

## TOPICS IN LANGLANDS PROGRAM (II)

The first half of the course is taught by Yihang Zhu, on Eisenstein series. The second half is taught by Bin Xu, on trace formulas.

**References.** For background on automorphic forms/representations and the Langlands program, we recommend [GH23], or Yihang Zhu's notes [Zhu] which follow the former closely. The book [GH23] also contains an introduction to trace formulas.

For Eisenstein series, the original source is Langlands' paper [Lan06]. Most of the foundations are found in the comprehensive book [MW95], which also contains proofs of all the main results in the subject. Also useful is the book [LW13]. For the proof of the meromorphic continuation of the Eisenstein series, we will explain the new and simpler proof in [BL24].

### 1. WEEK 1

- General motivations of trace formulas [Art05].
- The idea of discrete and continuous spectrum [GH23, §3.8], [Zhu, §4.1].
- Reductive groups, root data [Zhu, §1, §2].

### 2. WEEK 2

- Parabolic subgroups and Levi components. Classification of standard parabolic subgroups. [Zhu, §3.1].
- The analytic topology on points of a reductive group, especially the adelic group [Zhu, §3.2].
- The Harish-Chandra homomorphism and related definitions [BL24, §2.1], [MW95, §I.1.4].
- Siegel sets [MW95, §I.2.1], see also [LW13, §3.5].

### 3. WEEK 3

- Example of Siegel sets for  $SL_2$  over  $\mathbb{Q}$ . See for instance [Art05, Figure 8.3].
- Relative Siegel sets [MW95, §I.2.1].
- Functions of moderate growth [MW95, §I.2.5] and functions of rapid decay [MW95, §I.2.12].
- Definition of automorphic forms [MW95, §I.2.17].
- Main theorem about Eisenstein series [BL24, §2.3].
- Proof of convergence of Eisenstein series in the case of minimal parabolic [God67, §3].

**Exercise 3.1.** Let  $G$  be a Hausdorff locally compact topological group, and  $\Gamma$  a closed subgroup.

- (1) Assume that  $\Gamma \backslash G$  is compact. Show that there exists a relatively compact open subset  $R \subset G$  such that  $G = \Gamma R$ .
- (2) The compactness of  $\Gamma \backslash G$  is equivalent to the existence of a compact set  $S \subset G$  such that  $G = \Gamma S$ .

**Exercise 3.2.** Let  $U$  be a connected unipotent linear algebraic group over  $\mathbb{Q}$ . By general theory of linear algebraic groups, we know there exist closed subgroups

$$1 = U_0 \subset U_1 \subset \cdots \subset U_n = U(\mathbb{A})$$

such that  $U_{i-1}$  is normal in  $U_i$  and  $U_i/U_{i-1} \cong \mathbb{A}$ . Moreover, the image of  $U_i \cap U(\mathbb{Q}) \rightarrow U_i/U_{i-1} \cong \mathbb{A}$  is  $\mathbb{Q} \subset \mathbb{A}$ .

- (1) Using this, show that  $U(\mathbb{Q}) \backslash U(\mathbb{A})$  is compact.
- (2) Exhibit a choice of the  $U_i$ 's when  $U$  is the subgroup of  $\mathrm{GL}_n$  consisting of upper triangular matrices with 1's on the diagonal. Justify your answer.

**Exercise 3.3.** Let  $G$  be a reductive group over  $\mathbb{Q}$ , with fixed minimal parabolic and Levi decomposition  $P_0 = M_0 U_0$ . Use the usual Siegel sets and the previous two exercises to prove the following statement about relative Siegel sets: For every standard parabolic  $P = MU$ , there is a compact subset  $\omega \subset P(\mathbb{A})$ , and  $t_0 > 0$  such that

$$G(\mathbb{A}) = P(\mathbb{Q})\omega\{a \in A_{M_0} \mid \langle \alpha, a \rangle > t_0, \forall \alpha \in \Delta_0^M\}K.$$

(In addition, show that  $\omega$  can be chosen inside  $P_0(\mathbb{A})$ , using that  $U \subset U_0$ .)

#### 4. WEEK 4

(Sep 30.)

- Continuation of the proof of convergence. Proof of moderate growth. For the two proofs we mainly follow [God67, §3]. Some comments on this reference:
  - Note that in [God67], automorphic forms are right  $P(\mathbb{Q})$ - or  $G(\mathbb{Q})$ -invariant, and also the Siegel set  $S$  is such that  $SG(\mathbb{Q}) = G(\mathbb{A})$ . Thus the order of the three factors whose product is  $S$  is reversed. The Eisenstein series is defined as a summation over  $G(\mathbb{Q})/P(\mathbb{Q})$ , as opposed to  $P(\mathbb{Q}) \backslash G(\mathbb{Q})$ . His convention and ours can be easily translated into each other by replacing  $g$  by  $g^{-1}$ .
  - Above [God67, (3.9)], the assertion about  $L(gg')/L(gg'')$  contains typos. The correct assertion should be: The function  $L(g'g)/L(g''g)$  depends on  $g$  only via  $gP(\mathbb{A})$ , and hence it is uniformly bounded in  $g$  since  $g'$  and  $g''$  move in a compact set. (In class, we had this assertion with the reversed order of multiplication, since our  $L$  is related to Godement's  $L$  by  $g \mapsto g^{-1}$ .)
  - We only follow the proof in [God67, §3] in the special case where  $P$  is minimal. In [God67, §3],  $P$  can be non-minimal, but there is the assumption in Theorem 3 that  $L$  is bounded on every subset of  $G(\mathbb{A})$  which is compact modulo  $P(\mathbb{A})^1$ . If  $P$  is minimal, then the last assumption is automatic. Indeed, as we saw in class, the minimality of  $P$  implies that  $L$  is majorized by  $m_P(\cdot)^\lambda$ . But  $m_P(\cdot)^\lambda$  factors through  $P(\mathbb{A})^1 \backslash G(\mathbb{A})$ , and so it is bounded on every subset which is compact modulo  $P(\mathbb{A})^1$  on the left. (Godement's  $L(g)$  is our  $L(g^{-1})$ , and for Godement, “modulo  $P(\mathbb{A})^1$ ” means “modulo  $P(\mathbb{A})^1$  on the right”.)
- Classical Eisenstein series for  $\mathrm{SL}_2$ . See [Gar18, §2.8].

**Exercise 4.1.** Assume  $G$  is a split reductive group over  $\mathbb{Q}$ , with fixed minimal parabolic and Levi decomposition  $P_0 = M_0 U_0$ . Let  $\varpi : M_0 \rightarrow \mathbb{G}_m$  be a fundamental weight. Let  $\rho : G \rightarrow \mathrm{GL}_n$  be the corresponding highest weight representation, with highest weight vector  $e_1$ . Thus  $\rho(P_0)$  stabilizes  $e_1$ , and the function  $L = m_{P_0}(\cdot)^\varpi : G(\mathbb{A}) \rightarrow \mathbb{R}_{>0}$  sends every  $p \in P_0(\mathbb{A})$  to the idelic norm of the  $(1, 1)$ -th entry of  $\rho(p) \in \mathrm{GL}_n(\mathbb{A})$ .

- (1) Show that there exist constants  $A, B > 0$  such that

$$A\|\rho(h^{-1})e_1\| \leq L(h)^{-1} \leq B\|\rho(h^{-1})e_1\|$$

for all  $h \in G(\mathbb{A})$ . Here, for a (column) vector  $v = (a_1, \dots, a_n)^t \in \mathbb{A}^n$ , we define  $\|v\| = \prod_v \max_i |a_{i,v}|_v$ , where  $v$  runs over all places of  $\mathbb{Q}$ .

- (2) Show that  $\|\mathrm{GL}_n(\mathbb{Q})e_1\| \subset [1, +\infty)$ .  
 (3) Show that for every compact set  $U$  in  $\mathrm{GL}_n(\mathbb{A})$ , there exist constants  $B > A > 0$  such that for every  $v \in \mathbb{A}^n$ , we have  $\|Uv\| \subset [A\|v\|, B\|v\|]$ .  
 (4) Use the previous parts to give a new proof of the following statement (which strengthens the Key Lemma in class): For each compact set  $C$  in  $G(\mathbb{A})$ , there exists  $D > 0$  such that  $L(G(\mathbb{Q})C) \subset (0, D)$ .

## 5. WEEK 5

(Oct 9.)

- Classical Eisenstein series for  $\mathrm{SL}_2$ , convergence and moderate growth (in the non-adelic language). The proofs are essentially equivalent to our more general proofs using the adelic language. See [Gar18, §2.8] for the proof of convergence in non-adelic language.
- Convergence and moderate growth of Eisenstein series for non-minimal  $P$ , see [MW95, §II.1.5].
- The intertwining operators: beginning. [BL24, §2.2], [MW95, §II.1.6].

**Exercise 5.1.** Let  $U$  be a compact set in the upper half plane  $\mathcal{H}$ . Show that there exist  $A, B > 0$  such that

$$A(c^2 + d^2) \leq |c\tau + d|^2 \leq B(c^2 + d^2)$$

for all  $c, d \in \mathbb{R}, \tau \in U$ .

**Exercise 5.2.** Let  $P$  be the upper triangular subgroup in  $\mathrm{SL}_2$ . Consider the Eisenstein series

$$E(\tau, \lambda) = \sum_{\gamma \in P(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(\gamma\tau)^\lambda = \frac{1}{2} \sum_{c, d \in \mathbb{Z}, \gcd(c, d)=1} \frac{\mathrm{Im}(\tau)^\lambda}{|c\tau + d|^{2\lambda}}, \quad \tau \in \mathcal{H}, \lambda \in \mathbb{C}.$$

In class we showed that the series converges absolutely and locally uniformly if  $\mathrm{Re}(\lambda) > 1$ . In the following we always assume  $\lambda$  satisfies this.

- (1) Show that for fixed  $y > 0$ , the function

$$\mathbb{R} \rightarrow \mathbb{C}, x \mapsto E(x + iy, \lambda)$$

is periodic of period 1.

- (2) Consider the constant term of the Fourier expansion of the above function, defined as

$$\int_0^1 E(x + iy, \lambda) dx.$$

Show that it is of the form  $y^\lambda + c(\lambda)y^{1-\lambda}$ , where  $c(\lambda)$  is a function in  $\lambda$ , independent of  $x, y$ .

- (3) Show that

$$c(\lambda) = \int_{\mathbb{R}} (x^2 + 1)^{-\lambda} dx \zeta(2\lambda - 1) \zeta(2\lambda)^{-1}.$$

(In fact  $\int_{\mathbb{R}} (x^2 + 1)^{-\lambda} dx = \sqrt{\pi} \Gamma(\lambda - \frac{1}{2}) \Gamma(\lambda)^{-1}$ .)

**Hint:** you may need to use the following simple observation: For fixed  $c \in \mathbb{Z}$ , an integer  $d$  is coprime to  $c$  if and only if  $d \pm c$  are coprime to  $c$ .

**Exercise 5.3.** Classify all pairs of standard parabolic subgroups  $P = MU, P' = M'U'$  in  $\mathrm{GL}_n$  (over any field  $k$ ) such that  $M$  and  $M'$  are conjugate under  $\mathrm{GL}_n(k)$ .

## 6. WEEK 6

(Oct 14)

- Convergence of the integral defining the intertwining operator [MW95, §II.1.6].
- Statement of meromorphic continuation of intertwining operators [BL24, §2.3].
- Meromorphic functions in locally convex topological vector spaces [BL24, §3.1].

(Oct 16)

- Principle of Meromorphic Continuation (PMC) [BL24, §3, Appendix A].
- $\mathrm{SL}_2$  case: eigen property of Eisenstein series under convolution [BL24, §5, Claim 1 (1)]

## REFERENCES

- [Art05] James Arthur. An introduction to the trace formula. *Harmonic analysis, the trace formula, and Shimura varieties*, 4:1–263, 2005. [1](#)
- [BL24] Joseph Bernstein and Erez Lapid. On the meromorphic continuation of eisenstein series. *Journal of the American Mathematical Society*, 37(1):187–234, 2024. [1](#), [3](#), [4](#)
- [Gar18] Paul Garrett. *Modern analysis of automorphic forms by example*, volume 1. Cambridge University Press, 2018. available at [https://www-users.cse.umn.edu/~garrett/m/v/current\\_version.pdf](https://www-users.cse.umn.edu/~garrett/m/v/current_version.pdf). [2](#), [3](#)
- [GH23] Jayce R Getz and Heekyoung Hahn. *An Introduction to Automorphic Representations: With a View Toward Trace Formulae*, volume 300. Springer Nature, 2023. [1](#)
- [God67] Roger Godement. Introduction à la théorie de langlands. *Séminaire Bourbaki*, 10:115–144, 1967. [1](#), [2](#)
- [Lan06] Robert P Langlands. *On the functional equations satisfied by Eisenstein series*, volume 544. Springer, 2006. available at <https://publications.ias.edu/rpl/paper/39>. [1](#)
- [LW13] Jean-Pierre Labesse and Jean-Loup Waldspurger. *La formule des traces tordue d'après le Friday Morning Seminar*, volume 31. American Mathematical Soc., 2013. [1](#)
- [MW95] Colette Moeglin and Jean-Loup Waldspurger. *Spectral decomposition and Eisenstein series: a paraphrase of the scriptures*. Number 113. Cambridge University Press, 1995. [1](#), [3](#), [4](#)
- [Zhu] Yihang Zhu. Introduction to the Langlands program. <https://yhzmath.github.io/IL2024.pdf>. [1](#)