1

解

$$P_l'(x)=\sum_{n=1}^{l}\frac{(l+n)!}{2(n!)^2(l-n)!}\left(\frac{x-1}{2}\right)^{n-1}$$

故

$$\begin{aligned}P_l'(1)&=\frac{(l+1)!}{2(1!)^2(l-1)!}\\&=\frac{l(l+1)}{2}\end{aligned}$$

l 为奇数时,$P_l(x)$ 为奇函数,故 $P_l(0)=0$,$l=0$ 时,$P_l(0)=1$,l 为偶数时,则 $P_l(x)$ 在 0 处展开的常数项为

$$\begin{aligned}P_l(0)&=\frac{(-1)^n(2l-l)!}{2^l(\frac{l}{2})!(l-\frac{l}{2})!(l-l)!}\\&=\frac{(-1)^{\frac{l}{2}}l!}{2^l[(\frac{l}{2})!]^2}\end{aligned}$$

故

$$P_l(0)=\begin{cases}0 & (l\text{为奇数})\\ 1 & (l=0)\\ \dfrac{(-1)^{\frac{l}{2}}l!}{2^l[(\frac{l}{2})!]^2} & (l\text{为偶数})\end{cases}$$

$$P_l'(0)=\frac{(-1)^{\frac{l-1}{2}}(l+2)!}{2^l(\frac{l-1}{2})!(\frac{l+1}{2})!}$$

2

解 (1)$n=0$ 时,$P_0(x)=1$,$P_1(x)=x$,故

$$P_0(x)=P_1'(x)+P_0'(x)$$

设 $n=m-1$ 时该命题成立,则有

$$\sum_{l=0}^{m-1}(2l+1)P_l(x)=P_m'(x)+P_{m-1}'(x)$$

又由递推关系 $P_{m+1}'(x)-P_{m-1}'(x)=(2m+1)P_m(x)$,故有

$$\sum_{l=0}^{m-1}(2l+1)P_l(x)+(2m+1)P_m(x)=P_m'(x)+P_{m-1}'(x)+P_{m+1}'(x)-P_{m-1}'(x)$$

即 $n=m$ 时该命题也成立，原命题得证。

(2)

$$xP_m(x)=\frac{(m+1)P_{m+1}(x)+mP_{m-1}(x)}{2m+1}$$

故

$$\begin{aligned}I&=\frac{m+1}{2m+1}\int_{-1}^{1}P_{m+1}(x)P_n(x)\,\mathrm{d}x+\frac{m}{2m+1}\int_{-1}^{1}P_{m-1}(x)P_n(x)\,\mathrm{d}x\\&=\frac{2(m+1)}{(2m+1)(2n+1)}\delta_{m+1,n}+\frac{2m}{(2m+1)(2n+1)}\delta_{m-1,n}\end{aligned}$$

3

解　(1)

$$\begin{aligned}I&=\int_0^{\pi}2P_n(\cos\theta)\sin\theta\cos\theta\,\mathrm{d}\theta\\&=-2\int_0^{\pi}P_n(\cos\theta)\cos\theta\,\mathrm{d}\cos\theta\\&=2\int_{-1}^{1}xP_n(x)\,\mathrm{d}x\\&=2\int_{-1}^{1}\frac{(n+1)P_{n+1}(x)+nP_{n-1}(x)}{2n+1}\,\mathrm{d}x\\&=\frac{2(n+1)}{2n+1}\int_{-1}^{1}P_{n+1}(x)\,\mathrm{d}x+\frac{2n}{2n+1}\int_{-1}^{1}P_{n-1}(x)\,\mathrm{d}x\\&=\frac{2(n+1)}{2n+1}\int_{-1}^{1}P_{n+1}(x)P_0(x)\,\mathrm{d}x+\frac{2n}{2n+1}\int_{-1}^{1}P_{n-1}(x)P_0(x)\,\mathrm{d}x\\&=\frac{4(n+1)}{2n+1}\delta_{n+1,0}+\frac{4n}{2n+1}\delta_{n-1,0}\end{aligned}$$

(2)

$$\begin{aligned}I&=\int_{-1}^{1}(1+x)^k\frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\,\mathrm{d}x\\&=\frac{1}{2^l l!}\left[(1+x)^k\frac{d^{l-1}}{\mathrm{d}x^{l-1}}(x^2-1)^l|_{-1}^{1}-\int_{-1}^{1}\frac{\mathrm{d}(1+x)^k}{\mathrm{d}x}\frac{d^{l-1}}{\mathrm{d}x^{l-1}}(x^2-1)^l\mathrm{d}x\right]\\&\overset{\text{再进行}l-1\text{次分部积分}}{=}\frac{(-1)^n}{2^l l!}\int_{-1}^{1}(x^2-1)^l\frac{\mathrm{d}^l(1+x)^k}{\mathrm{d}x^l}\,\mathrm{d}x\end{aligned}$$

当 $k<l$ 时易知 $I=0$。

$k\geqslant l$ 时

$$I=\frac{1}{2^l l!}\int_{-1}^{1}(1-x^2)^l\frac{\mathrm{d}^l(1+x)^k}{\mathrm{d}x^l}\,\mathrm{d}x$$

$$
\begin{aligned}
&= \frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}(1-x^2)^l(1+x)^{k-l}\,\mathrm{d}x \\
&= \frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}(1-x)^l(1+x)^{k}\,\mathrm{d}x
\end{aligned}
$$

设 $t=\dfrac{x+1}{2}$,则

$$
\begin{aligned}
I &= \frac{k!}{2^l l!(k-l)!}\int_{0}^{1}[2(1-t)]^l(2t)^k\,\mathrm{d}(2t-1) \\
&= \frac{k!2^{k+1}}{l!(k-l)!}\int_{0}^{1}(1-t)^l t^k\,\mathrm{d}t \\
&= \frac{k!2^{k+1}}{l!(k-l)!}B(l+1,k+1) \\
&= \frac{k!2^{k+1}}{l!(k-l)!}\frac{k!l!}{(k+l+1)!} \\
&= \frac{2^{k+1}(k!)^2}{(k-l)!(k+l+1)!}
\end{aligned}
$$

4

解　设 $f(x)=a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+a_3P_3(x)$,对比系数可得方程组

$$
\begin{cases}
a_0-\dfrac{a_2}{2}=1 \\
a_1-\dfrac{3a_3}{2}=2 \\
\dfrac{3a_2}{2}=3 \\
\dfrac{5a_3}{2}=5
\end{cases}
$$

解得

$$
\begin{cases}
a_0=2 \\
a_1=5 \\
a_2=2 \\
a_3=2
\end{cases}
$$

故

$$
f(x)=2P_0(x)+5P_1(x)+2P_2(x)+2P_3(x)
$$

5

解 定解条件为

$$\begin{cases} \dfrac{1}{r^2}\partial_r(r^2\partial_r u) + \dfrac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta u) = 0 \\ u|_{r=1} = \cos\theta \\ u|_{r=2} = 1 + \cos^2\theta \end{cases}$$

分离变量 $u = R(r)\Theta(\theta)$ 得

$$\begin{cases} r^2R'' + 2rR' - l(l+1)R = 0 \\ \Theta'' + \cot\theta\Theta + l(l+1)\Theta = 0 \end{cases}$$

通解为

$$\begin{cases} R = A_l r^l + B_l r^{-l-1} \\ \Theta = P_l(\cos\theta) \end{cases}$$

则

$$u = \sum_{l=0}^{\infty} P_l(\cos\theta)(A_l r^l + B_l r^{-l-1})$$

代入边界条件得

$$\sum_{l=0}^{\infty} P_l(\cos\theta)(A_l + B_l) = \cos\theta$$

对比系数得到

$$\begin{cases} A_0 + B_0 = 0 \\ A_1 + B_1 = 1 \\ A_l + B_l = 0(l > 1) \end{cases}$$

又有

$$\sum_{l=0}^{\infty} P_l(\cos\theta)(A_l 2^l + B_l 2^{-l-1}) = 1 + \cos^2\theta$$

对比系数得到

$$\begin{cases} A_0 + \dfrac{B_0}{2} - \dfrac{1}{2}(4A_2 + \dfrac{B_2}{8}) = 1 \\ 2A_1 + \dfrac{B_1}{4} = 0 \\ \dfrac{3}{2}(4A_2 + \dfrac{B_2}{4}) = 1 \\ A_l 2^l + B_l 2^{-l-1} = 0(l > 2) \end{cases}$$

解得

$$\begin{cases} A_0 = \dfrac{5}{3} \\ B_0 = -\dfrac{2}{3} \\ A_1 = -\dfrac{1}{7} \\ B_1 = \dfrac{8}{7} \\ A_2 = \dfrac{16}{93} \\ B_2 = -\dfrac{16}{93} \\ A_l = B_l = 0 (l > 2) \end{cases}$$

故

$$u = \left(\frac{8}{7r^2} - \frac{r}{7}\right)\cos(\theta) + \frac{1}{2}\left(\frac{16r^2}{93} - \frac{16}{93r^3}\right)\left(3\cos^2(\theta) - 1\right) - \frac{2}{3r} + \frac{5}{3}$$