1

解 (1)

$$\begin{aligned} I &= \int_0^1 x\,\mathrm{d}x + \int_0^1 (1-\mathrm{i}y)\mathrm{i}\,\mathrm{d}y \\ &= \frac{1}{2} + \mathrm{i} + \frac{1}{2} \\ &= 1 + \mathrm{i} \end{aligned}$$

(2)

$$\begin{aligned} I &= \int_0^1 -\mathrm{i}y(\mathrm{i}\,\mathrm{d}y) + \int_0^1 (x-\mathrm{i})\,\mathrm{d}x \\ &= \frac{1}{2} + \frac{1}{2} - \mathrm{i} \\ &= 1 - \mathrm{i} \end{aligned}$$

(3)

$$\begin{aligned} I &= \int_0^{2\pi} e^{-\mathrm{i}\theta}\,\mathrm{d}e^{\mathrm{i}\theta} \\ &= \int_0^{2\pi} e^{-\mathrm{i}\theta}\mathrm{i}e^{\mathrm{i}\theta}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \mathrm{i}\,\mathrm{d}\theta \\ &= 2\pi\mathrm{i} \end{aligned}$$

(4)

$$\begin{aligned} I &= \int_{-1}^1 (x-\mathrm{i})\,\mathrm{d}x + \int_{-1}^1 (1-\mathrm{i}y)\mathrm{i}\,\mathrm{d}y + \int_1^{-1} (x+i)\,\mathrm{d}x + \int_1^{-1} (-1-\mathrm{i}y)\mathrm{i}\,\mathrm{d}y \\ &= \int_{-1}^1 -\mathrm{i}\,\mathrm{d}x + \int_{-1}^1 \mathrm{i}\,\mathrm{d}y + \int_1^{-1} \mathrm{i}\,\mathrm{d}x + \int_1^{-1} -\mathrm{i}\,\mathrm{d}y \\ &= -2\mathrm{i} + 2\mathrm{i} + -2\mathrm{i} + 2\mathrm{i} \\ &= 0 \end{aligned}$$

2

解 (1)

$$\begin{aligned} I &= \int_0^{2\pi} R\,\mathrm{d}\theta \\ &= 2\pi R \end{aligned}$$

(2) 积分区域包含奇点 $z=0$，由高阶导数公式知

$$\begin{aligned} I &= \frac{2\pi\mathrm{i}}{2!}\frac{\mathrm{d}^2 e^{iz}}{\mathrm{d}z^2}|_{z=0} \\ &= -\pi\mathrm{i} \end{aligned}$$

(3)

$$
\begin{aligned}
I &= \oint\limits_{|z|=R} \frac{Re^z}{z^2}\,\mathrm{d}z \\
&= \frac{2\pi\mathrm{i}}{1!} R \frac{\mathrm{d}e^z}{\mathrm{d}z}|_{z=0} \\
&= 2\pi\mathrm{i}R
\end{aligned}
$$

(4)

$$
\begin{aligned}
I &= \int_0^{2\pi} (\ln R + \mathrm{i}\theta)\,\mathrm{d}Re^{\mathrm{i}\theta} \\
&= \int_0^{2\pi} (\ln R + \mathrm{i}\theta) R\mathrm{i}e^{\mathrm{i}\theta}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} R\mathrm{i}\ln Re^{\mathrm{i}\theta}\,\mathrm{d}\theta - R\int_0^{2\pi} \theta e^{\mathrm{i}\theta}\,\mathrm{d}\theta \\
&= 0 - R(-\mathrm{i}\theta e^{\mathrm{i}\theta} + e^{\mathrm{i}\theta})|_0^{2\pi} \\
&= 2\pi\mathrm{i}R
\end{aligned}
$$

3

解 在该积分路径上

$$
\begin{aligned}
z &= be^{\mathrm{i}\theta} \\
\mathrm{d}z &= b\mathrm{i}e^{\mathrm{i}\theta}\,\mathrm{d}\theta \\
|\,\mathrm{d}z| &= b\,\mathrm{d}\theta \\
&= -\mathrm{i}b\frac{\mathrm{d}z}{z}
\end{aligned}
$$

故

$$
\begin{aligned}
I &= \oint\limits_{|z|=b} -\frac{\cos z}{(z-a)^2}\mathrm{i}b\frac{\mathrm{d}z}{z} \\
&= -\mathrm{i}b \oint\limits_{|z|=b} \frac{\cos z}{z(z-a)^2}\,\mathrm{d}z
\end{aligned}
$$

当 $b < a$ 时该曲线内部只有 $z = 0$ 一个奇点故此时

$$
\begin{aligned}
I &= -\mathrm{i}b \oint\limits_{|z|=b} \frac{\cos z}{(z-a)^2}\frac{1}{z}\,\mathrm{d}z \\
&= -\mathrm{i}b2\pi\mathrm{i}\frac{\cos z}{(z-a)^2}|_{z=0} \\
&= \frac{2\pi b}{a^2}
\end{aligned}
$$

当 $b>a$ 时该曲线内部有 $z=0, z=a$ 两个奇点故此时

$$\begin{aligned}
I &= -\mathrm{i}b[\oint\limits_{C_1} \frac{\cos z}{(z-a)^2}\frac{1}{z}\,\mathrm{d}z + \oint\limits_{C_2} \frac{1}{(z-a)^2}\frac{\cos z}{z}\,\mathrm{d}z] \\
&= -\mathrm{i}b2\pi\mathrm{i}\frac{\cos z}{(z-a)^2}|_{z=0} - \mathrm{i}b(\frac{2\pi\mathrm{i}}{1!}\frac{\mathrm{d}}{\mathrm{d}z}\frac{\cos z}{z})|_{z=a} \\
&= \frac{2\pi b}{a^2} + 2\pi b\frac{-a\sin a - \cos a}{a^2} \\
&= \frac{2\pi b(1 - a\sin a - \cos a)}{a^2}
\end{aligned}$$

4

解　(1) 该曲线内只有 $z=0$ 一个奇点故此时

$$\begin{aligned}
I &= \frac{2\pi\mathrm{i}}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{1}{z^{10}-2}|_{z=0} \\
&= \pi\mathrm{i}\frac{10z^8\left(11z^{10}+18\right)}{\left(z^{10}-2\right)^3}|_{z=0} \\
&= 0
\end{aligned}$$

(2) 被积函数有 11 个奇点，由于这些奇点均在曲线 $|z|=2$ 内部，故取 $R>2$ 则有

$$I = \oint\limits_{|z|=R} \frac{\mathrm{d}z}{z^3(z^{10}-2)}$$

又因为 R 是任取的，故令 $R\to\infty$ 有

$$I = \lim_{R\to\infty}\oint\limits_{|z|=R} \frac{\mathrm{d}z}{z^3(z^{10}-2)}$$

因为 $z\to\infty$ 时

$$z\frac{1}{z^3(z^{10}-2)} = \frac{1}{z^2(z^{10}-2)}$$

该式趋于 0，故由大圆弧引理可知

$$I = 0$$