

Abelian varieties.

§ Basic defns & facts.

S scheme, a group scheme over S is a scheme

$G \rightarrow S$ together w/ gp str

$m: G \times_S G \rightarrow G$ mult'n

$e: S \rightarrow G$

neutral section

$i: G \rightarrow G$ inversion

} all S -morphs.

satisf. usual axioms for a group.

Eg, $\forall S$ -sch T , $G(T)$ has gp str.

functorial in T .

(Yoneda).

Def. An abelian scheme / S is a proper sm.

gp sch. whose geom. fibers are connected.

(If $S = \text{Spec } k$, say abelian variety).

Thm (Rigidity). A/S ab sch. G/S gp sch. separated / S

Any S -sch map $A \rightarrow G$ preserving neutral sections

" $1 \mapsto 1$ " is a gp hom.

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Cor. Ab schs are commutative. Pf. i is homo \mathbb{D} .

Write gp str as +.

Ex. An elliptic curve / S is a proper sm S -sch E , whose geom. fibers are conn. proj sm curves of genus 1, together w/ distinguished section

$$e: S \rightarrow E.$$

Fact: E has unique str. of ab sch w/ e the neutral section.

$$S = \text{Spec } k, k \cong \bar{k}. P + Q = R$$

$$(\Leftrightarrow [P] + [\bar{Q}] \sim [R] + [e],$$

Fact ("Abel") this gives gp structure.

$$\begin{aligned} (E &\xrightarrow{\sim} \text{Jac } E = D_n^0(E)/\mathbb{G}_m) \\ P &\mapsto [P] - [e]. \end{aligned}$$

Ex. Jacobian of a sm. proj. curve.

Fact. If S is spec of a field, or normal, ^(Weil) or ^(Groth.)

Thm. If S is spec of a field, or normal,
then ab schs / S are projective.
Not true in general.

We'll need "polarized ab. schs". By def'n
they are (locally) proj. / S .

Thm (hard) $A/k = \overline{k}$ L ample line bdry $\Rightarrow L^3$ very ample
 S isogenies.

From now on, S is loc. noeth.

Def. $\varphi: A \rightarrow B$ hom. of ab. sch. is called
an isogeny, if it's quasi-finite & surj.
Lem. Isogenies are finite flat.

$\hookrightarrow \varphi: A \rightarrow B$ is

$\ker(\varphi)$ is \subset finite flat gp sch / S .

$\tilde{\Gamma}$
of locally constant order d

Rep. by a gp sch
for any hom of
gps / S .

(i.e. locally, $\mathcal{O}_{\ker(\varphi)}$ is a

a finite free \mathcal{O}_S -mod.
of rk d).

$d =: \deg(\varphi)$.

Rmk. $S = \text{Spec } k$. $k = \bar{k}$, $\varphi: A \rightarrow B$ is $\bar{\varphi}$

1) $\Rightarrow \dim A = \dim B$

2) $d = \deg$ of ext of fn fields

3)

If $d \neq 0$ in k

$\Rightarrow \ker(\varphi)(k)$ has d elts.

$\varphi: A(k) \rightarrow B(k)$ is d -to-1.

(finite étale).

Ex. $n \in \mathbb{Z} \setminus \{0\}$. $[n]: A \rightarrow A$, $x \mapsto \underbrace{x + \dots + x}_n$
($n < 0$, use $-$) is isof. $\deg = n^{2g}$. $g = \dim A$.

(1/6. $\mathbb{C}/1 \xrightarrow{n} \mathbb{C}/1$
kernel $\subseteq (\mathbb{Z}/n\mathbb{Z})^{2g}$).
pf later.

6 Line bds.

$A/k = \bar{k}$

Thm of Cube:

1. line bdl on A .

L line ball on A.

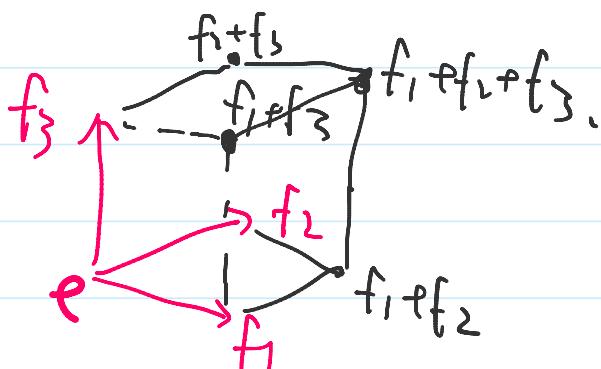
$T \xrightarrow{f_i} A$ k-maps $i=1, 2, 3,$

Then

$$(f_1 + f_2 + f_3)^* L^{-1} \otimes (f_1 + f_2)^* L \otimes$$

$$(f_1 + f_3)^* L \otimes (f_2 + f_3)^* L \otimes f_1^* L^{-1} \otimes \\ f_2^* L^{-1} \otimes f_3^* L^{-1}$$

is trivial.



$$\frac{n-n^2}{2}$$

Cor. L on A , $[n]^* L \cong L^{n^2} \otimes (L \otimes (-)^* L^{-1})$.

Def. We call L symm if $(-)^* L \cong L$.

$$\text{Thm } [n]^* L \cong L^{n^2}.$$

Pf. $n=0$ or 1 ✓

use induction to go up and down.

Apply Thm of cube to $T=A$

$$f_1 = [n+1] \quad f_2 = [1] \quad f_3 = [-1],$$

(ExG) \square

Pf that $[n]$ is isof of $\deg n^{2g}$:

Take ample L on A (b/c A is proj.)

Replace L by $L \otimes [1]^* L$.

\Rightarrow WMA L is symmetric.

$$\Rightarrow [n]^* L \cong L^{n^2}. \quad (\star)$$

If we know $[n]$ is isof - (\star) gives

$$\deg[n] = n^{2g} \quad \text{by standard tools}$$

(Hilbert Poly or Intersection Theory)

To show: $\sum \ker[n]$ is finite.

$$[n]^* L \cong L^{n^2}, \text{ ample. so its restr. to } Z$$

is ample. But this restr is trivial since

$$[n]: Z \rightarrow \{*\} \subset A. \quad \square.$$

Thm of Square. $a \in A(k)$, $t_a : A \rightarrow A$ transl.

for a . L b.b. on A . $\forall a, b \in A$.

Propn: L b.b. on A . $\forall a, b \in A$,

$t_{a+b}^* L \otimes t_a^* L^\perp \otimes t_b^* L^\perp \otimes L$ is trivial.

$$\begin{array}{ccc} t_a & \xrightarrow{\quad} & t_{a+b} \\ \downarrow & & \downarrow \\ t_e = id & \xrightarrow{\quad} & t_b \end{array}$$

E.g. define $\Lambda(L) : A(k) \rightarrow \text{Pic } A = \{1.b. \text{ in } A\}/\sim$

$$a \mapsto t_a^* L \otimes L^\perp$$

then $\Lambda(L)$ is a homo.

(by PFS) 1) In form of chbe, take

$$T = A \cdot f_1 = t_e, f_2 = t_a, f_3 = t_b \quad \square$$

2) Pic A has str. of sep gp sch/ k . Check: $\Lambda(L) : 0 \rightarrow 0$.

Rigidity $\Rightarrow \Lambda(L)$ is homo.

Rmk,

$\Lambda \text{ comm} \Rightarrow \Lambda(L)$ factors thru central conn. comp of $\text{Pic } A$,
which is an ab.var! A^\vee ($=$ dual of A).

\S Dual ab. sch.

A/S . Define $\text{Pic}_{A/S} : (S\text{-Schs}) \rightarrow (\text{Ab Gps})$

$$T \mapsto \left\{ (L, \rho) \mid L : \text{l. b. on } A_T := A_S^{\times} T, \rho: e_T^* L \hookrightarrow \mathcal{O}_T \right\}$$

$e_T: T \rightarrow A_T$ neutral section

((L, ρ) is called a rigidified line bundle).

ex. $T \rightarrow S$ a geometric point.

$$\text{Pic}_{A/S}(T) = \text{Pic}(A_T).$$

Thm. $\text{Pic}_{A/S}$ is representable by a group scheme / S .

let $\text{Pic}_{A/S}^0 = A^\vee$ be the max'l subgp sch. which has conn. geom. fibers. Then A^\vee is an ab. sch / S .

Pf. Three constructions.

① Mumford [Abelian Varieties]. Explicitly construct A^\vee by dividing A by a finite subgp (Scheme)

works only for $S = \text{Spec } k$. and specifically for A.V.

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(2) Grothendieck: projective methods, works for A/S

(locally) projective

$\hookrightarrow A^\vee$ is also projective (Rep. of $\mathbb{P}^1_{\mathbb{Z}} \times S$.)

(3) General:

$X \rightarrow S$ proj., flat
geom fibers integral

Artin: A/S abelian algebraic space

$\hookrightarrow \text{Pic}_{A/S}, A^\vee$ Rep. by alg. sp.

Raynaud, Automatically, $A, \text{Pic}_{A/S}, A^\vee$ are all schemes. $A \otimes A^\vee$ one ab. sch.

Ref. [Faltung-Chai].

3.