Mathematics UN1102
Section 1, Spring 2020
Midterm 2
April 13, 2020
Time Limit: 75 Minutes

Name: ____________________________

UNI: ____________________________

Instructions: This exam contains 6 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. Please **do not write on the back side of pages**.

You will have 75 minutes to complete this exam. You may choose any 75 minute period between 12 noon and 12 midnight Eastern time on April 13, 2020 to take the exam. You must scan and upload the completed exam to Gradescope by **12 midnight Eastern time on April 13, 2020**. Please write the 75 minute period you took the exam below.

Start Time: ____________________________

End Time: ____________________________

The use of outside material including books, notes, calculators, and electronic devices is not allowed. Due to the coronavirus situation, this exam will be take-home, meaning that these rules will be enforced by the honor code. Please sign below to affirm that you have followed these rules.

Signature: ____________________________

Formulas

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\quad R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\quad R = \infty$
- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ $\quad R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$ $\quad R = 1$

Problem 1 (15 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (5 points) If $\{a_n\}$ satisfies $a_n > \frac{1}{n^2}$ for all n, then $\sum_{n=1}^{\infty} a_n$ diverges.

Answer:

(b) (5 points) If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=10}^{\infty} a_n$ diverges.

Answer:

(c) (5 points) If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 2, then the power series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ converges at $x = 1$.

Answer:

Problem 2 (10 points) Determine whether the following sequences are convergent or divergent. If the sequence is convergent, determine its limit as $n \to \infty$. Justify your answer.

(a) (5 points) The sequence $\{a_n\}$, where $a_n = \frac{n^n}{n!}$.

Answer:

(b) (5 points) The sequence $\{b_n\}$, where $b_n = \frac{n^2+n}{3n^2+\sqrt{n}}$.

Answer:

Problem 3 (20 points) Determine whether the following series are convergent or divergent and **explain your reasoning**. If the series is convergent, you **do not need to** determine its sum.

(a) (10 points) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3n}}$.

Answer:

(b) (10 points) The series $\sum_{n=1}^{\infty} ne^{-n}$.

Answer:

Problem 4 (20 points) Consider the series $X = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n+1}$.

(a) (5 points) Show that the series converges. You **do not** need to determine the sum.

Answer:

(b) (5 points) Is the series absolutely convergent?

Answer:

(c) (10 points) We wish to approximate X by the partial sum $s_k := \sum_{n=0}^{k} (-1)^n \frac{1}{2^n+1}$. What is the minimum value of k for which $|s_k - X| < \frac{1}{30}$?

Answer:

Problem 5 (15 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \ln(n) \cdot 2^n (x-1)^n.$$

Answer:

Problem 6 (20 points)

(a) (10 points) Find the Taylor series about $x = 0$ for the function

$$f(x) = \frac{1}{1+4x^2}.$$

What is its radius of convergence?

Answer:

(b) (10 points) Using your answer above, compute

$$\lim_{x \to 0} \frac{\frac{1}{1+4x^2} - 1 + 4x^2}{x^4}.$$

Answer: