1

证明 (1)

$$\begin{aligned}\text{原式} &= \varepsilon_{ijk}a_ib_j\varepsilon_{klm}c_ld_m \\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})a_ib_jc_ld_m \\ &= \delta_{il}\delta_{jm}a_ib_jc_ld_m-\delta_{im}\delta_{jl}a_ib_jc_ld_m \\ &= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})\end{aligned}$$

(2)

$$\begin{aligned}\text{原式} &= \varepsilon_{krm}\varepsilon_{ijk}a_ib_j\varepsilon_{pqr}c_pd_q \\ &= -\varepsilon_{pqr}\varepsilon_{rkm}\varepsilon_{ijk}a_ib_jc_pd_q \\ &= (\delta_{pm}\delta_{qk}-\delta_{pk}\delta_{qm})\varepsilon_{ijk}a_ib_jc_pd_q \\ &= \delta_{pm}\delta_{qk}\varepsilon_{ijk}a_ib_jc_pd_q-\delta_{pk}\delta_{qm}\varepsilon_{ijk}a_ib_jc_pd_q \\ &= \varepsilon_{ijk}a_ib_jc_md_k-\varepsilon_{ijk}a_ib_jc_kd_m \\ &= [\vec{A}\cdot(\vec{B}\times\vec{D})]\vec{C}-[\vec{A}\cdot(\vec{B}\times\vec{C})]\vec{D}\end{aligned}$$

(3)

$$\begin{aligned}\text{右边} &= b_i\partial_ia_j+a_i\partial_ib_j-\varepsilon_{klm}b_l\varepsilon_{ijk}\partial_ia_j-\varepsilon_{klm}a_l\varepsilon_{ijk}\partial_ib_j \\ &= b_i\partial_ia_j+a_i\partial_ib_j+(\delta_{im}\delta_{jl}-\delta_{il}\delta_{jm})b_l\partial_ia_j+(\delta_{im}\delta_{jl}-\delta_{il}\delta_{jm})a_l\partial_ib_j \\ &= b_i\partial_ia_j+a_i\partial_ib_j+b_j\partial_ma_j-b_i\partial_ia_m+a_j\partial_mb_j-a_i\partial_ib_m \\ &= b_j\partial_ma_j+a_j\partial_mb_j \\ &= \text{左边}\end{aligned}$$

(4) 由 (3) 知 $\nabla(\vec{A}\cdot\vec{A})=2\vec{A}\cdot\nabla\vec{A}+2\vec{A}\times(\nabla\times\vec{A})$ 故

$$\vec{A}\times(\nabla\times\vec{A})=\frac{1}{2}\nabla(\vec{A}\cdot\vec{A})-\vec{A}\cdot\nabla\vec{A}$$

(5)

$$\begin{aligned}\text{左边} &= \nabla(\vec{r}\cdot(\vec{A}\times\vec{B})) \\ &= \nabla\vec{r}\cdot(\vec{A}\times\vec{B})+\vec{r}\cdot\nabla(\vec{A}\times\vec{B}) \\ &= \nabla\vec{r}\cdot(\vec{A}\times\vec{B}) \\ &= \delta_{ij}\cdot(\vec{A}\times\vec{B})\end{aligned}$$

$$= \vec{A} \times \vec{B}$$

(6)

$$\begin{aligned}
\text{左边} &= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) + \vec{C}(\vec{B}\cdot\vec{A}) - \vec{A}(\vec{B}\cdot\vec{C}) + \vec{A}(\vec{C}\cdot\vec{B}) - \vec{B}(\vec{C}\cdot\vec{A}) \\
&= 0
\end{aligned}$$

■

2

解 (1)

$$\begin{aligned}
\mathrm{d}x &= a\sinh u\cos v\,\mathrm{d}u - a\cosh u\sin v\,\mathrm{d}v \\
\mathrm{d}y &= a\cosh u\sin v\,\mathrm{d}u + a\sinh u\cos v\,\mathrm{d}v \\
\mathrm{d}z &= \mathrm{d}z
\end{aligned}$$

故

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\
&= \frac{a^2}{2}(\cosh 2u - \cos 2v)\,\mathrm{d}u^2 + \frac{a^2}{2}(\cosh 2u - \cos 2v)\,\mathrm{d}v^2 + \mathrm{d}z^2
\end{aligned}$$

没有交叉项，故是正交曲面。(2)

$$\begin{aligned}
h_1 &= \sqrt{(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2 + (\frac{\partial z}{\partial u})^2} \\
&= \sqrt{a^2(\cosh^2 u\sin^2 v + \cos^2 v\sinh^2 u)} \\
h_2 &= \sqrt{(\frac{\partial x}{\partial v})^2 + (\frac{\partial y}{\partial v})^2 + (\frac{\partial z}{\partial v})^2} \\
&= \sqrt{a^2(\cosh^2 u\sin^2 v + \cos^2 v\sinh^2 u)} \\
h_3 &= \sqrt{(\frac{\partial x}{\partial z})^2 + (\frac{\partial y}{\partial z})^2 + (\frac{\partial z}{\partial z})^2} \\
&= 1
\end{aligned}$$

故

$$\nabla^2 = \frac{1}{a^2(\cosh^2 u\sin^2 v + \cos^2 v\sinh^2 u)}[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \frac{\partial}{\partial z}(a^2(\cosh^2 u\sin^2 v + \cos^2 v\sinh^2 u)\frac{\partial}{\partial z})]$$

3

解

$$\frac{\mathrm{d}T}{\mathrm{d}x} = -\rho\omega^2 x$$

$$T(l) = 0$$

故

$$T = \frac{1}{2}\rho\omega^2(l^2 - x^2)$$

又

$$(T\frac{\partial u}{\partial x})|_{x+\Delta x} - (T\frac{\partial u}{\partial x})|_x = \rho\Delta x\overline{\frac{\partial^2 u}{\partial t^2}}$$

取 $\Delta x \to 0$ 有

$$\frac{\omega^2}{2}\frac{\partial}{\partial x}((l^2 - x^2)\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial t^2}$$

4

解

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0(0 < x < l)$$

$$u|_{x=0} = 0$$

$$u_t|_{x=l} = \frac{q}{k}$$

$$u|_{t=0} = x(l - x)$$

$$\begin{cases} \frac{d}{d\tau}\left(g_{\mathrm{tt}}\frac{dt}{d\tau} + g_{t\phi}\frac{d\phi}{d\tau}\right) & = 0, \\ \frac{d}{d\tau}\left(g_{\phi\phi}\frac{dt}{d\tau} + g_{\phi t}\frac{d\phi}{d\tau}\right) & = 0, \\ \frac{d}{d\tau}\left(g_{\mathrm{rr}}\frac{dr}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial r}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} & = 0, \\ \frac{d}{d\tau}\left(g_{\theta\theta}\frac{d\theta}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial \theta}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} & = 0. \end{cases} \tag{1}$$