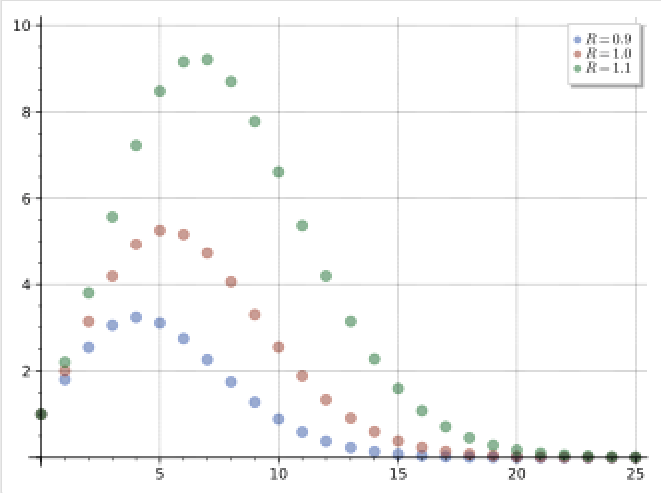
Volume of an n-ball

In geometry, a ball is a region in a space comprising all points within a fixed distance, called the radius, from a given point; that is, it is the region enclosed by a sphere or hypersphere. An n-ball is a ball in an n-dimensional Euclidean space. The **volume of a n-ball** is the Lebesgue measure of this ball, which generalizes to any dimension the usual volume of a ball in 3-dimensional space. The volume of a n-ball of radius R is $R^n V_n$, where V_n is the volume of the unit n-ball, the n-ball of radius 1.



Volumes of balls in dimensions 0 through 25; unit ball in red.

The real number V_n can be expressed via a two-dimension recurrence relation. Closed-form expressions involve the gamma, factorial, or double factorial function. The volume can also be expressed in terms of A_n, the area of the unit n-sphere.

Formulas

The first volumes are as follows:

Dimension	Volume of a ball of radius R	Radius of a ball of volume V
0	1	(all 0-balls have volume 1)
1	$2R$	$\frac{V}{2} = 0.5 \times V$
2	$\pi R^2 \approx 3.142 \times R^2$	$\frac{V^{1/2}}{\sqrt{\pi}} \approx 0.564 \times V^{\frac{1}{2}}$
3	$\frac{4\pi}{3} R^3 \approx 4.189 \times R^3$	$\left(\frac{3V}{4\pi}\right)^{1/3} \approx 0.620 \times V^{1/3}$
4	$\frac{\pi^2}{2} R^4 \approx 4.935 \times R^4$	$\frac{(2V)^{1/4}}{\sqrt{\pi}} \approx 0.671 \times V^{1/4}$
5	$\frac{8\pi^2}{15} R^5 \approx 5.264 \times R^5$	$\left(\frac{15V}{8\pi^2}\right)^{1/5} \approx 0.717 \times V^{1/5}$
6	$\frac{\pi^3}{6} R^6 \approx 5.168 \times R^6$	$\frac{(6V)^{1/6}}{\sqrt{\pi}} \approx 0.761 \times V^{1/6}$
7	$\frac{16\pi^3}{105} R^7 \approx 4.725 \times R^7$	$\left(\frac{105V}{16\pi^3}\right)^{1/7} \approx 0.801 \times V^{1/7}$
8	$\frac{\pi^4}{24} R^8 \approx 4.059 \times R^8$	$\frac{(24V)^{1/8}}{\sqrt{\pi}} \approx 0.839 \times V^{1/8}$
9	$\frac{32\pi^4}{945} R^9 \approx 3.299 \times R^9$	$\left(\frac{945V}{32\pi^4}\right)^{1/9} \approx 0.876 \times V^{1/9}$
10	$\frac{\pi^5}{120} R^{10} \approx 2.550 \times R^{10}$	$\frac{(120V)^{1/10}}{\sqrt{\pi}} \approx 0.911 \times V^{1/10}$
11	$\frac{64\pi^5}{10395} R^{11} \approx 1.884 \times R^{11}$	$\left(\frac{10395V}{64\pi^5}\right)^{1/11} \approx 0.944 \times V^{1/11}$
12	$\frac{\pi^6}{720} R^{12} \approx 1.335 \times R^{12}$	$\frac{(720V)^{1/12}}{\sqrt{\pi}} \approx 0.976 \times V^{1/12}$
13	$\frac{128\pi^6}{135135} R^{13} \approx 0.911 \times R^{13}$	$\left(\frac{135135V}{128\pi^6}\right)^{1/13} \approx 1.007 \times V^{1/13}$
14	$\frac{\pi^7}{5040} R^{14} \approx 0.599 \times R^{14}$	$\frac{(5040V)^{1/14}}{\sqrt{\pi}} \approx 1.037 \times V^{1/14}$
15	$\frac{256\pi^7}{2027025} R^{15} \approx 0.381 \times R^{15}$	$\left(\frac{2027025V}{256\pi^7}\right)^{1/15} \approx 1.066 \times V^{1/15}$
n	***$V_n(R)$***	***$R_n(V)$***

Closed form

The n-dimensional volume of a Euclidean ball of radius R in n-dimensional Euclidean space is:[1]

$$V_n(R) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} R^n,$$

where Γ is Euler's gamma function. The gamma function is offset from but otherwise extends the factorial function to non-integer arguments. It satisfies $\Gamma(n) = (n-1)!$ if n is a positive integer and $\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2}) \cdot (n-\frac{3}{2}) \cdot \ldots \cdot \frac{1}{2} \cdot \pi^{1/2}$ if n is a non-negative integer.

Two-dimension recurrence relation

The volume can be computed without use of the Gamma function. As is proved below using a vector-calculus double integral in polar coordinates, the volume V of an n-ball of radius R can be expressed recursively in terms of the volume of an $(n-2)$-ball, via the interleaved recurrence relation:

$$V_n(R) = \begin{cases} 1 & \text{if } n = 0, \\ 2R & \text{if } n = 1, \\ \dfrac{2\pi}{n} R^2 \times V_{n-2}(R) & \text{otherwise.} \end{cases}$$

This allows computation of $V_n(R)$ in approximately $n / 2$ steps.

Alternative forms

The volume can also be expressed in terms of an $(n-1)$-ball using the one-dimension recurrence relation:

$$\begin{aligned} V_0(R) &= 1, \\ V_n(R) &= \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}+1\right)} R\, V_{n-1}(R). \end{aligned}$$

Inverting the above, the radius of an n-ball of volume V can be expressed recursively in terms of the radius of an $(n-2)$- or $(n-1)$-ball:

$$\begin{aligned} R_n(V) &= \left(\tfrac{1}{2}n\right)^{1/n} \left(\Gamma\left(\tfrac{n}{2}\right)V\right)^{-2/(n(n-2))} R_{n-2}(V), \\ R_n(V) &= \frac{\Gamma\left(\frac{n}{2}+1\right)^{1/n}}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)^{1/(n-1)}} V^{-1/(n(n-1))} R_{n-1}(V). \end{aligned}$$

Using explicit formulas for particular values of the gamma function at the integers and half-integers gives formulas for the volume of a Euclidean ball in terms of factorials. For non-negative integer k, these are:

$$\begin{aligned} V_{2k}(R) &= \frac{\pi^k}{k!} R^{2k}, \\ V_{2k+1}(R) &= \frac{2(k!)(4\pi)^k}{(2k+1)!} R^{2k+1}. \end{aligned}$$

The volume can also be expressed in terms of double factorials. For a positive odd integer $2k + 1$, the double factorial is defined by

$$(2k+1)!! = (2k+1)\cdot(2k-1)\cdots 5\cdot 3\cdot 1.$$

The volume of an odd-dimensional ball is

$$V_{2k+1}(R) = \frac{2(2\pi)^k}{(2k+1)!!}R^{2k+1}.$$

There are multiple conventions for double factorials of even integers. Under the convention in which the double factorial satisfies

$$(2k)!! = (2k)\cdot(2k-2)\cdots 4\cdot 2\cdot\sqrt{2/\pi} = 2^k\cdot k!\cdot\sqrt{2/\pi},$$

the volume of an n-dimensional ball is, regardless of whether n is even or odd,

$$V_n(R) = \frac{2(2\pi)^{(n-1)/2}}{n!!}R^n.$$

Instead of expressing the volume V of the ball in terms of its radius R, the formulas can be inverted to express the radius as a function of the volume:

$$\begin{aligned} R_n(V) &= \frac{\Gamma\left(\frac{n}{2}+1\right)^{1/n}}{\sqrt{\pi}}V^{1/n} \\ &= \left(\frac{n!!V}{2(2\pi)^{(n-1)/2}}\right)^{1/n} \\ R_{2k}(V) &= \frac{(k!V)^{1/(2k)}}{\sqrt{\pi}}, \\ R_{2k+1}(V) &= \left(\frac{(2k+1)!V}{2(k!)(4\pi)^k}\right)^{1/(2k+1)}. \end{aligned}$$

Approximation for high dimensions

Stirling's approximation for the gamma function can be used to approximate the volume when the number of dimensions is high.

$$\begin{aligned} V_n(R) &\sim \frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2}R^n. \\ R_n(V) &\sim (\pi n)^{1/(2n)}\sqrt{\frac{n}{2\pi e}}V^{1/n}. \end{aligned}$$

In particular, for any fixed value of R the volume tends to a limiting value of 0 as n goes to infinity. Which value of n maximizes $V_n(R)$ depends upon the value of R; for example, the volume $V_n(1)$ is increasing for $0 \le n \le 5$, achieves its maximum when $n = 5$, and is decreasing for $n \ge 5$.[2]

Also, there is an asymptotic formula for the surface area[3]

$$\lim_n \frac{1}{n} \ln A_{n-1}(\sqrt{n}) = \frac{1}{2}(\ln(2\pi) + 1)$$

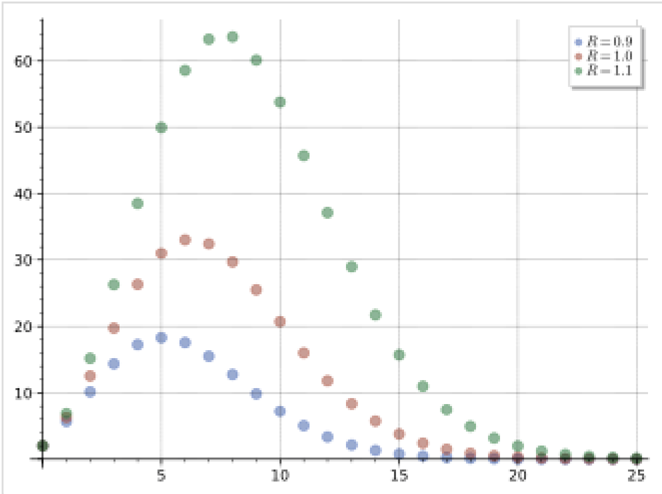
Relation with surface area

Let $A_{n-1}(R)$ denote the hypervolume of the $(n-1)$-sphere of radius R. The $(n-1)$-sphere is the $(n-1)$-dimensional boundary (surface) of the n-dimensional ball of radius R, and the sphere's hypervolume and the ball's hypervolume are related by:

$$A_{n-1}(R) = \frac{d}{dR} V_n(R) = \frac{n}{R} V_n(R).$$

Thus, $A_{n-1}(R)$ inherits formulas and recursion relationships from $V_n(R)$, such as

$$A_{n-1}(R) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} R^{n-1}.$$



Surface areas of hyperspheres in dimensions 0 through 25

There are also formulas in terms of factorials and double factorials.

Proofs

There are many proofs of the above formulas.

The volume is proportional to the *n*th power of the radius

An important step in several proofs about volumes of n-balls, and a generally useful fact besides, is that the volume of the n-ball of radius R is proportional to R^n:

$$V_n(R) \propto R^n.$$

The proportionality constant is the volume of the unit ball.

This is a special case of a general fact about volumes in n-dimensional space: If K is a body (measurable set) in that space and RK is the body obtained by stretching in all directions by the factor R then the volume of RK equals R^n times the volume of K. This is a direct consequence of the change of variables formula:

$$V(RK) = \int_{RK} dx = \int_K R^n \, dy = R^n V(K)$$

where $dx = dx_1 \ldots dx_n$ and the substitution $x = Ry$ was made.

Another proof of the above relation, which avoids multi-dimensional integration, uses induction: The base case is $n = 0$, where the proportionality is obvious. For the inductive step, assume that proportionality is true in dimension $n - 1$. Note that the intersection of an n-ball with a hyperplane is an $(n - 1)$-ball. When the volume of the n-ball is written as an integral of volumes of $(n - 1)$-balls:

$$V_n(R) = \int_{-R}^{R} V_{n-1}\left(\sqrt{R^2 - x^2}\right) dx,$$

it is possible by the inductive hypothesis to remove a factor of R from the radius of the $(n - 1)$-ball to get:

$$V_n(R) = R^{n-1} \int_{-R}^{R} V_{n-1}\left(\sqrt{1 - \left(\frac{x}{R}\right)^2}\right) dx.$$

Making the change of variables $t = \frac{x}{R}$ leads to:

$$V_n(R) = R^n \int_{-1}^{1} V_{n-1}\left(\sqrt{1 - t^2}\right) dt = R^n V_n(1),$$

which demonstrates the proportionality relation in dimension n. By induction, the proportionality relation is true in all dimensions.

The two-dimension recursion formula

A proof of the recursion formula relating the volume of the n-ball and an $(n - 2)$-ball can be given using the proportionality formula above and integration in cylindrical coordinates. Fix a plane through the center of the ball. Let r denote the distance between a point in the plane and the center of the sphere, and let θ denote the azimuth. Intersecting the n-ball with the $(n - 2)$-dimensional plane defined by fixing a radius and an azimuth gives an $(n - 2)$-ball of radius $\sqrt{R^2 - r^2}$. The volume of the ball can therefore be written as an iterated integral of the volumes of the $(n - 2)$-balls over the possible radii and azimuths:

$$V_n(R) = \int_0^{2\pi} \int_0^{R} V_{n-2}\left(\sqrt{R^2 - r^2}\right) r\, dr\, d\theta,$$

The azimuthal coordinate can be immediately integrated out. Applying the proportionality relation shows that the volume equals

$$V_n(R) = 2\pi V_{n-2}(R) \int_0^{R} \left(1 - \left(\frac{r}{R}\right)^2\right)^{(n-2)/2} r\, dr.$$

The integral can be evaluated by making the substitution $u = 1 - (\frac{r}{R})^2$ to get

$$\begin{aligned} V_n(R) &= 2\pi V_{n-2}(R) \cdot \left[-\frac{R^2}{n} \left(1 - \left(\frac{r}{R}\right)^2\right)^{n/2} \right]_{r=0}^{r=R} \\ &= \frac{2\pi R^2}{n} V_{n-2}(R), \end{aligned}$$

which is the two-dimension recursion formula.

The same technique can be used to give an inductive proof of the volume formula. The base cases of the induction are the 0-ball and the 1-ball, which can be checked directly using the facts $\Gamma(1) = 1$ and $\Gamma(\frac{3}{2}) = \frac{1}{2} \cdot \Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$. The inductive step is similar to the above, but instead of applying proportionality to the volumes of the $(n-2)$-balls, the inductive hypothesis is applied instead.

The one-dimension recursion formula

The proportionality relation can also be used to prove the recursion formula relating the volumes of an n-ball and an $(n-1)$-ball. As in the proof of the proportionality formula, the volume of an n-ball can be written as an integral over the volumes of $(n-1)$-balls. Instead of making a substitution, however, the proportionality relation can be applied to the volumes of the $(n-1)$-balls in the integrand:

$$V_n(R) = V_{n-1}(R) \int_{-R}^{R} \left(1 - \left(\frac{x}{R}\right)^2\right)^{(n-1)/2} dx.$$

The integrand is an even function, so by symmetry the interval of integration can be restricted to $[0, R]$. On the interval $[0, R]$, it is possible to apply the substitution $u = (\frac{x}{R})^2$. This transforms the expression into

$$V_{n-1}(R) \cdot R \cdot \int_0^1 (1-u)^{(n-1)/2} u^{-\frac{1}{2}}\, du$$

The integral is a value of a well-known special function called the beta function $\mathrm{B}(x, y)$, and the volume in terms of the beta function is

$$V_n(R) = V_{n-1}(R) \cdot R \cdot \mathrm{B}\left(\tfrac{n+1}{2}, \tfrac{1}{2}\right).$$

The beta function can be expressed in terms of the gamma function in much the same way that factorials are related to binomial coefficients. Applying this relationship gives

$$V_n(R) = V_{n-1}(R) \cdot R \cdot \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

Using the value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ gives the one-dimension recursion formula:

$$V_n(R) = R\sqrt{\pi}\frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} V_{n-1}(R).$$

As with the two-dimension recursive formula, the same technique can be used to give an inductive proof of the volume formula.

Direct integration in spherical coordinates

The volume of the n-ball $V_n(R)$ can be computed by integrating the volume element in spherical coordinates. The spherical coordinate system has a radial coordinate r and angular coordinates $\varphi_1, \ldots, \varphi_{n-1}$, where the domain of each φ except φ_{n-1} is $[0, \pi)$, and the domain of φ_{n-1} is $[0, 2\pi)$. The spherical volume element is:

$$dV = r^{n-1} \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \cdots \sin(\varphi_{n-2})\, dr\, d\varphi_1\, d\varphi_2 \cdots d\varphi_{n-1},$$

and the volume is the integral of this quantity over r between 0 and R and all possible angles:

$$V_n(R) = \int_0^R \int_0^\pi \cdots \int_0^{2\pi} r^{n-1} \sin^{n-2}(\varphi_1) \cdots \sin(\varphi_{n-2})\, d\varphi_{n-1} \cdots d\varphi_1\, dr.$$

Each of the factors in the integrand depends on only a single variable, and therefore the iterated integral can be written as a product of integrals:

$$V_n(R) = \left(\int_0^R r^{n-1}\, dr\right) \left(\int_0^\pi \sin^{n-2}(\varphi_1)\, d\varphi_1\right) \cdots \left(\int_0^{2\pi} d\varphi_{n-1}\right).$$

The integral over the radius is $\frac{R^n}{n}$. The intervals of integration on the angular coordinates can, by the symmetry of the sine about $\frac{\pi}{2}$, be changed to $[0, \frac{\pi}{2}]$:

$$V_n(R) = \frac{R^n}{n} \left(2\int_0^{\pi/2} \sin^{n-2}(\varphi_1)\, d\varphi_1\right) \cdots \left(4\int_0^{\pi/2} d\varphi_{n-1}\right).$$

Each of the remaining integrals is now a particular value of the beta function:

$$V_n(R) = \frac{R^n}{n} \mathrm{B}\left(\tfrac{n-1}{2}, \tfrac{1}{2}\right) \mathrm{B}\left(\tfrac{n-2}{2}, \tfrac{1}{2}\right) \cdots \mathrm{B}\left(1, \tfrac{1}{2}\right) \cdot 2\,\mathrm{B}\left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

The beta functions can be rewritten in terms of gamma functions:

$$V_n(R) = \frac{R^n}{n} \cdot \frac{\Gamma\left(\frac{n}{2} - \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\frac{n}{2} - 1\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} - \frac{1}{2}\right)} \cdots \frac{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \cdot 2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}.$$

This product telescopes. Combining this with the values $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(1) = 1$ and the functional equation $z\Gamma(z) = \Gamma(z + 1)$ leads to

$$V_n(R) = \frac{2\pi^{n/2} R^n}{n\,\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{n/2} R^n}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

Gaussian integrals

The volume formula can be proven directly using Gaussian integrals. Consider the function:

$$f(x_1,\ldots,x_n) = \exp\left(-\tfrac{1}{2}\sum_{i=1}^{n} x_i^2\right).$$

This function is both rotationally invariant and a product of functions of one variable each. Using the fact that it is a product and the formula for the Gaussian integral gives:

$$\int_{\mathbf{R}^n} f\,dV = \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}x_i^2\right)dx_i\right) = (2\pi)^{n/2},$$

where dV is the n-dimensional volume element. Using rotational invariance, the same integral can be computed in spherical coordinates:

$$\int_{\mathbf{R}^n} f\,dV = \int_0^{\infty}\int_{S^{n-1}(r)}\exp\left(-\tfrac{1}{2}r^2\right)dA\,dr,$$

where $S^{n-1}(r)$ is an $(n-1)$-sphere of radius r (being the surface of an n-ball of radius r) and dA is the area element (equivalently, the $(n-1)$-dimensional volume element). The surface area of the sphere satisfies a proportionality equation similar to the one for the volume of a ball: If $A_{n-1}(r)$ is the surface area of an $(n-1)$-sphere of radius r, then:

$$A_{n-1}(r) = r^{n-1}A_{n-1}(1).$$

Applying this to the above integral gives the expression

$$(2\pi)^{n/2} = \int_0^{\infty}\int_{S^{n-1}(r)}\exp\left(-\tfrac{1}{2}r^2\right)dA\,dr = A_{n-1}(1)\int_0^{\infty}\exp\left(-\tfrac{1}{2}r^2\right)r^{n-1}\,dr.$$

Substituting $t = \frac{r^2}{2}$:

$$\int_0^{\infty}\exp\left(-\tfrac{1}{2}r^2\right)r^{n-1}\,dr = 2^{(n-2)/2}\int_0^{\infty}e^{-t}t^{(n-2)/2}\,dt.$$

The integral on the right is the gamma function evaluated at $\frac{n}{2}$.

Combining the two results shows that

$$A_{n-1}(1) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

To derive the volume of an n-ball of radius R from this formula, integrate the surface area of a sphere of radius r for $0 \le r \le R$ and apply the functional equation $z\Gamma(z) = \Gamma(z+1)$:

$$V_n(R) = \int_0^{R}\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}r^{n-1}\,dr = \frac{2\pi^{n/2}}{n\,\Gamma\left(\frac{n}{2}\right)}R^n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}R^n.$$

Geometric proof

The relations $V_{n+1}(R) = \frac{R}{n+1} A_n(R)$ and $A_{n+1}(R) = (2\pi R) V_n(R)$ and thus the volumes of *n*-balls and areas of *n*-spheres can also be derived geometrically. As noted above, because a ball of radius R is obtained from a unit ball B_n by rescaling all directions in R times, $V_n(R)$ is proportional to R^n, which implies $\frac{dV_n(R)}{dR} = \frac{n}{R} V_n(R)$. Also, $A_{n-1}(R) = \frac{dV_n(R)}{dR}$ because a ball is a union of concentric spheres and increasing radius by *ε* corresponds to a shell of thickness *ε*. Thus, $V_n(R) = \frac{R}{n} A_{n-1}(R)$; equivalently, $V_{n+1}(R) = \frac{R}{n+1} A_n(R)$.

$A_{n+1}(R) = (2\pi R) V_n(R)$ follows from existence of a volume-preserving bijection between the unit sphere S_{n+1} and $S_1 \times B_n$:

$$(x, y, \vec{z}) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, \vec{z} \right)$$

($\vec{z}$ is an *n*-tuple; $|(x, y, \vec{z})| = 1$; we are ignoring sets of measure 0). Volume is preserved because at each point, the difference from isometry is a stretching in the *xy* plane (in $1/\sqrt{x^2+y^2}$ times in the direction of constant $x^2 + y^2$) that exactly matches the compression in the direction of the gradient of $|\vec{z}|$ on S_n (the relevant angles being equal). For S_2, a similar argument was originally made by Archimedes in *On the Sphere and Cylinder*.

Balls in L^p norms

There are also explicit expressions for the volumes of balls in L^p norms. The L^p norm of the vector $x = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$ is

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p},$$

and an L^p ball is the set of all vectors whose L^p norm is less than or equal to a fixed number called the radius of the ball. The case $p = 2$ is the standard Euclidean distance function, but other values of p occur in diverse contexts such as information theory, coding theory, and dimensional regularization.

The volume of an L^p ball of radius R is

$$V_n^p(R) = \frac{\left(2\,\Gamma\left(\frac{1}{p}+1\right)\right)^n}{\Gamma\left(\frac{n}{p}+1\right)} R^n.$$

These volumes satisfy recurrence relations similar to those for $p = 2$:

$$V_n^p(R) = \frac{\left(2\,\Gamma\left(\frac{1}{p}+1\right)\right)^p p}{n} R^p\, V_{n-p}^p(R)$$

and

$$V_n^p(R) = 2\frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{n-1}{p}+1\right)}{\Gamma\left(\frac{n}{p}+1\right)} R\, V_{n-1}^p(R),$$

which can be written more concisely using a generalized binomial coefficient,

$$V_n^p(R) = \frac{2}{\binom{n/p}{1/p}} R\, V_{n-1}^p(R).$$

For $p = 2$, one recovers the recurrence for the volume of a Euclidean ball because $2\Gamma(\frac{3}{2}) = \sqrt{\pi}$.

For example, in the cases $p = 1$ (taxicab norm) and $p = \infty$ (max norm), the volumes are:

$$V_n^1(R) = \frac{2^n}{n!} R^n,$$
$$V_n^\infty(R) = 2^n R^n.$$

These agree with elementary calculations of the volumes of cross-polytopes and hypercubes.

Relation with surface area

For most values of p, the surface area $A_{n-1}^p(R)$ of an L^p sphere of radius R (the boundary of an L^p n-ball of radius R) cannot be calculated by differentiating the volume of an L^p ball with respect to its radius. While the volume can be expressed as an integral over the surface areas using the coarea formula, the coarea formula contains a correction factor that accounts for how the p-norm varies from point to point. For $p = 2$ and $p = \infty$, this factor is one. However, if $p = 1$ then the correction factor is $\sqrt{n}$: the surface area of an L^1 sphere of radius R in $\mathbf{R}^n$ is $\sqrt{n}$ times the derivative of the volume of an L^1 ball. This can be seen most simply by applying the divergence theorem to the vector field $\mathbf{F}(\mathrm{x}) = \mathrm{x}$ to get

$$nV_n^1(R) = \iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \oiint_S \frac{1}{\sqrt{n}}(|x_1| + \cdots + |x_n|)\, dS = \frac{R}{\sqrt{n}} \oiint_S dS = \frac{R}{\sqrt{n}} A_{n-1}^1(R).$$

For other values of p, the constant is a complicated integral.

Generalizations

The volume formula can be generalized even further. For positive real numbers $p_1, \ldots, p_n$, define the $(p_1, \ldots, p_n)$ ball with limit $L \geq 0$ to be

$$B_{p_1,\ldots,p_n}(L) = \left\{ x = (x_1, \ldots, x_n) \in \mathbf{R}^n : |x_1|^{p_1} + \cdots + |x_n|^{p_n} \leq L \right\}.$$

The volume of this ball has been known since the time of Dirichlet:[4]

$$V\left(B_{p_1,\ldots,p_n}(L)\right) = \frac{2^n\Gamma\left(\frac{1}{p_1}+1\right)\cdots\Gamma\left(\frac{1}{p_n}+1\right)}{\Gamma\left(\frac{1}{p_1}+\cdots+\frac{1}{p_n}+1\right)} L^{\frac{1}{p_1}+\cdots+\frac{1}{p_n}}.$$

Comparison to L^p norm

Using the harmonic mean $p = \frac{n}{\frac{1}{p_1}+\cdots\frac{1}{p_n}}$ and defining $R = \sqrt[p]{L}$, the similarity to the volume formula for the L^p ball becomes clear.

$$V\left(\left\{x \in \mathbf{R}^n : \sqrt[p]{|x_1|^{p_1}+\cdots+|x_n|^{p_n}} \leq R\right\}\right) = \frac{2^n\Gamma\left(\frac{1}{p_1}+1\right)\cdots\Gamma\left(\frac{1}{p_n}+1\right)}{\Gamma\left(\frac{n}{p}+1\right)} R^n.$$

See also

- n-sphere
- Sphere packing
- Hamming bound

References

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3. Song Mei (2021-02-10). "Lecture 7: Concentration Inequalities and Field theoretic calculations" (https://www.stat.berkeley.edu/~songmei/Teaching/STAT260_Spring2021/Lecture_notes/scribe_lecture7.pdf) (PDF). *www.stat.berkeley.edu*.
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External links

- Derivation in hyperspherical coordinates (http://www.brouty.fr/Maths/sphere.html) (in French)
- Hypersphere (http://mathworld.wolfram.com/Hypersphere.html) on Wolfram MathWorld
- Volume of the Hypersphere (http://www.mathreference.com/ca-int,hsp.html) at Math Reference