EIN 5226

Exponential Distribution

Chapter 7 Sections 10-11

Note: Need Calculator

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Uses of Exponential Distribution in Quality

Reliability studies

Reliability: The probability of a product's performing its intended function under stated conditions without failure for a given period of time.

(https://asq.org/quality-resources/quality-glossary/)

Addresses performance with use

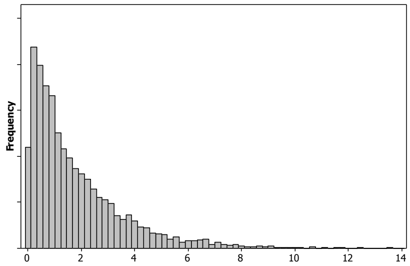
Performance metrics associated with time

Examples: Waiting times
Response times
Time to task completion

Exponential Distribution

- Think about waiting for an elevator and how long it takes for the elevator to arrive.
- Typically, it frequently comes in a short time but once in a while, it takes a really long time.

If you were to take a very large sample of the time for the elevator to arrive, and plotted the data on a histogram you might get a graph that looks like this.



Minutes between elevator arrivals

Exponential Distribution

Using a random variable to measure the time for the elevator to arrive.

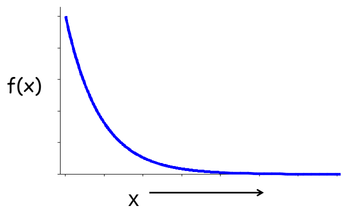
1. The variable is continuous.
2. Smaller values have larger probability.
3. Larger values have smaller probability.

This would follow an exponential distribution.

Exponential Distribution

- Continuous distribution
- Used primarily to model time between events
  - Waiting time
  - Time between arrivals
  - Time to failure for product
  - Time between failures for repairable equipment

What the exponential distribution should look like:



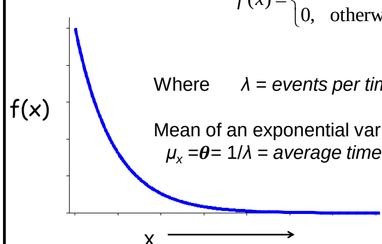
The distribution of observations in the exponential distribution will be

A. Negatively skewed. B. Positively skewed.

Exponential Distribution

Probability Density Function for random variable x

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x>0\\ 0, & \text{otherwise}\end{cases}.$$



Where λ = *events per time period = 1/ μ_x*

Mean of an exponential variable is

$\mu_x = \boldsymbol{\theta} = 1/\lambda$ = *average time until event occurs*

Exponential Distribution

If the average time waiting for elevators is 2 minutes,

$$\boldsymbol{\mu}_x = \boldsymbol{\theta} = 2 \text{ minutes}$$

Then the rate of arrival is

$$\lambda = 1/2 = .5 \text{ arrivals/minute}$$

λ (lambda) is always a rate - #events per time period

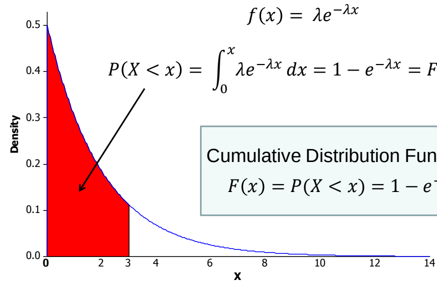
You just missed the elevator. What is the probability that the time to the next elevator is 3 minutes or less?

$$f(x) = \lambda e^{-\lambda x}$$

$$P(X < x) = \int_0^x \lambda e^{-\lambda x}\, dx = 1 - e^{-\lambda x} = F(x)$$

Cumulative Distribution Function

$$F(x) = P(X < x) = 1 - e^{-\lambda x}$$



For an Exponential Random Variable

Probability density function (pdf)

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Cumulative density function (cdf)

$$F(x) = \begin{cases} 0, & x \le 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}.$$

The mean of an exponential random variable is

$$\mu_x = \boldsymbol{\theta} = 1/\lambda$$

The variance of an exponential random variable is

$$\sigma_x^2 = 1/\lambda^2 = \mu_x^2 = \boldsymbol{\theta}^2$$

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For an Exponential Random Variable

Probability density function (pdf)

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Cumulative density function (cdf)

$$F(x) = \begin{cases} 0, & x \le 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}.$$

$$f(x) = \begin{cases} (\frac{1}{u})\, e^{-x/u}, x > 0 \\ 0, x \le 0 \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-x/u}, x > 0 \\ 0, x \le 0 \end{cases}$$

$$f(x) = \begin{cases} (\frac{1}{\theta})\, e^{-x/\theta}, x > 0 \\ 0, x \le 0 \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-x/\theta}, x > 0 \\ 0, x \le 0 \end{cases}$$

$$\mu_x = 1/\lambda = \boldsymbol{\theta}$$

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Exponential
Cumulative Distribution Function
$F(x) = P(X < x) = 1 - e^{-\lambda x}$

You just missed the elevator. The average time between elevators is 2 minutes. What is the probability that the time to the next elevator is 3 minutes or less?

$\boldsymbol{\mu}_x = \boldsymbol{\theta} = 2$ minutes
$\lambda = 1/2 = .5$ arrivals/minute
$x = 3$ minutes

$$P(X < 3) = F(3)$$
$$= 1 - e^{-(.5)3} = 0.777$$

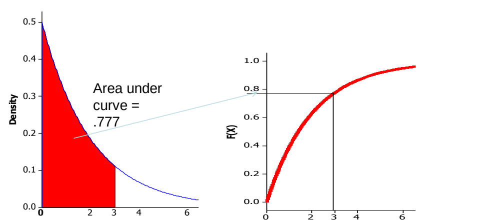
The average time between elevators is 2 minutes. What is the probability that the time to the next elevator is 3 minutes or less?

Probability Density Function

$f(x) = \lambda e^{-\lambda x}$

Cumulative Distribution Function

$F(x) = P(X < x) = 1 - e^{-\lambda x}$



Exponential Cumulative Distribution
$F(x) = P(X < x) = 1 - e^{-\lambda x}$

You just missed the elevator. The average time between elevators is 2 minutes.

$\boldsymbol{\mu}_x = \boldsymbol{\theta} = 2$ minutes
$\lambda = 1/2 = .5$ arrivals/minute

What is the probability that you will wait less than 4 minutes for the elevator?

A. 0.261 B. 0.865 C. 0.135 D. 0.789

What is the probability that you will wait longer than 5 minutes for the elevator?

A. 0.918 B.0.952 C. 0.120 D. 0.082

Problem 1

Exponential
Cumulative Distribution Function
$F(x) = P(X < x) = 1 - e^{-\lambda x}$

The life of standard light bulbs is know to be exponentially distributed with a mean of 1000 hours.

1a. What is the probability that a bulb will last less than 900 hours?

A) 0.593 B) 0.407 C) 0.655 D) 0.368

1b. What proportion of standard bulbs will last more than 1200 hours?

A) 0.699 B) 0.453 C) 0.301 D) 0.555

Problem 1

Exponential
Cumulative Distribution Function
$F(x) = P(X < x) = 1 - e^{-\lambda x}$

The life of standard light bulbs is know to be exponentially distributed with a mean of 1000 hours.

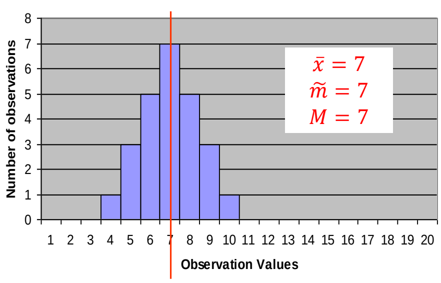
1c. What proportion of standard bulbs will last less than 1000 hours?

A) 0.408 B) 0.500 C) 0.755 D) 0.632

1d. What proportion of standard bulbs will last more than 693 hours?

A) 0.625 B) 0.500 C) 0.432 D) 0.555

Shape of the distribution



With a symmetrical distribution,
the mean, medium and mode are the same

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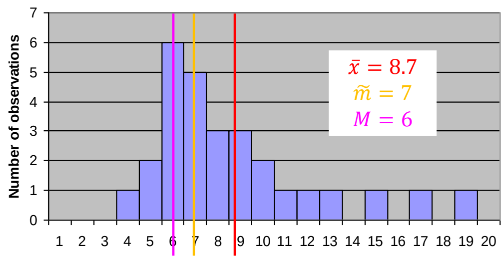

Skewed data – non-symmetrical - more data to one side of mean

Skewed to the right or positively skewed – more data on the right
Mean > Medium

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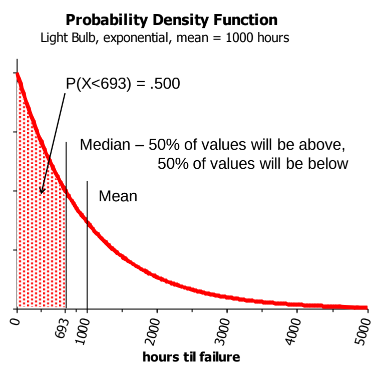

Exponential Cumulative Distribution
$F(x) = P(X < x) = 1 - e^{-\lambda x}$

The life of standard light bulbs is know to be exponentially distributed with a mean of 1000 hours.

$\boldsymbol{\mu}_x = \boldsymbol{\theta} = 1000 =$ **Mean Time To Failure (MTTF)**

$\lambda = 1/1000 = 0.001$ failures/hour

What is the probability that a bulb will last less than 900 hours?

$P(X < 900) = F(900) \quad = 1 - e^{-(.001)900} = 0.777$

What is the probability that a bulb will still be operating at 900 hours?

$P(X \geq 900) = 1 - P(X < 900) = 1 - 0.777 = 0.223$

Exponential Cumulative Distribution
$F(x) = P(X < x) = 1 - e^{-\lambda x}$

The life of standard light bulbs is know to be exponentially distributed with a mean of 1000 hours.

$\mu_x = \theta = 1000 =$ Mean Time To Failure (MTTF)

$\lambda = 1/1000 = 0.001$ failures/hour

What is the probability that a bulb will still be operating at 900 hours?

$P(X \geq 900) = 1 - P(X < 900) \quad = 1 - (1 - e^{-(.001)900})$

$= 1 - 0.777 = 0.223 \quad = + e^{-(.001)900} = +e^{-\lambda x}$

Reliability: The probability of a product's performing its intended function under stated conditions without failure for a given period of time.

$R(t) = e^{-\lambda t}$

(https://asq.org/quality-resources/quality-glossary/)

Reliability

Reliability – probability that a product will perform successfully under specified operating conditions for a given period of time

- **Probability** – How likely an event is to occur.
  - The event: the product will be performing successfully
- **Specified Operating Conditions** – must be defined
- **Time** – Can be actual time measurement
  - May also be other measure of product use
    Number of cycles, miles driven, missions

Problem 2

Reliability
$R(t) = e^{-\lambda t}$

MTTF: Mean Time To Failure for non-repairable item
$\mu_x = \theta = \text{MTTF} \qquad \lambda = 1/\mu_x = 1/\theta = 1/\text{MTTF}$

MTBF: Mean Time Between Failures for repairable item
$\mu_x = \theta = \text{MTBF} \qquad \lambda = 1/\mu_x = 1/\theta = 1/\text{MTBF}$

a. The MTTF of a system is 3000 hours. What is the reliability of the system for a 4000 hour mission?

A. 0.748 B. 0.451 C. 0.264 D. 0.305

b. The MTBF of a copy machine is 100 hours. What is the reliability of the machine at 60 hours?

A. 0.451 B. 0.624 C. 0.549 D. 0.376

Lack of Memory Property

Lack of Memory Property/Forgetfulness Property

Really interesting and useful property
of the Exponential Distribution

- The following have the same probability!
  - ❖ Probability that the elevator will come 2 minutes after you arrive.
  - ❖ Probability that the elevator will come 2 minutes after the last time it left.
  - ❖ Probability that the elevator will come 2 minutes after you have already been waiting 3 minutes.
- The distribution does not “remember” anything prior to now.

Lack of Memory Property

Lack of memory property:

If $T \sim \text{Exp}(\lambda)$, and t and s are positive numbers, then

$$P(T > t + s \mid T > s) = P(T > t).$$

The probability that we have to wait an additional t units given that we have already waited s units is equal to the probability that we must wait t units from the start.

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Problem 3

Exponential
Cumulative Distribution Function
$F(x) = P(X < x) = 1 - e^{-\lambda x}$

The lifetime of a transistor in a particular circuit has an exponential distribution with mean 1.5 years.

a. What is the probability that one of the transistors will fail within 1 year?

A) 0.4976 B) 0.3788 C) 0.4865 D) 0.5812

b. You have purchase some surplus transistors and do not know how old they are. The probability that a transistor will last more than one year is

A) 0.5135 B) 0.4378 C) 0.4865 D) 0.6123

True/False If a transistor is still working at 1 year, the probability that it will not be working when it is two years old is 0.4865.

Related Assignments

Please see Blackboard for related assignments