



1

解 设球外电势

$$\varphi = \frac{Q_f}{4\pi\epsilon R} + \varphi'$$

其中 φ' 是导体球表面电荷产生的电势, 故在球外满足拉普拉斯方程

$$\nabla^2 \varphi' = 0$$

解得

$$\varphi' = \sum_{l=0}^{\infty} A_l r^{-l-1} + B_0$$

因为无穷远处电势为 0, 故 $B_0 = 0$ 。代入球体表面边界条件有

$$\begin{aligned} \left(\frac{Q_f}{4\pi\epsilon R} + \varphi'\right)|_{r=R_0} &= 0 \\ \frac{Q_f}{4\pi\epsilon a} \sum_{l=0}^{\infty} \left(\frac{R_0}{a}\right)^l P_l(\cos\theta) + \sum_{l=0}^{\infty} A_l R_0^{-l-1} &= 0 \end{aligned}$$

对比 $P_l(\cos\theta)$ 系数得

$$A_l = -\frac{Q_f R_0^{2l+1}}{4\pi\epsilon a^{l+1}}$$

像电荷为 $Q' = -\frac{R_0 Q_f}{a}$, 距球心 $\frac{R_0^2}{a}$ 。故电像法得到的电势为

$$\begin{aligned} \varphi &= \frac{Q_f}{4\pi\epsilon} \left(\frac{R_0}{aR_1} + \frac{1}{R_2} \right) \\ &= \frac{Q_f}{4\pi\epsilon} \left(\frac{R_0}{a\sqrt{r^2 + \left(\frac{R_0^2}{a}\right)^2 - 2r\frac{R_0^2}{a}\cos\theta}} + \frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} \right) \end{aligned}$$

展开为勒让德级数后与分离变量法得到的结果是一致的。

2

解 像电荷大小为 $-\frac{R_1}{a}Q$, 距球心 $\frac{R_1^2}{a}$ 。故电势为

$$\varphi = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{QR_1}{a\sqrt{r^2 + \frac{R_1^4}{a^2} - 2\frac{2R_1^2 r}{a}\cos\theta}} \right]$$

由于导体球接地, 故外壳电势为 0, 由唯一性定理可知球外无电场, 故球壳外表面电荷为 0。作一个半径在 R_1, R_2 之间的高斯球面可易知, 感应电荷只分布在球壳内表面, 且总量为 $-Q$ 。



3

解 球壳内部边界条件与上题相同, 故内部电场强度一致, 即内部电势至多相差一常数。

当球壳有电势 φ_0 时, 易知球壳内部电势为

$$\varphi = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} - \frac{QR_1}{a\sqrt{r^2 + \frac{R_1^4}{a^2} - 2\frac{2R_1^2 r}{a} \cos \theta}} \right] + \varphi_0$$

当球壳带电 Q_0 时, 易知球壳电势为 $\frac{Q + Q_0}{4\pi\epsilon_0 R_2}$, 故球内电势为

$$\varphi' = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} - \frac{QR_1}{a\sqrt{r^2 + \frac{R_1^4}{a^2} - 2\frac{2R_1^2 r}{a} \cos \theta}} \right] + \frac{Q + Q_0}{4\pi\epsilon_0 R_2}$$

欲使

$$\varphi' = \varphi$$

即

$$\varphi_0 = \frac{Q + Q_0}{4\pi\epsilon_0 R_2}$$

4

解 本题有三个像电荷, 分别是 Q 与球面形成的像电荷 q_1 , Q 与平面形成的像电荷 q_2, q_1 与平面形成的像电荷 q_3 。故

$$q_1 = -\frac{a}{b}Q$$

$$q_2 = -Q$$

$$q_3 = \frac{a}{b}Q$$

以球心为原点, 平面为 $x - y$ 平面建立直角坐标系可以得到空间中电势为

$$\varphi = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{x^2 + y^2 + (z - b)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + b)^2}} - \frac{a}{b\sqrt{x^2 + y^2 + \left(z - \frac{a^2}{b}\right)^2}} + \frac{a}{b\sqrt{x^2 + y^2 + \left(z + \frac{a^2}{b}\right)^2}} \right)$$

5



解 将两电极以一个小球面包围起来, 故球面的电通量为

$$\begin{aligned}\iint_S \vec{E} \cdot d\vec{S} &= \iint_S \frac{\vec{j}}{\sigma} \cdot d\vec{S} \\ &= \frac{1}{\sigma} \iint_S \vec{j} \cdot d\vec{S} \\ &= \frac{I}{\sigma}\end{aligned}$$

又由唯一性定理知道, 这等效于一个电荷量为 Q 的电荷, 且满足

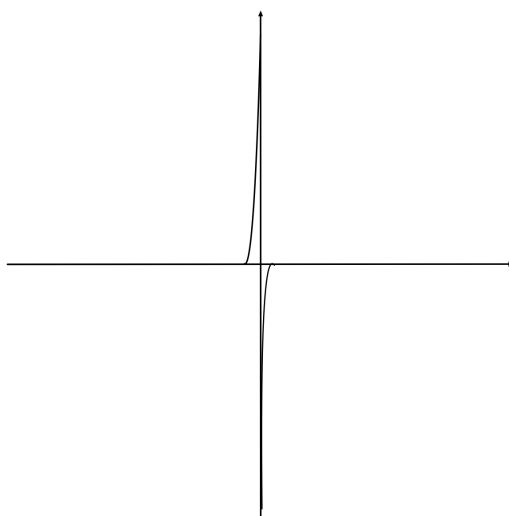
$$\begin{aligned}\frac{I}{\sigma} &= \frac{Q}{\varepsilon_0} \\ Q &= \frac{I\varepsilon_0}{\sigma}\end{aligned}$$

故会产生六个像电荷, 空间中电势由这八个电荷叠加而成, 故

$$\begin{aligned}\varphi &= \frac{I}{4\pi\sigma} \left(\frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}} \right. \\ &\quad + \frac{1}{\sqrt{(x+x_0)^2 + (y-y_0)^2 + (z+z_0)^2}} - \frac{1}{\sqrt{(x+x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \\ &\quad + \frac{1}{\sqrt{(x-x_0)^2 + (y+y_0)^2 + (z+z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2}} \\ &\quad \left. + \frac{1}{\sqrt{(x+x_0)^2 + (y+y_0)^2 + (z-z_0)^2}} - \frac{1}{\sqrt{(x+x_0)^2 + (y+y_0)^2 + (z+z_0)^2}} \right)\end{aligned}$$

6

解





不妨将 \vec{p} 的方向取为沿 x 轴正方向, 则电偶极子的电荷密度可写为

$$\begin{aligned}\rho &= q\delta(x - \frac{l}{2}) - q\delta(x + \frac{l}{2}) \\ &= -ql \frac{\delta(x + \frac{l}{2}) - \delta(x - \frac{l}{2})}{l}\end{aligned}$$

取 $l \rightarrow 0$ 的极限, 则

$$\rho = -ql \frac{d\delta(x)}{dx}$$

故

$$\rho = -(p \cdot \nabla)\delta(x)$$

7

解 定解条件为

$$\begin{cases} \nabla^2 \varphi_{in} = 0 \\ \nabla^2 \varphi_{out} = 0 \\ \lim_{r \rightarrow \infty} \varphi_{out} = 0 \\ \lim_{r \rightarrow 0} \varphi_{in} \text{ 有限} \end{cases}$$

故解得

$$\begin{aligned}\varphi_{out} &= \sum_{l=0}^{\infty} A_l r^{-l-1} P_l(\cos \theta) \\ \varphi_{in} &= \sum_{l=1}^{\infty} B_l r^l P_l(\cos \theta)\end{aligned}$$

代入球体表面边界条件有

$$\begin{aligned}\sum_{l=0}^{\infty} A_l R_0^{-l-1} P_l(\cos \theta) &= \begin{cases} \varphi_0 (0 < \theta < \frac{\pi}{2}) \\ -\varphi_0 (\frac{\pi}{2} < \theta < \pi) \end{cases} \\ \sum_{l=0}^{\infty} B_l R_0^l P_l(\cos \theta) &= \begin{cases} \varphi_0 (0 < \theta < \frac{\pi}{2}) \\ -\varphi_0 (\frac{\pi}{2} < \theta < \pi) \end{cases}\end{aligned}$$

解得

$$\begin{aligned}A_l &= \varphi_0 R_0^{l+1} [P_{l-1}(0) - P_{l+1}(0)] \\ B_l &= \frac{\varphi_0 [P_{l-1}(0) - P_{l+1}(0)]}{R_0^l}\end{aligned}$$



8

解 格林函数可取为

$$G(r, \theta, \phi, r', \theta', \phi') = \frac{1}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}} - \frac{1}{\sqrt{\left(\frac{rr'}{R_0}\right)^2 + R_0^2 - 2rr' \cos(\theta - \theta')}} \right]$$

故

$$\varphi = \epsilon_0 \int_{r'=R_0} \varphi(R_0, \theta, \phi) \frac{\partial G}{\partial n} d\vec{S}$$

可记 $\frac{r}{R_0} = R$, 故最后结果为

$$\begin{aligned} \varphi &= \begin{cases} -\frac{\varphi_0}{2} \left(\int_0^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\pi} \right) \frac{R^2 - 1}{[1 + R^2 - 2R \cos(\theta - \theta')]^{\frac{3}{2}}} \sin \theta' d\theta' & R > 1 \\ \frac{\varphi_0}{2} \left(\int_0^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\pi} \right) \frac{R^2 - 1}{[1 + R^2 - 2R \cos(\theta - \theta')]^{\frac{3}{2}}} \sin \theta' d\theta' & R < 1 \end{cases} \\ &= -\frac{\varphi_0}{2} \left(\int_0^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\pi} \right) \frac{|R^2 - 1|}{[1 + R^2 - 2R \cos(\theta - \theta')]^{\frac{3}{2}}} \sin \theta' d\theta' \end{aligned}$$

这是一个椭圆积分, 用勒让德级数展开后与上题结果一致。