Torque-Free Motion of a Rigid Body

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July 6, 2024

1 Introduction

The goal of this study is to analyze the motion of a torque-free rigid body and determine its trajectory in the inertial frame. To start, we focus on the case of a symmetric rigid body where two of the three principal moments of inertia are equal, eliminating the presence of an intermediate axis. The precession of such a body can be visualized as two circular cones rolling on top of each other. Next, we investigate the stability of non-axisymmetric rigid-body rotation about principal axes via linearization. This analysis provides insight into the conditions under which a non-axisymmetric rigid body can maintain stable rotation around its major and minor principal axes. Finally, we explore the concept of dual-spin stabilization, which builds on the stability analysis of torque-free rotation. This approach uses the dual-spin idea to stabilize spin about any principal axis, providing a practical application of spacecraft attitude control.

2 Euler's Equation of rigid-body rotation

Euler's equations are a vectorial first-order ordinary differential equation describing all rotational rigid-body dynamics [1]:

$$\frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{I}}\mathbf{h}_G\right) + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_G = \mathbf{M}_G \tag{1}$$

where ${}^{\mathcal{I}}\mathbf{h}_G$ is the angular momentum about the center of mass (CoM), $\mathbf{M}_G$ is the total external torque applied to the CoM, and the body-fixed frame $\mathcal{B}$ is rotating with respect to the inertial frame with $\mathcal{I}$ with angular velocity ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$. If the moment of inertia is constant and a set of principal axes is chosen as the body axes, the rotational equations of motion for the rigid body can be greatly simplified into the three scalar equations:

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 &= M_1 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_1\omega_3 &= M_2 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 &= M_3
\end{aligned} \tag{2}$$

, which are called Euler's equations. Here, the moment of inertia matrix is diagonal, $[\mathbb{I}_G]_{\mathcal{B}_p} = \text{diag}(I_1, I_2, I_3)$, and M_i, $i = 1, 2, 3$ are the components of $\mathbf{M}_G$ in $\mathcal{B}_p$.

In the absence of applied moments or torques, Euler's equations become

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 &= 0 \end{aligned} \tag{3}$$

Note that when the net torque acting on a body is zero, its angular momentum is conserved and remains unchanged in both direction and magnitude in absolute space. However, the body frame's orientation and the angular velocity can change with respect to the angular momentum [1].

2.1 Euler-angle based attitude representation

Let us consider the 3-2-3 rotation with Euler angles $(\psi, \theta, \phi)_{\mathcal{B}}^{\mathcal{I}}$ to describe the orientation in frame $\mathcal{I}$. The angular velocity ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_2\mathbf{b}_3$ can be related to the Euler angle rate by the following kinematic equations:

$$\dot{\psi} = (-\omega_1\cos\phi + \omega_2\sin\phi)\csc\theta \qquad \omega_1 = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \tag{4}$$

$$\dot{\theta} = \omega_1\sin\phi + \omega_2\cos\phi \qquad \omega_2 = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \tag{5}$$

$$\dot{\phi} = (\omega_1\cos\phi - \omega_2\sin\phi)\cot\theta + \omega_3 \qquad \omega_3 = \dot{\phi} + \dot{\psi}\cos\theta \tag{6}$$

Notice that gimbal lock occurs at $\theta = 0$ or π, which should be avoided in simulation.

3 Torque-free motion of an axisymmetric rigid body

Consider a rigid body that is axisymmetric with transverse moments of inertia $I_x = I_y = I_T$ and axial moment of inertia I_3. Plugging these into (3) yields

$$I_T\dot{\omega}_1 + (I_3 - I_T)\,\omega_2\omega_3 = 0 \tag{7}$$

$$I_T\dot{\omega}_2 - (I_3 - I_T)\,\omega_1\omega_3 = 0 \tag{8}$$

$$I_3\dot{\omega}_3 = 0 \tag{9}$$

Equation (9) means that ω_3 is constant, i.e., $\omega_3(t) = \omega_3(0)$. Also combining (7) and (8) yields the simple harmonic motion [2]

$$\begin{cases} \ddot{\omega}_1 + \omega_n^2 = 0 \\ \ddot{\omega}_2 + \omega_n^2 = 0 \end{cases}, \quad \text{where } \omega_n = \left(1 - \frac{I_3}{I_T}\right)\omega_3$$

with the general solution $\omega_i = A\cos(\omega_n t) + B\sin(\omega_n t)\,,\ i = 1, 2,$ where A, B are undetermined coefficients depending on initial conditions. This implies that the magnitude of the resultant angular velocity is a constant. For example, $\omega_1(0) = -\omega_o,\ \omega_2(0) = 0,\ \omega_3(0) = \omega_3$, we will have the solution $\omega_1(t) = -\omega_o\cos(\omega_n t)\,,\ \omega_2(t) = \omega_o\sin(\omega_n t)$ and the angular momentum about G written in body frame is ${}^{\mathcal{I}}\mathbf{h}_G = -I_T\omega_o\cos(\omega_n t)\,\mathbf{b}_1 + I_T\omega_o\sin(\omega_n t)\,\mathbf{b}_2 + I_3\mathbf{b}_3$, which can be interpreted as the $\mathbf{b}_3$ symmetric axis of the rigid body traces out a cone (Body Cone) along the inertial direction defined by ${}^{\mathcal{I}}\mathbf{h}_G$, i.e., $\mathbf{e}_3 =^{\mathcal{I}} \mathbf{h}_G / \left\| {}^{\mathcal{I}}\mathbf{h}_G \right\|$. The angle between the angular momentum vector ${}^{\mathcal{I}}\mathbf{h}_G$ and $\mathbf{b}_3$ symmetric axis can be found by inner product:

$$\cos\theta = \frac{{}^{\mathcal{I}}\mathbf{h}_G \cdot \mathbf{b}_3}{\|{}^{\mathcal{I}}\mathbf{h}_G\|} = \frac{I_3\omega_3}{\sqrt{I_T^2\omega_o^2 + I_3^2\omega_3^2}} \tag{10}$$

Since $I_T, I_3, \omega_o, \omega_3$ are all constant, the nutation angle θ will also be constant in absolute space, i.e., $\dot{\theta} = 0$. That is, the angular velocity vector The angular velocity can be represented in terms of Euler angle rate by using (4)-(6):

$$\omega_1 = -\omega_o \cos(\omega_n t) = -\dot{\psi}\sin\theta\cos\phi \tag{11}$$

$$\omega_2 = \omega_o \sin(\omega_n t) = \dot{\psi}\sin\theta\sin\phi \tag{12}$$

$$\omega_3 = \dot{\phi} + \dot{\psi}\cos\theta \tag{13}$$

The percession rate $\dot{\psi}$ is derived by taking a square root of the sum of (11) and (12), and the spin rate $\dot{\phi}$ can found [1, 3]:

$$\dot{\psi} = \frac{\omega_o}{\sin\theta} \quad \text{and} \quad \dot{\phi} = \left(1 - \frac{I_3}{I_T}\right)\omega_3 \tag{14}$$

3.1 Geometric interpretation: Space Cone and Body Cone

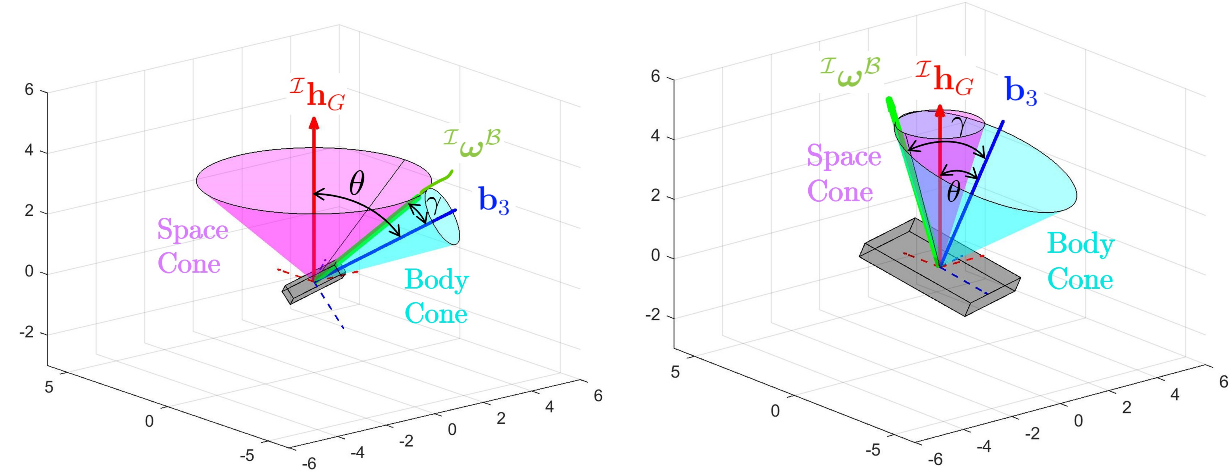
In this section, we will explore the movement of prolate and oblate objects in the inertial frame and examine how the space and body cones interact. As seen in Fig. 1, the "Space Cone" surrounds the fixed angular momentum in absolute space and is defined by the angle between ${}^{\mathcal{I}}\mathbf{h}_G$ and ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$, which describes how ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$ moves in inertial frame; while the "Body Cone" surrounds the $\mathbf{b}_3$ symmetric axis and is defined by the angle between ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$ and $\mathbf{b}_3$ denoted as γ, which describes how ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$ moves with respect to the body. We now are interested in two angles, θ and γ, defined with respect to the $\mathbf{b}_3$ body symmetric axis. Since $I_x = I_y = I_T$, we can again write the angular momentum in the body frame:

$${}^{\mathcal{I}}\mathbf{h}_G = I_T\omega_1\mathbf{b}_1 + I_T\omega_2\mathbf{b}_2 + I_3\omega_3\mathbf{b}_3 = I_T{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + (I_3 - I_T)\,\omega_3\mathbf{b}_3 \;\Rightarrow\; {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} = \frac{1}{I_T}{}^{\mathcal{I}}\mathbf{h}_G + \frac{(I_T - I_3)}{I_T}\omega_3\mathbf{b}_3$$

It implies that ${}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}$, ${}^{\mathcal{I}}\mathbf{h}_G$, and $\mathbf{b}_3$ are coplanar vectors in the three-dimensional space. Thus, the two angles, θ and γ can be found via the geometry of vectors,

$$\tan\theta = \frac{I_T\sqrt{\omega_1^2 + \omega_2^2}}{I_3\omega_3} \quad \text{and} \quad \tan\gamma = \frac{\sqrt{\omega_1^2 + \omega_2^2}}{\omega_3} \tag{15}$$

Thus we have the relationship $\tan\theta = \frac{I_T}{I_3}\tan\gamma$. For the prolate rigid body with $I_T > I_3$, we have ($\theta > \gamma$) implying that the body cone rolls around the outside of the space cone, as seen in Fig. 1(a); Similarly, for the oblate rigid body with $I_T < I_3$, we have ($\theta < \gamma$) implying that the inside of the body cone rolls around the space cone, as seen in Fig. 1(b).



(a) Prolate case $I_T > I_3$ $(\theta > \gamma)$.

(b) Oblate case $I_T < I_3$ $(\theta < \gamma)$.

Figure 1: Space and body cones for the precession of a rigid body.

4 Torque-free motion of a non-axisymmetric rigid body

In this section, we study the torque-free motion of a general rigid body with $I_1 > I_2 > I_3$, spinning about any of principal axis. An arbitrary rigid body is in equilibrium when it spins around any of its principal axes, and the stability of an equilibrium point is investigated by linearizing the dynamics about it.

4.1 Stability analysis on principal axes rotation

We first choose an arbitrary equilibrium point $\boldsymbol{\omega}_o = (0, 0, n)^T$. Linearizing around $\boldsymbol{\omega}_o$ using small disturbance theory, $\boldsymbol{\omega} = \boldsymbol{\omega}_o + \Delta\boldsymbol{\omega}$, yields

$$\Delta\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}_o = \begin{bmatrix} \frac{(I_2 - I_3)}{I_1}\left(\omega_{2,o} + \Delta\omega_2\right)\left(\omega_{3,o} + \Delta\omega_3\right) \\ \frac{(I_3 - I_1)}{I_2}\left(\omega_{3,o} + \Delta\omega_3\right)\left(\omega_{1,o} + \Delta\omega_1\right) \\ \frac{(I_1 - I_2)}{I_3}\left(\omega_{1,o} + \Delta\omega_1\right)\left(\omega_{2,o} + \Delta\omega_2\right) \end{bmatrix} = \begin{bmatrix} \frac{(I_2 - I_3)}{I_1} n \Delta\omega_2 \\ \frac{(I_3 - I_1)}{I_2} n \Delta\omega_1 \\ 0 \end{bmatrix} \tag{16}$$

where the products of disturbances are negligible. Rewrite the first two equations into the matrix form:

$$\begin{bmatrix} \Delta\dot{\omega}_1 \\ \Delta\dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{(I_2 - I_3)}{I_1} n \\ \frac{(I_3 - I_1)}{I_2} n & 0 \end{bmatrix} \begin{bmatrix} \Delta\omega_1 \\ \Delta\omega_2 \end{bmatrix} \tag{17}$$

The corresponding characteristic polynomial equation and roots are

$$s^2 - \frac{(I_2 - I_3)}{I_1}\frac{(I_3 - I_1)}{I_2} n^2 = 0 \;\Rightarrow\; s = \pm\sqrt{\frac{(I_2 - I_3)}{I_1}\frac{(I_3 - I_1)}{I_2}}\, n \tag{18}$$

If the spin axis is the major axis (axis of maximum moment of inertia) with $I_3 > I_1$ and $I_2 > I_1$ or the minor axis (axis of minimum moment of inertia) with $I_3 < I_1$ and $I_2 < I_1$, the product

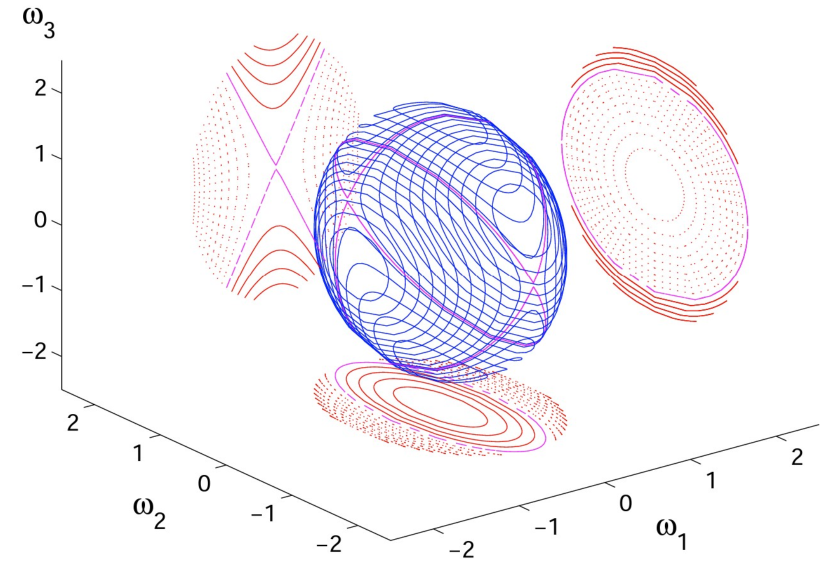


Figure 2: Polhode curves on the Poinsot Ellipsoid [2]. The shape of polhodes predict the stability of the equilibria associated with principal axes. The major and minor axes are centers (stable), while the intermediate axis is a saddle point (unstable).

$(I_2 - I_3)(I_3 - I_1)$ is negative, then the roots will pure imaginary and the resulting motion is marginally stable and is bounded by $\sqrt{\frac{(I_3-I_2)}{I_1}\frac{(I_3-I_1)}{I_2}}n$. On the other hand, if the spin axis is the intermediate one with $I_3 > I_1$ and $I_3 < I_1$, the product $(I_2 - I_3)(I_3 - I_1)$ is positive, then we will get one negative and one positive real poles, and the motion is unstable due to the presence of RHP pole, see Fig. 3.

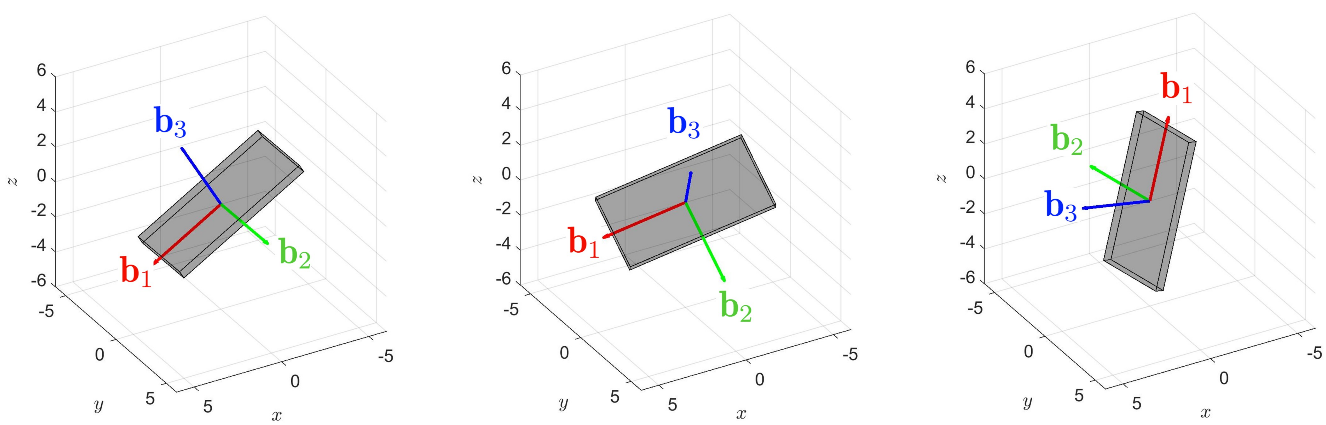


Figure 3: Instability of rotating about the intermediate axis.

5 A practical application: Dual-spin Stabilization

Dual-spin stabilization is a method of stabilizing a spacecraft's orientation in space by adding a rotor/flywheel inside of the spacecraft [2, 4]. Our goal is to achieve stable spinning of the spacecraft around an intermediate axis by making the flywheel spin at the proper speed. Consider a wheel locked to the satellite platform and rotating about the body axis $\mathbf{b}_1$ with respect to the platform at speed Ω, then the total angular momentum of the satellite is

$$
{}^{\mathcal{I}}\mathbf{h}_G = (\mathbb{I}_s + \mathbb{I}_w)\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + I_{w_s}\Omega\mathbf{b}_1 := [\mathbb{I}]_{\mathcal{B}_p}\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} + h\mathbf{b}_1 \tag{19}
$$

where $\mathbb{I}_s, \mathbb{I}_w$ are the inertia matrices of spacecraft and flywheel, respectively, I_{w_s} is the scalar moment of inertia of flywheel about $\mathbf{b}_1$. Based on Euler's second law, we have

$$
\begin{aligned}
&\frac{{}^{\mathcal{I}}d}{dt}\left({}^{\mathcal{I}}\mathbf{h}_G\right) = \frac{{}^{\mathcal{B}}d}{dt}\left({}^{\mathcal{I}}\mathbf{h}_G\right) + {}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} \times {}^{\mathcal{I}}\mathbf{h}_G = \mathbf{0} \\
\Rightarrow\;& [\mathbb{I}]_{\mathcal{B}_p}\,{}^{\mathcal{I}}\dot{\boldsymbol{\omega}}^{\mathcal{B}} = -\left[{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}}\times\right]_{\mathcal{B}_p}[\mathbb{I}]_{\mathcal{B}_p}\,{}^{\mathcal{I}}\boldsymbol{\omega}^{\mathcal{B}} - \dot{h}\mathbf{b}_1 - h\omega_3\mathbf{b}_2 + h\omega_2\mathbf{b}_3
\end{aligned} \tag{20}
$$

Rewrite the matrix equation (20) into the three scalar equations for the angular velocity components in $\mathcal{B}_p$.

$$
\begin{aligned}
\dot{\omega}_1 &= \frac{(I_2 - I_3)}{I_1}\omega_2\omega_3 - \frac{I_{w_s}}{I_1}\dot{\Omega} \\
\dot{\omega}_2 &= \frac{(I_3 - I_1)}{I_2}\omega_1\omega_3 - \frac{I_{w_s}}{I_2}\omega_3\Omega \\
\dot{\omega}_3 &= \frac{(I_1 - I_2)}{I_3}\omega_1\omega_2 + \frac{I_{w_s}}{I_3}\omega_2\Omega
\end{aligned} \tag{21}
$$

Consider the flywheel rotates at the constant speed, i.e., $\dot{\Omega}$=0, and the equilibrium point $\boldsymbol{\omega}_e = (\omega_{e,1}, 0, 0)^T$. Similarly, linearizing around $\boldsymbol{\omega}_e$ using small disturbance theory, we have the linearized equations of motion:

$$
\Delta\dot{\omega}_1 = 0 \tag{22}
$$

$$
\Delta\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\omega_{e,1} - \frac{I_{w_s}}{I_2}\Omega\right)\Delta\omega_3 \tag{23}
$$

$$
\Delta\dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\omega_{e,1} + \frac{I_{w_3}}{I_2}\Omega\right)\Delta\omega_2 \tag{24}
$$

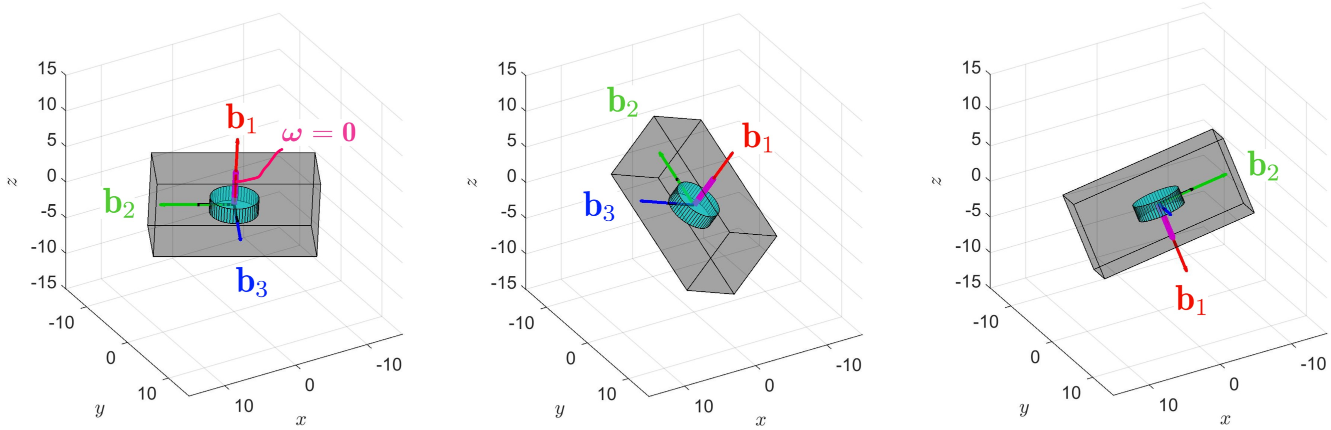
Combining (23) and (24) yields: $\Delta\omega_1 = \text{const.}$, $\Delta\ddot{\omega}_i + k\Delta\omega_i = 0, i = 1, 2$, where

$$
k = \frac{\omega_{e_1}^2}{I_2 I_3}\left(I_1 - I_3 + I_{w_s}\hat{\Omega}\right)\left(I_1 - I_2 + I_{w_s}\hat{\Omega}\right),\; \hat{\Omega} := \frac{\Omega}{\omega_{e_1}}
$$

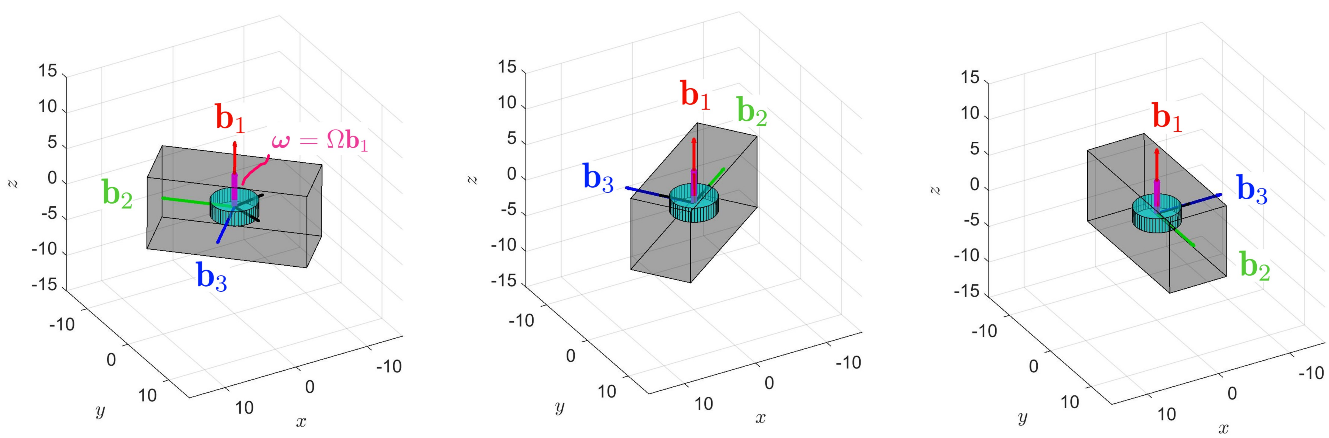
Thus, stability of spin about $\mathbf{b}_1$ requires $k > 0$.

5.1 Numerical Simulation

Consider a spacecraft including flywheel with moments of inertia $I_1 = 350, I_2 = 300, I_3 = 400, I_{w_s} = 10$ in the units of kg·m^2. We want the spacecraft to rotate about the intermediate axis $\mathbf{b}_1$ at 60 RPM. The positive condition for k ensures the stability of rotation, and the range that satisfies $k > 0$ is $\hat{\Omega} > 5$ or $\hat{\Omega} < -5$. Thus, the corresponding rotating speeds can be found as $\Omega > 5\omega_{e_1} \approx 31.4$(rad/s) or $\Omega < -31.4$, as seen in Fig. 4.



(a) Unstable spin of intermediate axis for the locked flywheel with $\hat{\Omega} = 0$



(b) Stable spin of intermediate axis for the spinning flywheel with $\hat{\Omega} = 20$

Figure 4: Dual-spin stabilization of a satellite.

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