AN INTRODUCTION TO THE LANGLANDS PROGRAM
PKU 2024 SUMMER SCHOOL

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1. Lecture 1

1.1. **Overview.** By class field theory, for F a global field we have the artin map $F^\times\backslash\mathbb{A}_F^\times \to \Gamma_F^{\mathrm{ab}}$, identifying Γ_F^{ab} with the maximal totally disconnected quotient of $F^\times\backslash\mathbb{A}_F^\times = \mathrm{GL}_1(F)\backslash\mathrm{GL}_1(\mathbb{A}_F)$. This suggests that one-dimensional representations of Γ_F are closely related to $\mathrm{GL}_1(F)\backslash\mathrm{GL}_1(\mathbb{A}_F)$. The Langlands conjectures suggest that n-dimensional representations of Γ_F are closely related to $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$. Similarly, generalizing local class field theory, n-dimensional representations of W_F (or rather Weil–Deligne representations) are closely related to $\mathrm{GL}_n(F)$.

To make these ideas precise, we need the notion of automorphic representations of G in the global case. Here G is a reductive group over a global field F. We will define a space $\mathcal{A}(G)$ of automorphic forms on G, which are certain functions on $G(F)\backslash G(\mathbb{A}_F)$. Roughly speaking, an automorphic representation is an irreducible subquotient representation of the $G(\mathbb{A}_F)$-representation on $\mathcal{A}(G)$ given by right translation. In the local case, the role of automorphic representations is played by all irreducible (smooth) representations of $G(F)$, for F a local field. The global and local theories are related, in a way similar to how global and local class field theories are related.

The Langlands program concerns, in both the global and local case, how these representations are related to the Galois side, and how these representations for *different* reductive groups G are related with each other. In the global case, these two questions are referred to as "reciprocity" and "functoriality".

The following two cases are the neatest to state and have been proven:

Theorem 1.1.1 (Local Langlands Correspondence for GL_n. Laumon–Rapoport–Stuhler for positive characteristic, Henniart, Harris–Taylor, and Scholze for characteristic zero)**.** *Let F be a local field. There is a canonical bijection between isomorphism classes of irreducible smooth representations of* $\mathrm{GL}_n(F)$ *and isomorphism classes of n-dimensional Frobenius semi-simple Weil–Deligne representations.*

Theorem 1.1.2 (Global Langlands Correspondence for GL_n over a function field. Drinfeld for $n = 2$, L. Lafforgue for general n)**.** *Let F be a global function field. Let ℓ be a prime unequal to* $\mathrm{char}(F)$. *There is a canonical bijection between isomorphism classes of cuspidal automorphic representations of* $\mathrm{GL}_n(\mathbb{A}_F)$ *and isomorphism classes of n-dimensional irreducible $\mathbb{Q}_\ell$-representations of Γ_F.*

The situation becomes much more complicated when F is a number field, or when G is a more general reductive group.

- For F local and G general, one only expects a finite-to-one map from the set of irreducible $G(F)$-representations to the set of certain Galois-theoretic data called L-parameters. When G is a classical group and $\mathrm{char}(F) = 0$, there have been various classical approaches (including global methods). Recently, such a map has been constructed unconditionally for all G, by Genestier–V. Lafforgue for positive characteristic local fields and by Fargues–Scholze for all local fields (but the latter work only constructs a weakened version, namely L-parameters are replaced by their semi-simplifications).
- For F a global function field and G general, the "automorphic-to-Galois" direction has been established by V. Lafforgue.
- The remaining case of a number field is perhaps the most profound part of the Langlands program!

The goal of the course is to discuss the fundamental concepts related to automorphic representations, state the main conjectures in the Langlands program, and survey the current status of these conjectures, mostly focusing on characteristic zero local and global fields. We will only consider the so-called arithmetic or classical Langlands program. The following topics are important in current research but will not be discussed:

- geometric Langlands in various settings (including the Fargues–Scholze setting, over the Fargues–Fontaine curve).
- mod p or p-adic local Langlands.

The main reference for the course is [1]. Another useful source is [3].

1.2. **Linear algebraic groups.** We formally develop the theory only over characteristic zero, and occasionally comment on some subtleties over positive characteristic.

Let k be a field of characteristic zero. A linear algebraic group over k is an affine k-variety G (i.e. an affine scheme of finite type over k which is geometrically reduced) equipped with morphisms $m : G \times_k G \to G$, $e : \operatorname{Spec} k \to G$, $i : G \to G$ satisfying the usual axioms for the multiplication, identity, and inversion in a group. For any k-algebra R, the set $G(R)$ is a group under these operations, and this defines a functor from k-algebras to groups.

Remark 1.2.1. In fact, over k of characteristic zero, every affine scheme of finite type equipped with a group structure is automatically geometrically reduced, thus a linear algebraic group. It is also automatically smooth. Over arbitrary k, geometric reducedness is an important axiom in the theory of linear algebraic groups, and it implies smoothness.

Example 1.2.2. $G = \mathrm{GL}_n = \{(g_{ij}, t) \in \mathbb{A}^{n^2+1} \mid \det(g_{ij}) \cdot t = 1\}$. We write $\mathbb{G}_m$ for GL_1, so $\mathbb{G}_m(R) = (R^\times, \times)$.

Example 1.2.3. $G = \mathbb{G}_a = \mathbb{A}^1$, $G(R) = (R, +)$.

Example 1.2.4. If l/k is a finite extension and G is a linear algebraic group over l, then there is a linear algebraic group $\mathrm{Res}_{l/k} G$ over l, called the Weil restriction of scalars of G, characterized by $(\mathrm{Res}_{l/k} G)(R) \cong G(R \otimes_k l)$ for any k-algebra R.

In the sequel, by a subgroup we always mean a closed subvariety (required to be geometrically reduced) which is also a subgroup.

By a finite dimensional linear representation of G (or simply a representation of G), we mean a homomorphism $\phi : G \to \mathrm{GL}(V) = \mathrm{GL}_n$ for some finite dimensional k-vector space V. It is called faithful if ϕ is a closed immersion.

Fact 1.2.5. *Any linear algebraic group G admits a faithful representation, i.e., it can be realized as a subgroup of GL_n for some n.*

The tangent space of G at the neutral element e has the structure of a Lie algebra over k of dimension equal to $\dim G$. Denote it by $\mathrm{Lie}\, G$. The construction $G \mapsto \mathrm{Lie}\, G$ is functorial. Moreover, it induces an injection (but not bijection) from the set of connected subgroups of G to the set of Lie subalgebras of $\mathrm{Lie}\, G$. See [2] §II.3, especially Prop. 3.22, for a discussion.

There is a natural adjoint representation $G \to \mathrm{GL}(\mathrm{Lie}\, G)$.

Let $\phi : G \to H$ be a homomorphism of linear algebraic groups. Then there is a normal subgroup $K = \ker(\phi)$ of G such that $K(R)$ is the kernel of $\phi(R) : G(R) \to H(R)$ for any k-algebra R. However, even if ϕ is surjective (equivalently $\phi(\bar{k}) : G(\bar{k}) \to H(\bar{k})$ is surjective), it does not follow that $\phi(k) : G(k) \to H(k)$ is surjective.

For any normal subgroup N of G (where normal means that $N(R)$ is normal in $G(R)$ for all k-algebras R), one can form the quotient group G/N such that $G \to G/N$ is surjective with kernel N. For instance, the center Z_G of G is a normal subgroup, characterized as the unique subgroup such that $Z_G(\bar{k})$ is the center of $G(\bar{k})$. The quotient G/Z_G is denoted by G^{ad}, called the adjoint group. For another example, the neutral connected component G^0 is always a normal subgroup, and G/G^0 is denoted by $\pi_0(G)$.

1.3. **Solvable and unipotent groups.**

Definition 1.3.1. Let G be a linear algebraic group. The derived subgroup G_{der} is the intersection of the kernels of all homomorphisms from G to commutative linear algebraic groups. (In fact G/G_{der} is a commutative linear algebraic group.) We say G is solvable, if taking successive derived subgroups of G leads to the trivial group after finitely many steps.

Let G be a linear algebraic group and $g \in G(\bar{k})$. There is a canonical decomposition $g = su = us$ with $s, u \in G(\bar{k})$ such that under every representation $\phi : G_{\bar{k}} \to \mathrm{GL}_n$ (defined over $\bar{k}$), $\phi(s)$ is semi-simple and $\phi(u)$ is unipotent (meaning that $\phi(u) - I_n$ is a nilpotent matrix). This is called the Jordan decomposition. If $g = s$ then we call g semi-simple, and if $g = u$ then we call g unipotent.

Definition 1.3.2. A linear algebraic group G is called unipotent if every element of $G(\bar{k})$ is unipotent.

Fact 1.3.3. *Let $\mathbb{U}_n$ be the subgroup of GL_n consisting of upper triangular matrices with 1's on the diagonal. Then a linear algebraic group is unipotent if and only if it is isomorphic (over k or over $\bar{k}$) to a subgroup of $\mathbb{U}_n$ for some n. Note that $\mathbb{U}_n$ is solvable, so every unipotent group is solvable.*

1.4. **Reductive groups.** Let G be a connected linear algebraic group. Suppose $\mathscr{P}$ is a property of subgroups of G, such as being normal in G or being solvable. Then by dimension considerations we know that every subgroup satisfying $\mathscr{P}$ is contained in a maximal subgroup satisfying $\mathscr{P}$, and contains a minimal subgroup satisfying $\mathscr{P}$.

Definition-Proposition 1.4.1. There is a unique maximal subgroup of G which is normal, connected, and solvable (resp. unipotent), called the radical (resp. unipotent radical), denoted by $R(G)$ (resp. $R_u(G)$). We call G semi-simple (resp. reductive) if $R(G) = 1$ (resp. $R_u(G) = 1$).

We have $R_u(G) \subset R(G)$, so semi-simple implies reductive. We have $R_u(G)_{\bar{k}} = R_u(G_{\bar{k}})$ (which is not true for non-perfect k), so G is reductive if and only if $G_{\bar{k}}$ is reductive. (Over positive characteristic, $R_u(G)_{\bar{k}}$ can be smaller than $R_u(G_{\bar{k}})$. One defines G to be reductive if and only if $R_u(G_{\bar{k}}) = 1$.)

Fact 1.4.2. *If G is reductive, then $R(G) = Z(G)^0$.*

Theorem 1.4.3. *Let G be a connected linear algebraic group. Then G is reductive if and only if every (equivalently, one faithful) finite dimensional representation of G is semi-simple.* (Warning: not true over positive characteristic.)

Theorem 1.4.4 (See [2, II.4.1, 4.2])**.** *Let G be a connected linear algebraic group. Then G is semi-simple if and only if* $\mathrm{Lie}\, G$ *is a semi-simple Lie algebra. (Not true for "semi-simple" replaced by "reductive".)*

Example 1.4.5. Examples of reductive groups: $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n = \mathrm{GL}_n^{\mathrm{ad}}, \mathrm{Sp}(V,\psi) = \mathrm{Sp}_{2g}$ for a symplectic space (V,ψ) over k, $\mathrm{SO}(V,\psi)$ for a quadratic space (V,ψ) over k, $\mathrm{U}(V,\psi)$ for a hermitian space (V,ψ) over a quadratic extension l/k.

For any (finite dimensional) simple algebra D over k, we also have a reductive group G such that $G(R) = (D \otimes_k R)^\times$. One often denotes G by $D^\times$. Note that if l is the center of D (thus l is a finite degree field extension of k) and $\dim_l D = n^2$, then

$$D \otimes_k \bar{k} \cong \prod_{\sigma \in \mathrm{Hom}_k(l,\bar{k})} D \otimes_{l,\sigma} \bar{k} \cong \prod_{\sigma} M_n(\bar{k}),$$

and so $G_{\bar{k}} \cong \prod_\sigma \mathrm{GL}_n$.

Example 1.4.6. The Weil restriction of scalars of a reductive group is again reductive.

Example 1.4.7. Let $\mathbb{B}_n$ be the subgroup of GL_n consisting of upper triangular matrices. Then $R_u(\mathbb{B}_n) = R_u(\mathbb{U}_n) = \mathbb{U}_n$, and so $\mathbb{B}_n$ and $\mathbb{U}_n$ are not reductive if $n > 1$.

1.5. **Tori.**

Definition 1.5.1. A linear algebraic group T is called a torus, if $T_{\bar{k}} \cong \mathbb{G}_{m,\bar{k}}^n$ for some n. If we have $T \cong \mathbb{G}_m^n$ for some n, then we say T is a split torus.

Example 1.5.2. Every torus is reductive.

Definition 1.5.3. For a linear algebraic group G, define the sets

$$X^*(G) = \mathrm{Hom}(G, \mathbb{G}_m), \quad X_*(G) = \mathrm{Hom}(\mathbb{G}_m, G).$$

(Here the base field k is implicit, and we only consider k-homomorphisms.) The first is always a $\mathbb{Z}$-module, and the second is a $\mathbb{Z}$-module if G is commutative.

Note that $X^*(G_{\bar{k}})$ is a discrete $\mathbb{Z}[\Gamma_k]$-module, and $X^*(G_{\bar{k}})^{\Gamma_k} = X^*(G)$.

Fact 1.5.4. *The functor $T \mapsto X^*(T_{\bar{k}})$ is an anti-equivalence from the category of tori over k to the category of discrete $\mathbb{Z}[\Gamma_k]$-modules which are finite free over $\mathbb{Z}$. The dimension of T is equal to the $\mathbb{Z}$-rank of $X^*(T_{\bar{k}})$. We have T is split if and only if the Γ_k-action on $X^*(T_{\bar{k}})$ is trivial.*

By the last assertion, we see that every torus over k splits over a finite extension of k.

Example 1.5.5. Let l/k be a finite extension. Then $T = l^\times$ is a reductive group, since $T_{\bar{k}} \cong \mathbb{G}_m^{[l:k]}$ (see Example 1.4.5). The Γ_k-module $X^*(T_{\bar{k}})$ is identified with $\mathrm{Ind}_{\{1\}}^{\Gamma_k} \mathbb{Z}$.

Fact 1.5.6. *All maximal split tori in a connected linear algebraic group G are conjugate by elements of $G(F)$.*

In particular, they are all isomorphic to $\mathbb{G}_m^r$ for a common r. We call r the rank of G.

Fact 1.5.7. *For each maximal torus T in a connected linear algebraic group G, $T_{\bar{k}}$ is a maximal torus in $G_{\bar{k}}$.*

In other words, the maximal tori in G are exactly those maximal tori in $G_{\bar{k}}$ which are "defined over k". In particular, they all have the same dimension equal to the rank of $G_{\bar{k}}$ (called the absolute rank of G). However, the maximal tori in G need not be isomorphic to each other, as shown by the following example.

Example 1.5.8. In GL_n, the diagonal subgroup is a maximal torus and it is split. For any degree n field extension l/k, we have a torus $l^\times$ (see Example 1.5.5) and a faithful representation $\phi : l^\times \to \mathrm{GL}_n$ by considering the multiplication action of $l^\times$ on $l \cong k^n$. The image T of ϕ is also a maximal torus in GL_n since it has dimension n equal to the rank of GL_n, but it is not split.

2. Lecture 2

2.1. **The Weyl group.**

Definition 2.1.1. Let G be a reductive group over k and $T \subset G$ a torus. Define the Weyl group $W(G,T) = N_G(T)/C_G(T)$. Here $N_G(T)$ and $C_G(T)$ are the normalizer and centralizer of T in G, characterized as the unique subgroups of G such that $N_G(T)(\bar{k})$ and $C_G(T)(\bar{k})$ are the normalizer and centralizer of $T(\bar{k})$ in $G(\bar{k})$ respectively.

Fact 2.1.2. *A torus $T \subset G$ is maximal if and only if $C_G(T) = T$. (Clearly we always have $T \subset T' \subset C_G(T)$ for any maximal torus T' containing T.)*

Remark 2.1.3. The above fact crucially depends on that G is reductive. For instance, $G = \mathbb{G}_m \times \mathbb{G}_a$ is not reductive, as its unipotent radical is $1 \times \mathbb{G}_a$. Then $T = \mathbb{G}_m \times 1$ is the unique maximal torus in G, but $C_G(T) = G$.

Fact 2.1.4. *The group $W(G,T)$ is finite étale. If T is a maximal split torus, then $W(G,T)$ is constant, in the sense that there exists an abstract group Γ such that for any k-algebra R we have $W(G,T)(R) =$ the group of locally constant functions $\operatorname{Spec} R \to \Gamma$ (with the group structure given by Γ). Thus $\Gamma = W(G,T)(k) = W(G,T)(\bar{k})$. Moreover, in this case we have $W(G,T)(k) = N_G(T)(k)/C_G(T)(k)$. (In general, the surjection $N_G(T) \to W(G,T)$ may not induce a surjection on k-points.) In this case we identify $W(G,T)$ with the abstract group $W(G,T)(k)$.*

For T a maximal split torus, we have a natural action of $W(G,T)(k)$ on T, i.e. a homomorphism of abstract groups $W(G,T)(k) \to \operatorname{Aut}_k(T)$. In particular, $W(G,T)(k)$ also acts on $X^*(T)$ and $X_*(T)$.

2.2. **Root data, split case.**

Definition 2.2.1. A reductive group G over k is called split, if it contains a maximal torus which is split (equivalently, every maximal split torus is a maximal torus, and equivalently, there exists a split maximal torus).

Example 2.2.2. The groups $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n, \mathrm{Sp}_{2g}$ are split. For a simple k-algebra D, the group $D^\times$ is split if and only if $D \cong M_n(k)$, in which case $D^\times \cong \mathrm{GL}_n$.

Let G be a split reductive group over k, and let T be a maximal split torus. Thus T is a split maximal torus. Since $T \cong \mathbb{G}_m^n$, any representation of T decomposes into a direct sum of one-dimensional representations, i.e., a direct sum of characters in $X^*(T) = \operatorname{Hom}(T, \mathbb{G}_m)$. Consider the adjoint representation $G \to \mathrm{GL}(\operatorname{Lie} G)$ restricted to T.

Definition 2.2.3. The non-trivial characters in $X^*(T)$ that appear in the T-representation $\operatorname{Lie} G$ are called roots. The set of them is denoted by $\Phi = \Phi(G,T) \subset X^*(T) - \{0\}$.

Note that the trivial character $0 \in X^*(T)$, namely $T \to \mathbb{G}^m, z \mapsto 1$, also appears, since T acts trivially on $\operatorname{Lie} T \subset \operatorname{Lie} G$. In fact, $\operatorname{Lie} T$ is precisely the eigenspace for the trivial character. Thus we have

$$\mathfrak{g} = \operatorname{Lie} G = \operatorname{Lie} T \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the eigenspace corresponding to α, on which T acts via $\alpha : T \to \mathbb{G}_m$. It turns out that each $\mathfrak{g}_\alpha$ has dimension 1, i.e., every non-trivial character of T appears in $\mathfrak{g}$ with multiplicity at most 1.

The pair $(V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}, \Phi \subset V)$ is a root system. Recall that this means, among other things, that there exists a Euclidean space structure $\langle \cdot, \cdot \rangle$ on V such that for each $\alpha \in \Phi$, the reflection along α

$$s_\alpha : V \to V, x \mapsto x - 2\frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

(which is the unique linear map sending α to $-\alpha$ and fixing the orthogonal complement of α) stabilizes the set Φ. The Euclidean structure is not canonical, but there is a canonical way to define $s_\alpha : V \to V$ as follows. It even comes from an automorphism $s_\alpha : X^*(T) \to X^*(T)$. Let

$$G_\alpha = C_G(\ker(\alpha)^0).$$

This is a reductive subgroup of G containing T, and T is a maximal torus in G_α (so G_α is split). We have $W(G_\alpha, T) \cong \mathbb{Z}/2\mathbb{Z}$, and the action of the non-trivial element on $X^*(T)$ is our desired s_α. Clearly $s_\alpha^2 = 1$.

Fact 2.2.4. *The action map $W(G, T) \to \mathrm{Aut}(T) \cong \mathrm{Aut}(X^*(T))$ is injective, and its image is generated by $s_\alpha, \alpha \in \Phi$.*

Since T is split, there is perfect pairing $\langle , \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$, sending (λ, μ) to the integer n such that the homomorphism $\lambda \circ \mu : \mathbb{G}_m \to \mathbb{G}_m$ is $z \mapsto z^n$.

Definition-Proposition 2.2.5. For each $\alpha \in \Phi$, there exists a unique element $\alpha^\vee \in X_*(T) - \{0\}$ such that

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad \forall x \in X^*(T).$$

This is called the coroot corresponding to α. The set of coroots is denoted by $\Phi^\vee = \Phi^\vee(G, T)$, and the map $\alpha \mapsto \alpha^\vee$ is a bijection $\Phi \xrightarrow{\sim} \Phi^\vee$.

Fact 2.2.6. *The quadruple $(X, \Phi, Y, \Phi^\vee) = (X^*(T), \Phi(G, T), X_*(T), \Phi^\vee(G, T))$, together with the perfect pairing $X \times Y \to \mathbb{Z}$ and the bijection $\Phi \xrightarrow{\sim} \Phi^\vee, \alpha \mapsto \alpha^\vee$, is a root datum, characterized by the following axioms:*

- *For each $\alpha \in \Phi$, we have $\langle \alpha, \alpha^\vee \rangle = 2$.*
- *For each $\alpha \in \Phi$, define $s_\alpha : X \to X, x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$, and $s_{\alpha^\vee} : Y \to Y, y \mapsto y - \langle \alpha, y \rangle \alpha^\vee$. Then*

$$s_\alpha(\Phi) \subset \Phi, \quad s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee.$$

(Note that s_α and $s_{\alpha^\vee}$ are involutions, so we have equalities.)

Moreover, this root datum is reduced, in the sense that for each $\alpha \in \Phi$ the only multiples of α in Φ are $\pm\alpha$. (Note that $-\alpha = s_\alpha(\alpha) \in \Phi$.)

We write $\Psi(G, T)$ for the root datum arising from (G, T). Since $W(G, T)$ is identified with the subgroup of $\mathrm{Aut}(X^*(T))$ generated by the s_α's, it is completely determined by $\Psi(G, T)$ in a combinatorial way. For fixed G, the different choices of

T are conjugate by $G(k)$, and so the isomorphism class of $\Psi(G,T)$ depends only on G.

Theorem 2.2.7 (Chevalley, Demazur)**.** *We have a bijection from the set of isomorphism classes of split reductive groups over k to the set of isomorphism classes of reduced root data. (Note that the latter set does not depend on k.)*

Remark 2.2.8. One can ask whether there is an equivalence of categories from pairs (G,T) to reduced root data. This cannot be done a naive way. Firstly, the natural map $\mathrm{Aut}(G,T) \to \mathrm{Aut}(\Psi(G,T))^{\mathrm{op}}$ is not an isomorphism. It is surjective, and the kernel consists of those automorphisms of G induced by conjugation by elements of $(T/Z_G)(k)$. Secondly, it is not easy to capture all homomorphisms $(G,T) \to (G',T')$ by the root data, although one can (partially) capture central isogenies $(G,T) \to (G',T')$, i.e., surjective homomorphisms with finite kernels, by certain morphisms between root data.

Example 2.2.9. Consider $G = \mathrm{GL}_n$. It is split, and a maximal split torus is given by the diagonal subgroup $T = \{(\ddots)\}$. We have $X^*(T) \cong \mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$, where

$$e_i : T \longrightarrow \mathbb{G}_m, \quad \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \longmapsto t_i.$$

Also $X^*(T) \cong \mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i^\vee$, where

$$e_i^\vee : \mathbb{G}_m \longrightarrow T, \quad z \mapsto \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & z \ (i\text{-th}) & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

The pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ is given by $\langle e_i, e_j^\vee \rangle = \delta_{ij}$. We have $\mathfrak{g} = \mathrm{Lie}\, G = M_n(k)$, and the adjoint action of G on $\mathfrak{g}$ is given by the usual conjugation action. (More generally, for any linear algebraic group G, the adjoint representation of G on $\mathrm{Lie}\, G$ can be deduced from this case by embedding G into some GL_n.) The roots are

$$\Phi(G,T) = \{e_i - e_j \mid i \neq j\}.$$

The coroot corresponding to $\alpha = e_i - e_j$ is $\alpha^\vee = e_i^\vee - e_j^\vee$. The reflection s_α permutes the e_k's by the transposition $(ij) \in S_n$. The Weyl group is identified with S_n.

2.3. **Borel subgroups and quasi-splitness.** Let G be a non-trivial reductive group over k.

Definition 2.3.1. A maximal connected solvable subgroup of $G_{\bar{k}}$ is called a Borel subgroup. A subgroup of G is called Borel, if its base change to $\bar{k}$ is a Borel subgroup of $G_{\bar{k}}$.

For dimension reasons, $G_{\bar{k}}$ always contains a Borel subgroup B, and $B \subsetneq G_{\bar{k}}$ since $B = R_u B$ is not reductive. In fact, we also always have $B \neq 1$. However, a Borel subgroup of $G_{\bar{k}}$ may not be defined over k, so G may not contain any Borel subgroup.

Definition 2.3.2. If a Borel subgroup of G exists, then we call G quasi-split.

Over $\bar{k}$, or more generally in the split case, Borel subgroups are classified as follows.

Fact 2.3.3. *If G is split then it is quasi-split. In this case every Borel subgroup contains a maximal split torus in G, and conversely for every maximal split torus T in G, the set of Borel subgroups B of G containing T is non-empty and a torsor under $W(G,T)$. This set is in bijection with the set of choices of positive roots in $\Phi(G,T)$. (A choice of positive roots is a subset $\Phi^+ \subset \Phi$ such that $\Phi = \Phi^+ \sqcup -\Phi^+$ and such that $\forall \alpha, \beta \in \Phi^+, \alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Phi^+$.) The bijection $\{B\} \leftrightarrow \{\Phi^+\}$ is characterized by*

$$\operatorname{Lie} B = \operatorname{Lie} T \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

Fact 2.3.4. *The reductive group G is quasi-split if and only if for one (hence any) maximal split torus S, $C_G(S)$ is a maximal torus (or equivalently, a torus). Assume this is the case. We call $C_G(S)$ a Cartan torus. Every Borel subgroup of G contains a Cartan torus. Conversely, given a Cartan torus $T = C_G(S)$, a Borel subgroup of $G_{\bar{k}}$ containing $T_{\bar{k}}$ is defined over k if and only if the corresponding set of positive roots $\Phi^+ \subset \Phi(G_{\bar{k}}, T_{\bar{k}})$ is stable under the Γ_k-action on $X^*(T_{\bar{k}})$. This condition is always satisfied by some Φ^+. Thus the (non-empty) set of Borel subgroups of G containing T is in bijection with the set of Γ_F-stable sets of positive roots in $\Phi(G_{\bar{k}}, T_{\bar{k}})$.*

Example 2.3.5. If G is split, then every maximal split torus S is a maximal torus, and hence $C_G(S) = S$. Therefore G is quasi-split. In general, a maximal split torus S is always contained in a maximal torus T, and hence $C_G(S) \supset T$. Thus asking $C_G(S)$ is a maximal torus amounts to asking that "S is not too small".

Example 2.3.6. Let $G = \mathrm{GL}_n$ and T be the diagonal torus. Then T is a split maximal torus. The Γ_k-action on $X^*(T_{\bar{k}})$ is trivial, so the Borel subgroups containing T correspond to choices of positive roots in $\Phi(G,T)$. One such choice is $\Phi^+ = \{e_i - e_j \mid i < j\}$. The corresponding Borel subgroup is the upper triangular subgroup $\mathbb{B}_n$.

By a based root datum, we mean a root datum together with a choice of positive roots. By a Γ_k-action on a based root datum, we mean a continuous action on the root datum stabilizing the set of positive roots.

Theorem 2.3.7. *The isomorphism classes of quasi-split reductive groups over k are in bijection with the isomorphism classes of reduced based root data with Γ_k-action.*

Quasi-split reductive groups play a special role in the classification of all reductive groups, by the following fact.

Fact 2.3.8. *For any reductive group G over k, there is a quasi-split reductive group G^* over k which is an inner form of G, i.e., there is an isomorphism $\phi : G_{\bar{k}} \xrightarrow{\sim} G^*_{\bar{k}}$ such that for each $\sigma \in \Gamma_k$, the automorphism $\sigma(\phi^{-1}) \circ \phi : G_{\bar{k}} \to G_{\bar{k}}$ is inner, that is, of the form $\mathrm{Int}(g) : x \mapsto gxg^{-1}$ for some $g \in G(\bar{k})$. For fixed G^*, the pairs (G, ϕ) as above modulo a suitable equivalence relation are classified by the Galois cohomology set $\mathbf{H}^1(k, (H^*)^{\mathrm{ad}})$.*

Example 2.3.9. Let D be a central simple algebra over k of dimension n^2. Then the reductive group $D^\times$ over k is an inner form of GL_n.

3. Lecture 3

3.1. **Parabolic subgroups.** Let G be a reductive group over k.

Fact 3.1.1 (Relative root datum)**.** *Let S be a maximal split torus in G and let $M_0 := C_G(S)$. (Caution: M_0 may not be a torus.) Let $\Phi(G, S)$ be the non-trivial characters of S appearing in the S-representation $\mathfrak{g} = \mathrm{Lie}\, G$. Then we have*

$$\mathfrak{g} = \mathrm{Lie}\, M_0 \oplus \bigoplus_{\alpha \in \Phi(G,S)} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the α-eigenspace (whose dimension may be > 1). The triple $(X^*(S), \Phi(G, S), X_*(S))$ canonically extends to a (possibly non-reduced) root datum $(X^*(S), \Phi(G, S), X_*(S), \Phi^\vee(G, S))$.

The root datum $(X^*(S), \Phi(G, S), X_*(S), \Phi^\vee(G, S))$ can be constructed from $\Psi(G_{\bar{k}}, T_{\bar{k}})$ where T is a maximal torus in G containing S, essentially by considering the restriction from T to S. Thus it is sometimes called the restricted root datum, or the relative root datum, for (G, S).

Definition 3.1.2. A subgroup P of G is called parabolic, if $P_{\bar{k}}$ contains a Borel subgroup of $G_{\bar{k}}$.

Clearly G is a parabolic subgroup of G, but there may not exist a proper parabolic subgroup. Since G is noetherian, there exist minimal parabolic subgroups, and every parabolic subgroup contains a minimal one.

Fact 3.1.3. *The minimal parabolic subgroups in G are all conjugate by $G(k)$. Each of them contains $C_G(S)$ for some maximal split torus S in G. For a fixed S, the set of minimal parabolic subgroups P_0 containing $M_0 = C_G(S)$ is in bijection with the set of choices of positive roots $\Phi^+ \subset \Phi(G, S)$. The bijection is characterized by: $P_0 \leftrightarrow \Phi^+$ if and only if*

$$\mathrm{Lie}\, P_0 = \mathrm{Lie}\, M_0 \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

From now on we fix $P_0 \supset M_0 = C_G(S)$ as above. We call parabolic subgroups containing P_0 standard. It follows that every parabolic subgroup is conjugate under $G(k)$ to a standard one.

Let Δ be the set of non-decomposable elements of Φ^+ (called simple roots). Then Δ is a root basis for $\Phi(G, S)$, i.e., it is linearly independent in $X^*(S)$ and every element of $\Phi(G, S)$ is either a $\mathbb{Z}_{\geq 0}$-linear or $\mathbb{Z}_{\leq 0}$-linear combination of Δ. In fact, choosing a set of positive roots is equivalent to choosing a root basis.

Theorem 3.1.4. *There is an inclusion-preserving bijection $J \mapsto P_J$ between the set of subsets of Δ and the set of standard parabolic subgroups, characterized as follows. Let $\Phi(J) = \Phi(G,S) \cap \mathrm{Span}_{\mathbb{Z}} J$. Then*

$$\mathrm{Lie}\, P_J = \mathrm{Lie}\, M_0 \oplus \bigoplus_{\alpha \in \Phi^+ \cup \Phi(J)} \mathfrak{g}_\alpha.$$

Example 3.1.5. $P_\emptyset = P_0, P_\Delta = G$.

Remark 3.1.6. We have $\Delta = \emptyset$ if and only if $\Phi(G,S) = \emptyset$ if and only if S is central. In this case, $M_0 = P_0 = G$, and G does not have proper parabolic subgroups. We say that G is anisotropic-mod-center.

Definition 3.1.7. Let H be a connected linear algebraic group over k (of characteristic zero). By a Levi component of H, we mean a subgroup L such that $H = L \ltimes R_u H$. In particular, L is reductive.

Theorem 3.1.8 (Levi decomposition)**.** *The group P_J admits a Levi component M_J satisfying $\mathrm{Lie}\, M_J = \mathrm{Lie}\, M_0 \oplus \bigoplus_{\alpha \in \Phi(J)} \mathfrak{g}_\alpha$. Moreover, M_J is the unique Levi component of P_J which contains M_0.*

Write N_J for $R_u P_J$. We have

$$\mathrm{Lie}\, N_J = \bigoplus_{\alpha \in \Phi^+, \alpha \notin \Phi(J)} \mathfrak{g}_\alpha.$$

Example 3.1.9. In $G = \mathrm{GL}_n$, choose $P_0 = \mathbb{B}_n$ and $M_0 = T =$ the diagonal torus. Then

$$\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \cdots, \alpha_{n-1} = e_{n-1} - e_n\}.$$

A subset $J \subset \Delta$ corresponds to an ordered partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of n (i.e., an ordered tuple such that $\sum \lambda_i = n$) by the relation

$$J = \{\alpha_i \mid i \notin \{\lambda_1, \lambda_1 + \lambda_2, \cdots, \lambda_1 + \cdots + \lambda_k\}\}.$$

For example the partition $(2,1,2,3)$ of $n = 8$ corresponds to $J = \{\alpha_1, \alpha_4, \alpha_6, \alpha_7\}$. Then P_J consists of the invertible block upper triangular matrices where the diagonal block sizes are $\lambda_1, \cdots, \lambda_k$. The group M_J consists of the invertible block diagonal matrices and so $M_J \cong \mathrm{GL}_{\lambda_1} \times \cdots \times \mathrm{GL}_{\lambda_k}$, and N_J consists of the block upper triangular matrices with identity matrices on the block diagonal.

3.2. **The analytic topology.** Let F be a local or global field (of characteristic zero). Let R be an F-algebra which is a Hausdorff locally compact topological ring. In applications, in the local case we take $R = F$, and in the global case we take $R = \mathbb{A}_F^S$ (the adeles away from S) for a finite set S of places of F.

Fact 3.2.1. *Let X be an affine variety over F. Equip $X(R)$ with the coarsest topology such that for every morphism ϕ from X to the affine line (i.e. element $\phi \in \mathcal{O}_X(X)$), the resulting map $\phi(R) : X(R) \to R$ is continuous. Then $X(R)$ is Hausdorff and locally compact. If $X \to Y$ is any morphism of varieties, then $X(R) \to Y(R)$ is continuous. If $X \to Y$ is a closed immersion, then $X(R) \to Y(R)$ is a closed embedding (i.e. homeomorphism onto the image and the image is closed).*

If G is a linear algebraic group over F, then $G(R)$ is a Hausdorff locally compact topological group.

Example 3.2.2. For a linear algebraic group G over F, we can choose closed immersions $G \hookrightarrow \mathrm{GL}_n \hookrightarrow \mathbb{A}_F^{n^2+1}$ (the (n^2+1)-dimensional affine space over F), where the second map is $g \mapsto (g_{ij}, \det g^{-1})$. Then $G(R)$ has the subspace topology inherited from R^{n^2+1}

Example 3.2.3. If $F = \mathbb{R}$ or $\mathbb{C}$, then $G(F)$ is a Lie group over $\mathbb{R}$ or $\mathbb{C}$.

Example 3.2.4. Let E/F be a finite extension of local fields. Let G be a linear algebraic group over E, and let $H = \mathrm{Res}_{E/F}\, G$. Then the natural isomorphism $H(F) \cong G(E)$ is also a topological isomorphism. Similarly, in the global case, $H(\mathbb{A}_F) \cong G(\mathbb{A}_E)$ is a topological isomorphism.

Definition 3.2.5. A locally profinite group is a Hausdorff and locally compact topological group such that the compact open subgroups form a neighborhood basis of 1.

Remark 3.2.6. In a Hausdorff space, every compact set is closed. Hence every compact open set is a union of connected components. If G is a locally profinite group, then for every $g \in G$ the set $\{g\}$ is a connected component. (However G may not have the discrete topology, and $\{g\}$ may not be open.) This property is called totally disconnected.

Proposition 3.2.7. *Let F be a local non-archimedean field, and let G be a linear algebraic group over F. Then $G(F)$ is locally profinite.*

Proof. Note that any closed subgroup of a locally profinite topological group is locally profinite. Hence we may assume that $G = \mathrm{GL}_n$. Let $\pi \in F$ be a uniformizer. Then for each positive integer k, the subset $I_n + \pi^k M_n(\mathcal{O}_F)$ is a compact open subgroup of $\mathrm{GL}_n(F)$ (called the k-th principal congruence subgroup), and for all k they form a neighborhood basis. □

Let F be global and G a linear algebraic group over F. Fix a faithful representation $\phi : G \to \mathrm{GL}_n$. For each non-archimedean place v of F, let $K_v = G(F_v) \cap \phi^{-1}(\mathrm{GL}_n(\mathcal{O}_{F_v}))$. This is a compact open subgroup of $G(F_v)$. If we change ϕ, then K_v will change for only finitely many v.

Fact 3.2.8. *Let S be a finite set of places of F. The natural map $G(\mathbb{A}_F^S) \to \prod_{v \notin S} G(F_v)$, where v runs over all places of F outside S, identifies $G(\mathbb{A}_F)$ with the restricted product with respect to K_v's*

$$\prod_{v\notin S}{}' G(F_v) = \{(g_v) \in \prod_{v\notin S} G(F_v) \mid g_v \in K_v \textit{ for almost all } v\}.$$

Moreover, it is a topological isomorphism, where the restricted product topology is defined to be generated by open sets of the form $\prod_v U_v$ where each U_v is an open set in $G(F_v)$ and $U_v = K_v$ for almost all v.

Recall that on any Hausdorff locally compact group H, there exists a left Haar measure, i.e., a positive Radon measure (= Borel measure which is finite on compact sets, outer regular, and inner regular for open sets) invariant under left translation. It is unique up to a positive scalar. Similarly for right Haar measure. If one (and hence every) left Haar measure is right Haar, then we say the group is unimodular. In general, there is a canonical homomorphism, called the modulus character

$$\delta_H : H \longrightarrow \mathbb{R}_{>0}$$

such that for any right Haar measure $d_r h$ on H, we have

$$d_r(h_0 h) = \delta_H(h_0) d_r(h), \quad \forall h_0 \in H.$$

Thus H is unimodular if and only if δ_H is trivial.

Fact 3.2.9. *Let G be a reductive group over a local or global field F. Then $G(F)$ in the local case and $G(\mathbb{A}_F^S)$ in the global case is unimodular.*

In fact, there is a way of obtaining a Haar measure on $G(\mathbb{A}_F^S)$ from Haar measures on $G(F_v)$, by a certain product process. The unimodularity of $G(\mathbb{A}_F^S)$ follows from that of $G(F_v)$.

3.3. **The automorphic quotient.** Let F be a number field and G a reductive group over F.

Fact 3.3.1. *The subgroup $G(F)$ in $G(\mathbb{A}_F)$ is discrete and hence closed.*

Generalizing the idele class group $\mathrm{GL}_1(F)\backslash \mathrm{GL}_1(\mathbb{A}_F)$, we would like to consider the quotient $G(F)\backslash G(\mathbb{A}_F)$. Recall that the idele class group is not compact, but we can shrink it to the unit idele class group $F^\times\backslash\mathbb{A}_F^{\times,1}$, which is compact. Here we define the idelic norm

$$|\cdot|_{\mathbb{A}} : \mathbb{A}_F^\times \longrightarrow \mathbb{R}_{>0}, \quad x \longmapsto \prod_v |x|_v$$

where each $|\cdot|_v$ is the canonically normalized absolute value on F_v (so that $d(xy) = |x|_v dy$ for a Haar measure dy on F_v), and

$$\mathbb{A}_F^{\times,1} = \{(x_v) \in \mathbb{A}_F^\times \mid |x|_{\mathbb{A}} = 1\}.$$

Similarly, we need to modify $G(F)\backslash G(\mathbb{A}_F)$.

Definition 3.3.2. Let

$$G(\mathbb{A}_F)^1 := \bigcap_{\chi\in X^*(G)} \ker\left(G(\mathbb{A}_F) \xrightarrow{\chi} \mathbb{A}_F^\times \xrightarrow{|\cdot|_{\mathbb{A}}} \mathbb{R}_{>0}\right).$$

This is a closed subgroup of $G(\mathbb{A}_F)$, and hence is itself a Hausdorff locally compact group. In general it is not the $\mathbb{A}_F$-points of an algebraic group. Note that

$$G(F) \subset G(\mathbb{A}_F)^1,$$

since for any $g \in G(F)$ and $\chi \in X^*(G)$ we have $\chi(g) \in F^\times \subset \mathbb{A}_F^{\times,1}$.

Lemma 3.3.3. *There is a closed central subgroup A_G of $G(\mathbb{A}_F)$ such that $G(\mathbb{A}_F) \cong A_G \times G(\mathbb{A}_F)^1$. The group $G(\mathbb{A}_F)^1$ is unimodular.*

Proof. The second assertion follows from the first and the unimodularity of $G(\mathbb{A}_F)$ and A_G (which is abelian). To prove the first assertion, if we set $G' = \mathrm{Res}_{F/\mathbb{Q}} G$, then $G'(\mathbb{A}_{\mathbb{Q}}) = G(\mathbb{A}_F)$ and $G'(\mathbb{A}_{\mathbb{Q}})^1 = G(\mathbb{A}_F)^1$. Thus we may assume $F = \mathbb{Q}$. Let $\mathscr{A}_G$ be the maximal split torus in Z_G° (over $\mathbb{Q}$), and let A_G be the identity component (for the analytic topology) of $\mathscr{A}_G(\mathbb{R})$.

Note that $\mathscr{A}_G \cong \mathbb{G}_m^k$, so $\mathscr{A}_G(\mathbb{R}) \cong (\mathbb{R}^\times)^k$ and so $A_G \cong (\mathbb{R}_{>0})^k$. To prove that $G(\mathbb{A}) = A_G \times G(\mathbb{A})^1$, we use the fact that the restriction map $X^*(G) \to X^*(\mathscr{A}_G)$ induces an isomorphism $X^*(G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} X^*(\mathscr{A}_G) \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus for every coordinate projection $\chi_i : A_G \cong (\mathbb{R}_{>0})^k \to \mathbb{R}_{>0}$, there exists an integer n_i such that $\chi_i^{n_i}$ is induced by some $\phi_i \in X^*(G)$. For $g \in A_G \cap G(\mathbb{A})^1$, we have $|\chi_i^{n_i}(g)|_\infty = |\phi_i(g)|_{\mathbb{A}} = 1$, and it follows that $\chi_i(g) = 1$ and so $g = 1$. On the other hand, for any $g \in G(\mathbb{A})$, by the fact we know that $g \in G(\mathbb{A})^1$ if and only if $|\phi_i(g)|_{\mathbb{A}} = 1$ for each i. For general g, let $x = (|\phi_1(g)|_{\mathbb{A}}^{1/n_1}, \cdots, |\phi_k(g)|_{\mathbb{A}}^{1/n_k}) \in A_G$. Then $x^{-1}g \in G(\mathbb{A})^1$. Hence $G(\mathbb{A}) = A_G \times G(\mathbb{A})^1$. ☐

Definition 3.3.4. Let $[G] = G(F)\backslash G(\mathbb{A}_F)^1 = (G(F)A_G)\backslash G(\mathbb{A}_F)$. This is called the automorphic quotient for G.

4. Lecture 4

4.1. **The automorphic spectrum.** Fix a Haar measure dg on $G(\mathbb{A}_F)^1$, and equip $[G] = G(F)\backslash G(\mathbb{A}_F)^1$ with the quotient measure of dg by the counting measure on $G(F)$. This is the unique Radon measure $d\bar{g}$ on $[G]$ characterized by

$$\int_{[G]} \Big(\sum_{\gamma \in G(\mathbb{Q})} f(\gamma g) \Big) d\bar{g} = \int_G f(g) dg$$

for all compactly supported continuous functions f on G. (The existence depends on the fact that $G(\mathbb{A}_F)^1$ and $G(F)$ are both unimodular.) Clearly $d\bar{g}$ is invariant under the right translation action by $G(\mathbb{A}_F)^1$.

Fact 4.1.1. *The space $[G]$ has finite volume under $d\bar{g}$. It is compact if and only if G is anisotropic-mod-center, i.e., G does not contain any proper parabolic subgroup, or equivalently, every split torus in G is central.*

Consider $L^2([G])$, the space of square integrable functions $[G] \to \mathbb{C}$ defined with respect to $d\bar{g}$ (and completed with respect to the L^2-norm). This is a Hilbert space, and $G(\mathbb{A}_F)^1$ acts on it by right translation:

$$r_g(f)(x) = f(xg), \quad \forall g \in G(\mathbb{A}_F)^1, f \in L^2([G]).$$

Definition 4.1.2. Let H be a topological group.

(1) By a Hilbert representation of H, we mean a continuous linear representation $H \times V \to V$ on a Hilbert space V (over $\mathbb{C}$, having a countable Hilbert basis). We often write π for the map $H \to \mathrm{GL}(V)$, and denote the representation by the pair (π, V).

(2) A Hilbert representation is called unitary, if $\pi(g)$ is a unitary operator for each $g \in H$.
(3) A Hilbert representation is called irreducible, if there is no proper closed H-stable subspace.
(4) Isomorphisms between Hilbert representations are by definition topological vector space isomorphisms preserving the H-actions. They are not required to be isometries. For two unitary representations, we are interested in whether we can find an isomorphism between them which is an isometry. When this is the case we say that they are unitarily equivalent.

Proposition 4.1.3. *The $G(\mathbb{A}_F)^1$-action on $L^2([G])$ is a unitary representation.*

Proof. Write H for $G(\mathbb{A}_F)^1$. Clearly each $g \in H$ acts by a unitary operator, so only the continuity of the action is not obvious. Here, knowing that each group element acts by a unitary operator, the continuity is equivalent to the following condition

- For each fixed $f \in L^2([G])$, the map $H \to L^2([G]), g \mapsto r_g f$ is continuous.

By rather general considerations, we know that the space $C_c([G])$ of compactly supported functions on $[G]$ is dense in $L^2([G])$. Using this, we reduce to checking that for each fixed $f \in C_c([G])$, we have $\|r_g f - f\|_2 \to 0$ when $g \to 1$ in H. Let U be a relatively compact open neighborhood of 1 in H. Then there exists a compact subset W of $[G]$ containing $\mathrm{supp}(f) \cdot U$. For $g \in U^{-1}$, the function $r_g f - f$ is supported inside W, and so

$$\|r_g f - f\|_2 \leq \mathrm{vol}(W)^{1/2} \max_W |r_g f - f|.$$

It remains to prove that $\max_W |r_g f - f| \to 1$ as $U^{-1} \ni g \to 1$. Let $\epsilon > 0$. For each $x \in W$, there exists an open neighborhood V_x of 1 in U^{-1} such that $V_x \cdot V_x \subset U^{-1}$ and such that the variance of f on $x \cdot V_x \cdot V_x$ is less than ϵ. Extract from the open covering $W \subset \bigcup_{x \in W} xV_x$ a finite subcovering $W \subset \bigcup_{i=1}^n x_i V_{x_i}$. Let $V = \bigcap_i V_{x_i}$, which is an open neighborhood of 1 in U^{-1}. Now let $g \in V$ and $x \in W$ be arbitrary. We have $x \in x_i V_{x_i}$ for some i. Then x and xg are both in $x_i \cdot V_{x_i} \cdot V_{x_i}$, and hence

$$|f(xg) - f(x)| < \epsilon.$$

This shows that $\max_W |r_g f - f| < \epsilon$. ∎

Definition 4.1.4. By a discrete automorphic representation, we mean an irreducible unitary representation of $G(\mathbb{A}_F)^1$ that is unitarily equivalent to a sub-representation of $L^2([G])$.

Example 4.1.5. The S^1-representation $L^2(S^1)$ is a Hilbert direct sum of its irreducible sub-representations:

$$L^2(S^1) \cong \widehat{\bigoplus}_{n \in \mathbb{Z}} \chi_n,$$

where χ_n is the one-dimensional unitary representation $S^1 \to S^1 \subset \mathbb{C}^\times, z \mapsto z^n$. This isomorphism sends a function on S^1 to its Fourier series.

Example 4.1.6. The $\mathbb{R}$-representation $L^2(\mathbb{R})$ is not a Hilbert direct sum of its irreducible sub-representations. In fact, it does not have any irreducible sub-representation other than 0!

The correct way to decompose $L^2(\mathbb{R})$ is to express it as a direct integral of the unitary representations $\{\chi_t \mid t \in \mathbb{R}\}$, where χ_t is the one-dimensional unitary representation $\mathbb{R} \to S^1 \subset \mathbb{C}^\times, x \mapsto e^{itx}$. (These are up to equivalence all the irreducible unitary representations of $\mathbb{R}$.) Let $\mathbb{C}_t = \mathbb{C}$ be the space of the representation χ_t. An element of $\prod_{t\in\mathbb{R}} \mathbb{C}_t$ is the same as a function $\mathbb{R} \to \mathbb{C}$. Define the subspace

$$\int_{t\in\mathbb{R}} \mathbb{C}_t \subset \prod_{t\in\mathbb{R}} \mathbb{C}_t$$

consisting of square integrable functions $\mathbb{R} \to \mathbb{C}$ (for the usual Lebesgue measure on $\mathbb{R}$). We can then define a $\mathbb{R}$-action on $\int_t \mathbb{C}_t$ by "letting it act on each $\mathbb{C}_t$ via χ_t". Namely, for $g \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{C}$ inside $\int_t \mathbb{C}_t$, define

$$gf : \mathbb{R} \longrightarrow \mathbb{C}, \quad t \mapsto \chi_t(g)f(t) = e^{itg}f(t).$$

This is easily checked to be a unitary representation of $\mathbb{R}$ on $\int_t \mathbb{C}_t$. By Fourier transform, this is unitarily equivalent to the natural $\mathbb{R}$-representation on $L^2(\mathbb{R})$.

Example 4.1.7. More generally, let H be a Hausdorff locally compact abelian group, and let $\widehat{H}$ be the Pontryagin dual $\mathrm{Hom}_{\mathrm{cont}}(H, S^1)$. Thus $\widehat{H}$ is the set of all irreducible unitary representations of H up to equivalence. There is a natural structure of Hausdorff locally compact abelian group on $\widehat{H}$. Define the H-representation $\int_{\chi\in\widehat{H}} \mathbb{C}_\chi$ in the same way as for $H = \mathbb{R}$, using the Haar measure on $\widehat{H}$. Then the natural H-representation $L^2(H)$ is unitarily equivalent to $\int_{\chi\in\widehat{H}} \mathbb{C}_\chi$.

As the above examples show, each $\mathbb{C}_\chi$ may or may not be isomorphic to an actual sub-representation of $L^2(H)$.

For the so-called type I topological groups, there is a general result on decomposing an arbitrary unitary representation into a direct integral of irreducible unitary representations. For F local (resp. global) of characteristic zero and G a reductive group over F, the group $G(F)$ (resp. $G(\mathbb{A}_F), G(\mathbb{A}_F)^1$) is of type I.

Theorem 4.1.8. *Let H be of type I. There is a canonical topology, called Fell topology, on $\widehat{H}$, the set of irreducible unitary representations of H up to unitary equivalence. For every unitary representation V_0 of G admitting a countable Hilbert basis, there exists a Borel measurable function $m : \widehat{H} \to \mathbb{Z}_{\geq 0}$ and a positive Borel measure $d\mu$ on $\widehat{H}$ such that V_0 is equivalent to $\int_{V\in\widehat{H}} V^{\oplus m(V)} d\mu$.*

The theorem can be applied to the $G(\mathbb{A}_F)^1$-representation $L^2([G])$. There is a much deeper theorem by Langlands, describing explicitly how $L^2(G)$ decomposes (as a direct integral) into discrete automorphic representations of G and those of the Levi components of standard parabolic subgroups of G.

In the rest of the course, we will only consider the number field $\mathbb{Q}$ and the local fields $\mathbb{R}$ and $\mathbb{Q}_p$. The other cases are treated by Weil restriction of scalars.

If V is a discrete automorphic representation of G, then it is also a representation of $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$ by restriction. We now discuss basic representation theory of $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$.

4.2. **Archimedean representation theory.** Let G be a reductive group over $\mathbb{R}$.

Fact 4.2.1. *The topological group $G(\mathbb{R})$ is a Lie group with finitely many connected components. Every compact subgroup is contained in a maximal compact subgroup. All maximal compact subgroups are conjugate by $G(\mathbb{R})^0$. Every maximal compact subgroup meets every connected component of $G(\mathbb{R})$.*

Lemma 4.2.2. *Let K be a compact Hausdorff group. Every irreducible Hilbert representation of K is finite dimensional. Every Hilbert representation of K is isomorphic to a unitary representation, and every unitary representation is a Hilbert direct sum of some of its irreducible sub-representations.*

Let $\widehat{K}$ be the set of isomorphism classes of irreducible representations of K. Note that by Schur's lemma, on an irreducible representation of K there is up to scalar a unique inner product making the representation unitary. Hence $\widehat{K}$ is in fact the unitary dual of K.

Suppose V is any representation of K (with or without topology). For each $\sigma \in \widehat{K}$, we have

$$\{v \in V \mid \mathrm{Span} K v \cong \sigma\} = \sum_{W \subset V, W \cong \sigma} W,$$

and this is a sub-representation of V. Denote it by $V(\sigma)$, called the σ-isotypic part of V.

Definition-Proposition 4.2.3. Define V_{fin} be the subspace of V given by

$$V_{\mathrm{fin}} = \{v \in V \mid \dim \mathrm{Span} K v < \infty\} = \bigoplus_{\sigma \in \widehat{K}} V(\sigma).$$

(Here the direct sum is algebraic direct sum.) This is a K-stable (non-closed) subspace of V, called the K-finite part of V.

From now on, we fix a maximal compact open subgroup K of $G(\mathbb{R})$.

Definition 4.2.4. For any Hilbert representation (π, V) of $G(\mathbb{R})$, define $V_{\mathrm{fin}} = \bigoplus_{\sigma \in \widehat{K}} V(\sigma) \subset V$ by restricting the representation to K. We call (π, V) admissible, if $\dim V(\sigma) < \infty$ for each $\sigma \in \widehat{K}$.

Theorem 4.2.5 (Harish-Chandra)**.** *Every irreducible unitary representation of $G(\mathbb{R})$ is admissible.*

The irreducible admissible representations of $G(\mathbb{R})$ are much easier to study and classify than irreducible unitary representations. However, the "correct" notion of equivalence between them turns out to be infinitesimal equivalence, which is weaker than the usual notion of isomorphism of Hilbert representations. We now explain this.

Recall that $G(\mathbb{R})$ is a Lie group. In fact, there is a canonical smooth structure: For any faithful representation $\phi : G \to \mathrm{GL}_n$, we have a closed embedding $\phi(\mathbb{R}) : G(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$. The image of $\phi(\mathbb{R})$ is a smooth submanifold of $\mathrm{GL}_n(\mathbb{R})$ (with standard smooth structure). We require that $\phi(\mathbb{R})$ is a diffeomorphism onto its image. The Lie algebra of the algebraic group G is canonically identified with the Lie algebra of the Lie group $G(\mathbb{R})$. Denote it by $\mathfrak{g}$.

We have the exponential map $\exp : \mathfrak{g} \to G(\mathbb{R})$. For GL_n this is the usual exponential of matrices. In general this is defined either by general theory of Lie groups, or by fixing a faithful representation $G \to \mathrm{GL}_n$ and inheriting from GL_n.

Let (π, V) be a Hilbert representation of $G(\mathbb{R})$. For any $X \in \mathfrak{g}$ and $v \in V$, we define the derivative of v along X to be

$$\pi(X)v = Xv := \frac{d}{dt}|_{t=0}\pi(\exp(tX))v = \lim_{t\to 0} \frac{\pi(\exp(tX))v - v}{t} \in V,$$

if the limit exists. We say $v \in V$ is smooth, if for every sequence $X_1, \cdots, X_k \in \mathfrak{g}$, the successive derivative $X_1 \cdots X_k v \in V$ exists.

Definition 4.2.6. Let V_{sm} be the subspace of V consisting of smooth vectors.

For $v \in V_{\mathrm{sm}}, X \in \mathfrak{g}, g \in G(\mathbb{R})$, we have $X \cdot (gv)$ exists and

$$X \cdot (gv) = g \cdot (\mathrm{Ad}(g)(X)) \cdot v.$$

Similarly, arbitrary successive derivatives of gv exist. Hence V_{sm} is a $G(\mathbb{R})$-stable subspace of V.

Remark 4.2.7. There is a general notion of smooth map $G(\mathbb{R}) \to V$. A vector $v \in V$ is smooth if and only if the map $G(\mathbb{R}) \to V, g \mapsto \pi(g)v$ is smooth.

Fact 4.2.8. *The natural action of $\mathfrak{g}$ on V_{sm} is a Lie algebra representation (without any continuity conditions).*

Proposition 4.2.9 ([1] Prop. 4.4.7)**.** *For any admissible Hilbert representation (π, V) of $G(\mathbb{R})$, we have $V_{\mathrm{fin}} \subset V_{\mathrm{sm}}$. Moreover, V_{fin} is stable under the $\mathfrak{g}$-action.*

Note that V_{fin} is not $G(\mathbb{R})$-stable, but K-stable. Hence it carries two structures: the K-action and the $\mathfrak{g}$-action. The compatibility between the two structures is captured in the following definition.

Definition 4.2.10. A $(\mathfrak{g}, K)$-module is a $\mathbb{C}$-vector space W (with no topology) together with a linear representation by K and a Lie algebra representation by $\mathfrak{g}$, satisfying the following conditions:

(1) W is a countable direct sum of finite dimensional K-stable subspaces. In particular, W is the union of finite-dimensional K-stable subspaces.
(2) For any finite dimensional K-stable subspace $W_1 \subset W$, the K-action on W_1 is continuous and smooth, in the sense that every vector in W_1 is a smooth vector. (Here W_1 is equipped with the canonical topology on a finite dimensional vector space.) Moreover, the resulting $\mathrm{Lie}\, K$-action on W_1 agrees with the restriction of the $\mathfrak{g}$-action on W.
(3) For all $k \in K, X \in \mathfrak{g}, w \in W$, we have $k \cdot X \cdot k^{-1} \cdot w = (\mathrm{Ad}(k)(X)) \cdot w$.

By condition (1), we have $W = \bigoplus_{\sigma\in\widehat{K}} W(\sigma)$, and each $W(\sigma)$ is at most countable dimensional. We say that W is admissible, if each $W(\sigma)$ is finite dimensional.

Remark 4.2.11. Suppose $W = \bigoplus_{\sigma\in\widehat{K}} W(\sigma)$ and each $W(\sigma)$ is finite dimensional. Then this is already a decomposition of W into a countable direct sum of finite dimensional K-stable subspaces, since $\widehat{K}$ is a countable set (by the second countability of K and Peter–Weyl theorem).

Theorem 4.2.12. *For any admissible representation (π, V) of $G(\mathbb{R})$, V_{fin} is an admissible $(\mathfrak{g}, K)$-module. Moreover, V is irreducible if and only if V_{fin} is irreducible as a $(\mathfrak{g}, K)$-module.*

Theorem 4.2.13 (Harish-Chandra, see [4] Theorem 4.15)**.** *Every irreducible $(\mathfrak{g}, K)$-module is automatically admissible, and it is isomorphic to V_{fin} for an irreducible admissible Hilbert representation (π, V) of $G(\mathbb{R})$.*

Definition 4.2.14. We say two irreducible admissible Hilbert representations of $G(\mathbb{R})$ are infinitesimally equivalent, if their associated $(\mathfrak{g}, K)$-modules are isomorphic.

Corollary 4.2.15. *The set of irreducible admissible Hilbert representations of $G(\mathbb{R})$ modulo infinitesimal equivalence, is in bijection with the set of irreducible (admissible) $(\mathfrak{g}, K)$-modules modulo isomorphism. We call either of these two sets the admissible dual of $G(\mathbb{R})$.*

However, in general infinitesimal equivalence is weaker than actual isomorphism. Thus there exist non-isomorphic irreducible admissible representation with isomorphic $(\mathfrak{g}, K)$-modules. Nevertheless, one positive result is the following:

Theorem 4.2.16. *Any two infinitesimally equivalent irreducible unitary representations of $G(\mathbb{R})$ are unitarily equivalent.*

Thus the unitary dual of $G(\mathbb{R})$, namely the set of irreducible unitary representations of $G(\mathbb{R})$ up to uintary equivalence, injects into the admissible dual of $G(\mathbb{R})$.

More generally, one can consider continuous representations of $G(\mathbb{R})$ on arbitrary locally convex topological vector spaces. From such representations one can similarly produce $(\mathfrak{g}, K)$-modules. A very subtle problem is to find a suitable category of topological representations such that the functor from this category to the category of $(\mathfrak{g}, K)$-modules has nice properties (e.g., an equivalence of categories). This is referred to as the problem of globalization. For a discussion see [1, §4.4] and [4, §4].

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