1

解　以自转轴方向为轴向建立球坐标得定解条件

$$\nabla^2\varphi_1 = 0$$

$$\nabla^2\varphi_2 = 0$$

$$\frac{1}{R_0}\left(\frac{\partial\varphi_2}{\partial\theta} - \frac{\partial\varphi_1}{\partial\theta}\right) = -\frac{Q\omega\sin\theta}{4\pi R_0}|_{r=R_0}$$

$$\frac{\partial\varphi_1}{\partial r} = \frac{\partial\varphi_2}{\partial r}$$

$$\varphi_1|_{r\to 0}\text{有限}$$

$$\varphi_2|_{r\to\infty} = 0$$

通解为

$$\varphi_1 = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

$$\varphi_2 = \sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta)$$

代入边界条件得

$$\frac{1}{R_0}(\sum_{l=0}^{\infty} a_l R_0^l \frac{l\cos\theta P_l(\cos\theta) - lP_{l-1}(\cos\theta)}{\sin\theta} - \sum_{l=0}^{\infty} b_l R_0^{-l-1} \frac{l\cos\theta P_l(\cos\theta) - lP_{l-1}(\cos\theta)}{\sin\theta}) = -\frac{Q\omega\sin\theta}{4\pi R_0}$$

$$\sum_{l=0}^{\infty} l a_l R_0^{l-1} P_l(\cos\theta) = \sum_{l=0}^{\infty} (l+1) b_l R_0^{-l-2} P$$

解得

$$a_1 = \frac{-Q\omega}{6\pi R_0}$$

$$b_1 = \frac{Q\omega R_0^2}{12\pi}$$

$$a_l = b_l = 0 (l \neq 1)$$

$$\varphi_1 = \frac{-Q\omega}{6\pi R_0} r\cos\theta$$

$$\varphi_2 = \frac{Q\omega R_0^2}{12\pi r^2}\cos\theta$$

故

$$\vec{B_1} = -\mu_0\nabla\varphi_1$$

$$
\begin{aligned}
&= \frac{Q\vec{\omega}}{6\pi R_0} \\
\vec{B}_2 &= -\mu_0 \nabla \varphi_2 \\
&= \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right]
\end{aligned}
$$

2

解　(1)

$$
\begin{aligned}
\vec{m} &= \frac{1}{2}\iiint \vec{r}\times\vec{j}\,\mathrm{d}V \\
&= \frac{1}{2}\iiint \vec{r}\times\frac{3Q}{4\pi R_0^3}(\omega\times r)\,\mathrm{d}V \\
&= \frac{1}{2}\frac{3Q}{4\pi R_0^2}\iiint \vec{r}\times(\vec{\omega}\times\vec{r})r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{1}{2}\frac{3Q}{4\pi R_0^2}\iiint (\vec{e}_r\times\vec{e}_\phi)r^4\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= -\frac{1}{2}\frac{3Q}{4\pi R_0^2\omega}\iiint \vec{e}_\theta r^4\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{3Q\omega}{8\pi R_0^2}\iiint \left[\sin\theta\vec{e}_z+\cos\theta(-\cos\phi\vec{e}_x-\sin\phi\vec{e}_y)\right]r^4\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{3Q\omega}{8\pi R_0^2}\vec{e}_z\int_0^{2\pi}\int_0^{\pi}\int_0^{R_0} r^4\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{QR_0^2\vec{\omega}}{5}
\end{aligned}
$$

(2)

$$
\begin{aligned}
\vec{L} &= I\vec{\omega} \\
&= \frac{2m_0R_0^2}{5}\vec{\omega} \\
\frac{m}{L} &= \frac{\frac{QR_0^2\omega}{5}}{\frac{2m_0R_0^2}{5}\omega} \\
&= \frac{Q}{2m_0}
\end{aligned}
$$

2

解　该问题近似于电偶极子在无穷大导体平面边界的问题，故可使用电像法。令介质平面为 $z=0$ 平面。$\vec{m}$ 距其 d，与 z 轴夹角 θ 得到镜像磁矩 $\vec{m}'$ 产生的磁标势为

$$
\varphi' = \frac{\vec{m}'\cdot\vec{r}}{4\pi r^3}
$$

故

$$\begin{aligned}\vec{B}' &= \mu_0(-\nabla\varphi') \\ &= \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}'\cdot\vec{r})\vec{r}}{r^5}-\frac{\vec{m}'}{r^3}\right]\end{aligned}$$

则在 $\vec{m}$ 处产生的磁感应强度为

$$\vec{B}' = \frac{\mu_0}{4\pi z^3}(3m'\cos\theta\vec{e}_z-\vec{m}')$$

故

$$\begin{aligned}\vec{F} &= -\nabla(-\vec{m}\cdot\vec{B}') \\ &= \nabla[\frac{\mu_0 m^2}{4\pi z^3}(1+\cos^2\alpha)] \\ &= \vec{e}_z\frac{\partial\frac{\mu_0 m^2}{4\pi z^3}(1+\cos^2\theta)}{\partial z} \\ &= \frac{-3\mu_0 m^2}{64\pi d^4}(1+\cos^2\theta)\vec{e}_z\end{aligned}$$