1

解

$$\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} - k^2 g = \delta(t-x)$$

当 $t<x$ 时，$g=a(x)e^{kt}+b(x)e^{-kt}$，当 $t>x$ 时，$g=c(x)e^{kt}+d(x)e^{-kt}$ 代入 $g(x,0)=A, \frac{\mathrm{d}g}{\mathrm{d}t}(x,0)=B$ 可得

$$a=\frac{kA+B}{2k}$$
$$b=\frac{kA-B}{2k}$$

又要求函数值连续导函数差 1 有

$$\begin{cases} ae^{kx}+be^{-kx}=ce^{kx}+de^{-kx} \\ ake^{kx}-bke^{-kx}=cke^{kx}-dke^{-kx}-1 \end{cases}$$

解得

$$c=\frac{2ake^{kx}+1}{2ke^{kx}}$$
$$d=\frac{2bke^{-kx}-1}{2ke^{kx}}$$

故

$$y(t)=\int_0^t f(x)(ce^{kt}+de^{-kt})\,\mathrm{d}x+\int_t^\infty f(x)(ae^{kt}+be^{-kt})\,\mathrm{d}x$$

2

解

$$\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + k^2 g = \delta(x-t)$$

当 $x<t$ 时，$g=a_1(t)\sin kx+a_2(t)\cos kx$，当 $x>t$ 时，$g=b_1(t)\sin kx+b_2(t)\cos kx$ 代入 $g(0,t)=A, g(1,t)=B$ 以及连续条件和跃度条件可得

$$\begin{cases} a_2=A \\ b_1\sin k+b_2\cos k=B \\ (a_1-b_1)\sin kt+(a_2-b_2)\cos kt=0 \\ (a_1-b_1)k\sin kt-k(a_2-b_2)\cos kt=-1 \end{cases}$$

解得

$$a_1=\frac{B-A\cos k+\frac{\cos k}{2k\cos kt}}{\sin k}-\frac{1}{2k\sin kt}$$

$$a_2 = A$$

$$b_1 = \frac{B - A\cos k + \frac{\cos k}{2k\cos kt}}{\sin k}$$

$$b_2 = A - \frac{1}{2k\cos kt}$$

故

$$y(x) = \int_0^x f(t)(b_1 \sin kx + b_2 \cos kx)\,\mathrm{d}t + \int_x^1 f(t)(a_1 \sin kx + a_2 \cos kx)\,\mathrm{d}t$$

3

解

$$\nabla^2 G = -\delta(\vec{r} - \vec{r}_0)$$

$$G|_{r=a} = 0$$

由电像法可知,该方程解为

$$G(M, M_0) = \frac{1}{4\pi}\left(\frac{1}{r_0^2 + r^2 - 2rr_0\cos\gamma} - \frac{a}{\sqrt{r^2 r_0^2 + a^4 - 2a^2 r r_0 \cos\gamma}}\right)$$

故

$$u(\vec{r}) = -\iiint\limits_{\Omega} G f(\vec{r}_0)\,\mathrm{d}V_0 - \iint\limits_{\Gamma} g(\vec{r}_0)\frac{\partial G}{\partial n_0}\,\mathrm{d}S_0$$

在球面 Γ 上 $\mathrm{d}S_0$ 的法线方向与径向相同,故

$$\frac{\partial G}{\partial n_0}|_\Gamma = -\frac{1}{4\pi a}\frac{a^2 - r^2}{(a^2 + r^2 - 2ar\cos\gamma)^{\frac{3}{2}}}$$

引入在无穷远处 $u = 0$ 的边界条件可消去另一积分,故在球坐标系中

$$\begin{aligned} u(r, \theta, \phi) = &-\int_a^\infty \int_0^\pi \int_0^{2\pi} G(M, M_0) f(r_0, \theta_0, \phi_0) r_0^2 \sin\theta_0 \,\mathrm{d}r_0\,\mathrm{d}\theta_0\,\mathrm{d}\phi_0 \\ &+ \frac{a(a^2 - r^2)}{4\pi}\int_0^{2\pi}\int_0^\pi \frac{g(a, \theta_0, \phi_0)}{(a^2 + r^2 - 2ar\cos\gamma)^{\frac{3}{2}}}\sin\theta_0\,\mathrm{d}\theta_0\,\mathrm{d}\phi_0 \end{aligned}$$

其中 $\cos\gamma = \sin\theta\sin\theta_0\cos(\phi - \phi_0) + \cos\theta\cos\theta_0$