Probabilistic Deep Learning: Theoretical Part Assignment 1

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The distribution of the sum of two independent random variables is the convolution of their individual distributions. We start by defining the probability density function of $\mathbf{y}, p_{\mathbf{y}}(\mathbf{y})$, as the probability that $\mathbf{u}+\mathbf{v}=\mathbf{y}$. Since $\mathbf{u}$ and $\mathbf{v}$ are independent, the joint probability density function is the product of the individual density functions.

For a given value of $\mathbf{y}, \mathbf{u}$ can take any value and $\mathbf{v}$ must take the value $\mathbf{y}-\mathbf{u}$ to satisfy $\mathbf{u}+\mathbf{v}=\mathbf{y}$. We express this as an integral over all possible values of $\mathbf{u}$:

$$p_{\mathbf{y}}(\mathbf{y}) = \int p_{\mathbf{u}}(\mathbf{u}) p_{\mathbf{v}}(\mathbf{y}-\mathbf{u}) \mathrm{d}\mathbf{u}.$$

This integral is the convolution of $p_{\mathbf{u}}(\mathbf{u})$ and $p_{\mathbf{v}}(\mathbf{v})$, and it gives the probability density function of $\mathbf{y}$, the sum of the two independent random variables $\mathbf{u}$ and $\mathbf{v}$.

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Since x and z are statistically independent, we can use the properties of expectation and variance to show the desired results. First, let's show that the expectation of the sum is the sum of the expectations:

$$\mathbf{E}[x+z] = \mathbf{E}[x] + \mathbf{E}[z]$$

Next, we will show that the variance of the sum is the sum of the variances:

$$\mathrm{var}[x+z] = \mathrm{var}[x] + \mathrm{var}[z].$$

Using the definition of variance and the independence of x and z

$$\mathrm{Var}(x) = \mathbf{E}\left[x^2\right] - \mathbf{E}[x]^2$$

we get:

$$\begin{aligned}
\mathrm{var}[x+z] &= \mathbf{E}\left[(x+z)^2\right] - (\mathbf{E}[x+z])^2 \\
&= \mathbf{E}\left[x^2 + 2xz + z^2\right] - (\mathbf{E}[x] + \mathbf{E}[z])^2 \\
&= \mathbf{E}\left[x^2\right] + 2\mathbf{E}[xz] + \mathbf{E}\left[z^2\right] - \mathbf{E}[x]^2 - 2\mathbf{E}[x]\mathbf{E}[z] - \mathbf{E}[z]^2 \\
&= \mathrm{var}[x] + \mathrm{var}[z] + 2(\mathbf{E}[xz] - \mathbf{E}[x]\mathbf{E}[z]).
\end{aligned}$$

Since x and z are independent, $\mathbf{E}[xz] = \mathbf{E}[x]\mathbf{E}[z]$, and thus:

$$\mathrm{var}[x+z] = \mathrm{var}[x] + \mathrm{var}[z] + 2(\mathbf{E}[x]\mathbf{E}[z] - \mathbf{E}[x]\mathbf{E}[z]) = \mathrm{var}[x] + \mathrm{var}[z]$$

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The expectation $E[x]$ can be written as:

$$E[x] = \int xp(x)dx.$$

Using the law of total probability, the marginal distribution $p(x)$ can be expressed in terms of the joint distribution $p(x, y)$:

$$p(x) = \int p(x, y)dy.$$

Substituting this into the expectation, we have:

$$E[x] = \int x \left(\int p(x, y)dy \right) dx.$$

By the definition of conditional probability, $p(x, y) = p(x \mid y)p(y)$, so:

$$E[x] = \int x \left(\int p(x \mid y)p(y)dy \right) dx.$$

Reverse the order of integration:

$$E[x] = \int \left(\int xp(x \mid y)dx \right) p(y)dy.$$

The inner integral is the conditional expectation $E_x[x \mid y]$:

$$E[x] = \int E_x[x \mid y]p(y)dy.$$

Finally, this is the definition of the outer expectation with respect to y :

$$E[x] = E_y \left[E_x[x \mid y] \right].$$

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