测度与积分

李岳恒

November 10, 2021

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0.0 黎曼积分的局限性

Theorem 1. *Let $f_n, f \in C([0,1])$. If $f_n \rightrightarrows f$ on $[0,1]$, then*

$$\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$$

Uniform convergence is a sufficient condition for interchanging lim and $\int$, but not necessary.

Example 1. $f_n(x) := x^n, f := \chi_{\{1\}}$.

Sometimes, the point-wise limit of a sequence of Riemannian integrable functions is not Riemannian integrable. e.g. the Dirichlet function is the point-wise limit of a sequence of Riemannian integrable functions.

0.1 Measuring Sets

0.1.1 集合运算

Definition 1. ***Symmetric difference*** $A\Delta B := (B\backslash A) \cup (A\backslash B)$.

Proposition 1. *Properties of symmetric difference.*

- $(A\Delta B)\Delta C = A\Delta(B\Delta C)$
- $(A\Delta B) \cap C = (A\cap C)\Delta(B\cap C)$

Definition 2. ***Upper and Lower Limits***

$$\limsup_{k\to\infty} A_k := \bigcap_{k\in\mathbb{N}} \bigcup_{j\geq k} A_j$$

$$\liminf_{k\to\infty} A_k := \bigcup_{k\in\mathbb{N}} \bigcap_{j\geq k} A_j$$

Proposition 2. *Properties of upper and lower limits.*

- *The upper and lower limits exist.*
- $\limsup A_k = \{\omega : \omega \text{ lies in infinite many } A_k.\}$
- $\liminf A_k = \{\omega : \omega \text{ does not lie in at most finite } A_k\}$

0.1.2 环、半环和σ-代数

我们用集合的交和对称差两种运算在集合族上构造一个环.

Definition 3. 称一个集族$\mathfrak{R}$为环,若其对集合交和对称差两种运算封闭.

Remark 1. 根据命题*1*, $\mathfrak{R}$确实是一个环.

Remark 2. 由于$A \cup B = (A\Delta B)\Delta(A\cap B)$及$A \setminus B = (A\Delta B)\cap A$,环对于集合的并和减法封闭.

Definition 4. 称一个集族$\mathfrak{S}$为半环,若其包含空集,对交封闭,且仍取$A_1 \subset A$,$A_1, A \in \mathfrak{S}$,存在着分解

$$A = \bigcup_{k-1}^{n} A_k \quad A_k \in \mathfrak{S}$$

半环的有限分解性质可以加强:

Lemma 1. 设$\mathfrak{S}$为半环,若$A_1, \ldots, A_n, A \in \mathfrak{S}$,且诸$A_k$为$A$两两不交的子集,则可以向$\{A_k\}$中添加$\mathfrak{S}$中的集合$A_{n+1}, \ldots, A_s$使得

$$A = \bigcup_{k=1}^{s} A_k$$

Proof. 使用归纳法:$n = 1$时命题显然成立.下考虑$n + 1$时情形. □

Definition 5. ***σ-algebra****. Given a set X, $\mathcal{A} \subset P(X)$ is a σ-algebra if*

- $\emptyset \in \mathcal{A}$
- $A \in \mathcal{A} \to A^c \in \mathcal{A}$
- *$\mathcal{A}$ is closed under countable union.*

Chapter 1

Lebesgue Measure

1.1 Measure

Definition 6. *A measure is a set function* $\mu : \mathcal{P}(X) \to [0, \infty]$ *satisfying*

- $\mu(\phi) = 0$
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