1

解　定解条件为

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u|_{t=0} = 0 \\ \dfrac{\partial u}{\partial t}|_{t=0} = \dfrac{I}{\rho}\delta(x - x_0) \end{cases}$$

则

$$\begin{aligned} u(x,t) &= \frac{1}{2a}\int_{x-at}^{x+at} \frac{I}{\rho}\delta(x - x_0)\,\mathrm{d}x \\ &= \frac{I}{2a\rho}\int_{x-at}^{x+at} \delta(x - x_0)\,\mathrm{d}x \\ &= \frac{I}{2a\rho}\left[H(x - x_0 + at) - H(x - x_0 - at)\right] \end{aligned}$$

其中 $H(x)$ 为阶跃函数。

2

解　设

$$u(x,y,t) = X(x)Y(y)T(t)$$

由

$$\frac{\partial^2 u}{\partial t^2} - a^2\nabla^2 u = 0$$

知

$$\frac{T''}{T} = a^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right)$$

设

$$\begin{aligned} \frac{X''}{X} &= \lambda_1 \\ \frac{Y''}{Y} &= \lambda_2 \end{aligned}$$

则关于 X 的定解条件为

$$\begin{cases} X''(x) + \lambda_1 X = 0 \\ X(0) = 0 \\ X(l) = 0 \end{cases}$$

$\lambda_1 = 0$ 时通解为 $X = ax + b$,代入边界条件后得 $a = 0, b = 0$,但 X 不能恒为 0,故舍去。
$\lambda_1 \neq 0$ 时通解为

$$X = A\sin\sqrt{\lambda_1}x + B\cos\sqrt{\lambda_1}x$$

代入边界条件后得

$$\lambda_1 = \left(\frac{n\pi}{l}\right)^2$$

故本征值与本征函数为

$$\lambda_1 = \left(\frac{n\pi}{l}\right)^2$$
$$\sin\left(\frac{n\pi}{l}x\right)$$

同理关于 Y 的本征值与本征函数为

$$\lambda_2 = \left(\frac{m\pi}{l}\right)^2$$
$$\sin\left(\frac{m\pi}{l}y\right)$$

关于 T 的方程为

$$T'' + a^2(\lambda_1 + \lambda_2)T = 0$$

通解为

$$T = A\cos\sqrt{a^2(\lambda_1+\lambda_2)}t + B\sin\sqrt{a^2(\lambda_1+\lambda_2)}t$$

代入边界条件 $T'(0) = 0$ 后得 $B = 0$，故

$$u_{mn} = A_{mn}\cos(\frac{a\pi\sqrt{m^2+n^2}}{l}t)\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}y\right)$$

故

$$u = \sum_{m,n} A_{mn}\cos\left(\frac{a\pi\sqrt{m^2+n^2}}{l}t\right)\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}y\right)$$

代入初始条件 $u|_{t=0} = Axy(l-x)(l-y)$ 得

$$\sum_{m,n} A_{mn}\sin\left(\frac{n\pi}{l}x\right)\sin\sin\left(\frac{m\pi}{l}y\right) = Axy(l-x)(l-y)$$

故

$$\begin{aligned} A_{mn} &= \frac{A}{l^2}\int_{-l}^{l} x(l-x)\sin\left(\frac{n\pi}{l}x\right)\mathrm{d}x\int_{-l}^{l} y(l-y)\sin\left(\frac{m\pi}{l}y\right)\mathrm{d}y \\ &= \frac{4Al^4(-1)^{m+n}}{mn\pi^2} \end{aligned}$$

故

$$u = \sum_{m,n}\frac{4Al^4(-1)^{m+n}}{mn\pi^2}\cos\left(\frac{a\pi\sqrt{m^2+n^2}}{l}t\right)\sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}y\right)$$

3

解 设

$$u(x,y)=X(x)Y(y)$$

由

$$\nabla^2 u=0$$

知

$$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda$$

故 X 通解为

$$X=a\sin\sqrt{\lambda}x+b\cos\sqrt{\lambda}x$$

代入边界条件 $X(0)=0,X(a)=0$ 得 X 的本征值与本征函数为

$$\lambda=\left(\frac{n\pi}{a}\right)^2$$
$$\sin\left(\frac{n\pi}{a}x\right)$$

Y 通解为

$$Y=A\sinh\sqrt{\lambda}y+B\cosh\sqrt{\lambda}y$$

代入边界条件 $Y(0)=0$ 得

$$Y=A\cosh\left(\frac{n\pi}{a}y\right)$$

故

$$u_n=A_n\cosh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)$$

$$u=\sum_{n=1}^{\infty}A_n\cosh\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n\pi}{a}x\right)$$

代入 $u|_{y=b}=T$ 得

$$\sum_{n=1}^{\infty}A_n\cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi}{a}x\right)=T$$

故

$$\begin{aligned}A_n&=\frac{\frac{2}{a}\int_0^a T\sin\left(\frac{n\pi}{a}x\right)\mathrm{d}x}{\cosh\left(\frac{n\pi b}{a}\right)}\\&=\frac{2T(1-\cos n\pi)}{n\pi\cosh\left(\frac{n\pi b}{a}\right)}\\&=\begin{cases}\dfrac{4T}{n\pi\cosh\left(\frac{n\pi b}{a}\right)} & n\text{为奇数}\\ 0 & n\text{为偶数}\end{cases}\end{aligned}$$

设 $n = 2m-1$，故

$$u = \sum_{m=1}^{\infty} \frac{4T}{(2m-1)\pi \cosh\left(\frac{(2m-1)\pi b}{a}\right)} \cosh\left(\frac{(2m-1)\pi}{a} y\right) \sin\left(\frac{(2m-1)\pi}{a} x\right)$$

4

解　设

$$u(x,t) = X(x)T(t)$$

由

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

知

$$\frac{T'}{T} = a^2 \frac{X''}{X} = -\lambda$$

故 X 通解为

$$X = a \sin\sqrt{\lambda} x + b \cos\sqrt{\lambda} x$$

代入边界条件 $X(0) = 0, a^2 X''(l) = 0$ 得 X 的本征值与本征函数为

$$\lambda = \left(\frac{an\pi}{l}\right)^2$$

$$\sin\left(\frac{n\pi}{l} x\right)$$

T 通解为

$$T = Ae^{-\lambda t}$$

故

$$u = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{an\pi}{l}\right)^2 t} \sin\left(\frac{n\pi}{l} x\right)$$

代入边界条件 $u|_{t=0} = x$ 得

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l} x\right)$$

故

$$\begin{aligned} A_n &= \frac{1}{l} \int_{-l}^{l} x \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x \\ &= \frac{2l(-1)^n}{n\pi} \end{aligned}$$

故

$$u = \sum_{n=1}^{\infty} \frac{2l(-1)^n}{n\pi} e^{-\left(\frac{an\pi}{l}\right)^2 t} \sin\left(\frac{n\pi}{l} x\right)$$