SPRING 2025 TSINGHUA NUMBER THEORY LEARNING SEMINAR: CANONICAL INTEGRAL MODELS OF SHIMURA VARIETIES

Goal: Understand the construction of canonical integral models for Shimura varieties of exceptional type given by [BST24].

Recall that a Shimura datum is a pair (G, X) of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class of homomorphisms $h\colon \mathbb{S} = \mathbb{C}^\times \to G(\mathbb{R})$ satisfying certain assumptions. Let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. The double quotient

$$G(\mathbb{Q}) \setminus (X \times G(\mathbb{A}_f)/K) =: \mathrm{Sh}_K(G, X)(\mathbb{C})$$

has the structure of a complex algebraic variety, which we denote by $\mathrm{Sh}_K(G, X)$. The datum (G, X) has a naturally associated reflex field $E \subset \mathbb{C}$, which is a number field, and which acts a natural field of definition for $\mathrm{Sh}_K(G, X)$. Along these lines, there exists a canonical model, which we will also denote $\mathrm{Sh}_K(G, X)$, which is an algebraic variety defined over E whose $\mathbb{C}$ points are isomorphic to the double quotient. These models are called canonical because they satisfy certain Galois compatibility conditions relating to the Artin reciprocity map.

Shimura varieties naturally arise as moduli spaces of abelian varieties. The simplest example is if we take $G = \mathrm{GL}_2$ in which case $X = \mathbb{H}^\pm$ is the upper and lower half planes. Then the Shimura variety $\mathrm{Sh}_K(\mathrm{GL}_2, \mathbb{H}^\pm)$ is nothing more than the modular curve with the level structure given by the choice of compact open subgroup K. In higher dimensions, taking $G = \mathrm{GSp}_{2g}$ gives an algebraic variety which is the moduli space of g-dimensional principally polarized abelian varieties. One would like to study the $\mathbb{F}_q$ points of these Shimura varieties for various reasons. For instance, the Hasse–Weil zeta function for Sh_K is conjecturally a product of automorphic L-functions. In the case of the modular curve, the connection is given by the Eichler–Shimura congruence relation. However, in order to study the $\mathbb{F}_q$ points, one must first have a model of the Shimura variety over the completion $\mathcal{O}_{E,v}$ for some place v of E and then consider the mod p points.

Given a variety over a number field E, there are many ways to construct smooth varieties over $\mathcal{O}_E$ whose generic fibers agree. Thus, we need to impose some conditions on this integral model in order for it to be unique. The appropriate condition turns out to be the extension property given by Milne in [Mil92] and later refined by Moonen in [Moo98]. The property is as follows.

Definition 1. A model S of $\mathrm{Sh}_K(G, X)$ over $\mathcal{O}_E$ is said to have the extension property is for every test scheme $Y/\mathcal{O}_E$ such that $Y(E)$ is dense in Y, every E-morphism $Y_E \to \mathrm{Sh}_K(G, X)$ extends uniquely to an $\mathcal{O}_E$-morphism $Y \to S$.

A model S is said to be canonical if it is smooth and satisfies the extension property.

Langlands conjectured that integral canonical models of Shimura varieties exist whenever the level structure at p K_p is chosen to be hyperspecial. These integral canonical models were first studied by Kottwitz ([Kot92]) in the setting of Shimura varieties of PEL-type. These types of Shimura varieties are moduli spaces for abelian varieties as well as extra data about their endomorphism rings. By framing the Shimura variety as a moduli problem, he was able to show that the moduli functor was representable in finite characteristic when the level structure K is hyperspecial. Later, Kisin in [Kis10] was able to prove the existence of canonical integral models when (G, X) is of Hodge type (parametrizing abelian varieties along with some Hodge cycles) and of abelian type (ones whose data is very similar to Hodge type). All other pairs of Shimura datum are called exceptional type. For exceptional Shimura varieties, there is no moduli interpretation in terms of abelian varieties, making the construction of integral models much more difficult.

We will cover the main result of [BST24]: Over all finite places v sufficiently large, there exists an integral canonical model for $\mathrm{Sh}_K(G, X)$. The semester will be divided into two halves. The first half will be introductory and suitable for graduate students and advanced undergraduate students. We will introduce Shimura varieties as well as cover their canonical models over their reflex fields. We presume familiarity with the language of schemes and algebraic groups. We will primarily follow Deligne ([Del71]) and Milne ([Mil05]). During the second half of the semester, we will cover [BST24, §2-5].

(1) **Variation of Hodge Structures** Give the definition of Hodge structures as well as the definition for a family of Hodge structures over a variety. Discuss how variations of Hodge structures can be mapped into Grassmannians. This should cover [Mil05, §2].
(2) **Locally Symmetric Varieties** Cover [Mil05, §3]. Recall results from [Mil05, §1] as necessary, discuss definitions of arithmetic subgroups and prove that arithmetic quotients of Hermitian symmetric domains are algebraic varieties.
(3) **Definition of Shimura Varieties** Cover [Del71, §1] and [Mil05, §5] up to "Structure of Shimura Varieties". Give the definition of Shimura datum as well as several examples. Introduce the projective system depending on the level structure K as well as how it relates to the double coset structure.
(4) **Structure of Shimura Varieties** Cover [Del71, §2] and the rest of [Mil05, §5]. Cover the structure of the connected components of $\mathrm{Sh}_K(G, X)$ and prove results from [Mil05, §4] on the algebraicity of the connected components of Sh_K, thus proving that Sh_K is also an algebraic variety.
(5) **Siegel Modular Variety** Cover [Mil05, §6] covering in depth the example of the moduli space of abelian varieties $\mathcal{A}_g$. Give the proof that this represents the moduli functor of principally polarized abelian varieties.

(6) **Definition of Canonical Models** Cover [Del71, §3] and [Mil05, §12] which give the definition for the reflex field and what properties canonical models satisfy. Then, prove that zero dimensional Shimura varieties (torii) have unique canonical models.
(7) **Existence of Canonical Models** Cover [Del71, §4] and [Mil05, §14] which prove the existence of a canonical model for $G = \mathrm{GSp}$ and Sh_K is the moduli space of abelian varieties. If time you may mention how PEL-type, Hodge type, and abelian type are covered in [Del79].
(8) **Extension Property of Integral Models** Cover [BST24, §2.1-2.3]. Give the definition of admissible test schemes and cover [BST24, Ex. 2.12] about why we must allow for finite type flat schemes as models instead of requiring smooth models. Cover [BST24, Cor. 2.5, 2.9] and their proofs on the crucial induction step on how to reduce to a codimension 2 subset.
(9) **Extension Property of Integral Models** Cover [BST24, §2.5-2.6] by proving [BST24, Prop. 2.16] about how the existence of integral models is equivalent to the extension of a certain local system.
(10) **Inverse Cartier Transform** Cover [BST24, §3.1-3.2.1] by giving the definition and properties of flat sheaves, Higgs sheaves, the nonabelian Hodge correspondence, and inverse Cartier transform of pullbacks of Higgs sheaves.
(11) **Inverse Cartier Transform on Nodal Curves** Cover the rest of [BST24, §3.2] culminating in the proof of Proposition 3.7.
(12) **Application to Fontaine–Laffaille Modules** Cover the rest of [BST24, §3]. Give the definition of Fontaine–Laffaille modules and then prove the main result of Proposition 3.9 which is a degree bound.
(13) **Reduction to Fontaine–Laffaille Modules** Cover [BST24, §4.1] by proving Theorem 4.1 and Theorem 4.3.
(14) **Alternate Proof** Cover [BST24, §4.2] and give another proof of Theorem 4.1 and 4.3 by proving Proposition 4.7.
(15) **Proof of Integral Models** Cover [BST24, §5.1] proving the existence of integral canonical models.

References

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