1

解

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\vec{x}'\,\mathrm{d}V' \\ &= \int_V \frac{\mathrm{d}\rho\vec{x}'}{\mathrm{d}t}\,\mathrm{d}V' \\ &= \int_V (\frac{\partial\rho\vec{x}'}{\partial t} + \frac{\partial\rho\vec{x}'}{\partial\vec{x}'}\frac{\mathrm{d}\vec{x}'}{\mathrm{d}t})\,\mathrm{d}V' \\ &= \int_V \frac{\partial\rho}{\partial t}\vec{x}'\,\mathrm{d}V' \\ &= -\int_V \nabla'\cdot\vec{j}\vec{x}'\,\mathrm{d}V'\end{aligned}$$

又因为

$$\begin{aligned}\nabla'\cdot(\vec{x}'\vec{j}) &= \partial_i x'_j j_i \\ &= x'_j\partial_i j_i + j_i\partial_i x'_j \\ &= x'_j\partial_i j_i + j_i\delta_{ij} \\ &= x'_j\partial_i j_i + j_j \\ &= (\nabla'\cdot\vec{j})\vec{x}' + \vec{j}\end{aligned}$$

故

$$\begin{aligned}-\int_V \nabla'\cdot\vec{j}\vec{x}'\,\mathrm{d}V' &= \int_V \vec{j}\,\mathrm{d}V' - \int_V \nabla'\cdot(\vec{x}'\vec{j})\,\mathrm{d}V' \\ &= \int_V \vec{j}\,\mathrm{d}V' - \int_{\partial V} \vec{x}'\vec{j}\cdot\mathrm{d}\vec{S}\end{aligned}$$

又体系电荷守恒，故在 ∂V 上 $\vec{j}\cdot\mathrm{d}\vec{S}=0$，故

$$\nabla'\cdot(\vec{x}'\vec{j}) = \int_V \vec{j}\,\mathrm{d}V'$$

2

解

$$\begin{aligned}\nabla\times\vec{A} &= \nabla(\frac{1}{r^3})\times(\vec{m}\times\vec{r}) + \frac{1}{r^3}\nabla\times(\vec{m}\times\vec{r}) \\ &= \frac{-3\vec{r}}{r^5}\times(\vec{m}\times\vec{r}) + \frac{1}{r^3}[(\vec{r}\cdot\nabla)\vec{m} + (\nabla\cdot\vec{r})\vec{m} - (\vec{m}\cdot\nabla)\vec{r} - (\nabla\cdot\vec{m})\vec{r}] \\ &= \frac{-3[r^2\vec{m} - (\vec{r}\cdot\vec{m})\vec{r}]}{r^5} + \frac{2\vec{m}}{r^3} \\ &= \frac{-3r^2\vec{m}}{r^3} + \frac{3(\vec{r}\cdot\vec{m})\vec{r}}{r^5} + \frac{2\vec{m}}{r^3}\end{aligned}$$

$$= \frac{3(\vec{r}\cdot\vec{m})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}$$

$$\begin{aligned}
\nabla\varphi &= \frac{1}{r^3}\nabla(\vec{m}\cdot\vec{r}) + (\vec{m}\cdot\vec{r})\nabla\frac{1}{r^3} \\
&= \frac{1}{r^3}[\vec{m}\times(\nabla\times\vec{r}) + (\vec{m}\cdot\nabla)\vec{r} + \vec{r}\times(\nabla\times\vec{m}) + (\vec{r}\cdot\nabla)\vec{m}] - \frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} \\
&= \frac{\vec{m}}{r^3} - \frac{3(\vec{r}\cdot\vec{m})\vec{r}}{r^5}
\end{aligned}$$

故

$$\nabla\times\vec{A} = -\nabla\varphi$$

4

解

$$\begin{aligned}
\vec{F}_{12} &= \frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1I_2\,\mathrm{d}\vec{l}_1\times(\mathrm{d}\vec{l}_2\times\hat{\mathbf{r}}_{12})}{r_{12}^2} \\
&= \frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1I_2[(\mathrm{d}\vec{l}_1\cdot\hat{\mathbf{r}}_{12})\,\mathrm{d}\vec{l}_2 - (\mathrm{d}\vec{l}_1\cdot\mathrm{d}\vec{l}_2)\hat{\mathbf{r}}_{12}]}{r_{12}^2} \\
&= -\frac{\mu_0}{4\pi}\oint_{(L_1)}\oint_{(L_2)}\frac{I_1I_2(\mathrm{d}\vec{l}_1\cdot\mathrm{d}\vec{l}_2)\hat{\mathbf{r}}_{12}}{r_{12}^2}
\end{aligned}$$

又因为被积函数连续,故积分可交换顺序,即

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi}\oint_{(L_2)}\oint_{(L_1)}\frac{I_1I_2(\mathrm{d}\vec{l}_1\cdot\mathrm{d}\vec{l}_2)\hat{\mathbf{r}}_{12}}{r_{12}^2}$$

同理

$$\vec{F}_{21} = -\frac{\mu_0}{4\pi}\oint_{(L_2)}\oint_{(L_1)}\frac{I_1I_2(\mathrm{d}\vec{l}_1\cdot\mathrm{d}\vec{l}_2)\hat{\mathbf{r}}_{21}}{r_{21}^2}$$

又因为 $\hat{\mathbf{r}}_{21} = -\hat{\mathbf{r}}_{12}$, 故

$$\vec{F}_{12} = -\vec{F}_{21}$$

3

解　设抛物线方程为 $y = ax^2(z=0)$,则其焦点为 $(0, \dfrac{1}{4a}, 0)$。在其上一点 $(x, ax^2, 0)$ 的电流元为 $I\,\mathrm{d}\vec{l} = (I\,\mathrm{d}x, I2ax\,\mathrm{d}x, 0)$。故其焦点处的磁感应强度为

$$\vec{B} = \int\frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\vec{l}\times\vec{r}}{r^3}$$

$$
\begin{aligned}
&= -\int_{-\infty}^{\infty} \frac{\mu_0 I}{4\pi} \frac{ax^2 + \frac{1}{4a}}{\sqrt{x^2 + (ax^2 - \frac{1}{4a})^2}^3} \,\mathrm{d}x \vec{e}_z \\
&= -\frac{\mu_0 I}{4\pi} 4a\pi \vec{e}_z \\
&= -a\mu_0 I \vec{e}_z
\end{aligned}
$$