Matrix Completion Methods for Causal Panel Data Models

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2025.6.25 ミクロ計量経済学

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Introduction

Today’s Agenda ; Keyword : Imputation

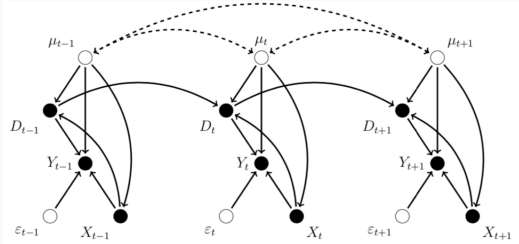
Imputation

- As many panel data methods, we want to know ATT : $\mathbb{E}[Y_{it}(1) - Y_{it}(0)|W_i = 1]$.
- Thus, it boils down to estimate (impute) the counterfactual $Y_{it}(0)$.
  - Horizontal : Under unconfoundedness, we can impute counterfactual PO using observed outcomes for control units.
  - Vertical : By SCM, we can also impute it using weighted average outcomes for control units with most predictive weights trained with pre-treatment datas.

$$\mathbf{Y} = \begin{pmatrix} \checkmark & \checkmark & \checkmark \\ \vdots & \vdots & \vdots \\ \checkmark & \checkmark & \checkmark \\ \checkmark & \checkmark & ? \\ \vdots & \vdots & \vdots \\ \checkmark & \checkmark & ? \end{pmatrix}. \qquad \mathbf{Y} = \begin{pmatrix} \checkmark & \checkmark & \dots & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \dots & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \dots & \checkmark & ? & \dots & ? \end{pmatrix}$$

Xu (2024): Counterfactual estimation

- functional form: $Y_{it}(0) = f(\mathbf{X_{it}}) + h(\mathbf{U_{it}}) + \epsilon_{it}$
  $\rightarrow$No anticipation, carryover, feedback, LDV
- strict exogeneity: $\forall i, j \in \{1, \ldots, N\},\ \forall s, t \in \{1, \ldots, T\}, \epsilon_{it} \perp\!\!\!\perp \{D_{js}, \mathbf{X_{js}}, \mathbf{U_{js}}\}$



- low-dimensional decomposition: $h(\mathbf{U_{it}}) = \{L_{it}\}, \text{rank}(\mathbf{L}_{N\times T}) \ll \min\{N, T\}$
  $\rightarrow$ The rank (= number of factors) is FIXED !!

Set Up

Notation and Estimand

- Consider a setting with N units observed over T periods characterized by a binary treatment W_{it} and hence two POs $Y_{it}(1), Y_{it}(0)$.
  - $\mathbf{X} \in \mathbb{R}^{N \times P}$, $\mathbf{Z} \in \mathbb{R}^{T \times Q}$: observe (unit / time)-specific covariance matrix
- Estimand : $\mathbf{Y} = \{Y_{it}(0)^{1}\} = \begin{pmatrix} Y_{11}(0) & \cdots & Y_{1T}(0) \\ \vdots & \ddots & \vdots \\ Y_{N1}(0) & \cdots & Y_{NT}(0) \end{pmatrix}$ (← Matrix!!)
- $W_{it} = \begin{cases} 1 & \text{if } (i,t) \in \mathcal{M} \\ 0 & \text{if } (i,t) \in \mathcal{O} \end{cases}$

[1] 以降は簡単のため，$Y_{it}(0) = Y_{it}$ とし，“(0)” を省略して表記する.

Patterns of data matrix

- Ordinary case (rich data wrt. units and times)

$$\mathbf{Y}_{N\times T} = \begin{pmatrix} \checkmark & \checkmark & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \checkmark & ? & \dots & ? \\ \checkmark & \checkmark & \checkmark & ? & \dots & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \checkmark & \checkmark & \checkmark & ? & \dots & ? \end{pmatrix}$$

- Staggered adoption

$$\mathbf{Y}_{N\times T} = \begin{pmatrix} \checkmark & \checkmark & \checkmark & \checkmark & \dots & \checkmark & \text{(never adopter)} \\ \checkmark & \checkmark & \checkmark & \checkmark & \dots & ? & \text{(late adopter)} \\ \checkmark & \checkmark & ? & ? & \dots & ? & \\ \checkmark & \checkmark & ? & ? & \dots & ? & \text{(medium adopter)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ \checkmark & ? & ? & ? & \dots & ? & \text{(early adopter)} \end{pmatrix}$$

Horizontal regression and unconfoundedness : thin matrix ($N \gg T$)

$$\mathbf{Y} = \begin{pmatrix} ✓ & ✓ & ✓ \\ \vdots & \vdots & \vdots \\ ✓ & ✓ & ✓ \\ ✓ & ✓ & ? \\ \vdots & \vdots & \vdots \\ ✓ & ✓ & ? \end{pmatrix}.$$

1. Regress the last period outcome on the lagged outcomes. (among untreated)
2. Predict the missing POs using the estimated regression.

$$\forall (i,T) \in \mathcal{M},\ \hat{Y}_{iT} = \hat{\beta}_0 + \sum_{t=1}^{T-1} \hat{\beta}_t Y_{it},\ \text{where } \hat{\beta} = \arg\min_{\beta} \sum_{i:(i,T)\in\mathcal{O}} (Y_{iT} - \beta_0 - \sum_{t=1}^{T-1} \beta_t Y_{it})^2.$$

$\rightarrow$ Nonparametrically,

Vertical regression and synthesis control : fat matrix ($T \gg N$)

$$\mathbf{Y} = \begin{pmatrix} \checkmark & \checkmark & \dots & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \dots & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \dots & \checkmark & ? & \dots & ? \end{pmatrix}$$

1. Regress the outcomes for treated unit prior to the treatment on the outcomes for the control units in the same periods.
2. Predct the missing POs using the estimated regression.

$$\forall (N,t) \in \mathcal{M},\ \hat{Y}_{Nt} = \hat{\gamma}_0 + \sum_{i=1}^{N-1} \hat{\gamma}_i Y_{it},\ \text{where } \hat{\gamma} = \arg\min_{\gamma} \sum_{t:(N,t)\in\mathcal{O}} (Y_{Nt} - \gamma_0 - \sum_{i=1}^{N-1} \gamma_i Y_{it})^2.$$

$\rightarrow$ Vertical regression is generalization of ADH(2010) in that it relaxes two restrictions :

- the coefficients $\hat{\gamma}$ are nonnegative. (Interpretability ; What is a negative weight?)
- the intercept in this regression is 0. (This is seen to be plausible in recent literatures.)

Matrix Completion

Model

- Under no covariates, we model the $N \times T$ matrix of complete matrix $\mathbf{Y}$ as

$$\mathbf{Y} = \mathbf{L}^* + \epsilon, \text{ where } \mathbb{E}[\epsilon|\mathbf{L}^*] = 0.$$

Assumption 1

- ϵ is independent of $\mathbf{L}^*$ (strict exogeneity)
- The element of ϵ are $\sigma - sub - Gaussian$ and independent each other.
  $\Leftrightarrow \forall t,\ \mathbb{E}[\exp(t\epsilon)] \leq \exp(\frac{\sigma^2 t^2}{2})$.

- The goal is to estimate the matrix $\mathbf{L}^*$. (low-rank assumption)
  $\rightarrow$ Note that two types[2] of fixed effects are included.

[2] これら以外にも Interactive fixed effect といったあらゆる factor を”少数まで”許容する

MC-NNM (Matrix Completion with Nuclear Norm Minimization) estimator

- MC-NNM estimator for $\mathbf{L}^*$ is given by $\hat{\mathbf{L}} + \hat{\Gamma}\mathbf{1}_T' + \mathbf{1}_N\hat{\Delta}'$

$$(\hat{\mathbf{L}}, \hat{\Gamma}, \hat{\Delta}) = \arg\min_{\mathbf{L},\Gamma,\Delta} \{\frac{1}{|\mathcal{O}|}||\mathbf{P}_{\mathcal{O}}(\mathbf{Y} - \mathbf{L} - \Gamma\mathbf{1}_T' - \mathbf{1}_N\Delta')||_F^2 + \lambda||\mathbf{L}||_*\}$$

  - $\Gamma \in \mathbb{R}^N$: unit-varying (and time-fixed) effect (individual effect)
  - $\Delta \in \mathbb{R}^T$: time-varying (and unit-fixed) effect (time effect)
  - matrix indicator function : $\mathbf{P}_{\mathcal{O}}(\mathbf{A}) = \begin{cases} A_{it} & \text{if } (i,t) \in \mathcal{O} \\ 0 & \text{if } (i,t) \notin \mathcal{O} \end{cases}$ (NA is regarded as 0)
  - Frobenius norm : $||\mathbf{A}||_F^2 = \sum_{i=1}^{N}\sum_{t=1}^{T} A_{it}^2$ (行列版の mean squared error を計算している)
- Regularization term $\lambda||L||_*$ leads to the low rank of $\mathbf{L}$.
  $\rightarrow$ minimize $\lambda||L||_* \Leftrightarrow$ the sparsity of Singular value $\sigma_i(\mathbf{L})(> 0) \Leftrightarrow$ low rank of $\mathbf{L}$

- **Fact 1.** (Singular value decomposition) *Every real matrix* $L \in \mathbb{R}^N \times \mathbb{R}^T$ *can be decomposed using a onthogonal matrix* $\mathbf{S} \in \mathbb{R}^N \times \mathbb{R}^{\min(N,T)}, \mathbf{R} \in \mathbb{R}^T \times \mathbb{R}^{\min(N,T)}$ *by*

$$\mathbf{L} = \mathbf{S}\mathbf{\Sigma}\mathbf{R}', \text{where } \mathbf{\Sigma} = \text{diag}(\sigma_1, \dots \sigma_{\min(N,T)}), \mathbf{S}'\mathbf{S} = \mathbf{I}_{\min(\mathbf{N},\mathbf{T})} = \mathbf{R}'\mathbf{R}$$

- **Fact 2.** *The number of non-zero singular value = rank* $\mathbf{L}$
  - Nuclear norm : $||L||_* = \sum_{i=1}^{\min(N,T)} \sigma_i(\mathbf{L})$

    $\rightarrow$ minimize $\lambda||L||_* \Leftrightarrow$ the sparsity of Singular value $\sigma_i(\mathbf{L})(> 0) \Leftrightarrow$ low rank of $\mathbf{L}$
- Since the rank of $\mathbf{L}$ corresponds to the number of factor, this assumption of low rank is quite plausible.
- Although the law rank matrix CAN include two fixed effects, these ”strong” factors are separately estimated for improving the quality of the practical imputations.

Algorithm for calculating $\hat{\mathbf{L}}$

- For simplicity, assume that there are no fixed effects. (only estimate $\mathbf{L}$)
- **Fact 3.** *For $\mathbf{A} = \mathbf{S\Sigma R}$, the minimizer is obtained analytically.*

  $$\mathbf{S\tilde{\Sigma}R}^{\mathsf{T}} = \arg\min_{\mathbf{A}}\{\frac{1}{2}||\mathbf{L}-\mathbf{A}||_F^2 + \lambda||\mathbf{A}||_*\}, \text{where } \tilde{\Sigma} = \text{diag}(\{\max(\sigma_i(\mathbf{A})-\lambda, 0)\}_i)$$

  $\rightarrow$ You can see elements with a small singular value (=weak factor) will be vanished.
- We perform this minimization over and over until the matrix converges.
  - Define $\text{shrink}_\lambda(\mathbf{A}) = \mathbf{S\tilde{\Sigma}R}^{\mathsf{T}}$ and start with the initial choice $\mathbf{L_1}(\lambda, \mathcal{O}) = \mathbf{P}_{\mathcal{O}}(\mathbf{Y})$ (The missing starts with 0.)

  $$\mathbf{L}_{k+1}(\lambda, \mathcal{O}) = \text{shrink}_{\frac{\lambda|\mathcal{O}|}{2}}\{\mathbf{P}_{\mathcal{O}}(\mathbf{Y}) + \mathbf{P}^{\perp}_{\mathcal{O}}(\mathbf{L}_k(\lambda, \mathcal{O}))\}$$

  - $\mathbf{P}_{\mathcal{O}}(\mathbf{A}) = \begin{cases} A_{it} & \text{if } (i,t) \in \mathcal{O} \\ 0 & \text{if } (i,t) \notin \mathcal{O} \end{cases}$, $\mathbf{P}^{\perp}_{\mathcal{O}}(\mathbf{A}) = \begin{cases} 0 & \text{if } (i,t) \in \mathcal{O} \\ A_{it} & \text{if } (i,t) \notin \mathcal{O} \end{cases}$
  - The limiting matrix $\hat{\mathbf{L}}(\lambda, \mathcal{O}) = \lim_{k\to\infty} \mathbf{L}_k(\lambda, \mathcal{O})$ is MC-NNM estimator given λ.

- For the case with fixed effects,

Inference : CV and CI

- The optimal value of λ is selected through cross-validation.
  -
- The theory of asymptotic distribution of $\mathbf{L}^* - \hat{\mathbf{L}}$ to construct CI has not yet developed.
- Instead, using resampling method, we see the fluctulation of the imputed matrix and construct CI like premutation methods in ADH-synthetic control method.

The relationship with horizontal and vertical regressions

- Let the estimand $\mathbf{Y}$ partition as $\begin{pmatrix} \mathbf{Y_0} & \mathbf{y_1} \\ \mathbf{y_2}' & ? \end{pmatrix}$, where $\mathbf{Y_0} \in \mathbb{R}^{N-1} \times \mathbb{R}^{T-1}, \mathbf{y_1} \in \mathbb{R}^{N-1}, \mathbf{y_2} \in \mathbb{R}^{T-1}$.
- For a given positive integer R, define an $N \times R$ matrix $\mathbf{A}$, an $T \times R$ matrix $\mathbf{B}$, a N-dim. vector γ and a R-dim. vector δ, then, the objective function w.r.t. MSE is

$$Q(\mathbf{Y}; R, \mathbf{A}, \mathbf{B}, \gamma, \delta) = \frac{1}{|\mathcal{O}|} ||\mathbf{P}_{\mathcal{O}}(\mathbb{Y} - \mathbf{A}\mathbf{B}' - \gamma \mathbf{1}_T' - \mathbf{1}_N \delta')||_F^2$$

MC-NNM estimator

- Nuclear norm matrix completion

$$(R_\lambda^{\text{mc-nnm}},\, \mathbf{A}_\lambda^{\text{mc-nnm}},\, \mathbf{B}_\lambda^{\text{mc-nnm}},\, \gamma_\lambda^{\text{mc-nnm}},\, \delta_\lambda^{\text{mc-nnm}}) = \arg\min_{R,A,B,\gamma,\delta} \Big\{ Q(\mathbf{Y}; R, \mathbf{A}, \mathbf{B}, \gamma, \delta) + \frac{\lambda}{2}\|\mathbf{A}\|_F^2 + \frac{\lambda}{2}\|\mathbf{B}\|_F^2 \Big\}.$$

- **Fact 3.** $\|\mathbf{L}\|_* = \min_{\mathbf{A},\mathbf{B}:\, \mathbf{L}=\mathbf{A}\mathbf{B}'} \frac{1}{2}(\|\mathbf{A}\|_F^2 + \|\mathbf{B}\|_F^2)$.

Horizontal regression (thin matrix, $N > T$)

- Horizontal regression estimator

$$(R^{hr},\, \mathbf{A}^{hr},\, \mathbf{B}^{hr},\, \gamma^{hr},\, \delta^{hr}) = \arg\min_{R,A,B,\gamma,\delta} Q(\mathbf{Y}; R, \mathbf{A}, \mathbf{B}, \gamma, \delta),$$

$$\text{subject to} \quad R = T-1, \quad \mathbf{A} = \begin{pmatrix} \mathbf{Y}_0 \\ y_2^\top \end{pmatrix}, \quad \gamma = 0, \quad \delta_1 = \cdots = \delta_{T-1} = 0.$$

- The solution for $\mathbf{B}$ is

$$\mathbf{B}^{hr\top} = \begin{pmatrix} E_{T-1} & \hat{\beta} \end{pmatrix}, \quad (\hat{\beta}, \hat{\delta}_T) = \arg\min_{\beta,\delta_T} \sum_{i=1}^{N-1} \big(Y_{iT} - \delta_T - \sum_{t=1}^{T-1} \beta_t\, Y_{it}\big)^2.$$

- Vertical regression, defined if $T > N$: fat matrix

Another methods

(iv) (synthetic control),

$$(R^{\text{sc-adh}}, \mathbf{A}^{\text{sc-adh}}, \mathbf{B}^{\text{sc-adh}}, \gamma^{\text{sc-adh}}, \delta^{\text{sc-adh}}) = \underset{R,\mathbf{A},\mathbf{B},\gamma,\delta}{\operatorname{argmin}} Q(\mathbf{Y}; R, \mathbf{A}, \mathbf{B}, \gamma, \delta),$$

subject to

$$R = N-1, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{Y}_0^\top \\ \mathbf{y}_1^\top \end{pmatrix}, \qquad \delta = 0, \qquad \gamma = 0,$$

$$\forall\, i, A_{iT} \geq 0, \ \sum_{i=1}^{N-1} A_{iT} = 1,$$

(v) (vertical regression, elastic net),

$$(R^{\text{vt-en}}, \mathbf{A}^{\text{vt-en}}, \mathbf{B}^{\text{vt-en}}, \gamma^{\text{vt-en}}, \delta^{\text{vt-en}}) = \underset{R,\mathbf{A},\mathbf{B},\gamma,\delta}{\operatorname{argmin}} \left\{ Q(\mathbf{Y}; R, \mathbf{A}, \mathbf{B}, \gamma, \delta) + \lambda \left[\frac{1-\alpha}{2} \left\| \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \right\|_F^2 + \alpha \left\| \begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} \right\|_1 \right] \right\},$$

subject to

$$R = N-1, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{Y}_0^\top \\ \mathbf{y}_1^\top \end{pmatrix},$$

$$\gamma_1 = \gamma_2 = \cdots = \gamma_{N-1} = 0, \qquad \delta = 0,$$

where $\mathbf{A}$ is partitioned as

$$\mathbf{A} = \begin{pmatrix} \tilde{\mathbf{A}} & \mathbf{a}_1 \\ \mathbf{a}_2^\top & \mathbf{a}_3 \end{pmatrix},$$

Theoritical Bounds
for the Estimation Error

Estimation error

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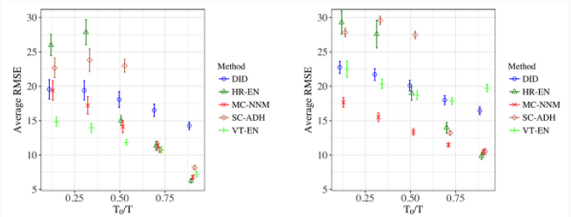
Two illustrations

Comparing each method

- In a real data matrix $\mathbf{Y}$ where no units is treated, we choose units as ”hypothetical treated” units (=regard as missing) and aim to predict (impute) their value.
  $\rightarrow$ Technically, compare the average root-mean-squared-error(RMSE) to assess which of the algorithm generally perform well.
- The compared estimators (algorithms) are as follows.
  - DiD : 2 回差分により counterfactual を補完する
  - VT-EN : elastics net で vertical regression function を作り、その推定値で補完
  - HR-EN : elastics net で horizontal regression function を作り、その推定値で補完
  - SC-ADH : classical approach to construct ”synthetic control”.
  - MC-NNM : 本日の主役

ADH(2010): California smoking data

- $N = 38, T = 31$
  - Simultaneous adoption : Let randomly selected $N_t = 8$ units "treated" in period $T_0 + 1$.
  - Staggered adoption : Let randomly selected $N_t = 35$ units "treated" in some periods after period T.



(a) Simultaneous adoption, $N_t = 8$ (b) Staggered adoption, $N_t = 35$

- (a) : VT-EN performs well on the whole, DiD is poor.
- (b) : MC-NNM is superior (large sample is favorable!), and VT-EN is generally poor.

Stock market data

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Final marks

Summary and Conclusion

- Matrix completion is a imputing method for the missing $Y_{it}(0)$ where $(i,t) \in \mathcal{M}$.
- MC-NNM estimator is the generalization of many estimators such as DiD, ADH-SC, vertical or horizontal (penalized) regression , and interactive fixed effect model.
- The critical differnce with previus estimators is that MC-NNM holistically restricts the factors by regularization with sparsity (Intrinsic factor vs Explicit factor)
- Practically, this estimator performs well with large N and T, and allows for a relatively large number of factors.
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