1

解

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2}J_2(x)) = -x^{-2}J_3(x)$$

故

$$\begin{aligned}\int J_3(x)\,\mathrm{d}x &= -\int x^2\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2}J_2(x))\,\mathrm{d}x \\ &= -\left[J_2(x) - \int x^{-2}J_2(x)2x\,\mathrm{d}x\right] \\ &= -J_2(x) + 2\int x^{-1}J_2(x)\,\mathrm{d}x \\ &= -J_2(x) - 2\frac{J_1(x)}{x} + C\end{aligned}$$

2

解 定解条件为

$$\begin{cases}\rho^2\Delta u - \dfrac{\partial u}{\partial t} = 0 \\ u|_{r=a} = 0 \\ u|_{r=0}\text{有限} \\ \dfrac{\partial u}{\partial z}|_{z=0} = \dfrac{\partial u}{\partial z}|_{z=h} = 0 \\ u|_{t=0} = u_0\end{cases}$$

分离变量后径向方程为

$$r^2R'' + rR' + \lambda_1 R = 0$$

代入边界条件后可知本征值为

$$k_m^0 = \frac{x_m^0}{a} = \sqrt{\lambda_1}$$

x_m^0 为方程 $J_0(x) = 0$ 的第 m 个非零根
通解为

$$J_0(k_m^0 r)$$

轴向方程为

$$Z'' + \lambda_2 Z = 0$$

本征值 $\sqrt{\lambda_2} = \dfrac{n\pi}{h}$,代入边界条件解为

$$b\cos\frac{n\pi}{h}z$$

时间方程为

$$T' + \rho^2(\lambda_1 + \lambda_2)T = 0$$

解为 $T = e^{-\rho^2(\lambda_1+\lambda_2)t}$ 故

$$u = \sum_{m,n} A_{m,n} J_0(\frac{x_m^0}{a}r) \cos\frac{n\pi}{h} z e^{-\rho^2[(\frac{x_m^0}{a})^2 + (\frac{n\pi}{h})^2]t} \tag{1}$$

代入边界条件 $u|_{t=0} = u_0$ 后得

$$A_{m,n} = \frac{2u_0}{x_m^0 J_1(x_m^0)} \delta_{n,0}$$

故

$$u = \sum_{m,n} = \frac{2u_0}{x_m^0 J_1(x_m^0)} \delta_{n,0} J_0(\frac{x_m^0}{a}r) \cos\frac{n\pi}{h} z e^{-\rho^2[(\frac{x_m^0}{a})^2 + (\frac{n\pi}{h})^2]t}$$

3

解 定解条件为

$$\begin{cases} v^2 \Delta u - \dfrac{\partial^2 u}{\partial t^2} = 0 \\ u|_{t=0} = 0 \\ \dfrac{\partial u}{\partial t}|_{t=0} = \dfrac{I}{\rho r}\delta(r - \dfrac{a}{2})\delta(\theta) \\ u|_{r=a} = 0 \end{cases}$$

分离变量后角向方程为

$$\Theta'' + m^2\Theta = 0$$

径向方程为

$$r^2 R'' + rR' + (k^2 r^2 - m^2)R = 0$$

时间方程为

$$T'' - k^2 v^2 T = 0$$

代入边界条件后可知解为

$$u = \sum_{m,n} (A_m \sin m\theta + B_m \cos m\theta) \sin\frac{x_m^n v}{a} t J_n(\frac{x_m^n}{a} r) \tag{2}$$

x_m^n 为方程 $J_n(x) = 0$ 的第 m 个非零根

代入边界条件 $\dfrac{\partial u}{\partial t}|_{t=0} = \dfrac{I}{\rho r}\delta(r - \dfrac{a}{2})\delta(\theta)$ 后得

$$\sum_{m,n} \frac{x_m^n v}{a} (A_m \sin m\theta + B_m \cos m\theta) J_n(\frac{x_m^n}{a} r) = \frac{I}{\rho r}\delta(r - \frac{a}{2})\delta(\theta)\frac{I}{\rho r}\delta(r - \frac{a}{2})\delta(\theta)$$

故

$$\frac{x_m^n v}{a}(A_m \sin m\theta + B_m \cos m\theta) = \frac{I}{\frac{a^2}{2}J_{n+1}^2(x_m^n)\rho}\int_0^a \frac{1}{r}\delta(r-\frac{a}{2})\delta(\theta)J_n(\frac{x_m^n}{a}r)\,\mathrm{d}r$$

故

$$A_m \sin m\theta + B_m \cos m\theta = \frac{2IJ_n(\frac{x_m^n}{2})I}{a^2 J_{n+1}^2 \rho x_m^n v}\delta(\theta)$$

故

$$A_m = 0$$

$$B_m = \frac{2IJ_n(\frac{x_m^n}{2})}{\pi\rho a x_m^n v J_{n+1}^2(x_m^n)(1+\delta_{m,0})}$$

故

$$u = \sum_{m,n}\frac{2IJ_n(\frac{x_m^n}{2})}{\pi\rho a x_m^n v J_{n+1}^2(x_m^n)(1+\delta_{m,0})}\cos m\theta \sin\frac{x_m^n v}{a}tJ_n(\frac{x_m^n}{a}r)$$

4

解　定解条件为

$$\begin{cases} \Delta u = 0 \\ u|_{z=0} = u|_{z=h} = 0 \\ u|_{r=a} = u_0 \sin\frac{2\pi}{h}z \end{cases}$$

轴向方程为

$$Z'' + \lambda Z = 0$$

径向方程为

$$r^2R'' + rR' + -\lambda R = 0$$

代入边界条件后可知解为

$$u = \sum_m A_m I_0(\frac{m\pi}{h}r)\sin\frac{m\pi}{h}z \tag{3}$$

代入边界条件 $u|_{r=a} = u_0 \sin\frac{2\pi}{h}z$ 后得

$$\sum_m A_m I_0(\frac{m\pi}{h}a)\sin\frac{m\pi}{h}z = u_0 \sin\frac{2\pi}{h}z$$

故

$$A_m = \frac{\delta_{m,2} u_0}{I_0(\frac{2a\pi}{h})}$$

故

$$u = \sum_m \frac{\delta_{m,2} u_0}{I_0(\frac{2a\pi}{h})} I_0(\frac{m\pi}{h} r) \sin \frac{m\pi}{h} z$$