Assignment 2

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Question 1

a

$$A := \{(x_1, x_2) : f(x_1) = f(x_2)\}$$

For any $y_1 \in Y$, $y_2 \in Y$ and $y_1 \neq y_2$, since f is onto, we may choose x_1, x_2 such that $f(x_i) = y_i$. Then $(x_1, x_2) \notin A$. Since A is closed, there exists an open neighborhood W of (x_1, x_2) such that $W \cap A = \emptyset$. From the definition of product topology on $X \times X$ we know there exists $U \times V$ such that

1. U and V are open neighborhoods of x_1 and x_2.
2. $U \times V \subset W \subset A^c$.

Since f is open, $f(U)$ and $f(V)$ are open neighborhoods of y_1 and y_2. They cannot intersect, otherwise there exist $z \in U$ and $w \in V$ such that $f(w) = f(z)$, thus $(z, w) \in A$, but $(z, w) \in U \times V \subset W \subset A^c$, a contradiction. Hence $f(U)$ and $f(V)$ are disjoint open neighborhoods of y_1 and y_2, so Y is Hausdorff.

b

$$B := \{x : f(x) = g(x)\}$$

For any $x \notin B$, $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open neighborhoods of $f(x)$ and $g(x)$, namely U and V. Since f and g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are open. Then since $W := f^{-1}(U) \cap g^{-1}(V)$ is the intersection of two open sets, W itself is also open. From $f(x) \in U$ and $g(x) \in V$ we find that $x \in W$.

For any $w \in W$, we have $f(w) \in U$ and $g(w) \in V$ while $U \cap V = \emptyset$. Thus $f(w) \neq g(w)$, so $w \notin B$. Therefore $W \cap B = \emptyset$ and $x \in W$. Hence B^c is open, or equivalently, B is closed.

Question 2

a

Suppose $f : [0,1] \to (0,1)$ is a homeomorphism, then $\tilde{f} := f|_{(0,1]}$ is a homeomorphism between $(0,1]$ and $(0,1) - \{f(0)\}$. But while $(0,1]$ is connected (because it is path connected), the other is not (because $(0,1) = (0, f(0)) \cup (f(0), 1)$), these two spaces cannot be homeomorphic since connectedness is invariant under homeomorphism. So, f must not exist.

Lemma 1 (I don't know if I need to prove this lemma or not)**.** *Connectedness is invariant under homeomorphism.*

Proof. Let $f : X \to Y$ be a homeomorphism. Suppose X is connected while Y is not, then $Y = U \sqcup V$ where U and V are disjoint non-empty open sets. Since f is a continuous surjection while U and V are non-empty, $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty open sets in X. Since $U \cap V = \emptyset$, they are disjoint. Since $U \sqcup V = Y$, we have $f^{-1}(U) \sqcup f^{-1}(V) = X$, which contradicts the fact that X is connected.

Suppose Y is connected and X isn't, the same argument applies since f^{-1} is also a continuous bijection. In summary, X and Y must have same connectedness should they be homeomorphic. $\square$

b

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism. Then $\tilde{f} := f|_{x \neq 0}$ is a homeomorphism between the punctured plane and $\mathbb{R} - \{f(0)\}$. The punctured plane is path connected since we may connect any two points on the punctured plane with a spiral, and thus it is also connected. While on the other hand $\mathbb{R} - \{f(0)\} = (-\infty, f(0),) \cup (f(0), \infty)$ isn't connected.

Hence f must not exist.

Question 3

Let A and B be two disjoint closed sets on the Sorgenfrey line.

For each $a \in A$, since $a \in B^c$ and B is a closed set, a has an open neighborhood that lies within B^c. Therefore, there exists an element of the topological base, $[a, a')$ such that $[a, a') \subset B^c$. Define an open neighborhood of A as

$$U := \bigcup_{a \in A} [a, a')$$

Similarly define an open neighborhood V for B:

$$V := \bigcup_{b \in B} [b, b')$$

where each $[b, b')$ is an open neighborhood of b that does not intersect A.

For any pair $a \in A, b \in B$, without loss of generality assume $a < b$. Since $[a, a') \subset B^c$, we have $[a, a') \cap [b, b') = \emptyset$. Thus $U \cap V = \emptyset$. Hence the Sorgenfrey line is T_4.

Question 4

a

Let $l_0 \in l^\infty$ and $\|l_0\|_\infty = 1$. $l_{n+1} := Tl_n$. $r_n := \|l_n\|_\infty$. Then

$$\|T^k\| = \max_{\|l_0\|_\infty=1} \|T^k l_0\| = \max_{\|l_0\|_\infty=1} r_k$$

Suppose

$$l_n = (a_1, a_2, a_3, \dots)$$

then

$$\begin{aligned} l_{n+1} &= (a_1 + a_2, a_1, a_2, \dots) \\ l_{n+2} &= (a_1 + a_2 + a_1, a_1 + a_2, a_1, \dots) \end{aligned}$$

Since every term of l_n present in l_{n+1}, we have

$$r_{n+1} = \max\{r_n, |a_1 + a_2|\} \leq 2r_n \tag{1}$$

Since all but the first term of l_{n+2} present in l_{n+1}, and the first term of l_{n+2} can be controlled by

$$|a_1 + a_2 + a_1| \leq |a_1| + |a_1 + a_2| \leq \|l_n\| + \|l_{n+1}\| = r_n + r_{n+1}$$

thus we have

$$r_{n+2} \leq r_{n+1} + r_n \tag{2}$$

Since $r_0 = 1$, it follows from inequality 1 that $r_1 \leq 2$.

With inequality 2, it is easy to prove using induction that r_n is always not greater than the $(n+1)th$ number in the Fibonacci sequence.

Notice that the above inequalities becomes equality when $l_0 = (1, 1, 0, \dots)$ and $\|l_0\| = 1$.

Therefore by the general formula of the Fibonacci sequence, (I believe I could assume this formula is known? As it's a common exercise in first-year linear algebra courses)

$$\|T^k\| = \max_{\|l_0\|=1} r_k = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{k+2} - (\frac{1-\sqrt{5}}{2})^{k+2})$$

and simple calculation yields

$$r(T) = \lim_{k\to\infty} \|T^k\|^{1/k} = \frac{1+\sqrt{5}}{2}$$

b

Suppose λ is an eigenvalue of T and v is its corresponding eigenvector with $\|v\|_\infty = 1$. Then

$$|\lambda| = \frac{\|Tv\|_\infty}{\|v\|_\infty} \leq \max_{\|v\|_\infty=1} \frac{\|Tv\|_\infty}{\|v\|_\infty} = \|T\|$$
$$|\lambda| = (|\lambda|^k)^{1/k} = (\frac{\|T^k v\|_\infty}{\|v\|_\infty})^{1/k} \leq \|T^k\|^{1/k}$$

taking limits on both sides yield $|\lambda| \leq r(T)$.

Question 5

Let $J = [a, b]$ be any bounded closed interval. Suppose $\{U_i\}_{i\in I}$ is an open cover of J but does not have a finite subcover. Divide J by half into two closed intervals

$$J_{11} = [a, \frac{a+b}{2}] \quad J_{12} = [\frac{a+b}{2}, b]$$

Then, at least one of these two intervals does not have a finite subcover. Again divide this interval without finite subcover by half and repeat to choose the one without finite subcover. Such process lasts endlessly and will result in a series of nested intervals, every one of them does not have a finite subcover. By the nested intervals theorem, there is a point x_0 that belongs to all the intervals. Since $\{U_i\}_{i\in I}$ covers J, there should be an open set U_{i_0} such that $x_0 \in U_{i_0}$. But as the lengths of the nested intervals tend to 0, at least one of them lies within U_{i_0}. This contradicts the fact that none of these intervals has a finite subcover. Hence $\{U_i\}_{i\in I}$ must have a finite subcover for J.

Question 6

Firstly, we show that the weak topology τ on X satisfies the given universal property:

Suppose g is continuous, since the weak topology makes all the f_is continuous, it follows that $f_i \circ g$ is continuous.

Then suppose $f_i \circ g$ is continuous for all is. Fix i, take any open set $U_i \in \tau_i$. it follows from the continuity of f_i and $f_i \circ g$ that $f_i^{-1}(U_i) \in \tau$ and $g^{-1}(f_i^{-1}(U_i)) \in \tau_Z$. Thus sets of the form $f_i^{-1}(U_i)$ are pulled back by g into τ_Z.

Notice that

$$\{f_i^{-1}(U_i) : i \in I \text{ and } U_i \in \tau_i\}$$

form a subbase of the weak topology τ. It follows that this subbase is pulled back by g into τ_Z. Therefore every element of τ is pulled back by g into τ_Z, that is, g is continuous.

For the other side of the proposition, assume τ' is a topology on X that satisfies the universal property. Let $(Z, \tau_Z) = (X, \tau)$ and $g =$id: $X \to X$. Since $f_i \circ g$ is continuous, $g =$id is continuous, thus every open set of τ' is an open set of τ, and $\tau' \subset \tau$.

$$\begin{tikzcd}
(X,\tau) \arrow[d, "g"] \arrow[dr, "f_i \circ g"] & \\
(X,\tau') \arrow[r, "f_i"] & (X_i,\tau_i)
\end{tikzcd}$$

We then let $(Z, \tau_Z) = (X, \tau')$, which makes $g =$id continuous. Therefore $f_i \circ g$ is continuous according to the universal property. Hence for any $U_i \in \tau_i$, $f_i^{-1}(U_i) \in \tau'$. Since sets of the form $f_i^{-1}(U_i)$ form a subbase of the weak topology τ, we may conclude that $\tau \subset \tau'$.

$$\begin{tikzcd}
(X,\tau') \arrow[d, "g"] \arrow[dr, "f_i \circ g"] & \\
(X,\tau') \arrow[r, "f_i"] & (X_i,\tau_i)
\end{tikzcd}$$

The arguments above shows that $\tau = \tau'$, thereby shows that the universal property characterizes the weak topology.