1

解　定解条件为

$$\begin{cases} \dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)+\dfrac{1}{r}\dfrac{\partial^2 u}{\partial \theta^2}=0, \\ u|_{r=1}=\begin{cases} 1 & (0<\theta<\pi), \\ 0 & (\pi<\theta<2\pi). \end{cases} \\ u(r,\theta)=u(r,\theta+2\pi) \end{cases}$$

设 $u=R(r)\Theta(\theta)$,则得到分离变量后的方程

$$\begin{cases} \Theta''+\lambda\Theta=0 \\ r^2R''+rR'-\lambda R=0 \end{cases}$$

对于 $\Theta''+\lambda\Theta=0$ 通解为

$$\Theta=a\sin\sqrt{\lambda}\theta+b\cos\sqrt{\lambda}\theta$$

本征值为

$$\lambda=n^2$$

将 $\lambda=n^2$ 代入 $r^2R''+rR'-\lambda R=0$ 得到

$$R=\begin{cases} c_0+d_0\ln r & n=0 \\ c_nr^n+d_nr^{-n} & n\neq 0 \end{cases}$$

故

$$u=C_0+D_0\ln r+\sum_{n=1}^{\infty}\left(a_n\sin n\theta+b_n\cos n\theta\right)\left(c_nr^n+d_nr^{-n}\right)$$

又因为 $r=0$ 时 u 应有界,故 $D_0=0,d_n=0$。故

$$u=C_0+\sum_{n=1}^{\infty}\left(A_n\sin n\theta+B_n\cos n\theta\right)r^n$$

代入

$$u|_{r=1}=\begin{cases} 1 & (0<\theta<\pi), \\ 0 & (\pi<\theta<2\pi). \end{cases}$$

得

$$\begin{aligned} C_0&=\frac{1}{2\pi}\int_0^{\pi}\mathrm{d}\theta \\ &=\frac{1}{2} \end{aligned}$$

$$
\begin{aligned}
A_n &= \frac{1}{\pi}\int_0^{\pi} \sin n\theta \,\mathrm{d}\theta \\
&= \begin{cases} \dfrac{2}{\pi} & n\text{为奇数} \\ 0 & n\text{为偶数} \end{cases} \\
B_n &= \frac{1}{\pi}\int_0^{\pi} \cos n\theta \,\mathrm{d}\theta \\
&= 0
\end{aligned}
$$

令 $n = 2m-1$，故

$$
u = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2}{\pi} \sin[(2m-1)\theta] r^{2m-1}
$$

2

解　定解条件为

$$
\begin{cases}
\dfrac{1}{a^2}\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) \\
u|_{r=1} = 0 \\
u|_{t=0} = 1 - r^2
\end{cases}
$$

设 $u = R(r)T(t)$，则得到分离变量后的方程

$$
\begin{cases}
T' - \lambda a^2 T = 0 \\
R' + rR'' - \lambda R = 0
\end{cases}
$$

3

证明

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}y}{\mathrm{d}x}\right] - qy + \lambda\rho y = 0 \tag{1}
$$

将 1两边取复共轭得到

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}y^*}{\mathrm{d}x}\right] - qy^* + \lambda^*\rho y^* = 0 \tag{2}
$$

$y^* \cdot$ 式1 $- y \cdot$ 式2 得

$$
\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}x}(yy^{*\prime} - y^*y') + p(yy^{*\prime\prime} - y^*y'') &= (\lambda - \lambda^*)\rho y^* y \\
\frac{\mathrm{d}}{\mathrm{d}x}[p(yy^{*\prime} - y^*y')] &= (\lambda - \lambda^*)\rho y^* y \\
\int_0^l \frac{\mathrm{d}}{\mathrm{d}x}[p(yy^{*\prime} - y^*y')]\,\mathrm{d}x &= \int_0^l (\lambda - \lambda^*)\rho |y|^2 \,\mathrm{d}x \\
[p(yy^{*\prime} - y^*y')]|_0^l &= (\lambda - \lambda^*)\int_0^l \rho |y|^2 \,\mathrm{d}x
\end{aligned}
$$

代入边界条件

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = \beta_1 y(l) + \beta_2 y'(l) = 0 \\ \alpha_1 y^*(0) + \alpha_2 {y'}^*(0) = \beta_1 y^*(l) + \beta_2 {y'}^*(l) = 0 \end{cases}$$

得到

$$(\lambda - \lambda^*) \int_0^l \rho |y|^2 \,\mathrm{d}x = [p(y{y^*}' - y^* y')]|_0^l = 0$$

又因为 $\int_0^l \rho |y|^2 \,\mathrm{d}x$ 不恒为 0，故 $\lambda - \lambda^* = 0$，即 λ 为实数。 ■

4

解 设 $y = a_0 \sum_{k=1}^{\infty} a_k x^k$，则有

$$y'' = 2a_2 + \sum_{k=1}^{\infty} (k+1)(k+2) a_{k+2} x^k$$

故

$$(k+1)(k+2) a_{k+2} + \omega^2 a_k = 0$$

可知

$$a_{2k} = (-1)^k \frac{\omega^{2k}}{(2k)!} a_0$$
$$a_{2k+1} = (-1)^k \frac{\omega^{2k}}{(2k+1)!} a_1$$

故

$$y = a_0 \cos \omega x + \frac{a_1}{\omega} \sin \omega x$$

5

解 (1) 在有限远处 p, q 均解析，令 $t = \frac{1}{x}$，则原方程可化为

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \left(\frac{2}{t} + \frac{2}{t^3}\right) \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2\lambda}{t^4} y$$

$t = 0$ 时 $t\left(\frac{2}{t} + \frac{2}{t^3}\right)$ 不解析，故无穷远点不为正则奇点。

(2) 原方程可化为

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1-2x}{2x(1-x)} \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\lambda + 2q - 4qx}{4x(1-x)} y = 0$$

故 $x=0$ 是正则奇点，$x=1$ 是正则奇点。因为 $t=0$ 时 $\frac{1}{t}\frac{1-\frac{2}{t}}{\frac{2}{t}(1-\frac{1}{t})}=\frac{t-2}{2(t-1)}$ 故无穷远点为正则奇点。