1

解

$$
\begin{aligned}
I=&\frac{(-1)^{m+n}}{(2^l l!)^2}\int_{-1}^{1}(1-x^2)^{\frac{m+n}{2}-1}\frac{\mathrm{d}^{l+m}(1-x^2)^l}{\mathrm{d}x^{l+m}}\frac{\mathrm{d}^{l+n}(1-x^2)^l}{\mathrm{d}x^{l+n}}\,\mathrm{d}x\\
=&\frac{(-1)^{m+n}}{(2^l l!)^2}(1-x^2)^{\frac{m+n}{2}-1}\frac{\mathrm{d}^{l+m}(1-x^2)^l}{\mathrm{d}x^{l+m}}\frac{\mathrm{d}^{l+n}(1-x^2)^l}{\mathrm{d}x^{l+n}}|_{-1}^{1}\\
-&\frac{(-1)^{m+n}}{(2^l l!)^2}\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^{\frac{m+n}{2}-1}\frac{\mathrm{d}^{l+m}(1-x^2)^l}{\mathrm{d}x^{l+m}}\right]\frac{\mathrm{d}^{l+n-1}(1-x^2)^l}{\mathrm{d}x^{l+n-1}}\,\mathrm{d}x\\
=&\frac{(-1)^{m+n-1}}{(2^l l!)^2}\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^{\frac{m+n}{2}-1}\frac{\mathrm{d}^{l+m}(1-x^2)^l}{\mathrm{d}x^{l+m}}\right]\frac{\mathrm{d}^{l+n-1}(1-x^2)^l}{\mathrm{d}x^{l+n-1}}\,\mathrm{d}x\\
&\overset{\text{再进行}n-1\text{次分部积分}}{=}\frac{(-1)^{m-1}}{(2^l l!)^2}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left[(1-x^2)^{\frac{m+n}{2}-1}\frac{\mathrm{d}^{l+m}(1-x^2)^l}{\mathrm{d}x^{l+m}}\right]\frac{\mathrm{d}^{l}(1-x^2)^l}{\mathrm{d}x^{l}}|_{-1}^{1}\\
-&\frac{(-1)^{m-1}}{(2^l l!)^2}\int_{-1}^{1}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x^2)^{\frac{m+n}{2}-1}\frac{\mathrm{d}^{l+m}(1-x^2)^l}{\mathrm{d}x^{l+m}}\right]\frac{\mathrm{d}^{l}(1-x^2)^l}{\mathrm{d}x^{l}}\,\mathrm{d}x
\end{aligned}
$$

现在考察减号后面一项

$$
\begin{aligned}
&\frac{(-1)^{m-1}}{(2^l l!)^2}\int_{-1}^{1}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x^2)^{\frac{m+n}{2}-1}\frac{\mathrm{d}^{l+m}(1-x^2)^l}{\mathrm{d}x^{l+m}}\right]\frac{\mathrm{d}^{l}(1-x^2)^l}{\mathrm{d}x^{l}}\,\mathrm{d}x\\
=&\frac{(-1)^{m-1}}{2^l l!}\int_{-1}^{1}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x^2)^{\frac{m+n}{2}-1}\frac{\mathrm{d}^{l+m}(1-x^2)^l}{\mathrm{d}x^{l+m}}\right]P_l(x)\,\mathrm{d}x
\end{aligned}
$$

$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x^2)^{\frac{m+n}{2}-1}\frac{\mathrm{d}^{l+m}(1-x^2)^l}{\mathrm{d}x^{l+m}}\right]$ 的次数为 $2l-(l+m)+2(\frac{m+n}{2}-1)-n=l-2<l$，故该积分值为 0。减号前面一项可写为

$$
\frac{1}{(2^l l!)^2}\delta_{mn}\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\left[(x^2-1)^{m-1}\frac{\mathrm{d}^{l+m}(x^2-1)^l}{\mathrm{d}x^{l+m}}\right]\frac{\mathrm{d}^{l}(x^2-1)^l}{\mathrm{d}x^{l}}\ |_{-1}^{1}
$$

该函数为奇函数，故只需考虑其在 1 处取值

$$
\begin{aligned}
&\frac{\mathrm{d}^l(x^2-1)^l}{\mathrm{d}x^l}|_{x=1}\\
=&\frac{\mathrm{d}^l(x-1)^l(x+1)^l}{\mathrm{d}x^l}|_{x=1}\\
=&\frac{\mathrm{d}^l(x-1)^l}{\mathrm{d}x^l}(x+1)^l|_{x=1}\\
=&l!2^l
\end{aligned}
$$

$$
\frac{\mathrm{d}^{l+m}(x^2-1)^l}{\mathrm{d}x^{l+m}}
$$

$$
\begin{aligned}
=&\frac{(l+m)!}{m!l!}[(x-1)^l]^{(l)}[(1+x)^l]^{(m)} \\
=&\frac{(l+m)!}{m!l!}l!\frac{l!}{(l-m)!}(x+1)^{l-m} \\
=&\frac{(l+m)!l!}{m!(l-m)!}(x+1)^{l-m}
\end{aligned}
$$

故

$$
\begin{aligned}
&\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\left[(x^2-1)^{m-1}\frac{\mathrm{d}^{l+m}(x^2-1)^l}{\mathrm{d}x^{l+m}}\right]\frac{\mathrm{d}^l(x^2-1)^l}{\mathrm{d}x^l}\ |_{-1}^{1} \\
=&2\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\left[(x^2-1)^{m-1}\frac{\mathrm{d}^{l+m}(x^2-1)^l}{\mathrm{d}x^{l+m}}\right]\frac{\mathrm{d}^l(x^2-1)^l}{\mathrm{d}x^l}\ |_{x=1} \\
=&\frac{2(l+m)!l!}{m!(l-m)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\left[(x-1)^{m-1}(x+1)^{l-1}\right]|_{x=1} \\
=&\frac{2(l+m)!l!}{m!(l-m)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}\left[(x-1)^{m-1}\right](x+1)^{l-1}\ |_{x=1} \\
=&\frac{2(l+m)!l!}{m!(l-m)!}(m-1)!2^{l-1}
\end{aligned}
$$

故

$$
\begin{aligned}
I &= \frac{1}{(2^l l!)^2}\delta_{mn}\frac{2(l+m)!l!}{m!(l-m)!}(m-1)!2^{l-1}l!2^l \\
&= \delta_{mn}\frac{(l+m)!}{m(l-m)!}
\end{aligned}
$$

2

解

$$
\begin{aligned}
&\sin^2\theta\cos^2\phi-1 \\
&=\sin^2\theta\left(\frac{e^{\mathrm{i}\phi}+e^{-\mathrm{i}\phi}}{2}\right)^2-1 \\
&=\frac{\sin^2\theta}{4}\left(e^{\mathrm{i}2\phi}+e^{-\mathrm{i}2\phi}\right)+\frac{1}{2}\sin^2\theta-1 \\
&=\frac{4\sqrt{\pi}}{3}Y_{0,0}-\frac{2}{3}\sqrt{\frac{\pi}{5}}Y_{2,0}(\theta,\phi)+\sqrt{\frac{2\pi}{15}}\left[Y_{2,2}(\theta,\phi)+Y_{2,-2}(\theta,\phi)\right]
\end{aligned}
$$

3

解 定解条件为

$$
\begin{cases}
\Delta u = 0 \\
u|_{r=R} = -\sin^2\theta\cos^2\phi+\dfrac{1}{3} \\
u|_{r=0}\text{有限} \\
u|_{r=\infty}\text{有限}
\end{cases}
$$

设 $u=R(r)\Theta(\theta)\Phi(\phi)$,则可得到通解

$$u=\sum_{l,m}(a_l r^l+b_l r^{-l-1})c_{l,m}Y_{l,m}(\theta,\phi)$$

$r<R$ 时,由于 $u|_{r=0}$有限 故 $b_l=0$,有

$$u=\sum_{l,m}C_{l,m}r^l Y_{l,m}(\theta,\phi)$$

代入边界条件 $u|_{r=R}=-\sin^2\theta\cos^2\phi+\dfrac{1}{3}$ 得

$$\begin{aligned}\sum_{l,m}C_{l,m}r^l Y_{l,m}(\theta,\phi)&=-\sin^2\theta\cos^2\phi+\frac{1}{3}\\&=-\sqrt{\frac{2\pi}{15}}\left[Y_{2,2}(\theta,\phi)+Y_{2,-2}(\theta,\phi)\right]+\frac{2}{3}\sqrt{\frac{\pi}{5}}Y_{2,0}(\theta,\phi)\end{aligned}$$

故

$$u=\left(-\sqrt{\frac{2\pi}{15}}\left[Y_{2,2}(\theta,\phi)+Y_{2,-2}(\theta,\phi)\right]+\frac{2}{3}\sqrt{\frac{\pi}{5}}Y_{2,0}(\theta,\phi)\right)\frac{r^2}{R^2}$$

$r>R$ 时,由于 $u|_{r=\infty}$有限 故 $a_l=0$,有

$$u=\sum_{l,m}C_{l,m}r^{-l-1}Y_{l,m}(\theta,\phi)$$

代入边界条件 $u|_{r=R}=-\sin^2\theta\cos^2\phi+\dfrac{1}{3}$ 得

$$\begin{aligned}\sum_{l,m}C_{l,m}r^{-l-1}Y_{l,m}(\theta,\phi)&=-\sin^2\theta\cos^2\phi+\frac{1}{3}\\&=-\sqrt{\frac{2\pi}{15}}\left[Y_{2,2}(\theta,\phi)+Y_{2,-2}(\theta,\phi)\right]+\frac{2}{3}\sqrt{\frac{\pi}{5}}Y_{2,0}(\theta,\phi)\end{aligned}$$

故

$$u=\left(-\sqrt{\frac{2\pi}{15}}\left[Y_{2,2}(\theta,\phi)+Y_{2,-2}(\theta,\phi)\right]+\frac{2}{3}\sqrt{\frac{\pi}{5}}Y_{2,0}(\theta,\phi)\right)\frac{r^{-3}}{R^{-3}}$$

4

证明

$$e^{\frac{x+y}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}J_n(x+y)t^n=\sum_{n=-\infty}^{\infty}J_n(x)t^n\sum_{n=-\infty}^{\infty}J_n(y)t^n$$

考察 t^n 的系数即可得到

$$J_n(x+y)=\sum_{k=-\infty}^{\infty}J_k(x)J_{n-k}(y)$$

$$e^{\frac{x}{2}(t-\frac{1}{t})}e^{\frac{x}{2}(\frac{1}{t})-t}=1$$

故

$$\sum_{n=-\infty}^{\infty}J_n(x)t^n\sum_{n=-\infty}^{\infty}J_n(x)t^{-n}=1$$

RHS 与 t 无关,故 LHS 中 t 的次数不为 0 的项和为 0,因此有

$$\sum_{n=-\infty}^{\infty}J_n^2(x)=1$$

又因为 $J_{-n}(x)=(-1)^nJ_n(x)$,故

$$J_0^2(x)+2\sum_{k=1}^{\infty}J_k^2(x)=\sum_{n=-\infty}^{\infty}J_n^2(x)=1$$

■

5

证明 (1)

$$\begin{aligned}\cos x+\mathrm{i}\sin x=e^{ix}&\\ &=\sum_{n=-\infty}^{\infty}J_n(x)\mathrm{i}^n\\ &=\sum_{n=-\infty}^{-1}J_n(x)\mathrm{i}^n+J_0(x)+\sum_{n=1}^{\infty}J_n(x)\mathrm{i}^n\\ &=\sum_{n=1}^{\infty}J_{-n}(x)\mathrm{i}^{-n}+J_0(x)+\sum_{n=1}^{\infty}J_n(x)\mathrm{i}^n\\ &=\sum_{n=1}^{\infty}\left[(-1)^n\frac{1}{\mathrm{i}^n}+\mathrm{i}^n\right]J_n(x)+J_0(x)\\ &=J_0(x)+2\sum_{k=1}^{\infty}J_{2k}(x)\mathrm{i}^{2k}+2\sum_{k=0}^{\infty}J_{2k+1}(x)\mathrm{i}^{2k+1}\end{aligned}$$

对比实部与虚部可得

$$\cos x=J_0(x)+2\sum_{k=1}^{\infty}(-1)^kJ_{2k}(x)$$

$$\sin x = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(x)$$

(2)

$$\begin{aligned}\cos(x\sin\theta) + \mathrm{i}\sin(x\sin\theta) = e^{\mathrm{i}x\sin\theta} \\ &= \sum_{n=-\infty}^{\infty} J_n(x)e^{\mathrm{i}n\theta} \\ &= \sum_{n=-\infty}^{\infty} J_n(x)\left(\cos n\theta + \mathrm{i}\sin n\theta\right)\end{aligned}$$

对比实部与虚部可得

$$\sin(x\sin\theta) = 2\sum_{m=0}^{\infty} J_{2m+1}(x)\sin(2m+1)\theta$$

等式左右同时对 θ 求导得到

$$x\cos\theta\cos(x\sin\theta) = 2\sum_{m=0}^{\infty} J_{2m+1}(x)(2m+1)\cos(2m+1)\theta$$

令 $\theta = 0$,则

$$x = 2\sum_{m=0}^{\infty}(2m+1)J_{2m+1}(x)$$

■