Chapter 3
The Time Value of Money

Learning Objectives

After studying Chapter 3, you should be able to:

1. Understand what is meant by "the time value of money."
2. Understand the relationship between present and future value.
3. Describe how the interest rate can be used to adjust the value of cash flows – both forward and backward – to a single point in time.
4. Calculate both the future and present value of: (a) an amount invested today; (b) a stream of equal cash flows (an annuity); and (c) a stream of mixed cash flows.
5. Distinguish between an “ordinary annuity” and an “annuity due.”
6. Use interest factor tables and understand how they provide a shortcut to calculating present and future values.
7. Use interest factor tables to find an unknown interest rate or growth rate when the number of time periods and future and present values are known.
8. Build an “amortization schedule” for an installment-style loan.

Topics

- The Interest Rate
- Simple Interest
- Compound Interest
- Amortizing a Loan
- Compounding More Than Once per Year

The Interest Rate

Which would you prefer -- $10,000 today or $10,000 in 5 years?

Obviously, $10,000 today.

You already recognize that there is *TIME VALUE TO MONEY*!!

Why TIME?

Why is TIME such an important element in your decision?

TIME allows you the *opportunity* to postpone consumption and earn INTEREST.

Types of Interest

- Simple Interest

  Interest paid (earned) on only the original amount, or principal, borrowed (lent).

- Compound Interest

  Interest paid (earned) on any previous interest earned, as well as on the principal borrowed (lent).

Simple Interest Formula

Formula $SI = P_0(i)(n)$

- SI: Simple Interest
- P_0: Deposit today (t = 0)
- i: Interest Rate per Period
- n: Number of Time Periods

Simple Interest Example

- Assume that you deposit $1,000 in an account earning 7% simple interest for 2 years. *What is the accumulated interest at the end of the 2nd year?*

- SI $= P_0(i)(n)$
  $= \$1{,}000(.07)(2)$
  $= \$140$

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Simple Interest (FV)

- What is the Future Value (FV) of the deposit?

$$FV = P_0 + SI = \$1{,}000 + \$140 = \$1{,}140$$

- Future Value is the value at some future time of a present amount of money, or a series of payments, evaluated at a given interest rate.

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Simple Interest (PV)

- What is the Present Value (PV) of the previous problem?

The Present Value is simply the $1,000 you originally deposited. That is the value today!

- Present Value is the current value of a future amount of money, or a series of payments, evaluated at a given interest rate.

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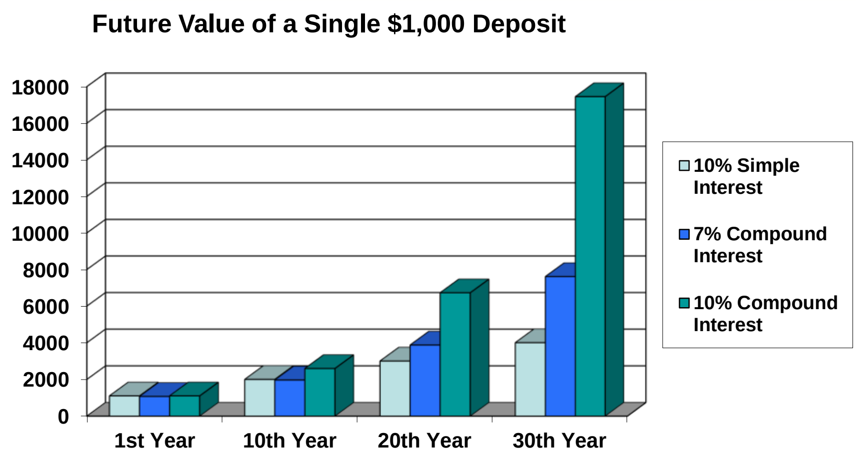
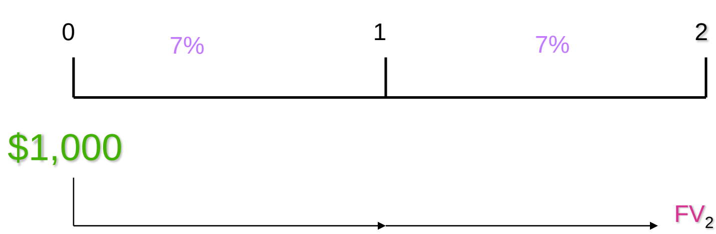
Why Compound Interest?



Future Value Single Deposit (Graphic)

Assume that you deposit $1,000 at a compound interest rate of 7% for 2 years.



Future Value Single Deposit (Formula)

$FV_1 = P_0 (1+i)^1 = \$1,000\ (1.07) = \$1,070$

Compound Interest

You earned $70 interest on your $1,000 deposit over the first year.

This is the same amount of interest you would earn under simple interest.

Future Value Single Deposit (Formula)

$FV_1 = P_0 (1+i)^1 \quad = \$1,000\ (1.07)$

$\quad = \$1,070$

$FV_2 = FV_1 (1+i)^1$

$\quad = P_0 (1+i)(1+i) \quad = \$1,000(1.07)(1.07)$

$\quad = P_0 (1+i)^2 \quad = \$1,000(1.07)^2$

$\quad = \$1,144.90$

You earned an *EXTRA* *$4.90* in Year 2 with compound over simple interest.

General Future Value Formula

$FV_1 = P_0(1+i)^1$

$FV_2 = P_0(1+i)^2$

etc.

General Future Value Formula:

$FV_n = P_0 (1+i)^n$

or $FV_n = P_0 (FVIF_{i,n})$ -- *See Table I*

or $FV_n = P_0 (F/P, i, n)$

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Valuation Using Table I

$FVIF_{i,n}$ is found on Table I at the end of the book.

Period	6%	7%	8%
1	1.060	1.070	1.080
2	1.124	1.145	1.166
3	1.191	1.225	1.260
4	1.262	1.311	1.360
5	1.338	1.403	1.469

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Using Future Value Tables

$FV_2 = \$1,000\ (FVIF_{7\%,2})$
$= \$1,000\ (1.145)$
$= \$1,145$ [Due to Rounding]

Period	6%	7%	8%
1	1.060	1.070	1.080
2	1.124	1.145	1.166
3	1.191	1.225	1.260
4	1.262	1.311	1.360
5	1.338	1.403	1.469

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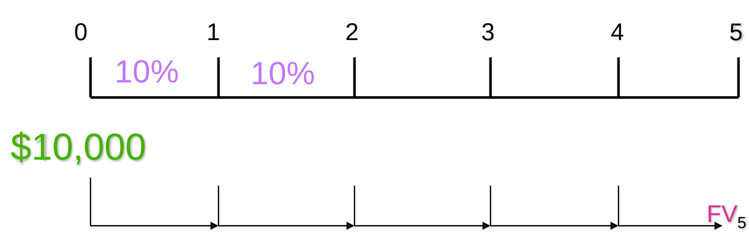
Story Problem Example

Julie Miller wants to know how large her deposit of $10,000 today will become at a compound annual interest rate of 10% for 5 years.



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Story Problem Solution

- Calculation based on general formula:

$$FV_n = P_0 (1+i)^n$$
$$FV_5 = \$10,000 (1+ 0.10)^5$$
$$= \$16,105.10$$

- Calculation based on Table I:

$$FV_5 = \$10,000 (FVIF_{10\%, 5})$$
$$= \$10,000 (1.611)$$
$$= \$16,110$$ [*Due to Rounding*]

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Double Your Money!!!

Quick! How long does it take to double $5,000 at a compound rate of 12% per year (approx.)?

We will use the "Rule-of-72".

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The "Rule-of-72"

Quick! How long does it take to double $5,000 at a compound rate of 12% per year (approx.)?

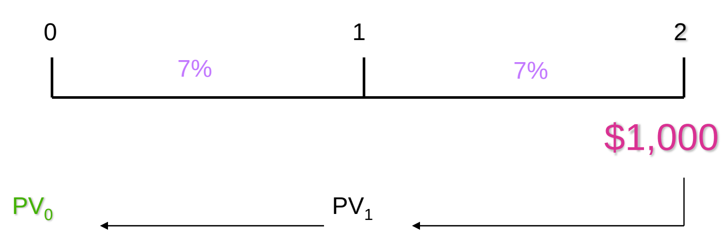
Approx. Years to Double = 72 / i%

72 / 12% = ***6 Years***

[Actual Time is 6.12 Years]

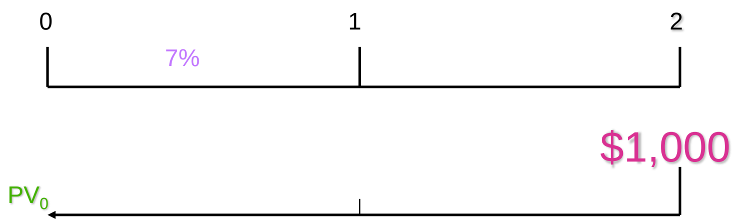
Present Value Single Deposit (Graphic)

Assume that you need $1,000 in 2 years. Let's examine the process to determine how much you need to deposit today at a discount rate of 7% compounded annually.



Present Value Single Deposit (Formula)

$$PV_0 = FV_2 / (1+i)^2 = \$1{,}000 / (1.07)^2$$
$$= FV_2 / (1+i)^2 = \$873.44$$



General Present Value Formula

$$PV_0 = FV_1 / (1+i)^1$$
$$PV_0 = FV_2 / (1+i)^2$$

etc.

General Present Value Formula:

$$PV_0 = FV_n / (1+i)^n$$

Or $PV_0 = FV_n (PVIF_{i,n})$ -- *See Table II*

Or $PV_0 = FV_n$ (P/F, i, n)

Valuation Using Table II

$PVIF_{i,n}$ is found on Table II at the end of the book.

Period	6%	7%	8%
1	.943	.935	.926
2	.890	**.873**	.857
3	.840	.816	.794
4	.792	.763	.735
5	.747	.713	.681

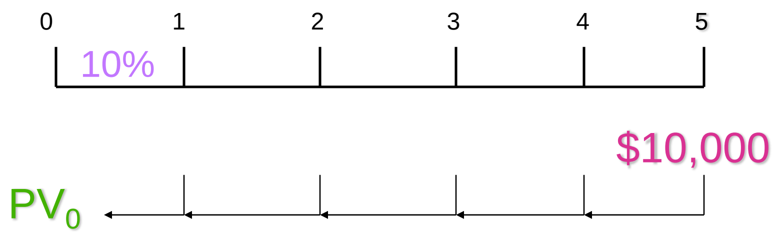
Using Present Value Tables

PV_2 = \$1,000 ($PVIF_{7\%,2}$)
= \$1,000 (.873)
= \$873 [Due to Rounding]

Period	6%	7%	8%
1	.943	.935	.926
2	.890	**.873**	.857
3	.840	.816	.794
4	.792	.763	.735
5	.747	.713	.681

Story Problem Example

Julie Miller wants to know how large of a deposit to make so that the money will grow to $10,000 in 5 years at a discount rate of 10%.



Story Problem Solution

- Calculation based on general formula:
  $PV_0 = FV_n / (1+i)^n$
  $PV_0 = \$10{,}000 / (1+ 0.10)^5$
  $= \$6{,}209.21$
- Calculation based on Table I:
  $PV_0 = \$10{,}000\ (PVIF_{10\%,\,5})$
  $= \$10{,}000\ (.621)$
  $= \$6{,}210.00$ [*Due to Rounding*]

Types of Annuities

- *An Annuity* represents a series of equal payments (or receipts) occurring over a specified number of equidistant periods.
- Ordinary Annuity: Payments or receipts occur at the end of each period.
- Annuity Due: Payments or receipts occur at the beginning of each period.

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Examples of Annuities

- Student Loan Payments
- Car Loan Payments
- Insurance Premiums
- Mortgage Payments
- Retirement Savings

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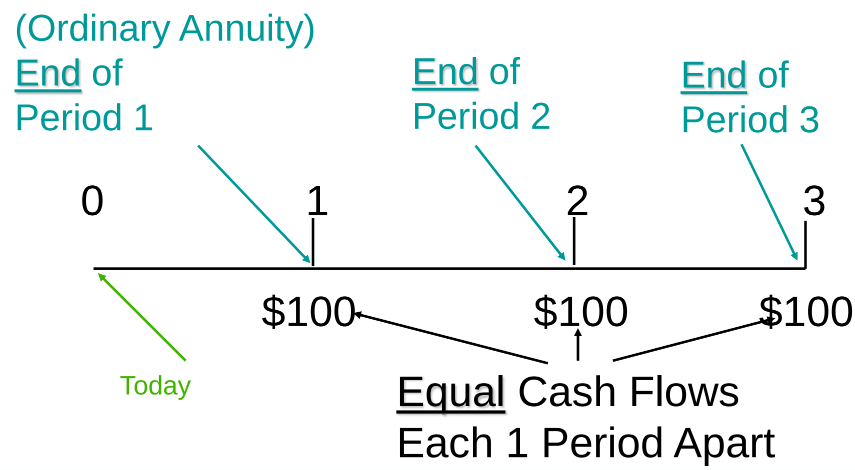
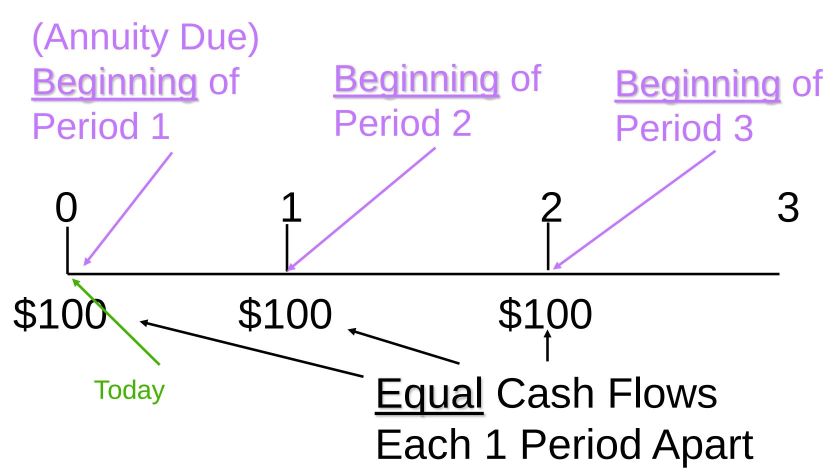
Parts of an Annuity

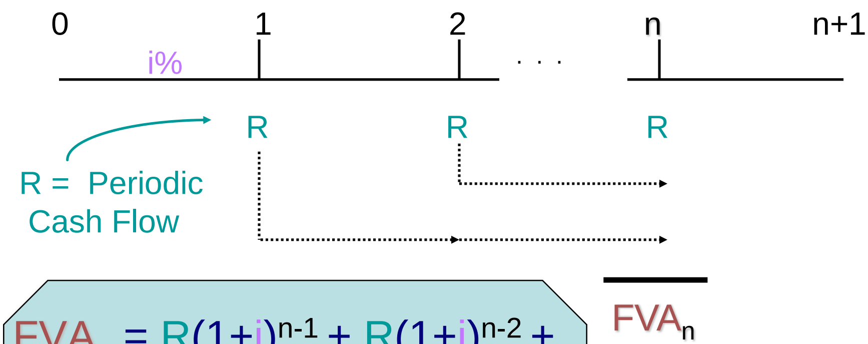


Parts of an Annuity



Overview of an Ordinary Annuity -- FVA

Cash flows occur at the end of the period



$$FVA_n = R(1+i)^{n-1} + R(1+i)^{n-2} + \ldots + R(1+i)^1 + R(1+i)^0$$

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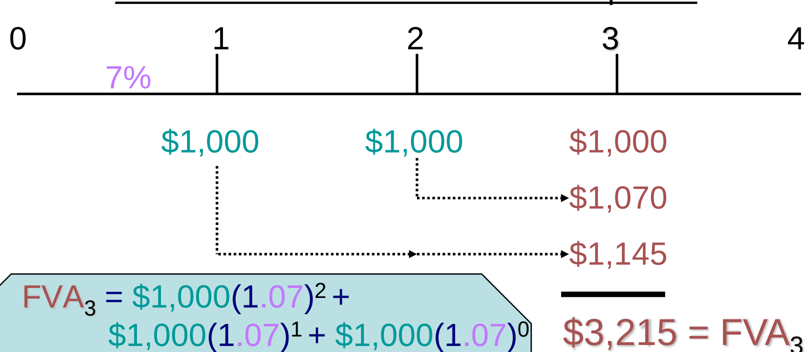
Example of an Ordinary Annuity -- FVA

Cash flows occur at the end of the period



$$FVA_3 = \$1{,}000(1.07)^2 + \$1{,}000(1.07)^1 + \$1{,}000(1.07)^0$$
$$= \$1{,}145 + \$1{,}070 + \$1{,}000$$
$$= \$3{,}215$$

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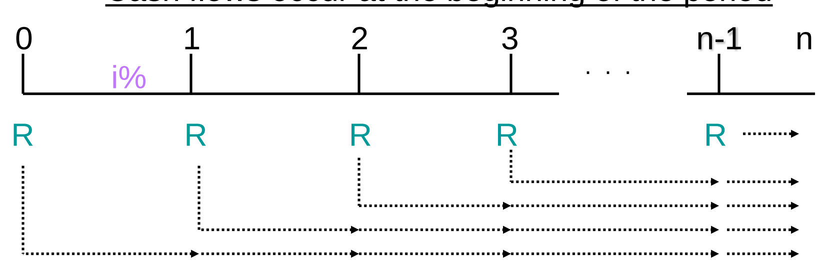
Valuation Using Table III

$$FVA_n = R\,(FVIFA_{i\%,n})$$
$$FVA_3 = \$1{,}000\,(FVIFA_{7\%,3})$$
$$= \$1{,}000\,(3.215) = \$3{,}215$$

Period	6%	7%	8%
1	1.000	1.000	1.000
2	2.060	2.070	2.080
3	3.184	**3.215**	3.246
4	4.375	4.440	4.506
5	5.637	5.751	5.867

Overview View of an Annuity Due -- FVAD

Cash flows occur at the beginning of the period



$$FVAD_n = R(1+i)^n + R(1+i)^{n-1} + \ldots + R(1+i)^2 + R(1+i)^1 = FVA_n\,(1+i)$$

$FVAD_n$

Example of an Annuity Due -- FVAD

Cash flows occur at the beginning of the period

0 1 2 3 4

7%

$1,000 $1,000 $1,000 → $1,070

→ $1,145

→ $1,225

$3,440 = $FVAD_3$

$$FVAD_3 = \$1{,}000(1.07)^3 + \$1{,}000(1.07)^2 + \$1{,}000(1.07)^1$$
$$= \$1{,}225 + \$1{,}145 + \$1{,}070$$
$$= \$3{,}440$$

Valuation Using Table III

$$FVAD_n = R\ (FVIFA_{i\%,n})(1+i)$$
$$FVAD_3 = \$1{,}000\ (FVIFA_{7\%,3})(1.07)$$
$$= \$1{,}000\ (3.215)(1.07) = \$3{,}440$$

Period	6%	7%	8%
1	1.000	1.000	1.000
2	2.060	2.070	2.080
3	3.184	**3.215**	3.246
4	4.375	4.440	4.506
5	5.637	5.751	5.867

Overview of an Ordinary Annuity -- PVA

Cash flows occur at the end of the period

0 1 2 . . . n n+1

i%

R R R

R = Periodic Cash Flow

PVA_n

$$PVA_n = R/(1+i)^1 + R/(1+i)^2 + \ldots + R/(1+i)^n$$

Example of an Ordinary Annuity -- PVA

Cash flows occur at the end of the period

0 1 2 3 4

7%

\$1,000 \$1,000 \$1,000

\$934.58

\$873.44

\$816.30

$\$2,624.32 = PVA_3$

$$PVA_3 = \$1,000/(1.07)^1 + \$1,000/(1.07)^2 + \$1,000/(1.07)^3$$

$$= \$934.58 + \$873.44 + \$816.30$$

$$= \$2,624.32$$

Valuation Using Table IV

$PVA_n = R\ (PVIFA_{i\%,n})$

$PVA_3 = \$1{,}000\ (PVIFA_{7\%,3})$

$= \$1{,}000\ (2.624) = \$2{,}624$

Period	6%	7%	8%
1	0.943	0.935	0.926
2	1.833	1.808	1.783
3	2.673	**2.624**	2.577
4	3.465	3.387	3.312
5	4.212	4.100	3.993

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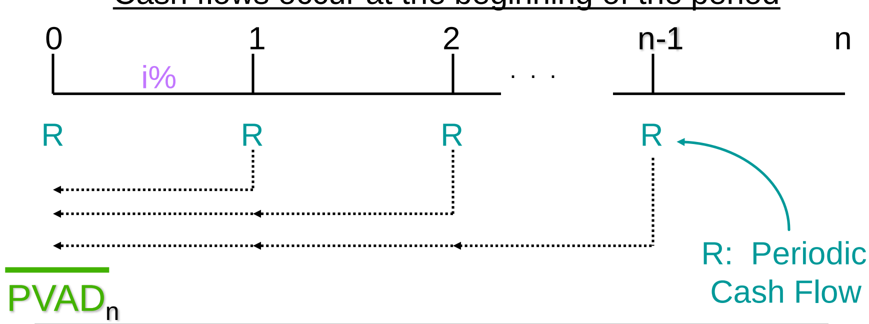
Overview of an Annuity Due -- PVAD

Cash flows occur at the beginning of the period



$PVAD_n = R/(1+i)^0 + R/(1+i)^1 + \ldots + R/(1+i)^{n-1}$

$= PVA_n\ (1+i)$

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Example of an Annuity Due -- PVAD

Cash flows occur at the beginning of the period

0	1	2	3	4
7%				
$1,000.00	$1,000	$1,000		

$ 934.58

$ 873.44

$2,808.02 = $PVAD_n$

$$PVAD_n = \$1{,}000/(1.07)^0 + \$1{,}000/(1.07)^1 + \$1{,}000/(1.07)^2 = \$2{,}808.02$$

Valuation Using Table IV

$$PVAD_n = R\ (PVIFA_{i\%,n})(1+i)$$

$$PVAD_3 = \$1{,}000\ (PVIFA_{7\%,3})(1.07)$$

$$= \$1{,}000\ (2.624)(1.07) = \$2{,}808$$

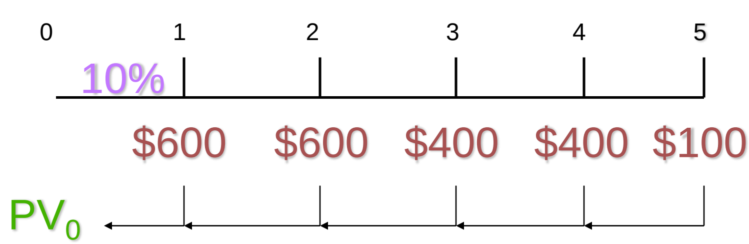
Period	6%	7%	8%
1	0.943	0.935	0.926
2	1.833	1.808	1.783
3	2.673	**2.624**	2.577
4	3.465	3.387	3.312
5	4.212	4.100	3.993

Steps to Solve Time Value of Money Problems

1. Read problem thoroughly
2. Create a time line
3. Put cash flows and arrows on time line
4. Determine if it is a PV or FV problem
5. Determine if solution involves a single CF, annuity stream(s), or mixed flow
6. Solve the problem
7. Check with financial calculator (optional)

Mixed Flows Example

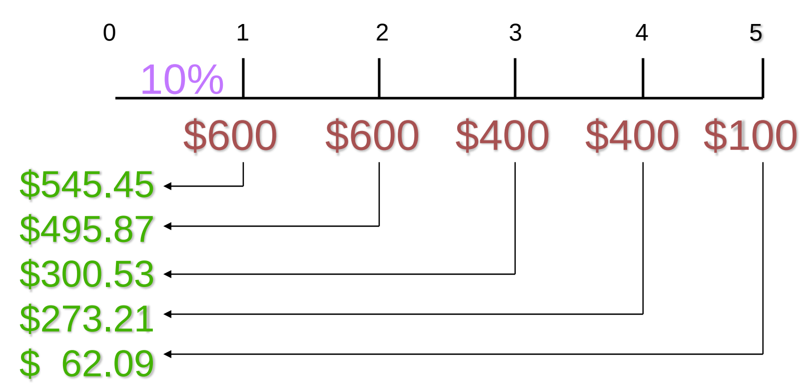
Julie Miller will receive the set of cash flows below. What is the Present Value at a discount rate of 10%.



How to Solve?

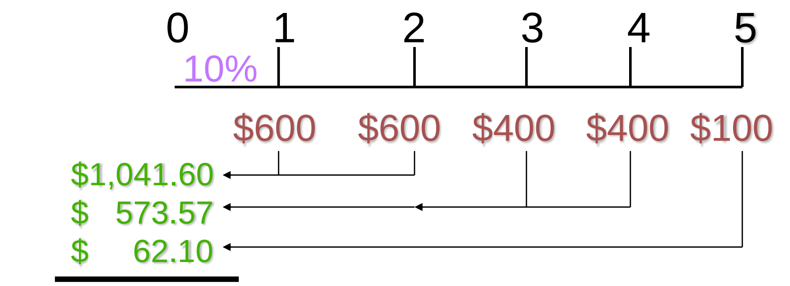
1. Solve a "*piece-at-a-time*" by discounting each *piece* back to t=0.
2. Solve a "*group-at-a-time*" by first breaking problem into <u>groups of annuity streams</u> and any <u>single cash flow groups</u>. Then discount each *group* back to t=0.

"Piece-At-A-Time"



$1677.15 = PV_0 of the Mixed Flow

"Group-At-A-Time" (#1)



$1,677.27 = PV_0 of Mixed Flow [Using Tables]

$600($PVIFA_{10\%,2}$) = $600(1.736) = $1,041.60

$400($PVIFA_{10\%,2}$)($PVIF_{10\%,2}$) = $400(1.736)(0.826) = $573.57

$100 ($PVIF_{10\%,5}$) = $100 (0.621) = $62.10

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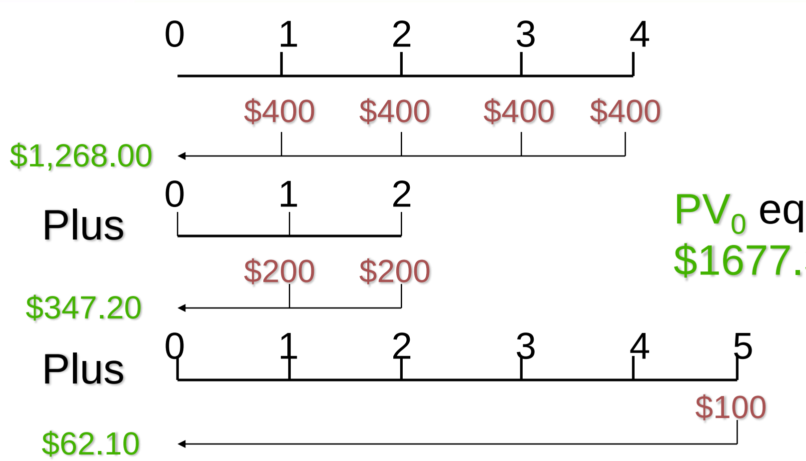
"Group-At-A-Time" (#2)



PV_0 equals $1677.30.

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Frequency of Compounding

General Formula:

$FV_n = PV_0(1 + [r/m])^{mn}$

n: Number of Years
m: Compounding Periods per Year
r: Nominal Annual Interest Rate
FV_n: FV at the end of Year n
PV_0: PV of the Cash Flow today

Impact of Frequency

Julie Miller has \$1,000 to invest for 2 Years at an annual interest rate of 12%.

Annual $FV_2 = 1{,}000(1+ [.12/1])^{(1)(2)}$
$= 1{,}254.40$

Semi $FV_2 = 1{,}000(1+ [.12/2])^{(2)(2)}$
$= 1{,}262.48$

Impact of Frequency

Qrtly $FV_2 = 1{,}000(1 + [.12/4])^{(4)(2)}$
$= 1{,}266.77$

Monthly $FV_2 = 1{,}000(1 + [.12/12])^{(12)(2)}$
$= 1{,}269.73$

Daily $FV_2 = 1{,}000(1+[.12/365])^{(365)(2)}$
$= 1{,}271.20$

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Effective Annual Interest Rate

Effective Annual Interest Rate

The actual rate of interest earned (paid) after adjusting the *nominal rate* for factors such as the number of compounding periods per year.

$$(1 + [\, r / m \,])^m - 1$$

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BWs Effective Annual Interest Rate

Basket Wonders (BW) has a $1,000 CD at the bank. The interest rate is 6% compounded quarterly for 1 year. What is the Effective Annual Interest Rate (EAR)?

EAR = (1 + 6% / 4)^4 - 1
= 1.0614 - 1 = .0614 or 6.14%!

Effective Annual Interest Rate for Continuous Compounding

Effective Annual Interest Rate

The actual rate of interest earned (paid) after adjusting the *nominal rate* for continuous compounding.

$$e^r - 1$$

Annual Percentage Yield (APY) Annual Percentage Rate (APR)

- APY: similar to the **effective annual interest rate,** based on the actual number of days for which the money is deposited in an account in a 365-day year (366 days in a leap year)
- APR: APR takes account of the interest rate and upfront charges paid by the borrower, whether expressed as a percent of the loan or in dollars.
  - Truth-in lending Act
  - Bank determines an effective periodic interest rate, based on usable funds, then simply multiply this rate by the number of such periods in a year.

Example of Annual Percentage Rate (APR)

- Mortgage: $200,000 for 10 years
- Mortgage rate: 3%
- Fees: 5%
- Monthly payment: $2,027.78
- Effective monthly rate: 0.3359%
- APR = 4.0302%

Steps to Amortizing a Loan

1. Calculate the payment per period.
2. Determine the interest in Period t.
   (Loan Balance at t-1) x (i% / m)
3. Compute principal payment in Period t.
   (Payment - Interest from Step 2)
4. Determine ending balance in Period t.
   (Balance - principal payment from Step 3)
5. Start again at Step 2 and repeat.

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Amortizing a Loan Example

Julie Miller is borrowing \$10,000 at a compound annual interest rate of 12%. Amortize the loan if annual payments are made for 5 years.

Step 1: Payment

$$PV_0 = R\ (PVIFA_{i\%,n})$$

$$\$10{,}000 = R\ (PVIFA_{12\%,5})$$

$$\$10{,}000 = R\ (3.605)$$

$$R = \$10{,}000 / 3.605 = \$2{,}774$$

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Amortizing a Loan Example

End of Year	Payment	Interest	Principal	Ending Balance
0	---	---	---	$10,000
1	$2,774	$1,200	$1,574	8,426
2	2,774	1,011	1,763	6,663
3	2,774	800	1,974	4,689
4	2,774	563	2,211	2,478
5	2,775	297	2,478	0
	$13,871	$3,871	$10,000	

[Last Payment Slightly Higher Due to Rounding]

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Usefulness of Amortization

1. Determine Interest Expense -- Interest expenses may reduce taxable income of the firm.
2. Calculate Debt Outstanding -- The quantity of outstanding debt may be used in financing the day-to-day activities of the firm.

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Summary of Interest Formulas

Name	Financial Symbols	Eng. Eco. Symbols	Equations
Future Value Interest Formula	$FVIF_{i,n}$	(F/P, i, n)	$F = P(1+i)^n$
Present Value Interest Formula	$PVIF_{i,n}$	(P/F, i, n)	$P = F/(1+i)^n$
Future Value Interest Formula of an Annuity	$FVIFA_{i,n}$	(F/A, i, n)	$F = A\left[\frac{(1+i)^n - 1}{i}\right]$
Present Value Interest Formula of an Annuity	$PVIFA_{i,n}$	(P/A, i, n)	$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$

Spreadsheet Annuity Functions

Excel Functions	Purpose
-PV (*i, n, A,* [F], [Type]**)**	To find P given i, n, and A (F is optional)
-PMT (*i, n, P,* [F], [Type]**)**	To find A given i, n, and P (F is optional)
-FV (*i, n, A,* [P], [Type]**)**	To find F given i, n, and A (P is optional)
NPER (*i, A, P,* [F], [Type]**)**	To find n given i, A, and P (F, optional)
RATE (*n, A, P,* [F], [Type], [guess]**)**	To find i given n, A, and P (F, optional)

Note:
1. Type is the number 0 or 1 and indicates when payments are due (0 for end-of-the-period, and 1 for beginning-of-the-period). If type is omitted, it is assumed to be 0.
2. Microsoft Excel solves for one financial argument in terms of the others. P(F/P, i, n) + A(F/A, i, n) + F = 0 Therefore, negative signs are added to find the equivalent values in P, A, and F.

Spreadsheet Block Functions

Excel Functions	Purpose
NPV (*i, range*)	To find net present value of a range of cash flows (from period 1 to n) at a given interest rate
IRR (range, [guess]**)**	To find internal rate of return from a range of cash flows (from period 0 to n)