

# Assignment 1

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## Problem 1

1

$g'(x) = -2\alpha(x - \eta)$  decreases monotonously. therefore  $g$  achieves a unique maximum  $r_0$  at  $x_{max} = \eta$ .

2

Set  $\dot{x} = 0$  to find three solutions  $\{0, \eta + \sqrt{\frac{r_0}{\alpha}}, \eta - \sqrt{\frac{r_0}{\alpha}}\}$ . Among which the first two are distinct non-negative solutions. They are the steady states of the system.

Near 0, we have  $x\dot{x} = x^2g(x) \leq 0$ .

Near  $\eta + \sqrt{\frac{r_0}{\alpha}}$ , we have  $(x - (\eta + \sqrt{\frac{r_0}{\alpha}}))\dot{x} \leq 0$ .

So, the system will return to these two steady states under perturbation. Therefore these steady states are stable.

3

Under the assumption  $\eta - \sqrt{\frac{r_0}{\alpha}} > 0$ , the arguments in the previous problem still hold. So 0 and  $\eta + \sqrt{\frac{r_0}{\alpha}}$  are stable steady states.

While near  $\eta - \sqrt{\frac{r_0}{\alpha}}$ ,  $(x - (\eta - \sqrt{\frac{r_0}{\alpha}}))\dot{x} \geq 0$ . If  $x \neq \eta - \sqrt{\frac{r_0}{\alpha}}$ , the inequality is strict. Therefore a perturbation will be magnified, and any perturbation of the system will result in it the system leaving a neighborhood of the steady state. This neighborhood is irrelevant with the perturbation, which means  $\eta - \sqrt{\frac{r_0}{\alpha}}$  is unstable.

4

$x_c := \eta - \sqrt{\frac{r_0}{\alpha}}$  is a critical value. When the initial population is smaller than  $x_c$ , the horde can no longer sustain itself and gradually evolves toward extinction (the steady state 0). While when the initial population is larger than  $x_c$ , the horde expands toward its maximum size  $\eta + \sqrt{\frac{r_0}{\alpha}}$ .

0 stands for extinction while  $\eta + \sqrt{\frac{r_0}{\alpha}}$  stands for ecological equilibrium.

## Problem 2

1

Solutions to the homogenous equation take the form  $x_{hom}(t) = C_1 e^t + C_2 e^{-t}$ , and a solution to the equation is  $x_p(t) = -\frac{1}{2} cost$ . Therefore a general solution would take the form

$$x(t) = \frac{1}{2}(x(0) + \dot{x}(0) + \frac{1}{2})e^t + \frac{1}{2}(x(0) - \dot{x}(0) + \frac{1}{2})e^{-t} - \frac{1}{2}cost$$

2

$$\begin{aligned} y &:= \dot{x} \\ t(\tau) &:= \tau \end{aligned}$$

then the system has an autonomous form

$$\frac{d}{d\tau} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} y \\ x + cost \\ 1 \end{bmatrix}$$

3

Given initial condition  $(\sigma_x, \sigma_y) \in \Sigma = \mathbb{R}^2$ , consider its image under the flow  $\phi_{2\pi}$ . Define the *Poincaré* map  $P$  as

$$\begin{aligned} P : \Sigma \times \{0\} &\rightarrow \Sigma \times \{0\} \\ (\sigma_x, \sigma_y, 0) &\mapsto (\phi_{2\pi}(\sigma_x, \sigma_y), 0) \end{aligned}$$

where

$$\begin{aligned} p_x \circ \phi_{2\pi}(\sigma_x, \sigma_y) &= \frac{1}{2}(\sigma_x + \sigma_y + \frac{1}{2})e^{2\pi} + \frac{1}{2}(\sigma_x - \sigma_y + \frac{1}{2})e^{-2\pi} - \frac{1}{2} \\ p_y \circ \phi_{2\pi}(\sigma_x, \sigma_y) &= \frac{1}{2}(\sigma_x + \sigma_y + \frac{1}{2})e^{2\pi} - \frac{1}{2}(\sigma_x - \sigma_y + \frac{1}{2})e^{-2\pi} \end{aligned}$$

where  $p_i$  are projections.