1

解

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\vec{x}'\,\mathrm{d}V' \\ &= \int_V \frac{\mathrm{d}\rho\vec{x}'}{\mathrm{d}t}\,\mathrm{d}V' \\ &= \int_V (\frac{\partial\rho\vec{x}'}{\partial t} + \frac{\partial\rho\vec{x}'}{\partial\vec{x}'}\frac{\mathrm{d}\vec{x}'}{\mathrm{d}t})\,\mathrm{d}V' \\ &= \int_V \frac{\partial\rho}{\partial t}\vec{x}'\,\mathrm{d}V' \\ &= -\int_V \nabla'\cdot\vec{j}\vec{x}'\,\mathrm{d}V'\end{aligned}$$

又因为

$$\begin{aligned}\nabla'\cdot(\vec{j}\vec{x}') &= \partial_i j_i x'_j \\ &= x'_j\partial_i j_i + j_i\partial_i x'_j \\ &= x'_j\partial_i j_i + j_i\delta_{ij} \\ &= x'_j\partial_i j_i + j_j \\ &= (\nabla'\cdot\vec{j})\vec{x}' + \vec{j}\end{aligned}$$

故

$$\begin{aligned}-\int_V \nabla'\cdot\vec{j}\vec{x}'\,\mathrm{d}V' &= \int_V \vec{j}\,\mathrm{d}V' - \int_V \nabla'\cdot(\vec{j}\vec{x}')\,\mathrm{d}V' \\ &= \int_V \vec{j}\,\mathrm{d}V' - \int_{\partial V}\mathrm{d}\vec{S}\cdot\vec{j}\vec{x}'\end{aligned}$$

又体系电荷守恒,故在 ∂V 上 $\vec{j}\cdot\mathrm{d}\vec{S}=0$,故

$$\nabla'\cdot(\vec{x}'\vec{j}) = \int_V \vec{j}\,\mathrm{d}V'$$

2

解

$$\begin{aligned}\nabla\times\vec{A} &= \nabla(\frac{1}{r^3})\times(\vec{m}\times\vec{r}) + \frac{1}{r^3}\nabla\times(\vec{m}\times\vec{r}) \\ &= \frac{-3\vec{r}}{r^5}\times(\vec{m}\times\vec{r}) + \frac{1}{r^3}[(\vec{r}\cdot\nabla)\vec{m} + (\nabla\cdot\vec{r})\vec{m} - (\vec{m}\cdot\nabla)\vec{r} - (\nabla\cdot\vec{m})\vec{r}] \\ &= \frac{-3[r^2\vec{m} - (\vec{r}\cdot\vec{m})\vec{r}]}{r^5} + \frac{2\vec{m}}{r^3} \\ &= \frac{-3\vec{m}}{r^3} + \frac{3(\vec{r}\cdot\vec{m})\vec{r}}{r^5} + \frac{2\vec{m}}{r^3}\end{aligned}$$

$$= \frac{3(\vec{r}\cdot\vec{m})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}$$

$$\begin{aligned}
\nabla\varphi &= \frac{1}{r^3}\nabla(\vec{m}\cdot\vec{r}) + (\vec{m}\cdot\vec{r})\nabla\frac{1}{r^3} \\
&= \frac{1}{r^3}[\vec{m}\times(\nabla\times\vec{r}) + (\vec{m}\cdot\nabla)\vec{r} + \vec{r}\times(\nabla\times\vec{m}) + (\vec{r}\cdot\nabla)\vec{m}] - \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} \\
&= \frac{\vec{m}}{r^3} - \frac{3(\vec{r}\cdot\vec{m})\vec{r}}{r^5}
\end{aligned}$$

故

$$\nabla\times\vec{A} = -\nabla\varphi$$

4

解

$$\begin{aligned}
\vec{F}_{12} &= \frac{\mu_0}{4\pi}\oint\limits_{(L_1)}\oint\limits_{(L_2)}\frac{I_1I_2\,\mathrm{d}\vec{l}_1\times(\mathrm{d}\vec{l}_2\times\hat{\mathbf{r}}_{12})}{r_{12}^2} \\
&= \frac{\mu_0}{4\pi}\oint\limits_{(L_1)}\oint\limits_{(L_2)}\frac{I_1I_2[(\mathrm{d}\vec{l}_1\cdot\hat{\mathbf{r}}_{12})\,\mathrm{d}\vec{l}_2 - (\mathrm{d}\vec{l}_1\cdot\mathrm{d}\vec{l}_2)\hat{\mathbf{r}}_{12}]}{r_{12}^2} \\
&= -\frac{\mu_0}{4\pi}\oint\limits_{(L_1)}\oint\limits_{(L_2)}\frac{I_1I_2(\mathrm{d}\vec{l}_1\cdot\mathrm{d}\vec{l}_2)\hat{\mathbf{r}}_{12}}{r_{12}^2}
\end{aligned}$$

又因为被积函数连续,故积分可交换顺序,即

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi}\oint\limits_{(L_2)}\oint\limits_{(L_1)}\frac{I_1I_2(\mathrm{d}\vec{l}_1\cdot\mathrm{d}\vec{l}_2)\hat{\mathbf{r}}_{12}}{r_{12}^2}$$

同理

$$\vec{F}_{21} = -\frac{\mu_0}{4\pi}\oint\limits_{(L_2)}\oint\limits_{(L_1)}\frac{I_1I_2(\mathrm{d}\vec{l}_1\cdot\mathrm{d}\vec{l}_2)\hat{\mathbf{r}}_{21}}{r_{21}^2}$$

又因为 $\hat{\mathbf{r}}_{21} = -\hat{\mathbf{r}}_{12}$, 故

$$\vec{F}_{12} = -\vec{F}_{21}$$

3

解　设抛物线方程为 $y = ax^2(z=0)$,则其焦点为 $(0,\dfrac{1}{4a},0)$。在其上一点 $(x,ax^2,0)$ 的电流元为 $I\,\mathrm{d}\vec{l} = (I\,\mathrm{d}x, I2ax\,\mathrm{d}x, 0)$。故其焦点处的磁感应强度为

$$\vec{B} = \int\frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{l}\times\vec{r}}{r^3}$$

$$
\begin{aligned}
&= -\int_{-\infty}^{\infty} \frac{\mu_0 I}{4\pi} \frac{ax^2 + \frac{1}{4a}}{\sqrt{x^2 + (ax^2 - \frac{1}{4a})^2}^3} \mathrm{d}x \vec{e}_z \\
&= -\frac{\mu_0 I}{4\pi} 4a\pi \vec{e}_z \\
&= -a\mu_0 I \vec{e}_z
\end{aligned}
$$

4

解　因为所有场量均只与 z, t 相关，故麦克斯韦方程组可简化为

$$
\begin{aligned}
\frac{\partial E_z}{\partial z} &= 0 \\
\frac{\partial E_y}{\partial z}\vec{e}_x - \frac{\partial E_x}{\partial z}\vec{e}_y &= \frac{\partial \vec{B}}{\partial t} \\
-\frac{\partial B_y}{\partial z}\vec{e}_x + \frac{\partial B_x}{\partial z}\vec{e}_y &= \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\
\frac{\partial B_z}{\partial z} &= 0
\end{aligned}
$$

因为 $\frac{\partial E_z}{\partial z} = 0, \frac{\partial B_z}{\partial z} = 0$ 故 E_z 为常数，同理 B_z 为常数，不妨将 E_z, B_z 均取为 0。故可得到两组独立方程

$$
\begin{cases}
\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t} \\
\frac{\partial B_x}{\partial z} = \mu_0\varepsilon_0 \frac{\partial E_y}{\partial t}
\end{cases}
$$

$$
\begin{cases}
-\frac{\partial E_x}{\partial z} = \frac{\partial B_y}{\partial t} \\
-\frac{\partial B_y}{\partial z} = \mu_0\varepsilon_0 \frac{\partial E_x}{\partial t}
\end{cases}
$$

消去第一组方程中的 B_x 可得到

$$
\mu_0\varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} = \frac{\partial^2 E_y}{\partial z^2}
$$

此是一波动方程，可解出 E_y，再代入原式，可解出 B_x。此是一组解。同理第二组方程可解出第二组解 (E_x, B_y)。故整个方程组的解由这两组独立的解组成。

4

解

$$
\begin{aligned}
\nabla \cdot \vec{E}' &= \cos\theta \nabla \cdot \vec{E} + c\sin\theta \nabla \cdot \vec{B} \\
&= 0 \\
\nabla \times \vec{E}' &= \cos\theta \nabla \times \vec{E} + c\sin\theta \nabla \times \vec{B} \\
&= -\cos\theta \frac{\partial \vec{B}}{\partial t} + c\mu_0\varepsilon_0 \sin\theta \frac{\partial \vec{E}}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\partial}{\partial t}(\cos\theta\vec{B} - c\mu_0\varepsilon_0\sin\theta\vec{E}) \\
&= -\frac{\partial}{\partial t}(\cos\theta\vec{B} - \sqrt{\mu_0\varepsilon_0}\sin\theta\vec{E}) \\
&= -\frac{\partial}{\partial t}(\cos\theta\vec{B} - \frac{1}{c}\sin\theta\vec{E}) \\
&= -\frac{\partial\vec{B}'}{\partial t}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\vec{B}' &= \frac{-\sin\theta}{c}\nabla\cdot\vec{E} + \cos\theta\nabla\cdot\vec{B} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\vec{B}' &= \frac{-\sin\theta}{c}\nabla\times\vec{E} + \cos\theta\nabla\times\vec{B} \\
&= \frac{\sin\theta}{c}\frac{\partial\vec{B}}{\partial t} + \cos\theta\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \\
&= \mu_0\varepsilon_0\frac{\partial}{\partial t}(\frac{\sin\theta}{\mu_0\varepsilon_0 c}\vec{B} + \cos\theta\vec{E}) \\
&= \mu_0\varepsilon_0\frac{\partial}{\partial t}(\frac{\sin\theta}{\sqrt{\mu_0\varepsilon_0}}\vec{B} + \cos\theta\vec{E}) \\
&= \mu_0\varepsilon_0\frac{\partial}{\partial t}(c\sin\theta\vec{B} + \cos\theta\vec{E}) \\
&= \mu_0\varepsilon_0\frac{\partial\vec{E}'}{\partial t}
\end{aligned}
$$

4

解

$$
\begin{aligned}
\vec{B} &= \nabla\times\vec{A} \\
&= (\nabla\frac{1}{r(r-\vec{r}\cdot\vec{n})})\times(\vec{r}\times\vec{n}) + \frac{1}{r(r-\vec{r}\cdot\vec{n})}\nabla\times(\vec{r}\times\vec{n}) \\
&= (\nabla\frac{1}{r^2}\frac{1}{1-\cos\theta})\times(\vec{r}\times\vec{n}) - \frac{2\vec{n}}{r(r-\vec{r}\cdot\vec{n})}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\frac{1}{r^2}\frac{1}{1-\cos\theta} &= \frac{1}{1-\cos\theta}\nabla\frac{1}{r^2} + \frac{1}{r^2}\nabla\frac{1}{1-\cos\theta} \\
&= \frac{1}{1-\cos\theta}\frac{-2\vec{r}}{r^4} + \frac{1}{r^2}\frac{1}{(1-\cos\theta)^2}\nabla\cos\theta \\
&= \frac{-2\vec{r}}{r^3(r-\vec{r}\cdot\vec{n})} + \frac{1}{(r-\vec{r}\cdot\vec{n})^2}\nabla\frac{\vec{r}\cdot\vec{n}}{r}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\frac{\vec{r}\cdot\vec{n}}{r} &= \frac{1}{r}\nabla(\vec{r}\cdot\vec{n}) + (\vec{r}\cdot\vec{n})\nabla\frac{1}{r} \\
&= \frac{1}{r}[\vec{n}\times(\nabla\times\vec{r}) + (\vec{n}\cdot\nabla)\vec{r} + \vec{r}\times(\nabla\times\vec{n}) + (\vec{r}\cdot\nabla)\vec{n}] - (\vec{r}\cdot\vec{n})\frac{\vec{r}}{r^3}
\end{aligned}
$$

$$= \frac{\vec{n}}{r} - \frac{(\vec{r}\cdot\vec{n})\vec{r}}{r^3}$$

故

$$\begin{aligned}
\vec{B} &= (\nabla\frac{1}{r^2}\frac{1}{1-\cos\theta}) \times (\vec{r}\times\vec{n}) - \frac{2\vec{n}}{r(r-\vec{r}\cdot\vec{n})} \\
&= (\frac{-2\vec{r}}{r^3(r-\vec{r}\cdot\vec{n})} + \frac{1}{(r-\vec{r}\cdot\vec{n})^2}\nabla\frac{\vec{r}\cdot\vec{n}}{r}) \times (\vec{r}\times\vec{n}) - \frac{2\vec{n}}{r(r-\vec{r}\cdot\vec{n})} \\
&= (\frac{-2\vec{r}}{r^3(r-\vec{r}\cdot\vec{n})} + \frac{1}{(r-\vec{r}\cdot\vec{n})^2}(\frac{\vec{n}}{r} - \frac{(\vec{r}\cdot\vec{n})\vec{r}}{r^3})) \times (\vec{r}\times\vec{n}) - \frac{2\vec{n}}{r(r-\vec{r}\cdot\vec{n})} \\
&= \frac{(\vec{r}\cdot\vec{n}-2r)\vec{r}+r^2\vec{n}}{r^3(r-\vec{r}\cdot\vec{n})^2} \times (\vec{r}\times\vec{n}) - \frac{2\vec{n}}{r(r-\vec{r}\cdot\vec{n})} \\
&= \frac{\vec{r}\cdot\vec{n}-2r}{r^3(r-\vec{r}\cdot\vec{n})^2}[(\vec{r}\cdot\vec{n})\vec{r}-r^2\vec{n}] + \frac{r^2}{r^3(r-\vec{r}\cdot\vec{n})^2}[\vec{r}-(\vec{n}\cdot\vec{r})\vec{n}] - \frac{2\vec{n}}{r(r-\vec{r}\cdot\vec{n})} \\
&= \frac{[(\vec{r}\cdot\vec{n})^2-2(\vec{r}\cdot\vec{n})r+r^2]\vec{r}}{r^3(r-\vec{r}\cdot\vec{n})^2} + \frac{\vec{n}}{r(r-\vec{r}\cdot\vec{n})}
\end{aligned}$$