1

解　设 $u = R(r)T(t)$,有

$$T' - m^2 T = 0$$

代入边界条件后可解得

$$T = A_m e^{m^2 t}$$

关于 R 的方程为

$$rR'' + R' + \frac{\beta - m^2}{\alpha} rR = 0$$

代入边界条件后可解得

$$R = j_0\left(\sqrt{\frac{\beta - m^2}{\alpha}} r\right)$$

代入 $R(a) = 0$ 可得

$$\sqrt{\frac{\beta - m^2}{\alpha}} = \frac{x_m^{\frac{1}{2}}}{a} \to a = \frac{x_m^{\frac{1}{2}}\sqrt{\alpha}}{\sqrt{\beta - m^2}}$$

又因为 $j_0(x) = \dfrac{\sin x}{x}$,故 $x_m^{\frac{1}{2}} = n\pi$,故

$$a = \frac{n\pi\sqrt{\alpha}}{\sqrt{\beta - m^2}}$$

又因为 $m^2 \geq 0$,故

$$a \geq \frac{\pi\sqrt{\alpha}}{\sqrt{\beta - m^2}} \geq \frac{\pi\sqrt{\alpha}}{\sqrt{\beta}}$$

2

解　记 $\hat{x}(p) = \mathcal{L}(x(t)), \hat{f}(p) = \mathcal{L}(f(t))$ 对方程两边进行拉普拉斯变换后得

$$p^2\hat{x} - px(0) - x'(0) + 2\gamma[p\hat{x} - x(0)] + \omega_0^2\hat{x} = \hat{f}$$

$$(p^2 + 2\gamma p + \omega_0^2)\hat{x} - p\phi - \psi - 2\gamma\phi = \hat{f}$$

$$\hat{x} = \frac{\hat{f} + p\phi + \psi + 2\gamma\phi}{p^2 + 2\gamma p + \omega_0^2}$$

$$\hat{x} = \frac{\hat{f} + p\phi + \psi + 2\gamma\phi}{(p+\gamma)^2 + \omega_0^2 - \gamma^2}$$

$$\hat{x} = \mathcal{L}\left(\frac{e^{-\gamma t}\sin\sqrt{\omega^2 - \gamma^2}t}{\sqrt{\omega^2 - \gamma^2}}\right)(\hat{f} + p\phi + \psi + 2\gamma\phi)$$

故

$$\mathcal{L}(x(t))=\mathcal{L}\left(\frac{e^{-\gamma t}\sin\sqrt{\omega^2-\gamma^2}t}{\sqrt{\omega^2-\gamma^2}}*f(t)\right)+\phi\mathcal{L}\left((\frac{e^{-\gamma t}\sin\sqrt{\omega^2-\gamma^2}t}{\sqrt{\omega^2-\gamma^2}})'\right)+(\psi+2\gamma\phi)\mathcal{L}(\frac{e^{-\gamma t}\sin\sqrt{\omega^2-\gamma^2}t}{\sqrt{\omega^2-\gamma^2}})$$

$$\begin{aligned}x(t)=&\int_0^t f(\tau)\frac{e^{-\gamma(t-\tau)}\sin\sqrt{\omega^2-\gamma^2}(t-\tau)}{\sqrt{\omega^2-\gamma^2}}\,\mathrm{d}\tau+\phi\frac{e^{\gamma(-t)}\cos\left(t\sqrt{\omega_0^2-\gamma^2}\right)}{\sqrt{\omega_0^2-\gamma^2}}\\&-\frac{\phi\gamma e^{\gamma(-t)}\sin\left(t\sqrt{\omega_0^2-\gamma^2}\right)}{\omega_0^2-\gamma^2}+(\psi+2\gamma\phi)\frac{e^{-\gamma t}\sin\sqrt{\omega^2-\gamma^2}t}{\sqrt{\omega^2-\gamma^2}}\end{aligned}$$

3

解 记 $\tilde{u}(\omega,t)=\mathcal{F}(u(x,t))$ 对方程两边进行傅里叶变换后得

$$\begin{aligned}\omega\frac{\partial\tilde{u}}{\partial t}&=-\omega^2\tilde{u}\\\frac{\partial\tilde{u}}{\partial t}&=\mathrm{i}\omega\tilde{u}\\\tilde{u}&=\phi(\omega)e^{\mathrm{i}\omega t}\end{aligned}$$

又

$$\begin{aligned}\tilde{u}(\omega,0)&=\mathcal{F}(e^{-|x|})\\&=\frac{2}{\omega^2+1}\end{aligned}$$

故

$$\begin{aligned}\tilde{u}&=\frac{2}{\omega^2+1}e^{\mathrm{i}\omega t}\\&=\mathcal{F}(e^{-|x+t|})\end{aligned}$$

故

$$u=e^{-|x+t|}$$

4

解 记 $\hat{u}(x,p)=\mathcal{L}(u(x,t)),\hat{f}(p)=\mathcal{L}(f(t))$ 对方程两边进行拉普拉斯变换后得

$$\begin{aligned}p^2\hat{u}-pu(x,0)-\frac{\partial u}{\partial t}(x,0)-a^2\frac{\partial^2\hat{u}}{\partial x^2}&=\hat{f}\\p^2\hat{u}-a^2\frac{\partial^2\hat{u}}{\partial x^2}&=\hat{f}\end{aligned}$$

$u(\infty, p)$ 有界,故

$$\hat{u} = \phi(p)e^{-\frac{p}{a}x} + \frac{\hat{f}}{p^2}$$

又

$$\hat{u}(0,p) = \phi(p) + \frac{\hat{f}}{p^2} = 0 \to \phi(p) = -\frac{\hat{f}}{p^2}$$

故

$$\hat{u} = -\frac{\hat{f}}{p^2}e^{-\frac{p}{a}x} + \frac{\hat{f}}{p^2}$$

故

$$\begin{aligned} u &= \int_0^t \int_0^\tau f(x)\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^{t-\frac{x}{a}} \int_0^\tau f(x)\,\mathrm{d}x\,\mathrm{d}\tau \\ &= \int_{t-\frac{x}{a}}^t \int_0^\tau f(x)\,\mathrm{d}x\,\mathrm{d}\tau \end{aligned}$$