Assignment 1

Li Yueheng s2306706

2022.1.24

Problem 1

1

$g'(x) = -2\alpha(x - \eta)$ decreases monotonously. therefore g achieves a unique maximum r_0 at $x_{max} = \eta$.

2

Set $\dot{x} = 0$ to find three solutions $\{0, \eta + \sqrt{\frac{r_0}{\alpha}}, \eta - \sqrt{\frac{r_0}{\alpha}}\}$. Among which the first two are distinct non-negative solutions. They, are the steady states of the system.

Near 0, we have $x\dot{x} = x^2 g(x) \leq 0$.

Near $\eta + \sqrt{\frac{r_0}{\alpha}}$, we have $(x - (\eta + \sqrt{\frac{r_0}{\alpha}}))\dot{x} \leq 0$.

So, the system will stay close to these two steady states under perturbation. Therefore the steady state is stable.

3

Under the assumption $\eta - \sqrt{\frac{r_0}{\alpha}} > 0$, the arguments in the previous problem still hold. So 0 and $\eta + \sqrt{\frac{r_0}{\alpha}}$ are stable steady states.

While near $\eta - \sqrt{\frac{r_0}{\alpha}}$, $(x - (\eta - \sqrt{\frac{r_0}{\alpha}}))\dot{x} \geq 0$. If $x \neq \eta - \sqrt{\frac{r_0}{\alpha}}$, the inequality is strict. Therefore a perturbation will be magnified, which means $\eta - \sqrt{\frac{r_0}{\alpha}}$ is unstable.

4

$x_c := \eta - \sqrt{\frac{r_0}{\alpha}}$ is a critical value. When the initial population is smaller than x_c, the horde can no longer sustain itself and gradually evolves toward extinction (the steady state 0). While when the initial population is larger than x_c, the horde expands toward its maximum size $\eta + \sqrt{\frac{r_0}{\alpha}}$.

0 and $\eta + \sqrt{\frac{r_0}{\alpha}}$ stand for extinction and ecological equilibrium.

Problem 2

1

Solutions to the homogenous equation take the form $x_{hom}(t) = C_1 e^t + C_2 e^{-t}$, and a solution to the equation is $x_p(t) = -\frac{1}{2}cost$. Therefore a general solution would take the form

$$x(t) = \frac{1}{2}(x(0) + \dot{x}(0) + \frac{1}{2})e^t + \frac{1}{2}(x(0) - \dot{x}(0) + \frac{1}{2})e^{-t} - \frac{1}{2}cost$$

2

$$\begin{aligned} y &:= \dot{x} \\ t(\tau) &:= \tau \end{aligned}$$

then the system has an autonomous form

$$\frac{d}{d\tau}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} y \\ x + cost \\ 1 \end{bmatrix}$$

3

Given initial condition $(\sigma_x, \sigma_y) \in \Sigma = \mathbb{R}^2$, consider its image under the flow $\phi_{2\pi}$. Define the *Poincaré* map P as

$$\begin{aligned} P : \Sigma \times \{0\} &\to \Sigma \times \{0\} \\ (\sigma_x, \sigma_y, 0) &\mapsto (\phi_{2\pi}(\sigma_x, \sigma_y), 0) \end{aligned}$$

where

$$\begin{aligned} p_x \circ \phi_{2\pi}(\sigma_x, \sigma_y) &= \frac{1}{2}(\sigma_x + \sigma_y + \frac{1}{2})e^{2\pi} + \frac{1}{2}(\sigma_x - \sigma_y + \frac{1}{2})e^{-2\pi} - \frac{1}{2} \\ p_y \circ \phi_{2\pi}(\sigma_x, \sigma_y) &= \frac{1}{2}(\sigma_x + \sigma_y + \frac{1}{2})e^{2\pi} - \frac{1}{2}(\sigma_x - \sigma_y + \frac{1}{2})e^{-2\pi} \end{aligned}$$

where p_i are projections.