# A Review of the Complex Logarithm

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# Outline

1. Definition and basic properties
2. Branches of the logarithm and branch cuts
3. Holomorphicity gained and continuity lost

# Definition of the complex logarithm

**Definition (1.7.1 on p.16, Gratwick)**

For $z \in \mathbb{C} - \{0\}$, define its *logarithm* by

$$log(z) := \{w \in \mathbb{C} : exp(w) = z\}$$

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## Remark

*The complex logarithm is a multivalued function. We call any element of the set $log(z)$ a logarithm of $z$.*

# Basic properties of the complex logarithm

## Proposition (1.7.3 on p.16, Gratwick)

*Let* $z, w \in \mathbb{C} - \{0\}$, *and* $z = re^{i\theta}$ *in exponential form. Then*

1. $\log(z) = \{\ln r + i\theta + i2\pi k : k \in \mathbb{Z}\}$ *(Depiction of the set* $\log(z)$*)*

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## Remark

*These statements should be understood as equalities between sets.*

# Sketch of proof

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3. Notice that $log(1) = \{i2\pi k : k \in \mathbb{Z}\}$ and use the law of multiplication.

# Brunches of the logarithm

## Definition (1.7.9 on p.17, Gratwick)

A *brunch* of the logarithm is defined as a function $Log_{\phi} : \mathbb{C} - \{0\} \to \mathbb{C}$,

$$Log_{\phi}(z) := \ln|z| + iArg_{\phi}(z)$$

where $Arg_{\phi}(z) \in arg(z) \cap (\phi, \phi + 2\pi]$ is a properly defined real number.

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## Example

*The principal brunch* of the logarithm is the function $Log : \mathbb{C} - \{0\} \to \mathbb{C}$ defined as

$$Log(z) := Log_{-\pi}(z) = \ln|z| + iArg(z)$$

# Brunch cuts and the cut plane

**Definition (1.7.7 on p.17, Gratwick)**

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## Remark

*Half-lines are an important class of branch cuts, so we introduce the following notation: For* $z_0 \in \mathbb{C}$ *and* $\phi \in \mathbb{R}$, *define*

$$L_{z_0,\phi} = \{z \in \mathbb{C} : z = z_0 + re^{i\phi}, r \geq 0\}$$

*and define its cut plane* $D_{z_0,\phi} := \mathbb{C} - L_{z_0,\phi}$. *In particular,* $D_\phi := D_{0,\phi}$.

# Holomorphicity gained

**Theorem (1.7.10 on p.18, Gratwick)**

*A brunch of the logarithm $Log_\phi$ is holomorphic on the cut plane $D_\phi$, and $Log'_\phi(z) = 1/z$ in $D_\phi$.*

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A sketch:

1. 1.Verify that $Log_\phi(z) = \ln(|z|) + iArg_\phi(z)$ has continuously differentiable real and imaginary parts on $D_\phi$.

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2. 2.Check that the Cauchy-Riemann equations are satisfied on $D_\phi$. These two steps combined show that $Log_\phi$ is holomorphic on $D_\phi$.
3. 3.Differentiate $z = exp(Log_\phi(z))$ using the chain rule to compute the derivative of $Log_\phi$ on $D_\phi$.

□

# Continuity lost

Since $Arg_\phi(z)$ takes a jump when $z$ crosses $L_{0,\phi}$, it isn't continuous there, and hence isn't $Log_\phi(z) = \ln(|z|) + iArg_\phi(z)$.

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## Example

Notice that

$$Arg_0(1) = 2\pi$$

while

$$Arg_0(e^{i\varepsilon}) = \varepsilon$$

Thus $2\pi = Arg_0(1) = Arg_0(\lim_{\varepsilon\to 0} e^{i\varepsilon}) \neq \lim_{\varepsilon\to 0} Arg_0(e^{i\varepsilon}) = 0$

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# References

1.Gratwick, R. (2022). Honours Complex Variables Lecture Notes 2021–2022. Accessed February 13th, 2022.