1

解 (1) 该级数为缺项的幂级数

$$\rho = \lim_{n\to\infty} \sqrt[n!]{1} = 1 \to r = \frac{1}{\rho} = 1$$

故收敛半径为 1

(2) 该级数为幂级数,故收敛条件为

$$|\frac{z}{1+z}| < 1$$
$$\frac{|z|}{|1+z|} < 1$$
$$|z| < |1+z|$$

故收敛条件为

$$\Re(z) > -\frac{1}{2}$$

2

解 (1) 设 $t = z - n\pi$ 则

$$\begin{aligned}\sin z &= \sin(t+n\pi)\\ &= \begin{cases} \sin t(n\text{为偶数}) \\ -\sin t(n\text{为奇数}) \end{cases}\\ &= \begin{cases} \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(n\text{为偶数}) \\ -\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}(n\text{为奇数}) \end{cases}\\ &= \begin{cases} \sum_{k=0}^{\infty} \frac{(z-n\pi)^{2k+1}}{(2k+1)!}(n\text{为偶数}) \\ -\sum_{k=0}^{\infty} \frac{(z-n\pi)^{2k+1}}{(2k+1)!}(n\text{为奇数}) \end{cases}\end{aligned}$$

(2) 设

$$\frac{1}{z^2+z+1} = \sum_{k=0}^{\infty} a_k z^k$$

则

$$\sum_{k=0}^{\infty} a_k z^{k+2} + \sum_{k=0}^{\infty} a_k z^{k+1} + \sum_{k=0}^{\infty} a_k z^k = 1$$

比较两边相同次幂系数可知 $a_{3k}=1, a_{3k+1}=-1, a_{3k+2}=0$，故

$$\frac{1}{z^2+z+1}=\sum_{k=0}^{\infty}z^{3k}-\sum_{k=0}^{\infty}z^{3k+1}$$

(3) 设 $t=z+1$，则

$$\begin{aligned}\frac{1}{z^2}&=\frac{1}{(t-1)^2}\\&=\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{1-t})\\&=\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{\infty}t^k\\&=\sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}t^k\\&=\sum_{k=0}^{\infty}(k+1)t^k\\&=\sum_{k=0}^{\infty}(k+1)(z+1)^k\end{aligned}$$

(4) 设 $t=\frac{1}{z}$，则

$$\begin{aligned}\ln\frac{1+z}{1-z}&=\ln\frac{1+\frac{1}{t}}{1-\frac{1}{t}}\\&=\ln\frac{t+1}{t-1}\\&=\ln(t+1)-\mathrm{i}\pi-\ln(1-t)(\text{此处规定单值分支}\ln(-1)=\mathrm{i}\pi)\\&=-2\sum_{n=1}^{\infty}\frac{t^{2n-1}}{(2n-1)!}-\mathrm{i}\pi\\&=-2\sum_{n=1}^{\infty}\frac{z^{-(2n-1)}}{(2n-1)!}-\mathrm{i}\pi\end{aligned}$$

3

解 (1)

$$\begin{aligned}\frac{1}{z^2-3z+2}&=\frac{1}{(z-1)(z-2)}\\&=\frac{1}{z-2}-\frac{1}{z-1}\\&=-\frac{1}{2}\frac{1}{1-\frac{z}{2}}-\frac{1}{z}\frac{1}{1-\frac{1}{z}}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \frac{1}{z}\sum_{n=0}^{\infty}z^{-n} \\
&= \sum_{n=0}^{\infty}\frac{-z^n}{2^{n+1}} - \sum_{n=0}^{\infty}z^{-n-1}
\end{aligned}
$$

(2)

$$
\begin{aligned}
\frac{1}{z^2-3z+2} &= \frac{1}{(z-1)(z-2)} \\
&= \frac{1}{z-2} - \frac{1}{z-1} \\
&= \frac{1}{z}\frac{1}{1-\frac{2}{z}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} \\
&= \frac{1}{z}\sum_{n=0}^{\infty}2^n z^{-n} - \frac{1}{z}\sum_{n=0}^{\infty}z^{-n} \\
&= \sum_{n=0}^{\infty}2^n z^{-n-1} - \sum_{n=0}^{\infty}z^{-n-1}
\end{aligned}
$$

(3) 设 $t = z - 1$

$$
\begin{aligned}
\frac{1}{z^2(z-1)} &= \frac{1}{(t+1)^2 t} \\
&= \frac{1}{t}\frac{1}{t+1} \\
&= -\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{1+t} \\
&= -\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty}(-1)^n t^n \\
&= -\frac{1}{t}\sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}(-1)^n t^n \\
&= -\frac{1}{t}\sum_{n=0}^{\infty}(-1)^n n t^{n-1} \\
&= \sum_{n=0}^{\infty}(-1)^{n+1} n t^{n-2} \\
&= \sum_{n=1}^{\infty}(-1)^{n+1} n z^{-(n-2)}
\end{aligned}
$$

(4)

$$
z^3 e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{z^{-n+3}}{n!}
$$

4

解 (1)$\cos z$ 在全平面解析，故 0 为二阶极点。

(2)$z=0$ 是可去奇点

$z=k\pi(k\neq 0)$ 是一阶极点

$z=\infty$ 是非孤立奇点

(3)$\ln z|_{z=1}=2n\pi\mathrm{i}$ 故对于 $n=0$ 的单值分支，1 是二阶极点。对 $n\neq 0$ 的单值分支，1 是一阶极点。

(4)$z=\sqrt{k\pi}$ 是一阶极点

$z=\infty$ 是非孤立奇点

5

解 (1) 设 $f(z)=(z-z_0)^m\phi(z)$，$\phi(z)$ 是全平面解析的。则

$$\frac{f''(z)}{f(z)}=(m-1)m(z-z_0)^{-2}+2m(z-z_0)^{-1}\frac{\phi'}{\phi}+\frac{\phi''}{\phi}$$

故 $\mathrm{res}(\dfrac{f''(z)}{f(z)},z_0)=2m\dfrac{\phi'(z_0)}{\phi(z_0)}$

(2) 设 $f(z)=(z-z_0)^{-m}\phi(z)$，$\phi(z)$ 是全平面解析的。则

$$\frac{f'(z)}{f(z)}=-m(z-z_0)^{-1}+\frac{\phi'}{\phi}$$

故 $\mathrm{res}(\dfrac{f'(z)}{f(z)},z_0)=-m$

6

解 (1)

$$\frac{1}{z^3-z^5}=\frac{1}{z^3(1-z)(1+z)}$$

故

$$\begin{aligned}\mathrm{res}(\frac{1}{z^3-z^5},0)&=\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(z^3\cdot\frac{1}{z^3-z^5})|_{z=0}\\&=-1\end{aligned}$$

$$\begin{aligned}\mathrm{res}(\frac{1}{z^3-z^5},1)&=\lim_{z\to 1}[(z-1)\frac{1}{z^3-z^5}]\\&=\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathrm{res}(\frac{1}{z^3-z^5},1)&=\lim_{z\to -1}[(z+1)\frac{1}{z^3-z^5}]\\&=\frac{1}{2}\end{aligned}$$

(2) 选定单值分支后只有 0 一个可去奇点，故

$$\mathrm{res}(\frac{\sqrt{z}}{\sinh\sqrt{z}},0)=0$$

(3)

$$\begin{aligned}\mathrm{res}(\frac{1}{z^2\sin z},0)&=\frac{1}{2!}\lim_{z\to0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(z^3\frac{1}{z^2\sin z})\\&=\frac{1}{6}\end{aligned}$$

$$\begin{aligned}\mathrm{res}(\frac{1}{z^2\sin z},k\pi)&=\lim_{z\to k\pi}[(z-k\pi)(z^3\frac{1}{z^2\sin z})]\\&=\frac{(-1)^k}{k^2\pi^2}\end{aligned}$$

(4)

$$\frac{1-e^{2z}}{z^4}=-\sum_{n=1}^{\infty}\frac{2^nz^{n-4}}{n!}$$

故

$$\mathrm{res}(\frac{1-e^{2z}}{z^4},0)=-\frac{2^3}{3!}=-\frac{4}{3}$$