1

解

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\,\mathrm{d}x = \frac{2\pi^2}{3}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi}x^2\cos nx\,\mathrm{d}x \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\sin nx\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi}x\sin nx\,\mathrm{d}x \\
&= \frac{2(-1)^n}{n}
\end{aligned}$$

故

$$g(x) = \frac{\pi^2}{3} + \sum_{n=0}^{\infty}(\frac{4(-1)^n}{n^2}\cos nx + \frac{2(-1)^n}{n}\sin nx)$$

$$g(\pi) = \frac{\pi^2}{3} + \sum_{n=0}^{\infty}\frac{4}{n^2}$$

又

$$\begin{aligned}
g(\pi) &= \frac{1}{2}\lim_{\delta\to0^+}g(\pi+\delta) + \frac{1}{2}\lim_{\delta\to0^-}g(\pi+\delta) \\
&= \frac{\pi^2-\pi}{2} + \frac{\pi^2+\pi}{2} \\
&= \pi^2
\end{aligned}$$

故

$$\frac{\pi^2}{3} + \sum_{n=0}^{\infty}\frac{4}{n^2} = \pi^2$$

$$\sum_{n=0}^{\infty}\frac{4}{n^2} = \frac{2}{3}\pi^2$$

$$\sum_{n=0}^{\infty}\frac{1}{n^2}=\frac{1}{6}\pi^2$$

2

解　(1)

$$\begin{aligned}\mathcal{F}(\frac{1}{x^2+1}) &= \int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}kx}}{x^2+1}\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}kz}}{z^2+1}\,\mathrm{d}z\end{aligned}$$

因为

$$\lim_{z\to\infty}\frac{z}{z^2+1}=0$$

1. 当 $k>0$ 时,取积分围道为下半平面以原点为圆心的无穷大半圆,有

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}kz}}{z^2+1}\,\mathrm{d}z\\ =&-2\pi\mathrm{i}\,\mathrm{res}(\frac{e^{-\mathrm{i}kz}}{z^2+1},-\mathrm{i})\\ =&-2\pi\mathrm{i}\lim_{z\to-\mathrm{i}}(z+\mathrm{i})\frac{e^{-\mathrm{i}kz}}{z^2+1}\\ =&\pi e^{-k}\end{aligned}$$

2. 当 $k<0$ 时,取积分围道为上半平面以原点为圆心的无穷大半圆,有

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}kz}}{z^2+1}\,\mathrm{d}z\\ =&2\pi\mathrm{i}\,\mathrm{res}(\frac{e^{-\mathrm{i}kz}}{z^2+1},\mathrm{i})\\ =&2\pi\mathrm{i}\lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{e^{-\mathrm{i}kz}}{z^2+1}\\ =&\pi e^{k}\end{aligned}$$

综上

$$\mathcal{F}(\frac{1}{x^2+1})=\pi e^{-|k|}$$

故

$$\mathcal{F}(\frac{x}{x^2+1})=\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}k}(\pi e^{-|k|})$$

$$= \begin{cases} \pi e^{k} & (k<0) \\ -\mathrm{i}\pi e^{-k} & (k>0) \end{cases}$$

(2)

$$\begin{aligned} \mathcal{F}(e^{-|x|}) &= \int_{-\infty}^{\infty} e^{-|x|}e^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \int_{-\infty}^{0} e^{x}e^{-\mathrm{i}kx}\,\mathrm{d}x + \int_{0}^{\infty} e^{-x}e^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \int_{0}^{\infty} e^{-x}e^{\mathrm{i}kx}\,\mathrm{d}x + \int_{0}^{\infty} e^{-x}e^{-\mathrm{i}kx}\,\mathrm{d} \\ &= 2\int_{0}^{\infty} e^{-x}\cos(kx)\,\mathrm{d}x \\ &= \frac{2}{k^2+1} \end{aligned}$$

3

解 令 $g(x)=\dfrac{1}{x^2+a^2}$,有

$$f(x)*g(x)=\frac{1}{x^2+b^2}$$

对方程两边进行傅里叶变换得

$$\begin{aligned} \mathcal{F}(f(x))\mathcal{F}(g(x)) &= \mathcal{F}(\frac{1}{x^2+b^2}) \\ \mathcal{F}(f(x)) &= \frac{\mathcal{F}(\frac{1}{x^2+b^2})}{\mathcal{F}(\frac{1}{x^2+a^2})} \\ f(x) &= \mathcal{F}^{-1}(\frac{\mathcal{F}(\frac{1}{x^2+b^2})}{\mathcal{F}(\frac{1}{x^2+a^2})}) \end{aligned}$$

$$\begin{aligned} &\mathcal{F}(\frac{1}{x^2+b^2}) \\ &= \int_{-\infty}^{\infty} \frac{1}{x^2+b^2}e^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{1}{z^2+b^2}e^{-\mathrm{i}kz}\,\mathrm{d}x \end{aligned}$$

因为

$$\lim_{z\to\infty}\frac{z}{z^2+b^2}=0$$

1. 当 $k>0$ 时,取积分围道为下半平面以原点为圆心的无穷大半圆,有

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}kz}}{z^2+b^2}\,\mathrm{d}z\\=&-2\pi\mathrm{i}\mathrm{res}(\frac{e^{-\mathrm{i}kz}}{z^2+b^2},-b\mathrm{i})\\=&-2\pi\mathrm{i}\lim_{z\to-b\mathrm{i}}(z+b\mathrm{i})\frac{e^{-\mathrm{i}kz}}{z^2+b^2}\\=&\frac{\pi e^{-bk}}{b}\end{aligned}$$

2. 当 $k<0$ 时,取积分围道为上半平面以原点为圆心的无穷大半圆,有

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}kz}}{z^2+b^2}\,\mathrm{d}z\\=&2\pi\mathrm{i}\mathrm{res}(\frac{e^{-\mathrm{i}kz}}{z^2+b^2},b\mathrm{i})\\=&2\pi\mathrm{i}\lim_{z\to b\mathrm{i}}(z-b\mathrm{i})\frac{e^{-\mathrm{i}kz}}{z^2+b^2}\\=&\frac{\pi e^{bk}}{b}\end{aligned}$$

综上

$$\mathcal{F}(\frac{1}{x^2+b^2})=\frac{\pi e^{-b|k|}}{b}$$

同理

$$\mathcal{F}(\frac{1}{x^2+a^2})=\frac{\pi e^{-a|k|}}{a}$$

故

$$\begin{aligned}\mathcal{F}(f(x))&=\frac{a}{b}e^{-(b-a)|k|}\\&=\frac{a}{b}\frac{\pi}{b-a}\frac{b-a}{\pi}e^{-(b-a)|k|}\\&=\frac{a(b-a)}{b\pi}\mathcal{F}(\frac{1}{x+(b-a)^2})\end{aligned}$$

故

$$f(x)=\frac{a(b-a)}{b\pi[x+(b-a)^2]}$$

4

解 (1)

$$\mathcal{L}(\sin 2t\cos 3t)=\mathcal{L}(\frac{\sin 5t-\sin t}{2})$$

$$=\frac{1}{2}\mathcal{L}(\sin 5t)-\frac{1}{2}\mathcal{L}(\sin t)$$

$$\begin{aligned}
&\int_0^\infty \sin ate^{-pt}\,\mathrm{d}t\\
=&\int_0^\infty \frac{e^{\mathrm{i}at}-e^{-\mathrm{i}at}}{2\mathrm{i}}e^{-pt}\,\mathrm{d}t\\
=&\frac{1}{2\mathrm{i}}(\int_0^\infty e^{\mathrm{i}at-pt}\,\mathrm{d}t-\int_0^\infty e^{-\mathrm{i}at-pt}\,\mathrm{d}t)\\
=&\frac{1}{2\mathrm{i}}(\frac{1}{p-a\mathrm{i}}-\frac{1}{p+a\mathrm{i}})\\
=&\frac{a}{p^2+a^2}
\end{aligned}$$

故

$$\begin{aligned}
&\frac{1}{2}\mathcal{L}(\sin 5t)-\frac{1}{2}\mathcal{L}(\sin t)\\
=&\frac{1}{2}\frac{5}{p^2+25}-\frac{1}{2}\frac{1}{p^2+1}
\end{aligned}$$

(2)

$$\begin{aligned}
\mathcal{L}(\sin^2 t)&=\int_0^\infty \sin^2 te^{-pt}\,\mathrm{d}t\\
&=\int_0^\infty (\frac{1-\cos 2t}{2})e^{-pt}\,\mathrm{d}t\\
&=\int_0^\infty \frac{1}{2}e^{-pt}\,\mathrm{d}t-\int_0^\infty \frac{\cos 2t}{2}e^{-pt}\,\mathrm{d}t\\
&=\frac{1}{2}(\frac{1}{p}-\int_0^\infty \cos 2t\,\mathrm{d}t)\\
&=\frac{1}{2}(\frac{1}{p}-\int_0^\infty \frac{e^{\mathrm{i}2t}+e^{-\mathrm{i}2t}}{2}\,\mathrm{d}t)\\
&=\frac{1}{2}(\frac{1}{p}-\frac{p}{p^2+4})
\end{aligned}$$

故

$$\mathcal{L}(e^{-\lambda t}\sin^2 t)=\frac{1}{2}(\frac{1}{p+\lambda}-\frac{p+\lambda}{(p+\lambda)^2+4})$$

5

解　(1)

$$\frac{1}{(p^2+\omega^2)(p^2+\nu^2)}=\frac{1}{p^2+\omega^2}\frac{1}{p^2+\nu^2}$$

$$= \frac{1}{\omega\nu}\mathcal{L}(\sin\omega t)\mathcal{L}(\sin\nu t)$$

由卷积定理得

$$\begin{aligned}\frac{1}{\omega\nu}\mathcal{L}(\sin\omega t)\mathcal{L}(\sin\nu t) &= \frac{1}{\omega\nu}\mathcal{L}(\int_0^t \sin[\omega\tau]\sin[\nu(t-\tau)]\,\mathrm{d}\tau) \\ &= \frac{1}{\omega\nu}\mathcal{L}(\frac{\nu\sin(t\omega)-\omega\sin(\nu t)}{\nu^2-\omega^2})\end{aligned}$$

故

$$\mathcal{L}^{-1}(\frac{1}{(p^2+\omega^2)(p^2+\nu^2)}) = \frac{1}{\omega\nu}\frac{\nu\sin(t\omega)-\omega\sin(\nu t)}{\nu^2-\omega^2}$$

(2)

$$\begin{aligned}\frac{1}{(p^2+\omega^2)(p^2+\nu^2)} &= \frac{p}{p^2+\omega^2}\frac{p}{p^2+\nu^2} \\ &= \mathcal{L}(\cos\omega t)\mathcal{L}(\cos\nu t)\end{aligned}$$

由卷积定理得

$$\begin{aligned}\mathcal{L}(\cos\omega t)\mathcal{L}(\cos\nu t) &= \mathcal{L}(\int_0^t \cos[\omega\tau]\cos[\nu(t-\tau)]\,\mathrm{d}\tau) \\ &= \mathcal{L}(\frac{\nu\sin(\nu t)-\omega\sin(t\omega)}{\nu^2-\omega^2})\end{aligned}$$

故

$$\mathcal{L}^{-1}(\frac{p^2}{(p^2+\omega^2)(p^2+\nu^2)}) = \frac{\nu\sin(\nu t)-\omega\sin(t\omega)}{\nu^2-\omega^2}$$

5

解　对方程两边进行拉普拉斯变换得

$$\begin{aligned}\mathcal{L}(y) &= a\frac{1}{p^2+1} - 2\mathcal{L}(y)\frac{p}{p^2+1} \\ \mathcal{L}(y) &= a\frac{1}{(p+1)^2} \\ \mathcal{L}(y) &= a(-1)^1\frac{\mathrm{d}}{\mathrm{d}p}\frac{1}{p+1} \\ \mathcal{L}(y) &= a\mathcal{L}(te^{-t}) \\ y &= \mathcal{L}^{-1}(a\mathcal{L}(te^{-t})) \\ y &= ate^{-t}\end{aligned}$$