1

解　(1)

$$
\begin{aligned}
&\frac{1-2\mathrm{i}}{3+4\mathrm{i}}+\frac{2-\mathrm{i}}{5\mathrm{i}}\\
=&\frac{(1-2\mathrm{i})(3-4\mathrm{i})}{25}-\frac{5\mathrm{i}(2-i)}{25}\\
=&\frac{3-4\mathrm{i}-6\mathrm{i}-8-10\mathrm{i}-5}{25}\\
=&-\frac{2}{5}-\frac{4}{5}\mathrm{i}
\end{aligned}
$$

故

$$
\begin{aligned}
Re(z)&=-\frac{2}{5}\\
Im(z)&=-\frac{4}{5}\\
|z|&=\frac{2\sqrt{5}}{5}\\
arg(z)&=\arctan(2)+\pi
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\sqrt[\mathrm{i}]{2\mathrm{i}}\\
=&e^{\frac{\ln(2i)}{\mathrm{i}}}\\
=&e^{\frac{\ln 2+\mathrm{i}(\frac{\pi}{2}+2k\pi)}{\mathrm{i}}}\\
=&e^{-\mathrm{i}\ln 2}e^{\frac{\pi}{2}+2k\pi}\\
=&e^{\frac{\pi}{2}+2k\pi}(\cos(\ln 2)-\mathrm{i}\sin(\ln 2))
\end{aligned}
$$

故

$$
\begin{aligned}
Re(z)&=e^{\frac{\pi}{2}+2k\pi}\cos(\ln 2)\\
Im(z)&=-e^{\frac{\pi}{2}+2k\pi}\sin(\ln 2)\\
|z|&=e^{\frac{\pi}{2}+2k\pi}\\
arg(z)&=-\ln 2
\end{aligned}
$$

(3)

$$
\begin{aligned}
& e^{\mathrm{i}e^{\mathrm{i}}} \\
=& e^{\mathrm{i}(\cos 1+i\sin 1)} \\
=& e^{\mathrm{i}\cos 1-\sin 1} \\
=& e^{-\sin 1}e^{\mathrm{i}\cos 1} \\
=& e^{-\sin 1}(\cos(\cos 1)+\mathrm{i}\sin(\cos 1))
\end{aligned}
$$

故

$$
\begin{aligned}
Re(z) &= e^{-\sin 1}\cos(\cos 1) \\
Im(z) &= e^{-\sin 1}\sin(\cos 1) \\
|z| &= e^{-\sin 1} \\
arg(z) &= \cos 1
\end{aligned}
$$

2

解

$$
\begin{aligned}
& (2+\mathrm{i})(3+\mathrm{i}) \\
=& 5+5\mathrm{i}
\end{aligned}
$$

记

$$
\begin{aligned}
z_1 &= 2+\mathrm{i} \\
z_2 &= 3+\mathrm{i}
\end{aligned}
$$

故 $arg(z_1)=\arctan(\frac{1}{2}), arg(z_2)=\arctan(\frac{1}{3}), arg(z_1z_2)=\frac{\pi}{4}$ 故

$$
\frac{\pi}{4}=\arctan(\frac{1}{2})+\arctan(\frac{1}{3})
$$

3

解

$$
\begin{aligned}
& \sum_{k=1}^{n} e^{\mathrm{i}(2k-1)\phi} \\
=& \frac{e^{\mathrm{i}\phi}(1-e^{\mathrm{i}2n\phi})}{1-e^{\mathrm{i}2\phi}} \\
=& \frac{e^{\mathrm{i}\phi}-e^{\mathrm{i}(2n+1)\phi}}{1-e^{\mathrm{i}2\phi}} \\
=& \frac{\cos\phi+\mathrm{i}\sin\phi-\cos(2n+1)\phi-\mathrm{i}\sin(2n+1)\phi}{1-\cos 2\phi-\mathrm{i}\sin 2\phi} \\
=& -\frac{1}{2}i\csc(\phi)\cos(2n\phi)+\frac{1}{2}\csc(\phi)\sin(2n\phi)+\frac{1}{2}i\csc(\phi)
\end{aligned}
$$

取实部之后得到

$$\sum_{k=1}^{n}\cos(2k-1)\phi=\frac{1}{2}\csc(\phi)\sin(2n\phi)$$

4

解

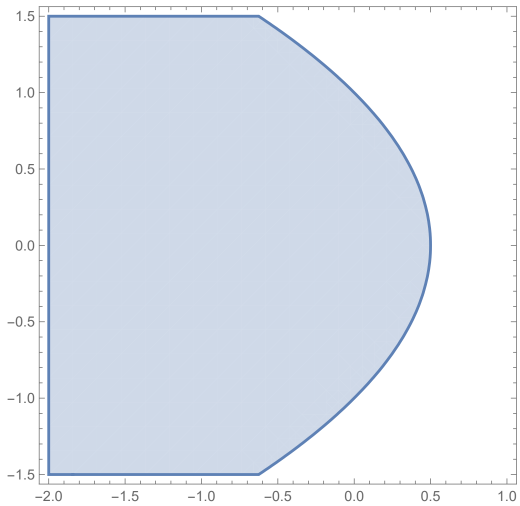

图 1: (1)

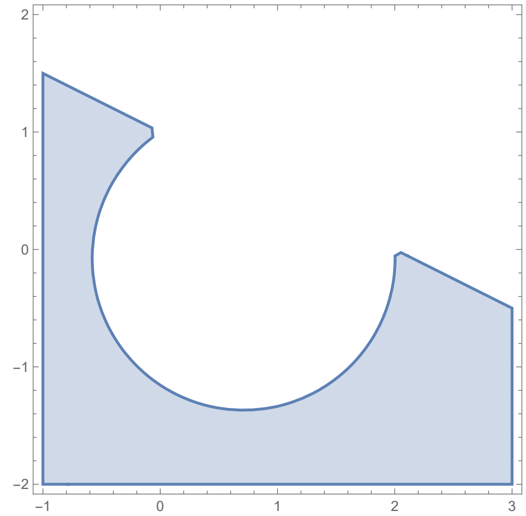

图 2: (2)

5

解

$$\begin{aligned}&e^{in\theta}\\=&(\cos\theta+\mathrm{i}\sin\theta)^n\\=&\sum_{k=0}^{n}\mathrm{i}^{n-k}C_n^k\cos^k\theta\sin^{n-k}\theta\end{aligned}$$

取实部之后得到

$$\cos n\theta=\cos^n\theta-C_n^2\cos^{n-2}\theta\sin^2\theta+C_n^4\cos^{n-4}\theta\sin^4\theta+\cdots$$

取虚部之后得到

$$\sin n\theta=C_n^1\cos^{n-1}\theta\sin\theta-C_n^3\cos^{n-3}\theta\sin^3\theta+C_n^5\cos^{n-5}\theta\sin^5\theta+\cdots$$

6

解 设 $z=\sin(\omega)$, 则 $\omega=\arcsin z$

$$
\begin{aligned}
z &= \frac{e^{\mathrm{i}\omega}-e^{-\mathrm{i}\omega}}{2\mathrm{i}} \\
2\mathrm{i}z &= e^{\mathrm{i}\omega}-e^{-\mathrm{i}\omega} \\
e^{\mathrm{i}2\omega}-2\mathrm{i}ze^{\mathrm{i}\omega}-1=0 \\
e^{\mathrm{i}\omega} &= \frac{2\mathrm{i}z+\sqrt{-4z^2+4}}{2} \\
e^{\mathrm{i}\omega} &= \mathrm{i}z+\sqrt{1-z^2} \\
\omega &= \frac{1}{\mathrm{i}}\ln(\mathrm{i}z+\sqrt{1-z^2})
\end{aligned}
$$

ω 还有另一解 $\frac{1}{\mathrm{i}}\ln(\mathrm{i}z-\sqrt{1-z^2})$,但是由于 $\sqrt{1-z^2}$ 是多值函数,因此两解等价。

7

解 由三角函数的级数定义知

$$
\begin{aligned}
\sin\sqrt{z} &= \sqrt{z}-\frac{z\sqrt{z}}{3!}+\cdots \\
&= \sqrt{z}(1-\frac{z}{3!}+\cdots)
\end{aligned}
$$

由于等式右边为一不恒为 0 的单值函数乘以一多值函数,故 $\sin\sqrt{z}$ 为多值函数。
同理 $\cos\sqrt{z}$ 为单值函数,$\frac{\sin\sqrt{z}}{\sqrt{z}}$ 为单值函数,$\frac{\cos\sqrt{z}}{\sqrt{z}}$ 为多值函数。

$$
\begin{aligned}
\sin(\mathrm{i}\ln z) &= \frac{e^{\mathrm{i}\mathrm{i}\ln z}-e^{-\mathrm{i}\mathrm{i}\ln z}}{2\mathrm{i}} \\
&= \frac{\frac{1}{z}-z}{2\mathrm{i}}
\end{aligned}
$$

故 $\sin(\mathrm{i}\ln z)$ 为单值函数。

8

解 (1)

$$
\begin{aligned}
&\sqrt[3]{(z-a)(z-b)} \\
=&\sqrt[3]{r_1r_2}e^{\frac{\mathrm{i}(\theta_1+\theta_2)}{3}}
\end{aligned}
$$

绕 a 转一圈后函数值变为

$$
\begin{aligned}
&\sqrt[3]{r_1r_2}e^{\frac{\mathrm{i}(\theta_1+\theta_2+2\pi)}{3}} \\
=&\sqrt[3]{r_1r_2}e^{\frac{\mathrm{i}(\theta_1+\theta_2)}{3}}e^{\mathrm{i}\frac{2\pi}{3}}
\end{aligned}
$$

故 a 为支点,同理 b 也是支点。
绕无穷远点逆时针绕一圈相当于顺时针绕有穷远所有点一圈,故绕无穷远点一圈后函数值

变为

$$\sqrt[3]{r_1r_2}e^{\frac{\mathrm{i}(\theta_1+\theta_2-4\pi)}{3}}$$
$$=\sqrt[3]{r_1r_2}e^{\frac{\mathrm{i}(\theta_1+\theta_2)}{3}}e^{\mathrm{i}(-\frac{4\pi}{3})}$$

故无穷远点也是支点。

(2)

$$\ln\frac{z-a}{z-b}$$
$$=\ln\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}$$
$$=\ln\frac{r_1}{r_2}+\mathrm{i}(\theta_1-\theta_2)$$

绕 a 转一圈后函数值变为

$$\ln\frac{r_1}{r_2}+\mathrm{i}(\theta_1-\theta_2+2\pi)$$

故 a 为支点。

绕 b 转一圈后函数值变为

$$\ln\frac{r_1}{r_2}+\mathrm{i}(\theta_1-\theta_2-2\pi)$$

故 b 为支点。

绕无穷远点逆时针绕一圈相当于顺时针绕有穷远所有点一圈，故绕无穷远点一圈后函数值变为

$$\ln\frac{r_1}{r_2}+\mathrm{i}(\theta_1-2\pi-\theta_2+2\pi)$$

故无穷远点不是支点。

9

解　记 $z-1=r_1e^{\mathrm{i}\theta_1}, z+1=r_2e^{\mathrm{i}\theta_2}$。在 $z=0$ 处 $\theta_1=\alpha,\theta_2=\beta$

$$\omega(0)=\ln(1)+\mathrm{i}(\alpha+\beta+\pi)$$

有 $\alpha+\beta+\pi=0$

(a) 取该割线时可沿逆时针由底下转到 $z=3$ 此时 $\Delta\theta_1=\pi,\Delta\theta_2=0$ 故在此时

$$\omega(3)=\ln 8+\mathrm{i}(\alpha+\pi+\beta+\pi)=3\ln 2+\mathrm{i}\pi$$

(b) 取该割线时可沿顺时针由上面转到 $z=3$ 此时 $\Delta\theta_1=-\pi,\Delta\theta_2=0$ 故在此时

$$\omega(3)=\ln 8+\mathrm{i}(\alpha-\pi+\beta+\pi)=3\ln 2-\mathrm{i}\pi$$

(c) 取该割线时 a，b 的路径取法都是可行的，故此割线上岸处

$$\omega(3)=3\ln 2-\mathrm{i}\pi$$

下岸处

$$\omega(3)=3\ln 2+\mathrm{i}\pi$$