1

解　设球外电势

$$\varphi = \frac{Q_f}{4\pi\varepsilon R} + \varphi'$$

其中 φ' 是导体球表面电荷产生的电势,故在球外满足拉普拉斯方程

$$\nabla^2\varphi' = 0$$

解得

$$\varphi' = \sum_{l=0}^{\infty} A_l r^{-l-1} + B_0$$

因为无穷远处电势为 0 ,故 $B_0 = 0$。代入球体表面边界条件有

$$(\frac{Q_f}{4\pi\varepsilon R} + \varphi')|_{r=R_0} = 0$$

$$\frac{Q_f}{4\pi\varepsilon a}\sum_{l=0}^{\infty}(\frac{R_0}{a})^l P_l(\cos\theta) + \sum_{l=0}^{\infty} A_l R_0^{-l-1} = 0$$

对比 $P_l(\cos\theta)$ 系数得

$$A_l = -\frac{Q_f R_0^{2l+1}}{4\pi\varepsilon a^{l+1}}$$

像电荷为 $Q' = -\dfrac{R_0 Q_f}{a}$,距球心 $\dfrac{R_0^2}{a}$。故电像法得到的电势为

$$\begin{aligned}\varphi &= \frac{Q_f}{4\pi\varepsilon}(\frac{R_0}{aR_1} + \frac{1}{R_2}) \\ &= \frac{Q_f}{4\pi\varepsilon}(\frac{R_0}{a\sqrt{r^2 + (\frac{R_0^2}{a})^2 - 2r\frac{R_0^2}{a}\cos\theta}} + \frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}})\end{aligned}$$

展开为勒让德级数后与分离变量法得到的结果是一致的。

2

解　像电荷大小为 $-\dfrac{R_1}{a}Q$,距球心 $\dfrac{R_1^2}{a}$。故电势为

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{QR_1}{a\sqrt{r^2 + \frac{R_1^4}{a^2} - 2\frac{2R_1^2 r}{a}\cos\theta}}\right]$$

由于导体球接地,故外壳电势为 0,由唯一性定理可知球外无电场,故球壳外表面电荷为 0. 作一个半径在 R_1,R_2 之间的高斯球面可易知,感应电荷只分布在球壳内表面,且总量为 $-Q$。

3

解 球壳内部边界条件与上题相同，故内部电场强度一致，即内部电势至多相差一常数。

当球壳有电势 φ_0 时，易知球壳内部电势为

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{QR_1}{a\sqrt{r^2+\frac{R_1^4}{a^2}-2\frac{2R_1^2 r}{a}\cos\theta}}\right]+\varphi_0$$

当球壳带电 Q_0 时，易知球壳电势为 $\dfrac{Q+Q_0}{4\pi\varepsilon_0 R_2}$，故球内电势为

$$\varphi' = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{QR_1}{a\sqrt{r^2+\frac{R_1^4}{a^2}-2\frac{2R_1^2 r}{a}\cos\theta}}\right]+\frac{Q+Q_0}{4\pi\varepsilon_0 R_2}$$

欲使

$$\varphi' = \varphi$$

即

$$\varphi_0 = \frac{Q+Q_0}{4\pi\varepsilon_0 R_2}$$

4

解 本题有三个像电荷，分别是 Q 与球面形成的像电荷 q_1，Q 与平面形成的像电荷 q_2,q_1 与平面形成的像电荷 q_3。故

$$q_1 = -\frac{a}{b}Q$$
$$q_2 = -Q$$
$$q_3 = \frac{a}{b}Q$$

以球心为原点，平面为 $x-y$ 平面建立直角坐标系可以得到空间中电势为

$$\varphi = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-b)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+b)^2}} - \frac{a}{b\sqrt{x^2+y^2+\left(z-\frac{a^2}{b}\right)^2}} + \frac{a}{b\sqrt{x^2+y^2+\left(z+\frac{a^2}{b}\right)^2}}\right)$$

5

解　将两电极以一个小球面包围起来，故球面的电通量为

$$\begin{aligned}\iint_S \vec{E}\cdot \mathrm{d}\vec{S} &= \iint_S \frac{\vec{j}}{\sigma}\cdot \mathrm{d}\vec{S}\\ &= \frac{1}{\sigma}\iint_S \vec{j}\cdot \mathrm{d}\vec{S}\\ &= \frac{I}{\sigma}\end{aligned}$$

又由唯一性定理知道，这等效于一个电荷量为 Q 的电荷，且满足

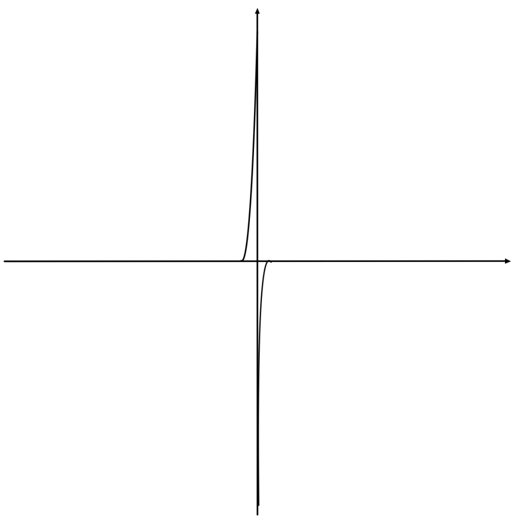
$$\begin{aligned}\frac{I}{\sigma} &= \frac{Q}{\varepsilon_0}\\ Q &= \frac{I\varepsilon_0}{\sigma}\end{aligned}$$

故会产生六个像电荷，空间中电势由这八个电荷叠加而成，故

$$\begin{aligned}\varphi = \frac{I}{4\pi\sigma}\Bigg(&\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\\ &+ \frac{1}{\sqrt{(x+x_0)^2+(y-y_0)^2+(z+z_0)^2}} - \frac{1}{\sqrt{(x+x_0)^2+(y-y_0)^2+(z-z_0)^2}}\\ &+ \frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2+(z+z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2+(y+y_0)^2+(z-z_0)^2}}\\ &+ \frac{1}{\sqrt{(x+x_0)^2+(y+y_0)^2+(z-z_0)^2}} - \frac{1}{\sqrt{(x+x_0)^2+(y+y_0)^2+(z+z_0)^2}}\Bigg)\end{aligned}$$

6

解



不妨将 $\vec{p}$ 的方向取为沿 x 轴正方向,则电偶极子的电荷密度可写为

$$\begin{aligned}\rho &= q\delta(x-\frac{l}{2}) - q\delta(x+\frac{l}{2}) \\ &= -ql\frac{\delta(x+\frac{l}{2})-\delta(x-\frac{l}{2})}{l}\end{aligned}$$

取 $l \to 0$ 的极限,则

$$\rho = -ql\frac{\mathrm{d}\delta(x)}{\mathrm{d}x}$$

故

$$\rho = -(p\cdot\nabla)\delta(x)$$

7

解　定解条件为

$$\begin{cases}\nabla^2\varphi_{in} = 0 \\ \nabla^2\varphi_{out} = 0 \\ \lim\limits_{r\to\infty}\varphi_{out} = 0 \\ \lim\limits_{[} r \to 0]\varphi_{in}\text{有限}\end{cases}$$

故解得

$$\varphi_{out} = \sum_{l=0}^{\infty} A_l r^{-l-1} P_l(\cos\theta)$$

$$\varphi_{in} = \sum_{l=1}^{\infty} B_l r^{l} P_l(\cos\theta)$$

代入球体表面边界条件有

$$\sum_{l=0}^{\infty} A_l R_0^{-l-1} P_l(\cos\theta) = \begin{cases}\varphi_0(0<\theta<\frac{\pi}{2}) \\ -\varphi_0(\frac{\pi}{2}<\theta<\pi)\end{cases}$$

$$\sum_{l=0}^{\infty} B_l R_0^{l} P_l(\cos\theta) = \begin{cases}\varphi_0(0<\theta<\frac{\pi}{2}) \\ -\varphi_0(\frac{\pi}{2}<\theta<\pi)\end{cases}$$

解得

$$A_l = \varphi_0 R_0^{l+1}[P_{l-1}(0) - P_{l+1}(0)]$$

$$B_l = \frac{\varphi_0[P_{l-1}(0) - P_{l+1}(0)]}{R_0^l}$$

7

解　格林函数可取为

$$G(r,\theta,\phi,r',\theta',\phi')=\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}}-\frac{1}{\sqrt{(\frac{rr'}{R_0})^2+R_0^2-2rr'\cos(\theta-\theta')}}\right]$$

故

$$\varphi=\varepsilon_0\int\limits_{r'=R_0}\varphi(R_0,\theta,\phi)\frac{\partial G}{\partial n}\,\mathrm{d}\vec{S}$$

可记 $\dfrac{r}{R_0}=R$,故最后结果为

$$\begin{aligned}\varphi&=\begin{cases}-\dfrac{\varphi_0}{2}\left(\displaystyle\int_0^{\frac{\pi}{2}}-\int_{\frac{\pi}{2}}^{\pi}\right)\dfrac{R^2-1}{[1+R^2-2R\cos(\theta-\theta')]^{\frac{3}{2}}}\sin\theta'\,\mathrm{d}\theta' & R>1\\ \dfrac{\varphi_0}{2}\left(\displaystyle\int_0^{\frac{\pi}{2}}-\int_{\frac{\pi}{2}}^{\pi}\right)\dfrac{R^2-1}{[1+R^2-2R\cos(\theta-\theta')]^{\frac{3}{2}}}\sin\theta'\,\mathrm{d}\theta' & R<1\end{cases}\\&=-\frac{\varphi_0}{2}\left(\int_0^{\frac{\pi}{2}}-\int_{\frac{\pi}{2}}^{\pi}\right)\frac{|R^2-1|}{[1+R^2-2R\cos(\theta-\theta')]^{\frac{3}{2}}}\sin\theta'\,\mathrm{d}\theta'\end{aligned}$$

这是一个椭圆积分,用勒让德级数展开后与上题结果一致。