

# Matrix Completion Methods for Causal Panel Data Models

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# Introduction

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## Today's Agenda ; Keyword : Imputation

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# Imputation

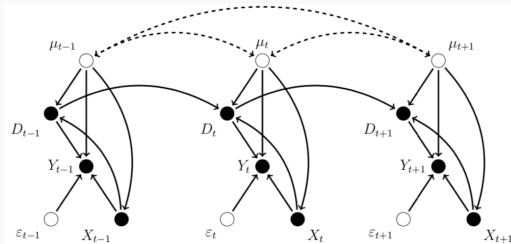
- As many panel data methods, we want to know ATT :  $\mathbb{E}[Y_{it}(1) - Y_{it}(0)|W_i = 1]$ .
- Thus, it boils down to estimate (impute) the counterfactual  $Y_{it}(0)$ .
  - Horizontal : Under unconfoundedness, we can impute counterfactual PO using observed outcomes for control units.
  - Vertical : By SCM, we can also impute it using weighted average outcomes for control units with most predictive weights trained with pre-treatment datas.

$$\mathbf{Y} = \begin{pmatrix} \checkmark & \checkmark & \checkmark \\ \vdots & \vdots & \vdots \\ \checkmark & \checkmark & \checkmark \\ \checkmark & \checkmark & ? \\ \vdots & \vdots & \vdots \\ \checkmark & \checkmark & ? \end{pmatrix}.$$

$$\mathbf{Y} = \begin{pmatrix} \checkmark & \checkmark & \dots & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \dots & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \dots & \checkmark & ? & \dots & ? \end{pmatrix}$$

## Xu (2024): Counterfactual estimation

- functional form:  $Y_{it}(0) = f(\mathbf{X}_{it}) + h(\mathbf{U}_{it}) + \epsilon_{it}$   
→ No anticipation, carryover, feedback, LDV
- strict exogeneity:  $\forall i, j \in \{1, \dots, N\}, \forall s, t \in \{1, \dots, T\}, \epsilon_{it} \perp\!\!\!\perp \{D_{js}, \mathbf{X}_{js}, \mathbf{U}_{js}\}$



- low-dimensional decomposition:  $h(\mathbf{U}_{it}) = \{L_{it}\}, \text{rank}(\mathbf{L}_{N \times T}) \ll \min\{N, T\}$   
→ The rank (= number of factors) is **FIXED !!**

## Set Up

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- Consider a setting with  $N$  units observed over  $T$  periods characterized by a binary treatment  $W_{it}$  and hence two POs  $Y_{it}(1), Y_{it}(0)$ .
  - $\mathbf{X} \in \mathbb{R}^{N \times P}$ ,  $\mathbf{Z} \in \mathbb{R}^{T \times Q}$  : observe (unit / time)-specific covariance matrix
- Estimand :  $\mathbf{Y} = \{Y_{it}(0)^1\} = \begin{pmatrix} Y_{11}(0) & \cdots & Y_{1T}(0) \\ \vdots & \ddots & \vdots \\ Y_{N1}(0) & \cdots & Y_{NT}(0) \end{pmatrix}$  ( $\leftarrow$  Matrix!!)
- $W_{it} = \begin{cases} 1 & \text{if } (i, t) \in \mathcal{M} \\ 0 & \text{if } (i, t) \in \mathcal{O} \end{cases}$

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<sup>1</sup>以降は簡単のため,  $Y_{it}(0) = Y_{it}$  とし, “(0)” を省略して表記する.

## Patterns of data matrix

- Ordinary case (rich data wrt. units and times)

$$\mathbf{Y}_{N \times T} = \begin{pmatrix} \checkmark & \checkmark & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \checkmark & ? & \dots & ? \\ \checkmark & \checkmark & \checkmark & ? & \dots & ? \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \checkmark & \checkmark & \checkmark & ? & \dots & ? \end{pmatrix}$$

- Staggered adoption

$$\mathbf{Y}_{N \times T} = \begin{pmatrix} \checkmark & \checkmark & \checkmark & \checkmark & \dots & \checkmark & \text{(never adopter)} \\ \checkmark & \checkmark & \checkmark & \checkmark & \dots & ? & \text{(late adopter)} \\ \checkmark & \checkmark & ? & ? & \dots & ? & \\ \checkmark & \checkmark & ? & ? & \dots & ? & \text{(medium adopter)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ \checkmark & ? & ? & ? & \dots & ? & \text{(early adopter)} \end{pmatrix}$$



## Horizontal regression and unconfoundedness : thin matrix ( $N \gg T$ )

$$\mathbf{Y} = \begin{pmatrix} \checkmark & \checkmark & \checkmark \\ \vdots & \vdots & \vdots \\ \checkmark & \checkmark & \checkmark \\ \checkmark & \checkmark & ? \\ \vdots & \vdots & \vdots \\ \checkmark & \checkmark & ? \end{pmatrix}.$$

1. Regress the last period outcome on the lagged outcomes. (among untreated)
2. Predict the missing POs using the estimated regression.

$$\forall (i, T) \in \mathcal{M}, \hat{Y}_{iT} = \hat{\beta}_0 + \sum_{t=1}^{T-1} \hat{\beta}_t Y_{it}, \text{ where } \hat{\beta} = \arg \min_{\beta} \sum_{i:(i,T) \in \mathcal{O}} (Y_{iT} - \beta_0 - \sum_{t=1}^{T-1} \beta_t Y_{it})^2.$$

→ Nonparametrically,

## Vertical regression and synthesis control : fat matrix ( $T \gg N$ )

$$\mathbf{Y} = \begin{pmatrix} \checkmark & \checkmark & \dots & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \dots & \checkmark & \checkmark & \dots & \checkmark \\ \checkmark & \checkmark & \dots & \checkmark & ? & \dots & ? \end{pmatrix}$$

1. Regress the outcomes for treated unit prior to the treatment on the outcomes for the control units in the same periods.
2. Predict the missing POs using the estimated regression.

$$\forall (N, t) \in \mathcal{M}, \hat{Y}_{Nt} = \hat{\gamma}_0 + \sum_{i=1}^{N-1} \hat{\gamma}_i Y_{it}, \text{ where } \hat{\gamma} = \arg \min_{\gamma} \sum_{t:(N,t) \in \mathcal{O}} (Y_{Nt} - \gamma_0 - \sum_{i=1}^{N-1} \gamma_i Y_{it})^2.$$

→ Vertical regression is generalization of ADH(2010) in that it relaxes two restrictions :

- the coefficients  $\hat{\gamma}$  are nonnegative. (Interpretability ; What is a negative weight?)
- the intercept in this regression is 0. (This is seen to be plausible in recent literatures.)

# Matrix Completion

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- Under no covariates, we model the  $N \times T$  matrix of complete matrix  $\mathbf{Y}$  as

$$\mathbf{Y} = \mathbf{L}^* + \epsilon, \text{ where } \mathbb{E}[\epsilon | \mathbf{L}^*] = 0.$$

## Assumption 1

- $\epsilon$  is independent of  $\mathbf{L}^*$
- The element of  $\epsilon$  are  $\sigma$ -sub-Gaussian and independent each other.  
 $\Leftrightarrow \forall t, \mathbb{E}[\exp(t\epsilon)] \leq \exp(\frac{\sigma^2 t^2}{2}).$

- The goal is to estimate the matrix  $\mathbf{L}^*$ . (low-rank assumption)  
→ Note that two types<sup>2</sup> of fixed effects are included.

<sup>2</sup>Interactive fixed effect も低ランクとしていいが, rank が明示的でないため具体的には考えない.



# **Theoretical Bounds for the Estimation Error**

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## Two illustrations

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## References

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# References

- Susan Athey, Mohsen Bayati, Nikolay Doudchenko, Guido Imbens, and Khashayar Khosravi (2021), *Matrix Completion Methods for Causal Panel Data Models*, Journal of the American Statistical Association.
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