1

解　电场强度可写为

$$E = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}$$

其法向导数为

$$\begin{aligned}\frac{\partial E}{\partial r} &= \frac{q}{4\pi\varepsilon_0}\frac{-2}{r^3}\\ &= \frac{-2E}{r}\end{aligned}$$

故

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\frac{2}{R}$$

2

解　设该电偶极子在直角坐标下的电偶极矩为 $\vec{p} = (ql, 0, 0)$,则产生的电势为

$$\begin{aligned}\varphi &= \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}\\ &= \frac{qlx}{4\pi\varepsilon_0(x^2+y^2+z^2)^{\frac{3}{2}}}\end{aligned}$$

故电场强度为

$$\begin{aligned}\vec{E} &= -\nabla\varphi\\ &= [\frac{3lqx^2}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{5/2}} - \frac{lq}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{3/2}}]\vec{e}_x +\\ &\frac{3lqxy}{(x^2+y^2+z^2)^{5/2}}\vec{e}_y + (3lqxz)/(x^2+y^2+z^2)^(5/2)\vec{e}_z\end{aligned}$$

故

$$\begin{aligned}\frac{\partial E_x}{\partial y} &= \frac{3lqy}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{5/2}} - \frac{15lqx^2y}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{7/2}}\\ \frac{\partial E_x}{\partial z} &= \frac{3lqz}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{5/2}} - \frac{15lqx^2z}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{7/2}}\\ \frac{\partial E_y}{\partial x} &= \frac{3lqy}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{5/2}} - \frac{15lqx^2y}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{7/2}}\\ \frac{\partial E_y}{\partial z} &= -\frac{15lqxyz}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{7/2}}\\ \frac{\partial E_z}{\partial x} &= \frac{3lqy}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{5/2}} - \frac{15lqx^2y}{4\pi\varepsilon_0\,(x^2+y^2+z^2)^{7/2}}\end{aligned}$$

$$\frac{\partial E_z}{\partial y} = -\frac{15lqxyz}{4\pi\varepsilon_0\left(x^2+y^2+z^2\right)^{7/2}}$$

因为

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}$$
$$\frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$
$$\frac{\partial E_y}{\partial z} = \frac{\partial E_z}{\partial y}$$

故

$$\nabla\times\vec{E} = 0$$

3

解 设该电荷位于坐标原点,则其产生的电势为

$$\begin{aligned}\varphi &= \frac{q}{4\pi\varepsilon_0 r}\\ &= \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+z^2}}\end{aligned}$$

考察其穿过平面 $z=a(-a<x<a,-a<y<a)$ 的电通量,只需计算电场的 z 方向分量

$$\begin{aligned}E_z &= \frac{\partial\varphi}{\partial z}\\ &= -\frac{q}{4\pi\varepsilon_0}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\end{aligned}$$

故其其穿过平面 $z=a(-a<x<a,-a<y<a)$ 的电通量为

$$\begin{aligned}I &= \iint\frac{q}{4\pi\varepsilon_0}\frac{a}{\left(x^2+y^2+a^2\right)^{3/2}}\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{aq}{4\pi\varepsilon_0}\int_{-a}^{a}\mathrm{d}y\int_{-a}^{a}\frac{1}{\left(x^2+y^2+a^2\right)^{3/2}}\,\mathrm{d}x\\ &= \frac{aq}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{2a}{\left(a^2+y^2\right)\sqrt{2a^2+y^2}}\,\mathrm{d}y\\ &= \frac{aq}{4\pi\varepsilon_0}\frac{2\pi}{3a}\\ &= \frac{q}{6\varepsilon_0}\end{aligned}$$

4

解 由对称性知,该点电场方向必为径向,故只需计算球上各点在该点产生电场的径向分量,取圆环微元该微元带电量为

$$q = 2\pi R^2\sigma\sin\theta\,\mathrm{d}\theta$$

其在考察点产生的电场强度为

$$\begin{aligned}
&= \frac{\mathrm{d}q}{4\pi\varepsilon_0}\frac{R - R\cos\theta}{[R^2\sin^2\theta + (R - R\cos\theta)^2]^{\frac{3}{2}}} \\
&= \frac{2\pi R^2\sigma\sin\theta\,\mathrm{d}\theta}{4\pi\varepsilon_0}\frac{R - R\cos\theta}{[R^2\sin^2\theta + (R - R\cos\theta)^2]^{\frac{3}{2}}}
\end{aligned}$$

故

$$\begin{aligned}
E &= \int \mathrm{d}E \\
&= \int_{\theta_0}^{\pi}\frac{2\pi R^2\sigma\sin\theta\,\mathrm{d}\theta}{4\pi\varepsilon_0}\frac{R - R\cos\theta}{[R^2\sin^2\theta + (R - R\cos\theta)^2]^{\frac{3}{2}}} \\
&= \frac{\sigma}{2\varepsilon_0}\int_{\theta_0}^{\pi}\frac{\sin\theta(1-\cos\theta)\,\mathrm{d}\theta}{[\sin^2\theta + (1-\cos\theta)^2]^{\frac{3}{2}}} \\
&= \frac{\sigma}{2\varepsilon_0}(1 - \sin\frac{\theta_0}{2})
\end{aligned}$$

故该点电场强度为

$$E = \frac{\sigma}{2\varepsilon_0}(1 - \sin\frac{\theta_0}{2})$$

方向沿径向