Assignment 2

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Question 1

a

$$A := \{(x_1, x_2) : f(x_1) = f(x_2)\}$$

For any $y_1 \in Y$, $y_2 \in Y$ and $y_1 \neq y_2$, choose x_1, x_2 such that $f(x_i) = y_i$. Then $(x_1, x_2) \notin A$. Since A is closed, there exists an open neighborhood W of (x_1, x_2) such that $W \cap A = \emptyset$. From the definition of product topology on $X \times X$ we know there exists $U \times V$ such that

1. U and V are open neighborhoods of x_1 and x_2.
2. $U \times V \subset W \subset A^c$.

Since f is open, $f(U)$ and $f(V)$ are open neighborhoods of y_1 and y_2. They cannot intersect, otherwise there exist $z \in U$ and $w \in V$ such that $f(w) = f(z)$, thus $(z, w) \in A$, but $(z, w) \in U \times V \subset W \subset A^c$, a contradiction. Hence $f(U)$ and $f(V)$ are disjoint open neighborhoods of y_1 and y_2, so Y is Huasdorff.

b

$$B := \{x : f(x) = g(x)\}$$

For any $x \notin B$, $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open neighborhoods of $f(x)$ and $g(x)$, namely U and V. Then $W := f^{-1}(U) \cap g^{-1}(V)$ is en open neighborhood of x and $W \cap B = \emptyset$. Hence B^c is open, i.e. B is closed.

Question 2

a

Suppose $f : [0, 1] \to (0, 1)$ is a homeomorphism, then $\tilde{f} := f|_{(0,1]}$ is a homeomorphism between $(0, 1]$ and $(0, 1) - \{f(0)\}$. While $(0, 1]$ is path connected, the other is not, which leads to a contradiction. So f must not exist.

b

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism. Then $\tilde{f} := f|_{x \neq 0}$ is a homeomorphism between the punctured plane and $\mathbb{R} - \{f(0)\}$. The punctured plane is path connected while $\mathbb{R} - \{f(0)\}$ isn't. Hence f must not exist.

Question 3

Let A and B be two disjoint closed sets on the Sorgenfrey line.

For each $a \in A$, since $a \in B^c$ and B is a closed set, a has an open neighborhood that lies within B^c. Therefore, there exists an element of the topological base, $[a, a')$ such that $[a, a') \subset B^c$. Define an open neighborhood of A as

$$U := \bigcup_{a \in A} [a, a')$$

Similarly define an open neighborhood V for B:

$$V := \bigcup_{b \in B} [b, b')$$

where each $[b, b')$ is an open neighborhood of b that does not intersect A.

For any pair $a \in A, b \in B$, without loss of generality assume $a < b$. Since $[a, a') \subset B^c$, we have $[a, a') \cap [b, b') = \emptyset$. Thus $U \cap V = \emptyset$. Hence the Sorgenfrey line is T_4.

Question 4

a

Let $l_0 \in l^\infty$ and $\|l_0\|_\infty = 1$. $l_{n+1} := Tl_n$. $r_n := \|l_n\|_\infty$. Then

$$\|T^k\| = \max_{\|l_0\|_\infty = 1} \|T^k l_0\| = \max_{\|l_0\|_\infty = 1} r_k$$

Suppose

$$l_n = (a_1, a_2, a_3, \dots)$$

then

$$\begin{aligned} l_{n+1} &= (a_1 + a_2, a_1, a_2, \dots) \\ l_{n+2} &= (a_1 + a_2 + a_1, a_1 + a_2, a_1, \dots) \end{aligned}$$

thus $r_{n+2} \leq r_{n+1} + r_n$ and with equality for $l_0 = (1, 1, 0, \dots)$. Therefore

$$\|T^k\| = \frac{1}{\sqrt{5}}((\frac{1 + \sqrt{5}}{2})^{k+2} - (\frac{1 - \sqrt{5}}{2})^{k+2})$$

and

$$r(T) = \lim_{k \to \infty} \|T^k\|^{1/k} = \frac{1 + \sqrt{5}}{2}$$

b

Suppose λ is an eigenvalue of T and v is its corresponding eigenvector with $\|v\|_\infty = 1$. Then

$$|\lambda| = \frac{\|Tv\|_\infty}{\|v\|_\infty} \leq \max_{\|v\|_\infty=1} \frac{\|Tv\|_\infty}{\|v\|_\infty} = \|T\|$$
$$|\lambda| = (|\lambda|^k)^{1/k} = (\frac{\|T^k v\|_\infty}{\|v\|_\infty})^{1/k} \leq \|T^k\|^{1/k}$$

taking limits on both sides yield $|\lambda| \leq r(T)$.

Question 5

Let $J = [a,b]$ be any bounded closed interval. Suppose $\{U_i\}_{i\in I}$ is an open cover of J but does not have a finite subcover. Divide J by half into two closed intervals

$$J_{11} = [a, \frac{a+b}{2}] \quad J_{12} = [\frac{a+b}{2}, b]$$

Then, at least one of these two intervals does not have a finite subcover. Again divide this interval without finite subcover by half and repeat to choose the one without finite subcover. Such process lasts endlessly and will result in a series of nested intervals, every one of them does not have a finite subcover. By the nested intervals theorem, there is a point x_0 that belongs to all the intervals. Since $\{U_i\}_{i\in I}$ covers J, there should be an open set U_{i_0} such that $x_0 \in U_{i_0}$. But as the lengths of the nested intervals tend to 0, at least one of them lies within U_{i_0}. This contradicts the fact that none of these intervals has a finite subcover. Hence $\{U_i\}_{i\in I}$ must have a finite subcover for J.

Question 6

Firstly, we show that the weak topology τ on X satisfies the given universal property:

Suppose g is continuous, since the weak topology makes all the f_is continuous, it follows that $f_i \circ g$ is continuous.

Then suppose $f_i \circ g$ is continuous for all is. Fix i, take any open set $U_i \in \tau_i$. it follows from the continuity of f_i and $f_i \circ g$ that $f_i^{-1}(U_i) \in \tau$ and $g^{-1}(f_i^{-1}(U_i)) \in \tau_Z$. Thus sets of the form $f_i^{-1}(U_i)$ are pulled back by g into τ_Z.

Notice that

$$\{f_i^{-1}(U_i) : i \in I \text{ and } U_i \in \tau_i\}$$

form a subbase of the weak topology τ. It follows that this subbase is pulled back by g into τ_Z. Therefore every element of τ is pulled back by g into τ_Z, that is, g is continuous.

For the other side of the proposition, assume τ' is a topology on X that satisfies the universal property. Let $(Z, \tau_Z) = (X, \tau)$ and $g =$id: $X \to X$. Since $f_i \circ g$ is continuous, $g =$id is continuous, thus every open set of τ' is an open set of τ, and $\tau' \subset \tau$.

$$\begin{tikzcd}
(X, \tau) \arrow[d, "g"] \arrow[dr, "f_i \circ g"] & \\
(X, \tau') \arrow[r, "f_i"] & (X_i, \tau_i)
\end{tikzcd}$$

We then let $(Z, \tau_Z) = (X, \tau')$, which makes $g =$id continuous. Therefore $f_i \circ g$ is continuous according to the universal property. Hence for any $U_i \in \tau_i$, $f_i^{-1}(U_i) \in \tau'$. Since sets of the form $f_i^{-1}(U_i)$ form a subbase of the weak topology τ, we may conclude that $\tau \subset \tau'$.

$$\begin{tikzcd}
(X, \tau') \arrow[d, "g"] \arrow[dr, "f_i \circ g"] & \\
(X, \tau') \arrow[r, "f_i"] & (X_i, \tau_i)
\end{tikzcd}$$

The arguments above shows that $\tau = \tau'$, thereby shows that the universal property characterizes the weak topology.