Topology

1. Introduction

양현서

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Manifolds란 무엇인가?

- 공간에서의 면을 일반화한 개념
- 예시: S2 (구면)
- 여기서의 함수의 최댓값, 최솟값 구하기는 평범한 평면에서 하던 것과는 다르다.
- A **manifold** is a **topological space** locally **homeomorphic** to $\mathbb{R}^n$.

Metric

- A non empty set X

- For $\forall x, y \in X$
  - $d(x, y) > 0 \Leftrightarrow x \neq y$
  - $d(x, y) = d(y, x)$
  - $d(x, y) \geq d(x, z) + d(z, y)$

Example of metrics

- Eucledian metric: $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y)^2}$

- Square metric: $d(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$

- Taxi-cab metric: $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

- Discrete metric: $d(x,y) = 1$ if $x \neq y$, 0 if $x = y$

Metric Space

- Set X

- Metric function $d : X^2 \to \mathbb{R}$

Open epsilon ball in metric space

- $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$

- It is an open set

Examples of balls in metric spaces

- inside of a circle with radius r where $X = \mathbb{R}^2, d = \text{Eucledian metric}$

- inside of a square where $X = \mathbb{R}^2, d = \text{Square metric}$

- Discrete metric space
  - $B_{\text{Discrete}}(p, 0.1) = \{p\}$
  - $B_{\text{Discrete}}(p, 1) = \{p\}$
  - $B_{\text{Discrete}}(p, 1.000001) = X$

Open sets in metric space

- if $U \in X$, U is an open set if $\forall x \in U, \exists \varepsilon > 0$ such that $B(x, \varepsilon) \subset U$

- Example (Eucledian space)
  - $U = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$
  - $\forall (x, y) \in U, B((x, y), x) \subset U$

Topology

- **Metric is not needed**

- Set X, $T \subseteq \mathcal{P}(X)$

- T is a topology on X if
  - $\emptyset, X \in T$
  - $U, V \in T \Rightarrow U \cap V \in T$
  - For any fixed index set I, $U_i \in T \Rightarrow \bigcup_{i \in I} U_i \in T$

Open sets

Elements of T are called open sets

Examples of topology

- Discrete Topology: $T = \mathcal{P}(X)$ is a topology on X

- Indiscrete Topology: $T = \{\emptyset, X\}$ is a topology on X

Open epsilon ball is an open set

- In a metric space (X, d), for any $x \in X$ and $\varepsilon > 0$, the set $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ is called an ε-ball around x.

- Proposition: Every ε-ball is an open set in the metric space (X, d).

Open epsilon ball is an open set: Proof

- Let $B(x, \varepsilon)$ be an ε-ball and let $p \in B(x, \varepsilon)$
- Then $d(x, p) < \varepsilon$
- Choose $\delta = \varepsilon - d(x, p)$, then $\delta > 0$.
- For any $q \in B(p, \delta)$, we have $d(p, q) < \delta$.
- By triangle ineq, $d(x, q) \leq d(x, p) + d(p, q) < d(x, p) + \delta = \varepsilon$
- Hence, $q \in B(x, \varepsilon)$
- Thus, $B(p, \delta) \subset B(x, \varepsilon)$, implying p is an interior point of $B(x, \varepsilon)$
- Since p was arbitrary, $B(x, \varepsilon)$ is an open set.

- Therefore, every ε-ball is an open set in the metric space (X, d).

Teaser: Closed set, continuous function