1

解

$$\begin{aligned}\psi(z) &= \frac{\Gamma'(z)}{\Gamma(z)} \\ &= \frac{\int_0^\infty -\ln t e^{-t} t^{1-z}\,\mathrm{d}t}{\int_0^\infty e^{-t} t^{1-z}\,\mathrm{d}t}\end{aligned}$$

故

$$\begin{aligned}\psi(1) &= \frac{-\int_0^\infty e^{-t}\ln t\,\mathrm{d}t}{\int_0^\infty e^{-t}\,\mathrm{d}t} \\ &= -\int_0^\infty e^{-t}\ln t\,\mathrm{d}t \\ &= -\gamma\end{aligned}$$

2

解

$$\begin{aligned}\psi(z+1) &= \psi(z) + \frac{1}{z} \\ \psi^{(m)}(z+1) &= \psi^{(m)}(z) + \frac{(-1)^m m!}{z^{m+1}}\end{aligned}$$

3

证明

$$\begin{aligned}B(a,b)B(a+b,c) &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\frac{\Gamma(a+b)\Gamma(c)}{\Gamma(a+b+c)} \\ &= \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)} \\ &= \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}\frac{\Gamma(a)\Gamma(b+c)}{\Gamma(a+b+c)} \\ &= B(b,c)B(a,b+c)\end{aligned}$$

■

4

解

$$\begin{aligned}\int_0^1 (1-x^a)^b\,\mathrm{d}x &= \frac{1}{a}\int_0^1 x^{a(\frac{1}{a}-1)}(1-x^a)^b\,\mathrm{d}x^a \\ &= \frac{1}{a}B(\frac{1}{a}, b+1)\end{aligned}$$

因为 $(1-x^2)^n$ 是偶函数故

$$I = 2\int_0^1 (1-x^2)^n\,\mathrm{d}x$$

$$
\begin{aligned}
&= B(\frac{1}{2}, n+1) \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(\frac{1}{2}+n+1)} \\
&= \frac{\sqrt{\pi}n!}{\frac{\sqrt{\pi}(2n+1)!!}{2^{n+1}}} \\
&= \frac{2(2n)!!}{(2n+1)!!}
\end{aligned}
$$

5

解 (1)

$$I = e^{-4}$$

(2)

$$
\begin{aligned}
I &= \int_{-4}^{7} \delta'(t)\cos(t-1)\,\mathrm{d}t \\
&= \int_{-4}^{7} \cos(t-1)\,\mathrm{d}\delta(t) \\
&= \cos(t-1)\delta(t)|_{-4}^{7} + \int_{-4}^{7} \sin(t-1)\delta(t)\,\mathrm{d}t \\
&= \sin(-1)
\end{aligned}
$$

6

解 (1) 因为 $\sin x = 0$ 时 $|\cos x| = 1$ 故

$$
\begin{aligned}
I &= \sum_{n=0}^{\infty} e^{-n\pi} \\
&= \frac{1}{1-e^{-\pi}}
\end{aligned}
$$

(2)

$$
\begin{aligned}
I &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \frac{r^2\cos^2\theta\delta(r^2-1)}{r^2+1} r\,\mathrm{d}r \\
&= \int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta \int_0^{\infty} \frac{r^2\delta(r^2-1)}{r^2+1} r\,\mathrm{d}r \\
&= \frac{\pi}{2}\int_0^{\infty} \frac{t\delta(t-1)}{t+1}\,\mathrm{d}t \\
&= \frac{\pi}{4}
\end{aligned}
$$