1

解　(1) 易知在 $0<r<r_1$ 处 $\vec{E}=0$

在 $r_1<r<r_2$ 处

$$\begin{aligned}4\pi r^2 D &= \frac{4\pi(r^3-r_1^3)\rho_f}{3}\\ D &= \frac{(r^3-r_1^3)\rho_f}{3r^2}\\ \varepsilon E &= \frac{(r^3-r_1^3)\rho_f}{3r^2}\\ E &= \frac{(r^3-r_1^3)\rho_f}{3\varepsilon r^2}\end{aligned}$$

故 $\vec{E}=\dfrac{(r^3-r_1^3)\rho_f\vec{r}}{3r^3}$

在 $r>r_2$ 处

$$\begin{aligned}4\pi r^2 D &= \frac{4\pi(r_2^3-r_1^3)\rho_f}{3}\\ D &= \frac{(r_2^3-r_1^3)\rho_f}{3r^2}\\ \varepsilon_0 E &= \frac{(r_2^3-r_1^3)\rho_f}{3r^2}\\ E &= \frac{(r_2^3-r_1^3)\rho_f}{3\varepsilon_0 r^2}\end{aligned}$$

故 $\vec{E}=\dfrac{(r_2^3-r_1^3)\rho_f\vec{r}}{3r^3}$

(2) 易知只在 $r_1<r<r_2$ 处有极化体电荷,有

$$\begin{aligned}\vec{P} &= \vec{D}-\varepsilon_0\vec{E}\\ &= \vec{D}-\varepsilon_0\frac{\vec{D}}{\varepsilon}\\ &= (1-\frac{\varepsilon_0}{\varepsilon})\vec{D}\end{aligned}$$

故

$$\begin{aligned}\rho_p &= \nabla\cdot\vec{P}\\ &= -(1-\frac{\varepsilon_0}{\varepsilon})\nabla\cdot\vec{D}\\ &= -(1-\frac{\varepsilon_0}{\varepsilon})\rho_f\end{aligned}$$

考虑内球壳，此时 $r=r_1$。

$$\sigma_p=-\frac{(\varepsilon-\varepsilon_0)(r^3-r_1^3)\rho_f}{3\varepsilon r^2}|_{r=r_1}=0$$

考虑外球壳，此时 $r=r_2$。

$$\sigma_p=\frac{(\varepsilon-\varepsilon_0)(r^3-r_1^3)\rho_f}{3\varepsilon r^3}|_{r=r_2}=(1-\frac{\varepsilon_0}{\varepsilon})\frac{r_2^3-r_1^3}{3r_2^3}\rho_f$$

2

解 易知在 $0<r<r_1$ 处 $\vec{B}=0$

在 $r_1<r<r_2$ 处

$$\begin{aligned}2\pi rH&=j_f\pi(r^2-r_1^2)\\H&=\frac{j_f(r^2-r_1^2)}{2r}\\\frac{B}{\mu}&=\frac{j_f(r^2-r_1^2)}{2r}\\B&=\frac{\mu j_f(r^2-r_1^2)}{2r}\end{aligned}$$

故 $\vec{B}=\dfrac{\mu(r^2-r_1^2)\vec{j}_f\times\vec{r}}{2r^2}$

在 $r>r_2$ 处

$$\begin{aligned}2\pi rH&=j_f\pi(r_2^2-r_1^2)\\H&=\frac{j_f(r_2^2-r_1^2)}{2r}\\\frac{B}{\mu_0}&=\frac{j_f(r_2^2-r_1^2)}{2r}\\B&=\frac{\mu_0 j_f(r_2^2-r_1^2)}{2r}\end{aligned}$$

故 $\vec{B}=\dfrac{\mu_0(r_2^2-r_1^2)\vec{j}_f\times\vec{r}}{2r^2}$

易知只在 $r_1<r<r_2$ 处有磁化电流，有

$$\begin{aligned}\vec{M}&=\frac{\vec{B}}{\mu_0}-\vec{H}\\&=\frac{\mu\vec{H}}{\mu_0}-\vec{H}\\&=(\frac{\mu}{\mu_0}-1)\vec{H}\end{aligned}$$

故

$$\begin{aligned}\vec{j}_m &= \nabla \times \vec{M} \\ &= (\frac{\mu}{\mu_0} - 1)\nabla \times \vec{H} \\ &= (\frac{\mu}{\mu_0} - 1)\vec{j}_f\end{aligned}$$

在内表面,此时 $r = r_1$。

$$\alpha_m = \vec{n} \times M|_{r=r_1} = 0$$

在外表面,此时 $r = r_2$。

$$\alpha_m = -\vec{n} \times M|_{r=r_2} = -(\frac{\mu}{\mu_0} - 1)\frac{r_2^2 - r_1^2}{2r^2}\vec{j}_f$$

3

解　(1)

$$E_1 l_1 + E_2 l_2 = \mathscr{E}$$

$$D_1 = D_2$$

$$D_1 = \varepsilon_1 E_1$$

$$D_2 = \varepsilon_2 E_2$$

解得

$$D = \frac{\varepsilon_1 \varepsilon_2 E}{l_1 \varepsilon_2 + l2 \varepsilon_1}$$

故

$$\omega_{f2} = -\omega_{f1} = D = \frac{\varepsilon_1 \varepsilon_2 E}{l_1 \varepsilon_2 + l2 \varepsilon_1}$$

(2)

$$\omega_{f3} = 0$$

若介质漏电,设漏电电流为 $\vec{j}_1, \vec{j}_2$ (1)

$$\vec{n} \cdot (\vec{j}_2 - \vec{j}_1) = 0$$

$$l_1 \frac{\vec{j}_1}{\sigma_1} + l_2 \frac{\vec{j}_2}{\sigma_2} = \mathscr{E}$$

解得

$$E_1 = \frac{\sigma_2 \mathscr{E}}{l_1 \sigma_2 + l_2 \sigma_1}$$

$$E_2 = \frac{\sigma_1 \mathscr{E}}{l_1\sigma_2 + l_2\sigma_1}$$

故

$$\begin{aligned}
\omega_{f1} &= D_1 = \frac{\varepsilon_1\sigma_2\mathscr{E}}{l_1\sigma_2 + l_2\sigma_1} \\
\omega_{f2} &= D_2 = \frac{\varepsilon_2\sigma_1\mathscr{E}}{l_1\sigma_2 + l_2\sigma_1} \\
\omega_{f3} &= D_2 - D_1 = \frac{\varepsilon_2\sigma_1\mathscr{E} - \varepsilon_1\sigma_2\mathscr{E}}{l_1\sigma_2 + l_2\sigma_1}
\end{aligned}$$

4

解 (1) 由 $\vec{D}$ 的法向连续条件知

$$D_1\cos\theta_1 = D_2\cos\theta_2$$

$$E_1\varepsilon_1\cos\theta_1 = E_2\varepsilon_2\cos\theta_2$$

又由 $\vec{E}$ 的切向连续条件知

$$E_1\sin\theta_1 = E_2\sin\theta_2$$

故有

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\varepsilon_2}{\varepsilon_1}$$

(2) 由 $\vec{j}$ 的法向连续条件知

$$j_1\cos\theta_1 = j_2\cos\theta_2$$

$$E_1\sigma_1\cos\theta_1 = E_2\sigma_2\cos\theta_2$$

又由 $\vec{E}$ 的切向连续条件知

$$E_1\sin\theta_1 = E_2\sin\theta_2$$

故有

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\sigma_2}{\sigma_1}$$

5

解 不妨设电场方程为

$$\vec{E} = \vec{E}_0\sin(\omega t - kx)$$

则有

$$\begin{aligned}\frac{\partial \vec{B}}{\partial t} &= -\nabla \times \vec{E} \\ &= -\vec{E}_0 \times \nabla(\sin(\omega t - kx)) \\ &= -\vec{E}_0 \times (-k\vec{e}_x \cos(\omega t - kx)) \\ &= k\cos(\omega t - kx)\vec{E}_0 \times \vec{e}_x\end{aligned}$$

故

$$\vec{B} = \int \cos(\omega t - kx)\,\mathrm{d}t k\vec{E}_0 \times \vec{e}_x$$

略去不属于电磁波部分的常数后可得

$$\vec{B} = \frac{k}{\omega}\sin(\omega t - kx)\vec{E}_0 \times \vec{e}_x$$

则能量密度为

$$\begin{aligned}w &= \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}) \\ &= \varepsilon_0 E_0^2 \sin^2(\omega t - kx)\end{aligned}$$

平均值为

$$\begin{aligned}\overline{w} &= \frac{1}{T}\int_0^T \varepsilon_0 E_0^2 \sin^2(\omega t - kx)\,\mathrm{d}t \\ &= \frac{1}{2}\varepsilon_0 E_0^2\end{aligned}$$

坡印廷矢量的瞬时值为

$$\begin{aligned}\vec{S} &= \Re E \times \Re H \\ &= \vec{E}_0\cos(\omega t - kx - \frac{\pi}{2}) \times (\frac{k}{\omega\mu_0}\cos(\omega t - kx - \frac{\pi}{2})\vec{E}_0 \times \vec{e}_x) \\ &= \frac{k\vec{e}_x}{\omega\mu_0}E_0^2\cos^2(\omega t - kx - \frac{\pi}{2})\end{aligned}$$

则均值为

$$\begin{aligned}\overline{S} &= \frac{1}{T}\int_0^T \vec{S}\,\mathrm{d}t \\ &= \frac{E_0^2 k\vec{e}_x}{2\omega\mu_0}\end{aligned}$$

6

解　(1) 静电条件下，导体内部电场为 0。由 $\vec{E}$ 的切向连续可知

$$\vec{n} \times \vec{E}_{外} = 0$$

故 $\vec{E}_{外}$ 垂直于导体表面。

(2) 稳恒电流条件下导体表面 $\sigma_f = 0$。故由 $\vec{D}$ 的切向连续可知

$$\vec{n} \cdot (\vec{D}_{内} - \vec{D}_{外}) = 0$$

又因为 $\vec{D}_{外} = 0$，故有

$$\vec{n} \cdot \vec{D}_{内} = 0$$

$$\vec{n} \cdot \vec{E}_{内} = 0$$

即电场方向平行于导体平面

7

解　(1) 由高斯定理知

$$\nabla \cdot \vec{D} = \rho_f$$

代入电荷守恒方程可得

$$\nabla \cdot \vec{j} + \frac{\partial \nabla \cdot \vec{D}}{\partial t} = 0$$

$$\nabla \cdot (\vec{j} + \frac{\partial \vec{D}}{\partial t}) = 0$$

$$\vec{j} + \frac{\partial \vec{D}}{\partial t} = 0$$

(2) 由高斯定理知

$$\vec{D} = \frac{\lambda_f}{2\pi r} \vec{e}_r$$

$$\vec{E} = \frac{\lambda_f}{2\pi \varepsilon r} \vec{e}_r$$

又

$$\vec{j} + \frac{\partial \vec{D}}{\partial t} = 0$$

$$\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} = 0$$

$$\vec{E} = \vec{E}_0 e^{-\frac{\sigma}{\varepsilon} t}$$

故

$$
\frac{\lambda_f}{2\pi\varepsilon r}\vec{e}_r = \frac{\lambda_{f0}}{2\pi\varepsilon r}e^{-\frac{\sigma}{\varepsilon}t}\vec{e}_r
$$
$$
\lambda_f = \lambda_{f0}e^{-\frac{\sigma}{\varepsilon}t}
$$

(3)

$$
\begin{aligned}
w &= \vec{j}^2\rho \\
&= \sigma^2 E^2 \frac{1}{\sigma} \\
&= (\frac{\lambda_f}{2\pi\varepsilon r})^2\sigma
\end{aligned}
$$

(4)

$$
\begin{aligned}
P &= \int_a^b (\frac{\lambda_f}{2\pi\varepsilon r})^2\sigma 2\pi r l \,\mathrm{d}r \\
&= \frac{\lambda_f^2\sigma l}{2\pi\varepsilon^2}\ln\frac{b}{a}
\end{aligned}
$$

静电能

$$
\begin{aligned}
W &= \int \frac{\vec{E}\cdot\vec{D}}{2}\,\mathrm{d}V \\
&= \int_a^b \frac{\lambda_f^2 l}{4\pi\varepsilon r}\,\mathrm{d}r \\
&= \frac{\lambda_f^2 l}{4\pi\varepsilon}\ln\frac{b}{a}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\mathrm{d}W}{\mathrm{d}t} &= -\frac{\lambda_f l}{2\pi\varepsilon}\ln\frac{b}{a}\frac{\mathrm{d}\lambda_f}{\mathrm{d}t} \\
&= \frac{\lambda_f^2\sigma l}{2\pi\varepsilon^2}\ln\frac{b}{a}
\end{aligned}
$$

8

解　(1)

$$
\begin{aligned}
\rho_p &= -\nabla\cdot\vec{P} \\
&= -[(\nabla\frac{K}{r^2})\cdot\vec{r} + \frac{K}{r^2}(\nabla\cdot\vec{r})] \\
&= -(\frac{K}{r^2} + 3\frac{K}{r^2}) \\
&= -\frac{K}{r^2}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_p &= -\vec{n} \cdot (\vec{P}_2 - \vec{P}_1)|_{r=R} \\
&= \vec{n} \cdot \vec{P}_1|_{r=R} \\
&= \vec{n} \cdot K\frac{\vec{r}}{r^2}|_{r=R} \\
&= \frac{K}{R}
\end{aligned}
$$

(2) 由

$$
\begin{aligned}
&\vec{D} = \varepsilon_0\vec{E} + \vec{P} \\
&\vec{D} = \varepsilon\vec{E}
\end{aligned}
$$

可得

$$
\vec{D} = \frac{\varepsilon\vec{P}}{\varepsilon - \varepsilon_0}
$$

故

$$
\begin{aligned}
\rho_f &= \nabla \cdot \vec{D} \\
&= \frac{\varepsilon}{\varepsilon - \varepsilon_0}\nabla \cdot \vec{P} \\
&= \frac{\varepsilon}{\varepsilon - \varepsilon_0}\frac{K}{r^2} \\
&= \frac{\varepsilon K}{(\varepsilon - \varepsilon_0)r^2}
\end{aligned}
$$

(3) 在球内部

$$
\begin{aligned}
\vec{E} &= \frac{\vec{P}}{\varepsilon - \varepsilon_0} \\
&= \frac{K\vec{r}}{(\varepsilon - \varepsilon_0)r^2}
\end{aligned}
$$

球外部

$$
\begin{aligned}
4\pi r^2 E &= \int_0^R 4\pi r^2 \frac{\rho_f}{\varepsilon_0}\,\mathrm{d}r \\
4\pi r^2 E &= \int_0^R 4\pi r^2 \frac{\varepsilon K}{\varepsilon_0(\varepsilon - \varepsilon_0)r^2}\,\mathrm{d}r \\
4\pi r^2 E &= \frac{4\pi\varepsilon K R}{\varepsilon_0(\varepsilon - \varepsilon_0)} \\
E &= \frac{\varepsilon K R}{\varepsilon_0(\varepsilon - \varepsilon_0)r^2}
\end{aligned}
$$

故

$$\vec{E} = \frac{\varepsilon KR}{\varepsilon_0(\varepsilon - \varepsilon_0)r^3}\vec{r}$$

故

$$\begin{aligned}\varphi &= \int_\infty^r \frac{\varepsilon KR}{\varepsilon_0(\varepsilon - \varepsilon_0)r^3}\vec{r} \cdot \mathrm{d}\vec{r} \\ &= \frac{\varepsilon KR}{(\varepsilon - \varepsilon_0)\varepsilon_0 r}\end{aligned}$$

在球内部

$$\begin{aligned}\vec{E} &= \frac{\vec{P}}{\varepsilon - \varepsilon_0} \\ &= \frac{K\vec{r}}{(\varepsilon - \varepsilon_0)r^2}\end{aligned}$$

故

$$\begin{aligned}\varphi &= \varphi(R) + \int_R^r \frac{K\vec{r}}{(\varepsilon - \varepsilon_0)r^2} \cdot \mathrm{d}\vec{r} \\ &= \frac{K}{\varepsilon - \varepsilon_0}(\ln\frac{R}{r} + \frac{\varepsilon}{\varepsilon_0})\end{aligned}$$

(4)

$$\begin{aligned}W &= \int_0^R \frac{\varepsilon E^2}{2}4\pi r^2 \,\mathrm{d}r + \int_R^\infty \frac{\varepsilon_0 E^2}{2}4\pi r^2 \,\mathrm{d}r \\ &= \int_0^R \frac{2\pi\varepsilon K^2}{(\varepsilon - \varepsilon_0)} \,\mathrm{d}r + \int_R^\infty \frac{2\pi\varepsilon^2 K^2 R}{\varepsilon_0(\varepsilon - \varepsilon_0)^2 r^2} \,\mathrm{d}r \\ &= 2\pi\varepsilon R(\frac{K}{\varepsilon - \varepsilon_0})^2 + \frac{2\pi\varepsilon^2 RK^2}{\varepsilon_0(\varepsilon - \varepsilon_0)^2} \\ &= 2\pi\varepsilon R(1 + \frac{\varepsilon}{\varepsilon_0})(\frac{K}{\varepsilon - \varepsilon_0})^2\end{aligned}$$