Cone Programming and Semidefinite Programming

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October 29, 2025

Contents

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Definition

- A *cone* is a set K such that for any $x \in K$ and any $\alpha \geq 0$, we have $\alpha x \in K$.
  - By definition, every cone contains 0.
- Simply put, a cone is a set of *directions*.

Examples

- nonnegative orthant $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq 0\}$;
- second-order cone (SOC) $\mathbf{L}_+^n := \left\{\boldsymbol{x} \in \mathbb{R}^n : x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}\right\}$
- positive semidefinite cone (PSD) $\mathbf{S}_+^n$.

Cone

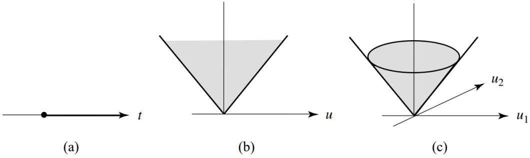

Fig. 13.3 Second-order cones of dimension **a** $n = 1$, **b** $n = 2$, and **c** $n = 3$

Figure 1: Second-order cone in $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$.[1]

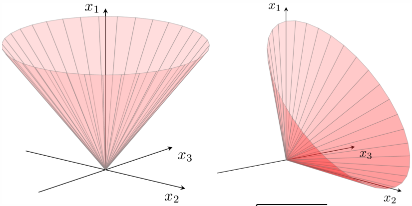

Figure 2: SOC can be rotated. Left: $x_1 \geq \sqrt{x_2^2 + x_3^2}$; Right: $2x_1x_2 \geq x_3^2 \ (x_1, x_2 \geq 0)$.[2]

[1] Image from Antoniou, A., & Lu, W. S. (2007). *Practical optimization*. Springer. 13.9.1.

[2] Image from `https://docs.mosek.com/modeling-cookbook/cqo.html`.

Cone

PSD:

$$\mathbf{S}_+^n := \{\boldsymbol{X} \in \mathbb{R}^{n\times n} : \boldsymbol{X} \succeq 0\} = \{\boldsymbol{X} \in \mathbb{R}^{n\times n} : \boldsymbol{v}^\top \boldsymbol{X}\boldsymbol{v} \geq 0, \forall \boldsymbol{v} \in \mathbb{R}^n\}$$

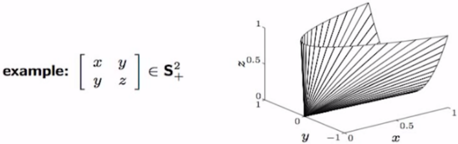

Figure 3: A 3D illustration of the PSD cone $\mathbf{S}_+^2$.[1]

[1] Image from https://math.stackexchange.com/questions/1875462/how-to-plot-the-psd-cone-in-matlab.

Linear Programming

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Cone Programming

Assume $K \subseteq \mathbb{R}^n$ is a cone,

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \in K \end{aligned}$$

Linear Programming

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq 0
\end{aligned}
$$

Cone Programming

Assume K is a cone,

$$
\begin{aligned}
\min \quad & \boldsymbol{c} \cdot \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{a}_k \cdot \boldsymbol{x} = b_k,\ k = 1, \ldots, m \\
& \boldsymbol{x} \in K
\end{aligned}
$$

Examples

- Linear Programming (LP): $K = \mathbb{R}^n_+$
- Second-Order Cone Programming (SOCP): $K = \mathbf{L}^n_+$
- Semidefinite Programming (SDP): $K = \mathbf{S}^n_+$

Note

- CP may cannot achieve optimal value, even if the objective function is bounded below.
- *Weak duality* always holds for CP, however, *Strong duality* may fail.
- Interior-point methods can be applied to solve CPs.

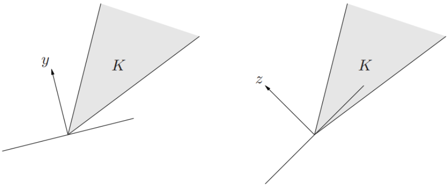

Figure 2.22 *Left.* The halfspace with inward normal y contains the cone K, so $y \in K^*$. *Right.* The halfspace with inward normal z does not contain K, so $z \notin K^*$.

Figure 4: Notice that the dual cone of K is the negative normal cone to K at 0.[1]

$$\max \boldsymbol{b} \cdot \boldsymbol{y}$$

$$\text{s.t.} \sum_{k=1}^{m} \boldsymbol{a}_k \cdot \boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c}$$

$$\boldsymbol{s} \in K^*$$

[1] Image from Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press. 2.6.1.

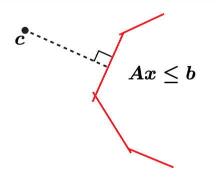

Figure 5: The Minimum Norm Problem.[1]

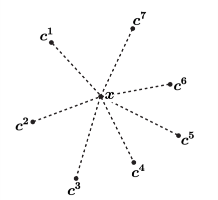

Figure 6: The Fermat-Weber Problem.[1]

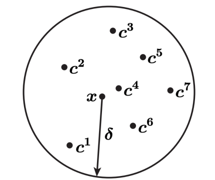

Figure 7: The Ball Circumscription Problem.[1]

[1] Image from Robert M. Freund. (2014). *Convex Conic Optimization, and SDP*. Lecture notes. http://s3.amazonaws.com/mitsloan-php/wp-faculty/sites/30/2016/12/15032100/Convex-Conic-Optimization-and-SDP-Semi-Definite-Programming.pdf/

SDP Problem

$$\begin{aligned} \min\ & \boldsymbol{C} \cdot \boldsymbol{X} \\ \text{s.t. } & \boldsymbol{A}_k \cdot \boldsymbol{X} = b_k,\ k = 1, \ldots, m \\ & \boldsymbol{X} \succeq 0 \end{aligned}$$

Some Applications

- Quadratic Constraint Quadratic Programming (QCQP)
- Calculate Matrix Norm
- Max-Cut Problem
- Matrix Completion Problem
- Graph Coloring Problem
- ...

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Semidefinite Relaxation (SDR)[1]

Notice that

$$x^\top A x = \mathrm{Tr}(A x x^\top) = A \cdot X$$

where $X = xx^\top \succeq 0$. More generally,

$$x^\top A x + 2b^\top x + c = \begin{pmatrix} A & b \\ b^\top & c \end{pmatrix} \cdot \begin{pmatrix} xx^\top & x \\ x^\top & 1 \end{pmatrix}$$

Spectral norm[2]

$$\|A\|_2 = \max \left\{ t : \begin{pmatrix} t\mathbf{I}, & A \\ A^\top, & t\mathbf{I} \end{pmatrix} \succeq 0 \right\}$$

[1] See Luo, Z. Q., Ma, W. K., So, A. M. C., Ye, Y., & Zhang, S. (2010). *Semidefinite relaxation of quadratic optimization problems*. IEEE Signal Processing Magazine, 27(3), 20-34.

[2] See Example 4.6.3 of the Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

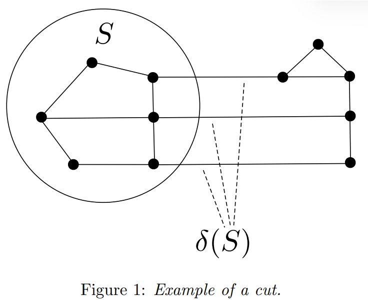

Figure 8: Example of a cut.[2]

(MAX-CUT)
$$\max \frac{1}{2} \sum_{(i,j)\in E} w_{ij}(1 - x_i x_j)$$
$$\text{s.t. } x_i \in \{1, -1\}, \quad i = 1, 2, \ldots, n$$

Goemans-Williamson Random Rounding

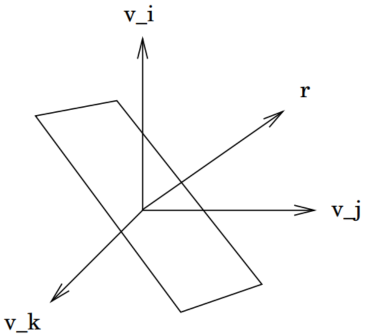

Figure 9: Using a plane randomly cut the sphere.[3]

[3] Image from David P. Williamson. (2014). *ORIE 6300: Mathematical Programming I*. Cornell University, https://people.orie.cornell.edu/dpw/orie6300/.

Goemans-Williamson Random Rounding

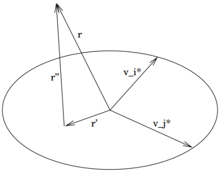

Figure 10: Illustration of the decomposition of r into r' and r''.[1]

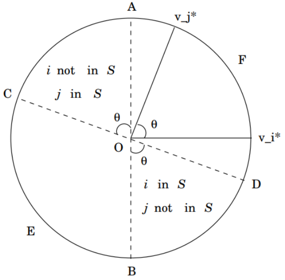

Figure 11: Illustration of the regions where exactly one of i and j are in the cut.[1]

[1] Image from David P. Williamson. (2014). *ORIE 6300: Mathematical Programming I*. Cornell University, https://people.orie.cornell.edu/dpw/orie6300/.

General Situations of MAX-CUT?

- [Nesterov, 1998]: $\frac{2}{\pi}$ approximation ratio for general weights.
  - Insight: Note that $\arcsin x - x$ has a series expansion with non-negative coefficients only. Thus if $X \succeq 0$ then $\arcsin[X] - X = \sum_{k=1}^{\infty} c_k X^{\odot k} \succeq 0$.
- Note that MAX-CUT is a special case of Binary Quadratic Programming (BQP).
- Note that MAX-CUT calculates the $\|\boldsymbol{L}\|_{\infty \to 1}$ norm of graph Laplacian $\boldsymbol{L}$.
- [D. Steinberg, 2005] Using same techniques to solve general matrix norm problems.
- [Alon and Naor, 2006] Improving the approximation ratio to 0.56 using Grothendieck's inequality.
- ...

Reference

Williamson, David P. (2014)
ORIE 6300: Mathematical Programming I
Cornell University, Fall 2014,
https://people.orie.cornell.edu/dpw/orie6300/.

Freund, Robert M. (2014)
Convex Conic Optimization, and SDP
Lecture Notes . https://s3.amazonaws.com/mitsloan-php/wp-faculty/sites/30/2016/12/15032100/Convex-Conic-Optimization-and-SDP-Semi-Definite-Programming.pdf

End

Thanks for your listening.