[M, M]$. We define, more generally, the *predictable quadratic variation* $\langle M, M\rangle$ of M as the predictable compensator of $[M, M]$ in the case where $[M, M]$ is locally integrable [24].

Proposition 9.7. Let M be a local martingale. Then $M - M_0$ and $[M, M]^{1/2}$ are locally integrable [25].

An $\mathbb{R}^d$-valued process X is called a $\mathbb{R}^d$-valued local martingale if every components of X is a local martingale.

Semimartingales

Definition 9.8. A semimartingale is a càdlàg adapted process X written in the form $X = X_0+M+A$ where

(1) M is a local martingale satisfying $M_0 = 0$,

(2) A is a càdlàg adapted process with $A_0 = 0$ whose paths have finite variation.

The decomposition of semimartingale into a local martingale and a process of finite variation is not unique. For example, a compensated Poisson process $X_t = N_t - \lambda t$ is both a martingale and a process of finite variation.

Definition 9.9. A semimartingale X is called a *special semimartingale* if we can choose a decomposition $X = X_0 + M + A$ such that A is predictable.

[24] Every adapted process of locally integrable variation has the predictable compensator. See, He, Wang, Yan [12] or Jacod and Shiryaev [13].

[25] Recall that a process $X = (X_t)_{t\in[0,T]}$ is locally integrable if there exists a localizing sequence (τ_n) such that $\sup_{t\in[0,T]}|X_t^{\tau_n}|$ is integrable for all n.

A decomposition of a special semimartingale $X = X_0 + M + A$ such that A is predictable is, indeed, unique. Consider two decompositions

$$X = X_0 + M + A = X_0 + \widetilde{M} + \widetilde{A}$$

where A and $\widetilde{A}$ are predictable. Then the process $M - \widetilde{M} = \widetilde{A} - A$ is a predictable local martingale of finite variation, and hence a constant process. Because the initial valued of this process is 0, we see that it is evanescent[26]. This unique decomposition is called the canonical decomposition of a special semimartingale.

Proposition 9.10. A semimartingale X is special if and only if $X - X_0$ is locally integrable. In particular every continuous semimartingale is special and both M and A of its canonical decomposition $X = X_0 + M + A$ are continuous.

Theorem 9.11 (Delbaen and Schachermayer [, Theorem 2.3], Delbaen and Schachermayer [8, Theorem 9.2.3])**.** Let X be a semimartingale such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\Delta X_t|^p\right] < \infty.$$

holds for some $p \in (0, \infty)$. Then X is a special martingale and it satisfies

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\Delta A_t|^p\right] \leq \frac{p}{p-1}\,\mathbb{E}\left[\sup_{t\in[0,T]} |\Delta X_t|^p\right]$$
$$\mathbb{E}\left[\sup_{t\in[0,T]} |\Delta M_t|^p\right] \leq \frac{2p-1}{p-1}\,\mathbb{E}\left[\sup_{t\in[0,T]} |\Delta X_t|^p\right]$$

for its canonical decomposition $X = X_0 + M + A$.

Proposition 9.12. Let X be a semimartingale on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$. If a probability measure $\mathbb{Q}$ is measure absolutely continuous with respect to $\mathbb{P}$, X is again a semimartingale under $\mathbb{Q}$.

An $\mathbb{R}^d$-valued process $X = (X^{(1)}, \ldots, X^{(d)})$ is called a $\mathbb{R}^d$-valued semimartingale if all $X^{(i)}$ are real semimartingales.

Stochastic Integration

We will define a metric on the space of all semimartingales. Let

$$d(X, Y) = \sup_{\substack{K:\,\text{predictable}\\ |K|\leq 1}} \mathbb{E}\left[1 \wedge |K \bullet (X - Y)_T|\right].$$

This is indeed a metric on the set of all semimartingales and the space endowed with this metric is complete. The topology generated by this metric is called the semimartingale topology or the Emery topology.

[26]i.e. indistinguishable from 0.

10 Settings

(1) $T \in \mathbb{R}_{++}$, $d \in \mathbb{N}$.

(2) $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$: a filtered probability space satisfying the usual hypothesis

(3) $S = (S_t)_{t\in[0,T]}$: an $\mathbb{R}^d$-valued semimartingale

(4) Let $a \in \mathbb{R}_{++}$. We say that a strategy $H = (H_t)_{t\in(0,T]}$ is *a-admissible for S* if it is S-integrable and its a.e. path satisfie

$$(H \bullet S)_t \geq 0 \quad \forall t \in [0, T].$$

H is *admissible for S* if there exists a constant a such that H is a-admissible for S .

(5) $\mathscr{K}_a = \{(H \bullet S)_T \mid H \text{ is } a\text{-admissible}\}$

(6) S satisfies the *No Arbitrage (NA)* condition if it satisfies $\mathscr{K}_1 \cap L^0_+ = \{0\}$

(7) We say that S satisfies the condition of *No Unbouded Profit with Bounded Risk (NUPBR)* if $\mathscr{K}_1$ is L^0-bounded.

(8) We say that S satisfies the condition of *No Free Lunch with Vanishing Risk (NFLVR)* if both (NA) and (NUPBR) are satisfied.

(9) An *equivalent martingale measure (EMM)* $\mathbb{Q}$ for S is a probability measure which is equivalent to $\mathbb{P}$ and under which S is a martingale. We also define an ELMM and an ESMM replacing the word "martingale" in the above condition with "local martingale" and "σ-martingale", respectively.

(10) An $\mathbb{R}_{++}$-valued process Z is called a *strict martingale density* if $Z_0 \in L^1(\mathbb{P})$ and ZS is a $\mathbb{R}^d$-valued σ-martingale [27].

11 The Choulli-Stricker Theorem for Continuous Processes

12 The Delbaen-Schachermayer Theorem

13 Change of Numeraire

A Complements to Functional Analysis

A.1 The Hahn-Banach Separation Theorem

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