Binomial and Hypergeometric Distributions

Chapter 7 Sections 4-7

Note: Need Calculator

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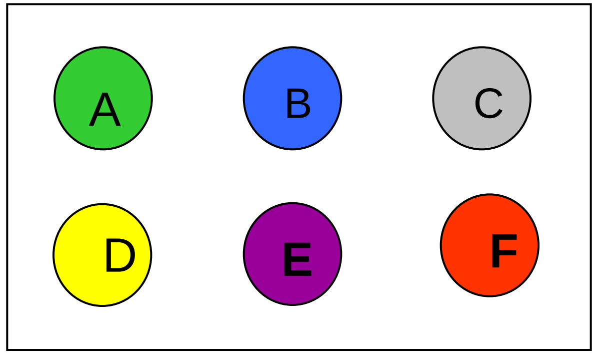
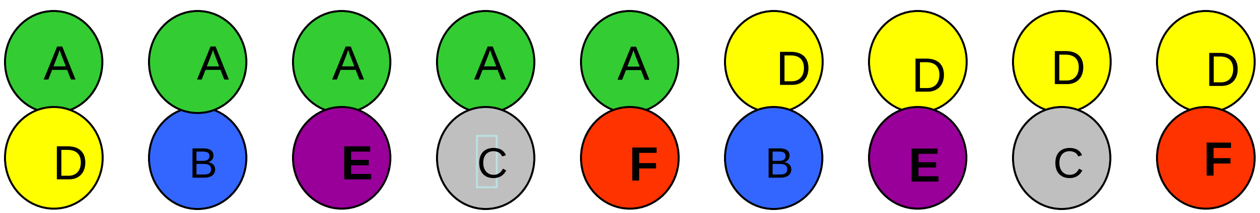
Probability Distributions

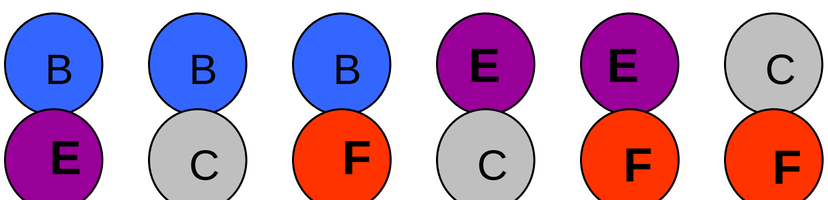
- Continuous
  - Normal,
  - Lognormal, Exponential, Weibull
  - Chi-Square, F, Student t
- **Discrete**
  - **Binomial**
  - **Hypergeometric**
  - Poisson

Combinations



Number of possible combinations of two balls:





15 different combinations

- Order unimportant -

Combinations

Number of ways in which a larger group of items can be arranged into smaller subgroups

Combinations

- where order does not matter

$$\binom{n}{x} = nCx = \frac{n!}{x!\,(n-x)!}$$

Where n = size larger group

x = subgroup size

note: n! = (n)(n-1)(n-2)(n-3)….(3)(2)(1)

Combinations

Combinations - where order does not matter

$$\binom{n}{x} = {}_nC_x = \frac{n!}{x!\,(n-x)!}$$

Where n = size larger group

x = subgroup size

note: n! = (n)(n-1)(n-2)(n-3)….(3)(2)(1)

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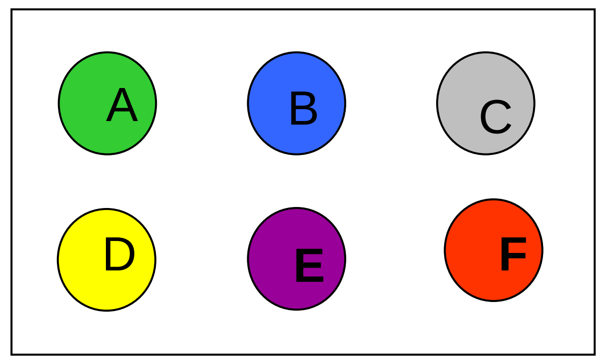
6 items taken 2 at a time,
order not important

n = 6 x = 2



$$\binom{6}{2} = {}_6C_2 = \frac{n!}{x!\,(n-x)!} = \frac{6!}{2!\,(6-2)!} = \frac{6*5*4*3*2*1}{(2*1)(4*3*2*1)} = 15$$

Combinations

Combinations - where order does not matter

$$\binom{n}{x} = nCx = \frac{n!}{x!\,(n-x)!}$$

Where n = size larger group

x = subgroup size

note: n! = (n)(n-1)(n-2)(n-3)….(3)(2)(1)

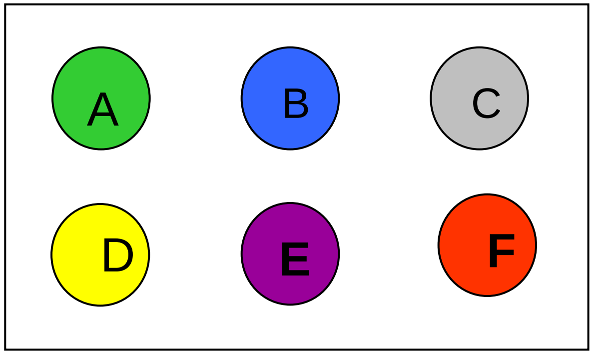
There are 10 runners in the first heat of a race. The top four will move on to the next heat of the race. How many different combinations of four runners could be are possible for moving to the next heat? (Order not important.)

A. 25,200 B. 1260 C. 360 D. 210 E. 40

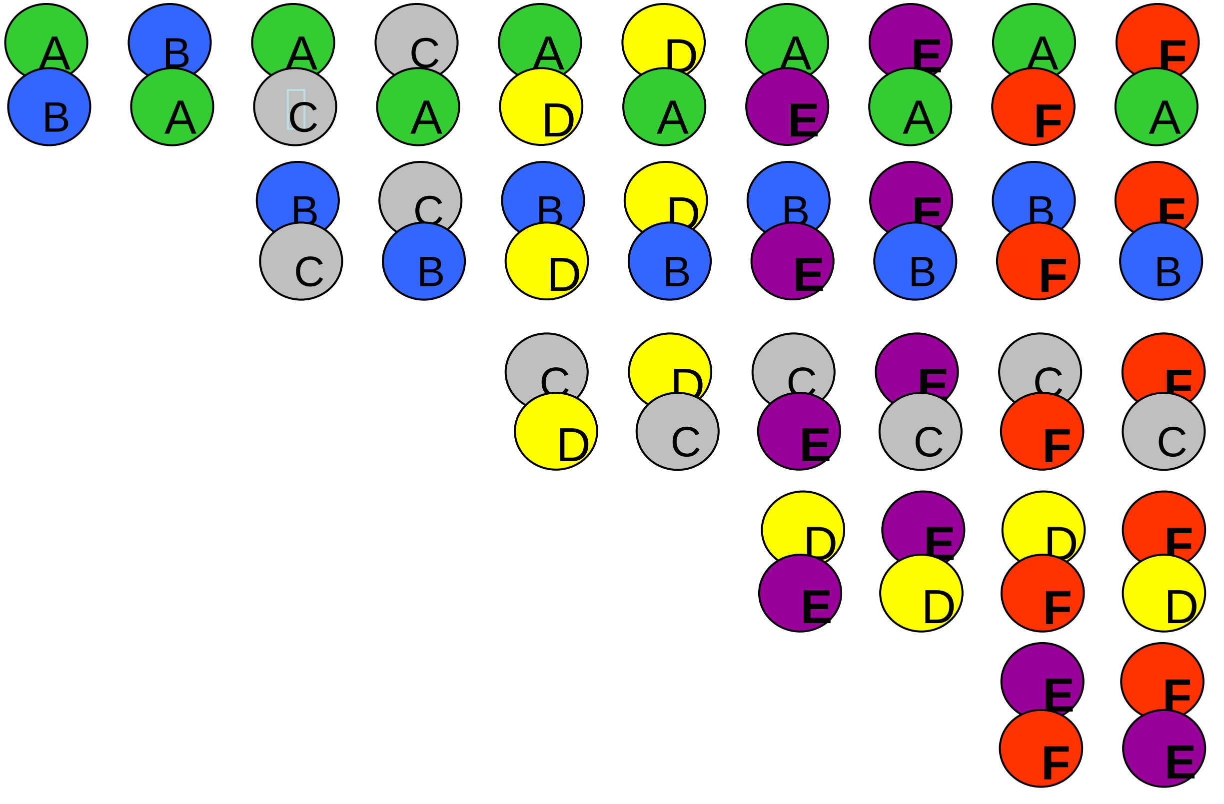
Permutations

30 different permutations

- Order important -



Number of possible permutations of two balls:



Combinations and Permutations

Number of ways in which a larger group of items can be arranged into smaller subgroups

Combinations

- where order does not matter

$$\binom{n}{x} = nCx = \frac{n!}{x!\,(n-x)!}$$

Permutations

- where order does matter

$$nPx = \frac{n!}{(n-x)!}$$

Where n = size larger group

x = subgroup size

note: n! = (n)(n-1)(n-2)(n-3)….(3)(2)(1)

Permutations

Permutations - where order does matter

$$nPx = \frac{n!}{(n-x)!}$$

Where n = size larger group

x = subgroup size

note: n! = (n)(n-1)(n-2)(n-3)….(3)(2)(1)

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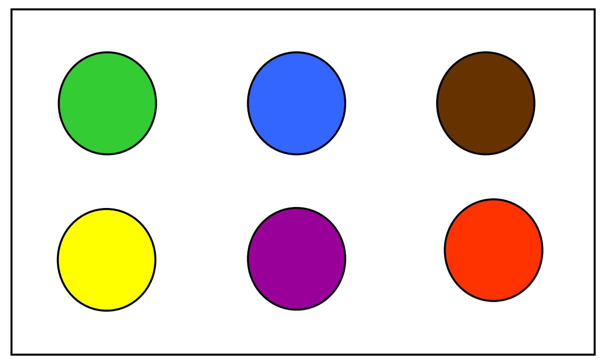
6 items taken 2 at a time, order important

n = 6 x = 2



$$nPx = \frac{n!}{(n-x)!} = \frac{6!}{(6-2)!} = \frac{6*5*4*3*2*1}{(4*3*2*1)} = 30$$

Permutations

Permutations - where order does matter

$$n\text{Pr} = \frac{n!}{(n-r)!}$$

Where n = size larger group

r = subgroup size

note: n! = (n)(n-1)(n-2)(n-3)….(3)(2)(1)

There are 8 runners in finals of a race. The top 3 are awarded medals How many different ways could the medals be distributed? (Order is important.)

A. 8064 B. 6270 C. 336 D. 210 E. 20

The Binomial Distribution

We use the Binomial Distribution when we have an experiment which can result in one of two outcomes.

Bernoulli Process - Output falls into one of only two categories

Examples:

- Heads or tails
- Inspecting items - defective or not
- Vote on issue - yes or no
- Persons who use/don’t use product

The Binomial Distribution

- A number of trials are made with outcomes observed.
- One outcome is labeled "success," and the other outcome is labeled "failure."
- The probability of a success is denoted by p. The probability of a failure (often denoted by q) is then $1 – p$.

Binomial Distribution

A Binomial Distribution shows the proportion of success/failure for a given number of identical trials from a Bernoulli process where

p = probability of success
q = probability of failure = 1-p
n = number of trials
x = number of successes in a given trial

Binomial Distribution

A Binomial Distribution shows the proportion of success/failure for a given number of identical trials from a Bernoulli process where

p = probability of success
q = probability of failure = 1-p
n = number of trials
x = number of successes in a given trial

What type of numerical data are we dealing with when using the binomial distribution?

A. Continuous

B. Discrete

The Binomial Distribution

Binomial distribution assumptions:

1. All trials are identical.
2. Each outcome is either a “success” or “failure”.
3. The probability of a success is constant.
4. All trials are independent

Example

The simplest Bernoulli trial is the toss of a coin.

The two outcomes are heads and tails.

If we define heads to be the success outcome, then p is the probability that the coin comes up heads.

For a fair coin, $p = ½ = .5$

If we toss the coin 4 times, our number of trials n=4.

If we are interested in the probability that heads will show up 4 times, then we are looking for the probability that x=4.

Example. Flip coin 4 times. $n = 4$ Number of trials

$p = .5$ Heads = success (X)

$q = 1\text{-}p = .5$ Tails = failure (0)

Possible outcomes in 4 flips: 0 heads, 1 head, 2 heads, 3 heads, 4 heads

$x=0$ or $x=1$ or $x=2$ or $x=3$ 0r $x=4$

<u>What is the probability of getting exactly 4 heads? P (x=4) = ?</u>

Possible combinations: XXXX

Probability of getting a head on 1 flip: $p = .50$

P(two times in a row) = $(p)(p)=(.50)(.50) = .25$

P(3 times in a row) = $(p)(p)(p)= (.50)^3 = .125$

P(4 times in a row) = $(p)(p)(p)(p)= (.50)^4 = .0625 = p^4$

Example. Flip coin 4 times. $n = 4$ Number of trials

$p = .5$ Heads = success (X)

$q = 1\text{-}p = .5$ Tails = failure (0)

What is the probability of getting exactly 1 heads? $P\ (x=1) = ?$

Possible combinations: XOOO OXOO OOXO OOOX

$$P(XOOO) = (p)(q)(q)(q) = (p)(1-p)(1-p)(1-p) = p(1-p)^3$$
$$P(OXOO) = (q)(p)(q)(q) = (1-p)\ (p)(1-p)(1-p) = p(1-p)^3$$
$$P(OOXO) = (q)\ (q)(p)(q) = (1-p)(1-p)\ (p)(1-p) = p(1-p)^3$$
$$P(OOOX) = (q)(q)(q)(p) = (1-p)(1-p)(1-p)\ (p) = p(1-p)^3$$

$$p(1-p)^3=(.5)\ (1-.5)^3=.0625$$

$$P(x=1) = P(XOOO) + P(OXOO) + P(OOXO) + P(OOOX)$$

P (x=1)= (# possible combinations) (probability of the combination)

$= 4\ (.0625) = 0.25$

Example. Flip coin 4 times. P=.5, n=4 Heads = success (X)
Tails = failure (0)

What is the probability of getting exactly 2 heads? P (x=2) = ?

Possible combinations: 00XX 0X0X 0XX0 X00X X0X0 XX00

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{4*3*2*1}{(2*1)(2*1)} = 6$$

The probability of any one of those combinations is

$P(XXOO) = (p)(p)(q)(q) = (p)(p)\,(1-p)(1-p) = p^2(1-p)^2 = .5^2(.5)^2 = .0625$

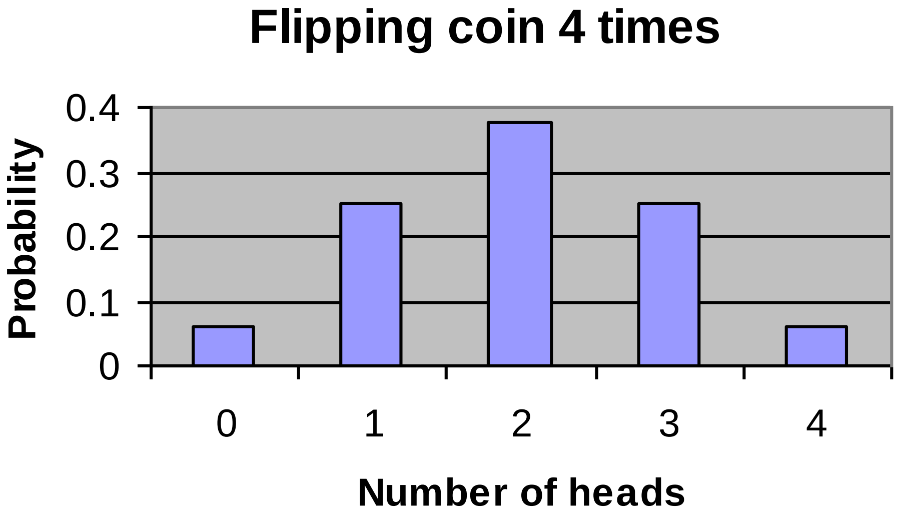
P (x=2) = (# possible combinations) (probability of the combination)

$$= \binom{4}{2} p^2(1-p)^2 = (6)\ .5^2(.5)^2 = 0.375$$

Example. Flip coin 4 times. p=.5, n=4 Heads = success (X)
Tails = failure (0)

P (x=0) =0.0625
P (x=1) =0.25
P (x=2) =0.375
P (x=3) =0.25
P (x=4) =0.0625

Probability Distribution



What if our coin is weighted? The probability of landing on heads is 25% and we will flip 8 times.

What is the probability of getting exactly 2 heads? P (x=2) = ?

P(Success)=P(Heads)=.25=p

Possible combinations:

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!} \qquad \binom{8}{2} = \frac{8!}{2!(8-2)!} = 28$$

Probability of getting 2 heads and 6 tails in any combination is:

(p)(p)(1-p) (1-p) (1-p) (1-p) (1-p) (1-p) $= p^{x}(1-p)^{n-x}$

$= .25^{2}(.75)^{6}$ $\qquad = .25^{2}(.75)^{(8-2)}$

$= .01112$

P (x=2) = (28) (.01112) = 0.3114

$$p(x) = \binom{n}{x} p^{x}(1-p)^{n-x}$$

Binomial Distribution

The probability of exactly x occurrences in n trials in an event that has a constant probability of occurrence

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

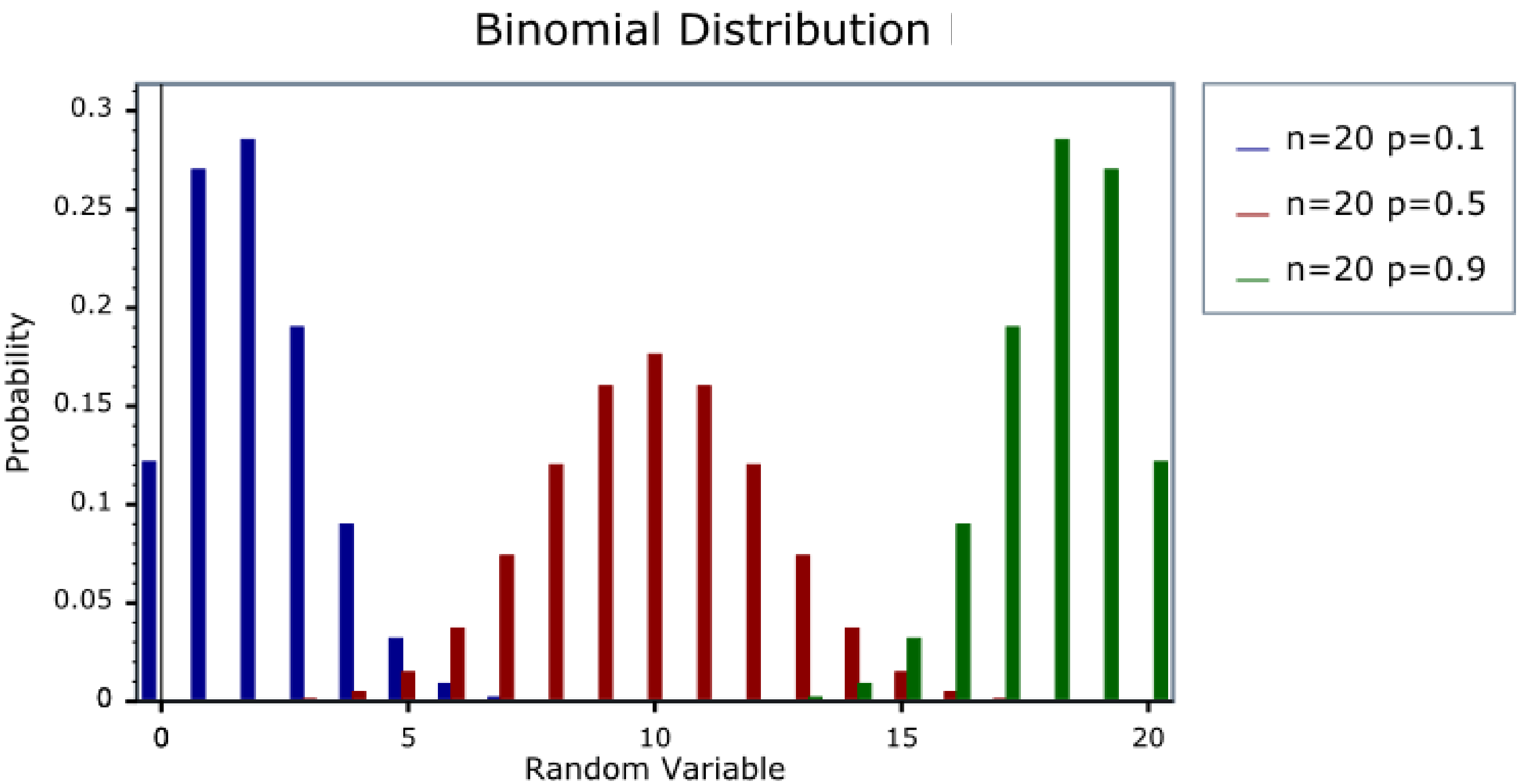
Probability Mass Function

where p = probability of success
n = the number of trials
x = the number of successes

and $$\binom{n}{x} = \frac{n!}{x!(n-x)!} = nCx$$

with $n!$ = (n)(n-1)(n-2)(n-2)(…….)(1)

Binomial Probability Histogram (Probability Mass Function)



http://www.boost.org/doc/libs/1_51_0/libs/math/doc/sf_and_dist/graphs/binomial_pdf_2.png

What if our coin is weighted? The probability of landing on heads is 25% and we will flip 8 times.

We are interested in the probability of getting exactly 3 heads?

If you define P(success)=P(heads)=.25 , what is x for this problem?

A. 8 B. 5 C. 3

How many combinations can we have for this problem?

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

A. 56 B. 86 C. 112 D. 1344

What is the probability of getting exactly 3 heads?

$\binom{n}{x} p^{x}(1-p)^{n-x}$

A. 2.3% B. 16.1% C. 18.2% D. 20.8%

What if our coin is weighted? The probability of landing on heads is 25% and we will flip 8 times.

<u>We are interested in the probability of getting less than or equal to 2 heads?</u>

n=8

P(success)=P(heads)=.25

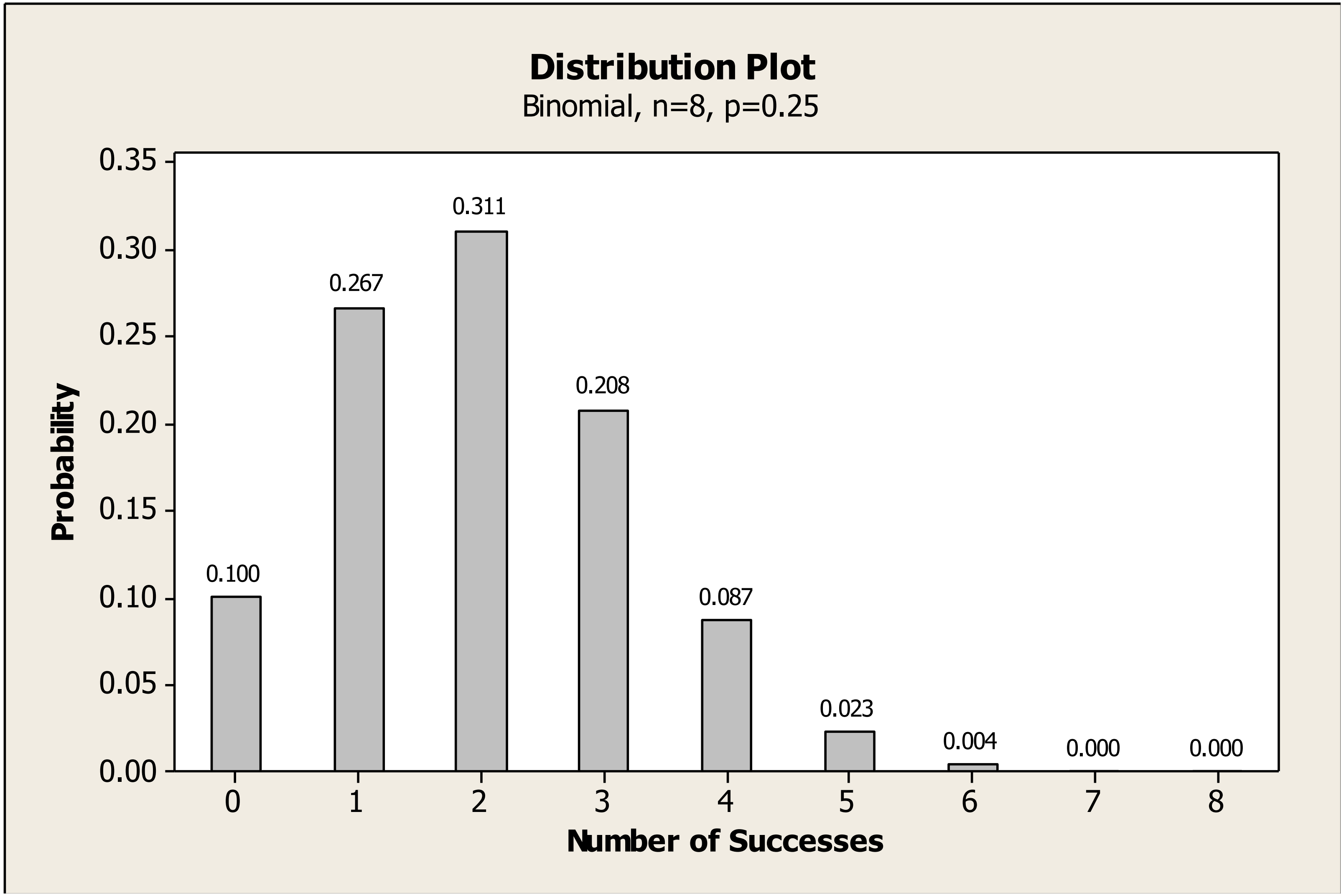
$$\binom{n}{x} p^{x}(1-p)^{n-x}$$

P(x≤2) = P(x=0) + P(x=1) + P(x=2)

$$= \binom{8}{0}(.25)^{0}(1-.25)^{8-0} + \binom{8}{1}(.25)^{1}(1-.25)^{8-1} + \binom{8}{2}(.25)^{2}(1-.25)^{8-2}$$

$$= .1011 + .2670 + .3115 = .6785$$

What if our coin is weighted with the probability of having 1 success (head) is p=.25 and we flip 8 times.



What if our coin is weighted? The probability of landing on heads is 25% and we will flip 8 times.

<u>What is the probability of getting exactly 5 tails?</u>

If you define P(success)=P(heads) which is correct in applying the binomial distribution formula

A. n=8, x=5, p= 0.25
B. n=8, x=5, p= 0.75
C. n=8, x=3, p= 0.75
D. n=8, x=3, p= 0.25

If you define P(success)=P(tails) which is correct in applying the binomial distribution formula

A. n=8, x=5, p= 0.25
B. n=8, x=5, p= 0.75
C. n=8, x=3, p= 0.75
D. n=8, x=3, p= 0.25

True / False Either way you work the problem you will come up with the same answer.

$$\binom{n}{x} p^{x}(1-p)^{n-x} \qquad \binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

A baseball player has a batting average of .200. (He averages about 1 hit about every 5 times up to bat.) Over a three game series he gets up to bat 12 times.

What is the probability that he will

Exactly 4 hits?

A. 0.275 B. 0.132 C. 0.842 D. 0.927

Less than 2 hits?

A. 0.275 B. 0.206 C. 0.725 D. 0.558

Get at least 1 hits?

A. 0.275 B. 0.206 C. 0.795 D. 0.931

The Binomial Distribution

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Mean

$$\mu = np$$

Variance

$$\sigma^2 = np(1-p)$$

The Binomial Distribution

Mean

$$\mu = np$$

Variance

$$\sigma^2 = np(1 - p)$$

A baseball player has a batting average of .200. (He averages about 1 hit about every 5 times up to bat.)

On average, how many hits will the player have if he is up to bat 12 times and what is the standard deviation of his number of successes?

A. 1.92, 2.4

B. 2.4, 1.92

C. 2.4, 1.386

D. 12, 1.92

The Binomial Distribution

Binomial distribution assumptions:

1. All trials are identical.
2. Each outcome is either a “success” or “failure”.
3. The probability of a success is constant.
4. All trials are independent

Hypergeometic Distribution

Hypergeometric distribution assumptions:

1. All trials are identical.
2. Each outcome is either a “success” or “failure”.
3. The probability of a success is not constant.
   Sampling without replacement
4. All trials are independent

Hypergeometic Distribution

Example: 6 men and 12 women place one ticket each in a raffle.

Five tickets will be randomly drawn without replacement to determine the winners.

What is the probability that the winners are 2 men and 3 women?

Any sample of 5 names is equally likely, therefore

P(2 men and 3 women) =

$$\frac{\text{\# of outcomes that result in 2 men and 3 women}}{\text{\# of possible samples of size 5}}$$

For the denominator:

# of possible samples of size 5 is $\binom{18}{5}$

Any sample of 5 names is equally likely, therefore

P(2 men and 3 women) =

$$\frac{\text{\# of outcomes that result in 2 men and 3 women}}{\text{\# of possible samples of size 5}}$$

For the numerator

combinations resulting in 2 men is $\binom{6}{2}$

combinations resulting in 3 women is $\binom{12}{3}$

Multiply to get the # combinations with 2 men and 3 women

Any sample of 5 names is equally likely, therefore

P(2 men and 3 women) =

$$\frac{\underline{\text{\# of outcomes that result in 2 men and 3 women}}}{\text{\# of possible samples of size 5}}$$

P(2 men and 3 women) =

$$\frac{\binom{6}{2}\binom{12}{3}}{\binom{18}{5}} = \frac{15 * 220}{8568} = .385$$

7.7 Hypergeometric Distribution

$$p(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

Where *N* = *total number of elements in the population*
r = *number of success in the population*
N-r = *number of failures in the population*
n = *number of trials (sample size)*
x = *number of successes in trial*
n-x = *number of failures in n trials*

Let p=r/N, then $\mu_x = np$, and $\sigma_x{}^2 = np(1-p)(\frac{N-n}{N-1})$.

Hypergeometic Distribution

6 men and 12 women place one ticket each in a raffle. Five tickets will be randomly drawn without replacement to determine the winners. What is the probability that the winners are 2 men and 3 women?

Define: “Men’s ticket” as success
“Women's ticket” as failure.

N	= 18	*total number of elements in the population*
r	= 6	*number of success in the population*
N-r	= 12	*number of failures in the population*
n	= 5	*number of trials (sample size)*
x	= 2	*number of successes in trial*
n-x	= 3	*number of failures in n trials*

Hypergeometic Distribution

6 men and 12 women place one ticket each in a raffle. Five tickets will be randomly drawn without replacement to determine the winners. What is the probability that the winners are 2 men and 3 women?

Define: "Men's ticket" as success
"Women's ticket" as failure.

N = *18*
r = *6*
N-r = *12*
n = *5*
x = *2*
n-x = *3*

$$p(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

$$P(2) = \frac{\binom{6}{2}\binom{12}{3}}{\binom{18}{5}} = \frac{15*220}{8568} = .385$$

6 men and 12 women place one ticket each in a raffle. Five tickets will be randomly drawn without replacement to determine the winners. What is the probability that the winners are 2 men and 3 women?

Define: “Men’s ticket” as success
“Women's ticket” as failure.

	Successes	Failures
Population		
Sample		

6 men and 12 women place one ticket each in a raffle. Five tickets will be randomly drawn without replacement to determine the winners. What is the probability that the winners are 2 men and 3 women?

Define: "Men's ticket" as success
"Women's ticket" as failure.

	Successes *# Men's tickets*	Failures *# Women's tickets*
Population	6	12
Sample		

6 men and 12 women place one ticket each in a raffle. Five tickets will be randomly drawn without replacement to determine the winners. What is the probability that the winners are 2 men and 3 women?

Define: "Men's ticket" as success
"Women's ticket" as failure.

	Successes *# Men's tickets*	Failures *# Women's tickets*
Population	6	12
Sample	2	3

6 men and 12 women place one ticket each in a raffle. Five tickets will be randomly drawn without replacement to determine the winners. What is the probability that the winners are 2 men and 3 women?

Define: “Men’s ticket” as success
“Women's ticket” as failure.

	Successes # Men’s tickets	Failures # Women’s tickets
Population	6	12
Sample	2	3

Totals	
Population	18
Sample	5

6 men and 12 women place one ticket each in a raffle. Five tickets will be randomly drawn without replacement to determine the winners. What is the probability that the winners are 2 men and 3 women?

Define: “Men’s ticket” as success
“Women's ticket” as failure.

	Successes	Failures
Population Sample	$\binom{6}{2}$	$\binom{12}{3}$

Totals	Population Sample	$\binom{18}{5}$

N = 18
r = 6
N-r = 12
n = 5
x = 2
n-x = 3

$$p(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

$$P(2) = \frac{\binom{6}{2}\binom{12}{3}}{\binom{18}{5}} = \frac{15 * 220}{8568} = .385$$

	Successes	Failures
Population	6	12
Sample	2	3

Totals	Population	18
	Sample	5

$$\binom{6}{2} \quad \binom{12}{3} \quad \binom{18}{5}$$

Hypergeometic Distribution

6 men and 12 women place one ticket each in a raffle. Five tickets will be randomly drawn without replacement to determine the winners. On average, how many men would hold winning tickets?

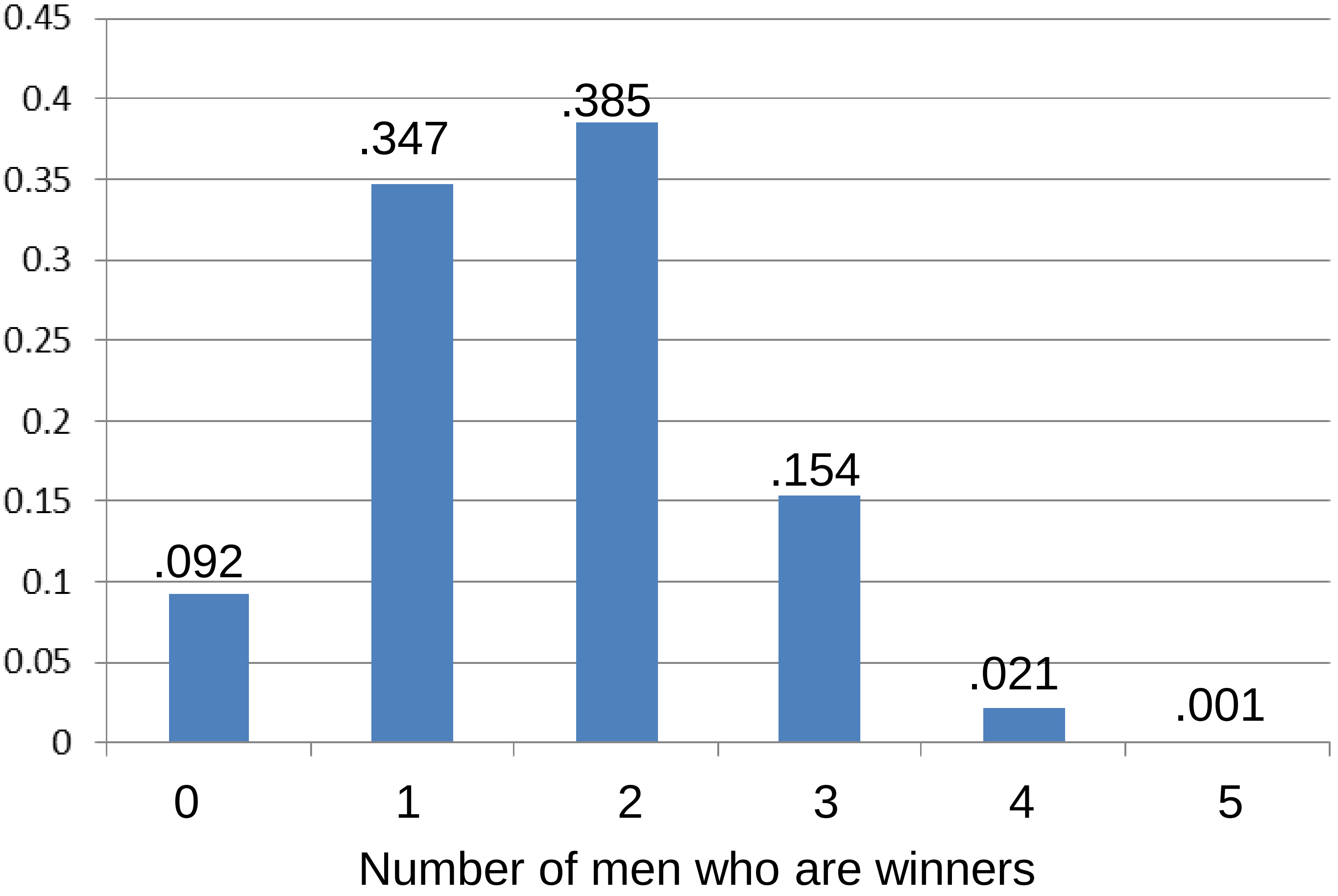
N = *18*
r = *6*
N-r = *12*
n = *5*
x = *2*
n-x = *3*

p=r/N = 6/18 = .333

$\mu_x = np$ = (5)(.333) = 1.665

$\sigma_x{}^2 = np(1-p)(\frac{N-n}{N-1})$.= .849

6 men and 12 women place one ticket each in a raffle. Five tickets will be randomly drawn without replacement to determine the winners.



A piggy bank contains 3 pennies and 5 dimes. If you shake out 4 coins, what is the probability that you will get exactly 2 dimes?

The population size, N is

A) 2 B) 3 C) 4 D) 5 E) 8

The sample size, n is

A) 2 B) 3 C) 4 D) 5 E) 8

If “success” is defined as getting a dime, how many successes are we looking for in the sample?

A) x=2 B) x=3 C) x= 4 D) x=5

A piggy bank contains 3 pennies and 5 dimes. If you shake out 4 coins, what is the probability that you will get exactly 2 dimes?

Using "hyper geometric without formulas", how would this problem be set up? (Assume dimes=success)

A)

$$\begin{array}{l|c|c} & \text{Successes} & \text{Failures} \\ \begin{array}{l}\text{Population}\\ \text{Sample}\end{array} & \dbinom{5}{2} & \dbinom{2}{3} \\ \hline \text{Totals} \begin{array}{l}\text{Population}\\ \text{Sample}\end{array} & \dbinom{8}{4} & \end{array}$$

B)

$$\begin{array}{l|c|c} & \text{Successes} & \text{Failures} \\ \begin{array}{l}\text{Population}\\ \text{Sample}\end{array} & \dbinom{5}{2} & \dbinom{3}{2} \\ \hline \text{Totals} \begin{array}{l}\text{Population}\\ \text{Sample}\end{array} & \dbinom{8}{4} & \end{array}$$

$$p(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

A piggy bank contains 3 pennies and 5 dimes. If you shake out 4 coins, what is the probability that you will get exactly 2 dimes?

A) 0.3581   B) 0.6122   C) 0.5013   D) 0.4286

Hypergeometric Probability Distribution Example

Suppose we select 5 cards from an ordinary deck of playing cards. What is the probability of obtaining 2 or fewer hearts?

Solution:

N = 52; since there are 52 cards in a deck.
r = 13; since there are 13 hearts in a deck.
n = 5; since we randomly select 5 cards from the deck.
x = 0 to 2; since our selection includes 0, 1, or 2 hearts.

We plug these values into the hypergeometric formula as follows:

$$p(0) = \frac{\binom{13}{0}\binom{39}{5}}{\binom{52}{5}} = .2215, \qquad p(1) = \frac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} = .4114, \qquad p(2) = \frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} = .2743$$

$p(x \leq 2) = p(0) + p(1) + p(2) = .2215 + .4114 + .2743 = .9072$

A lot contains 1000 components, 5% of which are defective. Ten components are sampled from the lot. What is the probability that exactly 1 is defective?

N=1000
n=10

$$p(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

Define success as defect
r= successes in population = 5% times 1000 = 50
x= successes in sample = 1

$\binom{r}{x}$	$\binom{N-r}{n-x}$	$\binom{N}{n}$	P(1)
50	1.67E+21	2.63E+23	0.317

Large Finite Populations

A lot contains 1000 components, 5% of which are defective. Ten components are sampled from the lot. What is the probability that exactly 1 is defective?

If the sample size is no more than 10% of a large population, the binomial distribution may be used to model the number of successes.

Large Finite Populations

A lot contains 1000 components, 5% of which are defective. Ten components are sampled from the lot. What is the probability that exactly 1 is defective?

<u>Trials not truly independent, p not constant</u>

5% defective: 50 out of 1000 are defective.

P(1^{st} component defective) = 50/1000 = .05

P(2^{nd} component defective/1^{st} component defective)

= 49/999 = .049

Large Finite Populations

A lot contains 1000 components, 5% of which are defective. Ten components are sampled from the lot. What is the probability that exactly 1 is defective?

$$p(x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

n=10
p=.05
x=1

$$p(1) = \binom{10}{1} .05^1 (1 - .05)^{10-1} = 0.3151$$

With hypergeometric : p(1) = 0.317

$$p(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

$$\binom{n}{x}\text{p}^{\,x}(1\text{-}p)^{\,n\text{-}x}$$

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

Practice Problem

A new medicine was developed to ease the symptoms of the common cold. Studies showed that 60% of the people that take the medicine obtain significant relief. 400 people are prescribed the medicine. A sample of 15 of these people was taken. What is the probability that 7 people in the sample obtained significant relief?

True / False For this problem I can use the binomial approximation to the hypergeometric.

P(x=7) = ?

A. 0.118 B. 0.205 C. 0.175 D. 0.301

$$p(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

$$\binom{n}{x} p^{x} (1-p)^{n-x}$$

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

Practice Problem

At a career fair, 65 companies had a booth? 25 of these had a candy dish at the booth. You only have time to visit 12 booths. What is the probability that you will visit exactly 8 booths with a candy dish?

True / False For this problem I can use the binomial approximation to the hypergeometric.

P(x=8) = ?

A. 0.2013 B. 0.0515 C. 0.0245 D. 0.0345

7.7 Hypergeometric Distribution

- A sample of size n is randomly selected without replacement from a population of N items.
- In the population, r items can be classified as successes, and N - r items can be classified as failures.
- A hypergeometric random variable, x, is the number of successes that result from a hypergeometric experiment
- May be estimated by the binomial where the sample size is small relative to the population size.

Related Assignments

Please see Blackboard for related assignments.