The Contest Behind the Feed: Optimal Contest for Recommender Systems

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Abstract

We study the contest design problem for an online platform's recommender system (RS) to incentivize costly efforts from n content producers, given a cost function $c(x) = x^\beta$ for $\beta > 0$ governing the convexity/concavity of the cost function. We analyze a class of rank-based policy that recommends i-th highest quality with probability p_i for $i \in [n]$. The RS aims to design a policy that maximizes a convex combination over two metrics, user welfare and platform quality parameterized by the weight $\alpha \in [0, 1]$, at equilibrium.

We characterize the exact structure of the optimal policy across the full range of weight parameter α and the cost parameter β. Interestingly, in the single-minded case ($\alpha \in \{0, 1\}$) where the RS either tries to solely maximize user welfare or platform quality, we derive a stark *phase transition* in the optimal structure, where the optimal policy turns out to be either HardMax policy that always recommends the highest quality content or UniformButLast policy that uniform randomly recommends the contents except the last-ranked one, depending on cost function. Thus, *fairness* arises endogenously. As a byproduct, we generalize the result by Glazer and Hassin (EI'88) beyond linear cost function as well as the complete information setting of Moldovanu and Sela (AER'01) beyond top-two prize structure. We extend this result to the convex-minded setting ($\alpha \in (0, 1)$), showing that the optimal policy is still highly structured: $p_1 \geq p_2 = \ldots = p_{n-1} \geq p_n = 0$ for any $\alpha \in [0, 1]$ and any $\beta > 0$. In words, any optimal policy assigns potentially high probability to p_1, zero to p_n, and all the other intermediate probabilities are all equal.

Our technical results rely on a Bernstein basis polynomials-weighted function's Schur-convexity (concavity) as well as a Bernstein basis polynomial matrix's total positivity via generalized Vandermonde matrix and its variation diminishing property, which might be of independent interest.

1 Introduction

Recommendation algorithms are ubiquitous in online platforms such as video streaming services, e-commerce sites, and social media. These algorithms aim to increase user engagement by recommending content that users are most likely to prefer, based on estimates of user preferences and content features. The creation of this content is typically handled by individuals known as content producers. However, producing high-quality content requires significant time and effort. To keep content producers engaged in the ecosystem, platforms often provide direct monetary compensation or encourage them to embed advertisements in their content to generate third-party revenue. Crucially, the incentives for content producers are closely tied to the recommendation algorithm itself, as greater visibility to users translates to higher potential earnings.

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Despite the strategic nature of content production, most recommendation algorithms do not account for the producers' strategic behavior—namely, their efforts to maximize revenue while managing production costs. This interaction is critical to designing an efficient ecosystem, yet it remains underexplored in existing models up to a few recent papers (Jagadeesan et al., 2024; Immorlica et al., 2024; Dai et al., 2024; Jagadeesan et al., 2023; Yao et al., 2024b,c,a, 2023).[1]

We consider a setting where an online platform operates a recommender system (RS) with n content producers who strategically decide their costly effort levels to create content of quality x, where the cost function is $c(x) = x^\beta$ for $\beta > 0$. We analyze a class of rank-based[2] policies $\boldsymbol{p}$ in which the RS recommends the i-th highest-quality content with probability p_i, such that $\sum_{i\in[n]} p_i = 1$. In this framework, the RS's objective is to maximize a convex combination of user welfare (derived from the recommended content) and the overall quality of the content, parameterized by $\alpha \in [0, 1]$, at an equilibrium strategy given the policy. Notably, our setting generalizes the seminal work by Glazer and Hassin (1988) beyond $\alpha = 0$ and $\beta = 1$.

To hight our main technical contributions:

1. We first establish the existence of a symmetric mixed Nash equilibrium (MNE) for a nontrivial class of policies using topological arguments on discontinuous game from Reny (1999).

2. Towards characterizing the optimal structure, we start with the single-minded case where $\alpha \in \{0, 1\}$. When $\alpha = 1$ (i.e., when we maximize for the user welfare), we show that the optimal policy is the HardMax policy, which always recommends the highest-quality content. That is, $p_1 = 1$ and $p_2 = \ldots = p_n = 0$. When $\alpha = 0$, however, we uncover a sharp *phase transition* between the HardMax policy and the UniformButLast policy, where $p_1 = \ldots = p_{n-1} = \frac{1}{n-1}$ and $p_n = 0$, depending on whether the cost function is sublinear or not. An interesting implication of this result is that *fairness* in the optimal recommendation algorithm emerges endogenously when the RS maximizes its efficiency against strategic content producers.

3. Finally, we extend our analysis to the convex-minded case where $\alpha \in (0, 1)$, which results in solving an optimization problem that consists of a summation over convex and concave functions that does not typically admit explicit optimal solution. Nevertheless, we identify the *structure of the optimal policy*: $p_1 \geq p_2 = \ldots = p_{n-1} \geq p_n = 0$. We finally verify these structures numerically using a brute-force search over a discretized policy space, and validate that our structural characterization is *tight*.

In what follows, we give an overview of our results and techniques. Section 1.2 discuss related works and compare our setting and results, and we present formal problem setup in Section 2.

1.1 Results and Techniques

Before presenting a summary of our results and techniques for the proofs, we briefly introduce our setting. More details can be found in Section 2. Looking forward, the structural characterization of our results can be depicted in Figure 1.

There are n content producers, each of whom decides a quality q_i for its content. Producing a content with quality x requires cost of $c(x) = x^\beta$ for $\beta > 0$.[3] The recommender system (RS) designs a rank-based recommendation policy $\mathbf{p} = (p_1, \ldots, p_n) \in \Delta$ where $\Delta = \{\mathbf{p} \in \mathbb{R}^n_{\geq 0} : p_1 \geq$

[1] We refer to Section 1.2 for more detailed discussion.

[2] The study of rank-based contest design dates back to Lazear and Rosen (1981), and has been received significant attention (Glazer and Hassin, 1988; Barut and Kovenock, 1998; Che and Gale, 2003; Moldovanu and Sela, 2008; Siegel, 2009). We refer to Section 1.2 for more detailed comparison.

[3] All our results readily adapts to linear transformation $c(x) = \gamma x^\beta$.

$\ldots \geq p_n \geq 0, \sum_{i=1}^n p_i = 1\}$. Then, whenever a user joins the platform, RS sorts the existing contents in decreasing order of quality, and recommends i-th highest quality content with probability p_i. Producer i gets revenue p_k if its quality q_i is k-th highest among $(q_i)_{i\in[n]}$, where the overall utility is therefore the revenue minus the cost. The RS aims to maximize a convex combination over user welfare $\mathcal{W}(\boldsymbol{\mu}, \mathbf{p})$ and platform quality $\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p})$ at a symmetric MNE $\boldsymbol{\mu}$, where the user welfare is the expected quality of the contents shown to the user, and platform quality is the average of expected quality of all the registered contents in the system. That is, it aims to design a policy $\mathbf{p}$ that has maximal $\alpha\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) + (1-\alpha)\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p})$ at the MNE $\boldsymbol{\mu}$.[4] We say that the RS is *single-minded* if $\alpha = 0$ or 1, *i.e.,* it is interested in solely maximizing either of the user welfare or the platform quality, and *convex-minded* if $\alpha \in (0,1)$, *i.e.,* when the objective function is truly a convex combination over two.

Existence of symmetric MNE. First, we show that the pure Nash equilibrium (PNE) does not exist for any *nontrivial* policy such that $p_i \neq p_j$ for some $i, j \in [n]$, generalizing the non-existential result by Jagadeesan et al. (2024) beyond the HardMax policy. Thus, we turn our attention to the existence of MNE. Due to the homogeneous nature of our setting, it is natural to expect the existence of the symmetric MNE where all the content producers play the same MNE.

On the other hand, it is not straightforward to apply standard existential theorems due to the discontinuity of the payoff of the content producers as increasing the content quality may induce discontinuous jump from p_{i+1} to p_i for some i. To overcome this challenge, we use the seminal result by Reny (1999) on the existence of symmetric MNE for quasi-symmetric games with diagonally better-reply secureness and upper semi-continuity of the producer's payoff function as a function of the mixed strategy, which is a probability measure on the space of pure strategies. We prove that the induced game satisfies these conditions by metrizing the underlying space of mixed strategy (weak* topology) with Lévy-Prokhorov metric equipped with ℓ_2 norm and dealing with the distance in the space of pure strategies rather than handling the underlying topology of mixed strategies directly.

Our existential result generalizes that of Jagadeesan et al. (2024) who prove the existence of symmetric MNE when the platform runs HardMax policy into broader class of policies, and complements the results of Glazer and Hassin (1988) which *assume* the existence of symmetric MNE and derive the optimal policy for specific case where $\alpha = 0$ and $\beta = 1$.

Optimization over Bernstein basis polynomials. Hence, we focus on the scenario where the producers decide the quality level according to the symmetric MNE. Then, the RS's objective can be written as the following optimization problem:

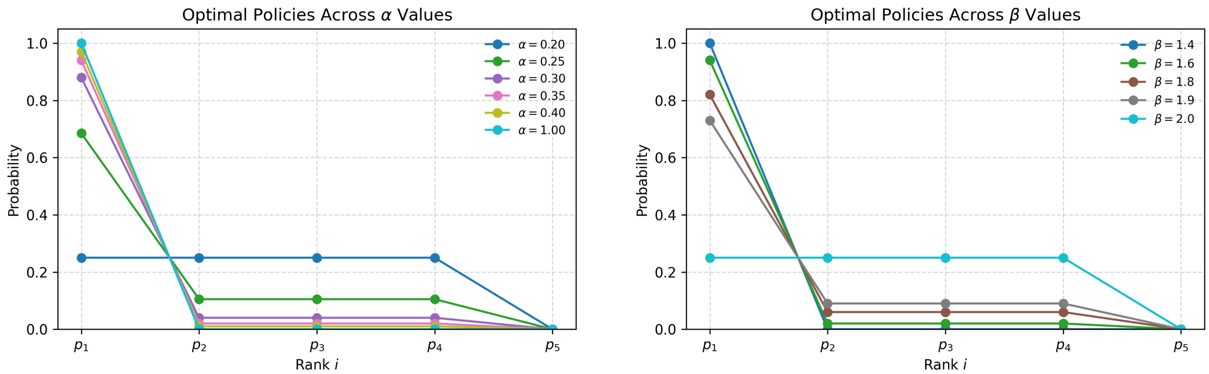
$$(\textsc{Opt}_1) \quad \underset{\mathbf{p}\in\Delta}{\text{maximize}} \quad \alpha\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) + (1-\alpha)\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p})$$
$$\text{subject to} \quad \boldsymbol{\mu} \text{ is a symmetric MNE}$$

To solve the optimization problem in its current form, however, this requires us to analyze the exact symmetric MNE $\boldsymbol{\mu}$. Using the property of the MNE $\boldsymbol{\mu}$ that every action in the support of $\boldsymbol{\mu}$, one can write down the following equation for $\boldsymbol{\mu}$ to be an equilibrium, for every $q \in \operatorname{supp}(\boldsymbol{\mu})$:[5]

$$u_0 + c(q) = \sum_{i=1}^n p_i \binom{n-1}{i-1} F(q)^{n-i}(1-F(q))^{i-1},$$

[4] It is worth noting that by setting $\alpha = 0$, our model subsumes the setting of Glazer and Hassin (1988) who prove the optimality of the UniformButLast policy when $\beta = 1$.

[5] This holds if F does not have a point mass, which is proven formally in Proposition 3.3.



(a) Optimal Policy with respect to α

(b) Optimal Policy with respect to β

Figure 1: Structures of optimal policy with respect to different α and β values. We compute the optimal policy using a brute-force search over every possible policy after discretizing the policy space with granularity 0.005, and using trapezoidal rule to approximate integral values over the grid $[0, 0.001, 0.002, \ldots, 1]$. For the left figure, we set $\beta = 2$ and for the right figure, we set $\alpha = 0.24$. This yields that our characterization is tight in a sense that the structures $p_1 > p_2 = \ldots = p_{n-1} \geq p_n = 0$ or $p_1 = p_2 = \ldots = p_{n-1} \geq p_n = 0$ indeed appear in the optimal policies.

where F is the cumulative distribution function (c.d.f.) corresponding to the mixed strategy $\boldsymbol{\mu}$. This is even intractable for $n \geq 6$ as high-degree polynomial does not typically admit explicit solutions.

Surprisingly, however, we show that one does not need to calculate the exact equilibrium strategy to solve the RS's problem. We first show that any optimal policy should have the last-rank probability to be zero, *i.e.*, $p_n = 0$.[6] Then, we show that the RS's problem can be rewritten as the following reduced optimization problem using properties of the symmetric MNE and Bernstein basis polynomials:

$$
\begin{aligned}
(\textsc{Opt}) \quad & \underset{\mathbf{p}}{\text{maximize}} \quad \alpha n \int_0^1 h(x,\mathbf{p})^{1+1/\beta} + (1-\alpha)\int_0^1 h(x,\mathbf{p})^{1/\beta} dx \\
& \text{subject to} \quad p_1 \geq p_2 \geq \ldots \geq p_n = 0 \\
& \qquad\qquad\qquad \sum_{i=1}^{n} p_i = 1
\end{aligned}
$$

where $h(x, \mathbf{p}) = \sum_{i=1}^n a_i(x) p_i$ for $(n-1)$-dimensional Bernstein basis polynomial $a_i(x) = \binom{n-1}{i-1} x^{n-i}(1-x)^{i-1}$.[7] Thus, we do not need to characterize the exact equilibrium to analyze the optimal policy.

Optimal structure for single-minded case. Despite the simplified optimization formulation, solving the optimization problem directly is challenging as it involves a summation over two functions each of which is again an integral over nontrivial function $h(x, \mathbf{p})$. A key analytical feature in these functions is that they share the same function $h(x, \mathbf{p})^r$ with different exponents r as integrands, which is the inner product between Bernstein basis polynomials and $\mathbf{p}$. Thus, it is essential to first characterize some properties of $h(x, \mathbf{p})^r$ for $r \geq 0$.

[6] This is consistent with results in literature on rank-based contest design with different models (Barut and Kovenock, 1998; Siegel, 2009).

[7] The standard i-th n-dimensional Bernstein basis polynomial is defined by $b_{i,n} = \binom{n}{i} x^i (1-x)^{n-i}$ for $i = 0, \ldots, n$.

Interestingly, if we consider an extension of the domain of the function $h(x, \mathbf{p})$ so that given any vector in n-dimensional probability simplex, we sort the values in decreasing order and apply $h(x, \mathbf{p})$, then the resulting symmetric function is *Schur-convex* if $r \geq 1$ or $r \leq 0$ and *Schur-concave* otherwise.[8] This immediately implies the Schur-convexity of the objective function if $\alpha = 1$, *i.e.,* the RS aims to maximize the user welfare, as $1 + 1/\beta > 1$. Recalling that $p_n = 0$, this shows that HardMax is optimal in this case. However, if $\alpha = 0$, *i.e.,* the RS tries to maximize the platform quality, the objective function becomes Schur-convex if $\beta \leq 1$ and Schur-concave if $\beta \geq 1$. This implies the optimality of HardMax with sublinear cost function and that of UniformButLast with superlinear cost function. In fact, in the sublinear cost function case, both the user welfare and the platform quality becomes Schur-convex function, implying the optimality of HardMax policy. We note that these results generalize the result by Glazer and Hassin (1988), who prove the optimality of UniformButLast for linear cost function.

Optimal structure for convex-minded case. On the other hand, in general case where $\alpha \in (0, 1)$ and $\beta > 1$, the objective function actually becomes a convex combination of Schur-convex function and Schur-concave function, which is technically involved to deal with. In general, a summation over convex and concave function usually does admit an exact characterization of the optimal solution.

Despite such challenges, we prove that the optimal policy is *highly structured*, characterized by $p_1 \geq p_2 = \ldots p_{n-1} \geq p_n = 0$. That is, it is piecewise constant function with potentially changing values in p_1 and p_n. En route to this result, we prove several technical results on the Bernstein basis polynomials as well as Bernstein polynomial matrix, which might be of independent interest.

Formally, we show that for any real numbers $0 < x_1 < \ldots < x_k < 1$ and integers $0 \leq i_1 < \ldots < i_k \leq n$, the matrix $(a_{i_s}(1 - x_l))_{s,l \in [k]}$ enjoys a nice property called *totally positive*, introduced by Karlin (1964), in a sense that its every minor has a positive determinant, by expressing each determinant as a generalized *Vandermonde* matrix (Yang et al., 2001).[9] Then, we use the seminal result on the theory of total positivity and variation diminishing property by Karlin (1964) and Karp et al. (2024) to show that the partial derivatives of the objective function with respect to p_i for $i \in [n-1]$ exhibits a *quasiconvexity* along with some other structures,[10] *i.e.,* it decreases and then increases after some index as i increases, by bounding the number of sign changes of these sequences by the number of sign changes in a quasi-convex function that consists of $h(x, \mathbf{p})$. Finally, we write the Lagrangian of the optimization problem and apply the *Karush–Kuhn–Tucker* (KKT) condition to derive the necessary condition of the optimal policy using the derived geometric properties of the partial derivatives.

1.2 Related Works

Our model is closely related to the literature on rank-based contests, as each probability p_i assigned to i-th highest quality content can be viewed as a prize p_i assigned to i-th highest quality contestant when n contestants compete to win a prize in a contest.[11] We refer to Sisak (2009) for a comprehensive survey on classic literature on contest design, including rank-based contest and Tullock contest.

[8]A function $f : \mathbb{R}^n \to \mathbb{R}$ is Schur-convex (concave) if $f(x) \geq f(y)$ when x majorizes y (y majorizes x). The definition of majorization and more details can be found in Section 5.

[9]We refer to Section 6 for more technical details.

[10]It is quasiconvex in a sense that it decreases then increases with respect to i. Formal definition can be found in Definition 6.2.

[11]In the contest design literature, the summation of prizes may not be 1, but one can simply normalize the prize to derive analogous results.

Rank-based contest. The literature on rank-based contest design, which studies the design of prize structures as a function of the rank of each contestant, dates back to Lazear and Rosen (1981) and Glazer and Hassin (1988). Lazear and Rosen (1981) study a stylized model where two contestants compete with each other for prizes where a designer designs the rates between higher and lower prizes to incentivize costly efforts. The most relevant to our model is Glazer and Hassin (1988) and Moldovanu and Sela (2008). Glazer and Hassin (1988) study complete information and incomplete information settings with linear cost function, where each contestant is equipped with a common utility function on prize that captures its behavior against the risk. They prove the optimality of UniformButLast policy in the complete information setting with linear cost, which can be deemed as a special case with $\beta = 1$ and $\alpha = 0$ in our results. Moldovanu and Sela (2008) consider an incomplete information setting with general (convex or concave) cost function, and analyze a class of top-two prize structures, *i.e.,* $p_1 \geq p_2 \geq p_3 = \dots p_n = 0$. They derive conditions under which the HardMax policy that does not assign prize to the second-best, *i.e.,* $p_2 = 0$, is optimal or not among such class. Barut and Kovenock (1998) analyze several properties of equilibria when the policy is given in an exogenous manner if the cost function is linear. Che and Gale (2003) consider a model with applications to research contests.

Building on classic rank-based prize design, recent literature explores various aspects of contest design, *e.g.,* examining how prize structures influence contestant strategies beyond just effort. Fang et al. (2024) show how a convex prize structure can induce contestants to choose riskier quality distributions. Kirkegaard (2023) analyze the optimal number of winners to reward with fixed prizes when contestant action yields stochastic quality. Diverging slightly, Chawla et al. (2019) model crowdsourcing contest with endogenous rewards. Liu et al. (2025) investigate two-stage contests with initial ability filtering followed by rank-based effort decisions. Complementing design, the Tullock contest literature analyzes equilibrium behavior under specific success functions, initiated by Tullock (2008). Recent works further analyze its computational complexity (He et al., 2024).

Strategic content production in recommender system. Jagadeesan et al. (2024) investigate a competition between multiple content producers, each of whom decide an effort level to create contents of certain quality. Their single-dimensional model yields the same expected payoff structure with our model, though they only analyze the equilibrium under HardMax policy. Immorlica et al. (2024) consider strategic behavior of content producers who may be gaming the platform to artificially boost the user engagement by clickbait. They prove that quality of contents and gaming are positively correlated, and thus the engagement-based optimization may reduce user welfare due to increment of gaming behavior. Jagadeesan et al. (2023) study a duopoly market where two platforms running multi-armed bandit algorithms compete for user participation. Ghosh and McAfee (2011) consider strategic content production of content producers who determine whether to participate in the platform and decide the effort level. Yao et al. (2023) analyze the inefficiency of top-k-style recommendation policies when content creators compete for exposure and show that such policies can induce suboptimal equilibria. Yao et al. (2024a) challenge the conventional wisdom that monotone reward structures are always desirable, demonstrating that non-monotone policies can improve user welfare by better aligning creator incentives. Yao et al. (2024b) investigate the inherent trade-off between maximizing user satisfaction and maintaining creator productivity, highlighting that aggressive optimization for user metrics can suppress content supply. Yao et al. (2024c) propose a Stackelberg framework to optimize user welfare in the presence of strategic creators, and design efficient algorithms for policy optimization under such endogenous response. Shin et al. (2022, 2025) analyze a scenario where content producers may adversarially replicate their own content to abuse the recommendation algorithm, and devise an algorithm that disincentivizes such behavior. Many other recent works (Ben-Porat and Tennenholtz, 2018; Ghosh and Hummel, 2013; Zhu et al., 2023; Esmaeili et al., 2025) introduced stylized models to study strategic interaction in online platforms, but we will not discuss the details due to the significant difference to our model.

Fairness in recommender system. There are significant interests in designing fairness-aware recommendation algorithm for online platforms in the past few years. Chen et al. (2024) study the two-sided fairness for both the item and user. Golrezaei et al. (2024) propose a general framework to incorporate fairness into sequential decision making problem. Chen et al. (2022) investigate a fair assortment planning problem to retrieve a set of items with fixed cardinality to maximize expected revenue while satisfying the pairwise fairness. We refer to Li et al. (2023) for a survey. Our results on the optimality beyond the HardMax complements these recent line of works, suggesting that fairness can *endogenously arise* in solely optimizing the platform's objective, and thus one may not need to explicitly incorporate in the platform's objective in an exogenous manner.

2 Model

There $n \geq 2$ content producers (henchforth, producer), each of which creates a single content at costly efforts. Each producer i strategically decides its own effort level to create a competitive content i with quality q_i, which will be made clear shortly.[12]

Recommender system. The produced contents are registered in an online platform with recommender system (RS). Whenever a user joins the RS, it recommends a content to the user among the set of registered contents. The RS's recommendation *policy* is characterized by an allocation vector[13] $\mathbf{p} \in \Delta = \{(p_i)_{i\in[n]} : p_1 + \ldots + p_n \leq 1, p_1 \geq p_2 \geq \ldots \geq p_n \geq 0\}$, *i.e.,* ordered n-dimensional probability simplex. We consider a class of rank-based policy for the RS in which the RS commits to an allocation vector $\mathbf{p} \in \Delta$, and recommends the content with i-th highest quality with probability p_i for $i \in [m]$, where the ties are broken in uniformly at random. Thus, a policy $\mathbf{p}$ first sorts the contents based on their qualities so that $q_{(1)} \geq q_{(2)} \geq \ldots \geq q_{(m)}$[14], and display the content with the i-th highest quality $q_{(i)}$ with probability p_i. We write $\mathbf{q} = (q_1, \ldots, q_n)$ to denote the vector of contents' qualities.

We introduce two specific policies, which will appear vastly throughout the paper.

1. HardMax policy: $p_1 = 1$.
2. UniformButLast policy: $p_1 = p_2 = \ldots = p_{n-1} = \frac{1}{n-1}$, and $p_n = 0$.

In words, HardMax recommends the highest quality, alternatively the most preferred, content to the user deterministically. On the other hand, UniformButLast uniformly randomly selects any content except the one with the lowest quality. Note that these two policies constitute two extreme points in the policy space.

Induced game and equilibrium. Given a policy $\mathbf{p}$, we consider a game *induced by* $\mathbf{p}$ given n producers. Each producer i strategically decides the quality q_i of the content it creates from the action space $\mathbb{R}_{\geq 0}$. Each producer i may play a mixed strategy that randomizes over the action space. In this case q_i is a random variable and we write μ_i to denote the associated probability measure over $\mathbb{R}_{\geq 0}$ and $F_i : \mathbb{R}_{\geq 0} \to [0, 1]$ to denote the corresponding cumulative distribution function (c.d.f.). We say that a producer plays a pure strategy if μ_i has a singleton support. We

[12] We assume that the RS has an external module to estimate such qualities before recommending the contents.

[13] We use a boldface to denote a vector.

[14] Given a set of random variables (or static values) $x_1, \ldots, x_n$, we write $x_{(i)}$ to denote the i-th largest order statistics (values).

write $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and $\boldsymbol{\mu}_{-i} = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n)$ to denote the strategy profiles except i's strategy.

Producer i's expected utility $\mathcal{U}_i$ is a function over $\boldsymbol{\mu}$ and the policy $\mathbf{p}$, which will be defined shortly. Given a policy $\mathbf{p}$, a strategy profiles $\boldsymbol{\mu}$ is called an (Nash) equilibrium if no provider has incentive to deviate from $\boldsymbol{\mu}$, *i.e.,* for any $i \in [m]$:

$$\mathcal{U}_i(\mu_i, \boldsymbol{\mu}_{-i}, \mathbf{p}) \geq \mathcal{U}_i(\mu_i', \boldsymbol{\mu}_{-i}, \mathbf{p}),$$

for any mixed strategy μ_i'. If every strategy μ_i in $\boldsymbol{\mu}$ is the pure strategy, the resulting equilibrium is pure Nash equilibrium (PNE), otherwise it is mixed Nash equilibrium (MNE). An equilibrium is called *symmetric* if every player is playing the same strategy $\mu = \mu_i$ for $i \in [n]$ under the equilibrium. In this case, we write F to denote the corresponding c.d.f. In particular, throughout, we focus on symmetric MNE of the induced game, which will formally be shown to always exist in Section 3.[15]

Producer's utility. Producer i's expected utility consists of two parts: (i) the *revenue*[16] from the user recommended its content with quality q_i and (ii) the *cost* of generating the content with quality q_i.

Each producer realizes a unit revenue of 1 whenever its content is recommended to the user. producer i's expected revenue $\mathcal{R}_i$ of playing μ_i can be written as

$$\mathcal{R}_i(\mu_i, \boldsymbol{\mu}_{-i}, \mathbf{p}) = \mathbb{E}_{\mathbf{q}\sim\boldsymbol{\mu}}\left[\mathbb{1}\left\{ I_t = i \right\}\right],$$

where I_t denotes the index of the content producer whose content is recommended, which is a function of the policy $\mathbf{p}$ and the registered contents' qualities.

To create a competitive content, however, each provider exerts a certain level of effort (or opportunity cost). Formally, creating a content with quality q incurs a cost $c(q)$ that captures the effort exerted by the producer, where $c(x) = q^\beta$ for some $\beta > 0$.

Given strategy profiles $\boldsymbol{\mu}_{-i}$ of other providers, provider i's expected utility of playing a mixed strategy μ_i can be written as:

$$\mathcal{U}_i(\mu_i, \boldsymbol{\mu}_{-i}, \mathbf{p}) = \mathcal{R}_i(q, \boldsymbol{\mu}_{-i}, \mathbf{p}) - \mathbb{E}_{q_i\sim\mu_i}\left[c(q)\right]$$

User welfare. The platform's objective, *e.g.,* long-term revenue or lifespan, is largely dependent on the *user welfare* (user engagement). To formally define the user welfare, we assume that once a user watches a content of quality q, the user realizes linear utility q. Then, given a policy $\mathbf{p}$ and corresponding producers' strategies $\boldsymbol{\mu}$, we write $\mathcal{W}(\boldsymbol{\mu}, \mathbf{p})$ to denote the resulting user welfare, defined formally as follows:

$$\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) = \mathbb{E}_{\mathbf{q}\sim\boldsymbol{\mu}}\left[\sum_{i=1}^{m} \mathbb{1}\left\{ I_t = i \right\} \cdot q_i\right].$$

We often write $\mathcal{W}(\boldsymbol{\mu})$ if the policy $\mathbf{p}$ that induces $\boldsymbol{\mu}$ is clear from the context.

Platform quality. The RS's primary goal is to maximize the user welfare $\mathcal{W}$ given the strategic behavior of the providers. On the other hand, to maintain its overall quality that might affect the

[15] In fact, pure Nash equilibrium does not always exist even in simple case when the platform runs HardMax policy and given certain choice of parameters, as shown in Proposition 3.1.

[16] The revenue may originate from advertisements that the producer includes in the content by itself or that the platform inserts before (or in the middle of) the content plays. Also, the platform may compensate the producer based on its outcomes, *e.g.,* views, clicks or user retention.

platform's reputation in the long run, the RS also desires to maintain the average quality of the registered contents. Formally, we are interested in the *platform quality* as an average quality of the contents at the symmetric equilibrium $\boldsymbol{\mu}$ given $\mathbf{p}$, *i.e.,*

$$\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p}) = \sum_{i=1}^{m} \mathbb{E}_{q_i \sim \mu_i}\left[q_i\right]/m.$$

Likewise, we often write $\mathcal{Q}(\boldsymbol{\mu})$ if the policy $\mathbf{p}$ that induces $\boldsymbol{\mu}$ is clear from the context.

Objective function. Finally, the RS's objective is to maximize the weighted summation of induced user welfare $\mathcal{W}(\boldsymbol{\mu})$ and the average quality $\mathcal{Q}(\boldsymbol{\mu})$:

$$\mathrm{OBJ}(\boldsymbol{\mu}, \mathbf{p}) = \alpha\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) + (1-\alpha)\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p}),$$

where $\alpha \geq 0$ governs the importance between the user welfare and the platform's quality.

Essentially, we are interested in a policy $\mathbf{p}$ that maximizes the objective function when the producers are playing symmetric equilibrium strategies $\boldsymbol{\mu}$. Thus, we aim to find a policy $\mathbf{p}$ that solves the following optimization problem:

$$\begin{aligned} (\textsc{Opt}) \quad & \underset{\mathbf{p}\in\Delta}{\text{maximize}} \quad \alpha\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) + (1-\alpha)\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p}) \\ & \text{subject to} \quad \boldsymbol{\mu} \text{ is an equilibrium} \end{aligned}$$

We say the RS is *single-minded* if $\alpha = 0$ or 1, *i.e.,* it aims to solely maximize either the user welfare or the platform quality, and *convex-minded* if $\alpha \in (0,1)$. One might notice that in the optimal policy, it is required to have $p_i \neq p_j$ for some $i, j \in [m]$ since otherwise the dominant strategy is play $q = 0$. Thus, we focus on the class of *nontrivial* policy in Δ such that $\mathbf{p} \neq (1/n, 1/n, \ldots, 1/n)$.

2.1 Expanding producer's expected utility and user welfare

Before getting into the main results, we provide some further preliminaries on our problem. We first provide some explanations on how one calculate the producer's expected utility as well as the user welfare. Consider a symmetric strategies $\boldsymbol{\mu}$ such that corresponding cumulative distribution function $F(\cdot)$ does not have a point mass. For any action q in the support of μ_i, we have

$$\begin{aligned} \mathcal{U}_i(q, \boldsymbol{\mu}_{-i}, \mathbf{p}) &= \mathcal{R}_i(q, \boldsymbol{\mu}_{-i}, \mathbf{p}) - c(q) \\ &= \mathbb{E}\left[\left(\sum_{t\in[n]} \mathbb{1}\left\{ I_t = i \right\}\right) - c(q)\right] \\ &= \left(\sum_{t\in[n]} \Pr\left[\text{Provider } i\text{'s content is recommended to user } t\right]\right) - c(q) \\ &= \left(\sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q)^{n-i}(1-F(q))^{i-1}\right) - c(q). \end{aligned}$$

For the user welfare, let E_i be the random event such that the chosen content has the i-th highest quality among $[n]$. Expanding using the definition of $\mathbf{p}$, we obtain:

$$\begin{aligned}\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) &= \mathbb{E}_{\mathbf{q}\sim\boldsymbol{\mu}}\left[\sum_{i=1}^{m} \mathbb{1}\{\, I_t = i \,\} \cdot q_i\right] \\ &= \mathbb{E}_{\mathbf{q}\sim\boldsymbol{\mu}}\left[\sum_{i=1}^{n} \mathbb{1}\{\, E_i \,\} \cdot q_{(i)}\right] \\ &= \sum_{i=1}^{n} p_i \mathbb{E}_{\mathbf{q}\sim\boldsymbol{\mu}}\left[q_{(i)}\right].\end{aligned}$$

2.2 Comparison with Glazer and Hassin (1988) and Jagadeesan et al. (2024)

As we briefly mentioned thus far, our model is closely related to that by Glazer and Hassin (1988) and Jagadeesan et al. (2024). We discuss several technical differences and how our model generalizes and subsumes their settings, respectively.

Comparison with Glazer and Hassin (1988). Our model generalizes Glazer and Hassin (1988) by setting $\alpha = 0$.[17] They also consider a rank-based policy to maximize the platform's quality $\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p})$, assuming the existence of symmetric MNE. In particular, they prove the following theorem when the cost function is linear:

Theorem 2.1 (Glazer and Hassin (1988))**.** *Consider a linear cost function ($c(q) = q$) and $\alpha = 0$,* i.e., *the objective function is $\mathcal{Q}(\boldsymbol{\mu}, \boldsymbol{p})$. Then the optimal policy is* UniformButLast*.*

Importantly, their proof follows from the following observations: (i) p_n should always set to be zero to avoid free-riding effect, (ii) in the linear cost case, we can rewrite the objective function as a concave function purely over $\mathbf{p}$, and finally (iii) the UniformButLast policy binds the first-order condition, thereby implying its optimality. If the cost function becomes nonlinear, their analysis does not work and even the objective function becomes nonconcave, implying that verifying the first-order condition does not anymore guarantee the optimality.

Nevertheless, we obtain the optimal policy for any cost function parameterized by $c(x) = x^{\beta}$ for $\beta > 0$, generalizing the linear cost case by Glazer and Hassin (1988). Notably, our analysis does not rely on the first-order condition, but rather exactly characterize the stronger conditions that the objective function satisfy. More details can be found in Section 5.

Finally, it is worth noting that Glazer and Hassin (1988) *assume* the existence of a symmetric MNE given any policy without proving it formally, then deal with its characteristics given such equilibrium. On the other hand, we formally prove the existence of symmetric MNE in Theorem 3.2, complementing the missing gap in Glazer and Hassin (1988).

Comparison with Jagadeesan et al. (2024). Our model, up to different perspective, also subsumes Jagadeesan et al. (2024) who characterize the equilibrium under the specific HardMax policy. We emphasize that they analyze equilibria under the given policy, and does not aim to consider a design problem that induces a good equilibrium for the platform. Notably, Jagadeesan et al. (2024) starts with quantifying the equilibrium outcome for the HardMax policy as follows.[18]

[17] Precisely, they consider a utility function over the p_i such that each contestant realizes utility $u(p_i)$. This feature is not captured in our setting since p_i is a probability rather than a prize in our setting.

[18] They in fact generalize this result into multi-dimensional setting, which would be an intriguing direction for contest design perspective, too.

Theorem 2.2 (Jagadeesan et al. (2024)). *Given* HardMax *policy, there exists a unique symmetric equilibrium* $\boldsymbol{\mu}$ *whose cumulative distribution function (c.d.f.) is* $F(q) = \left(q^{\beta}\right)^{1/(n-1)}$.[19]

Notably, the proof of Theorem 2.2 by Jagadeesan et al. (2024) follows from the characteristics of the HardMax policy that significantly simplifies the equilibrium condition, which leads to a simple equation of degree one over the cumulative distribution function $F(\cdot)$. Solving such equation immediately yields Theorem 2.2.

In this case, we observe that the induced user welfare of HardMax policy is as follows.

Proposition 2.3. *Given* HardMax *policy, consider the unique symmetric mixed equilibrium* $\boldsymbol{\mu}$ *characterized by Theorem 2.2. Then, it follows that*

$$\mathrm{OBJ}(\boldsymbol{\mu}; \textsc{HardMax}) = \frac{\beta n}{\beta n + n - 1} + \frac{\beta}{\beta + n - 1}. \tag{2.1}$$

Proof. By Theorem 2.2, we have $F(q) = q^{\beta/(n-1)}$. Let $F_{\max}$ be the c.d.f. of the largest order statistics over the random contents. Then, observe that the induced user welfare of HardMax is equivalent to the expectation of the largest order statistics among $q_1, \ldots, q_n$. Thus, we can write

$$\begin{aligned}
\mathrm{OBJ}(\boldsymbol{\mu}; \textsc{HardMax}) &= \alpha \mathbb{E}_{\mathbf{q} \sim \boldsymbol{\mu}}\left[\max_{j \in [n]} q_j\right] + (1-\alpha)\mathbb{E}_{q \sim F}\left[q\right] \\
&= \alpha \int_0^1 (1 - F_{\max}(x))dx + (1-\alpha)\int_0^1 (1 - F(x))dx.
\end{aligned}$$

To compute $\int_0^1 (1 - F_{\max}(x))dx$, notice that

$$F_{\max}(x) = \Pr\left[\, q_j \le x, \forall j \in [n] \,\right] = \left(x^{\beta}\right)^{n/(n-1)},$$

concluding

$$\begin{aligned}
\int_0^1 (1 - F_{\max}(x))dx &= \int_0^1 \left(1 - \left(x^{\beta}\right)^{n/(n-1)}\right)dx \\
&= 1 - \frac{n-1}{\beta n + n - 1} \\
&= \frac{\beta n}{\beta n + n - 1},
\end{aligned}$$

For $\int_0^1 1 - F(x)dx$, we have

$$\begin{aligned}
\int_0^1 (1 - F(x))dx &= \int_0^1 1 - x^{\beta/(n-1)}dx \\
&= 1 - \frac{1}{\beta/(n-1) + 1}\left[x^{\beta/(n-1)+1}\right]\Big|_0^1 \\
&= \frac{\beta}{n - 1 + \beta},
\end{aligned}$$

and combining two terms finish the proof. □

[19] Note that this is different from what is written as-is in Jagadeesan et al. (2024). We privately confirmed with one of the authors of Jagadeesan et al. (2024) that their statement has a minor typo, and we provide the corrected version here.

On the other hand, if one consider arbitrary policy beyond HardMax, it becomes significantly challenging to characterize the exact equilibrium since the equilibrium condition becomes an equation of degree $\Theta(n)$ over $F(q)$ as well as q, which makes it difficult to exactly analyze the equilibrium as obtaining the exact solution for high-degree polynomial equation is impossible in general.

Nonetheless, in our analysis, we show that one can characterize the value of the objective function purely as a function over $\mathbf{p}$ *without characterizing* the equilibrium. This enables us to re-obtain Proposition 2.3 without using the structure of $F(\cdot)$, as can be seen in Section 4.

3 Existence of Symmetric Mixed Nash Equilibrium

To analyze the induced user welfare, it is crucial to analyze what the equilibrium point is. We first prove that there exists no PNE for any nontrivial policy.

Proposition 3.1. *There exists no PNE for any nontrivial policy* $\boldsymbol{p}$.[20]

Proof. Given a policy $\mathbf{p}$, suppose there exists a PNE $q_1 \geq q_2 \geq \ldots \geq q_n$. If $q_i \neq 0$ but $p_i = 0$, producer i can effectively deviates to $q_i = 0$ while decreasing the cost, which is a contradiction. Thus, if $q_i \neq 0$ then we always have $p_i \neq 0$. Let i be the largest index where $q_i \neq 0$. Suppose there exists no tie among producers $j \in [i-1]$, *i.e.,* $q_1 > q_2 > \ldots > q_i$. Any producer $j \in [i]$ can deviate to $q_j(1-\varepsilon)$ for sufficiently small $\varepsilon > 0$ while respecting $q_j(1-\varepsilon) > q_{j+1}$, which only decreases the cost. Thus, this cannot be an equilibrium. Suppose now that there exists a tie at j such that $q_j = q_{j+1}$. Suppose there exists $k \geq 2$ such agents where j be the smallest index, *i.e.,* $q_j = q_{j+1} = \ldots = q_{j+k-1}$. Due to the uniformly random tie-breaking rule, all the producers from j to $j+k-1$ obtain revenue of $(p_j + p_{j+1} + \ldots p_{j+k-1})/k$. If $p_j = p_{j+k-1}$, then producer j can slightly decrease the quality while still receiving the revenue of $p_{j+k-1} = (p_j + p_{j+1} + \ldots p_{j+k-1})/k$ and decreasing the cost. Thus, $p_j \neq p_{j+k-1}$. In this case, producer j can instead increase the quality from q_j to $q_j + \varepsilon$ and solely obtain $p_j > (p_j + p_{j+1} + \ldots p_{j+k-1})/k$ while only increasing cost function sufficiently small so that the overall utility increases due to the continuity of the payoff function. This finishes the proof. □

Therefore, we aim to characterize the MNE instead of PNE. Specifically, we show that a symmetric MNE always exists for any nontrivial policy $\mathbf{p}$ in the following theorem.

Theorem 3.2. *For any nontrivial policy* $\mathbf{p}$*, there exists a symmetric MNE in the induced game.*

Intuitively speaking, the symmetry of the game, *i.e.,* symmetry of the provider's payoff and the action space, supports the existence of the symmetric equilibrium. The formal proof, however, requires a subtle argument based on the notion of *better-reply secure* and topological arguments, as the action space is continuous but the payoff is discontinuous over the pure strategies due to the thresholded nature of the rank-based policy. That is, increasing the quality increases the revenue in a potentially discontinuous manner from $p_n, p_{n-1}, \ldots$ to p_1.

The proof follows a similar structure by Jagadeesan et al. (2024), but generalizes their analysis into broader policies beyond HardMax which requires further technical arguments to reason the diagonally better-reply secureness of the game given any nontrivial policy $\mathbf{p}$.

Given the existence of the symmetric MNE, the following proposition states that we can effectively wipe out the tie-breaking scenario and solely represent the user welfare and producer utility as presented in Section 2.1.

[20] Note that this result strictly generalizes the nonexistence of PNE by Jagadeesan et al. (2024) (Proposition 1) under HardMax rule.

Proposition 3.3. *Given a nontrivial policy $\mathbf{p}$ and symmetric MNE $\boldsymbol{\mu}$, then the corresponding c.d.f. F of $\boldsymbol{\mu}$ is atomless and strictly increasing in some interval $[q_{\min}, q_{\max}]$.*

Proof. To prove that it is atomless, fix an agent i. Assume, for sake of contradiction, that there exists p such that an agent plays p with probability $\alpha > 0$. Let $p' = p + \varepsilon$ for $\varepsilon > 0$. Let E be the event that producers $j \in [n] \setminus \{i\}$ play p, *i.e.,* $\Pr[E] = \alpha^{n-1}$. Conditioned on E, playing p yields the expected revenue of $1/n$. On the other hand, playing p' yields the expected utility of at least $p_1 > 1/n$ for any $\varepsilon > 0$ as $\mathbf{p}$ is nontrivial. Thus, due to the continuity of the cost function, producer i can deviate to $p+\varepsilon$ for sufficiently small ε and strictly increases the revenue from $1/n$ to p_1 under the event E. Since this would not decrease the revenue under E^c, one can find sufficiently small ε to strictly increase producer i's utility, which is a contradiction.

To prove that it is strictly increasing, assume $(a, b) \in [q_{\min}, q_{\max}]$ be the subinterval with the maximal length in $[0, q_{\max}]$ such that F remains a constant, where $q_{\max} = \sup\{y : F(y) < 1\}$. Note that $q_{\min} \geq 0$ and $q_{\max} \leq 1$ as playing 1 induces nonpositive payoff. Obviously, any agent can deviate from playing b to a (moving probability mass) and strictly increasing the utility, which is a contradiction. □

This implies that the c.d.f. F of the symmetric MNE is differentiable almost everywhere (a.e.) by Lesbesgue's theorem, once it exists.

To prove the existence of a symmetric MNE, we first introduce a preliminary on relevant notions, where readers of whom are familiar with the topology and better-reply secureness in discontinuous game by Reny (1999) might skip the details.

3.1 Preliminaries

Technically speaking, to argue the existence of a symmetric MNE, we need to analyze the expected payoff of mixed strategies that belong to an open neighbourhood of a symmetric strategy to apply the argument by Reny (1999). To this end, we rigorously define the notion of open neighbourhood of a mixed strategy (a probability measure on a set X) using weak* topology. Since it is difficult to directly deal with the topological space itself, we use the metrizability of the space of probability measure on a compact Hausdorff space with respect to Lévy-Prokhorov metric, which allows us to treat the topological space as a metric space, *i.e.,* dealing the topological space of the set of probability measures on X equipped with weak* topology using the metric in the original topological (metric) space X. For readers who might not be familiar with the basic concepts in topology, we refer to Appendix A for some preliminaries on topology.

Game and mixed extension. A game $G = (X_i, u_i)_{i\in[n]}$ consists of n players, a set of pure strategy X_i and payoff functions $u_i : X \to \mathbb{R}$ where $X = \times_{i\in[n]} X_i$ for $i \in [n]$. We assume throughout that X_i is a compact Hausdorff space, and we say the game is *Hausdorff* game in such case. Throughout, the product of any number of sets is endowed with the product topology. Let M denote the set of (Borel) probability measures[21] on X, *i.e.,* the set of mixed strategies over X.

Given a game $G = (X_i, u_i)_{i\in[n]}$ and the set of mixed strategies M_i defined above, we extend each u_i to have its domain as $M = \times_{i\in[n]} M_i$ by defining

$$u_i(\mu) = \int_X u_i(x) d\mu,$$

for all $\mu \in M$. We denote by $\bar{G} = (M_i, u_i)_{i\in[n]}$ the mixed extension of G.

[21] A Borel measure on a topological space is a measure that is defined on all open sets.

Quasi-symmetric game. Consider a *game* $G = (X_i, u_i)_{i\in[n]}$ (which is possibly a mixed extension of another original game G'). We say that G is a *quasi-symmetric* game if $X = X_1 = \dots X_n$ and for any $x, y \in X$, it follows that

$$u_1(x, y, \dots, y) = u_2(y, x, y, \dots, y) = \dots = u_n(y, \dots, y, x).$$

Define a quasi-symmetric game's *diagonal* payoff function $v : X \to \mathbb{R}$ as $v(x) = u_1(x, \dots, x) = u_n(x, \dots, x)$ for every $x \in X$. In proving the existence of a symmetric MNE, the notion of better-reply secure is crucial. To that end, we first introduce a notion of securing a payoff.

Definition 3.4 (Secure a payoff)**.** Given a quasi-symmetric game $G = (X, u_i)_{i\in[n]}$, player i can *secure a payoff of* $\alpha \in \mathbb{R}$ *along the diagonal* at $(x, \dots, x)$ if there exists $\bar{x} \in X$ such that $u_i(x', \dots, \bar{x}, \dots, x') \geq \alpha$ for every x' in some open neighborhood of $x \in X$.[22]

The following diagonally better-reply secureness guarantees that any player could secure a strictly larger payoff along the diagonal, given a symmetric strategy profile which is not an equilibrium.

Definition 3.5 (Diagonally better-reply secure)**.** Given a quasi-symmetric game $G = (X, u_i)_{i\in[n]}$, let $(x, v(x)) \in X \times \mathbb{R}$ be the pair of an action $x \in X$ and diagonal payoff function $v(x)$. Then, G is *diagonally better-reply secure* if for any $(x, v(x))$ with $x \in X$ such that $(x, \dots, x)$ is not an equilibrium, there exists a player i who can secure a payoff strictly above $v(x)$ along the diagonal at $(x, \dots, x)$.

The seminal result by Reny (1999) prove that any quasi-symmetric game possesses a symmetric MNE under the diagonally better-reply secureness and semi-continuity of the payoff function.

Theorem 3.6 (Corollary 5.3, Reny (1999))**.** *Suppose $G = (X_i, u_i)_{i\in[n]}$ is a quasi-symmetric game with compact Hausdorff action space. Then, G has a symmetric MNE if its mixed extension $\bar{G}$ is diagonally better-reply secure and each $u_i(\mu, \dots, \mu)$ is upper semi-continuous over μ.*

3.2 Proof of Theorem 3.2

To prove Theorem 3.2, we will sequentially show that our induced game given a nontrivial policy $\mathbf{p}$ satisfies the conditions listed in Theorem 3.6. We first start with the following result that the induced game is quasi-symmetric and Hausdorff game.

Lemma 3.7. *The induced game given a nontrivial policy is quasi-symmetric Hausdorff game.*

Proof. The quasi-symmetry immediately follows from the fact that every producer is homogeneous in a sense that they have the same cost function and thereby the same payoff as a function over the quality.

To prove that it is Hausdorff game, it suffices to prove that its action space is compact. Given any policy, for any MNE, each provider must receive the nonnegative expected utility for any action q in any support of the MNE, as playing $q = 0$ simply yields the utility zero. On the other hand, if the support contains $q > 1$, then the expected utility is strictly negative. Thus, it is without loss of generality to restrict the support of action space to be $[0, 1]$, which is compact. $\square$

Next, we prove that the utility function of each producer as a function over the mixed strategy is upper semi-continuous at any symmetric strategy profiles.

[22] When dealing with G as a mixed extension of another original game, the open neighborhood refers to the open neighborhood with respect to the underlying topology over the space of mixed strategies.

Lemma 3.8. *For any nontrivial policy $\mathbf{p}$, at any symmetric mixed strategy profiles $\boldsymbol{\mu} = (\mu, \dots, \mu)$, the payoff function $\mathcal{U}(\mu; \boldsymbol{\mu}_{-i})$ is upper semi-continuous in μ.*

Proof. Due to the symmetric of the game, for any symmetric mixed strategy, it is straightforward to see that the overall expected revenue of each producer is simply $1/n$. Thus, the expected utility can be written as

$$\mathcal{U}_i(\mu; \boldsymbol{\mu}_{-i}) = \frac{1}{n} - \int_0^1 c(x) d\mu.$$

As the space of set of mixed strategies μ is the weak* topology as discussed in Appendix A, the utility function is continuous in μ, implying the upper semi-continuity.

□

Now, for ease of exposition, we define

$$R(q; \boldsymbol{\mu}_{-i}) = \sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q)^{m-i}(1 - F(q))^{i-1}. \tag{3.1}$$

Note that this is the expected revenue in playing q when the others play mixed strategy according to F, *i.e.*,

$$\mathcal{U}_i(q; \boldsymbol{\mu}_{-i}) = R(q; \boldsymbol{\mu}_{-i}) - c(q),$$

but *assuming* that there is no point mass in F. We often write $R(q)$ if the other producers' strategies are clear from the context.

The following lemma will be useful in proving diagonally better-reply secureness.

Lemma 3.9. *For any nontrivial policy $\mathbf{p}$, others' symmetric mixed strategies $\boldsymbol{\mu}_{-i}$ and corresponding c.d.f. F, $R(q; \boldsymbol{\mu}_{-i})$ is monotone increasing in q.*

Proof. Fix q, and consider a random variable $Y_q \sim \text{Binomial}(n-1, 1-F(q))$. Then

$$\Pr[Y_q = i-1] = \sum_{i=1}^{n} \binom{n-1}{i-1} F(q)^{n-i}(1 - F(q))^{i-1}.$$

Therefore,

$$\begin{aligned} R(q; \boldsymbol{\mu}_{-i}) &= \sum_{i=1}^{n} p_i \Pr[Y_q = i-1] \\ &= \sum_{i=1}^{n} p_i \Pr[i = Y_q + 1] \\ &= \mathbb{E}\left[p_{Y_q+1}\right]. \end{aligned}$$

Note that if $q \geq q'$, then $F(q) \geq F(q')$, and thus Y_q is stochastically dominated by $Y_{q'}$. Hence, we have $\mathbb{E}[f(Y_q)] \geq \mathbb{E}\left[f(Y_{q'})\right]$ for any nonincreasing function f. Since p_{Y+1} is nonincreasing in Y as $p_1 \geq p_2 \geq \dots \geq p_n$, we have $\mathbb{E}\left[p_{Y_q+1}\right] \geq \mathbb{E}\left[p_{Y_{q'}+1}\right]$, which finishes the proof. □

To prove the existence of the symmetric MNE, the final step is to prove the diagonally better-reply secureness by Reny (1999).

Lemma 3.10. *For any policy $\boldsymbol{p}$ with $p_1 \geq p_2 \geq \ldots \geq p_n$, the induced game is diagonally better-reply secure.*

Proof. Let μ be a mixed strategy which is not a symmetric MNE, and let u be the corresponding payoff at the diagonal, *i.e.,* $u = \mathcal{U}(\mu; \mu, \ldots, \mu)$. It's sufficient to show that there exists another strategy μ' such that $\mathcal{U}(\mu', (\mu'', \ldots, \mu'')) > u$ for any μ'' in some open neighborhood $B_\delta(\mu)$ with respect to the Lévy-Prokhorov metric given the ℓ_2 normed space for pure strategies.

Since μ is not an equilibrium, there exists q such that $\mathcal{U}(q; \mu, \ldots, \mu) > \mathcal{U}(\mu; \mu, \ldots, \mu) = u$. Now we will show that there exists a pure strategy q^{sec} that yields strictly larger payoff than any point on an open neighborhood of μ. To this end, we will properly perturb q.

Let F be the corresponding c.d.f. of μ. Let $r(q)$ denote the expected revenue in playing q when other producers play according to F. As any probability distribution can have at most countably many atoms, we can slightly perturb (increase) q to $\tilde{q}$ such that μ does not have a point mass at $\tilde{q}$ and satisfies $\mathcal{U}(\tilde{q}; \mu, \ldots, \mu) > u$.[23] Since there is no point mass at $\tilde{q}$, we can write

$$r(\tilde{q}) = R(\tilde{q}) = \sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(\tilde{q})^{n-i}(1 - F(\tilde{q}))^{i-1}.$$

Using similar arguments, in fact, we can further find q^{sec} having a sufficiently small slack ε such that:

$$\begin{aligned}\mathcal{U}(q^{sec}, \boldsymbol{\mu}_{-i}) &= \sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q^{sec})^{n-i}(1 - F(q^{sec}))^{i-1} - c(q^{sec}) &\quad (3.2)\\ &\geq \sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q^{sec} - \varepsilon)^{n-i}(1 - F(q^{sec} - \varepsilon))^{i-1} - c(q^{sec}) > u, &\quad (3.3)\end{aligned}$$

and that F does not have point mass at both $q^{sec} - \varepsilon$ and q^{sec} due to the continuity of F. Note that such ε can be set arbitrarily small by observing arbitrarily sufficient small neighborhood of q^{sec}. Specific choice of ε can be made later. We will prove that q^{sec} is the desired strategy that secures payoff larger than u along the diagonal at $\boldsymbol{\mu}$.

Define the event A as follows:

$$A = \{q' : q^{sec} > q'\},$$

and A^ε as

$$A^\varepsilon = \{q' : q^{sec} > q' + \varepsilon\}.$$

We can rewrite (3.3) as follows:

$$\Gamma := \sum_{i=1}^{n} p_i \binom{n-1}{i-1} \mu(A^\varepsilon)^{n-i}(1 - \mu(A^\varepsilon))^{i-1} - c(q^{sec}) > u.$$

For any δ, let $B_\delta(\mu)$ be the δ-open ball with respect to the Lévy-Prokhorov metric on the space of the probability measure, where the metric is given by the ℓ_2 norm in the original metric space over $[0, 1]$. Note that $B_\delta(\mu)$ is an open set (neighborhood) with respect to the weak* due to Theorem A.3.

[23] One can think about arbitrarily small continuous neighborhood of q that guarantees the expected utility larger than u, then such perturbed point can be found in this interval.

For every $q' \in A^\varepsilon$, notice that A contains the open neighborhood $B_\varepsilon(q')$ with respect to the ℓ_2 norm. Thus, by the definition of the Lévy-Prokhorov metric, for any $\mu' \in B_\varepsilon(\mu)$, we have

$$\mu'(A) \geq \mu(A^\varepsilon) - \varepsilon.$$

Further, for any $q' \in A^c$, observe that $(A^\varepsilon)^c$ contains the open neighborhood $B_\varepsilon(q')$ with respect to the ℓ_2 norm. Thus, again by the definition of the Lévy-Prokhorov metric, for any $\mu' \in B_\varepsilon(\mu)$, we have

$$\mu'((A^\varepsilon)^c) \geq \mu(A^c) - \varepsilon,$$

or equivalently

$$1 - \mu'(A^\varepsilon) \geq 1 - \mu(A) - \varepsilon,$$

concluding that

$$\mu(A^\varepsilon) - \varepsilon \leq \mu'(A) \leq \mu(A^\varepsilon) + \varepsilon.$$

Thus, we have that

$$\begin{aligned}
\mathcal{U}(q^{sec}; \mu', \ldots, \mu') &= \sum_{i=1}^{n} p_i \binom{n-1}{i-1} \mu'(A)^{n-i} (1 - \mu'(A))^{i-1} v(q^{sec}) - c(q^{sec}) \\
&\geq \sum_{i=1}^{n} p_i \binom{n-1}{i-1} (\mu(A^\varepsilon) - \varepsilon)^{n-i} (1 - \mu(A^\varepsilon) - \varepsilon)^{i-1} v(q^{sec}) - c(q^{sec}) \\
&= \Gamma + \underbrace{\left(\sum_{i=1}^{n} p_i \binom{n-1}{i-1} (\mu(A^\varepsilon) - \varepsilon)^{n-i} (1 - \mu(A^\varepsilon) - \varepsilon)^{i-1} v(q^{sec}) - c(q^{sec}) - \Gamma \right)}_{(\Delta)}.
\end{aligned}$$

Note that

$$\begin{aligned}
\Delta = \sum_{i=1}^{n} p_i \binom{n-1}{i-1} (\mu(A^\varepsilon) - \varepsilon)^{n-i} (1 - \mu(A^\varepsilon) - \varepsilon)^{i-1} v(q^{sec}) - c(q^{sec}) \\
- \sum_{i=1}^{n} p_i \binom{m-1}{i-1} \mu(A^\varepsilon)^{n-i} (1 - \mu(A^\varepsilon))^{i-1} v(q^{sec}) - c(q^{sec}),
\end{aligned}$$

which converges to 0 as $\varepsilon \to 0$ due to the continuity of F. Thus, we can pick sufficiently small ε such that

$$\Delta < \Gamma - u,$$

as $\Delta \to 0$ while Γ is always larger than u as $\varepsilon \to 0$. Finally, for any $\mu' \in B_\varepsilon(\mu)$, we obtain

$$\mathcal{U}(q^{sec}; \mu', \ldots, \mu') \geq \Gamma - \Delta > u,$$

which finishes the proof.

□

Proof of Theorem 3.2. Finally, the proof immediately follows from Theorem 3.6 combined with lemmas 3.7, 3.9, 3.8, and 3.10. □

4 Reduced Optimization Problem

Recall the RS's problem stated as an optimization problem OPT. In this section, we will show that the RS's objective can be written as a pure optimization problem over $\mathbf{p}$ without involving terms related to the symmetric MNE.

4.1 Notations and lemmas

To this end, we first introduce several preliminary lemmas that will be used throughout the analysis.

Lemma 4.1. *Let $u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an absolutely continuous nondecreasing function that satisfies $u(0) = 0$, and X be a nonnegative random variable with c.d.f. F with no point mass. Then $\mathbb{E}[\, u(X) \,] = \int_0^\infty (1 - F(x))u'(x)dx$ if the integral is finite.*

Proof.

$$
\begin{aligned}
\mathbb{E}[\, u(X) \,] &= \int_0^\infty \Pr[\, u(X) > t \,]\, dt && \text{(layer-cake representation)} \\
&= \int_0^\infty \Pr\left[\, X > u^{-1}(t) \,\right] dt && \text{(since } u \text{ is nondecreasing and invertible on } [0, \infty)) \\
&= \int_0^\infty \left[1 - F\big(u^{-1}(t)\big)\right] dt \\
&= \int_0^\infty \left[1 - F(x)\right] u'(x)\, dx && \text{(change of variable } t = u(x) \text{, so } dt = u'(x)\, dx).
\end{aligned}
$$

□

The following is a well-known characterization on the c.d.f. of order statistics for identically and independently distributed (i.i.d.) random variables.

Lemma 4.2 (Casella and Berger (2024))**.** *Given n i.i.d. random variables $X_1, .., X_n$, let F be its c.d.f. and $F_{(i)}$ be the c.d.f. of the i-th largest order statistic. Then, for any $i \in [n]$,*

$$
F_{(i)}(x) = \sum_{j=1}^{i} \binom{n}{j-1} (1 - F(x))^{j-1} F(x)^{n-j+1}.
$$

We omit the proof, which is routine and covered in standard texts. To present our main results, we introduce several notations for ease of exposition.

$$\psi(x,\mathbf{p}) = \sum_{i=2}^{n-1} p_i(n-1)\left(\binom{n-2}{i-1}x^{n-i-1}(1-x)^{i-1} - \binom{n-2}{i-2}x^{n-i}(1-x)^{i-2}\right) \\ + p_1(n-1)x^{n-2} - p_n(n-1)(1-x)^{n-2}, \tag{4.1}$$

$$\begin{aligned}\phi(x,\mathbf{p}) &= \sum_{i=1}^{n}\left(1-\sum_{j=1}^{i}\binom{n}{j-1}(1-x)^{j-1}(x)^{n-j+1})\right)p_i \\ &= 1-\sum_{i=1}^{m}\binom{n}{i-1}(1-x)^{i-1}x^{n-i+1}(\sum_{j=i}^{n}p_j),\end{aligned} \tag{4.2}$$

$$a_i(x) = \binom{n-1}{i-1}x^{n-i}(1-x)^{i-1} \tag{4.3}$$

$$h(x,\mathbf{p}) = \sum_{i=1}^{n} a_i(x)p_i. \tag{4.4}$$

Note that $a_i(x)$ is exactly the *Bernstein basis polynomial* of degree $n-1$, where the Bernstein polynomial $b_{\nu,n}$ is typically defined as:

$$b_{\nu,n}(x) = \binom{n}{\nu}x^{\nu}(1-x)^{n-\nu},$$

for $\nu = 0,\ldots,n$. Due to our construction of $a_i(x)$, observe that

$$a_i(x) = b_{n-i,n-1}(x),$$

for $i \in [n]$.

We will now provide some connections between the defined notations.

Lemma 4.3. *$\psi(x,\boldsymbol{p})$ is a derivative of $h(x,\boldsymbol{p})$ with respect to x.*

Proof. Note that

$$\frac{dh(x,\mathbf{p})}{dx} = \frac{d\sum_{i=1}^{n} p_i a_i(x)}{dx}. \tag{4.5}$$

If $i = 2,\ldots,n-1$, we have

$$\begin{aligned}\frac{da_i(x)}{dx} &= \binom{n-1}{i-1}(n-i)x^{n-i-1}(1-x)^{i-1} - \binom{n-1}{i-1}(i-1)x^{n-i}(1-x)^{i-2} \\ &= \frac{(n-1)!}{(i-1)!(n-i-1)!}x^{n-i-1}(1-x)^{i-1} - \frac{(n-1)!}{(i-2)!(n-i)!}x^{n-i}(1-x)^{i-2} \\ &= (n-1)\left(\binom{n-2}{i-1}x^{n-i-1}(1-x)^{i-1} - \binom{n-2}{i-2}x^{n-i}(1-x)^{i-2}\right).\end{aligned}$$

For $a_1(x)$ and $a_n(x)$, we have

$$\begin{aligned}\frac{da_1(x)}{dx} &= (n-1)x^{n-2} \\ \frac{da_n(x)}{dx} &= -(n-1)(1-x)^{n-2}.\end{aligned}$$

Plugging into (4.5) finishes the proof. □

The following is a well-known property of Bernstein polynomials.

Lemma 4.4 (Lorentz (2012))**.** *$\int_0^1 a_i(x)dx = 1/(n-1)$ for any $i \in [n]$.*

The proof is omitted as it follows from standard textbook arguments. Finally, we require the following relation between $\phi(x, \mathbf{p})$ and $h(x, \mathbf{p})$.

Lemma 4.5. *It follows that*

$$\frac{d\phi(x, \boldsymbol{p})}{dx} = -nh(x, \boldsymbol{p}).$$

Proof. We have

$$\begin{aligned}\frac{d\phi(x, \mathbf{p})}{dx} = -nx^{n-1} &+ \underbrace{\sum_{i=2}^{n} \binom{n}{i-1}(i-1)(1-x)^{i-2}x^{n-i+1}\sum_{j=i}^{n} p_j}_{(A)} \\ &- \underbrace{\sum_{i=2}^{n} \binom{n}{i-1}(n-i+1)(1-x)^{i-1}x^{n-i}\sum_{j=i}^{n} p_j}_{(B)}.\end{aligned}$$

Note that

$$\begin{aligned}(A) &= \sum_{i=2}^{n} \frac{n!}{(i-1)!(n-i+1)!}(i-1)(1-x)^{i-2}x^{n-i+1}\sum_{j=i}^{n} p_j \\ &= \sum_{i=2}^{n} \frac{n!}{(i-2)!(n-i+1)!}(1-x)^{i-2}x^{n-i+1}\sum_{j=i}^{n} p_j \\ &= n\sum_{i=2}^{n} \frac{(n-1)!}{(i-2)!(n-i+1)!}(1-x)^{i-2}x^{n-i+1}\sum_{j=i}^{n} p_j \\ &= n\sum_{i=1}^{n-1} \frac{(n-1)!}{(i-1)!(n-i)!}(1-x)^{i-1}x^{n-i}\sum_{j=i+1}^{n} p_j\end{aligned}$$

On the other hand,

$$\begin{aligned}(B) &= \sum_{i=2}^{n} \frac{n!}{(i-1)!(n-i+1)!}(n-i+1)(1-x)^{i-1}x^{n-i}\sum_{j=i}^{n} p_j \\ &= \sum_{i=2}^{n} \frac{n!}{(i-1)!(n-i)!}(1-x)^{i-1}x^{n-i}\sum_{j=i}^{n} p_j \\ &= n\sum_{i=2}^{n} \frac{(n-1)!}{(i-1)!(n-i)!}(1-x)^{i-1}x^{n-i}\sum_{j=i}^{n} p_j.\end{aligned}$$

Hence, we have

$$
\begin{aligned}
(A)-(B) &= nx^{n-1}\sum_{j=2}^{n} p_j - n(1-x)^{n-1}p_n + n\sum_{i=2}^{n-1}\frac{(n-1)!}{(i-1)!(n-i)!}(1-x)^{i-1}x^{n-i}\left(\sum_{j=i+1}^{n} p_j - \sum_{j=i}^{n} p_j\right) \\
&= nx^{n-1}\sum_{j=2}^{n} p_j - n(1-x)^{n-1}p_n - n\sum_{i=2}^{n-1}\frac{(n-1)!}{(i-1)!(n-i)!}(1-x)^{i-1}x^{n-i}p_i \\
&= nx^{n-1}\sum_{j=1}^{n} p_j - nx^{n-1}p_1 - n(1-x)^{n-1}p_n - n\sum_{i=2}^{n-1}\frac{(n-1)!}{(i-1)!(n-i)!}(1-x)^{i-1}x^{n-i}p_i \\
&= nx^{n-1}\sum_{i=1}^{n} p_i - n\sum_{i=1}^{n}\frac{(n-1)!}{(i-1)!(n-i)!}(1-x)^{i-1}x^{n-i}p_i
\end{aligned}
$$

Thus, we have

$$
(A)-(B) = nx^{n-1} - n\sum_{i=1}^{n} a_i(x)p_i,
$$

concluding that

$$
\frac{d\phi(x,\mathbf{p})}{dx} = -n\sum_{i=1}^{n} a_i(x)p_i = -nh(x,\mathbf{p}) \tag{4.6}
$$

$\square$

4.2 Optimization formulation

Using the property of the MNE such that any pure strategy in the support of the MNE should yield the same expected payoff given that the others are playing equilibrium strategies, we will now prove that the RS's problem can be written as an optimization problem over $\mathbf{p}$. Overall, we first prove that any optimal policy requires $p_n = 0$, and then transform the RS's objective purely into a function over $\mathbf{p}$ without introducing the equilibrium strategies $\boldsymbol{\mu}$ using algebraic manipulations. In particular, surprisingly, the user welfare $\mathcal{W}(\boldsymbol{\mu}, \mathbf{p})$ and the platform quality $\mathcal{Q}(\boldsymbol{\mu}, \mathcal{P})$ exhibits a significantly close structure up to the exponent.

Theorem 4.6. *The platform's problem can be written as the following optimization problem:*

$$
\begin{aligned}
(\textsc{Opt}) \quad & \underset{\boldsymbol{p}}{\textit{maximize}} && \alpha n\int_0^1 h(x,\boldsymbol{p})^{1+1/\beta} + (1-\alpha)\int_0^1 h(x,\boldsymbol{p})^{1/\beta}dx \\
& \textit{subject to} && p_1 \geq p_2 \geq \ldots \geq p_n = 0 \\
& && \sum_{i=1}^{n} p_i = 1
\end{aligned}
$$

Proof. Let $\boldsymbol{\mu}$ be the symmetric MNE and F be corresponding c.d.f. of the induced game given a nontrivial policy $\mathbf{p}$. For $\boldsymbol{\mu}$ to be the equilibrium, the following condition must hold:

$$
u_0 + c(q) = \sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q)^{n-i}(1-F(q))^{i-1},
$$

for some $u_0 \geq 0$ and for any $q \in \text{supp}(F)$, by Proposition 3.3. Note that u_0 is the expected payoff at the equilibrium. In fact, once this condition holds, the resulting strategy profiles will be the symmetric equilibrium.

Let $X_{(i)}$ denotes the i-th largest order statistics among $(X_i)_{i\in[n]}$ where $X_i \sim \mu$ for $i \in [n]$. Let $F_{(i)}$ be the c.d.f. of $X_{(i)}$. Using the derivation presented in Section 2.1, the RS's problem to maximize the objective function can be written as the following optimization problem:

$$\begin{aligned}
(\textsc{Opt}_1) \quad \underset{\mathbf{p}\in\Delta_m}{\text{maximize}} \quad & \alpha\sum_{i=1}^{n} p_i \mathbb{E}\left[\, X_{(i)} \,\right] + (1-\alpha)\mathbb{E}\left[\, X_i \,\right] \\
\text{subject to} \quad & u_0 + c(q) = \sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q)^{n-i}(1-F(q))^{i-1} \qquad \forall q \in \text{supp}(F) \\
& p_1 \geq p_2 \geq \ldots \geq p_n,\ \exists i,j : p_i \neq p_j \\
& \sum_{i=1}^{n} p_i = 1
\end{aligned}$$

Let l be the minimum value in $\text{supp}(F)$. Note that $F(l) = 0$. By plugging l in the constraint, we obtain

$$u_0 + c(l) = p_n.$$

Thus, we have $u_0 = p_n - c(l)$. Plugging $c(q) = q^\beta$ back in the constraint, we obtain

$$p_n - c(l) + q^\beta = \sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q)^{n-i}(1-F(q))^{i-1},$$

for every $q \in \text{supp}(F)$. Thus, we have

$$q = \left(\sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q)^{n-i}(1-F(q))^{i-1} - p_n + c(l) \right)^{1/\beta}. \tag{4.7}$$

Define

$$g(F(q), \mathbf{p}) = \sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q)^{n-i}(1-F(q))^{i-1} - p_n + c(l).$$

Then we have $q = g(F(q), \mathbf{p})^{1/\beta}$. By Lemma 4.1, we have

$$\begin{aligned}
\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) &= \sum_{i=1}^{n} p_i \mathbb{E}\left[\, X_{(i)} \,\right] \\
&= \sum_{i=1}^{n} \int_{q\in\text{supp}(F)} (1 - F_{(i)}(x)) dq.
\end{aligned}$$

Let $f(q)$ be the derivative of $F(q)$ with respect to q.[24] [25]

[24] Since F is monotone and atomless, it is differentiable almost every by Lebesgue's differention theorem, implying that the derivative of F exists almost everywhere. Thus, since measure zero set will has zero effect in the integral throughout the analysis as the objective function is always finite, it is without loss of generality to assume the differentiability of F. Alternatively speaking, one can consider integrals and corresponding constraints over the set of differentiable points, which would exactly coincide with the original objective function.

[25] Remark that since F is not guaranteed to be absolute continuous, f does not necessarily coincide with the density function. However, our analysis does not require f to be the density function.

Thus using Lemma 4.2, we can rewrite the user welfare function as:[26]

$$\begin{aligned}
\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) &= \int_{q \in \text{supp}(F)} \sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i} \binom{n}{j-1} (1-F(q))^{j-1} F(q)^{n-j+1} \right) p_i dq \\
&= \int_{q \in \text{supp}(F)} \sum_{i=1}^{n} \left(1 - \sum_{j=1}^{i} \binom{n}{j-1} (1-F(q))^{j-1} F(q)^{n-j+1} \right) p_i \frac{dq}{dF(q)} dF(q) \\
&= \int_{q \in \text{supp}(F)} \phi(F(q), \mathbf{p}) \frac{1}{f(q)} dF(q).
\end{aligned}$$

From the constraints on the condition for being equilibrium, by differentiating over q, we have

$$\beta q^{\beta-1} = f(q)\psi(F(q), \mathbf{p}), \tag{4.8}$$

since $\psi(x, \mathbf{p})$ is a derivative of $\left(\sum_{i=1}^{n} p_i \binom{n-1}{i-1} x^{n-i}(1-x)^{i-1}\right)$ with respect to x due to Lemma 4.3.

Finally, plugging $f(q)$ yields

$$\begin{aligned}
\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) &= \int_{q \in \text{supp}(F)} \phi(F(q), \mathbf{p}) \psi(F(q), \mathbf{p}) \frac{1}{\beta q^{\beta-1}} dF(q) \\
&= \frac{1}{\beta} \int_{q \in \text{supp}(F)} \phi(F(q), \mathbf{p}) \psi(F(q), \mathbf{p}) q^{-\beta+1} dF(q) \\
&= \frac{1}{\beta} \int_{q \in \text{supp}(F)} \phi(F(q), \mathbf{p}) \psi(F(q), \mathbf{p}) g(F(q), \mathbf{p})^{-1+1/\beta} dF(q) \\
&= \frac{1}{\beta} \int_{0}^{1} \phi(x, \mathbf{p}) \psi(x, \mathbf{p}) g(x, \mathbf{p})^{1/\beta-1} dx
\end{aligned}$$

Thus, this is entirely a function of $\mathbf{p}$ whose integrand does not explicitly depend on the cumulative distribution function $F(q)$.

On the other hand, for the quality, we have

$$\begin{aligned}
\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p}) &= \mathbb{E}\left[X_i \right] \\
&= \int_{q \in \text{supp}(F)} q dF(q) \\
&= \int_{q \in \text{supp}(F)} \left(\sum_{i=1}^{n} p_i \binom{n-1}{i-1} F(q)^{n-i} (1-F(q))^{i-1} - p_n + c(l) \right)^{1/\beta} dF(q) \\
&= \int_{0}^{1} g(x, \mathbf{p})^{1/\beta} dx.
\end{aligned}$$

Thus, since the coefficient of p_n is negative, decreasing p_n only increases $\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p})$.

Notice that if $p_n > 0$, then $c(l) < p_n$. In $\phi(x, \mathbf{p})$, we know that the coefficients are ordered, *i.e.*, for $i < k$:

$$1 - \sum_{j=1}^{i} \binom{n}{j-1} (1-x)^{j-1} x^{n-j+1} \geq 1 - \sum_{j=1}^{k} \binom{n}{j-1} (1-x)^{j-1} x^{n-j+1}.$$

[26] Note that $f(q) \neq 0$ with probability 1 by Proposition 3.3.

Hence, we can consider an operation such that given $\mathbf{p}$ with $p_n \neq 0$, transform this into $p_n = 0$ and distribute $p_n/(n-1)$ to each of $p_1, \dots, p_{n-1}$. By doing such operation, $\phi(x, \mathbf{p})$ only increases due to the sorted nature of coefficients in $\phi(x, \mathbf{p})$. Further, in $g(x, \mathbf{p})$, the coefficient of p_n is negative, yielding that decreasing p_n always increase $g(x, \mathbf{p})$.[27] Finally, recall the definition of $\psi(x, \mathbf{p})$, its coefficient on p_n is negative, so decreasing p_n increases $\psi(x, \mathbf{p})$. This concludes that decreasing p_n only increases $\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p})$, and thus the optimal point always have $p_n = 0$.[28]

Thus, in the optimal policy, we have $c(l) = u_0 = 0$, and further $g(x, \mathbf{p}) = h(x, \mathbf{p})$, and thus the user welfare simplifies to

$$\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) = \frac{1}{\beta} \int_0^1 \phi(x, \mathbf{p}) \psi(x, \mathbf{p}) h(x, \mathbf{p})^{1/\beta - 1} dx.$$

Note, however, that

$$\begin{aligned} \frac{dh(x, \mathbf{p})^{1/\beta}}{dx} &= \frac{dh(x, \mathbf{p})^{1/\beta}}{dh(x, \mathbf{p})} \frac{dh(x, \mathbf{p})}{dx} \\ &= \frac{1}{\beta} h(x, \mathbf{p})^{1/\beta - 1} \psi(x, \mathbf{p}). \end{aligned}$$

Hence, integrating by parts, user welfare can be written as:

$$\beta \mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) = \int_0^1 \phi(x, \mathbf{p}) \cdot \int_0^x h(t, \mathbf{p})^{1/\beta - 1} \cdot \psi(t, \mathbf{p}) dt \Big|_0^1 - \int_0^1 \frac{d\phi(x, \mathbf{p})}{dx} \left(\int_0^x h(t, \mathbf{p})^{1/\beta - 1} \cdot \psi(t, \mathbf{p}) dt \right) dx.$$

Since we have

$$\begin{aligned} h(0, \mathbf{p}) &= p_n = 0 \\ h(1, \mathbf{p}) &= p_1 \\ \phi(1, \mathbf{p}) &= 0 \\ \phi(0, \mathbf{p}) &= 1 \end{aligned}$$

we obtain

$$\begin{aligned} \beta \mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) &= \left(\beta \phi(x, \mathbf{p}) h(x, \mathbf{p})^{1/\beta} \Big|_0^1 - \beta \int_0^1 \frac{d\phi(x, \mathbf{p})}{dx} h(x, \mathbf{p})^{1/\beta} dx \right) \\ &= \beta \int_0^1 \frac{-d\phi(x, \mathbf{p})}{dx} h(x, \mathbf{p})^{1/\beta} dx. \end{aligned}$$

Finally by Lemma 4.5, we obtain

$$\mathcal{W}(\boldsymbol{\mu}, \mathbf{p}) = n \int_0^1 h(x, \mathbf{p})^{1 + 1/\beta} dx.$$

Recalling that

$$\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p}) = \int_0^1 h(x, \mathbf{p})^{1/\beta} dx,$$

we finish the proof. □

[27] Note that $c(l)$ would change by such operation, but since $p_n > c(l)$ if $p_n > 0$ and $p_n = c(l) = 0$ after the operation, it still increases.

[28] By doing such operation, the corresponding c.d.f. F would change, but as we write the objective function purely as a function over $\mathbf{p}$ while replacing integral of $F(q)$ over the support of F into that of x over $[0, 1]$, we do not need to consider the change of F.

As a side product, notice that one can directly reobtain Proposition 2.3 without characterizing the equilibrium.

5 Optimal Structure for Single-minded RS

Thanks to Theorem 4.6, the platform's objective boils down to the optimization problem Opt. This involves integrals of $h(x,\mathbf{p}) = \sum_{i=1}^{n} a_i(x)p_i$, which is not straightforward to analyze. To reveal the optimal structure, we first characterize several functional properties of $h(x,\mathbf{p})^r$ with respect to $r \in \mathbb{R}$. Eventually, we will prove that a symmetric version of the function $h(x,\mathbf{p})^r$ is Schur-convex or Schur-concave with respect to the parameter r.

To this end, we first rewrite $h(x,\mathbf{p})$ as a function over the general probability n-dimensional simplex. Recall that $p_n = 0$. Let Δ_n be a n-dimensional probability simplex. Note that this is different from Δ as it may violate the ordering condition $p_1 \geq p_2 \geq \ldots \geq p_{n-1} \geq p_n$. Let $O(\mathbf{p}) : [0,1] \times \Delta_n \to \mathbb{R}_{\geq 0}$ be a function that maps a probability vector in n-dimensional probability simplex to a nonnegative real number, defined as

$$O(x,\mathbf{p}) = \sum_{i=1}^{n} a_i(x)p_{(i)},$$

where $p_{(i)}$ is the i-th highest element in $(p_1,\ldots,p_n)$. Thus, $O(x,\mathbf{p})$ is a *symmetric extension* of $h(x,\mathbf{p})$ to have its domain to be the entire $n-1$-dimensional probability simplex.

To analyze the optimal policy, we first introduce the formal notion of *symmetric* function.

Definition 5.1 (Symmetric function)**.** Given a set X, a function $f : X^n \to \mathbb{R}$ is symmetric if for any permutation $\sigma : [n] \to [n]$ and any $x \in X^n$ we have

$$f(x_1,\ldots,x_n) = f(x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

As an acute reader might have noticed that, due to the construction of our $O(x,\mathbf{p})$, it is apparently a symmetric function over $\mathbf{p}$ given any $x \in [0,1]$. Thus, we will eventually prove that $O(x,\mathbf{p})$ is Schur-convex or Schur-concave depending on the parameter r, all of which will be defined formally in what follows. We first introduce the notion of *majorization*.

Definition 5.2 (Majorization)**.** Let $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ be two n-dimensional real vectors. We say x majorizes y ($x \succ y$) if the following holds:

$$\begin{gathered} x_1 \geq y_1 \\ x_1 + x_2 \geq y_1 + y_2 \\ \vdots \\ x_1 + \ldots + x_{n-1} \geq y_1 + \ldots + y_{n-1} \\ x_1 + \ldots + x_n = y_1 + \ldots + y_n \end{gathered}$$

The following notion of *Schur-convexity* (concavity) is an analogous notion of convexity (concavity) on the partially ordered set with respect to majorization.

Definition 5.3 (Schur-convexity)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is Schur-convex (-concave) if for any two $x, y \in \mathbb{R}^n$ with $x \succ y$, $f(x) \geq f(y)$ ($f(x) \leq f(y)$).

A convenient tool in proving Schur-convexity (concavity) is to directly prove its convexity (concavity) and symmetry, proven by Ando (1989) (see Chapter 86 in Peajcariaac and Tong (1992) for more details):

Theorem 5.4 (Ando (1989))**.** *A convex (concave) function $f : \mathbb{R}^n \to \mathbb{R}$ is Schur-convex (concave) if and only if it is symmetric.*

Now, our main result on $O(\mathbf{p})$ can be formalized as follows:

Theorem 5.5. *For any $\boldsymbol{p} \in \Delta_n$, the function $\int_0^1 O(x, \boldsymbol{p})^r dx$ is Schur-concave over $\boldsymbol{p}$ if $r \in [0, 1]$ and Schur-convex over $\boldsymbol{p}$ if $r \geq 1$ or $r \leq 0$.*[29]

Proof of Theorem 5.5. We will in fact prove a stronger statement such that for any $x \in [0, 1]$, the integrand $O(x, \mathbf{p})^r$ is either Schur-convex or concave with respect to r. Let σ be the permutation associated with the ordering of $\mathbf{p}$ such that $p_{\sigma(1)} \geq p_{\sigma(2)} \geq \dots p_{\sigma(n)}$. Note that

$$\frac{\partial O(x, \mathbf{p})^r}{\partial p_i} = rO(x, \mathbf{p})^{r-1} a_{\sigma^{-1}(i)}(x),$$

and thus the second derivative can be written as:

$$\frac{\partial^2 O(x, \mathbf{p})^r}{\partial p_i \partial p_j} = r(r-1)O(x, \mathbf{p})^{r-2} a_{\sigma^{-1}(i)}(x) a_{\sigma^{-1}(j)}(x).$$

Thus, the Hessian $H(x, \mathbf{p})$ of $O(x, \mathbf{p})^r$ with respect to $\mathbf{p}$ can be written as

$$H_{ij}(\mathbf{p}) = rO(x, \mathbf{p})^{r-2} a_{\sigma^{-1}(i)}(x) a_{\sigma^{-1}(j)}(x).$$

Now consider any nonzero n-dimensional real-vector $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$, and consider the following quadratic form:

$$\begin{aligned} v^\top H(\mathbf{p}) v &= r(r-1) \sum_{i,j \in [n]} v_i v_j a_{\sigma^{-1}(i)}(x) a_{\sigma^{-1}(j)}(x) O(x, \mathbf{p})^{r-2} \\ &= r(r-1) O(x, \mathbf{p})^{r-2} \left(\sum_{i \in [n]} v_i a_{\sigma^{-1}(i)}(x) \right)^2. \end{aligned}$$

Thus, if $r \in (0, 1)$, we have $v^\top H v \leq 0$ for any $v \in \mathbb{R}^n$, implying the concavity of $O(x, \mathbf{p})^r$. Otherwise, we have $v^\top H v \geq 0$ for any $v \in \mathbb{R}^n$, implying the convexity. Finally, as the pointwise integral over x preserves the convexity or concavity, we finish the proof.

□

Now we spell out an immediate implication of the Schur-convexity (concavity) of $O(\mathbf{p})^r$ in boundary regimes where $\alpha \in \{0, 1\}$.

Theorem 5.6. *Depending on the weight parameter α and the cost parameter β, the following is true:*

1. *If $\alpha = 1$ or $\beta \leq 1$, then* HardMax *is optimal.*

[29] An alternative standard way to prove a Schur-convexity (concavity) is to use the following Schur-Ostrowski criterion. In Appendix B, we provide another proof using Schur-Ostrowski criterion.

2. *If $\alpha = 0$ and $\beta < 1$, then* HardMax *is the optimal policy.*

3. *If $\alpha = 0$ and $\beta > 1$, then* UniformButLast *is the optimal policy.*

4. *If $\alpha = 0$ and $\beta = 1$, then any policy $\boldsymbol{p} \in \Delta$ such that $p_n = 0$ is optimal.*

Proof. Note first that the Opt in Theorem 4.6 enforces $p_n = 0$. To prove the first condition, if $\alpha = 1$, then the objective function becomes $\mathcal{W}(\boldsymbol{\mu}, \mathbf{p})$, and this is always Schur-convex function for any $\beta > 0$. Then, HardMax majorizes every other $\mathbf{p} \in \Delta_n$, and thus HardMax is optimal. Likewise, if $\beta \leq 1$, the objective function becomes a summation over two Schur-convex functions, and thus HardMax is optimal.

For the remaining conditions, notice that the objective function becomes $\mathcal{Q}(\boldsymbol{\mu}, \mathbf{p})$ if $\alpha = 0$. If $\beta > 1$, then this is Schur-convex function, implying the optimality of HardMax. If $\beta < 1$, then this is Schur-concave function. In this case, UniformButLast is majorized by every policy $\mathbf{p} \in \Delta_n$ such that $p_n = 0$, and thus UniformButLast is optimal.

Finally, if $\alpha = 0$ and $\beta = 1$, then the objective function simply reduces to (up to constant)

$$\int_0^1 h(x, \mathbf{p}) dx = \int_0^1 a_i(x) p_i dx.$$

Note that

$$\int_0^1 a_i(x) dx = 1/(n-1),$$

by Lemma 4.4, the objective function is always a constant. Since the sufficient condition to write the optimization problem is to have reasonable policy such that $p_i \neq p_j$ for some $i, j \in [n]$, we finish the proof. □

A direct corollary of this theorem is the optimality of UniformButLast for contest design if $\beta = 1$ and $\alpha = 0$ by Glazer and Hassin (1988).

6 Optimal Structure for Convex-minded RS

In this section, we characterize the optimal structure in the convex-minded setting where $\alpha \in (0, 1)$. Let $G(\mathbf{p})$ be the objective function given α in Opt in Theorem 4.6. We will treat the domain of the objective function to be ordered family Δ as originally defined in Section 2 in case a reader might be confused with the symmetric extension $O(x, \mathbf{p})$. Henceforth, we will focus on the case where $n \geq 3$ since if $n = 2$, we always have $p_n = 0$ for the optimal policy from the proof of Theorem 4.6, implying that the optimal policy for $n = 2$ is always HardMax.

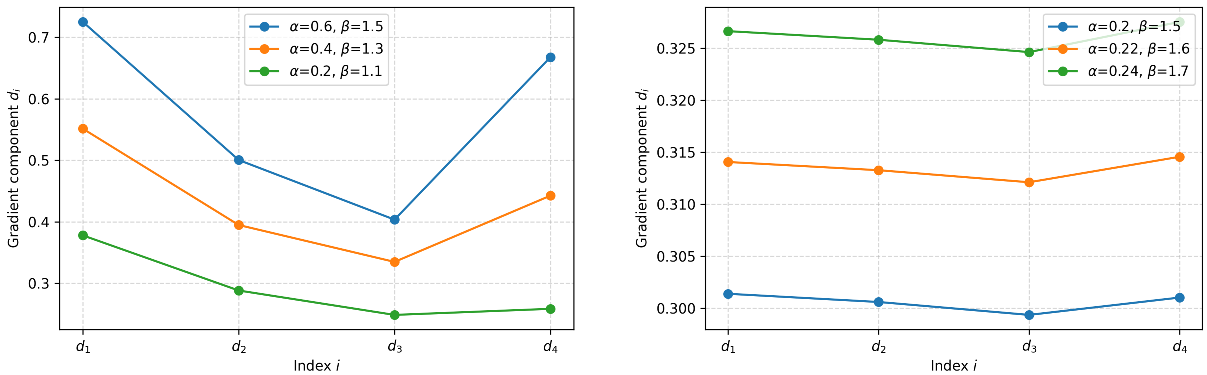
Overall, we will prove the following theorem.

Theorem 6.1. *For any $\alpha \in [0, 1]$, any optimal policy has a structure $p_1 \geq p_2 = p_3 = \ldots = p_{n-1} \geq p_n = 0$.*

To this end, we will analyze some geometric properties of the partial derivatives $\partial G(\mathbf{p})/\partial p_i$ for $i \in [n-1]$, and use KKT conditions to characterize the optimal structure. Define

$$d_i := \frac{\partial G(\mathbf{p})}{\partial p_i} = n\alpha(1 + 1/\beta) \int_0^1 h(x, \mathbf{p})^{1/\beta} a_i(x) dx + (1-\alpha)/\beta \int_0^1 h(x, \mathbf{p})^{1/\beta - 1} a_i(x) dx.$$

for $i \in [n-1]$. Looking forward, we will prove that the sequences $(d_i)_{i \in [n-1]}$ behaves in a quasiconvex manner in a strong sense, where the quasiconvexity for a sequence is defined as follows:



(a) d_i given HardMax policy

(b) d_i given UniformButLast policy

Figure 2: Behavior of d_i's given $n = 5$. Note that $p_5 = 0$, and thus d_5 is omitted. Note that such the quasiconvexity of d_i's follows from the quasiconvexity of $q(x)$ in Figure 3, as shown in Section 6.

Definition 6.2. A sequence $x_1, \ldots, x_n$ is quasiconvex if there exists $k \in [n]$ such that $x_1 \geq x_2 \geq \ldots \geq x_k \leq x_{k+1} \leq \ldots \leq x_n$. Similarly, a real-valued function over $X \subseteq \mathbb{R}$ is quasiconvex if there exists $x^* \in \mathbb{R}$ such that $f(x) \geq f(y)$ if $x < y \leq x^*$ and $f(x) \leq f(y)$ if $x^* \leq x < y$.

Then, our main lemma can be stated as follows:

Lemma 6.3. *The sequence $d_1, d_2, \ldots, d_{n-1}$ has the following properties:*

1. *It is quasiconvex.*
2. *Let k be the index that the slope changes,* i.e., $p_{k-1} \geq p_k \leq p_{k+1}$. *Then, $d_i = d_{i+1}$ only if $i = k-1$ or $i = k$ for $i \in [n-2]$.*
3. *If $d_k = d_{k+1}$ then $d_k \neq d_{k-1}$, and if $d_k = d_{k-1}$ then $d_k \neq d_{k+1}$.*

To prove the lemmas, we require several technical tools, namely the total positivity and variation diminishing property. The first is the notion of totally positive function.

Definition 6.4 (Total positivity)**.** Let $K(x, y)$ be a function over ordered sets X and Y. K is totally positive of order r (TP_r) if for all $1 \leq k \leq r$, and any ordered sequences $x_1 < \ldots < x_k$ and $y_1 < \ldots < y_k$ such that $x_i \in X$ and $y_j \in Y$ for $i, j \in [k]$, we have[30]

$$\det \begin{pmatrix} K(x_1, y_1) & \ldots & K(x_1, y_k) \\ \vdots & & \vdots \\ K(x_k, y_1) & \ldots & K(x_k, y_k) \end{pmatrix} \geq 0.$$

Further, K is said to be strictly totally positive of order r (STP_r) if the inequality is strict.

Now we connect the TP with a notion of variation diminishing property, which provides a connection between two sequences' (functions') geometric characteristics. We first present the formal definition of variation of a sequence as a number of sign changes.

[30] A typical choice of X or Y is an interval over the real line or a finite set of integers.

Definition 6.5 (Number of sign changes)**.** Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Define $S^-(x)$ the number of sign changes of sequences $x_1, x_2, \ldots, x_n$, ignoring zeros. Similarly, $S^+(x)$ denotes the maximal number of sign changes of sequences including zeroes, *i.e.,* zero terms being permitted to take on arbitrary signs. We similarly define S^- and S^+ for an infinite (possibly uncountable) sequence.

For instance, if $x = (1, -2, 3, 0, 4)$, we have $S^-(x) = 2$ as there are two strict sign changes excluding zeroes, and $S^+(x) = 4$ as if we assign $-$ to the zero, we have four changes of signs.

The following notion of variation diminishing property relates the number of sign changes in two functions.

Definition 6.6 (Variation diminishing property)**.** Let X and Y be ordered sets and $K(x, y) : X \times Y \to \mathbb{R}$ be a integrable real-valued function. Given an integrable function $q : X \to \mathbb{R}$, let

$$Q(y) = \int K(x, y)q(x)dx.$$

Then, we say $K(x, y)$ has variation diminishing (VD) property if

$$S^-(Q(y)) \leq S^-(q(x)),$$

and strong variation diminishing (SVD) property if

$$S^+(Q(y)) \leq S^-(q(x)).$$

The following seminal theorem in linear algebra and functional analysis states that totally positive matrix has a variation diminishing property.

Theorem 6.7 (Chapter 5, Theorem 3.1 in Karlin (1964))**.** *Consider $Q(y), q(x)$ and $K(x, y)$ defined in Definition 6.6. If $K(x, y)$ is TP_r, then it has VD,* i.e.,

$$S^-(Q(y)) \leq S^-(q(x)),$$

for any $q(x)$ with $S^-(q(x)) \leq r - 1$.[31] *Further, if both the LHS and RHS are exactly $k \leq r - 1$, then its sign pattern coincide. Under the stronger condition that $K(x, y)$ is STP_r, it has SVD,* i.e.,

$$S^+(Q(y)) \leq S^-(q(x)).$$

Recall that our primary objective is to prove the U-shapeness of the sequence $(d_i)_{i \in [m-1]}$. Intuitively, quasiconvexity in a single-dimension is closely related to the number of sign changes as it needs to change its sign at most two times up to proper scaling. The following lemma formalizes such intuitive connection between the quasiconvexity and the number of sign changes:

Lemma 6.8 (Karp et al. (2024))**.** *Given a function $g : I \to \mathbb{R}$, g is quasiconvex if and only if for every $\lambda \in \mathbb{R}$, the function $x \to f_\lambda(x) = g(x) - \lambda$ has no more than 2 sign changes on I excluding zeroes, and its sign pattern remains $(+, -, +)$ for a subset of λ where it has exactly 2 sign changes.*

We also need the following technical lemma on the monotonicity of $h(x, \mathbf{p})$ with respect to x.

Lemma 6.9. *$h(x, \boldsymbol{p})$ is monotone increasing in x.*

Proof. Observe that $h(x, \mathbf{p}) = R(F^{-1}(q), \boldsymbol{\mu}_{-i})$ as presented in Claim 3.9. Then, the lemma immediately follows from Claim 3.9 as F^{-1} is monotone increasing in x. □

[31] Karlin (1964) actually prove a stronger version of theorem using the notion of sign regularity, which is a generalization of total positivity.

We finally need the following theorem on the total positivity of a generalized Vandermonde matrix.[32]

Theorem 6.10 (Generalized Vandermonde Matrix, Yang et al. (2001))**.** *Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be a real vector and $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ be a vector of nonnegative integers. If $a_n > a_{n-1} > \ldots > a_1$ and $x_n > x_{n-1} > \ldots x_1 > 0$, we have that*

$$\det \begin{pmatrix} x_1^{a_1} & \ldots & x_1^{a_n} \\ \vdots & & \vdots \\ x_n^{a_1} & \ldots & x_n^{a_n} \end{pmatrix} > 0.$$

Proof of Lemma 6.3. Having defined the technical tools, we now prove Lemma 6.3.

Proof of Lemma 6.3. We will first prove that $K(x, i) = a_i(1-x)$ is TP_{m-1}. That is, for any $k \in [n-1]$, it suffices to prove that

$$\det \begin{pmatrix} a_{i_1}(1-x_1) & \ldots & a_{i_1}(1-x_k) \\ \vdots & & \vdots \\ a_{i_k}(1-x_1) & \ldots & a_{i_k}(1-x_k) \end{pmatrix} > 0,$$

for all $1 \le i_1 < \ldots < i_k \le n-1$ and $0 \le x_1 < \ldots < x_k \le 1$. Denote the matrix inside the determinant by M for simplicity.

Consider $r, s \in [k]$. Note first that

$$a_{i_r}(1-x_s) = \binom{n-1}{i_r-1}(1-x_s)^{n-1}\left(\frac{x_s}{1-x_s}\right)^{i_r-1}.$$

Define $y_s = x_s/(1-x_s)$, then

$$a_{i_r}(1-x_s) = \binom{n-1}{i_r-1}(1-x_s)^{n-1} y_s^{i_r-1}.$$

Factoring out the row constants, we have

$$\det(M) = \left(\prod_{r=1}^{k} \binom{n-1}{i_r-1}\right) \det \begin{pmatrix} (1-x_1)^{n-1} y_1^{i_1-1} & \ldots & (1-x_k)^{n-1} y_k^{i_1-1} \\ \vdots & & \vdots \\ (1-x_1)^{n-1} y_1^{i_k-1} & \ldots & (1-x_k)^{n-1} y_k^{i_k-1} \end{pmatrix}$$

Factoring out the column constants, we have

$$\det(M) = \left(\prod_{r=1}^{k} \binom{n-1}{i_r-1}\right)\left(\prod_{s=1}^{k}(1-x_s)^{n-1}\right) \det \begin{pmatrix} y_1^{i_1-1} & \ldots & y_k^{i_1-1} \\ \vdots & & \vdots \\ y_1^{i_k-1} & \ldots & y_k^{i_k-1} \end{pmatrix}$$

By taking transpose in the remaining matrix and applying Theorem 6.10, we conclude that $\det(M) > 0$. Since this holds for arbitrary choice of $x_1, \ldots, x_k$ and $i_1, \ldots, i_k$, this concludes that $K(x, i)$ is STP_{n-1}.

[32] The standard Vandermonde matrix has $a_i = i-1$, and its determinant is shown to be $\prod_{0 \le i < j \le n-1}(x_j - x_i)$. See Golub and Van Loan (2013) for more details.

Now we write $q(x) = f(x) + g(x)$, where

$$f(x) = n\alpha(1 + 1/\beta)h(x, \mathbf{p})^{1/\beta}$$
$$g(x) = (1 - \alpha)/\beta h(x, \mathbf{p})^{1/\beta - 1}.$$

Then we have

$$\begin{aligned} d_i &= \int_0^1 q(x)a_i(x)dx \\ &= \int_0^1 q(1-y)a_i(1-y)dy. \end{aligned}$$

Now we will prove that $q(x)$ is quasiconvex over x, which implies that $q(1-y)$ is also quasiconvex on y. Recall that $\psi(x, \mathbf{p})dh(x, \mathbf{p})/dx \geq 0$ from Lemma 6.9. Differentiating over x, we obtain

$$\begin{aligned} \frac{dq(x)}{dx} &= n\alpha(1 + 1/\beta)(1/\beta)h(x, \mathbf{p})^{1/\beta-1}\psi(x, \mathbf{p}) + (1 - \alpha)(\frac{1}{\beta} - 1)\frac{1}{\beta}h(x, \mathbf{p})^{1/\beta-2}l(x) \\ &= \psi(x, \mathbf{p})h(x, \mathbf{p})^{1/\beta-2}(w_1 h(x, \mathbf{p}) + w_2), \end{aligned}$$

where we write $w_1 = n\alpha(1 + 1/\beta)(1/\beta)$ and $w_2 = (1 - \alpha)(\frac{1}{\beta} - 1)\frac{1}{\beta}$.

Note that $w_1 \geq 0$ and $w_2 \leq 0$. Thus $dq(x)/dx$ is nonnegative if and only if $w_1 h(x, \mathbf{p}) + w_2 \geq 0$. Since $h(x, \mathbf{p})$ is nonnegative monotone increasing function over x, this concludes that $q(x)$ is either nonnegative or decreasing then increasing, *i.e.,* quasiconvex. This immediately implies that $q(1-y)$ is also quasiconvex.

Thus, by Lemma 6.8, $q(1 - y) - \lambda$ has at most two sign changes for every $\lambda \in \mathbb{R}$, and whenever it has exactly two sign changes, it must be $(+, -, +)$.

Recalling the definition of d_i,

$$\begin{aligned} d_i &= \int_0^1 q(1-y)a_i(1-y)dy \\ &= \int_0^1 K(y, i)q(1-y)dy. \end{aligned}$$

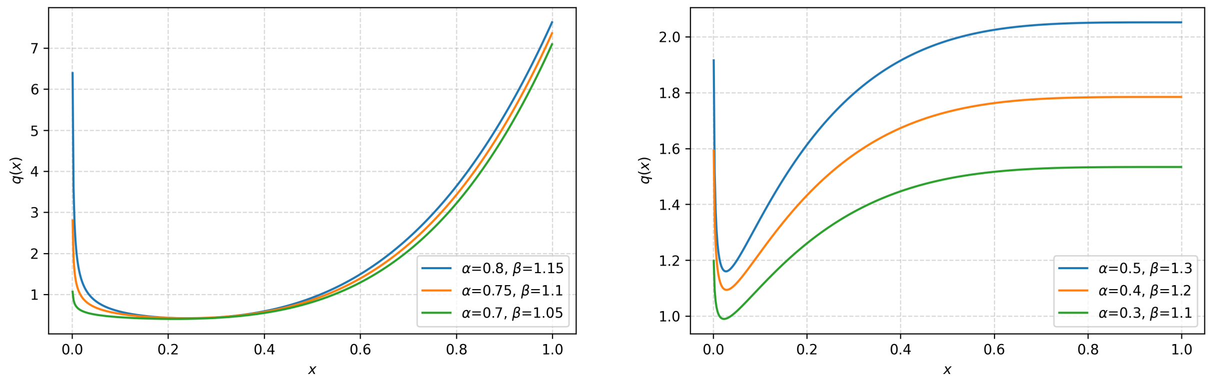
Subtracting $\lambda' \in \mathbb{R}$:

$$\begin{aligned} d_i - \lambda' &= -\lambda' + \int_0^1 K(y, i)q(1-y)dy \\ &= \int_0^1 K(y, i)(q(1-y) - (n-1)\lambda')dy, \end{aligned}$$

where the second equality follows from Lemma 4.4.

Taking $\lambda = (n-1)\lambda'$, we know that $q(1-y)-\lambda$ has at most two sign changes, and the pattern is exactly $(+, -, +)$ if it has exactly two sign changes. Further, since we know that $K(y, i)$ is STP_{n-1}, we can apply Theorem 6.7 as $S^-(q(1-y)) \leq 2$ if $n \geq 3$ and conclude that $S^+(d_i) \leq 2$. This concludes that $d_i - \lambda'$ always have at most two sign changes assigning zeroes arbitrary signs, and the pattern is $(+, -, +)$ if it has exactly two sign changes.

Let k be the indices where the sign changes, *i.e.,* $d_1 \geq d_2 \geq \ldots \geq d_k \leq d_{k+1} \leq \ldots \leq d_{n-1}$. If there exists $i \leq k-2$ or $i \geq k+1$ such that $d_i = d_{i+1}$, then by taking $\lambda' = d_i$, it follows that $S^+(d_i) \geq 4$ by assigning zeros appropriately, which is a contradiction. Thus, $d_i = d_{i+1}$ can only



(a) $q(x)$ given HardMax policy (b) $q(x)$ given UniformButLast policy

Figure 3: Behavior of $q(x)$ given $n = 5$. The quasiconvexity of $q(x)$ transfers to the quasiconvexity of d_i in Figure 2, thanks to the variation diminishing property that arises from the total positivity of the matrix consists of $a_i(x)$.

happen if $i = k - 1$ or $i = k + 1$. If $d_{k-1} = d_k = d_{k+1}$, then we can set $\lambda' = d_k$ and take the sign of the zeroes at $d_{k-1} - \lambda' = d_k - \lambda' = d_{k+1} - \lambda'$ as $(-, +, -)$. Then, the eventual pattern of sign changes is $(+, -, +, -, +)$, which is a contradiction. Thus, there can be at most two consecutive elements in either of $(k - 1, k)$ or $(k, k + 1)$, and otherwise it is strictly decreasing or increasing. This finishes the proof.

□

Proof of Theorem 6.1. We finally provide the proof of main theorem.

Proof of Theorem 6.1. For a policy $\mathbf{p}$ to be optimal, the Karush-Kuhn-Tucker (KKT) condition should hold as the constraints are linear. We will ignore p_n as it is always zero in the optimum. First, the Lagrangian can be written as:

$$L(\mathbf{p}, \nu, \lambda) = G(\mathbf{p}) + \lambda(1 - \sum_{i=1}^{n-1} p_i) + \sum_{i=1}^{n-2} \nu_i (p_i - p_{i+1}) + \nu_{n-1} p_{n-1},$$

where $\nu_i \geq 0$ for $i \in [n-1]$ and $\lambda \in \mathbb{R}$. Writing down KKT condition, we obtain

$$\begin{aligned} d_1 &= \frac{\partial G(\mathbf{p})}{\partial p_1} = \lambda - \nu_1 \\ d_i &= \frac{\partial G(\mathbf{p})}{\partial p_i} = \lambda + \nu_{i-1} - \nu_i, \ i = 2, 3, \ldots, n-1 \\ \nu_i(p_i - p_{i+1}) &= 0, \ i = 1, 2, \ldots, n-2 \\ \nu_{n-1} p_{n-1} &= 0 \\ \lambda(p_1 + \ldots + p_n - 1) &= 0 \\ p_1 \geq p_2 \geq \ldots \geq p_{n-1} &\geq 0 \\ \nu_i &\geq 0, \ i = 1, 2, \ldots, n-1 \end{aligned}$$

According to Lemma 6.3, the sequence $(d_1, \ldots, d_{m-1})$ can only have either of the following two structure for some $k \in [n-1]$:

1. $d_1 > d_2 > \ldots > d_k < d_{k+1} < \ldots < d_{n-1}$
2. $d_1 > d_2 > \ldots > d_k = d_{k+1} < \ldots < d_{n-1}$
3. $d_1 > d_2 > \ldots > d_{n-1}$
4. $d_1 > d_2 > \ldots > d_{n-2} = d_{n-1}$

Let k be the slope-changing index in the former two cases.

(Case 1: $p_1 \neq p_2$) Assume first that $p_1 \neq p_2$. Then, we have $\nu_1 = 0$. Thus, we have

$$\begin{aligned} d_1 &= \lambda \\ d_2 &= \lambda - \nu_2. \end{aligned}$$

Suppose further that $p_2 \neq p_3$. Then, we have $\nu_2 = 0$, implying

$$\begin{aligned} d_1 &= \lambda \\ d_2 &= \lambda \\ d_3 &= \lambda - \nu_3. \end{aligned}$$

Thus, it should be the case that the slope-changing index $k = 1$. Note, however, that then we should have $d_3 > 0$, which is a contradiction. Thus, if $p_1 \neq p_2$ then $p_2 = p_3$.

Now, suppose that

$$d_1 \neq d_2 = d_3 = \ldots = d_i \neq d_{i+1},$$

for some $i \geq 3$. Then $\nu_i = 0$ and we have a series of equations:

$$\begin{aligned} d_1 &= \lambda \\ d_2 &= \lambda - \nu_2 \\ d_3 &= \lambda + \nu_2 - \nu_3 \\ &\vdots \\ d_i &= \lambda + \nu_{i-1} \\ d_{i+1} &= \lambda - \nu_{i+1}. \end{aligned}$$

If $\nu_{i+1} \neq 0$ and $\nu_{i-1} \neq 0$, then it is a contradiction as increases then decreases. Thus, either of ν_{i-1} or ν_{i+1} is zero.

If $\nu_{i+1} = 0$, then we should again have $\nu_{i-1} = 0$ as otherwise it increases from λ then decreases. Then, notice that

$$\begin{aligned} d_1 &= \lambda \\ d_2 &= \lambda - \nu_2 \\ d_3 &= \lambda + \nu_2 - \nu_3 \\ &\vdots \\ d_{i-1} &= \lambda + \nu_{i-2} - \nu_{i-1} \\ d_i &= \lambda \\ d_{i+1} &= \lambda, \end{aligned}$$

which is a contradiction as i should be the slope-changing index and thus $\lambda = d_1 > d_2 > \ldots > d_i = d_{i+1} = \lambda$.

Therefore, we have $\nu_{i-1} = 0$. In this case, again, we have

$$\begin{aligned} d_1 &= \lambda \\ d_2 &= \lambda - \nu_2 \\ d_3 &= \lambda + \nu_2 - \nu_3 \\ &\vdots \\ d_i &= \lambda \\ d_{i+1} &= \lambda - \nu_{i+1}, \end{aligned}$$

and thus it should be the case that $\lambda = d_1 > d_2 > \ldots \geq d_k < \ldots < d_i = \lambda < d_{i+1}$ since slope-changing needs to occur strictly before the index i, which is a contradiction as $\lambda = d_i > d_{i+1} = \lambda - \nu_{i+1}$. Thus, if $p_1 \neq p_2$, then it should be the case that $p_1 \neq p_2 = \ldots = p_{n-1} \geq p_n = 0$.

(Case 2: $p_1 = p_2$) Suppose otherwise that $p_1 = p_2$, and let $i \geq 2$ be the first index such that $p_i \neq p_{i+1}$, *i.e.*, $\nu_i = 0$. If $i \geq 3$, we have

$$\begin{aligned} d_1 &= \lambda - \nu_1 \\ d_2 &= \lambda + \nu_1 - \nu_2 \\ d_3 &= \lambda + \nu_2 - \nu_3 \\ &\vdots \\ d_{i-1} &= \lambda + \nu_{i-2} - \nu_{i-1} \\ d_i &= \lambda + \nu_{i-1} \\ d_{i+1} &= \lambda - \nu_{i+1}. \end{aligned}$$

For these equations to be held under the condition on d_i's, it should be the case that either of ν_{i-1} and ν_{i+1} is zero as otherwise it increases from d_1 then decreases to d_{i+1}. If $\nu_{i-1} = 0$, we have $d_{i-1} = \lambda + \nu_{i-2}$, which is again a contradiction. If $\nu_{i+1} = 0$, then we should have $d_i = d_{i+1}$ as otherwise it increases from d_1 then decreases to λ. Thus, $d_i = d_{i+1} = \lambda$. Again, in this case, we have $d_{i-1} = \lambda + \nu_{i-2}$, which is a contradiction.

Thus, the only remaining case is $p_1 = p_2 > p_3$. In this case, $\nu_2 = 0$ and we have

$$\begin{aligned} d_1 &= \lambda - \nu_1 \\ d_2 &= \lambda + \nu_1 \\ d_3 &= \lambda - \nu_3 \\ d_4 &= \lambda + \nu_3 - \nu_4 \end{aligned}$$

Thus, it should be the case that $\nu_1 = 0$ as otherwise $d_1 < d_2 > d_3$. Then, the resulting equations are:

$$\begin{aligned} d_1 &= \lambda \\ d_2 &= \lambda \\ d_3 &= \lambda - \nu_3 \\ d_4 &= \lambda + \nu_3 - \nu_4, \end{aligned}$$

and thus the slope-changing index $k = 1$. Thus, after $k + 1 = 2$, it should always strictly increase, which is a contradiction to $d_3 = \lambda - \nu_3$. Therefore, if $p_1 = p_2$, we always have $p_1 = p_2 = \ldots = p_{n-1}$.

Combining, the only possible cases are:

$$p_1 \neq p_2 = \ldots = p_{n-1} \geq p_n = 0$$
$$p_1 = p_2 = \ldots = p_{n-1} \geq p_n = 0,$$

and it finishes the proof.

□

7 Conclusion

We study a strategic competition between content producers in online platforms, and characterize the optimal rank-based recommendation policy to optimally incentivize the strategic content production of producers. We observe a stark phase transition between HardMax that always recommend the highest quality content and UniformButLast that uniformly randomizes among every content but the lowest quality one when the platform is single-minded to either optimize user welfare or platform quality. In the convex-minded case when the platform tries to maximize the convex combination over user welfare and platform quality, we reveal that the optimal policy is still highly structured: potentially high probability to the highest quality content, zero probability to the lowest quality content, and equal probabilities to the intermediate quality contents.

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A Preliminary on Topology for Section 3

A topology on X is a collection (topology) $\mathcal{T}$ of subsets of X, called open sets satisfying the following:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. $\mathcal{T}$ is closed under any (possibly infinite) union of members of $\mathcal{T}$.
3. $\mathcal{T}$ is closed under any finite intersection of members of $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*, where we often write X as a topological space if underlying topology $\mathcal{T}$ is clear from the context. A subset $C \subset X$ is closed in $\mathcal{T}$ if $X \setminus C \in \mathcal{T}$. Given a topological space X and $p \in X$, a neighbourhood of p is a subset $V \subseteq X$ that includes an open set $U \in \mathcal{T}$ containing p, *i.e.,* $p \in U \subseteq V \subseteq X$.

We say two points $x, y \in X$ can be separated by neighbourhoods if there exists a neighbourhood U of x and V of y such that $U \cap V = \emptyset$. X is *Hausdorff space* if any two distinct points $x, y \in X$ are separated by neighbourhoods. A function $f : X \to Y$ between topological spaces is called *continuous* if for every $x \in X$ and every neighbourhood N of $f(x)$, there is a neighbourhood M of x such that $f(M) \subseteq N$. In our problem setup, the action space X is an Euclidean metric space, which is a well-known to be a Hausdorff space.

A basis $\mathcal{B}$ of a topological space X is a collection of subsets of X such that

1. (Covering) Every element of X is contained in at least one basis element, *i.e.,* $\cup_{B \in \mathcal{B}} B = X$.
2. (Minimal) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Note that the basis of a topological space X naturally generates the collection of open sets by taking

$$\mathcal{T} = \{U \subseteq X : U = \cup\{B \in \mathcal{B} : B \subseteq U\}\}$$

For a natural number n, let X_i be a topological space for $i \in [n]$. Let $X := \prod_{i \in [n]} X_i$. The product topology on X is the topology whose basis consists of all sets of the form $U_1 \times U_2 \times \ldots \times U_n$, where U_i is an open set in X_i for $i \in [n]$.

Metrizability. Our result relies on the *metrizability* of a space of every possible mixed strategies on a topological space by the Lévy-Prokhorov metric (to be defined shortly). Intuitively, it's technically difficult to directly deal with a open neighborhood of a mixed strategy (probabiliy measure on pure strategy space) given underlying topology. A convenient tool in this context is to metrize the topology with a proper metric that can effectively be represented as a distance over the space of original pure strategy, and deal with such metric to analyze the open neighborhood of a mixed strategy.

First, we use the fact that if X is a compact Hausdorff space and $C(X) = \{f : X \to \mathbb{R} : f \text{ is continuous and linear}\}$, then every positive bounded linear functional on $C(X)$ can completely be represented by a finite Borel measure on X by the seminal Riesz representation theorem. Formally, let $C(X)$ be the space of all real-valued, continuous, and bounded linear functions on X. Then, it is known that $C(X)$ is a complete normed vector space, *i.e.,* a Banach space, with the supremum norm: $||f|| = \sup_{x \in X} |f(x)|$, *e.g.,* see (Parthasarathy, 2005). Let $C(X)^*$ be the dual space of $C(X)$, *i.e.,* the space of all real-valued continuous linear functionals $\phi : C(X) \to \mathbb{R}$, and the corresponding supremum norm:

$$\|\phi\| = \sup\{|\phi(f)| : f \in C(X), \|f\| \leq 1\}.$$

Then, the Riesz representation theorem can be written as follows:

Theorem A.1 (Riesz representation theorem, Parthasarathy (2005))**.** *Let X be a compact Hausdorff space and $C(X)$ be the real vector space of all real-valued continuous bounded linear functionals on X. If $\phi \in C(X)^*$ is a positive linear functional such that $\phi(f) \geq 0$ for all $f \in C(X)$ with $f(x) \geq 0$ for every $x \in X$ and $||\phi|| = 1$ with the supremum norm, then there exists a unique Borel probability measure μ on X such that*

$$\phi(f) = \int_X f d\mu, \forall f \in C(X).$$

Let M denote the set of (Borel) probability measures over a compact Hausdorff space X, *i.e.,* the space of mixed strategies. Then, the Riesz representation theorem asserts the one-to-one correspondence between the space of positive linear functional ϕ on $C(X)$ with $\|\phi\| = 1$ and M, *i.e.,* the space of mixed strategy. Such equivalence will be helpful in endowing the space of mixed strategies with a proper topology:

Definition A.2 (Weak* topology)**.** The weak* topology on X^* on the dual space X^* of a normed vector space X is the topology where a sequence $\{f_\alpha\} \subset X^*$ converges to $f \in X^*$ if and only if $f_\alpha(x) \to f(x)$ for every $x \in X$.

The terminology weakness stems from the fact that it is the coarsest (weakest) topology such that the mapping f remains continuous. Throughout we consider the set of probability measures equipped with the weak* topology.

Further, if X is a metric space, it is known that $C(X)^*$ is metrizable with a proper metric.

Theorem A.3 (Narici and Beckenstein (2010))**.** *If X is a metric space and M is a bounded subset of its dual $C(X)^*$ of the set of bounded linear continuous functionals, then M endowed with the weak* topology is a metrizable topological space,* i.e., *homeomorphic to a metric space, with respect to the Lévy-Prokhorov metric with metric over X.*

Note that the Lévy-Prokhorov metric is defined as follows.

Definition A.4 (Lévy-Prokhorov metric)**.** Let (X, d) be a metric space with its Borel sigma algebra $\mathcal{B}(X)$. Let $\mathcal{P}(X)$ denote the collection of all Borel probability measures on the measurable space $(M, \mathcal{B}(X))$. For a subset $A \subseteq X$, define the ε-neighborhood of A by

$$A^\varepsilon = \{p \in X : \exists q \in A, d(p, q) < \varepsilon\} = \cup_{p \in A} B_\varepsilon(p),$$

where $B_\varepsilon(p)$ is the open ball of radius ε at p. Then, the Lévy-Prokhorov metric $\pi : \mathcal{P}(M)^2 \to [0, \infty]$ is defined by setting the distance between two probability measures μ and ν to be

$$\pi(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon, \quad \nu(A) \leq \mu(A^\varepsilon) + \varepsilon, \forall A \in \mathcal{B}(M)\}.$$

Thus, Theorem A.3 immediately asserts that Lévy-Prokhorov metric metrizes the weak* topology on the space of bounded linear functionals, which in fact is the space of mixed strategies by Theorem A.1, *i.e.,* any open ball with respect to the Lévy-Prokhorov metric is an open set in the weak* topology.

B Alternative Proof of Theorem 5.5

We require the following result.

Theorem B.1 (Schur-Ostrowski criterion)**.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is syemmtric and all first partial derivative exists, then f is Schur-convex (concave) if and only if for every $x \in \mathbb{R}^n$, it satisfies*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq (\leq)\ 0,$$

for any $1 \leq i, j \leq n$.

Proof of Theorem 5.5. Define a generalized notion of $a_i(x)$ as follows: $a_{i,n}(x) := \binom{n}{i} x^{n-i+1}(1-x)^{i-1}$. Note that $a_i(x) = a_{i,n-1}(x)$. Then, we use the following well-known fact from Bernstein basis polynomials Lorentz (2012):

$$n(a_i(x) - a_{i+1}(x)) = -\frac{da_{i+1,n}(x)}{dx}.$$

Consider, without loss of generality, $x \in \Delta$. An argument for any other probability vectors in Δ_n will simply follow from similar arguments, as can be seen shortly.

By Theorem B.1, it suffices to prove that

$$\int_0^1 \left(\frac{\partial O(x,\mathbf{p})^r}{\partial p_i} - \frac{\partial O(x,\mathbf{p})^r}{\partial p_j}\right) dx \geq 0,$$

if $r \geq 1$ or $r \leq 1$, and that left hand side is ≤ 0 if $r \in [0,1]$, for any $i \leq j$. Since we are considering $\mathbf{p} \in \Delta$, it suffices to show this inequality for any two consecutive indices $i, i+1$.

Notice that the left hand side is equivalent to

$$\int_0^1 \big(rO(x,\mathbf{p})^{r-1} a_i(x) - rO(x,\mathbf{p})^{r-1} a_{i+1}(x)\big) dx = \int_0^1 \big(rO(x,\mathbf{p})^{r-1}(a_i(x) - a_{i+1}(x))\big) dx.$$

Integrating by parts, we obtain

$$rO(x,\mathbf{p})^{r-1} \frac{-a_{i+1,n}(x)}{n}\bigg|_0^1 + r\int_0^1 \frac{\partial O(x,\mathbf{p})^{r-1}}{\partial x} \frac{a_{i+1,n}(x)}{n} dx,$$

for $i \in [n-1]$. Note that $a_{i+1,n}(x)$ is always zero for $i \in [n-1]$ if $x = 0$ or $x = 1$. Further, we have

$$\frac{\partial O(x,\mathbf{p})^{r-1}}{\partial x} = (r-1)O(x,\mathbf{p})^{r-2}\psi(x,\mathbf{p}) \geq 0,$$

and $\psi(x,\mathbf{p}) \geq 0$ due to Lemma 6.9. Thus, it becomes equivalent to

$$r(r-1)\int_0^1 O(x,\mathbf{p})^{r-2}\psi(x,\mathbf{p}) a_{i+1,n}(x)/n dx,$$

which is ≥ 0 if $r \geq 1$ or $r \leq 0$, and otherwise ≤ 0. ∎