Simultaneous Confidence Intervals

Consider the following data corresponding to a randomized block design with $k = 4$ treatments and $b = 5$ blocks.

	Block 1	Block 2	Block 3	Block 4	Block 5
Treatment 1	-3.0	-5.0	-5.5	-6.4	-5.5
Treatment 2	-2.3	-2.3	-1.8	-2.8	-1.6
Treatment 3	-0.3	1.6	1.5	-1.6	0.3
Treatment 4	6.2	4.6	5.5	5.5	3.7

We calculated the following the sums of squares in an earlier example:

A $100(1-\alpha)\%$ confidence interval for the difference between τ_i and τ_j is given by

$$\bar{Y}_{i\bullet} - \bar{Y}_{j\bullet} \pm t_{\alpha/2} S \sqrt{\frac{2}{b}},$$

$S^2 = \text{MSE}$ gives an unbiased estimate of σ^2 with $n - b - k + 1 = 12$ degrees of freedom.

Note that we want to compare $\binom{4}{2} = 6$ pairs of the form $\tau_i - \tau_j$. If we want a collection of confidence intervals for $\tau_i - \tau_j$ for each (i, j) pair with $100(1-\alpha)\%$ *simultaneous confidence coefficient* then it suffices to find a two-sided $100(1-\alpha/6)\%$ confidence interval for $\tau_i - \tau_j$ for each pair (i, j).

Recall that we calculated the observed value of $S = \sqrt{\text{MSE}}$ in an earlier example:

$$S = \sqrt{\text{MSE}} = 1.0227414.$$

The following code calculates the required value of $t_{\alpha/12}$.

```
b <- 5
k <- 4
alpha <- 0.05
tval <- qt(p = alpha/12, df = (b-1)*(k-1), lower.tail = FALSE)
```

This gives

$$t_{\alpha/12} = t_{0.00417} = 3.1526813.$$

Putting everything together, we obtain the set of confidence intervals with 0.95 simultaneous confidence coefficient using the formula

$$\bar{Y}_{i\bullet} - \bar{Y}_{j\bullet} \pm t_{\alpha/12} S \sqrt{\frac{2}{b}} = \bar{Y}_{i\bullet} - \bar{Y}_{j\bullet} \pm (3.1526813)(1.0227414) \sqrt{\frac{2}{5}} = \bar{Y}_{i\bullet} - \bar{Y}_{j\bullet} \pm 2.0392755.$$

To calculate the individual confidence intervals for $\tau_i - \tau_j$, we first calculate the means $\bar{Y}_i$ using the `rowMeans` function and then calculate the end points of each confidence interval using a for-loop. The following code performs the necessary calculations.

```
# Create matrix storing observations.
y <- matrix(data = c(-3.0, -2.3, -0.3, 6.2,
                     -5.0, -2.3, 1.6, 4.6,
```

```r
                     -5.5, -1.8, 1.5, 5.5,
                     -6.4, -2.8, -1.6, 5.5,
                     -5.5, -1.6, 0.3, 3.7),
            nrow = 4, ncol = 5)

# Calculate row averages.
rm <- rowMeans(y)

# Initialize matrices of lower and upper bounds for confidence intervals.
lbs <- matrix(data = 0, nrow = 4, ncol = 4)
ubs <- matrix(data = 0, nrow = 4, ncol = 4)

# Calculate confidence intervals for each (i,j)
rad <- 2.0392755 # Radius calculated above.

for (i in 1:3){
  for (j in (i+1):4){

    # lower bound for (i,j) CI
    lbs[i,j] <- rm[i] - rm[j] - rad

    # upper bound for (i,j) CI
    ubs[i,j] <- rm[i] - rm[j] + rad

  } # end for j.
} # end for i.
```

This code yields two matrices. One whose (i, j)th entry gives the lower bound for the desired confidence interval for $\tau_i - \tau_j$ if $i < j$, and with (i, j)th entry equal to 0 if $i \geq j$. Similarly, the second matrix provides the corresponding upper bounds for the desired confidence intervals. This yields the following collection of confidence intervals for $\tau_i - \tau_j$ for $i < j$ with simultaneous confidence coefficient 0.95:

$$\begin{aligned}
-4.9592755 &< \tau_1 - \tau_2 < -0.8807245, \\
-7.4192755 &< \tau_1 - \tau_3 < -3.3407245, \\
-12.2192755 &< \tau_1 - \tau_4 < -8.1407245, \\
-4.4992755 &< \tau_2 - \tau_3 < -0.4207245, \\
-9.2992755 &< \tau_2 - \tau_4 < -5.2207245, \\
-6.8392755 &< \tau_3 - \tau_4 < -2.7607245.
\end{aligned}$$

Note that this suggests that we have significant evidence that $\tau_1 < \tau_2 < \tau_3 < \tau_4$.