TOPICS IN LANGLANDS PROGRAM (II)

The first half of the course is taught by Yihang Zhu, on Eisenstein series. The second half is taught by Bin Xu, on trace formulas.

References. For background on automorphic forms/representations and the Langlands program, we recommend [GH23], or Yihang Zhu's notes [Zhu] which follow the former closely. The book [GH23] also contains an introduction to trace formulas.

For Eisenstein series, the original source is Langlands' paper [Lan06]. Most of the foundations are found in the comprehensive book [MW95], which also contains proofs of all the main results in the subject. Also useful is the book [LW13]. For the proof of the meromorphic continuation of the Eisenstein series, we will explain the new and simpler proof in [BL24].

1. Week 1

- General motivations of trace formulas [Art05].
- The idea of discrete and continuous spectrum [GH23, §3.8], [Zhu, §4.1].
- Reductive groups, root data [Zhu, §1, §2].

2. Week 2

- Parabolic subgroups and Levi components. Classification of standard parabolic subgroups. [Zhu, §3.1].
- The analytic topology on points of a reductive group, especially the adelic group [Zhu, §3.2].
- The Harish-Chandra homomorphism and related definitions [BL24, §2.1], [MW95, §I.1.4].
- Siegel sets [MW95, §I.2.1], see also [LW13, §3.5].

3. Week 3

- Example of Siegel sets for SL_2 over $\mathbb{Q}$. See for instance [Art05, Figure 8.3].
- Relative Siegel sets [MW95, §I.2.1].
- Functions of moderate growth [MW95, §I.2.5] and functions of rapid decay [MW95, §I.2.12].
- Definition of automorphic forms [MW95, §I.2.17]
- Main theorem about Eisenstein series [BL24, §2.3].
- Proof of convergence of Eisenstein series in the case of minimal parabolic [God67, §3].

Exercise 3.1. Let G be a Hausdorff locally compact topological group, and Γ a closed subgroup.

(1) Assume that $\Gamma\backslash G$ is compact. Show that there exists a relatively compact open subset $R \subset G$ such that $G = \Gamma R$.
(2) The compactness of $\Gamma\backslash G$ is equivalent to the existence of a compact set $S \subset G$ such that $G = \Gamma S$.

Exercise 3.2. Let U be a connected unipotent linear algebraic group over $\mathbb{Q}$. By general theory of linear algebraic groups, we know there exist closed subgroups

$$1 = U_0 \subset U_1 \subset \cdots \subset U_n = U(\mathbb{A})$$

such that U_{i-1} is normal in U_i and $U_i/U_{i-1} \cong \mathbb{A}$. Moreover, the image of $U_i \cap U(\mathbb{Q}) \to U_i/U_{i-1} \cong \mathbb{A}$ is $\mathbb{Q} \subset \mathbb{A}$.

(1) Using this, show that $U(\mathbb{Q})\backslash U(\mathbb{A})$ is compact.
(2) Exhibit a choice of the U_i's when U is the subgroup of GL_n consisting of upper triangular matrices with 1's on the diagonal. Justify your answer.

Exercise 3.3. Let G be a reductive group over $\mathbb{Q}$, with fixed minimal parabolic and Levi decomposition $P_0 = M_0U_0$. Use the usual Siegel sets and the previous two exercises to prove the following statement about relative Siegel sets: For every standard parabolic $P = MU$, there is a compact subset $\omega \subset P(\mathbb{A})$, and $t_0 > 0$ such that

$$G(\mathbb{A}) = P(\mathbb{Q})\omega\{a \in A_{M_0} \mid \langle \alpha, a\rangle > t_0, \forall \alpha \in \Delta_0^M\}K.$$

(In addition, show that ω can be chosen inside $P_0(\mathbb{A})$, using that $U \subset U_0$.)

4. Week 4

(Sep 30.)

- Continuation of the proof of convergence. Proof of moderate growth. For the two proofs we mainly follow [God67, §3]. Some comments on this reference:
  - Note that in [God67], automorphic forms are right $P(\mathbb{Q})$- or $G(\mathbb{Q})$-invariant, and also the Siegel set S is such that $SG(\mathbb{Q}) = G(\mathbb{A})$. Thus the order of the three factors whose product is S is reversed. The Eisenstein series is defined as a summation over $G(\mathbb{Q})/P(\mathbb{Q})$, as opposed to $P(\mathbb{Q})\backslash G(\mathbb{Q})$. His convention and ours can be easily translated into each other by replacing g by g^{-1}.
  - Above [God67, (3.9)], the assertion about $L(gg')/L(gg'')$ contains typos. The correct assertion should be: The function $L(g'g)/L(g''g)$ depends on g only via $gP(\mathbb{A})$, and hence it is uniformly bounded in g since g' and g'' move in a compact set. (In class, we had this assertion with the reversed order of multiplication, since our L is related to Godement's L by $g \mapsto g^{-1}$.)
  - We only follow the proof in [God67, §3] in the special case where P is minimal. In [God67, §3], P can be non-minimal, but there is the assumption in Theorem 3 that L is bounded on every subset of $G(\mathbb{A})$ which is compact modulo $P(\mathbb{A})^1$. If P is minimal, then the last assumption is automatic. Indeed, as we saw in class, the minimality of P implies that L is majorized by $m_P(\cdot)^\lambda$. But $m_P(\cdot)^\lambda$ factors through $P(\mathbb{A})^1\backslash G(\mathbb{A})$, and so it is bounded on every subset which is compact modulo $P(\mathbb{A})^1$ on the left. (Godement's $L(g)$ is our $L(g^{-1})$, and for Godement, "modulo $P(\mathbb{A})^1$" means "modulo $P(\mathbb{A})^1$ on the right".)
- Classical Eisenstein series for SL_2. See [Gar18, §2.8].

Exercise 4.1. Assume G is a split reductive group over $\mathbb{Q}$, with fixed minimal parabolic and Levi decomposition $P_0 = M_0U_0$. Let $\varpi : M_0 \to \mathbb{G}_m$ be a fundamental weight. Let $\rho : G \to \mathrm{GL}_n$ be the corresponding highest weight representation, with highest weight vector e_1. Thus $\rho(P_0)$ stabilizes e_1, and the function $L = m_{P_0}(\cdot)^\varpi : G(\mathbb{A}) \to \mathbb{R}_{>0}$ sends every $p \in P_0(\mathbb{A})$ to the idelic norm of the $(1,1)$-th entry of $\rho(p) \in \mathrm{GL}_n(\mathbb{A})$.

(1) Show that there exist constants $A, B > 0$ such that
$$A\|\rho(h^{-1})e_1\| \le L(h)^{-1} \le B\|\rho(h^{-1})e_1\|$$
for all $h \in G(\mathbb{A})$. Here, for a (column) vector $v = (a_1, \cdots, a_n)^t \in \mathbb{A}^n$, we define $\|v\| = \prod_v \max_i |a_{i,v}|_v$, where v runs over all places of $\mathbb{Q}$.

(2) Show that $\|\operatorname{GL}_n(\mathbb{Q})e_1\| \subset [1, +\infty)$.

(3) Show that for every compact set U in $\operatorname{GL}_n(\mathbb{A})$, there exist constants $B > A > 0$ such that for every $v \in \mathbb{A}^n$, we have $\|Uv\| \subset [A\|v\|, B\|v\|]$.

(4) Use the previous parts to give a new proof of the following statement (which strengthens the Key Lemma in class): For each compact set C in $G(\mathbb{A})$, there exists $D > 0$ such that $L(G(\mathbb{Q})C) \subset (0, D)$.

5. Week 5

(Oct 9.)

- Classical Eisenstein series for SL_2, convergence and moderate growth (in the non-adelic language). The proofs are essentially equivalent to our more general proofs using the adelic language. See [Gar18, §2.8] for the proof of convergence in non-adelic language.
- Convergence and moderate growth of Eisenstein series for non-minimal P, see [MW95, §II.1.5].
- The intertwining operators: beginning. [BL24, §2.2], [MW95, §II.1.6].

Exercise 5.1. Let U be a compact set in the upper half plane $\mathcal{H}$. Show that there exist $A, B > 0$ such that
$$A(c^2 + d^2) \le |c\tau + d|^2 \le B(c^2 + d^2)$$
for all $c, d \in \mathbb{R}, \tau \in U$.

Exercise 5.2. Let P be the upper triangular subgroup in SL_2. Consider the Eisenstein series
$$E(\tau, \lambda) = \sum_{\gamma \in P(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} \operatorname{Im}(\gamma\tau)^\lambda = \frac{1}{2} \sum_{c,d \in \mathbb{Z}, \gcd(c,d)=1} \frac{\operatorname{Im}(\tau)^\lambda}{|c\tau + d|^{2\lambda}}, \quad \tau \in \mathcal{H}, \lambda \in \mathbb{C}.$$
In class we showed that the series converges absolutely and locally uniformly if $\operatorname{Re}(\lambda) > 1$. In the following we always assume λ satisfies this.

(1) Show that for fixed $y > 0$, the function
$$\mathbb{R} \to \mathbb{C}, x \mapsto E(x + iy, \lambda)$$
is periodic of period 1.

(2) Consider the constant term of the Fourier expansion of the above function, defined as
$$\int_0^1 E(x + iy, \lambda) dx.$$
Show that it is of the form $y^\lambda + c(\lambda)y^{1-\lambda}$, where $c(\lambda)$ is a function in λ, independent of x, y.

(3) Show that
$$c(\lambda) = \int_{\mathbb{R}} (x^2 + 1)^{-\lambda} dx \; \zeta(2\lambda - 1)\zeta(2\lambda)^{-1}.$$
(In fact $\int_{\mathbb{R}} (x^2 + 1)^{-\lambda} dx = \sqrt{\pi}\Gamma(\lambda - \frac{1}{2})\Gamma(\lambda)^{-1}$.)

Hint: you may need to use the following simple observation: For fixed $c \in \mathbb{Z}$, an integer d is coprime to c if and only if $d \pm c$ are coprime to c.

Exercise 5.3. Classify all pairs of standard parabolic subgroups $P = MU, P' = M'U'$ in GL_n (over any field k) such that M and M' are conjugate under $\mathrm{GL}_n(k)$.

6. Week 6

(Oct 14)

- Convergence of the integral defining the intertwining operator [MW95, §II.1.6].
- Statement of meromorphic continuation of intertwining operators [BL24, §2.3].
- Meromorphic functions in locally convex topological vector spaces [BL24, §3.1].

(Oct 16)

- Principle of Meromorphic Continuation (PMC) [BL24, §3, Appendix A].
- SL_2 case: eigen property of Eisenstein series under convolution [BL24, §5, Claim 1 (1)]

7. Week 7

(Oct 21) Finish of proof of meromorphic continuation in the SL_2 case. [BL24, §5].

Exercise 7.1. Let $r \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ be such that $e^r > |\lambda|$. Prove the last statement in the proof of Claim 4 in [BL24, §5]. Namely, the operator

$$\begin{aligned} L^2(\mathbb{R}_{>0}, e^{-2rx}dx) &\longrightarrow L^2(\mathbb{R}_{>0}, e^{-2rx}dx) \oplus L^2([0,1]) \\ f &\longmapsto (f(x+1) - \lambda f(x), f|_{[0,1]}) \end{aligned}$$

is a strict embedding (i.e., an embedding inducing a homeomorphism onto the image).

(Oct 23)

- Cuspidal components and leading cuspidal components [BL24, §6.2, §6.3].
- Unique characterization of Eisenstein series in terms of leading cuspidal components (statement only) [BL24, §6.9].
- Preparations from Weyl sets, Bruhat decomposition, etc. [BL24, §6.7]. See also [GH23, §10.4], [Ren10, §V.4.6].
- Statement of Geometric Lemma (i.e. computation of constant term of Eisenstein series) [BL24, §6.10]. See also [GH23, §10.4].

Exercise 7.2. Consider $G = \mathrm{GL}_3$. Consider standard parabolic subgroups $P = P_{1,2}, Q = P_{2,1}$. Here $P_{m,n}$ is the block upper triangular subgroup with diagonal block sizes (m, n).

(1) Compute ${}_QW_P$ (under the standard identification $W \cong S_3$).
(2) Let $w \in {}_QW_P$. Find Q_w, the standard parabolic subgroup whose Levi component is $M_Q \cap wM_Pw^{-1}$. Also find P_w, the standard parabolic subgroup whose Levi component is $M_P \cap w^{-1}M_Qw$.
(3) Prove directly that $U_{Q_w} = (M_Q \cap wU_Pw^{-1}) \ltimes U_Q$.
(4) Prove directly that $U_Q \cap wPw^{-1} = U_Q \cap wU_{P_w}w^{-1}$.
(5) For any base field F, prove directly that $U_{Q_w}(F) = (U_{Q_w}(F) \cap wU_{P_w}(F)w^{-1}) \cdot U_Q(F)$.

8. Week 8

(Oct 28)

- Proof of Geometric Lemma [BL24, §6.10]. See also [GH23, §10.4].
- Sketch of the main theorem on meromorphic continuation. See [BL24, Thm. 7.2, §§8.1–8.4].
- Meromorphic continuation of intertwining operators (sketch) [BL24, §8.5].

(Oct 30)

- Langlands' description of the continuous spectrum in terms of discrete spectra of Levi subgroups [Art05, §7].

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