Assignment 2

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Workshop2

Problem 5

$\forall z \in A_{r,R}(z_0)$, suppose $z = z_0 + le^{i\theta}$. Define $d := \min\{\frac{1}{2}(l-r), \frac{1}{2}(R-l)\}$. We claim that the open neighborhood $B_d(z)$ of z that lies within $A_{r,R}(z_0)$.

Proof of the claim: $\forall w \in B_d(z)$, by the triangle inequality we have

$$r < l - d \leq ||z_0 - z| - |z - w|| \leq |z_0 - w| \leq |z_0 - z| + |z - w| \leq l + d < R.$$

Therefore every point in $B_d(z)$ lies in $A_{r,R}(z_0)$.

Therefore $A_{r,R}(z_0)$ is open.

Problem 6

Notice that $\mathbb{Q} \times i\mathbb{Q}$ is a countable dense set in $\mathbb{C}$ due to the following reasons:

1.Q is dense in $\mathbb{R}$, so every complex number can be approximated by a complex number in $\mathbb{Q} \times i\mathbb{Q}$ in the sense that both its real and imaginary parts are approximated to any accuracy.

2. the inequality $|z| \leq |Rez| + |Imz|$ shows the approximation in 1. is also an approximation in the sense of the metric topology on the complex plane.

3.$\mathbb{Q} \times i\mathbb{Q}$ is a countable set, which is due to the diagonal rule of Cantor and the countability of $\mathbb{Q}$ (which is also due to the diagonal rule)

Workshop 3

Problem 6

a

$$\frac{g_1(z+h) - g_1(z)}{h} = \frac{\overline{f(\overline{z+h})} - \overline{f(\overline{z})}}{h} = \overline{\frac{f(\overline{z+h}) - f(\overline{z})}{\overline{h}}} \overset{\|h\| \to 0}{\longrightarrow} \overline{f'(\overline{z})}.$$

So g_1 is differentiable everywhere, which means it is holomorphic.

b

Cauchy-Riemann equations of $f = u + iv$ and $g_2 = u - iv$ give

$$\begin{aligned}\partial_x u &= \partial_y v\\ \partial_y u &= -\partial_x v\\ \partial_x u &= -\partial_y v\\ \partial_y u &= \partial_x v.\end{aligned}$$

Thus

$$\begin{aligned}\partial_y v = -\partial_y v = 0\\ \partial_x v = -\partial_x v = 0.\end{aligned}$$

Which means v is a constant. A constant is naturally holomorphic, so $u = f - iv$ is also holomorphic because of the algebraic properties of holomorphic functions. A real holomorphic function is a constant, so u is also a constant. Together we have shown $f = u + iv$ is a constant.

c

Cauchy-Riemann equations of $f = u+iv$ and $g_3(x+iy) = u(x-iy)+iv(x-iy)$ give

$$\begin{aligned}\partial_x u &= \partial_y v\\ \partial_y u &= -\partial_x v\\ \partial_x u &= -\partial_y v\\ \partial_y u &= \partial_x v.\end{aligned}$$

and the remaining prove is similar to part b.

Problem 7

Should $f = u+iv$ be a holomorphic function, Cauchy-Riemann equations must be satisfied. Therefore

$$\begin{aligned}\partial_y v = \partial_x u = by^2 + 2cxy + 3dx^2\\ \partial_x v = -\partial_y u = -3ay^2 - 2bxy - cx^2.\end{aligned}$$

integration gives

$$v = \frac{1}{3}by^3 + cxy^2 + 3dx^2y + g(x) = -3axy^2 - bx^2y - \frac{1}{3}cx^3 + h(y)$$

compare these two expressions, the coefficients of the polynomial must satisfy

$$\begin{cases} c = -3a \\ 3d = -b. \end{cases}$$

This condition is also sufficient: when the polynomial satisfies this condition, its harmonic conjugate v can be constructed as discussed above. Both the polynomial and its conjugate are smooth functions, and $f = u + iv$ satisfies Cauchy-Riemann equations. Thus f is holomorphic.