# Contents

# Chapter 1

# 基本矩陣分析

## 1.3 線性系統的靈敏度

### 1.3.1 Forward Error

Given $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Consider the problem: Solve $x$ of the linear system

$$Ax = b.$$

**Lemma 1.3.1** (Invertibility of an Invertible Matrix Under Perturbation)**.** Suppose $A$ is invertible. Then $A+E$ is invertible if $\|A^{-1}E\| = r < 1$. Moreover,

$$\|(A+E)^{-1} - A^{-1}\| \leq \frac{\|E\| \cdot \|A^{-1}\|^2}{1-r}.$$

**Theorem 1.3.2** (Forward Error Estimation)**.** Suppose

$$\begin{cases} Ax = b, \\ (A+\Delta A)y = b + \Delta b, \end{cases} \tag{1.3.1}$$

where $\|\Delta A\| \leq \delta\|A\|$ and $\|\Delta b\| \leq \delta\|b\|$. If

$$\delta \cdot \kappa(A) = r < 1,$$

then $A + \Delta A$ is also nonsingular and

$$\frac{\|y\|}{\|x\|} \leq \frac{1+r}{1-r}, \quad \text{and} \quad \frac{\|x-y\|}{\|x\|} \leq \frac{2r}{1-r}. \tag{1.3.2}$$

where

$$\kappa(A) := \|A\| \cdot \|A^{-1}\|.$$

**Definition 1.3.1** (Condition Number)**.** The **condition number** [**條件數**] of a nonsingular matrix $A$ is defined by

$$\kappa(A) := \|A\| \cdot \|A^{-1}\|. \tag{1.3.3}$$

### 1.3.2 Backward Error

Suppose

$$\begin{cases} Ax = b, \\ (A+\Delta A)y = b + \Delta b. \end{cases}$$

- Forward Error $= \|y - x\|$. **This one considers the error between the real solution $x$ and the computed solution $y$.**
- **Backward Error:**
  (i) If we put the computed solution $y$ to the original equation $Ax - b = 0$, this one considers the difference from the equality.
  $$\text{Backward Error} = \|Ay - b\|. \tag{1.3.4}$$
  Equivalently, how large should $\widetilde{\Delta b}$ be for the following equation
  $$Ay = b + \widetilde{\Delta b}.$$
  So
  $$\text{Backward Error} = \|\widetilde{\Delta b}\|.$$
  **Equivalently, how must should we adjust the the coefficients in the original equation** $Ax = b$ to make $y$ be a real solution.

  (ii) Similarly, the backward error can be thought as how large should $\widetilde{\Delta A}$ be for the following equation
  $$(A + \widetilde{\Delta A})y = b.$$
  So
  $$\text{Backward Error} = \|\widetilde{\Delta A}\|.$$
  (iii) In general, we may define
  $$\text{Backward Error} = \min\left\{\epsilon \mid (A + \widetilde{\Delta A})y = b + \widetilde{\Delta b},\ \ \|\widetilde{\Delta A}\| < \epsilon\|A\|,\ \|\widetilde{\Delta b}\| < \epsilon\|b\|\right\}. \tag{1.3.5}$$

**Theorem 1.3.3** (Backward Error Estimation)**.** Suppose
$$\begin{cases} Ax = b, \\ (A + \Delta A)y = b + \Delta b. \end{cases}$$
Then
$$\text{Backward Error} = \frac{\|b - Ay\|}{\|A\| \cdot \|y\| + \|b\|}. \tag{1.3.6}$$