TOPICS IN LANGLANDS PROGRAM (II)

The first half of the course is taught by Yihang Zhu, on Eisenstein series. The second half is taught by Bin Xu, on trace formulas.

References. For background on automorphic forms/representations and the Langlands program, we recommend [GH23], or Yihang Zhu's notes [Zhu] which follow the former closely. The book [GH23] also contains an introduction to trace formulas.

For Eisenstein series, the original source is Langlands' paper [Lan06]. Most of the foundations are found in the comprehensive book [MW95], which also contains proofs of all the main results in the subject. Also useful is the book [LW13]. For the proof of the meromorphic continuation of the Eisenstein series, we will explain the new and simpler proof in [BL24].

1. Week 1

- General motivations of trace formulas [Art05].
- The idea of discrete and continuous spectrum [GH23, §3.8], [Zhu, §4.1].
- Reductive groups, root data [Zhu, §1, §2].

2. Week 2

- Parabolic subgroups and Levi components. Classification of standard parabolic subgroups. [Zhu, §3.1].
- The analytic topology on points of a reductive group, especially the adelic group [Zhu, §3.2].
- The Harish-Chandra homomorphism and related definitions [BL24, §2.1], [MW95, §I.1.4].
- Siegel sets [MW95, §I.2.1], see also [LW13, §3.5].

3. Week 3

- Example of Siegel sets for SL_2 over $\mathbb{Q}$. See for instance [Art05, Figure 8.3].
- Relative Siegel sets [MW95, §I.2.1].
- Functions of moderate growth [MW95, §I.2.5] and functions of rapid decay [MW95, §I.2.12].
- Definition of automorphic forms [MW95, §I.2.17]
- Main theorem about Eisenstein series [BL24, §2.3].
- Proof of convergence of Eisenstein series in the case of minimal parabolic [God67, §3].

Exercise 3.1. Let G be a Hausdorff locally compact topological group, and Γ a closed subgroup.

(1) Assume that $\Gamma\backslash G$ is compact. Show that there exists a relatively compact open subset $R \subset G$ such that $G = \Gamma R$.
(2) The compactness of $\Gamma\backslash G$ is equivalent to the existence of a compact set $S \subset G$ such that $G = \Gamma S$.

Exercise 3.2. Let U be a connected unipotent linear algebraic group over $\mathbb{Q}$. By general theory of linear algebraic groups, we know there exist closed subgroups

$$1 = U_0 \subset U_1 \subset \cdots \subset U_n = U(\mathbb{A})$$

such that U_{i-1} is normal in U_i and $U_i/U_{i-1} \cong \mathbb{A}$. Moreover, the image of $U_i \cap U(\mathbb{Q}) \to U_i/U_{i-1} \cong \mathbb{A}$ is $\mathbb{Q} \subset \mathbb{A}$.

(1) Using this, show that $U(\mathbb{Q})\backslash U(\mathbb{A})$ is compact.
(2) Exhibit a choice of the U_i's when U is the subgroup of GL_n consisting of upper triangular matrices with 1's on the diagonal. Justify your answer.

Exercise 3.3. Use the usual Siegel sets and the previous two exercises to prove the following statement about relative Siegel sets: For every standard parabolic $P = MU$ in a reductive group G over $\mathbb{Q}$, there is a compact subset $\omega \subset P$, and $t_0 > 0$ such that

$$G(\mathbb{A}) = P(\mathbb{Q})\omega\{a \in A_{M_0} \mid \langle \alpha, a\rangle > t_0, \forall \alpha \in \Delta_0^P\}K.$$

4. Week 4

(Sep 30.)

- Continuation of the proof of convergence. Proof of moderate growth. For the two proofs we mainly follow [God67, §3]. Some comments on this reference:
  - Note that in [God67], automorphic forms are right $P(\mathbb{Q})$- or $G(\mathbb{Q})$-invariant, and also the Siegel set S is such that $SG(\mathbb{Q}) = G(\mathbb{A})$. Thus the order of the three factors whose product is S is reversed. The Eisenstein series is defined as a summation over $G(\mathbb{Q})/P(\mathbb{Q})$, as opposed to $P(\mathbb{Q})\backslash G(\mathbb{Q})$. His convention and ours can be easily translated into each other by replacing g by g^{-1}.
  - Above [God67, (3.9)], the assertion about $L(gg')/L(gg'')$ contains typos. The correct assertion should be: The function $L(g'g)/L(g''g)$ depends on g only via $gP(\mathbb{A})$, and hence it is uniformly bounded in g since g' and g'' move in a compact set. (In class, we had this assertion with the reversed order of multiplication, since our L is related to Godement's L by $g \mapsto g^{-1}$.)
  - We only follow the proof in [God67, §3] in the special case where P is minimal. In [God67, §3], P can be non-minimal, but there is the assumption in Theorem 3 that L is bounded on every subset of $G(\mathbb{A})$ which is compact modulo $P(\mathbb{A})^1$. If P is minimal, then the last assumption is automatic. Indeed, as we saw in class, the minimality of P implies that L is majorized by $m_P(\cdot)^\lambda$. But $m_P(\cdot)^\lambda$ factors through $P(\mathbb{A})^1\backslash G(\mathbb{A})$, and so it is bounded on every subset which is compact modulo $P(\mathbb{A})^1$ on the left. (Godement's $L(g)$ is our $L(g^{-1})$, and for Godement, "modulo $P(\mathbb{A})^1$" means "modulo $P(\mathbb{A})^1$ on the right".)
- Classical Eisenstein series for SL_2. See [Gar18, §2.8].

Exercise 4.1. Assume G is a split reductive group over $\mathbb{Q}$, with fixed minimal parabolic and Levi decomposition $P_0 = M_0U_0$. Let $\varpi : M_0 \to \mathbb{G}_m$ be a fundamental weight. Let $\rho : G \to \mathrm{GL}_n$ be the corresponding highest weight representation, with highest weight vector e_1. Thus $\rho(P_0)$ stabilizes e_1, and the function $L = m_{P_0}(\cdot)^\varpi : G(\mathbb{A}) \to \mathbb{R}_{>0}$ sends every $p \in P_0(\mathbb{A})$ to the idelic norm of the $(1,1)$-th entry of $\rho(p) \in \mathrm{GL}_n(\mathbb{A})$.

(1) Show that there exist constants $A, B > 0$ such that

$$A\|\rho(h^{-1})e_1\| \leq L(h)^{-1} \leq B\|\rho(h^{-1})e_1\|$$

for all $h \in G(\mathbb{A})$. Here, for a (column) vector $v = (a_1, \cdots, a_n)^t \in \mathbb{A}^n$, we define $\|v\| = \prod_v \max_i |a_{i,v}|_v$, where v runs over all places of $\mathbb{Q}$.

(2) Show that $\| \mathrm{GL}_n(\mathbb{Q})e_1\| \subset [1, +\infty)$.
(3) Show that for every compact set U in $\mathrm{GL}_n(\mathbb{A})$, there exist constants $B > A > 0$ such that for every $v \in \mathbb{A}^n$, we have $\|Uv\| \subset [A\|v\|, B\|v\|]$.
(4) Use the previous parts to give a new proof of the following statement (which strengthens the Key Lemma in class): For each compact set C in $G(\mathbb{A})$, there exists $D > 0$ such that $L(G(\mathbb{Q})C) \subset (0, D)$.

5. Week 5

(Oct 9.)

- Classical Eisenstein series for SL_2, convergence and moderate growth (in the non-adelic language). The proofs are essentially equivalent to our more general proofs using the adelic language. See [Gar18, §2.8] for the proof of convergence in non-adelic language.
- Convergence and moderate growth of Eisenstein series for non-minimal P, see [MW95, II.1.5].
- The intertwining operators: beginning.

References

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