1

解 (1)

$$\frac{\partial u}{\partial x} = 2x$$
$$\frac{\partial u}{\partial y} = 2$$
$$\frac{\partial v}{\partial x} = 2x$$
$$\frac{\partial v}{\partial y} = 2y$$

由柯西黎曼条件知该函数在 $(1,1)$ 处可导,全平面不解析。

(2)

$$\frac{\partial u}{\partial x} = 2x$$
$$\frac{\partial u}{\partial y} = -2y$$
$$\frac{\partial v}{\partial x} = 2y$$
$$\frac{\partial v}{\partial y} = 2x$$

由柯西黎曼条件知该函数在全平面可导,全平面解析。

(3)

$$\frac{\partial u}{\partial x} = 2y$$
$$\frac{\partial u}{\partial y} = 2x$$
$$\frac{\partial v}{\partial x} = 2x$$
$$\frac{\partial v}{\partial y} = 2y$$

由柯西黎曼条件知该函数在 $x = 0$ 线上可导,全平面不解析。

(4)

$$\frac{\partial u}{\partial x} = y^2$$
$$\frac{\partial u}{\partial y} = 2xy$$
$$\frac{\partial v}{\partial x} = 2xy$$
$$\frac{\partial v}{\partial y} = x^2$$

由柯西黎曼条件知该函数在 $(0,0)$ 处可导,全平面不解析。

2

解 (1)

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=\cos x\sinh y$$
$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=-\sin x\cosh y$$

取积分路径 $(0,0)\to(x,0)\to(x,y)$，则虚部为

$$\begin{aligned}v&=\int_{(0,0)}^{(x,y)}\frac{\partial v}{\partial x}\,\mathrm{d}x+\frac{\partial v}{\partial y}\,\mathrm{d}y\\&=\int_0^x\cos x\sinh 0\,\mathrm{d}x-\int_0^y\sin x\cosh y\,\mathrm{d}y+C\\&=-\sin x\sinh y+C'\end{aligned}$$

故

$$f(z)=\cos x\cosh y+\mathrm{i}(-\sin x\sinh y+C')$$

(2)

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\frac{x}{x^2+y^2}$$
$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=\frac{y}{x^2+y^2}$$

取积分路径 $(1,1)\to(x,1)\to(x,y)$，则虚部为

$$\begin{aligned}u&=\int_{(1,1)}^{(x,y)}\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y\\&=\int_1^x\frac{1}{x\left(\frac{y^2}{x^2}+1\right)}\,\mathrm{d}x+\int_1^y\frac{y}{x^2\left(\frac{y^2}{x^2}+1\right)}\,\mathrm{d}y+C\\&=\frac{1}{2}\ln(x^2+y^2)+C'\end{aligned}$$

故

$$f(z)=\frac{1}{2}\ln(x^2+y^2)+C'+\mathrm{i}(\arctan\frac{y}{x})$$

3

解 (1)

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=\frac{-2y}{x^2+y^2}$$
$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2}$$

取积分路径 $(1,1) \to (x,1) \to (x,y)$，则虚部为

$$\begin{aligned} v &= \int_{(1,1)}^{(x,y)} \frac{\partial v}{\partial x}\,\mathrm{d}x + \frac{\partial v}{\partial y}\,\mathrm{d}y \\ &= \int_1^x \frac{-2y}{x^2+y^2}\,\mathrm{d}x - \int_1^y \frac{2x}{x^2+y^2}\,\mathrm{d}y + C \\ &= 2\arctan\frac{y}{x} + C' \end{aligned}$$

故

$$f(z) = \ln(x^2+y^2) + \mathrm{i}(2\arctan\frac{y}{x} + C')$$

(2)

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = 6x^2 - 6xy - 6y^2 \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = -(3x^2 + 12xy - 3y^2) \end{aligned}$$

取积分路径 $(0,0) \to (x,0) \to (x,y)$，则虚部为

$$\begin{aligned} u &= \int_{(0,0)}^{(x,y)} \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y \\ &= \int_0^x 6x^2\,\mathrm{d}x - \int_0^y (3x^2+12xy-3y^2)\,\mathrm{d}y + C \\ &= 2x^3 - 3x^2y - 6xy^2 + y^3 + C' \end{aligned}$$

故

$$f(z) = 2x^3 - 3x^2y - 6xy^2 + y^3 + C' + \mathrm{i}(x^3 + 6x^2y - 3xy^2 - 2y^3)$$