Finite group schemes

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① Finite gp sch.

S base scheme. (usually $S = \mathrm{Spec}(R)$).

Def: $f: G \to S$ is a gp scheme if.

$\exists\ \mu: G \times_S G \to G$, $s: S \to G$, $i: G \to G$ $/S$.

1) $\mu \circ (\mu \times id) = \mu \circ (id \times \mu)$ 2). $id_G = \mu \circ (id \times s) = \mu \circ (s \times id)$.

3). $\mu \circ (id \times i) = s \circ f = \mu \circ (i \times id)$.

Commutative ....

Def: A gp schere $f: G \to S$ is finite of order n if.

$f: G \to S$ is finite. (as schemes). &

$f_*(\mathcal{O}_G)$ is locally free of rank. n. $\mathcal{O}_S$-mod.

$n = \mathrm{ord}(G)$.

rem: $G \to S$ finite commutative gp schem of order n.

$[n]: G \to G$. annihilates. G, i.e.

$$\begin{tikzcd} & S \arrow[d, "s"] \\ G \arrow[ur] \arrow[r, "{[n]}"'] & G \end{tikzcd}$$

e.g. 1). $\mathbb{G}_a = \mathrm{Spec}(R[t])$. $\mu: t \mapsto 1 \otimes t + t \otimes 1$ $B \mapsto (B, +)$

2) $\mathbb{G}_m = \mathrm{Spec}(R[t^{\pm 1}])$ $\mu: t \mapsto t \otimes t$. $B \mapsto (B, \times)$.

3). $\mu_n = \mathrm{Spec}(R[t]/(t^n - 1))$. $= \ker(\mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m)$.

4). Δ $\overset{\text{finite}}{gp}$. $\underline{\Delta}$: $\mathrm{Spec}(R^{\Delta} = \mathrm{Fun}(\Delta, R))$. $\mu f(g, h) = f(g \cdot h)$.

5). ζ $(p-1)$-th root of unity. $\in \mathbb{Z}_p$. s.t. $\zeta^m \equiv m \bmod p$,

$\Lambda_p := \mathbb{Z}[\zeta, \frac{1}{p(p-1)}] \cap \mathbb{Z}_p \subseteq \mathbb{Q}_p$.

5). ζ $(p-1)$-th root of unity $\in \mathbb{Z}_p$, s.t. $\zeta \equiv m \bmod p$,

$$\Lambda_p := \mathbb{Z}[\zeta, \tfrac{1}{p(p-1)}] \cap \mathbb{Z}_p \subseteq \mathbb{Q}_p.$$

$B = \Lambda_p[z]/(z^p - 1)$,

$$w_i = \frac{\left(\sum_{m=1}^{p-1} \zeta^{-m}(1-z^m)\right)^i}{\left(\sum_{m=1}^{p-1} \zeta^{-im}(1-z^m)\right)}$$ is unit in Λ_p.

$\varphi : \Lambda_p \to R$.

$G^p_{a,b}$. $a, b \in R$ with $a \cdot b = p$.

$A = R[t]/(t^p - at)$

$$\mu : t \longmapsto 1 \otimes t + t \otimes 1 - \frac{b}{1-p} \sum_{i=1}^{p-1} \frac{1}{\varphi(w_i w_{p-i})} t^i \otimes t^{p-i}$$

s : $t \longmapsto 0$.

i : $t \longmapsto -t$.

Then $G^p_{a,b}$ is finite commutative gp sch of order p.

(When R is complete noetherian local alg, res char $= p$, naturally $\mathbb{Z}_p = \Lambda_p$-alg).

Thm. (Oort–Tate). Any G/R ord $p \simeq G^p_{a,b}$.

$G^p_{a,b} \simeq G^p_{c,d}$ iff $\exists u \in R^\times$ s.t. $c = u^{p-1} a$, $d = u^{1-p} b$.

Def: $0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 0$ is exact if.

1). g is fully faithful. (.g is surj).

2). f is closed immersion and $G_1 \simeq \{e\} \times_{G_3} G_2$.

Then, $\mathrm{ord}(G_2) = \mathrm{ord}(G_1) \cdot \mathrm{ord}(G_3)$.

Cartier dual. G finite commutative gp sch $/R$.

$G = \mathrm{Spec}(A) \to S = \mathrm{Spec}(R)$ finite gp scheme.

Def $G' = \mathrm{Spec}(A')$, $A' = \mathrm{Hom}_R(A, R)$

$m' : A' \otimes_R A' \to A'$ $\quad$ $\mu' : A' \to A' \otimes_R A'$.

$(\varphi, \psi) \longmapsto \varphi \otimes \psi(\mu(-))$ $\quad$ $\varphi \longmapsto \varphi(m(-))$.

e.g. 1) $(\mathbb{Z}/n\mathbb{Z})' \simeq \mu_n$.

2). $(G^p_{a,b})' \simeq G^p_{b,a}$.

pairing. $G_{a,b} \times G_{b,a} \to \mu_p$.

$$1 + \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{(x \otimes x')^i}{w_i} \longleftarrow\!\mapsto z.$$

differentials.

Prop. R an Λ_p-alg. $\forall a, b \in R$, $a \cdot b = p$. $G = G^p_{a,b}$.

then 1) $\Omega^1_{G/R} \cong R[t]/\langle t^p - at, pt^{p-1} - a\rangle \, dt$.

2). $s^*\Omega^1_{G/R} = R/a\cdot R$.

pf 1) $A = R[t]/\langle t^p - at\rangle$.

$$I/I^2 \xrightarrow{d} \Omega_{R[t]/R} \otimes A \to \Omega^1_{A/R} \to 0.$$

$$t^p - at \mapsto (pt^{p-1} - a)\, dt.$$

$t \mapsto 0$. $\Rightarrow$ 2).

Prop. $\#(s^*\Omega^1_{G/R}) \cdot \#(s^*\Omega^1_{G'/R}) = \# R/pR$.

pf. use. $(G^p_{a,b})^\vee \cong G^p_{b,a}$.

multiplicative type. $\Rightarrow$ $\# s^*\Omega^1_{G/R} = \#(R/pR)$
($G^\vee$ is étale).

Thm. R. DVR. with quotient field. K char $= 0$. Let.

$$0 \to G_1 \to G_2 \to G_3 \to 0.$$

Then, $0 \to s^*\Omega^1_{G_3/R} \to s^*\Omega^1_{G_2/R} \to s^*\Omega^1_{G_1/R} \to 0$.

pf ①. Left exact $\Rightarrow$ right exact on. $s^*\Omega^1_{\cdot/R}$. (work for any gp schms.)

$$A_1 \twoheadleftarrow A_2 \leftarrow A_3$$

$\Rightarrow$ $\Omega^1_{A_2/R} \otimes_{A_2} A_1 \to \Omega^1_{A_1/R}$ is surj.

$R \to A_2 \to A_3$ $\rightsquigarrow$.

$$\underset{\otimes A_1}{\Omega^1_{A_3/R} \otimes_{A_3} A_2} \to \Omega^1_{A_2/R} \overset{\otimes A_1}{\longrightarrow} \Omega_{A_2/A_3} \overset{\otimes A_1}{\longrightarrow} 0.$$

$$\begin{tikzcd}
& A_2 \otimes_{A_3} R \arrow[dl, "\theta", "\sim"'] & R \arrow[l] \\
A_1 & A_2 \arrow[u] \arrow[l, "\varphi"] & A_3 \arrow[u] \arrow[l, "\psi"]
\end{tikzcd}$$

$\rightsquigarrow$. $0 \leftarrow \Omega^1_{A_2/A_3} \otimes_{A_2} A_1 \leftarrow \Omega^1_{A_2/R} \otimes_{A_2} A_1 \leftarrow \cdots$

$s \uparrow \wr$ $\qquad\qquad \downarrow \varphi$.

$$\begin{array}{ccc} \cdots & & \cdots \\ s \uparrow\downarrow s & & \downarrow \varphi \\ \Omega^1_{A_2 \otimes_{A_3} B/R} & \xleftarrow[\sim]{\Theta^{-1}} & \Omega^1_{A_1/R} \end{array}$$

$\simeq$ base change.

(2)' injectivity at left.

Thm. (Raynaud). $0 \to G \hookrightarrow A \xrightarrow{\varphi} B \to 0$.

$$0 \textcolor{red}{\to} s^*\Omega^1_{B/R} \xrightarrow{\tilde\varphi} s^*\Omega_{A/R} \to s^*\Omega^1_{G/R} \to 0.$$

$$s^*\Omega^1_{B/R} \cong R^g, \qquad s^*\Omega_{A/R} \cong R^g.$$

$\tilde\varphi$ has $\deg\varphi$ as determinant. $\xRightarrow{+\,\mathrm{char}\,k=0}$ $\tilde\varphi$ is injective.

$G_2 \hookrightarrow A$, $B = A/G_{\#}$, $C = A/G_1$.

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ 0 & \to & G_1 & \to & A & \to & B \\ & & \downarrow & & & & \\ 0 & \to & G_2 & \to & A & \to & C \\ & & \downarrow & & & & \\ 0 & \to & G_3 & \to & B & \to & C \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

+ diagram chase. $\Rightarrow$ thm.

Thm. Same assumption as above. G finite gp of order n/R, G' dual.

$$\mathcal{L}(s^*\Omega^1_{G/R}) + \mathcal{L}(s^*\Omega^1_{G'/R}) = \mathcal{L}(R/nR).$$

Pf. $0 \to G \to A \to B \to 0$.

$0 \to G' \to A' \to B' \to 0$. A', B' are dual of A, B.

$\Rightarrow$ $0 \to s^*\Omega^1_B \to s^*\Omega_A \to s^*\Omega_G \to 0$.

$\Rightarrow$ $s^*\Omega_G = \mathrm{Coker}(s^*\Omega_B \xrightarrow{\varphi} s^*\Omega_A)$.

$s^*\Omega_{G'} = \mathrm{Coker}(H^1(B, \mathcal{O}_B) \xrightarrow{\varphi} H^1(A, \mathcal{O}_A))$.

(duality. $H^1(A, \mathcal{O}) = (s^*\Omega_A)^\vee$).

$$\mathcal{L}(s^*\Omega^1_G) = \mathcal{L}(\wedge^g\varphi : \Gamma(B, \Omega^g_B) \to \Gamma(A, \Omega^g_A)).$$

Duality.

$$\mathcal{L}(s^*\Omega^1_G) = \mathcal{L}(\wedge^g\varphi : P(B,\Omega^g_B) \to P(A,\Omega^g_A)).$$

$$\mathcal{L}(s^*\Omega^1_{G'}) = \mathcal{L}(\wedge^g\psi : H^g(B,\mathcal{O}_B) \to H^g(A,\mathcal{O}_A)).$$

Serre duality:

$$\begin{tikzcd} H^g(A,\mathcal{O}_A) \times P(A,\Omega^g_A) \arrow[r] & R \\ H^g(B,\mathcal{O}_B) \times P(B,\Omega^g_B) \arrow[r] \arrow[u, "\psi"] & R \arrow[u, "\deg(\varphi)"'] \end{tikzcd}$$

$+ \quad \deg\varphi = \operatorname{ord} n. \implies$ thm $\square$

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p-divisible gp.

R complete Noetherian local ring with residue field k char $p>0$

Def. A p-divisible gp $\underline{G}$ over R of height h is a system $\underline{G} = (G_k, i_k)$, $k \geq 0$.

1) G_k is finite commutative gp schem of order p^{kh}.

2) $\forall k \geq 0$,
$$0 \to G_k \xrightarrow{i_k} G_{k+1} \xrightarrow{\times p^k} G_{k+1} \quad \text{is exact.}$$

Set $i_{k,\ell} : G_k \hookrightarrow G_{k+\ell}$ by composition.

Prop. 1). The sequence $0 \to G_k \xrightarrow{i_{k,\ell}} G_{k+\ell} \xrightarrow{\times p^\ell} G_{k+\ell}$ exact.

2). G_k is annihilated by p^k.

3). $\exists$
$$\begin{tikzcd} 0 \arrow[r] & G_k \arrow[r, hook, "i_{k,\ell}"] & G_{k+\ell} \arrow[r, "\times p^\ell"] \arrow[dr, dashed, "j_{k,\ell}"'] & G_{k+\ell} \\ & & & G_\ell \arrow[u, hook, "i_{\ell,k}"'] \end{tikzcd}$$

4). $0 \to G_k \xrightarrow{i_{k,\ell}} G_{k+\ell} \xrightarrow{j_{k,\ell}} G_\ell \to 0$ is exact.

Pf. 1). induction on ℓ. $\qquad p^k \circ i_k = i_k \circ p^k$.

Pf. 1). induction on ℓ.

2).

$$\begin{tikzcd} A_k & A_{k+1} \arrow[l, two heads, "i_k"'] \arrow[r, bend right, phantom, ""] & \\ A_k \arrow[u, "p^k"] \arrow[rr, dashed, bend right] & A_{k+1} \arrow[u, "p^k"] \arrow[l, two heads, "i_k"'] \arrow[r, "s"] & R \arrow[ull, bend right] \end{tikzcd}$$

$p^k \circ i_k = i_k \circ p^k$.

factor through zero

see th $s: A_{k+1} \to R$.

3). by 2).

$$\begin{tikzcd} G_{k+\ell} \arrow[r, "p^k"] \arrow[dr, dotted, "j_{k,\ell}"'] & G_{k+\ell} \arrow[r, "p^\ell"] & G_{k+\ell} \\ & G_\ell \arrow[u, "i_{\ell,k}"'] & \end{tikzcd}$$

is zero. $\exists$. $j_{k,\ell}$.

4). Need to show $j_{k,\ell}$ is faithfully flat. $[p^k]: G_{k+\ell} \to G_{k+\ell}$ is flat.

$$\begin{tikzcd} A_k & A_{k+\ell} \arrow[l, "i_{k,\ell}"'] & A_{k+\ell} \arrow[l, "p^k"'] \arrow[d, "i_{\ell,k}"] \\ & & A_\ell \arrow[ul, "j_{k,\ell}"] \end{tikzcd}$$

flat $\to$

injective: reduce to show $rk_R(p^k(A_{k+\ell})) = p^{h\ell}$. (base change to $K = \mathrm{Frac}(R)$)

faithfully flat: finite + going-up. $j_{k,\ell}$ is surjective on closed pts.

e.g. 1) $\mathbb{G}_m / R$. $\mathbb{G}_m(p) := (\mu_{p^k}, i_k)$.

2). $\mathbb{Q}_p/\mathbb{Z}_p = (\frac{1}{p^k}\mathbb{Z}/\mathbb{Z}, i_k)$.

3). A abelian sch $/R$. $(A[p^k], i_k)$.

Def. $0 \to \underline{F} \to \underline{G} \to \underline{H} \to 0$ a seq of p-div gps.

is exact if $\forall k$.

$0 \to F_k \to G_k \to H_k \to 0$ is exact.

Étale finite (commutative) gp sch $/R$. $K = R/m$. $\mathcal{G} = \mathrm{Gal}(K^{sep}/K)$.

1) (R henselian local). étale finite gp sch$/R$ $\xrightarrow{\sim}$ étale finite gp sch$/K$. $G \mapsto G_K$.

2). A finite gp G/K is étale $\Leftrightarrow$ $G_{K^{sep}}$ is constant.

(R henselian local)

2). Δ finite gp. G/κ is étale $\Leftrightarrow$ $G_{\kappa^{sep}}$ is constant.

étale finite gp sch $/\kappa$ $\xrightarrow{\sim}$ finite. $\mathcal{G}$-modules.

$G \longmapsto G(\kappa^{sep}) \supseteq \mathcal{G}$.

$\left(\mathrm{Spec}(A),\ A = \mathrm{Map}_{\mathcal{G}}(M, \kappa^{sep})\right) \longleftarrow M$

[When char $\kappa = 0$, every finite gp sch. is étale]

char p.
$\alpha_p = \mathrm{Spec}(k[T]/T^p)$
$T \mapsto T\otimes 1 + 1 \otimes T$
is not étale.

Étale and connected.

G. finite gp sch $/R$.

$\exists$ $0 \to G^0 \to G \to G^{et} \to 0$.

where G^{et} is étale $/R$ and G^0 is connected.

Construction. $G = \mathrm{Spec}(A)$. Since R henselian, $A = \prod_{\text{finite}}$ local R-algs.

A^0 local quotient of A s.t. $s: A \to R$ factors through A^0.

$G^0 = \mathrm{Spec}(A^0)$ has a subgp structure.

G/G^0 is étale. $G/G^0 = \mathrm{Spec}(B)$ with

$B = A^G = \{a \in A \mid m(a) = 1 \otimes a\}$, $m: A \to A \otimes A$.

$G \mapsto G^{et}$, $G \mapsto G^0$ is exact functor.

$\underline{G}$ - p-divisible gp $/R$

$\rightsquigarrow$ $0 \to \underline{G}^0 \to \underline{G} \to \underline{G}^{et} \to 0$

Formal Lie gps & connected p-divisible gp.

Def: An n-dim formal Lie gp. $/R$

$F(\underline{Y}, \underline{Z}) = (F_1(\underline{Y}, \underline{Z}), \dots, F_n(\underline{Y}, \underline{Z}))$

: $\mathcal{A} \to \mathcal{A} \hat{\otimes}_R \mathcal{A}$ $X_i \mapsto F_i(\underline{Y}, \underline{Z})$, $\mathcal{A} = R[[X_1, \dots, X_n]]$.

satisfying 1). $\underline{X} = \underline{F(X, 0)} = \underline{F(0, X)}$.

satisfying 1). $\underline{X} = \underline{F(X,0)} = \underline{F(0,X)}$.

2). $F(X, F(Y,Z)) = F(F(X,Y),Z)$.

3). $F(X,Y) = F(Y,X)$.

$\exists !\ \alpha : \mathcal{A} \to \mathcal{A}$ s.t. $F(\underline{X}, \alpha(\underline{X})) = 0$. (inverse).

$s : \mathcal{A} \to R$ $X_i \mapsto 0$. (zero section).

$\rightsquigarrow (\mathfrak{m}^{\oplus n}, +_F)$ a gp structure. $(y_i) +_F (z_i) = (F(\underline{y}, \underline{z}))$.

We write. $X * Y = F(X,Y) = X + Y +$ higher terms

$\psi(X) = X * \cdots * X$ (p-times).

$\rightsquigarrow \psi : \mathcal{A} \to \mathcal{A}$. $X_i \overset{\psi}{\mapsto} \psi_i(\underline{X})$ ([p] map on $\mathcal{A}$).

$\psi^k : \mathcal{A} \to \mathcal{A}$. k-th composition, $\psi^k(X_i) = p^k X_i +$ higher ....

Def. F is divisible if ψ is injective & makes $\mathcal{A}$ into a free module of finite rk over itself.

rmk: $[m]$ is invertible on $\mathcal{A}$ if $(m,p) = 1$.

F divisible commutative formal Lie gps. $\rightsquigarrow \widetilde{F}$ p-divisible gps.

$\forall k \geq 1$. $A_k := R[[\underline{X}_i]] / (\psi^k(X_i))$.

$G_k = \operatorname{Spec}(A_k)$. form an inductive system.

$(\psi^k(X_i)) \supseteq (\psi^{k+1}(X_i)) \Rightarrow i_k : G_k \hookrightarrow G_{k+1}$.

Prop. 1) G_k is a connected finite gp sch / R.

2). $\widetilde{F} = (G_k, i_k)$ forms a connected p-divisible gp.

Thm. $F \mapsto \widetilde{F}$ is an equivalence between.

Thm. $F \longmapsto \tilde{F}$ is an equivalence between

divisible formal Lie gps $\simeq$ connected p-divisible gps.

Sketch of 1). $k=1$. Since

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\psi"] \arrow[d, "\mu"] & \mathcal{A} \arrow[d, "\mu"] \\ \mathcal{A}\otimes\mathcal{A} \arrow[r, "\psi\otimes\psi"] & \mathcal{A} \end{tikzcd}$$

formal gp law $\rightsquigarrow$ a gp law on $\overset{A_1}{\overset{\|}{A}} = \mathcal{A}/(\psi(x_i))$.

$\mathfrak{m}$ max ideal of R, $(\mathfrak{m}, X_i)$ is still the max ideal of A $\Rightarrow$ connected.

divisible : $\mathcal{A}$ is free $/\mathcal{A}$ via ψ with a basis $a_1, \cdots, a_m$

$\Longrightarrow$ A is finite free R-mod with a basis $\bar{a}_1, \cdots, \bar{a}_m$

then $m = p^n$ as G_1 is connected.

$\Longrightarrow \operatorname{rk}_R A_k = p^{nk}$ with a basis $\psi^{k-1}(a_{i_{k-1}}) \cdots \psi(a_{i_1}) a_{i_0}$.

Moreover $[p^k]$ annihilates $G_k = \operatorname{Spec}(A_k)$

Def $\omega \in \hat{\Omega}_{G/R}$ is called invariant if $\forall g \in G(S)$, $\lambda_g : G \to G/S$, we have $\lambda_g^*(\omega) = \omega$.

Fact - $\hat{\Omega}^{inv}_{G/R}$ is a free R-mod of rank $= \dim G$.

- $m : G \hat{\times}_R G \to G \Longrightarrow m^*\omega = \omega\otimes 1 + 1\otimes\omega$, $\omega \in \hat{\Omega}^1_{G/R}$

then $[N]^*\omega = N\cdot\omega$.

e.g.1) $F(X,Y) = X+Y+XY$ $\psi(X) = (1+X)^p - 1$. $(1, X, \cdots, X^{p-1})$ basis.

$\rightsquigarrow$ $(\mathfrak{m}, F) \xrightarrow{\sim} (1+\mathfrak{m}, \times)$ as gps.

$a \longmapsto 1+a$

F is divisible. $\omega(Y) = \frac{\partial F}{\partial X}(0,Y)^{-1} dY = \frac{dY}{1+Y}$.

2). $F(X,Y) = X+Y$. F is not divisible.

$\psi(X) = pX$.

$\psi: \mathcal{A} \to \mathcal{A}$, $X \mapsto pX$. is not free.

étale p-divisible gp. $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$.

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e.g. A/R abelian scheme $\rightsquigarrow$ $A(p) = (A[p^k], \iota_k)$. $ht = 2g$.

$E/\mathbb{Z}_p$ elliptic curve.

$E(p)$ is connected. $\iff$ $E_{\mathbb{F}_p}$ is supersingular

$\underline{\dim = 2}$.

$E_{\mathbb{F}_p}$ ordinary. $\rightsquigarrow$

$$0 \to \underset{\substack{\uparrow \\ \dim = 1}}{E(p)^0} \to E(p) \to \underset{\substack{\uparrow \\ ht = 1}}{E(p)^{et}} \to 0.$$

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discriminant ideal. A finite free R-mod. $\alpha_1, \cdots, \alpha_n$ a basis$/R$.

$Tr_{A/R}: A \to R$, $a \mapsto Tr(xa: A \to A)$. $\quad Nm_{A/R}: A \to R$, $a \mapsto \det(xa: A \to A)$.

$disc(A/R) = (\det.(Tr(\alpha_i \alpha_j))_{ij}) \subseteq R$.

$\mathcal{D}_{A/R}^{-1} = \{a \in A[\frac{1}{p}] \mid tr_{A/R}(a \cdot A) \subset R\}$. is fractional ideal of A.

$\mathcal{D}_{A/R} \subseteq A$ is an ideal. inverse.

— $disc(A/R) := Nm_{A/R}(\mathcal{D}_{A/R})$.

Prop $G = (G_k, \iota_k)$. p-divisible gp. $G_k = Spec(A_k)$.

Prop $G = (G_k, i_k)$. p-divisible gp. $G_k = \mathrm{Spec}(A_k)$.
Then $\mathrm{disc}(A_k/R)$ is gen. by $(p^{k\cdot n \cdot p^{h\cdot k}})$.

$n = \dim G$. $h = ht(G)$.

Pf. $0 \to G_k^{\circ} \to G_k \to G_k^{\acute{e}t} \to 0$.

$$\mathrm{disc}(G_k) = \mathrm{disc}(G_k^{\circ})^{\mathrm{ord}(G_k^{\acute{e}t})} \cdot \mathrm{disc}(G_k^{\acute{e}t})^{\mathrm{ord}(G_k^{\circ})}$$

$\mathrm{disc}\ G_k^{\acute{e}t} = 1$. WMA. G is connected i.e. $n = h$.

$G_k = \mathrm{Spec}(A_k)$. $A_k = \mathcal{A}/(\phi(X_i))$ Set $\phi_i(\underline{X}) = \phi(X_i)$.

Fact: $\mathfrak{D}_{A_k/R} = \det\left(\frac{\partial \phi_i}{\partial X_j}\right) A_k$.

($\phi_1, \cdots, \phi_n$ forms a regular sequence in $R[\![X_1, \cdots, X_n]\!]$)

$$0 \to G_k \to G \xrightarrow{\phi} H\ (\simeq G) \to 0.$$

$$\phi^*(\hat{\Omega}^n_{H/R}) \longrightarrow \hat{\Omega}^n_{G/R}.$$

$$\wedge dX_i \longmapsto \underbrace{\det\left(\frac{\partial \phi_i}{\partial X_j}\right)}_{= a} \cdot \wedge dX_i.$$

$\Rightarrow \mathfrak{D}_{A_k/R} = (a) \cdot A_k$.

$\hat{\Omega}_{G/R}$ admits a basis of invariant differentials ω_i s.t.

$[m]^* \omega_i = m \cdot \omega_i$.

$[p^k]^* \wedge \omega_i = p^{k\cdot n} \cdot \wedge \omega^i$. i.e. $a = p^{k\cdot n}$.

A_k/R is finite of rk. $p^{k\cdot n}$

$\Rightarrow \mathrm{Nm}_{A_k/R}(p^{kn}) = p^{kn \cdot p^{kn}}$ $\square$.

Tate module. G p-divisible gp/R.

$\forall\, k \geq 1$. $\quad j_{k,1}: G_{k+1} \twoheadrightarrow G_k$.

$\rightsquigarrow j_k: G_{k+1}(\bar{K}) \longrightarrow G_k(\bar{K})$.

Def. $T\underline{G} := \varprojlim G_k(\bar{K})$ $\curvearrowleft G_K$. Tate module.

As $\mathbb{Z}_p$-mod. $T\underline{G} \simeq \mathbb{Z}_p^n$, where $G_k(\bar{K})$ is a gp order p^n.
$n = \mathrm{ht}(\underline{G})$.

$\rightsquigarrow \rho_{\underline{G}}: G_K \longrightarrow GL_n(\mathbb{Z}_p)$.

$\det \rho_{\underline{G}} := \chi_{\underline{G}}: G_K \to \mathbb{Z}_p^\times$.

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e.g. 1. $\underline{G} = \mathbb{G}_m(p)$. $\quad T\underline{G} \simeq \varprojlim \mu_{p^n}(\bar{K})$.

$\rightsquigarrow \chi_{cyc}: G_K \to \mathbb{Z}_p^\times$. $\quad g(\zeta_{p^n}) = \zeta_{p^n}^{\chi_{cyc}(g)}$.

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2. étale gp.

$C = \hat{\bar{K}}$ $\curvearrowleft G_K$. continuous action.

$\forall\, \psi: G_K \to \mathbb{Z}_p^\times \rightsquigarrow$, G_K-action on C by.
$\sigma(x) = \psi(\sigma)\cdot\sigma(x)$. (twisted by ψ).

$C(\chi_0^t) := C(t)$. $\quad \forall\, t \in \mathbb{Z}$.

Def: $\forall\, L/K$ ext. $t_{\underline{G}}(L) := \mathrm{Hom}_R(I/I^2, L)$. tangent space of $\underline{G}$

I $\quad$ + 1 ideals of the forml Liegp Γ associated to G°.

I augmented ideal$/_R$ of the formal Lie gp Γ associated to G°.

$\underline{\dim t_G} = \dim \underline{G}$.

Tate. (Hodge-Tate decomposition) $\exists$ G_K-equiv isom.

$$\mathrm{Hom}(T(G), C) \simeq t_{G^\vee}(C) \oplus \Big(\underbrace{(t_G(C))^\vee \otimes C(-1)}_{\dim = d}\Big).$$

$(\Rightarrow$ HT decomposition for Abelian schs $/R$

Coro. $\Lambda^h T(G) \otimes_{\mathbb{Z}_p} C \simeq C(d)$.