1

解 (1) 在积分回路中有 $2n$ 个奇点且均为一阶极点，故

$$\begin{aligned}I &= 4n\pi\mathrm{i}\,\mathrm{res}(\tan\pi z,\frac{\pi}{2})\\&= 4n\pi\mathrm{i}\lim_{z\to\frac{\pi}{2}}(z-\frac{\pi}{2})\frac{\sin\pi z}{\cos\pi z}\\&= 4n\pi\mathrm{i}(-\frac{1}{\pi})\\&= -4n\mathrm{i}\end{aligned}$$

(2) 令 $z=e^{\mathrm{i}x}$，则 $\mathrm{d}x=-\mathrm{i}\frac{\mathrm{d}z}{z}$，$\cos x=\frac{z^2+1}{2z}$。

$$\begin{aligned}I &= \oint_{|z|=1}\frac{-\mathrm{i}\,\mathrm{d}z}{[a+\frac{b(z^2+1)}{2z}]^2 z}\\&= -\mathrm{i}\oint_{|z|=1}\frac{4z\,\mathrm{d}z}{b^2(z+\frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1})^2(z+\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1})^2}\end{aligned}$$

记 $z_1=-\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}$，$z_1=-\frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1}$。因为 $|z_2|>1>|z_1|$，故

$$\begin{aligned}I &= 2\pi\mathrm{i}\,\mathrm{res}(\frac{4z}{b^2(z+\frac{a}{b}-\sqrt{\frac{a^2}{b^2}-1})^2(z+\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1})^2},z_1)\\&= \frac{2a\pi}{(a^2-b^2)^{\frac{3}{2}}}\end{aligned}$$

(3)

$$\lim_{z\to\infty} z\frac{1}{(z^2+a^2)(z^2+b^2)}=0$$

故

$$\begin{aligned}I &= 2\pi\mathrm{i}[\mathrm{res}(\frac{1}{(z^2+a^2)(z^2+b^2)},ai)+\mathrm{res}(\frac{1}{(z^2+a^2)(z^2+b^2)},-ai)\\&+\mathrm{res}(\frac{1}{(z^2+a^2)(z^2+b^2)},bi)+\mathrm{res}(\frac{1}{(z^2+a^2)(z^2+b^2)},-bi)]\\&= \frac{\pi}{ab(a+b)}\end{aligned}$$

(4)

$$I=\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}mx}}{(x+a)^2+b^2}\,\mathrm{d}x$$

因为 $\lim\limits_{x\to\infty}\dfrac{1}{(x+a)^2+b^2}=0$ 故

$$\begin{aligned}I&=2\pi\mathrm{i}\mathrm{res}(\frac{e^{\mathrm{i}mx}}{(x+a)^2+b^2},-a+|b|\mathrm{i})\\&=\frac{e^{(b-a\mathrm{i})}\pi}{b}\end{aligned}$$

(4)

$$I=\frac{1}{2\mathrm{i}}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}mx}}{x(x^2+a^2)}\,\mathrm{d}x$$

因为 $\lim\limits_{x\to\infty}\dfrac{1}{x(x^2+a^2)}=0$ 故

$$\begin{aligned}I&=\frac{1}{2\mathrm{i}}[2\pi\mathrm{i}\mathrm{res}(\frac{e^{\mathrm{i}mx}}{x(x^2+a^2)},a\mathrm{i})+2\pi\mathrm{i}\mathrm{res}(\frac{1}{x(x^2+a^2)},0)]\\&=\frac{(1-e^{-am})\pi}{2a^2}\end{aligned}$$

2

解 在 $|z-\mathrm{i}|<1$ 时

$$\begin{aligned}f_1(z)&=\frac{\frac{1}{\mathrm{i}}}{1-\mathrm{i}(z-\mathrm{i})}\\&=\frac{1}{z}\end{aligned}$$

$$f_2(z)=\int_0^{\infty}e^{-zt}\,\mathrm{d}t=\frac{1}{z}=f_1(z)$$

故 $f_1(z)$ 和 $f_2(z)$ 互为解析延拓。

3

证明

$$\begin{aligned}&|\Gamma(x+iy)|\\=&|\int_0^{\infty}e^{-t}t^{1-x-\mathrm{i}y}\,\mathrm{d}t|\\\leq&\int_0^{\infty}e^{-t}t^{1-x}|t^{-\mathrm{i}y}|\,\mathrm{d}t\end{aligned}$$

$$|t^{-\mathrm{i}y}| = |e^{\mathrm{i}(-y\ln t)}| = 1$$

故

$$|\Gamma(x+iy)| \le \int_0^\infty e^{-t}t^{1-x}\,\mathrm{d}t = \Gamma(x)$$

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