Assignment 1

Li Yueheng s2306706

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Question 1

We need to assume that X and V are normed linear spaces, otherwise boundedness would be ambiguous.

Define addition and multiplication by numbers on $Hom(X,V)$ as follows:

$$\begin{aligned}
&\forall f,g \in Hom(X,V), \forall x \in X, \forall \lambda \in \mathbb{R},\\
&(f+g)(x) := f(x)+g(x),\\
&(\lambda f)(x) := \lambda f(x)
\end{aligned}$$

Thus, $Hom(X,V)$ becomes a linear space. $B(X,V)$ inherits these operations. To prove $B(X,V)$ is a subspace, we need only verify it is closed under such operations.

Suppose $||f|| = M$, $||g|| = N$. By triangle inequality we have the following estimates

$$\begin{aligned}
&||(f+g)(x)|| \le ||f(x)|| + ||g(x)|| \le (M+N)||x||\\
&i.e. ||f+g|| \le M+N\\
&||(\lambda f)(x)|| = |\lambda|||f(x)|| \le |\lambda|M||x||\\
&i.e. ||\lambda f|| \le |\lambda M|
\end{aligned}$$

Therefore $B(X,V)$ is closed under addition and multiplication by numbers, so it is a linear space.

Question 2

Sets of the form $\widetilde{v} := v + S = \{v + w : w \in S\}$ are elements of X/S.

Define addition and scalar multiplication on X/S as

$$\begin{aligned}
&\widetilde{u} + \widetilde{v} := \widetilde{u+v}\\
&\lambda\widetilde{u} := \widetilde{\lambda u}
\end{aligned}$$

Since S is closed under addition and scalar multiplication, these operations are well-defined (do not depend on the choice of v that represents $\widetilde{v}$ in the operations)

Simple calculation yields that these operations satisfy the axioms of linear space. So X/S is a linear space under such operations.

$$\begin{aligned}
&\widetilde{u}+\widetilde{v}=\widetilde{v}+\widetilde{u}\\
&(\widetilde{u}+\widetilde{v})+\widetilde{w}=\widetilde{u}+(\widetilde{v}+\widetilde{w})\\
&\widetilde{0}+\widetilde{u}=\widetilde{u}\\
&\widetilde{u}+\widetilde{-u}=\widetilde{0}\\
&1\widetilde{u}=\widetilde{u}\\
&\lambda(\widetilde{u}+\widetilde{v})=\lambda\widetilde{u}+\lambda\widetilde{v}\\
&(\lambda+\eta)\widetilde{u}=\lambda\widetilde{u}+\eta\widetilde{u}
\end{aligned}$$

Question 3

$|x(t)-y(t)|\geq 0$ implies $\rho(x,y)\geq 0$. From the property of integration and the fact that both x and y are continuous, $\rho(x,y)=0$ iff $|x(t)-y(t)|=0$, that is, x=y.

$|x(t)-y(t)|=|y(t)-x(t)|$ implies $\rho(x,y)=\rho(y,x)$

$|x(t)-y(t)|\leq|x(t)-z(t)|+|z(t)-y(t)|$ implies $\rho(x,y)\leq\rho(x,z)+\rho(z,y)$

Therefore ρ is a metric.

Question 4

4.1

Note that if $f(x)=y$, then $f^{-1}(B_\varepsilon(y))=B_\varepsilon(x)$ as f is a one-to-one isomorphism. Same argument applies to f^{-1}, so both f and f^{-1} are continuous.

4.2

Suppose f is an isomorphism between $\mathbb{R}$ and $\mathbb{R}^2$. Let $f(0)=y$.

Notice that

$$f^{-1}(\partial B_1(y))=\{-1,1\}$$

$\partial B_1(y)$ is an infinite set and $\{-1,1\}$ is finite. This contradicts the fact that f is one-to-one.

Question 5

$$
\begin{aligned}
&\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2) \\
=&\frac{1}{4}(<x+y, x+y> - <x-y, x-y> + i<x+iy, x+iy> - i<x-iy, x-iy>) \\
=&\frac{1}{4}(<x,x> + <x,y> + \overline{<x,y>} + <y,y> - <x,x> + <x,y> + \overline{<x,y>} - <y,y> \\
&+ i<x,x> + <x,y> - <y,x> - i<y,y> - i<x,x> + <x,y> - <y,x> + i<y,y>) \\
=& <x,y>
\end{aligned}
$$

This identity associates an inner product with the norm it induces by directly representing the former with the latter.

Question 6

6.1

$\forall (x,y) \notin A$, $y \neq tanx$. By the continuity of the tangent function, there is a neighborhood U of x and V of y such that $f(U) \cap V = \emptyset$. Therefore $\forall (x,y) \in U \times V$, $(x,y) \notin A$, which means A is closed.

6.2

Suppose W is an open set in $\mathbb{R}^2$, $\forall z \in W$, there exists $B_r(z) \subset W$. So $(\pi_1(z) - r, \pi_1(z) + r) \subset \pi_1(W)$. Since $\pi_1(z)$ can be any point in $\pi_1(W)$, $\pi_1(W)$ is open. So π_1 is an open map.

π_1 is the first coordinate of the identity map on $\mathbb{R}^2$. Since the latter is obviously continuous, it follows that π_1 is continuous as well.

$\pi_1(A) = (-\frac{\pi}{2}, \frac{\pi}{2})$ is not a closed set while A is. So π_1 is not closed.