1

解 (1) 选用球坐标系，以外电场方向为极轴，选取地面为势能零点则定解条件为

$$\begin{cases}\dfrac{1}{r^2}\partial_r(r^2\partial_r\varphi)+\dfrac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\varphi)=0(r>R_0)\\ \varphi|_{r=R_0}=\phi_0\\ \lim\limits_{r\to\infty}\varphi=-E_0r\cos\theta+\varphi_0\end{cases}$$

其中 φ_0 为为放入导体球时原点处的电势。
分离变量 $\varphi=R(r)\Theta(\theta)$ 得

$$\begin{cases}r^2R''+2rR'-l(l+1)R=0\\ \Theta''+\cot\theta\Theta+l(l+1)\Theta=0\end{cases}$$

通解为

$$\begin{cases}R=A_lr^l+B_lr^{-l-1}\\ \Theta=P_l(\cos\theta)\end{cases}$$

则

$$\varphi=\sum_{l=0}^{\infty}P_l(\cos\theta)(A_lr^l+B_lr^{-l-1})$$

代入无穷远处边界条件得

$$\lim_{r\to\infty}\sum_{l=0}^{\infty}P_l(\cos\theta)(A_lr^l+B_lr^{-l-1})=-E_0r\cos\theta+\varphi_0$$

对比系数得到

$$A_0=\varphi_0$$

$$A_1=-E_0$$

$$A_l=0(l>1)$$

故

$$\varphi=\varphi_0-E_0r\cos\theta+\sum_{l=0}^{\infty}B_lr^{-l-1}$$

代入导体表面边界条件有

$$\varphi_0-E_0R_0\cos\theta+\sum_{l=0}^{\infty}B_lR_0^{-l-1}P_l(\cos\theta)=\phi_0$$

由于勒让德级数是正交的,故等式两端相同幂次的 $\cos\theta$ 的系数相等对比系数得到

$$\begin{cases} \varphi_0 + \dfrac{B_0}{R_0} = \phi_0 \\ -E_0R_0 + \dfrac{B_1}{R_0^2} = 0 \\ B_l = 0(l>1) \end{cases}$$

解得

$$B_0 = R_0(\phi_0 - \varphi_0)$$

$$B_1 = E_0R_0^3$$

$$B_l = 0(l>1)$$

故

$$\varphi = -E_0 r\cos\theta + \varphi_0 + \frac{R_0(\phi_0-\varphi_0)}{r} + \frac{E_0R_0^3\cos\theta}{r^2}$$

(2) 定解条件变为

$$\begin{cases} \dfrac{1}{r^2}\partial_r(r^2\partial_r\varphi) + \dfrac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\varphi) = 0(r>R_0) \\ \dfrac{1}{r^2}\partial_r(r^2\partial_r\varphi) + \dfrac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\varphi) = 0(r<R_0) \\ \lim\limits_{r\to\infty}\varphi = -E_0r\cos\theta + \varphi_0 \\ \lim\limits_{r\to 0}\varphi\text{有限} \\ \lim\limits_{r\to R_0^+}\varphi = \lim\limits_{r\to R_0^-}\varphi \\ -\oint\limits_S \varepsilon_0\dfrac{\partial\varphi}{\partial r}\,\mathrm{d}S = Q \end{cases}$$

故可得满足边界条件的解为

$$\varphi = \begin{cases} \sum\limits_{l=0}^{\infty} a_l r^l P_l(\cos\theta) & (r<R_0) \\ \varphi_0 - E_0 r\cos\theta + \sum\limits_{l=0}^{\infty} b_l r^{-l-1}P_l(\cos\theta) & (r>R_0) \end{cases}$$

又因为导体处于静电平衡,故 $r<R_0$ 时,φ 为常数,故 $a_l=0(l>0)$。同时球表面等势,有

$$\varphi_0 - E_0R_0\cos\theta + \sum_{l=0}^{\infty} b_l R_0^{-l-1}P_l(\cos\theta) = a_0$$

解得

$$b_0 = R_0(a_0 - \varphi_0)$$

$$b_1 = E_0 R_0^3$$

$$b_l = 0 (l > 1)$$

故

$$\varphi = \begin{cases} a_0 & (r < R_0) \\ \varphi_0 - E_0 r \cos\theta + \dfrac{R_0(a_0 - \varphi_0)}{r} + \dfrac{E_0 R_0^3 \cos\theta}{r^2} & (r > R_0) \end{cases}$$

故

$$\begin{aligned} \frac{\partial \varphi}{\partial r}|_{r=R_0} &= (-E_0\cos\theta - \frac{R_0(a_0-\varphi_0)}{r^2} - \frac{2E_0R_0^3}{r^3}\cos\theta)|_{r=R_0} \\ &= -E_0\cos\theta - \frac{a_0-\varphi_0}{R_0} - 2E_0\cos\theta \end{aligned}$$

$$\begin{aligned} -\oint_S \varepsilon_0 \frac{\partial\varphi}{\partial r}\,\mathrm{d}S &= -\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\varepsilon_0(-E_0\cos\theta - \frac{a_0-\varphi_0}{R_0} - 2E_0\cos\theta)R_0^2\sin\theta\,\mathrm{d}\theta \\ &= 4\pi\varepsilon_0 R_0(a_0-\varphi_0) \end{aligned}$$

$$4\pi\varepsilon_0 R_0(a_0-\varphi_0) = Q$$

$$a_0 = \frac{Q}{4\pi\varepsilon_0 R_0} + \varphi_0$$

故

$$\varphi = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 R_0} + \varphi_0 & (r < R_0) \\ \varphi_0 - E_0 r\cos\theta + \dfrac{Q}{4\pi\varepsilon_0 r} + \dfrac{E_0R_0^3\cos\theta}{r^2} & (r > R_0) \end{cases}$$

2

解 由对称性分析知,电势只依赖于 r。即

$$\varphi = \varphi(r)$$

又由叠加原理知,该电势可由中心电荷与球壳电荷叠加得到,即

$$\begin{aligned} \varphi &= \varphi_q + \varphi_s \\ &= \frac{Q_f}{4\pi\varepsilon r} + \varphi_s \end{aligned}$$

又易知 φ_s 的通解为

$$\varphi_s = \begin{cases} a + \dfrac{b}{r} & (r < R) \\ c + \dfrac{d}{r} & (r > R) \end{cases}$$

无穷远处电势为 0 有

$$\lim_{r\to\infty} \frac{Q_f}{4\pi\varepsilon r} + c + \frac{d}{r} = 0$$
$$c = 0$$

又球壳电荷均匀分布,故 $b = 0$。即

$$\varphi_s = \begin{cases} a & (r < R) \\ \dfrac{d}{r} & (r > R) \end{cases}$$

又在 R 处

$$a + \frac{Q_f}{4\pi\varepsilon R} = \frac{Q_f}{4\pi\varepsilon R} + \frac{d}{R}$$
$$a = \frac{d}{R}$$

$$D_1 = D_2$$
$$\frac{-\varepsilon Q_f}{4\pi\varepsilon R^2} = \frac{-\varepsilon_0 Q_f}{4\pi\varepsilon R^2} - \frac{\varepsilon_0 d}{R^2}$$
$$d = \frac{Q_f}{4\pi}(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon})$$

故

$$\varphi_s = \begin{cases} \dfrac{Q_f}{4\pi R}(\dfrac{1}{\varepsilon_0} - \dfrac{1}{\varepsilon}) & (r < R) \\ \dfrac{Q_f}{4\pi r}(\dfrac{1}{\varepsilon_0} - \dfrac{1}{\varepsilon}) & (r > R) \end{cases}$$

即

$$\varphi = \begin{cases} \dfrac{Q_f}{4\pi\varepsilon r} + \dfrac{Q_f}{4\pi R}(\dfrac{1}{\varepsilon_0} - \dfrac{1}{\varepsilon}) & (r < R) \\ \dfrac{Q_f}{4\pi r}\dfrac{1}{\varepsilon_0} & (r > R) \end{cases}$$

这与使用高斯定理的结果是一致的。

3

解　该电势可由电偶极子电势与球面极化电荷电势叠加得到,即

$$\begin{aligned}\varphi &= \varphi_p + \varphi_s \\ &= \frac{\vec{p}_f \cdot \vec{r}}{4\pi\varepsilon_1 r^3} + \varphi_s\end{aligned}$$

以 $\vec{p}$ 方向为极轴选取球坐标系,则 φ_s 满足拉普拉斯方程,且由对称性知,电势与 ϕ 无关。又由物理边界条件 (电势有限) 知

$$\varphi_s = \begin{cases} \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) & (r < R) \\ \sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta) & (r > R) \end{cases}$$

又

$$\varphi_{in}(R,\theta) = \varphi_{out}(R,\theta)$$

$$\varepsilon_1 \frac{\partial \varphi_{in}}{\partial r} = \varepsilon_1 \frac{\partial \varphi_{out}}{\partial r}$$

对比 $P_l(\cos\theta)$ 的系数得到

$$a_l = 0 (l \neq 1)$$

$$b_l = 0 (l \neq 1)$$

$$a_1 = \frac{(\varepsilon_1 - \varepsilon_2)p}{2\pi\varepsilon_1(\varepsilon_1 + \varepsilon_2)R^3}$$

$$b_1 = R^3 a_1$$

故

$$\varphi = \begin{cases} \dfrac{\vec{p}_f \cdot \vec{r}}{4\pi\varepsilon_1 r^3} + \dfrac{(\varepsilon_1 - \varepsilon_2)p_f r\cos\theta}{2\pi\varepsilon_1(\varepsilon_1 + \varepsilon_2)R^3} & (r < R) \\ \dfrac{\vec{p}_f \cdot \vec{r}}{4\pi\varepsilon_1 r^3} + \dfrac{(\varepsilon_1 - \varepsilon_2)p_f \cos\theta}{2\pi\varepsilon_1(\varepsilon_1 + \varepsilon_2)r^2} = \dfrac{3p\cos\theta}{4\pi(\varepsilon_1 + 2\varepsilon_2)r^2} & (r > R) \end{cases}$$

介质内部没有电荷的地方便没有极化电荷,故球心有极化偶极子

$$\vec{p}_p = (\frac{\varepsilon}{\varepsilon_0} - 1)\vec{p}_f$$

球面极化电荷密度为

$$\begin{aligned}\sigma_p &= -(\varepsilon_1 - \varepsilon_0)\frac{\partial \varphi_{in}}{\partial r}|_{r=R} + (\varepsilon_2 - \varepsilon_0)\frac{\partial \varphi_{out}}{\partial r}|_{r=R} \\ &= \frac{3(\varepsilon_1 - \varepsilon_2)\varepsilon_0 p_f \cos\theta}{2\pi\varepsilon_1(\varepsilon_1 + 2\varepsilon_2)R^3}\end{aligned}$$

4

解 设

$$\varphi = \begin{cases} \varphi_1 & (r < R_1) \\ \varphi_2 & (R_1 < r < R_2) \\ \varphi_3 & (r > R_2) \end{cases}$$

易知

$$\varphi_2 = \frac{Q}{4\pi\varepsilon_0 R_2}$$
$$\varphi_3 = \frac{Q}{4\pi\varepsilon_0 r}$$

设

$$\varphi_1 = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} + \varphi'$$

则 φ' 满足拉普拉斯方程，取 $\vec{p}$ 的方向为极轴，可且对称性知，电势与 ϕ 无关。又在 $r \to 0$ 时，电势应取为偶极子电势，故

$$\varphi' = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) + \frac{b_0}{r}$$

又

$$\varphi_1|_{r=R_1} = \varphi_2|_{r=R_1}$$
$$\frac{p\cos\theta}{4\pi\varepsilon_0 R_1^2} + \sum_{l=0}^{\infty} a_l R_1^l P_l(\cos\theta) + \frac{b_0}{r} = \frac{Q}{4\pi\varepsilon_0 R_2}$$

对比系数得到

$$a_l = 0(l > 1)$$
$$a_0 + \frac{b_0}{R_1} = \frac{Q}{4\pi\varepsilon_0 R_2}$$
$$a_1 = \frac{-p}{4\pi\varepsilon_0 R_1^3}$$

则

$$\varphi_1 = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + a_0 - \frac{pr\cos\theta}{4\pi\varepsilon_0 R_1^3} + \frac{b_0}{r}$$

又由高斯定理知球壳内表面电荷量为 0 即

$$\oint_S \frac{\partial \varphi_1}{\partial r}\,\mathrm{d}S = \oint_S \frac{-p\cos\theta}{2\pi\varepsilon_0 r^3} - \frac{p\cos\theta}{4\pi\varepsilon_0 R_1^3} + \frac{b_0}{r^2}\,\mathrm{d}S$$
$$= 0$$

解得 $b_0 = 0$,故 $a_0 = \dfrac{Q}{4\pi\varepsilon_0 R_2}$,故

$$\varphi_1 = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \frac{Q}{4\pi\varepsilon_0 R_2} - \frac{pr\cos\theta}{4\pi\varepsilon_0 R_1^3}$$

即

$$\varphi = \begin{cases} \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2} + \dfrac{Q}{4\pi\varepsilon_0 R_2} - \dfrac{pr\cos\theta}{4\pi\varepsilon_0 R_1^3} & (r < R_1) \\ \dfrac{Q}{4\pi\varepsilon_0 R_2} & (R_1 < r < R_2) \\ \dfrac{Q}{4\pi\varepsilon_0 r} & (r > R_2) \end{cases}$$

故在 $r = R_1$ 处

$$\sigma = \varepsilon_0 \frac{\partial \varphi_1}{\partial r}|_{r=R_1}$$
$$= -\frac{3p\cos\theta}{4\pi R_1^3}$$

在 $r = R_1$ 处

$$\sigma = -\varepsilon_0 \frac{\partial \varphi_3}{\partial r}|_{r=R_2}$$
$$= \frac{Q}{4\pi R_2^2}$$

5

解 定解条件为

$$\begin{cases} \nabla^2 \varphi_{in} = -\dfrac{\rho_f}{\varepsilon} \\ \nabla^2 \varphi_{out} = 0 \\ \varphi_{in}|_{r=R} = \varphi_{out}|_{r=R} \\ \varepsilon \dfrac{\partial \varphi_{in}}{\partial r}|_{r=R} = \varepsilon_0 \dfrac{\partial \varphi_{out}}{\partial r}|_{r=R} \\ \lim\limits_{r\to\infty} \varphi_{out} = -E_0 r\cos\theta \\ \lim\limits_{r\to 0} \varphi_{in} \text{有限} \end{cases}$$

设

$$\varphi_{in} = \varphi_1 + \varphi_1'$$

$$\varphi_{out} = \varphi_2 + \varphi_2'$$

由高斯定理可导出球对称部分的特解

$$\varphi_1 = \frac{\rho_f(R^2 - r^2)}{6\varepsilon} + \frac{\rho_f R^2}{3\varepsilon_0}$$
$$\varphi_2 = \frac{\rho_f R^3}{3\varepsilon_0 r}$$

则剩下的 φ_1', φ_2' 满足拉普拉斯方程则

$$\varphi_1' = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$
$$\varphi_2' = \sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta) + d_1 r\cos\theta$$

又

$$\lim_{r\to\infty} \varphi_{out} = -E_0 r\cos\theta$$

得到 $d_1 = -E_0$。

$$\frac{\rho_f R^2}{3\varepsilon_0} + \sum_{l=0}^{\infty} a_l R^l P_l(\cos\theta) + c_0 = \frac{\rho_f R^2}{3\varepsilon_0} + \sum_{l=0}^{\infty} b_l R^{-l-1} P_l(\cos\theta) + d_0 - E_0 R\cos\theta$$
$$\varepsilon \frac{\partial \varphi_{in}}{\partial r}|_{r=R} = \varepsilon_0 \frac{\partial \varphi_{out}}{\partial r}|_{r=R}$$

联立解得

$$a_1 = -\frac{3\varepsilon_0 E_0}{\varepsilon + 2\varepsilon_0}$$
$$b_1 = \frac{(\varepsilon - \varepsilon_0)E_0 R^3}{\varepsilon + 2\varepsilon_0}$$
$$a_l = 0(l > 1)$$
$$b_l = 0(l > 1)$$

故

$$\varphi = \begin{cases} \dfrac{\rho_f(R^2 - r^2)}{6\varepsilon} + \dfrac{\rho_f R^2}{3\varepsilon_0} - \dfrac{3\varepsilon_0 E_0 r\cos\theta}{\varepsilon + 2\varepsilon_0} & r < R \\ \dfrac{\rho_f R^3}{3\varepsilon_0 r} - E_0 r\cos\theta + \dfrac{(\varepsilon - \varepsilon_0)E_0 R^3\cos\theta}{(\varepsilon + 2\varepsilon_0)r^2} & r > R \end{cases}$$

6

解 取 $\vec{j}_{f0}$ 为轴向，球心为原点，在稳恒情况下，电势仍满足拉普拉斯方程。故可得解为

$$\varphi_1 = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

$$\varphi_2 = \sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta) - \frac{j_0 r\cos\theta}{\sigma_2}$$

选取 $r=0$ 处为势能零点故 $a_0=0$。在球面上有

$$\varphi_1|_{r=R} = \varphi_2|_{r=R}$$

$$\sigma_2 \frac{\partial\varphi_2}{\partial r}|_{r=R} = \sigma_1 \frac{\partial\varphi_1}{\partial r}|_{r=R}$$

解得

$$a_l = 0(l>1)$$

$$b_l = 0(l>1)$$

$$a_1 = \frac{3j_0}{\sigma_1+2\sigma_2}$$

$$b_1 = \frac{(\sigma_1-\sigma_2)j_0R^3}{(\sigma_1+2\sigma_2)\sigma_2}$$

故

$$\varphi_1 = -\frac{3j_0 r\cos\theta}{\sigma_1+2\sigma_2}$$

$$\varphi_2 = -\frac{j_0 r\cos\theta}{\sigma_2} + \frac{(\sigma_1-\sigma_2)j_0R^3\cos\theta}{(\sigma_1+2\sigma_2)\sigma_2 r^2}$$

$$\begin{aligned}\vec{j}_1 &= \sigma_1\vec{E}_1 \\ &= -\sigma_1\nabla\varphi_1 \\ &= \frac{3\sigma_1}{\sigma_1+2\sigma_2}\vec{j}_0\end{aligned}$$

同理

$$\vec{j}_2 = \vec{j}_0 + \frac{(\sigma_1-\sigma_2)R^3}{\sigma_1+2\sigma_2}\left(\frac{3(\vec{j}_0\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{j}_0}{r^3}\right)$$

故

$$\vec{j} = \begin{cases} \dfrac{3\sigma_1}{\sigma_1+2\sigma_2}\vec{j}_0 & r<R \\ \vec{j}_0 + \dfrac{(\sigma_1-\sigma_2)R^3}{\sigma_1+2\sigma_2}\left(\dfrac{3(\vec{j}_0\cdot\vec{r})\vec{r}}{r^5} - \dfrac{\vec{j}_0}{r^3}\right) & r>R \end{cases}$$

故交界面电荷密度为

$$
\begin{aligned}
\sigma &= \vec{e}_r \cdot (\vec{D}_2 - \vec{D}_1) \\
&= \vec{e}_r \cdot (\varepsilon_0 \vec{E}_2 - \varepsilon_0 \vec{E}_1) \\
&= \varepsilon_0 \vec{e}_r \cdot (\frac{\vec{j}_1}{\sigma_1} - \frac{\vec{j}_2}{\sigma_2}) \\
&= \frac{3(\sigma_1 - \sigma_2)\varepsilon_0 j_0 \cos\theta}{(\sigma_1 + 2\sigma_2)\sigma_2}
\end{aligned}
$$

当 $\sigma_1 \gg \sigma_2$ 时

$$
\vec{j} = \begin{cases} 3\vec{j}_0 & r < R \\ \vec{j}_0 + \frac{R^3}{r^3}(\frac{3(\vec{j}_0 \cdot \vec{r})\vec{r}}{r^2} - \vec{j}_0) & r > R \end{cases}
$$

$$
\sigma = \frac{3\varepsilon_0 j_0 \cos\theta}{\sigma_2}
$$

当 $\sigma_1 \ll \sigma_2$ 时

$$
\vec{j} = \begin{cases} 0 & r < R \\ \vec{j}_0 - \frac{R^3}{2r^3}(\frac{3(\vec{j}_0 \cdot \vec{r})\vec{r}}{r^2} - \vec{j}_0) & r > R \end{cases}
$$

$$
\sigma = \frac{-3\varepsilon_0 j_0 \cos\theta}{2\sigma_2}
$$

7

解 选取圆环轴向为 z 轴所处平面为 $x-y$ 平面,则空间中电势为

$$
\begin{aligned}
\varphi(x,y,z) &= \int_0^{2\pi} \frac{\lambda R\,\mathrm{d}\theta}{4\pi\varepsilon_0\sqrt{(x-R\cos\theta)^2+(y-R\sin\theta)^2+z^2}} \\
&= \frac{\lambda R}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{(x-R\cos\theta)^2+(y-R\sin\theta)^2+z^2}}
\end{aligned}
$$

其中积分 $\int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{(x-R\cos\theta)^2+(y-R\sin\theta)^2+z^2}}$ 是一椭圆积分,无解析解。