1

解　该区域的格林函数可以取为

$$G(\vec{r},\vec{r}')=\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}-\frac{1}{\sqrt{(x-x')^2+(y+y')^2+(z-z')^2}}\right)$$

又该空间没有电荷分布，故

$$\begin{aligned}
\varphi&=-\iint\limits_S V\frac{\partial G}{\partial y'}\,\mathrm{d}x'\,\mathrm{d}z'\\
&=\frac{V}{4\pi\varepsilon_0}\iint\limits_S\frac{y-y'}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}+\frac{y+y'}{((x-x')^2+(y+y')^2+(z-z')^2)^{3/2}}\,\mathrm{d}x'\,\mathrm{d}z'\\
&=\frac{V}{4\pi\varepsilon_0}\iint\limits_S\frac{y}{((x-x')^2+y^2+(z-z')^2)^{3/2}}+\frac{y}{((x-x')^2+y^2+(z-z')^2)^{3/2}}\,\mathrm{d}x'\,\mathrm{d}z'\\
&=\frac{V}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\mathrm{d}z'\int_{-a}^{a}\frac{y}{((x-x')^2+y^2+(z-z')^2)^{3/2}}+\frac{y}{((x-x')^2+y^2+(z-z')^2)^{3/2}}\,\mathrm{d}x'\\
&=\frac{V}{4\pi\varepsilon_0}4\left[\arctan\left(\frac{a-x}{y}\right)+\arctan\left(\frac{a+x}{y}\right)\right]\\
&=\frac{V}{\pi\varepsilon_0}\left[\arctan\left(\frac{a-x}{y}\right)+\arctan\left(\frac{a+x}{y}\right)\right]
\end{aligned}$$

故

$$\begin{aligned}
\vec{E}&=\nabla\varphi\\
&=\frac{V}{\pi\varepsilon_0}\left[\frac{1}{y\left(\frac{(a+x)^2}{y^2}+1\right)}-\frac{1}{y\left(\frac{(a-x)^2}{y^2}+1\right)}\right]\vec{e}_x\\
&+\frac{V}{\pi\varepsilon_0}\left[-\frac{a-x}{y^2\left(\frac{(a-x)^2}{y^2}+1\right)}-\frac{a+x}{y^2\left(\frac{(a+x)^2}{y^2}+1\right)}\right]\vec{e}_y
\end{aligned}$$

2

解　取轴向为 z 轴，由对称性有

$$D_{xx}=D_{yy}$$
$$D_{xz}=D_{yz}$$

$$\begin{aligned}
D_{xx}&=\iiint r\left(3r^2\cos^2\theta-r^2\right)\frac{q}{2\pi a}\delta(r-a)\delta(z)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z\\
&=-\frac{a^2q}{2\pi}
\end{aligned}$$

$$
\begin{aligned}
D_{xy} &= \iiint r(3r^2\cos\theta\sin\theta)\frac{q}{2\pi a}\delta(r-a)\delta(z)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
D_{xz} &= \iiint r(3r\cos\theta z)\frac{q}{2\pi a}\delta(r-a)\delta(z)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= 0
\end{aligned}
$$

故

$$
\begin{aligned}
&D_{xx} = D_{yy} = -\frac{a^2q}{2\pi} \\
&D_{zz} = -(D_{xx}+D_{yy}) = \frac{a^2q}{\pi} \\
&\text{其余元素均为 } 0
\end{aligned}
$$

其在远处产生的电势为

$$
\begin{aligned}
\varphi &= \frac{1}{24\pi\varepsilon_0}\left(D_{xx}\frac{\partial^2}{\partial x^2}\frac{1}{r}+D_{yy}\frac{\partial^2}{\partial x^2}\frac{1}{r}+D_{zz}\frac{\partial^2}{\partial x^2}\frac{1}{r}\right) \\
&= \frac{-a^2q}{48\pi^2\varepsilon_0}\frac{3\left(x^2+y^2-2z^2\right)}{\left(x^2+y^2+z^2\right)^{5/2}}
\end{aligned}
$$

3

解 取轴向为 z 轴,由对称性有

$$
\begin{aligned}
D_{xx} &= D_{yy} \\
D_{xz} &= D_{yz}
\end{aligned}
$$

$$
\begin{aligned}
D_{xx} &= \iiint r\left(3r^2\cos^2\theta - r^2\right)\frac{q}{\pi a^2}\delta(z)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= -\frac{a^2q}{4\pi}
\end{aligned}
$$

$$
\begin{aligned}
D_{xy} &= \iiint r(3r^2\cos\theta\sin\theta)\frac{q}{\pi a^2}\delta(z)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
D_{xz} &= \iiint r(3r\cos\theta z)\frac{q}{\pi a^2}\delta(z)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= 0
\end{aligned}
$$

故

$$D_{xx} = D_{yy} = -\frac{a^2 q}{4\pi}$$
$$D_{zz} = -(D_{xx} + D_{yy}) = \frac{a^2 q}{2\pi}$$

其余元素均为 0

其在远处产生的电势为

$$\begin{aligned}\varphi &= \frac{1}{24\pi\varepsilon_0}\left(D_{xx}\frac{\partial^2}{\partial x^2}\frac{1}{r} + D_{yy}\frac{\partial^2}{\partial x^2}\frac{1}{r} + D_{zz}\frac{\partial^2}{\partial x^2}\frac{1}{r}\right)\\ &= \frac{-a^2 q}{96\pi^2\varepsilon_0}\frac{3\left(x^2+y^2-2z^2\right)}{\left(x^2+y^2+z^2\right)^{5/2}}\end{aligned}$$

4

解 由对称性知 $\vec{A}$ 只与 r 有关,取 $\vec{A} = A(r)\vec{e}_z$ 在柱坐标下求解定解条件为

$$\nabla^2 A_{in} = -\mu_0 j$$
$$\nabla^2 A_{out} = 0$$
$$A_{in}|_{r=a} = A_{out}|_{r=a}$$
$$A_{in}|_{r=0}\text{有限}$$
$$\frac{1}{\mu_0}\nabla\times\vec{A}_{in} = \frac{1}{\mu}\nabla\times\vec{A}_{out}$$

故可得解系为

$$A_{in} = -\frac{1}{4}\mu j r^2 + C_1\ln r + C_2$$
$$A_{out} = C_3\ln r + C_4$$

由 $A_{in}|_{r=0}$有限 得 $C_1 = 0$, 由 $\frac{1}{\mu_0}\nabla\times\vec{A}_{in} = \frac{1}{\mu}\nabla\times\vec{A}_{out}$ 得 $C_3 = -\frac{\mu j a^2}{2}$。又 $A_{in}|_{r=a} = A_{out}|_{r=a}$ 故 $C_2 = \frac{\mu_0 j a^2}{4}$, $C_4 = \frac{\mu j a^2\ln a}{2}$。故

$$\vec{A}_{in} = \left(-\frac{1}{4}\mu j r^2 + \frac{\mu_0 j a^2}{4}\right)\vec{e}_z$$
$$\vec{A}_{out} = \left(-\frac{\mu j a^2}{2}\ln r + \frac{\mu j a^2\ln a}{2}\right)\vec{e}_z$$

5

解 取 $\vec{H}_0$ 方向为轴向建立球坐标得定解条件

$$\nabla^2\varphi_1 = 0$$

$$\nabla^2\varphi_2 = 0$$

$$\varphi_1|_{r=R_0} = \varphi_2|_{r=R_0}$$

$$\mu\frac{\partial\varphi_1}{\partial r}|_{r=R_0} = \mu_0\frac{\partial\varphi_2}{\partial r}|_{r=R_0}$$

$$\varphi_1|_{r=0}\text{有限}$$

$$\varphi_2|_{r\to\infty} = -H_0 r\cos\theta(\text{将未放球体之前的原点记为零点})$$

解得

$$\varphi_1 = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

$$\varphi_2 = -H_0 r\cos\theta + \sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta)$$

代入 $\varphi_1|_{r=R_0} = \varphi_2|_{r=R_0}$ 得

$$\sum_{l=0}^{\infty} a_l R_0^l P_l(\cos\theta) = -H_0 R_0\cos\theta + \sum_{l=0}^{\infty} b_l R_0^{-l-1} P_l(\cos\theta)$$

代入 $\mu\frac{\partial\varphi_1}{\partial r}|_{r=R_0} = \mu_0\frac{\partial\varphi_2}{\partial r}|_{r=R_0}$ 得

$$\mu\sum_{l=0}^{\infty} l a_l R_0^{l-1} P_l(\cos\theta) = \mu_0(-H_0\cos\theta + \sum_{l=0}^{\infty}(-l-1) b_l R_0^{-l-1} P_l(\cos\theta))$$

对比 $P_l(\cos\theta)$ 系数有

$$a_1 = -\frac{3\mu_0 H_0}{\mu + 2\mu_0}$$

$$b_1 = \frac{\mu-\mu_0}{\mu+2\mu_0} H_0 R_0^3$$

$$a_l = b_l = 0(l\neq 1)$$

故

$$\varphi_1 = -\frac{3\mu_0}{\mu+2\mu_0} H_0 r\cos\theta$$

$$\varphi_2 = -H_0 r\cos\theta + \frac{\mu-\mu_0}{\mu+2\mu_0}\frac{R_0^3 H_0}{r^2}\cos\theta$$

故

$$\vec{B}_1 = -\mu\nabla\varphi_1$$

$$= \frac{3\mu\mu_0}{\mu+2\mu_0}\vec{H}_0$$

$$\begin{aligned}\vec{B}_2 &= -\mu_0\nabla\varphi_2\\ &= \mu_0\vec{H}_0 + \frac{\mu-\mu_0}{\mu+2\mu_0}\mu_0 R_0^3\left[\frac{3(\vec{H}_0\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{H}_0}{r^3}\right]\end{aligned}$$

φ_2 中的第二项 $\frac{\mu-\mu_0}{\mu+2\mu_0}\frac{R_0^3H_0}{r^2}\cos\theta$ 可视为一磁偶极子产生的势故

$$\frac{\vec{m}\cdot\vec{r}}{4\pi r^3} = \frac{\mu-\mu_0}{\mu+2\mu_0}\frac{R_0^3H_0}{r^2}\cos\theta$$

$$\vec{m} = 4\pi\frac{\mu-\mu_0}{\mu+2\mu_0}R_0^3\vec{H}_0$$

6

解　以 $\vec{H}_0$ 为极轴建立球坐标系，由对称性知，磁标势与 ϕ 无关则定解条件为

$$\nabla^2\varphi_1 = 0$$

$$\nabla^2\varphi_2 = 0$$

$$\nabla^2\varphi_3 = 0$$

$$\varphi_1|_{r=R_1} = \varphi_2|_{r=R_1}$$

$$\mu_0\frac{\partial\varphi_1}{\partial r}|_{r=R_1} = \mu\frac{\partial\varphi_2}{\partial r}|_{r=R_1}$$

$$\varphi_2|_{r=R_2} = \varphi_3|_{r=R_2}$$

$$\mu\frac{\partial\varphi_2}{\partial r}|_{r=R_2R_2} = \mu_0\frac{\partial\varphi_3}{\partial r}|_{r=R_2}$$

$$\varphi_1|_{r\to 0}\text{有限}$$

$$\varphi_3|_{r\to\infty} = -H_0 r\cos\theta(\text{已将未放入时的原点取为势零点})$$

解得

$$\varphi_1 = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

$$\varphi_2 = \sum_{l=0}^{\infty}(c_l r^l + d^l r^{-l-1})P_l(\cos\theta)$$

$$\varphi_3 = \sum_{l=0}^{\infty} b_l r^{-l-1}P_l(\cos\theta) - H_0 r\cos\theta$$

代入边界条件有

$$\sum_{l=0}^{\infty} a_l R_1^l P_l(\cos\theta) = \sum_{l=0}^{\infty} (c_l R_1^l + d_l R_1^{-l-1}) P_l(\cos\theta)$$

$$\mu_0 \sum_{l=0}^{\infty} l a_l R_1^{l-1} P_l(\cos\theta) = \mu \sum_{l=0}^{\infty} (l c_l R_1^{l-1} - (l+1) d_l R_1^{-l-2}) P_l(\cos\theta)$$

$$\sum_{l=0}^{\infty} (c_l R_2^l + d_l R_2^{-l-1}) P_l(\cos\theta) = \sum_{l=0}^{\infty} b_l R_2^{-l-1} P_l(\cos\theta) - H_0 R_2 \cos\theta$$

$$\mu \sum_{l=0}^{\infty} (l c_l R_2^{l-1} - (l+1) d_l R_2^{-l-2}) P_l(\cos\theta) = -\mu_0 \sum_{l=0}^{infty} (l+1) b_l R_2^{-l-2} P_l(\cos\theta) - \mu_0 H_0 \cos\theta$$

对比 $P_l(\cos\theta)$ 系数可得

$$a_1 R_1 = b_1 R_1 + \frac{c_1}{R_1^2}$$

$$\mu_0 a_1 = \mu(b_1 - \frac{2c_1}{R_1^3})$$

$$b_1 R_2 + \frac{c_1}{R_2^2} = \frac{d_1}{R_2^2} - H_0 R_2$$

$$\mu \left(b_1 - \frac{2c_1}{R_2^3} \right) = \mu_0 \left(\frac{-2d_1}{R_2^3} - H_0 \right)$$

解得

$$a_1 = \frac{-H_0}{\frac{2(\mu-\mu_0)^2}{9\mu\mu_0} \left[\frac{(\mu+2\mu_0)(2\mu+\mu_0)}{2(\mu-\mu_0)^2} - \left(\frac{R_1}{R_2} \right)^2 \right]}$$

故

$$\begin{aligned}
\varphi_1 &= a_1 r \cos\theta \\
\vec{B}_1 &= -\mu_0 \nabla \varphi_1 \\
&= -a_1 \mu_0 \nabla (r \cos\theta) \\
&= -a_1 \mu_0 \vec{e}_z \\
&= \frac{\mu_0 H_0 \vec{e}_z}{\frac{2(\mu-\mu_0)^2}{9\mu\mu_0} \left[\frac{(\mu+2\mu_0)(2\mu+\mu_0)}{2(\mu-\mu_0)^2} - \left(\frac{R_1}{R_2} \right)^2 \right]}
\end{aligned}$$

当 $\mu \ll \mu_0$ 时,$\vec{B}_1 \to 0$,接近电场中的导体屏蔽作用。

7

解 以 $\vec{M}_0$ 方向为轴向建立球坐标得定解条件

$$\nabla^2 \varphi_1 = 0$$

$$\nabla^2\varphi_2 = 0$$

$$\varphi_1|_{r=R_0} = \varphi_2|_{r=R_0}$$

$$-\mu\frac{\partial\varphi_1}{\partial r} + \mu_0 M_0\cos\theta = -\mu'\frac{\partial\varphi_2}{\partial r}$$

$$\varphi_1|_{r\to 0}\text{有限}$$

$$\varphi_2|_{r\to\infty} = 0$$

通解为

$$\varphi_1 = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

$$\varphi_2 = \sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta)$$

代入边界条件得

$$\sum_{l=0}^{\infty} a_l R_0^l P_l(\cos\theta) = \sum_{l=0}^{\infty} b_l R_0^{-l-1} P_l(\cos\theta)$$

$$-\mu\sum_{l=0}^{\infty} l a_l R_0^{l-1} P_l(\cos\theta) + \mu_0 M_0\cos\theta = \mu'\sum_{l=0}^{\infty}(l+1) b_l R_0^{-l-2} P_l(\cos\theta)$$

解得

$$a_1 = \frac{\mu_0 M_0}{2\mu' + \mu}$$

$$b_1 = \frac{\mu_0 M_0}{2\mu' + \mu} R_0^3$$

$$a_l = b_l = 0 (l \neq 0)$$

$$\varphi_1 = \frac{\mu_0 M_0}{2\mu' + \mu} r\cos\theta$$

$$\varphi_2 = \frac{\mu_0 M_0 R_0^3}{(2\mu' + \mu) r^2}\cos\theta$$

故

$$\vec{B}_1 = -\mu\nabla\varphi_1 + \mu_0\vec{M}_0$$

$$= \frac{2\mu'\mu_0}{2\mu' + \mu}\vec{M}_0$$

$$\vec{B}_2 = -\mu'\nabla\varphi_2$$

$$
\begin{aligned}
&= \frac{\mu' \mu_0 R_0^3}{2\mu' + \mu} \left[\frac{3(\vec{M}_0 \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{M}_0}{r^3} \right] \\
\vec{\alpha}_M &= \frac{\vec{n} \times (\vec{B}_2 - \vec{B}_1)}{\mu_0}|_{r=R_0} - \vec{\alpha} \\
&= -\frac{3\mu'}{2\mu' + \mu_0} M_0 \sin\theta \vec{e}_\phi
\end{aligned}
$$