MATRIX MODELS FOR MULTILEVEL HECKMAN-OPDAM AND MULTIVARIATE BESSEL MEASURES

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Abstract. We study multilevel matrix ensembles at general β by identifying them with a class of processes defined via the branching rules for multivariate Bessel and Heckman-Opdam hypergeometric functions. For $\beta = 1, 2$, we express the joint multilevel density of the eigenvalues of a generalized Wishart matrix as a multivariate Bessel ensemble, generalizing a result of Dieker-Warren in [DW09]. In the null case, we prove the conjecture of Borodin-Gorin in [BG13] that the joint multilevel density of the β-Jacobi ensemble is given by a principally specialized Heckman-Opdam measure.

Contents

1. Introduction

The purpose of the present work is to provide a link between measures defined via the branching structure of certain multivariate hypergeometric functions at general β and the multilevel eigenvalue density of certain random matrix models at $\beta = 1, 2$. In particular, we consider multivariate Bessel and Heckman-Opdam processes defined in a way similar to the Schur process of [OR03] and the Macdonald process of [BC14]. At

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$\beta = 1$, we show that the multivariate Bessel process with general specializations is realized by the multilevel eigenvalue density of the generalized β-Wishart ensemble, generalizing a result of Dieker-Warren in [DW09] for $\beta = 2$. At $\beta = 1, 2$, we prove a conjecture of Borodin-Gorin in [BG13] that the Heckman-Opdam process at shifted principal specializations is realized by the multilevel eigenvalue density of the generalized β-Jacobi ensemble.

The motivation for our work stems from the work of Borodin-Gorin in [BG13], where they use degenerations of techniques from Macdonald processes to show that a rescaling of the Heckman-Opdam process converges to a 2 dimensional Gaussian free field. Combined with our identification of the Heckman-Opdam and β-Jacobi ensembles at $\beta = 1, 2$, this reveals a Gaussian free field structure in the eigenvalues of random matrices, as was first shown for Wigner random matrices in [Bor14].

In future work, we hope to extend this philosophy to the generalized β-Wishart case. A special case of this setting is the real spiked covariance model, which admits statistical applications (see [Joh01, OMH14]) and exhibits the Baik-Ben Arous-Péché phase transition for the largest eigenvalue (see [BBP05, Mo12, BV13]). Our hope is that applying techniques of integrable probability to multivariate Bessel measures will allow us to give a new proof of the BBP phase transition and understand its multilevel structure.

In the remainder of this introduction, we state our results more precisely and provide additional motivation and background. For convenience, all notations will be redefined in later sections.

1.1. **Results for generalized β-Wishart ensembles.** For $\beta > 0$, let $\mathcal{B}^{n,m}_\beta(\lambda, s)$ and $\widetilde{\mathcal{B}}^{n,m}_\beta(\lambda, s)$ denote the multivariate Bessel and dual multivariate Bessel functions with parameter β, defined in detail in Subsection 2.5. Fix $\theta = \beta/2$. The multivariate Bessel ensemble with parameters $\{\pi_i\}_{1\le i\le n}$ and $\{\widehat{\pi}_j\}_{j\ge 1}$ is the process on $\{\mu^m_i\}_{1\le i\le \min\{n,m\}}$ whose distribution on the first m levels is supported on

$$\mu^1 \prec \cdots \prec \mu^m, \qquad \mu^l_i \in [0, \infty)$$

with joint density proportional to

$$\Delta(\mu^m)^\theta e^{-\theta \sum_{l=1}^m \widehat{\pi}_l(|\mu^l| - |\mu^{l-1}|)} \prod_{i=1}^{\min\{n,m\}} (\mu^m_i)^{\theta(m-\min\{m,n\})} \frac{(\mu^{\min\{n,m\}}_i)^{\theta-1}}{(\mu^m_i)^{\theta-1}} \prod_{l=1}^{m-1} \frac{\Delta(\mu^l, \mu^{l+1})^{\theta-1}}{\Delta(\mu^l)^{\theta-1}\Delta(\mu^{l+1})^{\theta-1}} \widetilde{\mathcal{B}}^{n,n}_\beta(\mu^n, -\theta\pi),$$

where for sets of variables μ and λ we define $\Delta(\mu) = \prod_{i<j}(\mu_i - \mu_j)$ and $\Delta(\mu, \lambda) = \prod_{i,j} |\mu_i - \lambda_j|$. The marginal density on level m is proportional to

$$\Delta(\mu^m)^{2\theta} \prod_{i=1}^{\min\{m,n\}} (\mu^m_i)^{\theta(m-\min\{n,m\})} \mathcal{B}^{\min\{n,m\},m}_\beta(\mu^m, -\theta\widehat{\pi}) \widetilde{\mathcal{B}}^{\min\{n,m\},n}_\beta(\mu^m, -\theta\pi).$$

At $\beta = 2$, a multilevel matrix model was proposed in [BP08] and proven in [DW09] for the multivariate Bessel ensemble. Our first main result is a construction of such a matrix model at $\beta = 1$. Fix some $n \ge 1$. Let (A_{ij}) be an infinite matrix of independent real Gaussian random variables with mean 0 and variance $(\pi_i + \widehat{\pi}_j)^{-1}$, and let A_m be its top $m \times n$ corner. At $\beta = 1$, define the generalized β-Wishart process to be the sequence of eigenvalues of the $n \times n$ symmetric positive semi-definite matrices

$$M_m := A_m^T A_m.$$

Let $\{\mu^m_i\}_{1\le i\le \min\{m,n\}}$ be the non-zero eigenvalues of M_m and denote their joint distribution by $P^{\pi,\widehat{\pi}}$. The following two results show that the β-Wishart process gives a matrix model for the multivariate Bessel process at $\beta = 1$.

Theorem 3.3. The process $\{\mu^m_i\}_{m\ge 1}$ is a Markov chain under $P^{\pi,\widehat{\pi}}$ with transition kernel

$$Q^{\pi,\widehat{\pi}}_{m-1,m}(\mu^{m-1}, d\mu^m) = \frac{\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V \widetilde{\mu}^m V^T)} d\mathsf{Haar}_V}{\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V \widetilde{\mu}^{m-1} V^T)} d\mathsf{Haar}_V} \frac{1}{\Gamma(m/2)\Gamma(1/2)^m} \prod_{i=1}^n (\pi_i + \widehat{\pi}_m)^{\frac{1}{2}} e^{-\frac{1}{2}\widehat{\pi}_m(|\mu^m| - |\mu^{m-1}|)} \frac{\prod_{i=1}^m (\mu^m_i)^{\frac{n-m-1}{2}}}{\prod_{i=1}^{m-1} (\mu^{m-1}_i)^{\frac{n-m}{2}}} \Delta(\mu^m) \prod_{i,j} |\mu^m_i - \mu^{m-1}_j|^{-\frac{1}{2}} 1_{\mu^{m-1} \prec \mu^m} d\mu^m,$$

where $d\mathsf{Haar}_V$ denotes Haar measure on the orthogonal group.

Corollary 3.8. The joint density of $\{\mu_i^m\}_{1\le m\le n}$ under $P^{\pi,\widehat{\pi}}$ is given by

$$\frac{\Gamma(1/2)^{n-n(n-1)/2}}{\Gamma(n/2)\cdots\Gamma(1/2)}\prod_{i=1}^{n}\prod_{j=1}^{m}(\pi_i+\widehat{\pi}_j)^{1/2}$$
$$\Delta(\mu^n)^{1/2}e^{-\frac{1}{2}\sum_{l=1}^{n}\widehat{\pi}_l(|\mu^l|-|\mu^{l-1}|)}\prod_{l=1}^{m-1}\frac{\Delta(\mu^l,\mu^{l+1})^{-1/2}}{\Delta(\mu^l)^{-1/2}\Delta(\mu^{l+1})^{-1/2}}\widetilde{\mathcal{B}}_1^{n,n}(\mu^n,-\pi/2).$$

1.2. **Results for β-Jacobi ensembles.** For $\beta>0$, let $\mathcal{F}_\beta^{n,m}(\lambda,s)$ and $\widetilde{\mathcal{F}}_\beta^{n,m}(\lambda,s)$ denote the Heckman-Opdam and dual Heckman-Opdam hypergeometric functions, defined in detail in Subsection 2.4. The Heckman-Opdam ensemble with parameters $\{\pi_i\}_{1\le i\le n}$ and $\{\widehat{\pi}_i\}_{i\ge 1}$ is the process on $\{\mu_i^m\}_{1\le i\le \min\{n,m\}}$ whose joint distribution on the first m levels is supported on

$$\mu^1\prec\cdots\prec\mu^m \qquad \mu_i^l\in[0,\infty)$$

with density proportional to

$$\Delta^{\mathrm{trig}}(\mu^m)^\theta\prod_{i=1}^{\min\{m,n\}}(1-e^{-\mu_i^m})^{\theta(m-\min\{m,n\})}\prod_{i=1}^{\min\{m,n\}}\frac{(1-e^{-\mu_i^{\min\{m,n\}}})^{\theta-1}}{(1-e^{-\mu_i^m})^{\theta-1}}e^{-\theta\sum_{l=1}^{m}\widehat{\pi}_l(|\mu^l|-|\mu^{l-1}|)}$$
$$\prod_{l=1}^{m-1}\frac{\Delta(e^{-\mu^l},e^{-\mu^{l+1}})^{\theta-1}}{\Delta(e^{-\mu^l})^{\theta-1}\Delta(e^{-\mu^{l+1}})^{\theta-1}}e^{(\theta-1)|\mu^l|}\widetilde{\mathcal{F}}_\beta^{m,n}(\mu^m,-\theta\pi).$$

Its marginal density on level m is proportional to

$$\prod_{i=1}^{\min\{m,n\}}(1-e^{-\mu_i^m})^{\theta(m-\min\{m,n\})}\Delta^{\mathrm{trig}}(\mu^m)^{2\theta}\mathcal{F}_\beta^{\min\{m,n\},m}(\mu^m,-\theta\widehat{\pi})\widetilde{\mathcal{F}}_\beta^{\min\{m,n\},n}(\mu^m,-\theta\pi).$$

Our second main result is an identification of the Heckman-Opdam ensemble at $\beta=1,2$, $\pi=\theta(A-m+1,A-m+2,\ldots,A-m+n)$, and $\widehat{\pi}=\theta(0,1,\ldots)$ with the β-Jacobi ensemble, as conjectured in [BG13]. To define the β-Jacobi ensemble, let X and Y be infinite matrices of independent standard Gaussian random variables over $\mathbb{R}$ for $\beta=1$ and $\mathbb{C}$ for $\beta=2$. Choose $A\ge n$ and $m\le n$, and let X^{An} and Y^{mn} denote the left $A\times n$ and $m\times n$ corners of X and Y. Then the matrix

$$(X^{An})^*X^{An}((X^{An})^*X^{An}+(Y^{mn})^*Y^{mn})^{-1}$$

has $m=\min\{A,m,n\}$ non-zero eigenvalues $\lambda_1^m,\ldots,\lambda_m^m$ in $(0,1)$. The β-Jacobi ensemble is the joint distribution of the eigenvalues $\{\lambda_i^m\}_{1\le i\le m}$ for $1\le m\le n$. The following two theorems identify it with a principally specialized Heckman-Opdam measure.

Theorem 4.1. The eigenvalues $\{\lambda_i^l\}$ of (4.2) at $\beta=2$ are supported for $m\le n$ on interlacing sequences

$$\lambda^1\prec\cdots\prec\lambda^m, \qquad \lambda_i^l\in[0,1]$$

with joint density proportional to

$$\Delta(\lambda^m)\prod_{i=1}^{m}(\lambda_i^m)^{A-1}\prod_{l=1}^{m-1}\prod_{i=1}^{l}(\lambda_i^l)^{-2}\prod_{l=1}^{m}\prod_{i=1}^{l}d\lambda_i^l.$$

Theorem 4.2. The eigenvalues $\{\lambda_i^l\}$ of (4.2) at $\beta=1$ are supported for $m\le n$ on interlacing sequences

$$\lambda^1\prec\cdots\prec\lambda^m, \qquad \lambda_i^l\in[0,1]$$

with joint density proportional to

$$\Delta(\lambda^m)\prod_{i=1}^{m}(\lambda_i^m)^{A/2-1}(1-\lambda_i^m)^{-1/2}\prod_{l=1}^{m-1}\prod_{i=1}^{l}(\lambda_i^l)^{-1}\Delta(\lambda^l)\prod_{i,j}|\lambda_i^l-\lambda_j^{l+1}|^{-1/2}.$$

1.3. **Relation to the literature.** In [BP08], Borodin-Péché introduced the $\beta = 2$ generalized Wishart ensemble and conjectured that its joint eigenvalue structure is described by a $\beta = 2$ multivariate Bessel measure with corresponding specializations. This conjecture was proven for special values of parameters in [FN11] and in general in [DW09]. Our Theorem 3.3 and Corollary 3.8 generalize these results to the $\beta = 1$ case. This case is of particular interest because it contains the statistically important case of real sample covariance matrices under Johnstone's spiked covariance model; we refer the reader to [Joh01, OMH14] for some examples of such applications.

In [BG13], Borodin-Gorin identify the single-level marginal principally specialized Heckman-Opdam ensemble with the single level density of the β-Jacobi ensemble. They then proved that a rescaling of the multilevel Heckman-Opdam process (termed the β-Jacobi corners process in [BG13]) converges to the Gaussian free field in a large N limit. Borodin-Gorin conjectured that for $\beta = 1, 2, 4$, the principally specialized Heckman-Opdam ensemble corresponded to a multilevel eigenvalue process for real, complex, and symplectic random Jacobi random matrices, respectively. Our Theorems 4.1 and 4.2 prove this for $\beta = 1, 2$, providing an interpretation of the probabilistic limit theorems of [BG13] in terms of random matrix ensembles.

Remark. A different approach to this identification may be obtained via the random co-rank 1 projections discussed in [FR05], though we do not pursue this approach further here.

1.4. **Outline of method and organization.** The remainder of this paper is organized as follows. In Section 2, we fix our notations for Heckman-Opdam hypergeometric functions and multivariate Bessel functions and derive some identities for them as degenerations of the corresponding identities for Macdonald polynomials. In Section 3, we define the multivariate Bessel ensemble and use the generalized β-Wishart ensemble to give a multilevel matrix model for it at $\beta = 1, 2$. In Section 4, we define the Heckman-Opdam ensemble and prove the conjecture of Borodin-Gorin that its principal specialization has matrix model given by the β-Jacobi ensemble at $\beta = 1, 2$. In Appendix A, we collect some elementary computations of limits of different special functions which appear in our limit transitions.

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2. Multivariate Bessel and Heckman-Opdam functions

In this section, we fix our notations on multivariate Bessel functions, Heckman-Opdam hypergeometric functions, and Macdonald polynomials. We then describe scaling limits which transform Macdonald polynomials to Heckman-Opdam hypergeometric functions and then multivariate Bessel functions. Finally, we take limits of the Cauchy identity for Macdonald polynomials to prove Cauchy identities for multivariate Bessel and Heckman-Opdam hypergeometric functions.

2.1. **Notations.** Throughout this paper we denote the rational and trigonometric Vandermondes by

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) \qquad \text{and} \qquad \Delta^{\mathrm{trig}}(\lambda) = \prod_{i<j}\left(e^{\frac{\lambda_i-\lambda_j}{2}} - e^{\frac{\lambda_j-\lambda_i}{2}}\right).$$

Notice that $\Delta^{\mathrm{trig}}(\lambda) = e^{\frac{n-1}{2}|\lambda|}\Delta(e^{-\lambda})$. For a fixed $\beta > 0$, define $\theta := \frac{\beta}{2}$. For $\lambda_1 \geq \cdots \geq \lambda_n \in \mathbb{R}^n$, define the Gelfand-Tsetlin polytope to be

$$\mathrm{GT}_\lambda := \{(\mu_i^l)_{1\leq i\leq l, 1\leq l<n} \mid \mu_i^{l+1} \geq \mu_i^l \geq \mu_{i+1}^{l+1}\},$$

where we take $\mu_i^n = \lambda_i$. A point $\{\mu_i^l\}$ in GT_λ is called a Gelfand-Tsetlin pattern subordinate to λ.

2.2. **Macdonald polynomials.** We recall some identities for Macdonald polynomials; we refer the reader to the book of [Mac95] for a complete treatment. We will take scaling limits of these to obtain facts on Heckman-Opdam hypergeometric functions and multivariate Bessel functions. Let $P_\lambda(x; q, t)$ and $Q_\lambda(x; q, t)$ denote the Macdonald and dual Macdonald polynomials. Recall that

$$Q_\lambda(x; q, t) = b_\lambda(q, t) P_\lambda(q, t),$$

where $b_\lambda(q,t)$ is defined by taking $a(s) = \lambda_i - j$ and $l(s) = \lambda'_j - i$ for $s = (i,j)$ and setting

$$b_\lambda(q,t) := \langle P_\lambda, P_\lambda\rangle^{-1} = \prod_{s\in\lambda} \frac{1 - q^{a(s)}t^{l(s)+1}}{1 - q^{a(s)+1}t^{l(s)}} = \prod_{l=1}^{m}\prod_{i=0}^{l-1} \frac{(t^{i+1}q^{\lambda_{l-i}-\lambda_l};q)_{\lambda_l-\lambda_{l+1}}}{(t^i q^{\lambda_{l-i}-\lambda_l+1};q)_{\lambda_l-\lambda_{l+1}}}. \tag{2.1}$$

The Macdonald polynomials satisfy a Cauchy identity, evaluation identity, and branching rule.

Proposition 2.1 (Cauchy identity)**.** For any $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$, we have

$$\sum_{\ell(\lambda)\le\min\{m,n\}} P_\lambda(x_1,\ldots,x_m;q,t)Q_\lambda(y_1,\ldots,y_n;q,t) = \prod_{i=1}^{n}\prod_{j=1}^{m}\frac{(tx_iy_j;q)}{(x_iy_j;q)}.$$

Proposition 2.2 (Evaluation identity)**.** For any λ and m with $n = \ell(\lambda) \le m$, we have

$$P_\lambda(1,t,\ldots,t^{m-1};q,t) = t^{\sum_{i=1}^n (i-1)\lambda_i} \prod_{1\le i<j\le n} \frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)}\frac{(t^{j-i+1};q)}{(t^{j-i};q)} \prod_{i=1}^{n}\prod_{j=n+1}^{m}\frac{(q^{\lambda_i}t^{j-i};q)}{(q^{\lambda_i}t^{j-i+1};q)}\frac{(t^{j-i+1};q)}{(t^{j-i};q)}.$$

Proposition 2.3 (Branching rule)**.** For $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n, 0, \ldots, 0)$, we have

$$P_\lambda(x_1,\ldots,x_m;q,t) = \sum_{\mu\prec\lambda} \psi^m_{\lambda/\mu}(q,t)P_\mu(x_1,\ldots,x_{m-1};q,t)x_m^{|\lambda|-|\mu|}$$

for the branching coefficient defined for $\ell(\lambda) = m$ in terms of $f(u) := \frac{(tu;q)}{(qu;q)}$ by

$$\psi^m_{\lambda/\mu}(q,t) = \prod_{1\le i\le j\le m-1} \frac{f(q^{\mu_i-\mu_j}t^{j-i})f(q^{\lambda_i-\lambda_{j+1}}t^{j-i})}{f(q^{\mu_i-\lambda_{j+1}}t^{j-i})f(q^{\lambda_i-\mu_j}t^{j-i})}.$$

Corollary 2.4 (Truncated branching rule)**.** For $\ell(\lambda) = n < m$, we have for $f(u) := \frac{(tu;q)}{(qu;q)}$ that

$$\psi^m_{\lambda/\mu}(q,t) = \prod_{1\le i<j\le n} \frac{f(q^{\mu_i-\mu_j}t^{j-i})f(q^{\lambda_i-\lambda_j}t^{j-i-1})}{f(q^{\lambda_i-\mu_j}t^{j-i})f(q^{\mu_i-\lambda_j}t^{j-i-1})} \prod_{i=1}^{n}\frac{f(1)}{f(q^{\lambda_i-\mu_i})}\prod_{i=1}^{n}\frac{f(q^{\lambda_i}t^{n-i})}{f(q^{\mu_i}t^{n-i})},$$

where $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n, 0, \ldots, 0)$ and $\mu = (\mu_1 \ge \cdots \ge \mu_n, 0, \ldots, 0)$.

2.3. **Scaling limits of Macdonald polynomials.** We compute now some quasi-classical limits of Macdonald polynomials at both a general specialization and the principal specialization. These results overlap with those of [BG13, Section 6], but we include them here for the reader's convenience.

Lemma 2.5. For $q = e^{-\varepsilon}$ and $t = e^{-\theta\varepsilon}$, we have

$$\lim_{\varepsilon\to0}\varepsilon^{\theta((m-n)n+n(n-1)/2)}P_{\varepsilon^{-1}\lambda}(1,t,\ldots,t^{m-1};q,t) = \frac{\Gamma(\theta)^n}{\Gamma(m\theta)\cdots\Gamma((m-n+1)\theta)}\Delta(e^{-\lambda})^\theta\prod_{i=1}^{n}(1-e^{-\lambda_i})^{\theta(m-n)}.$$

Proof. Applying Lemmas A.1 and A.3 in Proposition 2.2, we find that

$$\begin{aligned}
&\lim_{\varepsilon\to0}\varepsilon^{\theta((m-n)n+n(n-1)/2)}P_{\varepsilon^{-1}\lambda}(1,t,\ldots,t^{m-1};q,t)\\
&= e^{-\theta\sum_{i=1}^n(i-1)\lambda_i}\prod_{1\le i<j\le n}(1-e^{\lambda_j-\lambda_i})^\theta\frac{\Gamma((j-i)\theta)}{\Gamma((j-i+1)\theta)}\prod_{i=1}^{n}\prod_{j=n+1}^{m}(1-e^{-\lambda_i})^\theta\frac{\Gamma((j-i)\theta)}{\Gamma((j-i+1)\theta)}\\
&= \frac{\Gamma(\theta)^n}{\Gamma(m\theta)\cdots\Gamma((m-n+1)\theta)}\Delta(e^{-\lambda})^\theta\prod_{i=1}^{n}(1-e^{-\lambda_i})^{\theta(m-n)}.
\end{aligned}$$

□

Lemma 2.6. For $q = e^{-\varepsilon}$ and $t = e^{-\theta\varepsilon}$, if $\ell(\lambda) = m$, we have

$$\lim_{\varepsilon\to0}\varepsilon^{(\theta-1)n}\psi^m_{\varepsilon^{-1}\lambda/\varepsilon^{-1}\mu}(q,t) = \Gamma(\theta)^{1-m}\frac{\Delta(e^{-\mu},e^{-\lambda})^{\theta-1}}{\Delta(e^{-\mu})^{\theta-1}\Delta(e^{-\lambda})^{\theta-1}}e^{(\theta-1)|\mu|}.$$

Proof. Taking the limit of Proposition 2.3, we find by applying Lemma A.5 that

$$\lim_{\varepsilon\to 0}\varepsilon^{(\theta-1)(m-1)}\psi^m_{\varepsilon^{-1}\lambda/\varepsilon^{-1}\mu}(q,t)=\Gamma(\theta)^{1-m}\prod_{1\le i<j\le m-1}(1-e^{\mu_j-\mu_i})^{1-\theta}\prod_{1\le i\le j\le m-1}\frac{(1-e^{\lambda_{j+1}-\lambda_j})^{1-\theta}}{(1-e^{\lambda_{j+1}-\mu_i})^{1-\theta}(1-e^{\mu_j-\lambda_i})^{1-\theta}}$$
$$=\Gamma(\theta)^{1-m}\frac{\Delta(e^{-\mu},e^{-\lambda})^{\theta-1}}{\Delta(e^{-\mu})^{\theta-1}\Delta(e^{-\lambda})^{\theta-1}}e^{(\theta-1)|\mu|}.\qquad\square$$

Lemma 2.7. For $q=e^{-\varepsilon}$ and $t=e^{-\theta\varepsilon}$, if $\ell(\lambda)=n<m$, we have

$$\lim_{\varepsilon\to 0}\varepsilon^{(\theta-1)n}\psi^m_{\varepsilon^{-1}\lambda/\varepsilon^{-1}\mu}(q,t)=\Gamma(\theta)^{-n}\frac{\Delta(e^{-\mu},e^{-\lambda})^{\theta-1}}{\Delta(e^{-\mu})^{\theta-1}\Delta(e^{-\lambda})^{\theta-1}}\prod_{i=1}^n\frac{(1-e^{-\mu_i})^{\theta-1}}{(1-e^{-\lambda_i})^{\theta-1}}e^{(\theta-1)|\mu|}.$$

Proof. Taking the limit of Corollary 2.4, we find by applying Lemma A.5 that

$$\lim_{\varepsilon\to 0}\varepsilon^{(\theta-1)n}\psi^m_{\varepsilon^{-1}\lambda/\varepsilon^{-1}\mu}(q,t)=\Gamma(\theta)^{-n}\prod_{1\le i<j\le n}\frac{(1-e^{\mu_j-\lambda_i})^{\theta-1}(1-e^{\lambda_j-\mu_i})^{\theta-1}}{(1-e^{\mu_j-\mu_i})^{\theta-1}(1-e^{\lambda_j-\lambda_i})^{\theta-1}}\prod_{i=1}^n\frac{(1-e^{\mu_i-\lambda_i})^{\theta-1}(1-e^{-\mu_i})^{\theta-1}}{(1-e^{-\lambda_i})^{\theta-1}}$$
$$=\Gamma(\theta)^{-n}\frac{\Delta(e^{-\mu},e^{-\lambda})^{\theta-1}}{\Delta(e^{-\mu})^{\theta-1}\Delta(e^{-\lambda})^{\theta-1}}\prod_{i=1}^n\frac{(1-e^{-\mu_i})^{\theta-1}}{(1-e^{-\lambda_i})^{\theta-1}}e^{(\theta-1)|\mu|}.\qquad\square$$

For $s=(s_1,\dots,s_m)$ and $\lambda=(\lambda_1,\dots,\lambda_n,0,\dots,0)$, define the integral formula

$$\Phi^{n,m}_\theta(\lambda,s)=\Gamma(\theta)^{-n(m-n)-n(n-1)/2}\int_{\mu\in\mathrm{GT}_\lambda}e^{\sum_{l=1}^m s_l(|\mu^l|-|\mu^{l-1}|)}$$
$$\prod_{i=1}^n\frac{(1-e^{-\mu_i^n})^{\theta-1}}{(1-e^{-\lambda_i})^{\theta-1}}\prod_{l=1}^{m-1}\frac{\Delta(e^{-\mu^l},e^{-\mu^{l+1}})^{\theta-1}}{\Delta(e^{-\mu^l})^{\theta-1}\Delta(e^{-\mu^{l+1}})^{\theta-1}}e^{(\theta-1)|\mu^l|}\prod_{l=1}^{m-1}\prod_{i=1}^{\min\{l,n\}}d\mu_i^l,$$

where the integral is over a space of dimension $(m-n)n+\frac{n(n-1)}{2}$.

Corollary 2.8 ([BG13, Propositions 6.2 and 6.4])**.** If $\lambda=(\lambda_1,\dots,\lambda_n,0,\dots,0)$ and $s=(s_1,\dots,s_m)$ with $\lambda_1>\dots>\lambda_n>0$, then

$$\lim_{\varepsilon\to 0}\varepsilon^{\theta(n(m-n)+n(n-1)/2)}P_{\varepsilon^{-1}\lambda}(e^{\varepsilon s_1},\dots,e^{\varepsilon s_m};e^{-\varepsilon},e^{-\theta\varepsilon})=\Phi^{n,m}_\theta(\lambda,s).$$

Proof. Combine Proposition 2.3 and Lemmas 2.6 and 2.7. $\square$

We now extend this scaling to $Q_\lambda(x;q,t)$.

Lemma 2.9. For $\lambda_1>\dots>\lambda_n>0$, we have the scaling limit

$$\lim_{\varepsilon\to 0}\varepsilon^{n(\theta-1)}b_{\lfloor\varepsilon^{-1}(\lambda_1,\dots,\lambda_n,0,\dots,0)\rfloor}(e^{-\varepsilon},e^{-\theta\varepsilon})=\Gamma(\theta)^{-n}\prod_{i=1}^n(1-e^{-\lambda_i})^{\theta-1}.$$

Proof. By Lemma A.2, for $q=e^{-\varepsilon}$, $t=e^{-\theta\varepsilon}$, and $i>0$ that

$$\lim_{\varepsilon\to 0}\frac{(t^{i+1}q^{\lambda_{l-i}-\lambda_l};q)_{\lambda_l-\lambda_{l+1}}}{(t^iq^{\lambda_{l-i}-\lambda_l+1};q)_{\lambda_l-\lambda_{l+1}}}=\frac{(1-e^{\lambda_l-\lambda_{l-i}})^{-\theta+1}}{(1-e^{\lambda_{l+1}-\lambda_{l-i}})^{-\theta+1}}$$

and

$$\lim_{\varepsilon\to 0}\varepsilon^{\theta-1}\frac{(t;q)_{\lambda_l-\lambda_{l+1}}}{(q;q)_{\lambda_l-\lambda_{l+1}}}=\Gamma(\theta)^{-1}(1-e^{\lambda_{l+1}-\lambda_l})^{\theta-1}.$$

We conclude that

$$\lim_{\varepsilon\to 0}\varepsilon^{n(\theta-1)}b_{\lfloor\varepsilon^{-1}(\lambda_1,\dots,\lambda_n,0,\dots,0)\rfloor}(e^{\varepsilon},e^{\theta\varepsilon})=\Gamma(\theta)^{-n}\prod_{l=1}^n(1-e^{\lambda_{l+1}-\lambda_l})^{\theta-1}\prod_{i=1}^{l-1}\frac{(1-e^{\lambda_l-\lambda_{l-i}})^{-\theta+1}}{(1-e^{\lambda_{l+1}-\lambda_{l-i}})^{-\theta+1}}$$
$$=\Gamma(\theta)^{-n}\prod_{i=1}^n(1-e^{-\lambda_i})^{\theta-1}.\qquad\square$$

Corollary 2.10. If $\lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)$ and $s = (s_1, \ldots, s_m)$ with $\lambda_1 > \cdots > \lambda_n > 0$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{\theta(n(m-n)+n(n-1)/2)+n(\theta-1)} Q_{\varepsilon^{-1}\lambda}(e^{\varepsilon s_1}, \ldots, e^{\varepsilon s_m}; e^{-\varepsilon}, e^{-\theta\varepsilon}) = \Gamma(\theta)^{-n} \prod_{i=1}^{n} (1 - e^{-\lambda_i})^{\theta-1} \Phi_\theta^{n,m}(\lambda, s).$$

Corollary 2.11. For $q = e^{-\varepsilon}$ and $t = e^{-\theta\varepsilon}$, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{\theta((m-n)n+n(n-1)/2)+n(\theta-1)} Q_{\varepsilon^{-1}\lambda}(1, t, \ldots, t^{m-1}; q, t)$$
$$= \frac{1}{\Gamma(m\theta) \cdots \Gamma((m-n+1)\theta)} \Delta(e^{-\lambda})^\theta \prod_{i=1}^{n} (1 - e^{-\lambda_i})^{\theta(m-n+1)-1}.$$

2.4. **Definition of the Heckman-Opdam hypergeometric function.** The Heckman-Opdam hypergeometric function is defined by

$$\mathcal{F}_\beta^{n,m}(\lambda, s) := \frac{\Gamma(m\theta) \cdots \Gamma((m-n+1)\theta)}{\Gamma(\theta)^n} \frac{\Phi_\theta^{n,m}(\lambda, s)}{\Delta^{\mathrm{trig}}(\lambda)^\theta \prod_{i=1}^{n} (1 - e^{-\lambda_i})^{\theta(m-n)}}. \tag{2.2}$$

Define the conjugate Heckman-Opdam hypergeometric function by

$$\widetilde{\mathcal{F}}_\beta^{n,m}(\lambda, s) := \Gamma(\theta)^{-n} \prod_{i=1}^{n} (1 - e^{-\lambda_i})^{\theta-1} \mathcal{F}_\beta^{n,m}(\lambda, s). \tag{2.3}$$

We may now translate the results of the previous section into this language.

Corollary 2.12. For $n \leq m$, we have the scalings

$$\lim_{\varepsilon \to 0} \frac{P_{\lfloor \varepsilon^{-1}(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) \rfloor}(e^{\varepsilon s_1}, \ldots, e^{\varepsilon s_m}; e^{-\varepsilon}, e^{-\theta\varepsilon})}{P_{\lfloor \varepsilon^{-1}(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) \rfloor}(1, e^{-\theta\varepsilon}, \ldots, e^{-(m-1)\theta\varepsilon}; e^{-\varepsilon}, e^{-\theta\varepsilon})} = e^{\frac{n-1}{2}\theta|\lambda|} \mathcal{F}_\beta^{n,m}(\lambda, s)$$

and

$$\lim_{\varepsilon \to 0} \frac{Q_{\lfloor \varepsilon^{-1}(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) \rfloor}(e^{\varepsilon s_1}, \ldots, e^{\varepsilon s_m}; e^{-\varepsilon}, e^{-\theta\varepsilon})}{P_{\lfloor \varepsilon^{-1}(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) \rfloor}(1, e^{-\theta\varepsilon}, \ldots, e^{-(m-1)\theta\varepsilon}; e^{-\varepsilon}, e^{-\theta\varepsilon})} = e^{\frac{n-1}{2}\theta|\lambda|} \widetilde{\mathcal{F}}_\beta^{n,m}(\lambda, s).$$

We now apply Corollary 2.12 to obtain a Cauchy identity for Heckman-Opdam hypergeometric functions.

Proposition 2.13. For $m \geq n$ and parameters $s = (s_1, \ldots, s_m)$ and $r = (r_1, \ldots, r_n)$, we have

$$\int_{\lambda_1 \geq \cdots \geq \lambda_n \geq 0} \mathcal{F}_\beta^{n,m}(\lambda, s) \widetilde{\mathcal{F}}_\beta^{n,n}(\lambda, r) \Delta(e^{-\lambda})^{2\theta} e^{(n-1)\theta|\lambda|} \prod_{i=1}^{n} (1 - e^{-\lambda_i})^{\theta(m-n)} d\lambda$$
$$= \frac{\Gamma(m\theta) \cdots \Gamma((m-n+1)\theta) \Gamma(n\theta) \cdots \Gamma(\theta)}{\Gamma(\theta)^{2n}} \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{\Gamma(-s_i - r_j)}{\Gamma(\theta - s_i - r_j)}.$$

Proof. We apply Corollary 2.12, Lemma 2.5, and Lemma A.3. □

2.5. **Definition of the multivariate Bessel function.** For $\lambda = (\lambda_1 > \cdots > \lambda_n)$ and $s = (s_1, \ldots, s_m)$ with $n \leq m$, define the integral formula

$$\phi_\theta^{n,m}(\lambda, s) = \Gamma(\theta)^{-\frac{n(n-1)}{2} - n(m-n)} \int_{\mu \in \mathrm{GT}_\lambda} e^{\sum_{l=1}^{m} s_l(|\mu^l| - |\mu^{l-1}|)}$$
$$\prod_{i=1}^{n} \frac{(\mu_i^n)^{\theta-1}}{\lambda_i^{\theta-1}} \prod_{l=1}^{m-1} \frac{\Delta(\mu^l, \mu^{l+1})^{\theta-1}}{\Delta(\mu^l)^{\theta-1} \Delta(\mu^{l+1})^{\theta-1}} \prod_{l=1}^{m-1} \prod_{i=1}^{\min\{l,n\}} d\mu_i^l. \tag{2.4}$$

It was shown in [GK02, Section V] that for $n \leq m$, the multivariate Bessel function is given by

$$\mathcal{B}_\beta^{n,m}(\lambda, s) := \frac{\Gamma(m\theta) \cdots \Gamma((m-n+1)\theta)}{\Gamma(\theta)^n} \frac{\phi_\theta(\lambda, s)}{\Delta(\lambda)^\theta \prod_i \lambda_i^{\theta(m-n)}}.$$

Define also the conjugate multivariable Bessel function by

$$\widetilde{\mathcal{B}}_\beta^{n,m}(\lambda, s) := \Gamma(\theta)^{-n} \prod_{i=1}^{n} \lambda_i^{\theta-1} \mathcal{B}_\beta^{n,m}(\lambda, s).$$

Remark. We have adjusted the normalization of $\mathcal{B}_\beta^{n,m}(\lambda, s)$ from [GK02] so that $\mathcal{B}_\beta^{n,m}(\lambda, 0) = 1$.

Proposition 2.14. For $m \geq n$, the multivariate Bessel function $\mathcal{B}_\beta^{n,m}(\lambda, s)$ satisfies

- $\mathcal{B}_\beta^{n,m}(0, s) = \mathcal{B}_\beta^{n,m}(\lambda, 0) = 1$;
- $\mathcal{B}_\beta^{n,m}(\lambda, s) = \mathcal{B}_\beta^{m,m}(s, \lambda)$;
- $c^{\theta(n(m-n)+n(n-1)/2)}\mathcal{B}_\beta^{n,m}(\lambda, cs) = \mathcal{B}_\beta^{n,m}(c\lambda, s)$.

Proposition 2.15. We have the scaling limits

$$\lim_{\varepsilon \to 0} \mathcal{F}_\beta^{n,m}(\varepsilon\lambda, \varepsilon^{-1}s) = \mathcal{B}_\beta^{n,m}(\lambda, s)$$
$$\lim_{\varepsilon \to 0} \varepsilon^{-(\theta-1)n}\widetilde{\mathcal{F}}_\beta^{n,m}(\varepsilon\lambda, \varepsilon^{-1}s) = \widetilde{\mathcal{B}}_\beta^{n,m}(\lambda, s)$$

Proof. By taking explicit scaling limits in the integral expressions for $\mathcal{F}_\beta^{n,m}(\varepsilon\lambda, \varepsilon^{-1}s)$ and $\mathcal{B}_\beta^{n,m}(\lambda, s)$. □

Proposition 2.16. For $n \leq m$, $s = (s_1, \ldots, s_m)$, and $r = (r_1, \ldots, r_n)$, we have

$$\int_{\lambda_1 \geq \cdots \geq \lambda_n \geq 0} \mathcal{B}_\beta^{n,m}(\lambda, -s)\widetilde{\mathcal{B}}_\beta^{n,n}(\lambda, -r)\Delta(\lambda)^{2\theta}\prod_{i=1}^n \lambda_i^{\theta(m-n)}d\lambda$$
$$= \frac{\Gamma(m\theta)\cdots\Gamma((m-n+1)\theta)\Gamma(n\theta)\cdots\Gamma(\theta)}{\Gamma(\theta)^{2n}}\prod_{i=1}^m\prod_{j=1}^n (s_i + r_j)^{-\theta}.$$

Proof. In Proposition 2.13, substitute $(\varepsilon\lambda, \varepsilon^{-1}s, \varepsilon^{-1}r)$ for (λ, s, r), multiply both sides by $\varepsilon^{-\theta mn}$, and take the limit as $\varepsilon \to 0$, applying Lemma A.4. □

3. The multivariate Bessel ensemble and the generalized β-Wishart ensemble

We define the multivariate Bessel ensemble and generalized β-Wishart ensemble. At $\beta = 1, 2$, we prove that the generalized β-Wishart ensemble gives a matrix model for the multivariate Bessel process. The $\beta = 2$ case was considered in [BP08] and [DW09], where the analysis hinged on the HCIZ integral in two ways. The paper [DW09] uses results of [Def10] which reduce to an evaluation of this integral, and our reduction to the Laplace transform on the Laguerre ensemble also uses it.

3.1. Definition of the multivariate Bessel ensemble.

The multivariate Bessel ensemble with parameters $\{\pi_i\}_{1 \leq i \leq n}$ and $\{\widehat{\pi}_j\}_{j \geq 1}$ is the process on $\{\mu_i^m\}_{1 \leq i \leq \min\{n,m\}}$ with level m marginal density given by

$$p_{\pi,\widehat{\pi}}^{m,n}(\mu^m) := \frac{\Gamma(\theta)^{2\min\{m,n\}}}{\Gamma(m\theta)\cdots\Gamma((m-\min\{n,m\}+1)\theta)\Gamma(\min\{n,m\}\theta)\cdots\Gamma(\theta)}\prod_{j=1}^m\prod_{i=1}^n(\widehat{\pi}_j + \pi_i)^\theta$$
$$\Delta(\mu^m)^{2\theta}\prod_{i=1}^{\min\{m,n\}}(\mu_i^m)^{\theta(m-\min\{n,m\})}\mathcal{B}_\beta^{\min\{n,m\},m}(\mu^m, -\theta\widehat{\pi})\widetilde{\mathcal{B}}_\beta^{\min\{n,m\},n}(\mu^m, -\theta\pi).$$

The joint distribution is supported on

$$\mu^1 \prec \cdots \prec \mu^m, \qquad \mu_i^l \in [0, \infty)$$

with μ^l of length $\min\{l, n\}$ and with density given by

$$p_{\pi,\widehat{\pi}}^{m,n}(\mu^1 \prec \cdots \prec \mu^m) := \frac{\Gamma(\theta)^{\min\{m,n\}-\min\{m,n\}(\min\{m,n\}-1)/2-\min\{m,n\}(m-\min\{m,n\})}}{\Gamma(\min\{n,m\}\theta)\cdots\Gamma(\theta)}\prod_{j=1}^m\prod_{i=1}^n(\widehat{\pi}_j+\pi_i)^\theta \tag{3.1}$$
$$\Delta(\mu^m)^\theta e^{-\theta\sum_{l=1}^m \widehat{\pi}_l(|\mu^l|-|\mu^{l-1}|)}\prod_{i=1}^{\min\{n,m\}}(\mu_i^m)^{\theta(m-\min\{m,n\})}\frac{(\mu_i^{\min\{n,m\}})^{\theta-1}}{(\mu_i^m)^{\theta-1}}$$
$$\prod_{l=1}^{m-1}\frac{\Delta(\mu^l, \mu^{l+1})^{\theta-1}}{\Delta(\mu^l)^{\theta-1}\Delta(\mu^{l+1})^{\theta-1}}\widetilde{\mathcal{B}}_\beta^{n,n}(\mu^n, -\theta\pi).$$

We will show that for $\beta = 1, 2$, the multivariate Bessel measure admits a matrix model in terms of a multilevel generalized β-Wishart ensemble.

3.2. **Matrix model at $\beta = 2$.** Fix two sets of non-negative real parameters $\{\pi_i\}_{1\le i\le N}$ and $\{\widehat{\pi}_i\}_{i\ge 1}$, and let (A_{ij}) be an infinite matrix of zero-mean complex Gaussian random variables with variance $(\pi_i + \widehat{\pi}_j)^{-1}$. Let $A_m = (A_{ij})_{1\le i\le m, 1\le j\le n}$ be the top $m\times n$ corner. In [BP08], the authors introduce the generalized Wishart random matrix process as the sequence of random matrices

$$M_m = A_m^* A_m.$$

Let $\{\mu_i^m\}$ be the non-zero eigenvalues of M_m. The joint density of the $\{\mu_i^m\}$ was determined in [DW09].

Theorem 3.1 ([DW09, Theorem 3.1])**.** The $\{\mu_i^m\}$ form a Markov process with transition kernel

$$Q^{\pi,\widehat{\pi}}_{m-1,m}(\mu^{m-1}, d\mu^m) = \prod_{i=1}^n (\pi_i + \widehat{\pi}_m) \frac{h_\pi(\mu^m)\Delta(\mu^m)}{h_\pi(\mu^{m-1})\Delta(\mu^{m-1})} e^{-\widehat{\pi}_m \sum_{i=1}^n (\mu_i^m - \mu_i^{m-1})} 1_{\mu^{m-1}\prec \mu^m} d\mu^m,$$

where $\Delta(\mu)$ denotes the Vandermonde determinant and $h_\pi(\mu) = \frac{\det(e^{-\pi_i\mu_j})}{\Delta(\pi)\Delta(\mu)}$ denotes the HCIZ integral.

3.3. **Matrix model at $\beta = 1$.** We now consider a matrix model for $\beta = 1$. As before, fix two sets of non-negative real parameters $\{\pi_i\}_{1\le i\le n}$ and $\{\widehat{\pi}_j\}_{j\ge 1}$. Let (A_{ij}) be an infinite matrix of independent real Gaussian random variables with mean 0 and variance $(\pi_i + \widehat{\pi}_j)^{-1}$, and let A_m be its top $m\times n$ corner. At $\beta = 1$, the generalized Wishart process is the sequence of $n\times n$ symmetric positive semi-definite matrices

$$M_m := A_m^T A_m.$$

Let $\{\mu_i^m\}_{1\le i\le \min\{m,n\}}$ be the non-zero eigenvalues of M_n. If $m \le n$, these are also the eigenvalues of $A_m A_m^T$.

Remark. Setting $\pi_i = 1$ and $\widehat{\pi}_j = 0$ recovers the standard Wishart process.

Let $P^{\pi,\widehat{\pi}}$ denote the joint distribution of $\{\mu_i^m\}$ under parameters $(\pi, \widehat{\pi})$ and let $P := P^{(1,\dots,1),(0,\dots,0)}$. Let a_m denote the m^{th} row of A, so that

$$M_m = M_{m-1} + a_m^T a_m = M_{m-1} + (\pi + \widehat{\pi}_m I)^{-1/2} z_m^T z_m (\pi + \widehat{\pi}_m I)^{-1/2},$$

where $z_m = a_m(\pi + \widehat{\pi}_m I)$ is a $1\times n$ vector of real standard Gaussian random variables.

Proposition 3.2. For $m \ge 1$, the $P^{\pi,\widehat{\pi}}$-law of $M_m - M_{m-1}$ is absolutely continuous with respect to the P-law of $M_m - M_{m-1}$ with Radon-Nikodyn derivative

$$\prod_{i=1}^n (\pi_i + \widehat{\pi}_m)^{1/2} \exp\Big(-\frac{1}{2}(\widehat{\pi}_m - 1)\mathrm{Tr}(M_m - M_{m-1}) - \frac{1}{2}\sum_{i=1}^n \pi_i (M_{m,ii} - M_{m-1,ii})\Big).$$

Proof. The difference $M_m - M_{m-1} = a_m^T a_m$ is determined by its diagonal entries, which have exponential distribution with parameter determined by π and $\widehat{\pi}$. Comparing densities for $\pi, \widehat{\pi}$ and $(1,\dots,1), (0,\dots,0)$ yields the conclusion. □

We now use a change of measure motivated by the proof of [DW09, Theorem 3.1] to derive the joint density of the $\beta = 1$ generalized Wishart ensemble. Let $P_n^{\pi,\widehat{\pi}}$ and $p_n^{\pi,\widehat{\pi}}$ be the distributions of $M_1,\dots,M_n$ and $\mu^1,\dots,\mu^n$ under $P^{\pi,\widehat{\pi}}$ and its restriction to the σ-field generated by $\mu^1,\dots,\mu^n$. Let P_n and p_n denote the corresponding distributions for $\pi_i = 1$ and $\widehat{\pi}_j = 0$.

Theorem 3.3. The process $\{\mu_i^n\}_{i\ge 1}$ is a Markov chain under $P^{\pi,\widehat{\pi}}$ with transition kernel

$$\begin{aligned} Q^{\pi,\widehat{\pi}}_{m-1,m}(\mu^{m-1}, d\mu^m) = &\frac{\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V \widetilde{\mu}^m V^T)} d\mathsf{Haar}_V}{\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V \widetilde{\mu}^{m-1} V^T)} d\mathsf{Haar}_V} \frac{1}{\Gamma(m/2)\Gamma(1/2)^m} \\ &\prod_{i=1}^n (\pi_i + \widehat{\pi}_m)^{\frac{1}{2}} e^{-\frac{1}{2}\widehat{\pi}_m(|\mu^m| - |\mu^{m-1}|)} \frac{\prod_{i=1}^m (\mu_i^m)^{\frac{n-m-1}{2}}}{\prod_{i=1}^{m-1}(\mu_i^{m-1})^{\frac{n-m}{2}}} \Delta(\mu^m) \prod_{i,j} |\mu_i^m - \mu_j^{m-1}|^{-\frac{1}{2}} 1_{\mu^{m-1}\prec\mu^m} d\mu^m, \end{aligned}$$

where we note that if $\pi_i = 1$ and $\widehat{\pi}_j = 0$, we obtain

$$Q_{m-1,m}(\mu^{m-1}, d\mu^m) = \frac{e^{-\frac{1}{2}(|\mu^m| - |\mu^{m-1}|)}}{\Gamma(m/2)\Gamma(1/2)^m} \frac{\prod_{i=1}^m (\mu_i^m)^{\frac{n-m-1}{2}}}{\prod_{i=1}^{m-1}(\mu_i^{m-1})^{\frac{n-m}{2}}} \Delta(\mu^m) \prod_{i,j} |\mu_i^m - \mu_j^{m-1}|^{-1/2} 1_{\mu^{m-1}\prec\mu^m} d\mu^m.$$

Proof. The result holds for p_n by [FR05, Corollary 3]. By Proposition 3.2, we have

$$\frac{dP_m^{\pi,\widehat{\pi}}}{dP_m}(M_1,\ldots,M_m)=\prod_{i=1}^n\prod_{j=1}^m(\pi_i+\widehat{\pi}_j)^{1/2}\exp\Big(-\frac{1}{2}\sum_{m=1}^m\widehat{\pi}_m\mathrm{Tr}(M_m-M_{m-1})-\frac{1}{2}\mathrm{Tr}((\pi-I)M_m)\Big).$$

We obtain that

$$\begin{aligned}\frac{dp_m^{\pi,\widehat{\pi}}}{p_m}(\mu^1,\ldots,\mu^m)&=\mathbb{E}_P\left[\frac{dP_m^{\pi,\widehat{\pi}}}{dP_m}(M_1,\ldots,M_m)\mid\mu^1,\ldots,\mu^m\right]\\&=\prod_{i=1}^n\prod_{j=1}^m(\pi_i+\widehat{\pi}_j)^{1/2}e^{-\frac{1}{2}\sum_{m=1}^m\widehat{\pi}_m(|\mu^m|-|\mu^{m-1}|)}e^{\frac{1}{2}|\mu^m|}\mathbb{E}_P\left[e^{-\frac{1}{2}\mathrm{Tr}(\pi M_m)}\mid\mu^1,\ldots,\mu^m\right],\end{aligned}$$

where we define $|\mu^l|:=\mu_1^l+\cdots+\mu_l^l$. Because the P-distribution of $M_1,\ldots,M_m$ conditional on $\mu^1,\ldots,\mu^m$ is invariant under simultaneous orthogonal group conjugation, we obtain by applying Lemma 3.5 that

$$\begin{aligned}\mathbb{E}_P\left[e^{-\frac{1}{2}\mathrm{Tr}(\pi M_m)}\mid\mu^1,\ldots,\mu^m\right]&=\mathbb{E}_P\left[\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V\widetilde{\mu}^mV^T)}d\mathsf{Haar}_V\mid\mu^1,\ldots,\mu^m\right]\\&=\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V\widetilde{\mu}^mV^T)}d\mathsf{Haar}_V.\end{aligned}$$

We conclude that $\{\mu_i^m\}$ is Markov with transition kernel

$$\begin{aligned}&Q_{m-1,m}^{\pi,\widehat{\pi}}(\mu^{m-1},d\mu^m)\\&=\frac{\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V\widetilde{\mu}^mV^T)}d\mathsf{Haar}_V}{\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V\widetilde{\mu}^{m-1}V^T)}d\mathsf{Haar}_V}\prod_{i=1}^n(\pi_i+\widehat{\pi}_m)^{1/2}e^{-\frac{1}{2}(\widehat{\pi}_m-1)(|\mu^m|-|\mu^{m-1}|)}Q_{m-1,m}(\mu^{m-1},d\mu^m)\\&=\frac{\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V\widetilde{\mu}^mV^T)}d\mathsf{Haar}_V}{\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V\widetilde{\mu}^{m-1}V^T)}d\mathsf{Haar}_V}\frac{1}{\Gamma(m/2)\Gamma(1/2)^m}\prod_{i=1}^n(\pi_i+\widehat{\pi}_m)^{\frac{1}{2}}\\&\qquad e^{-\frac{1}{2}\widehat{\pi}_m(|\mu^m|-|\mu^{m-1}|)}\frac{\prod_{i=1}^m(\mu_i^m)^{\frac{n-m-1}{2}}}{\prod_{i=1}^{m-1}(\mu_i^{m-1})^{\frac{n-m}{2}}}\Delta(\mu^m)\prod_{i,j}|\mu_i^m-\mu_j^{m-1}|^{-\frac{1}{2}}1_{\mu^{m-1}\prec\mu^m}d\mu^m.\end{aligned}$$

$\square$

Corollary 3.4. The joint density of $\{\mu_i^m\}_{m\le n}$ under $P^{\pi,\widehat{\pi}}$ is supported on

$$\mu^1\prec\cdots\prec\mu^n\qquad\mu_i^m\in[0,\infty)$$

and given by

$$\frac{\Gamma(1/2)^{-n(n-1)/2}}{\Gamma(n/2)\cdots\Gamma(1/2)}\prod_{i=1}^n\prod_{j=1}^m(\pi_i+\widehat{\pi}_j)^{\frac{1}{2}}\int_V e^{-\frac{1}{2}\mathrm{Tr}(\pi V\widetilde{\mu}^nV^T)}d\mathsf{Haar}_V$$
$$e^{-\frac{1}{2}\sum_m\widehat{\pi}_m(|\mu^m|-|\mu^{m-1}|)}\prod_{i=1}^n(\mu_i^n)^{-1/2}\prod_{m=1}^n\Delta(\mu^m)\Delta(\mu^m,\mu^{m-1})^{-1/2}.$$

3.4. **Identification with a multivariate Bessel ensemble.** To identify the result of Corollary 3.4 with the multivariate Bessel ensemble, we must relate orbital integrals for the orthogonal group with multivariate Bessel functions.

Lemma 3.5. For $n\times n$ real symmetric matrices A and B with eigenvalues $\{a_i\}$ and $\{b_i\}$ and a_i distinct, we have

$$\begin{aligned}&\int_{V\in O(n)}e^{-\mathrm{Tr}(VAV^TB)}d\mathsf{Haar}_V\\&\qquad=\frac{\Gamma(n/2)\cdots\Gamma(1/2)}{\Gamma(1/2)^{n(n+1)/2}}\int_{a^1\prec\cdots\prec a^n=a}e^{-\sum_{l=1}^n b_l(|a^l|-|a^{l-1}|)}\prod_{l=1}^{n-1}\Delta(a^l)\Delta(a^l,a^{l+1})^{-1/2}da.\end{aligned}$$

Proof. By restricting the $\theta=1/2$ case of [Ner03, Proposition 1.1] to matrices with eigenvalues $\{a_i\}$ and normalizing to obtain 1 for $B=0$. $\square$

Corollary 3.6. For $n \times n$ real symmetric matrices A and B with eigenvalues $\{a_i\}$ and $\{b_i\}$ and a_i distinct, we have

$$\int_{V \in O(n)} e^{-\mathrm{Tr}(VAV^TB)} d\mathsf{Haar}_V = \mathcal{B}_1^{n,n}(a, -b).$$

Proof. By Lemma 3.5 and the integral formula for $\mathcal{B}_1$. □

Proposition 3.7. The density of $\{\mu_i^n\}$ under $P^{\pi,\widehat{\pi}}$ is given by

$$\frac{\Gamma(1/2)^{2n}}{\Gamma(n/2)^2 \cdots \Gamma(1/2)^2} \prod_{i=1}^n \prod_{j=1}^n (\pi_i + \widehat{\pi}_j)^{1/2} \Delta(\mu^n) \mathcal{B}_1^{n,n}(\mu^n, -\widehat{\pi}/2) \widetilde{\mathcal{B}}_1^{n,n}(\mu^n, -\pi/2).$$

Proof. By applying Corollary 3.4 and Corollary 3.6. □

Corollary 3.8. The joint density of $\{\mu_i^m\}_{1 \le m \le n}$ under $P^{\pi,\widehat{\pi}}$ is given by

$$\frac{\Gamma(1/2)^{n-n(n-1)/2}}{\Gamma(n/2) \cdots \Gamma(1/2)} \prod_{i=1}^n \prod_{j=1}^n (\pi_i + \widehat{\pi}_j)^{1/2}$$
$$\Delta(\mu^n)^{1/2} e^{-\frac{1}{2}\sum_{m=1}^n \widehat{\pi}_m(|\mu^m| - |\mu^{m-1}|)} \prod_{m=1}^{n-1} \frac{\Delta(\mu^m, \mu^{m+1})^{-1/2}}{\Delta(\mu^m)^{-1/2}\Delta(\mu^{m+1})^{-1/2}} \widetilde{\mathcal{B}}_1^{n,n}(\mu^n, -\pi/2).$$

Proof. By applying Corollary 3.4 and Corollary 3.6 and noting that $\Gamma(1/2) = \pi^{1/2}$. □

Remark. This matches with the expression for the density of the multivariate Bessel ensemble from (3.1).

4. The Heckman-Opdam ensemble and the β-Jacobi ensemble

In this section, we define the Heckman-Opdam and β-Jacobi ensembles and prove that at $\beta = 1, 2$, the β-Jacobi ensemble gives a matrix model for the principally specialized Heckman-Opdam ensemble.

4.1. **Definition of the Heckman-Opdam ensemble.** Let the Heckman-Opdam ensemble with parameters $\{\pi_i\}_{1 \le i \le n}$ and $\{\widehat{\pi}_i\}_{i \ge 1}$ be the process on $\{\mu_i^m\}_{1 \le i \le \min\{n,m\}}$ with level m marginal density given by

$$\frac{\Gamma(\theta)^{2\min\{m,n\}}}{\Gamma(m\theta) \cdots \Gamma((m - \min\{m,n\} + 1)\theta)\Gamma(\min\{m,n\}\theta) \cdots \Gamma(\theta)} \prod_{i=1}^m \prod_{j=1}^n \frac{\Gamma(\theta + \theta\pi_j + \theta\widehat{\pi}_i)}{\Gamma(\theta\pi_j + \theta\widehat{\pi}_i)}$$
$$\prod_{i=1}^{\min\{m,n\}} (1 - e^{-\mu_i^m})^{\theta(m - \min\{m,n\})} \Delta^{\mathrm{trig}}(\mu^m)^{2\theta} \mathcal{F}_\beta^{\min\{m,n\},m}(\mu^m, -\theta\widehat{\pi}) \widetilde{\mathcal{F}}_\beta^{\min\{m,n\},n}(\mu^m, -\theta\pi)$$

and joint distribution supported on

$$\mu^1 \prec \cdots \prec \mu^n \qquad \mu_i^n \in [0, \infty)$$

with density proportional to

$$\frac{\Gamma(\theta)^{2\min\{m,n\} - \min\{n,m\}(\min\{n,m\}-1)/2 - (m - \min\{m,n\})\min\{m,n\}}}{\Gamma(m\theta) \cdots \Gamma((m - \min\{m,n\} + 1)\theta)\Gamma(\min\{m,n\}\theta) \cdots \Gamma(\theta)} \prod_{i=1}^m \prod_{j=1}^n \frac{\Gamma(\theta + \theta\pi_j + \theta\widehat{\pi}_i)}{\Gamma(\theta\pi_j + \theta\widehat{\pi}_i)} \Delta^{\mathrm{trig}}(\mu^m)^\theta \tag{4.1}$$
$$\prod_{i=1}^{\min\{m,n\}} (1 - e^{-\mu_i^m})^{\theta(m - \min\{m,n\})} \prod_{i=1}^{\min\{m,n\}} \frac{(1 - e^{-\mu_i^{\min\{m,n\}}})^{\theta-1}}{(1 - e^{-\mu_i^m})^{\theta-1}} e^{-\theta\sum_{l=1}^m \widehat{\pi}_l(|\mu^l| - |\mu^{l-1}|)}$$
$$\prod_{l=1}^{m-1} \frac{\Delta(e^{-\mu^l}, e^{-\mu^{l+1}})^{\theta-1}}{\Delta(e^{-\mu^l})^{\theta-1}\Delta(e^{-\mu^{l+1}})^{\theta-1}} e^{(\theta-1)|\mu^l|} \widetilde{\mathcal{F}}_\beta^{m,n}(\mu^m, -\theta\pi).$$

4.2. **The β-Jacobi ensemble.** Let X and Y be infinite matrices of independent Gaussian random variables with mean 0 and variance 1 over $\mathbb{R}$ for $\beta = 1$ and over $\mathbb{C}$ for $\beta = 2$. Let X^{AC} and Y^{BC} denote the left $A \times C$ and $B \times C$ corners of X and Y. Then the matrix

$$(X^{AC})^* X^{AC}((X^{AC})^* X^{AC} + (Y^{BC})^* Y^{BC})^{-1} \tag{4.2}$$

has $m = \min\{A, B, C\}$ eigenvalues $\lambda_1^m, \ldots, \lambda_m^m$ in $[0, 1]$.Choose $C = n$, $A \geq n$ and let $B = m$ vary in the range $1, \ldots, m$. The β-Jacobi ensemble is the joint density of the eigenvalues $\lambda^1 \prec \cdots \prec \lambda^m$.

4.3. **The conditioned β-Jacobi ensemble in the null case as a generalized Wishart ensemble.** Let $X := X^{An}$ and $Y_m := Y^{mn}$ and condition on the value of X. We may find some U_1 in $U(A)$ for $\beta = 2$ and $O(A)$ for $\beta = 1$ so that

$$X = U_1 \begin{pmatrix} X_1 \\ 0 \end{pmatrix},$$

where X_1 is $n \times n$ upper-triangular and a.s. invertible. For such a U_1, we have that

$$X^* X = X_1^* U_1^* U_1 X_1 = X_1^* X_1.$$

The eigenvalues $\{\lambda_i^m\}$ of $X^* X(X^* X + Y_m^* Y_m)^{-1}$ which are not equal to 1 are the solutions to

$$\det(\lambda(X^* X + Y_m^* Y_m) - X^* X) = 0 \qquad \Longleftrightarrow \qquad \det((1/\lambda - 1) + X_1^{-1} X_1^{-*} Y_m^* Y_m) = 0$$

which are not equal to 1. If τ_i^m are the eigenvalues of $X_1^{-1} X_1^{-*} Y_m^* Y_m$ not equal to 0, then $\lambda_i^m = (1 + \tau_i^m)^{-1}$. Notice now that $X_1^{-1} X_1^{-*} Y_m^* Y_m$ and $X_1^{-*} Y_m^* Y_m X_1^{-1} = (Y_m X_1^{-1})^* (Y_m X_1^{-1})$ have the same non-zero eigenvalues. Because $A^* A$ and AA^* have the same non-zero eigenvalues, this means that we can consider the nonzero eigenvalues of $(Y_m X_1^{-1})(Y_m X_1^{-1})^* = Y_m X_1^{-1} X_1^{-*} Y_m^*$. Let V in $U(n)$ for $\beta = 2$ or $O(n)$ for $\beta = 1$ be a unitary/orthogonal matrix so that $X_1^{-1} X_1^{-*} = V \Lambda_X^{-1} V^*$ for a positive real diagonal matrix Λ_X. The distribution of Y_m is unitarily/orthogonally invariant, so we find that

$$Y_m X_1^{-1} X_1^{-*} Y_m^* = Y_m V \Lambda_X^{-1} V^* Y_m^* \sim Y_m \Lambda_X^{-1} Y_m^* = (Y_m \Lambda_X^{-1/2})(Y_m \Lambda_X^{-1/2})^*.$$

This last product has the same non-zero eigenvalues as $(Y_m \Lambda_X^{-1/2})^* (Y_m \Lambda_X^{-1/2}) = \Lambda_X^{-1/2} Y_m^* Y_m \Lambda_X^{-1/2}$. Define now the matrix

$$A_{X,m} := \Lambda_X^{-1/2} Y_m^* Y_m \Lambda_X^{-1/2}.$$

The distribution of the $A_{X,m}$ is given by the recurrence

$$A_{X,m+1} = A_{X,m} + (\lambda_{X,i}^{-1/2} z_i^* z_j \lambda_{X,j}^{-1/2}),$$

where z_i is a $1 \times n$ vector of standard unit Gaussian random variables, real for $\beta = 1$ and complex for $\beta = 2$. This corresponds to the generalized Wishart distribution with $\pi_i = \lambda_{X,i}$ and $\widehat{\pi}_j = 0$.

4.4. **Matrix model for $\beta = 2$ null case.** For each realization of X, we showed the conditioned process is Markov and depends only on its eigenvalues $\lambda_{X,i}$. The unconditioned process is therefore a mixture of Markov processes; we wish to check that it is itself Markov and to identify it with the one in [BG13].

Theorem 4.1. The eigenvalues $\{\lambda_i^m\}$ of (4.2) are supported for $m \leq n$ on interlacing sequences

$$\lambda^1 \prec \cdots \prec \lambda^m \qquad \lambda_i^l \in [0, 1]$$

with joint density proportional to

$$\Delta(\lambda^m) \prod_{i=1}^{m} (\lambda_i^m)^{A-1} \prod_{l=1}^{m-1} \prod_{i=1}^{l} (\lambda_i^l)^{-2} \prod_{l=1}^{m} \prod_{i=1}^{l} d\lambda_i^l.$$

Proof. Change variables to $\tau_i^m := (\lambda_{m+1-i}^m)^{-1} - 1 \in [0, \infty)$. Note that

$$d\lambda_{m+1-i}^m = -\frac{1}{(1 + \tau_i^m)^2} d\tau_i^m = -(\lambda_{m+1-i}^m)^2 d\tau_i^m$$

and that $\tau^m \prec \tau^{m+1}$ if and only if $\lambda^m \prec \lambda^{m+1}$. Therefore, it suffices for us to check that $\{\tau_i^m\}$ are supported on $\tau^1 \prec \cdots \prec \tau^m$ with density proportional to

$$p_m(\tau^m, \ldots, \tau^1) \prod_{l=1}^{m} \prod_{i=1}^{l} d\tau_i^l. = \Delta(\lambda^m) \prod_{i=1}^{m} (\lambda_i^m)^{A+1} \prod_{l=1}^{m} \prod_{i=1}^{l} d\tau_i^l = \Delta(\tau^m) \prod_{i=1}^{m} (1 + \tau_i^m)^{-A-m} \prod_{l=1}^{m} \prod_{i=1}^{l} d\tau_i^l.$$

Note that $X^*X \sim \mathsf{Wishart}^2(I_{n\times n}, n, A)$, which means that its eigenvalues $\lambda_{X,i}$ have the density

$$p_X(\lambda_X) = C_{n,2}\Delta(\lambda_X)^2 e^{-\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{A-n}$$

for some constant $C_{n,2}$. Further, the conditioned process of eigenvalues is Markov with transition density

$$Q^{\lambda_X}_{m-1,m}(\tau^{m-1}, d\tau^m) = \prod_{i=1}^n \lambda_{X,i} \frac{h_{\lambda_X}(\tau^m)\Delta(\tau^m)}{h_{\lambda_X}(\tau^{m-1})\Delta(\tau^{m-1})} 1_{\tau^{m-1}\prec\tau^m} d\tau^m.$$

In particular, its density is

$$p^{\lambda_X}_m(\tau^m, \ldots, \tau^1) = \prod_{i=1}^n \lambda_{X,i}^m h_{\lambda_X}(\tau^m)\Delta(\tau^m) 1_{\tau^1\prec\cdots\prec\tau^m}$$

and its marginal density is

$$p^{\lambda_X}_m(\tau^m) = \prod_{i=1}^n \lambda_{X,i}^m \int_{\tau^1\prec\cdots\prec\tau^m} \det(e^{-\lambda_{X,i}\tau^m_j}) d\lambda = \frac{1}{\Gamma(1)\cdots\Gamma(m)} \prod_{i=1}^n \lambda_{X,i}^m \frac{\det(e^{-\lambda_{X,i}\tau^m_i})}{\Delta(\lambda_X)} \Delta(\tau^m),$$

where we recall that the volume of the Gelfand-Tsetlin polytope for τ is $\frac{\Delta(\tau)}{\Gamma(1)\cdots\Gamma(m)}$.

The transition density of the unconditioned process is therefore

$$Q_{m-1,m}(\tau^{m-1}, \ldots, \tau^1, d\tau^m) = \int_{\lambda_X} Q^{\lambda_X}_{m-1,m}(\tau^{m-1}, d\tau^m) p(\lambda_X \mid \tau^{m-1}, \ldots, \tau^1) d\lambda_X.$$

By Bayes' rule, we have for $\tau^1 \prec \cdots \prec \tau^m$ that

$$\begin{aligned}
p(\lambda_X \mid \tau^{m-1}, \ldots, \tau^1) &= \frac{p_X(\lambda_X)}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} p^{\lambda_X}_{m-1}(\tau^{m-1}, \ldots, \tau^1) \\
&= \frac{C_{n,2}\Delta(\lambda_X)^2 e^{-\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{A-n}}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} \prod_{i=1}^n \lambda_{X,i}^{m-1} h_{\lambda_X}(\tau^{m-1})\Delta(\tau^{m-1}) \\
&= C_{n,2} \frac{\Delta(\lambda_X)^2 e^{-\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{A-n+m-1}}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} h_{\lambda_X}(\tau^{m-1})\Delta(\tau^{m-1}).
\end{aligned}$$

We conclude for some constant C that

$$\begin{aligned}
&Q_{m-1,m}(\tau^{m-1}, \ldots, \tau^1, d\tau^m) \\
&= C_{n,2} \int_{\lambda_X} Q^{\lambda_X}_{m-1,m}(\tau^{m-1}, d\tau^m) \frac{\Delta(\lambda_X)^2 e^{-\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{A-n+m-1}}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} h_{\lambda_X}(\tau^{m-1})\Delta(\tau^{m-1}) d\lambda_X \\
&= C_{n,2} \int_{\lambda_X} \prod_{i=1}^n \lambda_{X,i} \frac{h_{\lambda_X}(\tau^m)\Delta(\tau^m)}{h_{\lambda_X}(\tau^{m-1})\Delta(\tau^{m-1})} \frac{\Delta(\lambda_X)^2 e^{-\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{A-n+m-1}}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} h_{\lambda_X}(\tau^{m-1})\Delta(\tau^{m-1}) d\lambda_X \, 1_{\tau^{m-1}\prec\tau^m} d\tau^m \\
&= C_{n,2} \int_{\lambda_X} h_{\lambda_X}(\tau^m)\Delta(\tau^m) \frac{\Delta(\lambda_X)^2 e^{-\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{A-n+m}}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} d\lambda_X \, 1_{\tau^{m-1}\prec\tau^m} d\tau^m \\
&= C_{n,2} \int_{\lambda_X} h_{\lambda_X}(\tau^m)\Delta(\lambda_X)^2 e^{-\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{A-n+m} d\lambda_X \, \frac{\Delta(\tau^m)}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} \, 1_{\tau^{m-1}\prec\tau^m} d\tau^m \\
&= \int_{X\sim(A+m)\times n} e^{-\mathrm{Tr}(X^*X\tau^m)} e^{-\frac{1}{2}||X||^2} dX \, \frac{\Delta(\tau^m)}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} \, 1_{\tau^{m-1}\prec\tau^m} d\tau^m \\
&= C \frac{\prod_i (1+\tau_i^{m-1})^{A+m-1}}{\prod_{i=1}^m (1+\tau_i^m)^{A+m}} \frac{\Delta(\tau^m)}{\Delta(\tau^{m-1})} \, 1_{\tau^{m-1}\prec\tau^m} d\tau^m,
\end{aligned}$$

where the fifth equality follows from the HCIZ integral and the sixth from [FR05, Theorem 6]. This shows that the level-to-level transitions are Markov and that the joint density takes the claimed form. ☐

4.5. **Matrix model for $\beta = 1$ null case.** We again need to check that the mixture of Markov processes is itself Markov. For this, we require an analogue of the previous argument at $\theta = 1/2$. In the proof of the theorem, all integrals over Gaussian random variables will be taken over real-valued Gaussians.

Theorem 4.2. The eigenvalues $\{\lambda_i^m\}$ of (4.2) at $\beta = 1$ are supported for $m \leq n$ on interlacing sequences

$$\lambda^1 \prec \cdots \prec \lambda^m \qquad \lambda_i^l \in [0, 1]$$

with joint density proportional to

$$\Delta(\lambda^m) \prod_{i=1}^m (\lambda_i^m)^{A/2-1}(1-\lambda_i^m)^{-1/2} \prod_{l=1}^{m-1} \prod_{i=1}^l (\lambda_i^l)^{-1} \Delta(\lambda^l) \prod_{i,j} |\lambda_i^l - \lambda_j^{l+1}|^{-1/2}.$$

Proof. As in Theorem 4.1, change variables to $\tau_i^m := (\lambda_{m+1-i}^m)^{-1} - 1 \in [0, \infty)$. We wish to check that $\{\tau_i^m\}$ are supported on $\{\tau^1 \prec \cdots \prec \tau^m\}$ with density proportional to

$$\begin{aligned} p_l(\tau^1, \ldots, \tau^m) \prod_{l,i} d\tau_i^l &= \Delta(\lambda^m) \prod_{i=1}^m (\lambda_i^m)^{A/2-1}(1-\lambda_i^m)^{-1/2} \prod_{l=1}^{m-1} \prod_{i=1}^l (\lambda_i^l)^{-1} \Delta(\lambda^l) \prod_{i,j} |\lambda_i^l - \lambda_j^{l+1}|^{-1/2} \prod_{l,i} d\lambda_i^l \\ &= \Delta(\tau^m) \prod_{i=1}^m (1+\tau_i^m)^{-l/2-A/2} (\tau_i^m)^{-1/2} \prod_{l=1}^{m-1} \Delta(\tau^l) \prod_{i,j} |\tau_i^l - \tau_j^{l+1}|^{-1/2} \prod_{l,i} d\tau_i^l. \end{aligned}$$

Note that $X^*X \sim \mathsf{Wishart}^1(I_{n\times n}, n, A)$, which means that its eigenvalues $\lambda_{X,i}$ have the density

$$p_X(\lambda_X) = C_{n,1} \Delta(\lambda_X) e^{-\frac{1}{2}\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{\frac{A-n-1}{2}}$$

for some constant $C_{n,1}$. The transition density of the unconditioned process is

$$Q_{m-1,m}(\tau^{m-1}, \ldots, \tau^1, d\tau^m) = \int_{\lambda_X} Q_{m-1,m}^{\lambda_X}(\tau^{m-1}, d\tau^m) p(\lambda_X \mid \tau^{m-1}, \ldots, \tau^1) d\lambda_X.$$

By Bayes' rule, for $\tau^1 \prec \cdots \prec \tau^m$ we have

$$\begin{aligned} p(\lambda_X \mid \tau^{m-1}, \ldots, \tau^1) &= \frac{p_X(\lambda_X)}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} p_{m-1}^{\lambda_X}(\tau^{m-1}, \ldots, \tau^1) \\ &= \frac{C_{n,1} \prod_{i<j} \Delta(\lambda_X) e^{-\frac{1}{2}\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{\frac{A-n-1}{2}}}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} \\ &\qquad \int_V e^{-\frac{1}{2}\mathrm{Tr}(\lambda_X V \widetilde{\tau}^{m-1} V^T)} d\mathsf{Haar}_V \pi^{-\frac{m(m-1)}{2}} \prod_i \lambda_{X,i}^{\frac{m-1}{2}} \prod_{i=1}^{m-1} (\tau_i^{m-1})^{\frac{n-m}{2}} \prod_{l=1}^{m-1} \Delta(\tau^l) \prod_{i,j} |\tau_i^l - \tau_j^{l-1}|^{-\frac{1}{2}} \\ &= \frac{C_{n,1} \prod_{i<j} \Delta(\lambda_X) e^{-\frac{1}{2}\sum_i \lambda_{X,i}} \prod_i \lambda_{X,i}^{\frac{A-n+m-2}{2}}}{p_{m-1}(\tau^{m-1}, \ldots, \tau^1)} \\ &\qquad \int_V e^{-\frac{1}{2}\mathrm{Tr}(\lambda_X V \widetilde{\tau}^{m-1} V^T)} d\mathsf{Haar}_V \pi^{-\frac{m(m-1)}{2}} \prod_{i=1}^{m-1} (\tau_i^{m-1})^{\frac{n-m}{2}} \prod_{l=1}^{m-1} \Delta(\tau^l) \prod_{i,j} |\tau_i^l - \tau_j^{l-1}|^{-\frac{1}{2}}. \end{aligned}$$

We conclude that

$$\begin{aligned}
&Q_{m-1,m}(\tau^{m-1},\dots,\tau^1,d\tau^m)\\
&\quad=\int_{\lambda_X}\frac{C_{n,1}\prod_{i<j}\Delta(\lambda_X)e^{-\frac{1}{2}\sum_i\lambda_{X,i}}\prod_i\lambda_{X,i}^{\frac{A-n+m-1}{2}}}{p_{m-1}(\tau^{m-1},\dots,\tau^1)}\\
&\qquad\int_V e^{-\frac{1}{2}\mathrm{Tr}(\lambda_X V\widetilde{\tau}^m V^T)}d\mathsf{Haar}_V\pi^{-\frac{m(m+1)}{2}}\prod_{i=1}^m(\tau_i^m)^{\frac{n-m-1}{2}}\prod_{l=1}^m\Delta(\tau^l)\prod_{i,j}|\tau_i^l-\tau_j^{l-1}|^{-\frac{1}{2}}d\lambda_X\\
&\quad=\int_{\lambda_X}\int_V e^{-\frac{1}{2}\mathrm{Tr}(\lambda_X V\widetilde{\tau}^m V^T)}C_{n,1}\prod_{i<j}\Delta(\lambda_X)e^{-\frac{1}{2}\sum_i\lambda_{X,i}}\prod_i\lambda_{X,i}^{\frac{A-n+m-1}{2}}d\mathsf{Haar}_V d\lambda_X\\
&\qquad\frac{\pi^{-\frac{m(m+1)}{2}}\prod_{i=1}^m(\tau_i^m)^{\frac{n-m-1}{2}}\prod_{l=1}^m\Delta(\tau^l)\prod_{i,j}|\tau_i^l-\tau_j^{l-1}|^{-\frac{1}{2}}}{p_{m-1}(\tau^{m-1},\dots,\tau^1)}.
\end{aligned}$$

Notice now that

$$\begin{aligned}
&\int_{\lambda_X}\int_V e^{-\frac{1}{2}\mathrm{Tr}(\lambda_X V\widetilde{\tau}^m V^T)}C_{n,1}\prod_{i<j}\Delta(\lambda_X)e^{-\frac{1}{2}\sum_i\lambda_{X,i}}\prod_i\lambda_{X,i}^{\frac{A-n+m-1}{2}}d\mathsf{Haar}_V d\lambda_X\\
&\qquad=\int_{Z\sim(A+m)\times n}e^{-\frac{1}{2}\mathrm{Tr}(Z^TZ\widetilde{\tau}^m)}e^{-\frac{1}{2}||Z||^2}dZ\\
&\qquad=\int_{Z_{ij}}e^{-\frac{1}{2}\sum_{j=1}^n\widetilde{\tau}_j^m\sum_{i=1}^{A+m}Z_{ij}^2}e^{-\frac{1}{2}\sum_{i,j}Z_{ij}^2}dZ_{ij}\\
&\qquad=\int_{Z_{ij}}e^{-\frac{1}{2}\sum_{i=1}^{A+m}\sum_{j=1}^n(\widetilde{\tau}_j^m+1)Z_{ij}^2}dZ_{ij}\\
&\qquad=\prod_{j=1}^m(1+\tau_j^m)^{-\frac{A+m}{2}}.
\end{aligned}$$

Substituting back in and applying induction, we obtain for some constant C the desired

$$Q_{m-1,m}(\tau^{m-1},\dots,\tau^1,d\tau^m)=C\frac{\prod_i(1+\tau_i^{m-1})^{\frac{A+m-1}{2}}}{\prod_i(1+\tau_i^m)^{\frac{A+m}{2}}}\frac{\prod_{i=1}^m(\tau_i^m)^{\frac{n-m-1}{2}}}{\prod_{i=1}^{m-1}(\tau_i^{m-1})^{\frac{n-m}{2}}}\Delta(\tau^m)\prod_{i,j}|\tau_i^m-\tau_j^{m-1}|^{-1/2}.\quad\square$$

4.6. **Identifying β-Jacobi ensembles as Heckman-Opdam ensembles in the null case.** In this subsection, we identify the measures we just studied with Heckman-Opdam ensembles at certain specializations. Because the β-Jacobi ensemble has eigenvalues in $[0,1]$, this identification will involve a change of variables.

4.6.1. *The case of $\beta=2$.* In this case, by Theorem 4.1, the joint density of $\lambda^1\prec\cdots\prec\lambda^m$ is

$$\Delta(\lambda^m)\prod_{i=1}^m(\lambda_i^m)^{A-1}\prod_{l=1}^{m-1}\prod_{i=1}^l(\lambda_i^l)^{-2}\prod_{l=1}^m\prod_{i=1}^l d\lambda_i^l$$

for λ_i^l supported in $[0,1]$. Changing variables to $\mu_i^l=-\log\lambda_i^l$, the density of $\mu^1\prec\cdots\prec\mu^m$ is

$$\Delta(e^{-\mu^m})e^{-A|\mu^m|}\prod_{l=1}^{m-1}e^{|\mu^l|}\prod_{l=1}^m\prod_{i=1}^l d\mu_i^l.$$

We would like to check that this is proportional to

$$\begin{aligned}
&\frac{1}{(m-1)_{\min\{m,n\}}(\min\{m,n\}-1)!}\prod_{i=1}^m\prod_{j=1}^n(\pi_j+\widehat{\pi}_i)\Delta^{\mathrm{trig}}(\mu^m)\\
&\quad\prod_{i=1}^{\min\{m,n\}}(1-e^{-\mu_i^m})^{m-\min\{m,n\}}\prod_{i=1}^{\min\{m,n\}}e^{-\sum_{l=1}^m\widehat{\pi}_l(|\mu^l|-|\mu^{l-1}|)}\prod_{l=1}^{m-1}\widetilde{\mathcal{F}}_2^{\min\{m,n\},n}(\mu^m,-\pi)\prod_{l,i}d\mu_i^l.
\end{aligned}$$

for the choices of $\pi = (A-m+1, A-m+2, \dots, A-m+n)$ and $\widehat{\pi} = (0, 1, \dots)$. For these choices of π and $\widehat{\pi}$, we obtain

$$\widetilde{\mathcal{F}}_2^{\min\{m,n\},n}(\mu^m, -\pi) = e^{-\frac{n-1}{2}|\mu^m| - (A-m+1)|\mu^m|}$$

so that the density is

$$\frac{\prod_{i=1}^m \prod_{j=1}^n (A+i+j-1)}{(m-1)_{\min\{m,n\}}(\min\{m,n\}-1)!} \Delta(e^{-\mu^m}) e^{-A|\mu^m|} \prod_{i=1}^{\min\{m,n\}} (1-e^{-\mu_i^m})^{m-\min\{m,n\}} \prod_{l=1}^{m-1} e^{|\mu^l|} \prod_{l,i} d\mu_i^l.$$

The two densities agree for $m \leq n$.

4.6.2. *The case of $\beta = 1$.* In this case, by Theorem 4.2, the joint density of $\lambda^1 \prec \cdots \prec \lambda^m$ is

$$\Delta(\lambda^m) \prod_{i=1}^m (\lambda_i^m)^{A/2-1}(1-\lambda_i^m)^{-1/2} \prod_{l=1}^{m-1} \prod_{i=1}^{l} (\lambda_i^l)^{-1} \Delta(\lambda^l) \Delta(\lambda^l, \lambda^{l+1})^{-1/2} \prod_{l,i} d\lambda_i^l$$

for λ_i^l supported in $[0,1]$. Changing variables to $\mu_i^l = -\log \lambda_i^l$, the density of $\mu^1 \prec \cdots \prec \mu^m$ is

$$\Delta(e^{-\mu^m}) e^{-A/2|\mu^m|} \prod_{i=1}^m (1-e^{-\mu_i^m})^{-1/2} \prod_{l=1}^{m-1} \Delta(e^{-\mu^l}) \Delta(e^{-\mu^l}, e^{-\mu^{l+1}})^{-1/2} \prod_{l,i} d\mu_i^l.$$

We would like to check that this is proportional to

$$\frac{\Gamma(1/2)^{2\min\{m,n\} - \min\{n,m\}(\min\{n,m\}-1)/2 - (m-\min\{m,n\})\min\{m,n\}}}{\Gamma(m/2)\cdots\Gamma((m-\min\{m,n\}+1)/2)\Gamma(\min\{m,n\}/2)\cdots\Gamma(1/2)} \prod_{i=1}^m \prod_{j=1}^n \frac{\Gamma(1/2+\pi_j/2+\widehat{\pi}_i/2)}{\Gamma(\pi_j/2+\widehat{\pi}_i/2)} \Delta^{\mathrm{trig}}(\mu^m)^{1/2}$$
$$\prod_{i=1}^{\min\{m,n\}} (1-e^{-\mu_i^m})^{(m-\min\{m,n\})/2} \prod_{i=1}^{\min\{m,n\}} \frac{(1-e^{-\mu_i^{\min\{m,n\}}})^{-1/2}}{(1-e^{-\mu_i^m})^{-1/2}} e^{-\frac{1}{2}\sum_{l=1}^m \widehat{\pi}_l(|\mu^l| - |\mu^{l-1}|)}$$
$$\prod_{l=1}^{m-1} \frac{\Delta(e^{-\mu^l}, e^{-\mu^{l+1}})^{-1/2}}{\Delta(e^{-\mu^l})^{-1/2}\Delta(e^{-\mu^{l+1}})^{-1/2}} e^{-\frac{1}{2}|\mu^l|} \widetilde{\mathcal{F}}_1^{\min\{m,n\},n}(\mu^m, -\pi/2) \prod_{l,i} d\mu_i^l$$

for the choices $\pi = (\frac{A-m+1}{2}, \frac{A-m+2}{2}, \dots, \frac{A-m+n}{2})$ and $\widehat{\pi} = (0, \frac{1}{2}, \dots)$. For these choices of π and $\widehat{\pi}$, we obtain

$$\widetilde{\mathcal{F}}_1^{\min\{m,n\},n}(\mu^m, -\pi) = \Gamma(1/2)^{-\min\{m,n\}} e^{-\frac{n-1}{4}|\mu^m| - \frac{A-m+1}{2}|\mu^m|} \prod_{i=1}^{\min\{m,n\}} (1-e^{-\mu_i^m})^{-1/2}$$

so that the density is

$$\frac{\Gamma(1/2)^{\min\{m,n\} - \min\{n,m\}(\min\{n,m\}-1)/2 - (m-\min\{m,n\})\min\{m,n\}}}{\Gamma(m/2)\cdots\Gamma((m-\min\{m,n\}+1)/2)\Gamma(\min\{m,n\}/2)\cdots\Gamma(1/2)} \prod_{i=1}^m \prod_{j=1}^n \frac{\Gamma(1/2+\pi_j/2+\widehat{\pi}_i/2)}{\Gamma(\pi_j/2+\widehat{\pi}_i/2)} \Delta(e^{-\mu^m})$$
$$e^{-\frac{A}{2}|\mu^m|} \prod_{i=1}^{\min\{m,n\}} (1-e^{-\mu_i^m})^{(m-\min\{m,n\}-1)/2} \prod_{i=1}^{\min\{m,n\}} \frac{(1-e^{-\mu_i^{\min\{m,n\}}})^{-1/2}}{(1-e^{-\mu_i^m})^{-1/2}}$$
$$\prod_{l=1}^{m-1} \Delta(e^{-\mu^l}) \Delta(e^{-\mu^l}, e^{-\mu^{l+1}})^{-1/2} \prod_{l,i} d\mu_i^l.$$

The two densities agree for $m \leq n$.

Appendix A. Elementary asymptotic relations

In this section we collect some elementary results on limits of various special functions.

Lemma A.1 ([BG13, Lemma 2.4])**.** For $a, b \in \mathbb{C}$ and $u(q)$ a complex-valued function defined in a neighborhood of 1 so that $\lim_{q\to 1} u(q) = u$ with $0 < u < 1$, we have that

$$\lim_{q\to 1} \frac{(q^a u(q); q)}{(q^b u(q); q)} = (1-u)^{b-a}.$$

Lemma A.2. For any a, b, $u(q)$ a complex-valued function defined in a neighborhood of 1 so that $\lim_{q\to 1} u(q) = u$ with $0 < u < 1$, and $m(q)$ with $\lim_{q\to 1} q^{m(q)} = e^m$, we have that

$$\lim_{q\to 1} \frac{(q^a u(q); q)_{m(q)}}{(q^b u(q); q)_{m(q)}} = \frac{(1-u)^{b-a}}{(1-ue^m)^{b-a}}.$$

Proof. By Lemma A.1, we obtain

$$\lim_{q\to 1} \frac{(q^a u(q); q)}{(q^a q^{m(q)} u(q); q)} \frac{(q^b q^{m(q)} u(q); q)}{(q^b u(q); q)} = \frac{(1-u)^{b-a}}{(1-ue^m)^{b-a}}.$$ □

Lemma A.3. We have

$$\lim_{q\to 1} \frac{(q^a; q)}{(q^b; q)} (1-q)^{a-b} = \frac{\Gamma(b)}{\Gamma(a)}.$$

Lemma A.4. We have

$$\lim_{\varepsilon\to 0} \varepsilon^{-\theta} \frac{\Gamma(\varepsilon^{-1}a)}{\Gamma(\varepsilon^{-1}a+\theta)} = a^{-\theta}.$$

Proof. By Stirling's approximation, we have $\Gamma(z) = z^{z-1/2} e^{-z} \sqrt{2\pi}(1 + O(z^{-1}))$, hence

$$\lim_{\varepsilon\to 0} \varepsilon^{-\theta} \frac{\Gamma(\varepsilon^{-1}a)}{\Gamma(\varepsilon^{-1}a+\theta)} = \lim_{\varepsilon\to 0} \frac{a^{a\varepsilon^{-1}-1/2} \varepsilon^{-a\varepsilon^{-1}+1/2} e^{-a\varepsilon^{-1}}}{a^{a\varepsilon^{-1}+\theta-1/2} \varepsilon^{\theta} (\varepsilon^{-1}+\theta a^{-1})^{a\varepsilon^{-1}-1/2+\theta} e^{-a\varepsilon^{-1}-\theta}}$$
$$= a^{-\theta} e^{\theta} \lim_{\varepsilon\to 0} \left(1 + \frac{\theta\varepsilon}{a}\right)^{-\frac{a}{\varepsilon}+1/2} = a^{-\theta}.$$ □

Lemma A.5. For $f(u) = \frac{(tu;q)}{(qu;q)}$, $q = e^{-\varepsilon}$, and $t = e^{-\theta\varepsilon}$, we have

(a) $\lim_{\varepsilon\to 0} f(uq^a) = (1-u)^{1-\theta}$
(b) $\lim_{\varepsilon\to 0} f(q^a)\varepsilon^{\theta-1} = \frac{\Gamma(1+a)}{\Gamma(\theta+a)}$.

Proof. Apply Lemma A.1 for (a) and Lemma A.3 for (b). □

References

[BBP05] J. Baik, G. Ben Arous, and S. Péché. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Ann. Probab.*, 33(5):1643–1697, 2005.

[BC14] A. Borodin and I. Corwin. Macdonald processes. *Probab. Theory Related Fields*, 158(1-2):225–400, 2014.

[BG13] A. Borodin and V. Gorin. General β Jacobi corners process and the Gaussian free field. *Comm. Pure Appl. Math.*, 2013.

[Bor14] A. Borodin. CLT for spectra of submatrices of Wigner random matrices. *Mosc. Math. J.*, 14(1):29–38, 170, 2014.

[BP08] A. Borodin and S. Péché. Airy kernel with two sets of parameters in directed percolation and random matrix theory. *J. Stat. Phys.*, 132(2):275–290, 2008.

[BV13] A. Bloemendal and B. Virág. Limits of spiked random matrices I. *Probab. Theory Related Fields*, 156(3-4):795–825, 2013.

[Def10] M. Defosseux. Orbit measures, random matrix theory and interlaced determinantal processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(1):209–249, 2010.

[DW09] A. Dieker and J. Warren. On the largest-eigenvalue process for generalized Wishart random matrices. *ALEA Lat. Am. J. Probab. Math. Stat.*, 6:369–376, 2009.

[FN11] P. Forrester and T. Nagao. Determinantal correlations for classical projection processes. *J. Stat. Mech.*, 2011(08):P08011, 2011.

[FR05] P. Forrester and E. Rains. Interpretations of some parameter dependent generalizations of classical matrix ensembles. *Probab. Theory Related Fields*, 131(1):1–61, 2005.

[GK02] T. Guhr and H. Kohler. Recursive construction for a class of radial functions. I. Ordinary space. *J. Math. Phys.*, 43(5):2707–2740, 2002.

[Joh01] I. Johnstone. On the distribution of the largest eigenvalue in principal components analysis. *Ann. Statistics*, 29(2):295–327, 2001.

[Mac95] I. Macdonald. *Symmetric functions and Hall polynomials.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Mo12] M. Mo. Rank 1 real Wishart spiked model. *Comm. Pure Appl. Math.*, 65(11):1528–1638, 2012.

[Mui82] R. Muirhead. *Aspects of multivariate statistical theory.* John Wiley & Sons, Inc., New York, 1982. Wiley Series in Probability and Mathematical Statistics.

[Ner03] Y. Neretin. Rayleigh triangles and nonmatrix interpolation of matrix beta integrals. *Mat. Sb.*, 194(4):49–74, 2003.

[OMH14] A. Onatski, M. Moreira, and M. Hallin. Signal detection in high dimension: The multispiked case. *Ann. Statistics*, 42(1):225–254, 2014.

[OR03] A. Okounkov and N. Reshetikhin. Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram. *J. Amer. Math. Soc.*, 16(3):581–603 (electronic), 2003.

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