Conditional Fatou's lemma の証明

Lemma 1. (conditional MCoT)

$X_n \geq 0$, $X_n \leq X_{n+1} \leq \cdots$, $\mathcal{G} \subset \mathcal{F}$: σ-alg.

$\Longrightarrow$ $E[X_n \mid \mathcal{G}] \to E[X_\infty \mid \mathcal{G}]$ a.s.

where $X_\infty := \lim_{n\to\infty} X_n$.

(Proof)

Let Y_n be a version of $E[X_n \mid \mathcal{G}]$.

Then we have $Y_0 \leq Y_1 \leq \cdots$ a.e.

Set
$$Y_\infty(\omega) = \begin{cases} \lim_{n\to\infty} Y_n(\omega) & \text{on } \{Y_0 \leq Y_1 \leq \cdots\} \\ 0 & \text{otherwise.} \end{cases}$$

Then Y_∞ is $\mathcal{G}$-mble and satisfies

$$Y_0 \leq Y_1 \leq \longrightarrow Y_\infty \quad P\text{-a.s.}$$

By the MCOT, we have

$$\lim_{n\to\infty} E[X_n 1_E] = E[X_\infty 1_E] \quad \cdots ①$$

$$\lim_{n\to\infty} E[Y_n 1_E] = E[Y_\infty 1_E] \quad \cdots ②$$

for all $E \in \mathcal{G}$.

Since $Y_n = E[X_n \mid \mathcal{G}]$ a.e.,

$$E[X_n 1_E] = E[Y_n 1_E] \quad \cdots ③$$

Combining ①, ②, ③, we obtain

$$\begin{aligned} E[Y_\infty 1_E] &= \lim_{n\to\infty} E[Y_n 1_E] \\ &= \lim_{n\to\infty} E[X_n 1_E] \\ &= E[X_\infty 1_E] \qquad \forall E \in \mathcal{G}. \end{aligned}$$

Hence, Y_∞ is a version of $E[X_\infty \mid \mathcal{G}]$. //

<u>Theorem</u> (Conditional Fatou's lemma)

Suppose $X_n \geq 0$ holds for all n.
Then

$$\liminf_{n\to\infty} E[X_n \mid \mathcal{G}] \geq E[\liminf_{n\to\infty} X_n \mid \mathcal{G}]$$

(Proof)

Let $X_\infty = \liminf_{n\to\infty} X_n$.

$$X_n' = \inf_{k\geq n} X_k. \quad Y_n = E[X_n' \mid \mathcal{G}]$$

Then we have $X_\infty = \lim_{n\to\infty} X_n'$ and

$X_0' \leq X_1' \leq \cdots$ pointwise

$Y_0 \leq Y_1 \leq \cdots$ P-a.e.

By the conditional MCOT.,

$$\lim_{n\to\infty} Y_n = E[X_\infty \mid \mathcal{G}] \text{ a.e.} \quad \cdots ④$$

Clearly we have $X_n \geq X_n'$, and therefore,

$$E[X_n \mid \mathcal{G}] \geq Y_n = E[X_n' \mid \mathcal{G}] \quad \text{a.e.} \quad .$$

Then,

$$\begin{aligned} \liminf_{n\to\infty} E[X_n \mid \mathcal{G}] &\geq \liminf_{n\to\infty} Y_n \\ &= \lim_{n\to\infty} Y_n \\ &= E[X_\infty \mid \mathcal{G}] \\ &= E[\liminf X_n \mid \mathcal{G}] . \end{aligned}$$

P-a.e.

This completes the proof. //