1

解 (1)

$$
\begin{aligned}
\text{右边} &= b_i\partial_i a_j + a_i\partial_i b_j - \varepsilon_{klm}b_l\varepsilon_{ijk}\partial_i a_j - \varepsilon_{klm}a_l\varepsilon_{ijk}\partial_i b_j \\
&= b_i\partial_i a_j + a_i\partial_i b_j + (\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm})b_l\partial_i a_j + (\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm})a_l\partial_i b_j \\
&= b_i\partial_i a_j + a_i\partial_i b_j + b_j\partial_m a_j - b_i\partial_i a_m + a_j\partial_m b_j - a_i\partial_i b_m \\
&= b_j\partial_m a_j + a_j\partial_m b_j \\
&= \text{左边}
\end{aligned}
$$

(2) 由 (1) 知 $\nabla(\vec{A}\cdot\vec{A}) = 2\vec{A}\cdot\nabla\vec{A} + 2\vec{A}\times(\nabla\times\vec{A})$ 故

$$
\vec{A}\times(\nabla\times\vec{A}) = \frac{1}{2}\nabla(\vec{A}\cdot\vec{A}) - \vec{A}\cdot\nabla\vec{A}
$$

2

解 (1)

$$
\begin{aligned}
\nabla r &= \frac{x-x'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\vec{e}_x + \frac{y-y'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\vec{e}_y \\
&+ \frac{z-z'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\vec{e}_z \\
&= \frac{(x-x')\vec{e}_x+(y-y')\vec{e}_y+(z-z')\vec{e}_z}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\
&= \frac{\vec{r}}{r}
\end{aligned}
$$

$$
\begin{aligned}
\nabla' r &= \frac{x'-x}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\vec{e}_x + \frac{y'-y}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\vec{e}_y \\
&+ \frac{z'-z}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\vec{e}_z \\
&= \frac{(x'-x)\vec{e}_x+(y'-y)\vec{e}_y+(z'-z)\vec{e}_z}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\
&= -\frac{\vec{r}}{r}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\frac{1}{r} =& -\frac{x-x'}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\vec{e}_x - \frac{y-y'}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\vec{e}_y \\
&- \frac{z-z'}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\vec{e}_x\vec{e}_z \\
&= \frac{(x'-x)\vec{e}_x+(y'-y)\vec{e}_y+(z'-z)\vec{e}_z}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}} \\
&= -\frac{\vec{r}}{r^3}
\end{aligned}
$$

$$
\begin{aligned}
\nabla'\frac{1}{r} =& \frac{x-x'}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\vec{e}_x + \frac{y-y'}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\vec{e}_x \\
&+ \frac{z-z'}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\vec{e}_x\vec{e}_z \\
&= -\frac{(x'-x)\vec{e}_x+(y'-y)\vec{e}_y+(z'-z)\vec{e}_z}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}} \\
&= \frac{\vec{r}}{r^3}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\frac{\vec{r}}{r^3} &= (\nabla\frac{1}{r^3})\times\vec{r} + \frac{1}{r^3}(\nabla\times r) \\
&= (\nabla\frac{1}{r^3})\times\vec{r} \\
&= -3\frac{\vec{r}}{r^5}\times r \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\frac{\vec{r}}{r^3} &= (\nabla\frac{1}{r^3})\cdot\vec{r} + \frac{1}{r^3}(\nabla\cdot r) \\
&= (\nabla\frac{1}{r^3})\cdot\vec{r} + \frac{3}{r^3} \\
&= -3\frac{\vec{r}}{r^5}\cdot r + \frac{3}{r^3} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla'\cdot\frac{\vec{r}}{r^3} &= (\nabla'\frac{1}{r^3})\cdot\vec{r} + \frac{1}{r^3}(\nabla'\cdot r) \\
&= (\nabla'\frac{1}{r^3})\cdot\vec{r} - \frac{3}{r^3} \\
&= 3\frac{\vec{r}}{r^5}\cdot r - \frac{3}{r^3} \\
&= 0
\end{aligned}
$$

(2)

$$\nabla \cdot \vec{r} = 3$$

$$\nabla \times \vec{r} = 0$$

$$(\vec{a} \cdot \nabla) \times \vec{r} = 0$$

$$\begin{aligned}
\nabla(\vec{a} \cdot \vec{r}) &= \vec{a} \times (\nabla \times \vec{r}) + (\vec{a} \cdot \nabla)\vec{r} + \vec{r} \times (\nabla \times \vec{a}) + (\vec{r} \cdot \nabla)\vec{a} \\
&= 0 + 0 + 0 + 3\vec{a} \\
&= 3\vec{a}
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot [\vec{E}_0 \sin(\vec{k} \cdot \vec{r})] &= \sin(\vec{k} \cdot \vec{r})(\nabla \cdot \vec{E}_0) + \vec{E}_0 \cdot (\nabla \sin(\vec{k} \cdot \vec{r})) \\
&= 0 + \vec{E}_0 \cdot \vec{k} \cos(\vec{k} \cdot \vec{r}) \\
&= \vec{E}_0 \cdot \vec{k} \cos(\vec{k} \cdot \vec{r})
\end{aligned}$$

$$\begin{aligned}
\nabla \times [\vec{E}_0 \sin(\vec{k} \cdot \vec{r})] &= (\nabla \sin(\vec{k} \cdot \vec{r})) \times \vec{E}_0 + \sin(\vec{k} \cdot \vec{r})(\nabla \times \vec{E}_0) \\
&= \vec{k} \cos(\vec{k} \cdot \vec{r}) \times \vec{E}_0 \\
&= \cos(\vec{k} \cdot \vec{r})\vec{k} \times \vec{E}_0
\end{aligned}$$

3

解 欲证 $\int\limits_V \vec{A}\,\mathrm{d}V = 0$,即证式

$$\vec{c} \cdot \int\limits_V \vec{A}\,\mathrm{d}V = 0$$

对于任意常矢量 $\vec{c}$ 成立,我们构造一个新矢量场 $\vec{F}$,定义为

$$\vec{F} = (\vec{c} \cdot \vec{r})\vec{A}$$

则

$$\nabla \cdot \vec{F} = (\nabla \cdot \vec{A})(\vec{c} \cdot \vec{r}) + \vec{A} \cdot \nabla(\vec{c} \cdot \vec{r})$$

又因为 $\nabla\cdot\vec{A}=0$,故

$$\begin{aligned}\nabla\cdot\vec{F}&=\vec{A}\cdot\nabla(\vec{c}\cdot\vec{r})\\&=\vec{A}\cdot\vec{c}\end{aligned}$$

故

$$\begin{aligned}\vec{c}\cdot\int\limits_V\vec{A}\,\mathrm{d}V&=\int\limits_V\vec{c}\cdot\vec{A}\,\mathrm{d}V\\&=\int\limits_V\nabla\cdot\vec{F}\,\mathrm{d}V\\&=\int\limits_S\vec{F}\cdot\mathrm{d}\vec{S}\\&=\int\limits_S(\vec{c}\cdot\vec{r})\vec{A}\cdot\mathrm{d}\vec{S}\end{aligned}$$

又因为 $\vec{A}\cdot\mathrm{d}\vec{S}=0$,故 $\vec{c}\cdot\int\limits_V\vec{A}\,\mathrm{d}V=0$,又因为此处 $\vec{c}$ 是任取的,故原式得证。

4

解　一维波:

$$\frac{\partial^2u}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2u}{\partial t^2}=0$$

令 $v=\dfrac{\partial u}{\partial t},w=c\dfrac{\partial u}{\partial x}$,则有

$$\begin{aligned}\frac{\partial v}{\partial t}&=c\frac{\partial w}{\partial x}\\c\frac{\partial v}{\partial x}&=\frac{\partial w}{\partial t}\end{aligned}$$

一维麦克斯韦方程可写为

$$\begin{aligned}\frac{\partial E}{\partial x}&=-\frac{\partial B}{\partial t}\\\frac{\partial B}{\partial x}&=-\varepsilon_0\mu_0\frac{\partial E}{\partial t}\end{aligned}$$

异:v,w 代表 u 的时空变化率,而 B,E 表示电磁场强度。
同:数学结构相同,均允许波动解。

5

解

$$|A-\lambda I| = \lambda^4 + \left(-a^2-b^2-c^2+d^2+e^2+f^2\right)\lambda^2 - a^2f^2 - 2abef - 2acdf - b^2e^2 - 2bcde - c^2d^2$$

$$|B-\lambda I| = \lambda^4 + \left(a^2+b^2+c^2+d^2+e^2+f^2\right)\lambda^2 + a^2f^2 + 2abef + 2acdf + b^2e^2 + 2bcde + c^2d^2$$

6

解　(a) 球坐标: 设 $\rho = c\delta(r-a)$, 则有

$$\begin{aligned}\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{R} r^2 c\delta(r-a)\,\mathrm{d}r &= q\\ 4\pi a^2 c &= q\\ c &= \frac{q}{4\pi a^2}\end{aligned}$$

即 $\rho = \dfrac{q}{4\pi a^2}\delta(r-a)$

(b) 柱坐标: 设 $\rho = c\delta(r-a)(0<z<\lambda)$, 则有

$$\begin{aligned}\int_0^{\lambda}\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\int_0^{R} cr\delta(r-a)\,\mathrm{d}r &= q\\ 2\pi\lambda a c &= q\\ c &= \frac{q}{2\pi\lambda a}\end{aligned}$$

即 $\rho = \dfrac{q}{2\pi\lambda a}\delta(r-a)(0<z<\lambda)$

(c) 柱坐标: 设 $\rho = c\delta(z)(0<r<a)$, 则有

$$\begin{aligned}\int_{-\lambda}^{\lambda} c\delta(z)\,\mathrm{d}z\int_0^{2\pi}\mathrm{d}\theta\int_0^{a} r\,\mathrm{d}r &= q\\ \pi a^2 c &= q\\ c &= \frac{q}{\pi a^2}\end{aligned}$$

即 $\rho = \dfrac{q}{\pi a^2}\delta(z)(0<r<a)$

7

解

$$\begin{aligned}I &= \frac{2}{|4-5|} + \frac{3}{|6-5|}\\ &= 5\end{aligned}$$

8

解

$$\begin{aligned} T_{ik}a_ib_k - T_{ik}a_kb_i &= T_{ik}a_ib_k - T_{ki}a_ib_k \\ &= (T_{ik} - T_{ki})a_ib_k \end{aligned}$$

$$\begin{aligned} 2\vec{\omega} \cdot (\vec{a} \times \vec{b}) &= 2\varepsilon_{ijk}\omega_i a_j b_k \\ &= 2\varepsilon_{jik}\omega_j a_i b_k \end{aligned}$$

欲使等式成立即使

$$\begin{aligned} 2\varepsilon_{jik}\omega_j &= T_{ik} - T_{ki} \\ \varepsilon_{jik}\omega_j &= \frac{T_{ik} - T_{ki}}{2} \end{aligned}$$

两边均为反对称矩阵故只需为 ω 选取合适的分量,即可使得等式成立。

9

解 设三条带电直线交点为 $(0,0),(a,0),(b,c)$,选取这三点组成的三角形角平分线的交点为零电势点,则平面上除直线上的任意一点电势可写为

$$\varphi = k \ln \frac{d_1 d_2 d_3}{r^3}$$

式子中的 k 为一与电荷线密度相关的常数,r 为零电势点距这三条直线的距离且为常数。为了方便,我们不妨只考察 $\ln d_1 d_2 d_3$。即

$$\varphi \cong \ln d_1 d_2 d_3 \cong \ln d_1^2 d_2^2 d_3^2 = \ln \left(y^2 \left(y - \frac{cx}{b} \right)^2 \left(\frac{y(b-a)}{c} + a - x \right)^2 \right)$$

则场强为

$$\begin{aligned} \vec{E} &= -\nabla\varphi \\ &= -\frac{2c(a(y-c) - 2by + 2cx)}{(cx - by)(a(y-c) - by + cx)}\vec{e}_x - \frac{a\left(4cy(b+x) - 6by^2 - 2c^2x\right) + 6b^2y^2 - 8bcxy + 2c^2x^2}{y(by - cx)(a(c-y) + by - cx)}\vec{e}_y \end{aligned}$$

令 $\vec{E} = 0$,则解得

$$x = \frac{a+b}{3}, y = \frac{c}{3}$$

即该点位于三线交点所组成的三角形的重心。