Workshop 1

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Problem 4(c)

Consider the Fourier series of $f(x) = (x - \frac{1}{2})^2$:

$$\begin{aligned} c_n &= \int_{\mathbb{T}} f(x) e^{-i2\pi nx} dx \\ &= \int_0^1 (x - \frac{1}{2})^2 e^{-i2\pi nx} dx \end{aligned}$$

by substitution of variables formula we have

$$c_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 e^{-i2\pi n(x+\frac{1}{2})} dx$$

notice that $e^{-in\pi} = (-1)^n$, $x^2 e^{-i2\pi nx}$ has even real part and odd imaginary part, and the integral domain $[-\frac{1}{2}, \frac{1}{2}]$ is symmetric with respect to the origin, we have

$$c_n = (-1)^n \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \cos(2\pi nx) dx$$

integrate by parts,

$$c_n = \frac{2}{(2\pi n)^2}, \forall n \neq 0$$

and

$$c_0 = \frac{1}{12}$$

therefore by Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, we have

$$f(x) \sim \frac{1}{12} + \sum_{n\in\mathbb{Z}} c_n e^{i2\pi nx} = \frac{1}{12} + \sum_{n\geq 1} \frac{\cos(2\pi nx)}{(\pi n)^2} \tag{1}$$

Notice that

$$0 \leq \frac{1}{n^2} \leq \int_{n-1}^{n} \frac{1}{x^2} dx$$

and

$$\int_{1}^{\infty} \frac{1}{x^2} dx = 1 < \infty$$

so the series $\sum_{n \geq 1} \frac{1}{n^2}$ converges absolutely.

Since the cosine functions are bounded, the right hand side series of formula 1 converges uniformly by the Weierstrass M test . Each of its term is a continuous function, which implies it converges to a continuous function. While f is continuous, f and the series are continuous functions with identical Fourier series. Therefore they coincide.

Problem 4(d)

take $x = 1$ in formula 1 to find

$$\frac{1}{4} = f(1) = \frac{1}{12} + \sum_{n \geq 1} \frac{1}{(\pi n)^2}$$

$$\frac{1}{6} = \sum_{n \geq 1} \frac{1}{(n\pi)^2}$$

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$