Assignment 2

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In this document $F(x)$ is the vector field of the dynamical system we discuss.

1

We first calculate the equilibriums of the system. Set

$$\begin{cases} \dot{x} = y - x^3 = 0 \\ \dot{y} = -x - y^3 = 0 \end{cases}$$

trivial calculation shows that there is a unique equilibrium at $(0, 0)$.

Define $V = x^2 + y^2$. Obviously this function is positive everywhere except at 0, which is the unique equilibrium, and $V(0) = 0$. One the other hand,

$$\langle \nabla V, F(x) \rangle = (2x, 2y) \cdot (y - x^3, -x - y^3)^T = -2(x^4 + y^4) \leq 0$$

and the inequality is strict everywhere but at $(0, 0)$. Thus V is a Lyapunov function.

Should the system has a closed orbit $\mathcal{O}$ except fixed point, then let $x_0 \in \mathcal{O}$, there exists $T > 0$ such that $\phi_T(x_0) = x_0$.

Set $x(t)$ to be the integral curve that starts at x_0.

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x(t)), F(x(t)) \rangle \leq 0$$

Since $x(t)$ does not pass 0 which is the only point where the above inequality isn't strict, we may conclude that $V(x(t))$ decreases strictly with respect to t. That leads to a contradiction:

$$V(x_0) = V(x(0)) > V(x(T)) = V(x_0)$$

Thus there is no closed orbit.

2

2.1

Set

$$\begin{cases} \dot{u} = -u + av + u^2 v = 0 \\ \dot{v} = b - av - u^2 v = 0 \end{cases}$$

the unique equilibrium is

$$\begin{cases} u^* = b \\ v^* = \frac{b}{a+b^2} \end{cases}$$

Set $x^* := (u^*, v^*)$. Compute the Jacobian $JF(x^*)$:

$$JF(x^*) = \begin{pmatrix} \partial_u F_u & \partial_v F_u \\ \partial_u F_v & \partial_v F_v \end{pmatrix}(x^*) = \begin{pmatrix} -1 + 2u^* v^* & a + u^{*2} \\ -2u^* v^* & -a - u^{*2} \end{pmatrix} = \begin{pmatrix} \frac{-a+b^2}{a+b^2} & a + b^2 \\ \frac{-2b^2}{a+b^2} & -a - b^2 \end{pmatrix}$$

Notice that

$$Tr(JF(x^*)) = \frac{-a + b^2 - (a+b^2)^2}{a+b^2}$$
$$Det(JF(x^*)) = a + b^2 > 0$$

Since the determinant is always positive, the equilibrium is stable iff $Tr(JF(x^*)) > 0$, and is unreachable, i.e. has no eigenvalue with non-negative real part, iff $Tr(JF(x^*)) < 0$. Thus,

- x^* is stable iff $-a + b^2 - (a+b^2)^2 < 0$
- x^* is unreachable (or totally unstable) iff $-a + b^2 - (a+b^2)^2 > 0$

In addition, x^* cannot be a saddle point. And solutions near it are spirals iff $Tr(JF(x^*))^2 - 4Det(JF(x^*)) < 0$.

2.2

First we compute the outer normal vectors for the boundaries given. For ∂S_i denote its outer normal vector as n_i. The computations are elementary, so detailed calculations are omitted.

$$n_1 = (0, -1)$$
$$n_2 = (1, 1)$$
$$n_3 = (0, 1)$$
$$n_4 = (-1, 0)$$

Then we estimate $\langle F(x), n_i \rangle$ on the boundaries.

$F(u, v) = (-u + av + u^2 v, b - av - u^2 v)^T$

$\langle F(x), n_1 \rangle = -\dot{v} = u - av - u^2 v = -b < 0$ since when $x \in \partial S_1$, $v \equiv 0$.

$\langle F(x), n_2 \rangle = \dot{u} + \dot{v} = b - u \leq 0$ since when $x \in \partial S_2$, $u \geq b$.

$\langle F(x), n_3 \rangle = \dot{v} = b - av - u^2 v = -u^2 v \leq 0$ since when $x \in \partial S_3$, $v \equiv \frac{b}{a}$

$\langle F(x), n_4 \rangle = -\dot{u} = u - av - u^2 v = -av \leq 0$ since when $x \in \partial S_4$, $u \equiv 0$

Hence S is a trapping region.

2.3

The only equilibrium of the system is x^*, which lies within S but is totally unstable. Thus we choose a small ball $B_r(x^*)$: for any $x \in \partial B_r(x^*)$,

$$\langle F(x), n_x \rangle = \langle JF(x^*)(x - x^*), x - x^* \rangle + o(\|x - x^*\|^2) \tag{1}$$

Since x^* is totally unstable, from the theory of linear ODE we know for $B_r(x^*)$ sufficiently small, the integral curve of the linear system $\dot{x} = JF(x^*)(x - x^*)$ that originates at x must bent strictly outward $B_r(x^*)$, that is

$$\langle JF(x^*)(x - x^*), x - x^* \rangle > 0$$

for any $x \in \partial B_r(x^*)$. Notice that

$$f(x - x^*) := \langle JF(x^*)(x - x^*), x - x^* \rangle$$

is a positive continuous function defined on a compact set $\partial B_r(x^*)$, so it achieves a positive minimum $m\|x - x^*\|^2$ on it. Notice also that f is a quadratic homogenous function, which means the above conclusion holds for $B_r(x^*)$ of any radius, that is, the constant m does not change with respect to r.

Therefore we may conclude that for r sufficiently small, the expression 1 is positive:

$$\begin{aligned} \langle F(x), n_x \rangle &= \langle JF(x^*)(x - x^*), x - x^* \rangle + o(\|x - x^*\|^2) \\ &\geq m\|x - x^*\|^2 - o\|x - x^*\|^2 \geq 0 \end{aligned}$$

Our final conclusion is $S - B_r(x^*)$ is positively invariant, as we've verified $\langle F(x), n \rangle \leq 0$ on its boundary, where n is its outer normal vector at x. By Poincaré-Bendixson theorem, $S - B_r(x^*)$ contains a closed orbit.