The Market for Volatility Trading; Vix Futures

Article · May 2007

3 authors:

Menachem Brenner
New York University
58 PUBLICATIONS   **2,441** CITATIONS

SEE PROFILE

Jinghong Shu
University of International Business and Economics
7 PUBLICATIONS   **305** CITATIONS

SEE PROFILE

Jin E. Zhang
University of Otago
85 PUBLICATIONS   **1,854** CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Project   "A Market-Based Approach to Sharing the Economic Benefit and Consequences of Aging in the PRC – A Conceptual Solution View project

Project   Macroeconomic News and Markets View project

The Market for Volatility Trading; VIX Futures

Menachem Brenner
Stern School of Business
New York University
New York, NY 10012, U.S.A.
Email: mbrenner@stern.nyu.edu
Tel: (212) 998 0323, Fax: (212) 995 4731

Jinghong Shu
School of International Trade and Economics
University of International Business and Economics
Huixindongjie, Beijing, P. R. China
Email: shujinghong@uibe.edu.cn

Jin E. Zhang
School of Economics and Finance
and School of Business
The University of Hong Kong
Pokfulam Road, Hong Kong
Email: jinzhang@hku.hk

First Version: August 2006

This Version: May 2007

Key words: Volatility Trading; VIX; VIX Futures

JEL Classification Code: G13

The Market for Volatility Trading; VIX Futures

Abstract

This paper analyses the new market for trading volatility; VIX futures. We first use market data to establish the relationship between VIX futures prices and the index itself. We observe that VIX futures and VIX are highly correlated; the term structure of VIX futures price is upward sloping while the term structure of VIX futures volatility is downward sloping. To establish a theoretical relationship between VIX futures and VIX, we model the instantaneous variance using a simple square root mean-reverting process. Using daily calibrated variance parameters and VIX, the model gives good predictions of VIX futures prices. These parameter estimates could be used to price VIX options.

1. Introduction

Stochastic volatility was ignored for many years by academics and practitioners. Changes in volatility were usually assumed to be deterministic (e.g. Merton (1973)). The importance of stochastic volatility and its potential effect on asset prices and hedging/investment decisions has been recognized after the crash of '87. The industry and academia have started to examine it in the late 80s, empirically as well as theoretically. The need to hedge potential volatility changes which would require a reference index has been first presented by Brenner and Galai (1989).

In 1993 the Chicago Board Options Exchange (CBOE) has introduced a volatility index based on the prices of index options. This was an implied volatility index based on option prices of the S&P100 and it was traced back to 1986. Until about 1995 the index was not a good predictor of realized volatility. Since then its forecasting ability has improve markedly (see Corrado and Miller (2005)) though it is biased upwards. Although many market participants considered the index to be a good predictor of short term volatility, daily or even intraday, it took many years for the market to introduce volatility products, starting with over the counter products like variance swaps. The first exchange traded product, VIX futures, was introduced in March 2004 followed by VIX options in February 2006. These volatility derivatives use the VIX index as their underlying. The current VIX is based on a different methodology than the previous VIX, renamed VXO, and uses the S&P500 European style options rather than the S&P100 American style options. Despite these two major differences the correlation between the levels of the two indices is about 98%. (see Carr and Wu (2006)).

VIX is computed from the option quotes of all available calls and puts on the S&P500 (SPX) with a non-zero bid price (see the CBOE white paper[1]) using following formula

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2}e^{RT}Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0}-1\right)^2, \qquad (1)$$

where the volatility σ times 100 gives the value of the VIX index level. T is the 30 day volatility estimate. In practice options with 30-day maturity might not exist. Thus, the variances of the two near-term options, with at least 8 days left to expiration, are combined to obtain the 30-day variance. F is the implied forward index level derived from the nearest to the money index option prices by using put-call parity. K_i is the strike price of ith out-of-money options, ΔK_i is the interval between two strikes, K_0 is the first strike that is below the forward index level. R is the risk-free rate to expiration. $Q(K_i)$ is the midpoint of the bid-ask spread of each option with strike K_i.

Carr and Madan (1998), and Demeterfi et al (1999) developed the original idea of replicating realized variance by a portfolio of European options. In September 2003, the CBOE used their theory to design a new methodology to compute VIX.

We now briefly review the theory behind equation (1). If we assume that the strike price is distributed continuously from 0 to $+\infty$ and neglect the discretizing error, equation (1) becomes

$$\sigma^2 = \frac{2}{T}\left[e^{RT}\int_0^{K_0}\frac{1}{K^2}p(K)dK + e^{RT}\int_{K_0}^{+\infty}\frac{1}{K^2}c(K)dK\right] + \frac{2}{T}\left[\ln\frac{F}{K_0} - \left(\frac{F}{K_0}-1\right)\right]. \quad (2)$$

By construction, K_0 is very close to F, hence $\dfrac{F}{K_0}-1$ is very small but always positive. With

a Taylor series expansion we obtain

$$\ln \frac{F}{K_0} = \ln\left[1 + \left(\frac{F}{K_0} - 1\right)\right] = \left(\frac{F}{K_0} - 1\right) - \frac{1}{2}\left(\frac{F}{K_0} - 1\right)^2 + O\left(\frac{F}{K_0} - 1\right)^3.$$

By omitting the third order terms, $O\left(\frac{F}{K_0} - 1\right)^3$, the last term of equation (2) becomes that of equation (1). Carr and Madan (1998) and Demeterfi et al (1999) show that due to the following mathematical identity,

$$\ln \frac{S_T}{K_0} = \left(\frac{S_T}{K_0} - 1\right) - \int_0^{K_0} \frac{1}{K^2} \max(K - S_T, 0)dK - \int_{K_0}^{+\infty} \frac{1}{K^2} \max(S_T - K, 0)dK,$$

the risk-neutral expectation of the log of the terminal stock price over strike K_0 is

$$E_0^Q\left[\ln \frac{S_T}{K_0}\right] = \left(\frac{F}{K_0} - 1\right) - e^{RT}\int_0^{K_0} \frac{1}{K^2} p(K)dK - e^{RT}\int_{K_0}^{+\infty} \frac{1}{K^2} c(K)dK.$$

Hence equation (2) can be written as

$$\sigma^2 = \frac{2}{T}\left[\ln \frac{F}{K_0} - E_0^Q\left(\ln \frac{S_T}{K_0}\right)\right] = \frac{2}{T}\left[\ln \frac{F}{S_0} - E_0^Q\left(\ln \frac{S_T}{S_0}\right)\right]$$

$$= \frac{2}{T} E_0^Q\left[\int_0^T \frac{dS_t}{S_t} - d(\ln S_t)\right] = \frac{1}{T} E_0^Q\int_0^T \sigma_t^2 dt,$$

where the last equal sign is due to Ito's Lemma $d(\ln S_t) = \frac{dS_t}{S_t} - \frac{1}{2}\sigma_t^2 dt$, under the assumption that the SPX index follows a diffusion process, $dS_t = \mu S_t dt + \sigma_t S_t dB_t$ with a general stochastic volatility process, σ_t. So VIX^2 represents 30-day S&P 500 variance swap rate[2].

---

[2] In practice, the variance swap rate is quoted as volatility instead of variance. It should be noted that the realized variance can be replicated by a portfolio of all out-of-money calls and puts but the VIX index itself cannot be replicated by a portfolio of options because the computation of the VIX involves a square root operation against the price of a portfolio of options and the square root function is nonlinear.

On March 26, 2004, the newly created CBOE Futures Exchange (CFE) started to trade an exchange listed volatility product; VIX futures, a futures contract written on the VIX index. It is cash settled with the VIX. Since VIX is not a traded asset, one cannot replicate a VIX futures contract using the VIX and a risk free asset. Thus a cost-of-carry relationship between VIX futures and VIX cannot be established.

Our objective is two fold; First, to use market data to analyse empirically the relationship between VIX futures prices and VIX, the term structure of VIX futures prices and to estimate the volatility of VIX futures prices. Second, to find parameter estimates, using a simple stochastic volatility model, that best describe the empirical relationships and could possibly be used to price VIX futures and options.

2. Data

In this paper, we use the daily VIX index and VIX futures data provided by the CBOE. The VIX index data, including open, high, low and close levels, are available from January 2 1990 to the present. The VIX futures data, including open, high, low, close and settle prices, trading volume and open interest, are available from March 26 2004 to the present.

For each day we have four futures contracts: two near term and two additional months on the February quarterly cycle. For example, on the first day of the listing, 26 March 2004, four contracts K4, M4, Q4 and X4 were traded which stand for the following futures expiration: May, June, August and November 2004 respectively. The first letter indicates the expiration month followed by the expiration year. The underlying value of the VIX futures contract is VIX times 10 under the symbol "VXB"

$$VXB = 10 \times VIX. \qquad (3)$$

The contract size is $100 times VXB. For example, with a VIX value of 17.33 on 26 March 2004, the VXB would be 173.3 and the contract size would be $17,330. The settlement date is usually the Wednesday prior to the third Friday of the expiration month.

Our empirical study covers the period of two years and eight months from March 16 2004 to November 21 2006, within which there were 34 contract months traded all together. Table 1 provides a summary statistics of all of them. The average open interest for each contract is 2784, which corresponds to a market value of 38 million dollars[3]. The average daily trading volume for each contract is 181, which corresponds to 2.5 million dollars. The shortest contract lasted 35 days, while the longest 188 days. The average futures price for each contract changed from 185.6 for contracts that matured in May 2004 to 164.1 for contracts that will mature in August 2007 (the average was taken for samples up to the maturity date or November 21 2006, whichever is earlier), while the VIX level ranged from 17.33 on March 26 2004 to 9.90 on November 21 2006. In general, the market expected future volatility decreased during this period.

3. Empirical evidence

3.1. The relation between VIX futures and VIX (VXB)

Because the underlying variable of VIX futures, i.e. VXB, is not a traded asset, we are not able to obtain a simple cost-of-carry relationship, arbitrage free, between the futures price, F_t^T, and its underlying, VXB_t. That is,

$$F_t^T \neq VXB_t e^{r(T-t)},$$

---

[3] Using the average VIX futures prices 135.5, we compute the market value as $135.5 \times 100 \times 2784 = 37,723,200$.

where r is the interest rate, and T is the maturity. Thus, we have gone to the data to see what we can learn about the relationship between VIX futures prices and VXB. We use this relationship to estimate the parameters, in a stochastic volatility model, that could be used to price volatility derivatives.

There are four futures contracts available on a typical day. For example, on 26 March 2004, we have four kind of VIX futures with maturities, in May, June, August and November, which corresponds to times to maturity of 53, 80, 142 and 231 days. We construct 30, 60 and 90-day futures prices by a linear interpolation technique. For example, the 30-day futures price is computed by using the market data of VXB and May futures on 26 March 2004. The 60-day futures price is computed by using the market data of May and June futures. The 90-day futures price is computed with June and August futures. We calculate these fixed time-to-maturity futures price on each day and obtain three time series of 30-, 60-, and 90-day futures prices. Figure 1 shows the time series of VXB and VIX futures for three fixed time-to-maturities. Intuitively the four time series are highly correlated. Table 2 presents the correlation matrix between the returns of S&P 500 index, VXB and VIX futures. The return is computed as the logarithm of the price relative on two consecutive ends of day prices. All of the four series are negatively correlated with the S&P 500 index. VXB and VIX futures with three different maturities are almost perfectly correlated. Figure 1 also shows that the trading volume of VIX futures has been gradually increasing.

Figure 2 shows the relationship between 30-day VIX futures and VXB for the market data from 26 March 2004 to 21 November 2006. In general, the VIX futures price is an increasing function of VXB. The higher the VIX (VXB), the higher is the price of VIX futures with a given maturity.

7

3.2. The term structure of VIX futures price

Over the period of March 26 2004 to 21 November 2006, the average VXB was 135.5. The average VIX futures prices were 144.8, 152.8 and 157.8 for 30-, 60- and 90-day maturities respectively. The term structure of the average VIX futures price is upward sloping, which is demonstrated graphically in Figure 3.

The upward sloping VIX futures term structure indicates that the current level of volatility is relatively low compared with the long-term mean level and that the volatility is increasing to the long-term high level.

3.3. The volatility of VXB and VIX futures

With the time series of VXB and fixed maturity VIX futures price, we compute the standard deviation of daily log price (index) relatives to obtain estimates of the volatility of these four series, assuming that these series follow a lognormal process. During the two year period of our study we estimated the volatility of VXB to be 84.4%, while the volatilities of VIX futures price are 35.7%, 30.0% and 25.8% for 30, 60 and 90 day maturities respectively. The longer the maturity, the lower is the volatility of volatility. Figure 4 shows the volatility of VXB and fixed maturity VIX futures price. The term structure of VIX futures volatility is downward sloping.

The phenomenon of downward sloping VIX futures volatility is consistent with the mean-reverting feature of the volatility. Since the long-term volatility approaches to a fixed level, long-tenor VIX futures should be less volatile than short-tenor ones.

4. A Theoretical Model of VIX Futures price

4.1. VIX futures price

We now use a simple theoretical model to price the futures contracts using parameter estimates obtained from market data. We then test the extent to which model prices can explain market prices.

In the physical measure, the SPX index, S_t, is assumed to follow Heston (1993) stochastic variance model

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_{1t}^P,$$

$$dV_t = \kappa^P(\theta^P - V_t)dt + \sigma_V \sqrt{V_t} dB_{2t}^P,$$

where μ is the expected return, θ^P is long-term mean level of the instantaneous variance, κ^P is the mean-reverting speed of the variance, σ_V measures the volatility of variance, dB_{1t}^P and dB_{2t}^P are increments of two Brownian motions that describe the random noises in SPX index return and variance. They are assumed to be correlated with a constant coefficient, ρ.

By changing probability measure from P to Q as follows,

$$dB_{1t}^P = dB_{1t}^Q - \frac{\mu - r}{\sqrt{V_t}} dt, \qquad dB_{2t}^P = dB_{2t}^Q - \frac{\lambda}{\sigma_V} \sqrt{V_t} dt,$$

where r is the risk-free rate, and λ is the market price of variance risk, we obtain the dynamics of the SPX index in the risk-neutral measure

$$dS_t = rS_t dt + \sqrt{V_t} S_t dB_{1t}^Q,$$

$$dV_t = [\kappa^P \theta^P - (\kappa^P + \lambda)V_t]dt + \sigma_V \sqrt{V_t} dB_{2t}^Q,$$

where dB_{1t}^Q and dB_{2t}^Q are increments of two Q Brownian motions with the correlation ρ.

We define risk-neutral long-term mean level θ and mean-reverting speed κ as

$$\theta = \frac{\kappa^P \theta^P}{\kappa^P + \lambda}, \qquad \kappa = \kappa^P + \lambda.$$

Then the risk-neutral dynamics of the instantaneous variance can be written as

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dB_{2t}^Q, \qquad (4)$$

The transition probability density, given by Cox, Ingersoll and Ross (1985), is

$$f(V_T \mid V_t) = ce^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q\left(2\sqrt{uv}\right), \qquad (5)$$

where

$$c = \frac{2\kappa}{\sigma_V^2[1 - e^{-\kappa(T-t)}]}, \quad u = cV_t e^{-\kappa(T-t)}, \qquad v = cV_T, \qquad q = \frac{2\kappa\theta}{\sigma_V^2} - 1,$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. The distribution function is the noncentral chi-square, $\chi^2(2v; 2q+2, 2u)$, with $2q+2$ degrees of freedom and parameter of noncentrality $2u$ proportional to the current variance, V_t.

With the risk-neutral dynamics of the variance, we can evaluate the first three conditional moments of the future variance, V_s, $0 < t < s$, as follows

$$E_t^Q(V_s) = \theta + (V_t - \theta)e^{-\kappa(s-t)},$$

$$E_t^Q\left[\left(V_s - E_t^Q(V_s)\right)^2\right] = \sigma_V^2 V_t e^{-\kappa(s-t)}\frac{1 - e^{-\kappa(s-t)}}{\kappa} + \sigma_V^2\theta\frac{\left(1 - e^{-\kappa(s-t)}\right)^2}{2\kappa},$$

$$E_t^Q\left[\left(V_s - E_t^Q(V_s)\right)^3\right] = \frac{3}{2}\sigma_V^4 V_t e^{-\kappa(s-t)}\frac{\left(1 - e^{-\kappa(s-t)}\right)^2}{\kappa^2} + \frac{1}{2}\sigma_V^4\theta\frac{\left(1 - e^{-\kappa(s-t)}\right)^3}{\kappa^2},$$

where E_t^Q stands for the conditional expectation in the risk-neutral measure.

10

The VIX index at current time t is defined as the variance swap rate over the next 30 calendar days. It is equal to the risk-neutral expectation of the future variance over the period of 30 days from t to $t + \tau_0$ with $\tau_0 = 30/365$,

$$\left(\frac{VIX_t}{100}\right)^2 = E_t^Q\left[\frac{1}{\tau_0}\int_t^{t+\tau_0} V_s ds\right] = \frac{1}{\tau_0}\int_t^{t+\tau_0} E_t^Q(V_s)ds$$

$$= \frac{1}{\tau_0}\int_t^{t+\tau_0}\left[\theta + (V_t - \theta)e^{-\kappa(s-t)}\right]ds = (1-B)\theta + BV_t, \qquad (6)$$

where $B = \dfrac{1 - e^{-\kappa\tau_0}}{\kappa\tau_0}$ is a number between 0 and 1. Hence $\left(\dfrac{VIX_t}{100}\right)^2$ is the weighted average between long-term mean level θ and instantaneous variance V_t with B as the weight. Notice that the correlation, ρ, does not enter into the VIX formula, hence the VIX values do not capture the skewness of stock return.

The price of VIX futures with maturity T is then determined by

$$F_t^T = E_t^Q(VXB_T) = E_t^Q(10 \times VIX_T) = 1000 \times E_t^Q\left[\sqrt{(1-B)\theta + BV_T}\right]$$

$$= 1000 \times \int_0^{+\infty}\sqrt{(1-B)\theta + BV_T} \times f(V_T \mid V_t)dV_T . \qquad (7)$$

Equations (6) and (7) determine the VIX futures price, F_t^T, as a function of its underlying, VXB_t ($10 \times VIX_t$) and time to maturity T-t with three parameters, κ, θ and σ_V as follows

$$F_t^T = F_t^T(VXB_t, T - t; \kappa, \theta, \sigma_V) . \qquad (8)$$

Given the values of parameters, κ, θ, σ_V and current level of VXB_t, we can back out V_t from equation (6) and then calculate with equation (7) the current VIX futures price, F_t^T, for different maturities T. Equation (7) can be regarded as a closed-form formula for the VIX futures price.

11

To further the relation between F_t^T and VXB_t, we expand $\sqrt{(1-B)\theta + BV_T}$ with Taylor expansion near the point of $E_t^Q(V_T)$ and obtain

$$[(1-B)\theta + BV_T]^{1/2} = [(1-B)\theta + BE_t^Q(V_T)]^{1/2} + \frac{1}{2}[(1-B)\theta + BE_t^Q(V_T)]^{-1/2} B[V_T - E_t^Q(V_T)]$$

$$-\frac{1}{8}[(1-B)\theta + BE_t^Q(V_T)]^{-3/2} B^2[V_T - E_t^Q(V_T)]^2$$

$$+\frac{1}{16}[(1-B)\theta + BE_t^Q(V_T)]^{-5/2} B^3[V_T - E_t^Q(V_T)]^3 + O([V_T - E_t^Q(V_T)]^4),$$

where $O(.)$ stands for the higher order terms. Taking expectation in the risk-neutral measure gives an approximate formula for the VIX futures price

$$\frac{F_t^T}{1000} = [\theta(1 - Be^{-\kappa(T-t)}) + V_t Be^{-\kappa(T-t)}]^{1/2}$$

$$-\frac{\sigma_V^2}{8}[\theta(1 - Be^{-\kappa(T-t)}) + V_t Be^{-\kappa(T-t)}]^{-3/2} B^2\left[V_t e^{-\kappa(T-t)}\frac{1 - e^{-\kappa(T-t)}}{\kappa} + \theta\frac{(1 - e^{-\kappa(T-t)})^2}{2\kappa}\right]$$

$$+\frac{\sigma_V^4}{16}[\theta(1 - Be^{-\kappa(T-t)}) + V_t Be^{-\kappa(T-t)}]^{-5/2} B^3\left[\frac{3}{2}V_t e^{-\kappa(T-t)}\frac{(1 - e^{-\kappa(T-t)})^2}{\kappa^2} + \frac{1}{2}\theta\frac{(1 - e^{-\kappa(T-t)})^3}{\kappa^2}\right] + O(\sigma_V^6)$$

(9)

Numerical results show that the approximate formula is very accurate for reasonable set of parameter values. Evaluating VIX futures price with the approximate formula is much faster. This proves to be very important in the model calibration exercises when large numbers of price calculation are required.

4.2. Estimating model parameters from VIX values

Given the physical process of the instantaneous variance we can obtain the likelihood function of the variance over one day. In equation (6) we relate the VIX index to the instantaneous variance; hence we would be able to evaluate the likelihood of the VIX over

the day. By maximizing the likelihood function over the last 42 trading days, we can estimate the four parameters, κ^P, θ^P, λ and σ_V for each day[4]. The daily estimated parameters are presented in Figure 5. Their average values are 6.9671, 0.02492, -1.02046 and 0.3501 respectively; their standard deviations are 0.9914, 0.01298, 0.7882 and 0.1375 respectively. The negative value of λ is consistent with the empirical phenomenon of negative volatility/variance risk premium[5].

We then price VIX futures by using equation (7) with the estimated parameters. The model price and market price are presented in Figure 6. In general, the model price seems to be highly correlated with the market price, but there is a non trivial gap between the prices The root mean squared error is about 10-15% of the VIX futures price.

4.3. Calibrating the VIX futures price model

We then used the term structure of VIX futures to get parameter estimates that may provide better model prices. For example, on 26 March 2004, the market prices are given by $VXB_0 = 173.3$, $(T_1, F_{0mkt}^{T_1}) = (30/365, 190.2)$, $(T_2, F_{0mkt}^{T_2}) = (60/365, 202.8)$ and $(T_3, F_{0mkt}^{T_3}) = (90/365, 201.5)$, we solve the following optimization problem

$$\min_{(\kappa,\theta,\sigma V)} \sum_{i=1}^{3} \left(F_{0mdl}^{T_i}(VXB_0, T_i; \kappa, \theta, \sigma V) - F_{0mkt}^{T_i} \right)^2$$

and obtain the following values of the three parameters[6]

---

[4] Zhang and Zhu (2006) use the same method to estimate the parameters for the whole 1990 to 2005 period.

[5] Coval and Shumway (2001) report a negative return of zero-beta, at-the-money straddle positions. Bakshi and Kapadia (2003) examine the negative market volatility risk premium by using Delta-hedged option portfolios. Carr and Wu (2003) study the variance risk premia on indexes and individual stocks by using the return of variance swaps.

[6] It takes 420 seconds for one calibration exercise if we use the exact formula (7), however it takes only one second if we use the approximate formula (9). The results from two formulas are very close each other. For efficiency, we will use the approximate formula in our calibration from now on.

$$\kappa = 7.6246, \qquad \theta = 0.04396, \qquad \sigma_V = 0.2005.$$

Figure 7 shows the daily calibrated set of parameters. When we do the calibration we set bounds for the three parameters $4 \le \kappa \le 8$, $0.01 \le \theta \le 0.25$, and $0.2 \le \sigma_V \le 0.8$. The calibrated values of κ and σ_V are somehow randomly distributed within the bounds. The calibrated values of θ seem to be more stable than the other two parameters. The average values of the three parameters κ, θ and σ_V are 5.5805, 0.03259 and 0.5885 respectively. Their standard deviations are 1.5246, 0.00909 and 0.2398 respectively.

With the daily calibrated set of parameters we can recover the latent instantaneous variance/volatility variable by using formula (6). The result is shown in Figure 8 together with the VIX index. The two time series seem to be highly correlated.

With the set of parameters and market value of VXB on day t-1 we can compute the model price of VIX futures on the next day, t. We then compare the market prices with the model prices. Figure 9 shows that the model prices are remarkably close to the market prices of 30-, 60- and 90-day VIX futures. The root of mean squared error between the two prices is around 2.3 dollars, which is less than 2% of the average VIX futures price of around 150. Our simple mean-reverting variance model captures the dynamics of VIX futures price very well.

4.4. Volatility of the VIX index and VIX futures

From equations (6) and (4), applying Ito's Lemma gives us a stochastic process for the VIX index as follows

$$\frac{dVIX_t}{VIX_t} = \left[\frac{1}{2}\left(\frac{100}{VIX_t}\right)^2 B\kappa(\theta - V_t) - \frac{1}{8}\left(\frac{100}{VIX_t}\right)^4 B^2\sigma_V^2 V_t \right] dt + \frac{1}{2}\left(\frac{100}{VIX_t}\right)^2 B\sigma_V \sqrt{V_t} dB_{2t}^Q.$$

The volatility of the VIX index (or VXB) is then given by

$$\sigma_{VIX} = \frac{1}{2}\left(\frac{100}{VIX_t}\right)^2 B\sigma_V \sqrt{V_t} \ .$$

With a reasonable set of parameters, for example, $\kappa = 8.0$, $\sigma_V = 0.3501$, $VIX_t = 13.55$, $V_t = 0.01489$, we obtain a volatility of 85.2%, which is close to that computed from the market values of the VIX index.

We now define volatility hedge ratio, Δ, to be the sensitivity of the VIX futures price with respect to VXB, i.e.,

$$\Delta = \frac{\partial F_t^T(VXB_t, T-t)}{\partial VXB_t} \ . \tag{10}$$

With the average $VXB_0 = 135.5$ and average values of the calibrated parameters $\kappa = 5.5805$, $\theta = 0.03259$ and $\sigma_V = 0.5885$, we may obtain Δ as a function of maturity T. As shown in Figure 10, the volatility hedge ratio is a decreasing function of time to maturity. It starts from 1 for the VXB, goes to zero for the VIX futures with a very long maturity.

The volatilities of the VIX futures and VXB are related by

$$\sigma_F = \Delta \frac{VXB_t}{F_t^T}\sigma_{VXB} \ . \tag{11}$$

Because F_t^T and VXB_t are very close to each other and $\Delta < 1$ the volatility of the VIX futures is smaller than the volatility of VXB (VIX). For example, if $\sigma_{VXB} = 0.844$, $VXB_0 = 135.5$, $\kappa = 5.5805$, $\theta = 0.03259$ and $\sigma_V = 0.5885$, we may compute the volatility of 30-day VIX futures price as follows

$$\sigma_{F30} = 0.5321 \times \frac{135.5}{144.8} \times 0.844 = 0.409,$$

$$\sigma_{F60} = 0.3042 \times \frac{135.5}{152.8} \times 0.844 = 0.228,$$

$$\sigma_{F90} = 0.1819 \times \frac{135.5}{157.8} \times 0.844 = 0.132,$$

which is close to the value, 0.357, 0.300 and 0.258 computed from the market prices. Our simple model also explains the downward sloping term structure of the volatility of VIX futures.

5. Conclusion

With the enormous increase in derivatives trading and the focus on volatility came the realization that stochastic volatility is an important risk factor affecting pricing and hedging. A new asset class, volatility instruments, is emerging and markets that trade these instruments are created. The first exchange traded instrument is, VIX futures. It has been trading on the CBOE Futures Exchange since 26 March 2004.

In this paper, we first study the behavior of VIX futures prices using the market data from March 2004 to November 2006. We observe three stylized facts:

1. The index, VIX, and the 3 VIX futures prices are negatively correlated with the S&P 500 index. VIX and VIX futures with three different maturities are almost perfectly correlated.

2. The term structure of the average VIX futures price is upward sloping.

3. The volatility term structure of VIX futures is downward sloping.

The first fact is not surprising. Traders often take long position in volatility derivatives to hedge the risk in stock position. The second fact shows that the long term mean level of volatility is higher than the current level. The third observation is not well-known.

In the second part of the paper we use a simple mean-reverting variance model to establish the theoretical relationship between VIX futures prices and its underlying spot index. Using the variance parameters calibrated from our model with the market data at t-1, we can price VIX futures at time t conditional on VIX at time t. An empirical study over the whole sample period shows that our model provides prices that are very close to the market prices. The root mean squared error between the market and model prices is around 2.3 dollars, which is less than 2% of the average VIX futures price of around 150. Our simple mean-reverting variance model captures the dynamics of VIX futures price very well.

References:

1. Bakshi, Gurdip, and Nikunji Kapadia, 2003, Delta-hedged gains and the negative market volatility risk premium, *Review of Financial Studies* 16, 527-566.

2. Brenner, Menachem, and Dan Galai, 1989, New Financial Instruments for Hedging Changes in Volatility, *Financial Analyst Journal*, July/August, 61-65.

3. Carr, Peter, and Dilip Madan, 1998, Towards a theory of volatility trading. In Robert Jarrow (Ed.), *Volatility estimation techniques for pricing derivatives*, London: Risk Books, pp. 417-427.

4. Carr, Peter, and Liuren Wu, 2003, Variance risk premia, Working paper, City University of New York and Bloomberg L. P.

5. Carr, Peter, and Liuren Wu, 2006, A tale of two indices, *Journal of Derivatives* 13, 13-29.

6. Corrado, Charles J., and Thomas W. Miller, Jr., 2005, The forecast quality of CBOE implied volatility indexes, *Journal of Futures Markets* 25, 339-373.

7. Coval, Joshua D., and Tyler Shumway, 2001, Expected option returns, *Journal of Finance* 56, 983-1009.

8. Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385-407.

9. Demeterfi, Kresimir, Emanuel Derman, Michael Kamal, and Joseph Zou, 1999, A guide to volatility and variance swaps, *Journal of Derivatives* 6, 9-32.

10. Heston, Steven L., 1993, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* 6, 327-343.

11. Merton, Robert C., 1973, Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science* 4, 141-183.

12. Zhang, Jin E., and Yingzi Zhu, 2006, VIX Futures, *Journal of Futures Markets* 26, 521-531.

Table 1: Summary Statistics for VIX Futures Contracts from 26 March 2004 to 21 November 2006

Code	Contract	No. of observations	Period covered		VIX Futures prices		Open interest	Volume
			Start	End	Mean	Std	Mean	Mean
K4	May 04	38	03/26/04	05/19/04	185.6	9.2	1166	148
M4	Jun 04	56	03/26/04	06/16/04	186.8	14.3	1256	170
N4	July 04	35	05/21/04	07/14/04	163.5	30.9	1926	186
Q4	Aug 04	100	03/26/04	08/18/04	193.2	12.3	1697	135
U4	Sep 04	42	07/19/04	09/15/04	178.6	22.8	1533	158
V4	Oct0 4	37	08/20/04	10/13/04	152.7	29.0	1226	124
X4	Nov 04	164	03/26/04	11/17/04	190.1	25.6	2794	145
F5	Jan 05	62	10/21/04	01/19/05	146.1	11.6	1210	75
G5	Feb 05	168	06/18/04	02/16/05	171.2	32.0	3074	124
H5	Mar 05	37	01/24/05	03/16/05	127.4	7.9	1181	154
K5	May 05	168	09/20/04	05/18/05	156.5	17.2	2629	157
M5	June 05	61	03/21/05	06/15/05	142.9	12.4	1178	138
Q5	Aug 05	186	11/19/04	08/17/05	148.6	18.9	4403	172
V5	Oct 05	86	06/20/05	10/19/05	143.0	5.4	826	66
X5	Nov 05	188	02/22/05	11/16/05	150.6	8.3	3303	129
Z5	Dec 05	42	10/21/05	12/21/05	123.7	23.0	928	84
F6	Jan 06	40	11/18/05	01/18/06	122.1	20.9	305	52
G6	Feb 06	185	05/23/05	02/15/06	150.7	11.6	3416	146
H6	Mar 06	43	01/20/06	03/22/06	122.8	20.6	677	64
J6	Apr 06	42	02/17/06	04/19/06	122.1	19.7	648	61
K6	May 06	187	08/19/05	05/17/06	148.4	19.1	5627	281
M6	Jun 06	43	04/21/06	06/14/06	146.2	32.1	1905	279
N6	Jul 06	42	05/22/06	07/19/06	154.5	27.5	1293	174
Q6	Aug 06	164	12/21/06	08/16/06	150.4	16.4	10487	467
U6	Sep 06	42	07/24/06	09/20/06	140.8	12.9	3788	412
V6	Oct 06	43	08/18/06	10/18/06	131.9	23.9	8220	685
X6	Nov 06	178	03/08/06	11/15/06	147.5	19.3	14957	604
Z6	Dec 06	45	09/21/06	12/20/06	134.2	15.2	6849	408
F7	Jan 07	24	10/20/06	01/17/07	124.7	27.1	406	72
G7	Feb 07	182	03/08/06	02/21/07	156.7	15.2	3839	190
H7	Mar 07	24	10/20/06	03/21/07	136.1	29.5	508	28
J7	Apr 07	24	10/20/06	04/18/07	140.5	30.3	36	7
K7	May 07	172	03/22/06	05/16/07	161.3	13.9	1183	49
Q7	Aug 07	109	06/21/06	08/15/07	164.1	4.1	177	6
	Average				150.5		2784	181

Note: The futures contract code is the expiration month code followed by a digit representing the expiration year. The expiration month codes follow the convention for all commodities futures, which is defined as follows: January-F, February-G, March-H, April-J, May-K, June-M, July-N, August-Q, September-U, October-V, November-X and December-Z.

Table 2: The correlation matrix between daily continuously compounded returns of S&P 500 index, VXB and fixed maturity VIX futures computed based on the market data from 26 March 2004 to 21 November 2006. The fixed maturity VIX futures prices are constructed by using the market data of available contracts with linear interpolation technique. The daily continuously compounded return is defined as the logarithm of the ratio between the price on next day and the price on current day. The upper triangle is the same as the lower triangle because the matrix is symmetric.

	S&P 500 index	VXB	30-day VIX futures	60-day VIX futures	90-day VIX Futures
S&P 500 index	1				
VXB	-0.7974	1			
30-day VIX futures	-0.7748	0.8140	1		
60-day VIX futures	-0.6827	0.6918	0.8420	1	
90-day VIX futures	-0.6975	0.6715	0.8075	0.8249	1
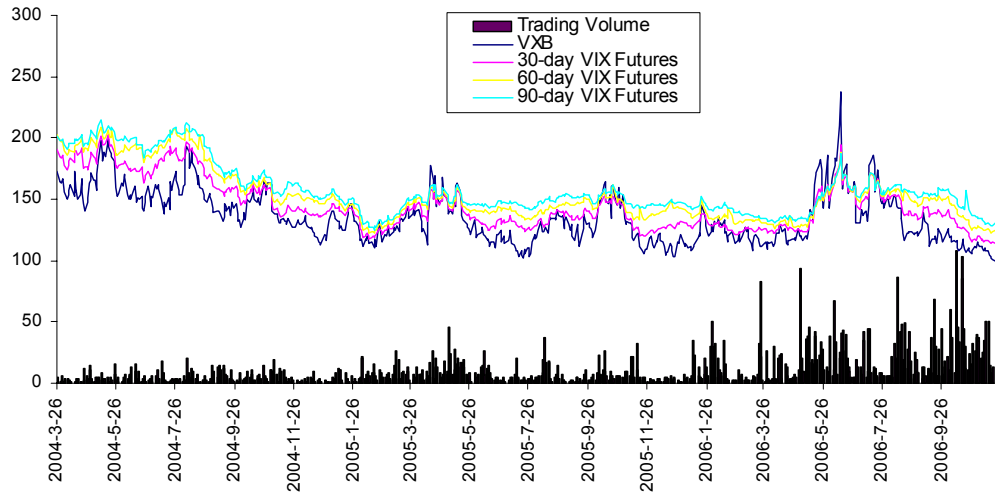
Figure 1. VXB and VIX futures price with three fixed time-to-maturities between 26 March 2004 and 21 November 2006. The VXB time series is from the CBOE. The fixed maturity VIX futures prices are constructed by using the market data of available contracts with linear interpolation technique. The bar chart shows the trading volume (normalized by 100 contracts) of futures of all maturity on the day.
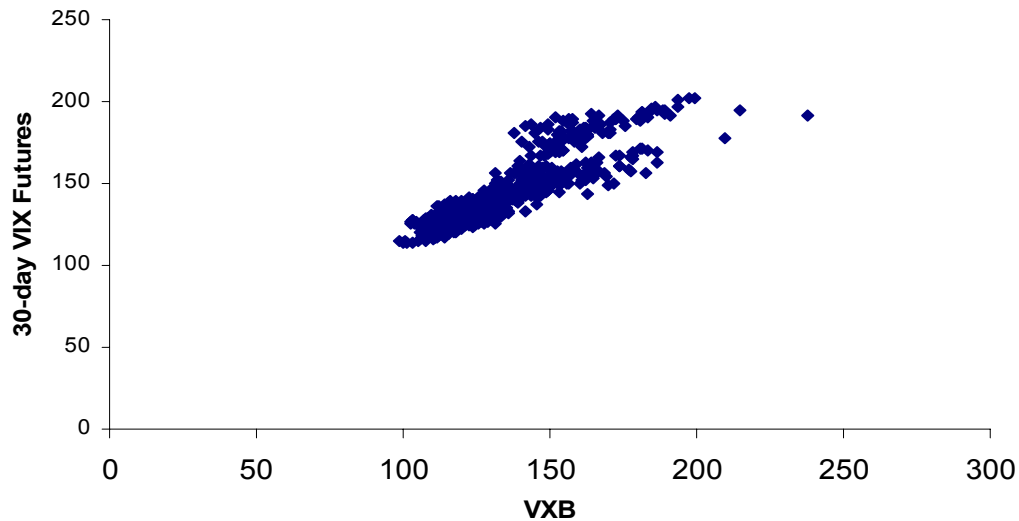
Figure 2. The relation between 30-day VIX futures and VXB (March 26 2004 – 21 November 2006). The VXB time series is from the CBOE. The 30-day VIX futures prices are constructed by using the market data of available contracts with linear interpolation technique.

21
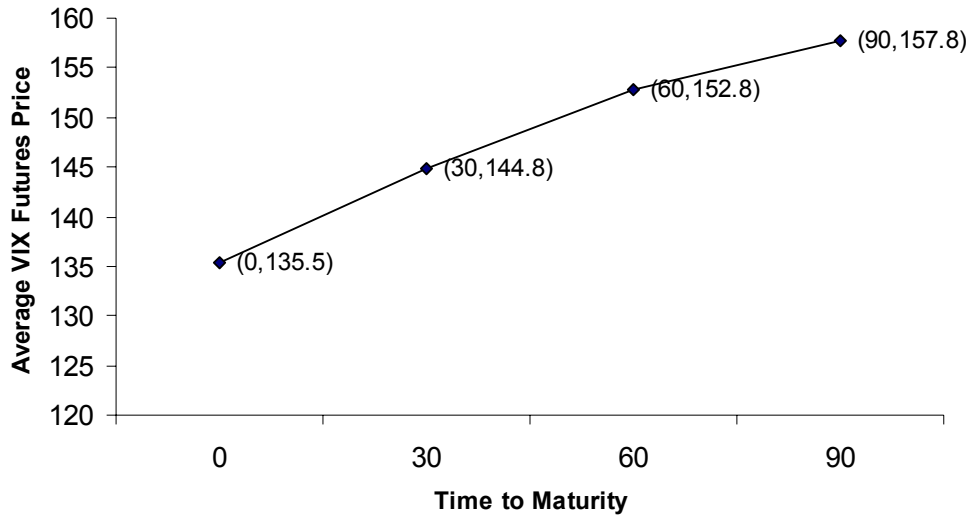
Figure 3. The term structure of average VIX futures price. The average VIX futures price is computed based on the fixed maturity data from 26 March 2004 to 21 November 2006. The fixed maturity VIX futures prices are constructed by using the market data of available contracts with linear interpolation technique.
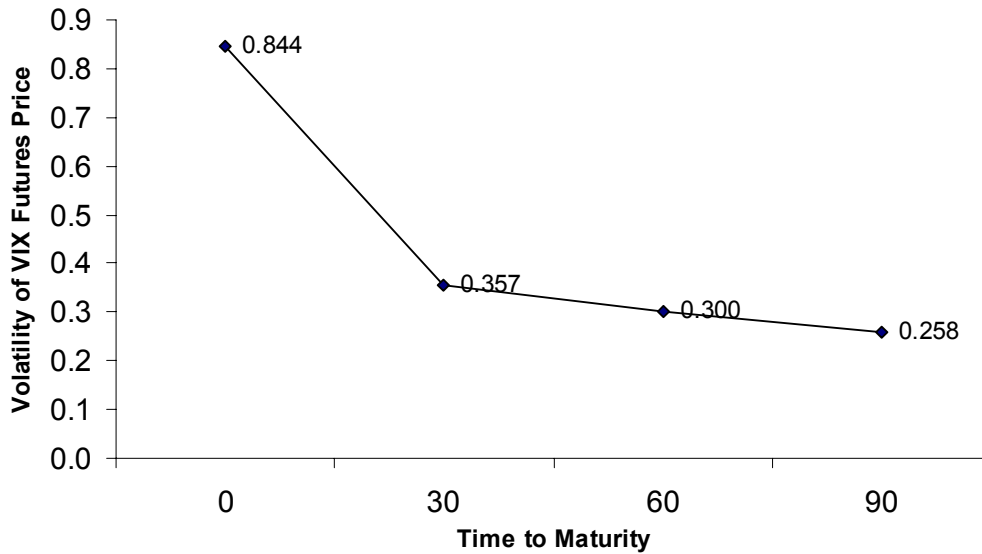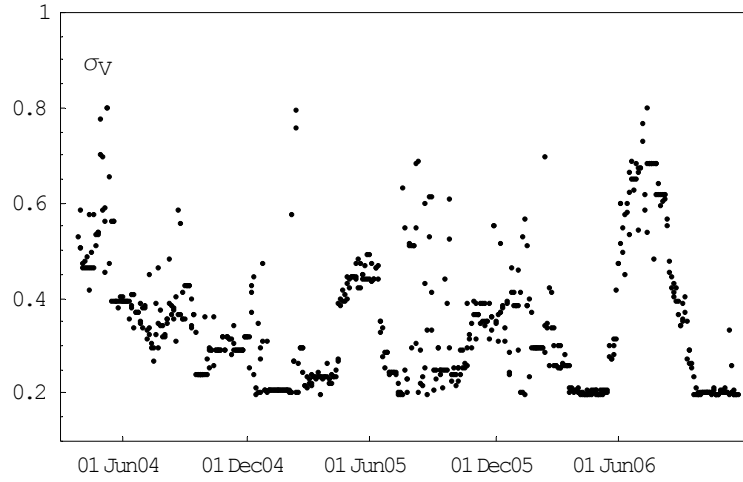

Figure 4. The annualized volatility of VIX futures price. The annualized volatility is computed based on the fixed maturity data from 26 March 2004 to 21 November 2006. The fixed maturity VIX futures prices are constructed by using the market data of available contracts with linear interpolation technique.
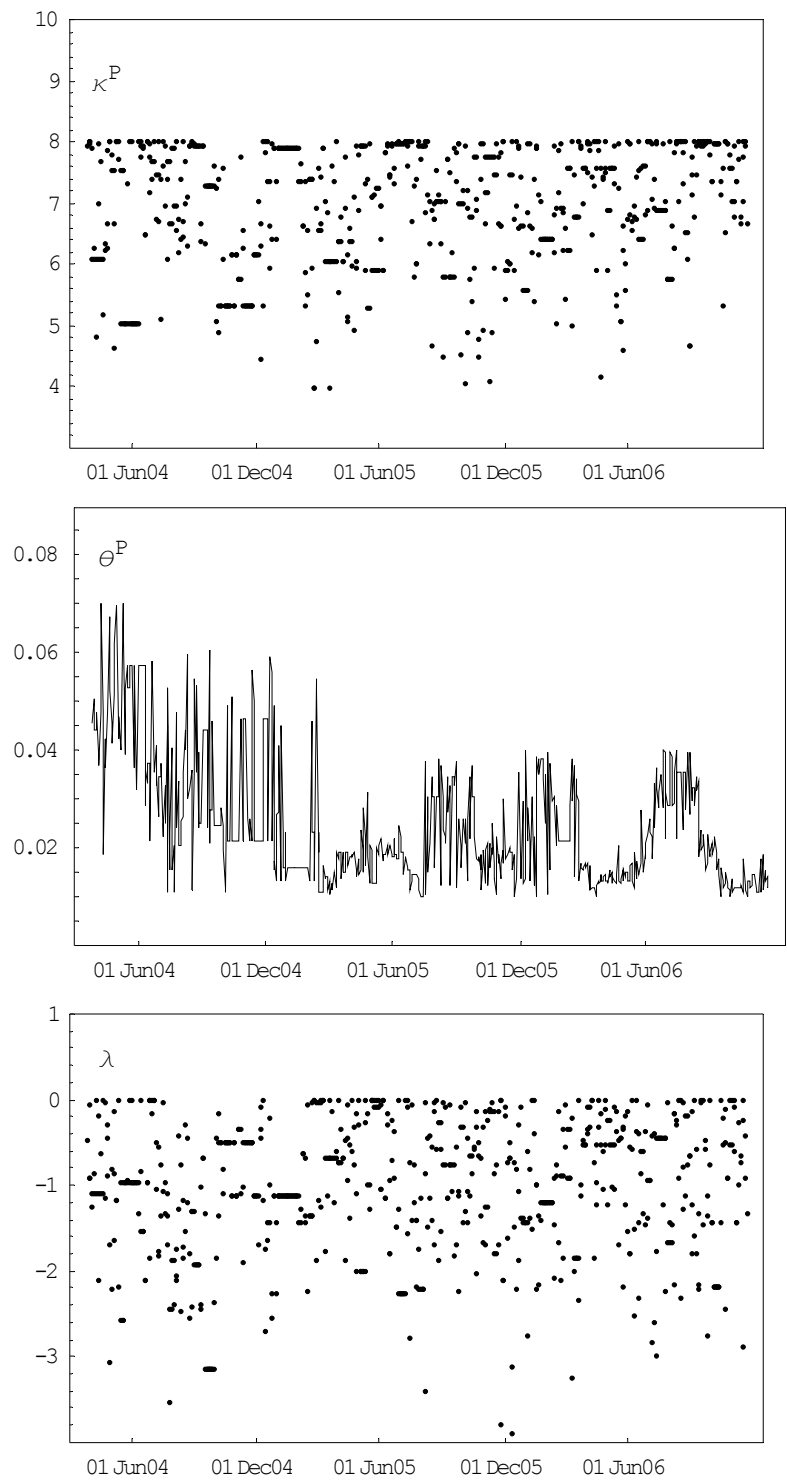
Figure 5. The variance parameters estimated from the historical VIX index of the last 42 trading days. The average values of the estimated four parameters κ^P, θ^P, λ and σ_V are 6.9671, 0.02492, -1.02046 and 0.3501 respectively. Their standard deviations are 0.9914, 0.01298, 0.7882 and 0.1375 respectively.
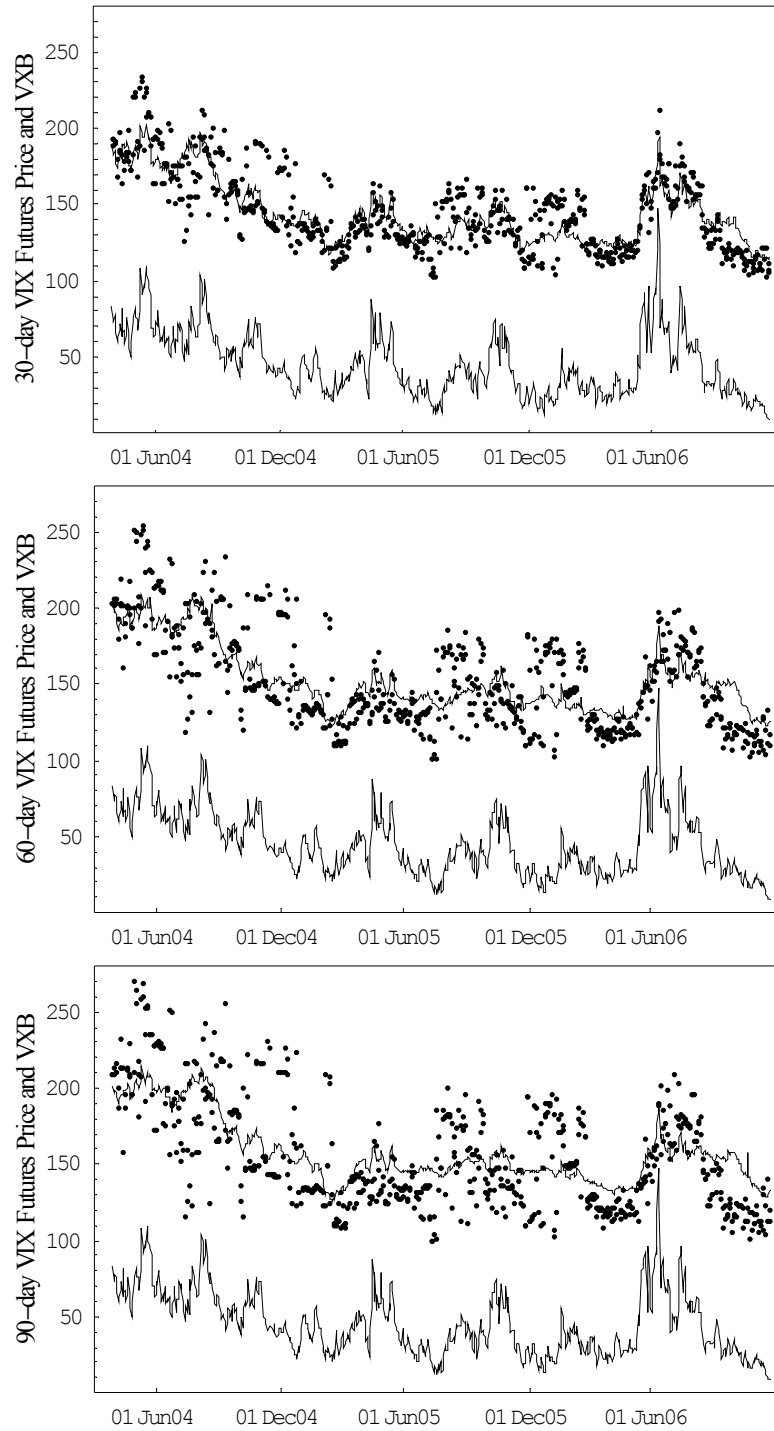
Figure 6. A comparison of model predicted prices and market prices of 30-, 60- and 90-day VIX futures. The upper solid line is the market price. The dots are model predicted prices based on the parameters estimated from the VIX index of the last 42 trading days. The lower solid line is the VXB value shifted downward by 90 for the purpose of an easy comparison. The average prices of 30-, 60- and 90-day VIX futures are (144.8, 152.8, 157.8). The roots of mean squared error between model predicted price and market price are (14.89, 21.97, 26.18).

25
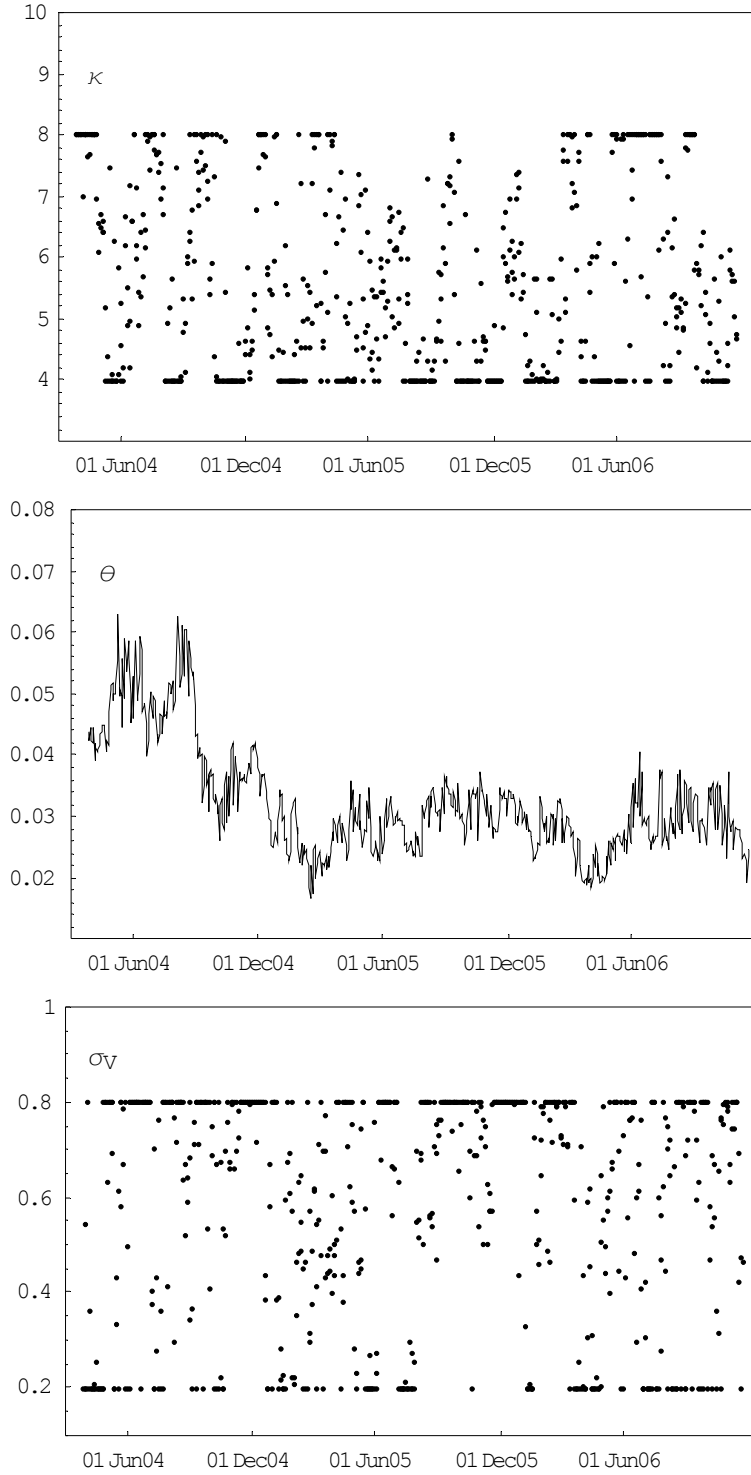
Figure 7. The model parameter values calibrated from the market prices of 30-, 60- and 90-day VIX futures. The average values of the three parameters κ, θ and σ_V are 5.5805, 0.03259 and 0.5885 respectively. Their standard deviations are 1.5246, 0.00909 and 0.2398 respectively.
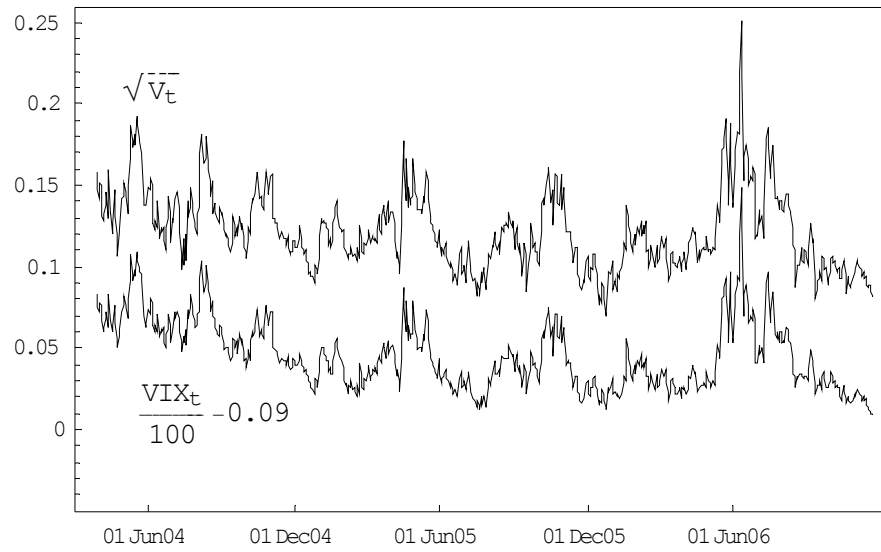
Figure 8. The instantaneous volatility, $\sqrt{V_t}$, recovered from VIX index and daily calibrated set of parameters. The upper line is the instantaneous variance, the lower line is VIX index divided by 100 and shifted down by 9% for the purpose of an easy comparison.
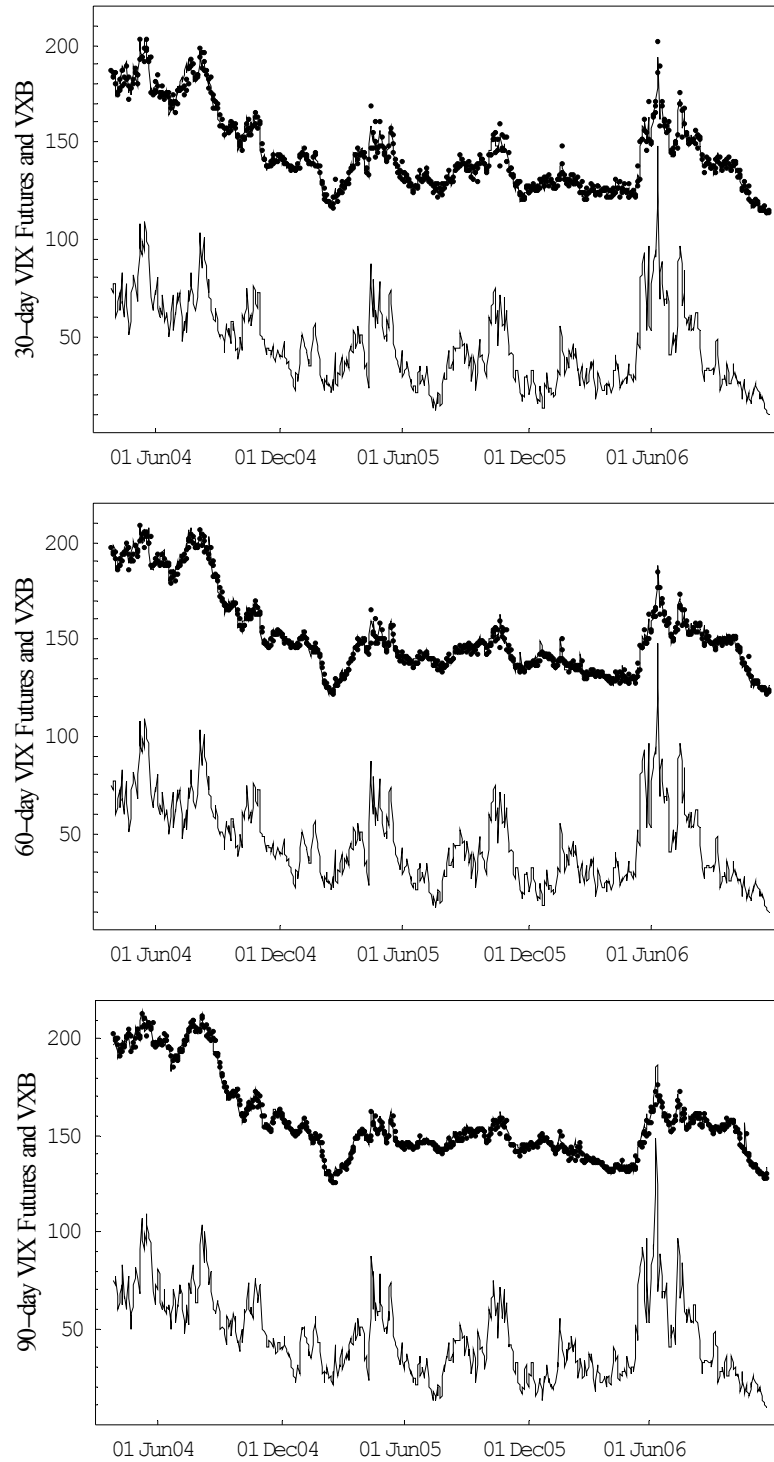
Figure 9. A comparison on model predicted prices and market prices of 30-, 60- and 90-day VIX futures. The upper solid line is the market price. The dots are model predicted prices based on the parameters calibrated from VIX futures prices on the previous day. The lower solid line is the VXB value shifted downward by 90 for the purpose of an easy comparison. The average prices of 30-, 60- and 90-day VIX futures are (144.8, 152.8, 157.8). The roots of mean squared error between model price and market price are (2.56, 2.34, 2.27).
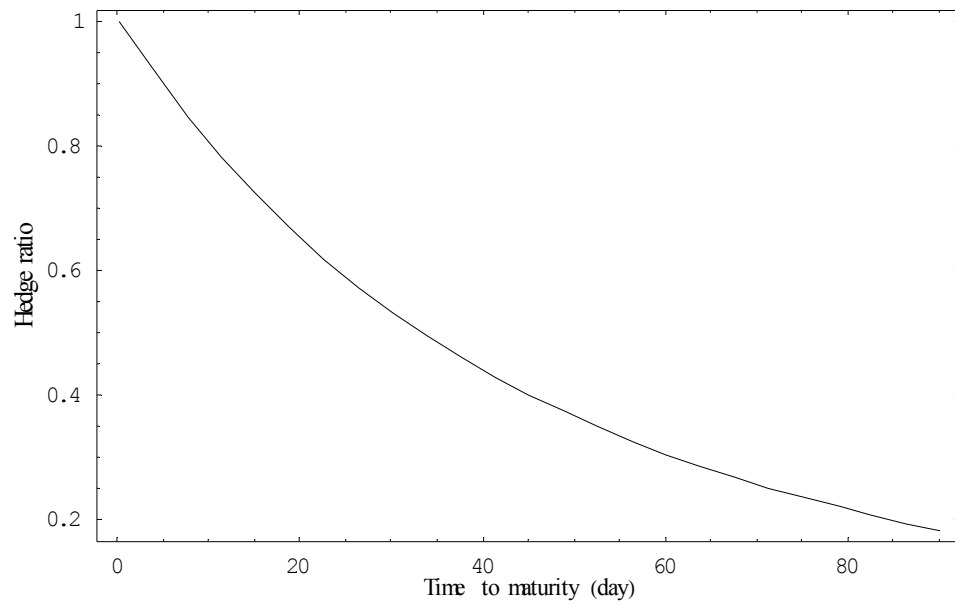
Figure 10. The volatility hedge ratio as a function of time to maturity. $VXB_0 = 135.5$, the set of parameters is taken to be the average calibrated values $\kappa = 5.5805$, $\theta = 0.03259$ and $\sigma_V = 0.5885$.

29