THE SATURATION CONJECTURE FOR LITTLEWOOD-RICHARDSON COEFFICIENTS

YI SUN

Abstract. This paper gives an overview of Knutson and Tao's proof of the Saturation Conjecture for Littlewood-Richardson coefficients. We begin by introducing the honeycomb model and discussing its connection with BZ patterns. We then sketch the proof of the Saturation Conjecture. We conclude with some remarks about the implications of this result.

1. Introduction

Recall that the Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ was defined to be the coefficient of s_λ in the expansion

$$s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda.$$

These coefficients have some structure; it is clear from the definition that $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$. In addition, recalling that s_λ was the character of the irreducible representation V_λ of $GL_n(\mathbb{C})$ with highest weight λ, we may view $c^{\lambda}_{\mu\nu}$ as the multiplicity of V_λ in the decomposition of $V_\mu \otimes V_\nu$ into irreducible representations. This implies that $c^{\lambda}_{\mu\nu} \geq 0$ for all λ, μ, ν.

Alternatively, the classical Littlewood-Richardson rule provides a combinatorial interpretation of $c^{\lambda}_{\mu\nu}$ as the number of semistandard Young tableaux of shape λ/μ and type ν with reverse reading word a lattice permutation (see [10], Theorem A1.3.3). Since the number of such tableaux (known as Littlewood-Richardson tableau) is by definition non-negative, we obtain an alternate proof of the non-negativity of the Littlewood-Richardson coefficients.

We may obtain more about the structure of the Littlewood-Richardson coefficients from this combinatorial interpretation. In particular, it is interesting to ask when $c^{\lambda}_{\mu\nu}$ is positive. It is evident from the definition we must have that $\mu + \nu = \lambda$, but we can say more. Define the set of triples of partitions of length at most n with non-zero Littlewood-Richardson coefficient by

$$T_n = \{(\lambda, \mu, \nu) \mid \ell(\lambda), \ell(\mu), \ell(\nu) \leq n, c^{\lambda}_{\mu,\nu} \neq 0\}.$$

We may then obtain the following proposition, which shows that T_n is closed under addition.

Proposition 1. If $(\lambda_1, \mu_1, \nu_1)$ and $(\lambda_2, \mu_2, \nu_2)$ are in T_n, then so is $(\lambda_1 + \lambda_2, \mu_1 + \mu_2, \nu_1 + \nu_2)$.

Proof. Since $c^{\lambda_1}_{\mu_1\nu_1} \neq 0$ and $c^{\lambda_2}_{\mu_2\nu_2} \neq 0$, we may choose Littlewood-Richardson tableaux T_1 of shape λ_1/μ_1 and type ν_1 and T_2 of shape λ_2/μ_2 and type ν_2. Now, form a new tableau T of shape $(\lambda_1 + \lambda_2)/(\mu_1 + \mu_2)$ and type $\nu_1 + \nu_2$ such that the entries each row of T are the entries in the corresponding rows of T_1 and T_2 arranged in non-decreasing order.

It is easy to check that T is a valid tableau. Now, by construction, the reverse reading word of T is formed by interleaving the reverse reading words of T_1 and T_2; hence, it is a lattice permutation as the interleaving of two lattice permutations. This means that T is a valid Littlewood-Richardson tableau of shape $(\lambda_1 + \lambda_2)/(\mu_1 + \mu_2)$ and type $\nu_1 + \nu_2$, and therefore $c^{\lambda_1+\lambda_2}_{\mu_1+\mu_2\,\nu_1+\nu_2} \neq 0$. $\square$

Thus we see that T_n forms an additive semigroup within $\mathbb{Z}^{3n}$. In [6], Klyachko uses the theory of stable bundles to give a system of linear inequalities on (λ, μ, ν) that are satisfied on T_n. In

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addition, if these inequalities are satisfied for the triple (λ, μ, ν), then $(N\lambda, N\mu, N\nu) \in T_n$ for some positive integer N. This result is almost a complete characterization of the semigroup T_n, leaving open only the question of whether it is saturated within $\mathbb{Z}^{3n}$. The following theorem, first proved by Knutson and Tao in [7], answers this question in the affirmative.

Theorem 2 (Saturation Conjecture)**.** If $(N\lambda, N\mu, N\nu) \in T_n$ for some $N \in \mathbb{N}$, then $(\lambda, \mu, \nu) \in T_n$.

Knutson and Tao's approach requires a small reformulation. Recall from basic representation theory that we have a canonical isomorphism $\mathrm{Hom}_{GL_n(\mathbb{C})}(V_\lambda, V_\mu \otimes V_\nu) \simeq (V_\lambda^* \otimes V_\mu \otimes V_\nu)^{GL_n(\mathbb{C})}$, where the second expression is the space of $GL_n(\mathbb{C})$-invariants. Recalling that $V_\lambda^* \simeq V_{\lambda^*}$ for some highest weight λ^* which depends linearly on λ, we may write

$$c_{\mu\nu}^\lambda = \dim \mathrm{Hom}_{GL_n(\mathbb{C})}(V_\lambda, V_\mu \otimes V_\nu) = \dim(V_{\lambda^*} \otimes V_\mu \otimes V_\nu)^{GL_n(\mathbb{C})}.$$

Now, define the *symmetric Littlewood-Richardson coefficient* to be $c_{\lambda\mu\nu} := \dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{GL_n(\mathbb{C})}$ and $T_n' := \{(\lambda, \mu, \nu) \mid c_{\lambda\mu\nu} \neq 0\}$. It is obvious from the definition that $c_{\lambda\mu\nu}$ is invariant under permutation of λ, μ, ν and that $c_{\mu\nu}^\lambda = c_{\lambda^*\mu\nu}$. Further, because λ^* is a linear function of λ, we see that $(N\lambda)^* = N\lambda^*$, meaning that Theorem 2 follows from the following theorem.

Theorem 3 (Knutson and Tao)**.** If $(N\lambda, N\mu, N\nu) \in T_n'$ for some $N \in \mathbb{N}$, then $(\lambda, \mu, \nu) \in T_n'$.

This paper will be devoted to sketching Knutson and Tao's proof of Theorem 3 and illustrating some of its implications. Note that other proofs of Theorem 2 have been found using approaches using representations of quivers in [4] and intersection theory in [1], but we choose to present a proof with more combinatorial content.

The remainder of this paper is structured as follows. In section 2, we describe the honeycomb model and show its equivalence to the BZ cone. In section 3, we give Knutson and Tao's application of honeycombs to prove the saturation conjecture. In section 4, we discuss some applications to Horn's conjecture on the spectra of sums of hermitian matrices.

We would like to note here that most of the figures in this paper were made using Knutson and Tao's online Honeycomb applet [9].

2. Honeycombs and BZ Patterns

In this section, we introduce the primary combinatorial object of this paper, the honeycomb, and provide the connection between it and the BZ cone introduced in [2]. We follow the notation and general approach of Knutson and Tao in [7] throughout this section, but we specialize immediately to the case of $GL_n(\mathbb{C})$ honeycombs only.

We begin with a notational note about graphs. In this paper we wish to consider directed graphs with some edges containing only one vertex. We think of these edges as coming from or pointing to infinity and call them *semi-infinite*. In this case, a subgraph Γ' of a graph $\Gamma = (V, E)$ is given by any pair of subsets of $V' \subset V$ and $E' \subset E$, where we consider any edge in E' with one end missing to be semi-infinite.

2.1. Tinkertoys and Honeycombs.

Consider the vector space $B := \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$. Notice that we have an embedding of the triangular lattice $B_\mathbb{Z} := \{(x, y, z) \in \mathbb{Z}^3 \mid x + y + z = 0\}$ inside B and that lines along the three directions $\{(-1, 1, 0), (0, -1, 1), (1, 0, -1)\}$ form a triangular grid in B. Moving along any line in each of these three directions fixes one of the three coordinates, we will call this the *constant coordinate* of this line. We then have the following definitions.

Definition 1. A *tinkertoy* τ on B is a pair (Γ, d), where $\Gamma = (V, E)$ is a directed graph with some edges possibly one-ended and $d : E \to \{(-1, 1, 0), (0, -1, 1), (1, 0, -1)\}$ is a map assigning to each edge a direction. Given a tinkertoy $\tau = (\Gamma, d)$, define a *subtinkertoy* τ' of τ to be a tinkertoy $\tau' = (\Gamma', d')$, where Γ' is a subgraph of Γ and d' the restriction of d to the edges of Γ'.

We wish to think of tinkertoys as abstract graphs which may be realized via embeddings into B so that the edges point in the specified directions. We formalize this as follows.

Definition 2. An embedding $h : \Gamma \to B$ is a (virtual) *configuration* for τ if, for any two-ended edge $e = (v_1, v_2) \in E$, the vector $h(v_2) - h(v_1)$ is a (possibly negative) non-negative multiple of $d(e)$. We will say that h is a *lattice configuration* if $h(v) \in B_{\mathbb{Z}}$ for all v and that h is *non-degenerate* if $h(v_2) - h(v_1) \neq 0$ for all two-headed edges $e = (v_1, v_2)$.

Since the conditions for a mapping $h : \Gamma \to B$ to be a virtual configuration are linear, we see that the space of virtual configuration is a linear subspace of the vector space of maps $\Gamma \to B$. Because the space of configurations is cut out by the half spaces guaranteeing that each two-ended edge has non-negative length, it is a closed cone in this subspace. Call this the *cone of configurations* of a tinkertoy.

Our goal will be to study a particular tinkertoy associated with $GL_n(\mathbb{C})$, which we will define as a a subtinkertoy of a larger tinkertoy. Define a graph as follows. Take the set of vertices

$$V = \{(x, y, z) \in B_{\mathbb{Z}} \mid 2i + j \not\equiv 0 \pmod 3\}.$$

Now, for each vertex (x, y, z) with $2x + y \equiv 2 \pmod 3$, create three directed edges from (x, y, z) to $(x - 1, y + 1, z)$, $(x, y - 1, z + 1)$, and $(x + 1, y, z - 1)$. We then obtain an infinite directed graph Γ embedded in B with all vertices in $B_{\mathbb{Z}}$; by inspection, each vertex of this graph has degree 3, and all edges of the graph are along one of the three chosen directions in B. Therefore, along with its embedding into B, the graph Γ defines a tinkertoy $\tau = (\Gamma, d)$ that we call the *infinite honeycomb tinkertoy*. We wish to study the configurations of the following subtinkertoys of τ.

Definition 3. The GL_n *tinkertoy* τ_n is the subtinkertoy of τ containing the set of vertices V_n within the triangle $\{(x, y, z) \in B \mid z + 3n \geq x \geq y \geq z\}$. Note that τ_n has $3n$ semi-infinite edges (n in each direction). We call these are the *boundary edges* of τ_n.

Definition 4. A *honeycomb* is a configuration $h : V_n \to B$ of τ_n. The *boundary conditions* of a honeycomb h are the constant coordinates of the $3n$ boundary edges of τ_n. Denote these by $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_n$, and $\nu_1, \ldots, \nu_n$ in clockwise order beginning with the leftmost edge in the direction $(0, -1, 1)$.

Write $\mathtt{HONEY}(\tau_n)$ for the space of honeycombs $h : V_n \to B$. Then, the boundary conditions give a map $\mathtt{BDRY} : \mathtt{HONEY}(\tau_n) \to (\mathbb{R}^n)^3$ that sends h to the corresponding triple (λ, μ, ν). Denote the image of this map by $\mathtt{BDRY}(\tau_n)$. For our purposes, we wish to think of $\mathtt{BDRY}(h)$ as a triple of weights for $GL_n(\mathbb{C})$; our goal will then be to classify the possible honeycombs h corresponding to a particular element of $\mathtt{BDRY}(\tau_n)$.

The notations that we have given so far are illustrated in Figure 1. In this figure, the labels on the edges denote their constant coefficient. Notice that for the given τ_4-honeycomb h, we have $\mathtt{BDRY}(h) = \Big((0, -25, -40, -55), (25, 5, -10, -25), (60, 40, 20, 5)\Big)$.

2.2. **The BZ Cone.** The goal of this construction is realize the Littlewood-Richardson coefficients as the number of lattice honeycombs in $\mathtt{HONEY}(\tau_n)$. To do so, we will relate $\mathtt{HONEY}(\tau_n)$ to the BZ cone. For this, we follow the approach of [11], which defines a BZ pattern as follows (note that we have changed the notation to fit the present paper).

Definition 5. Take vectors $l, m, r \in \mathbb{Z}^{n-1}$. A *BZ pattern* of type (n, l, m, r) is a collection of integers $\{y_{i,j,k}\}$ for $(i, j, k) \in \{(i, j, k) \in \mathbb{Z}^3 \mid 0 \leq i, j, k < n, i + j + k = n\}$ such that

- the sums in the $(0, -1, 1)$ direction are equal to $l_1, \ldots, l_{n_1}$ from top to bottom
- the sums in the $(1, 0, -1)$ direction are equal to $m_1, \ldots, m_{n-1}$ from left to right
- the sums in the $(1, -1, 0)$ direction are equal to $r_1, \ldots, r_{n-1}$, right to left

and in each case, the partial sums increasing in the given direction are all non-negative.

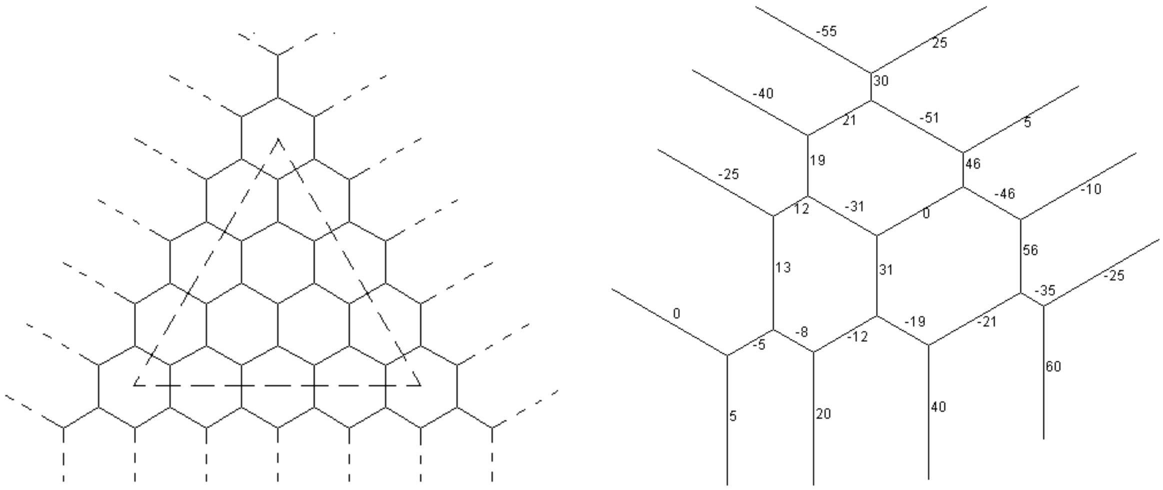

FIGURE 1. The GL_4 tinkertoy τ_4 (enclosed in the dotted triangle) and a τ_4-honeycomb with boundary labels.

We will also use the following related notions. Call the set of all real $\{y_{i,j,k}\}$ satisfying only the positivity constraints on the partial sums the *BZ cone* (denoted $\texttt{BZ}(n)$). Call an arbitrary point in $\texttt{BZ}(n)$ a quasi-BZ pattern and let its type be the sums along the specified directions.

The key theorem relating BZ-patterns and Littlewood-Richardson coefficients that we will use is the following, which we state without proof (see [2]).

Theorem 4. Given partitions λ, μ, ν with at most n parts, define $l, m, r \in \mathbb{Z}^{n-1}$ by $l_i = \lambda_i - \lambda_{i+1}$, $m_i = \mu_i - \mu_{i+1}$, and $r_i = \nu_i - \nu_{i+1}$. Then, the coefficient $c_{\lambda\mu\nu}$ is equal to the number of BZ patterns of type (n, l, m, r).

Notice here that BZ patterns of type (n, l, m, r) are simply integer points in $\texttt{BZ}(n)$ with row and column sums given by (l, m, r). Similarly, lattice honeycombs are integer points in the cone $\texttt{HONEY}(\tau_n)$. This suggests the following theorem, which provides the desired link.

Theorem 5. There is a $\mathbb{Z}$-linear map $\texttt{HONEY}(\tau_n) \to \texttt{BZ}(n)$ that maps a honeycomb h with boundary $\texttt{BDRY}(h) = (\lambda, \mu, \nu)$ to a quasi-BZ pattern of type (l, m, r) with $l_i = \lambda_i - \lambda_{i+1}$, $m_i = \mu_i - \mu_{i+1}$, and $r_i = \nu_i - \nu_{i+1}$.

Proof. Take any $h \in \texttt{HONEY}(\tau_n)$. First, given a (possibly degenerate) hexagon $\alpha \in h$, it is easy to check that the signed difference in length between the left and right vertical edges is invariant under $120°$ rotation of the hexagon (for instance, we may write each side of the hexagon as a vector and use the fact that the vectors sum to 0). Call this number the *torsion* of a hexagon.

Now, notice that the regions of the τ_n-tinkertoy in or bordering the interior are in a bijective correspondence with the indices that a BZ pattern is indexed over. Then, construct the map as follows.

- Assign to each interior hexagon of τ_n its torsion.
- Assign to each exterior wedge on the right the length of its left vertical border.
- Assign to each exterior wedge on the left the length of its bottom right border.
- Assign to each exterior wedge on the bottom the length of its upper right border.

Notice that this construction is obviously $\mathbb{Z}$-linear in the coordinates of h. An example of these construction for the τ_4-honeycomb given before is shown in Figure 2.

We now claim that this construction results in a valid quasi-BZ pattern of type (n, l, m, r). Since the construction was invariant under rotations, it suffices to check the properties of a BZ pattern along the horizontal sums. But notice that the horizontal partial sum from right to left ending at any hexagon is just the length of the left edge of the hexagon, which is evidently non-negative. Further, the total sum of each row is the sum of the lengths of the border of the left exterior wedge in that row, which is equal to the difference between the constant coordinates of its two semi-infinite edges, verifying the other condition. □

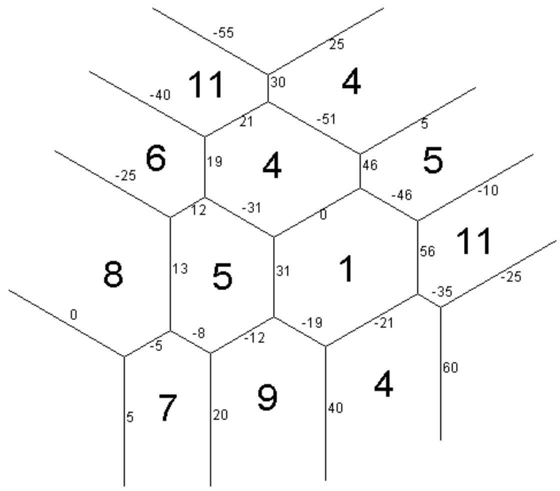

Figure 2. The BZ pattern corresponding to a τ_4-honeycomb.

Theorem 5 implies the following two corollaries, which provide the link between honeycombs and our original definition of the Littlewood-Richardson coefficients.

Corollary 6. There is a bijection between lattice τ_n-honeycombs h with boundary $\texttt{BDRY}(h) = (\lambda, \mu, \nu)$ and BZ patterns of type (n, l, m, r) with $l_i = \lambda_i - \lambda_{i+1}$, $m_i = \mu_i - \mu_{i+1}$ and $r_i = \nu_i - \nu_{i+1}$.

Proof. We already know that the map given in the proof of Theorem 5 takes honeycombs to quasi-BZ patterns, so it is enough to show that any BZ pattern in the image of this map is mapped to by a lattice honeycomb. But the proof of Theorem 5 showed that the lengths of the finite edges of a honeycomb were just sums and differences of the entries of the quasi-BZ pattern. Hence, if we have a bona fide BZ pattern, then all finite edges of the corresponding honeycomb must be integral. But then shifting the bottom left corner of the honeycomb to have integral coordinates makes the honeycomb a lattice honeycomb, as needed. □

Corollary 7. The symmetric Littlewood-Richardson coefficient $c_{\lambda\mu\nu}$ is equal to the number of τ_n-honeycombs h such that $\texttt{BDRY}(h) = (\lambda, \mu, \nu)$.

Proof. This follows immediately from Corollary 6 and Theorem 4. □

3. The Proof of the Saturation Conjecture

In this section, we use the machinery of honeycombs to prove Theorem 3 via Corollary 7. We continue to follow [7], but omit some of the more technical details. The general strategy will be as follows. First, we characterize the type of possible degeneracies of honeycombs that can occur. Then, we show that each point in $\texttt{BDRY}(\tau_n)$ has a corresponding preimage in $\texttt{HONEY}(\tau_n)$ with a "nice" set of degeneracies. Finally, we show that a honeycomb with degeneracies of this type has vertices whose coordinates are integer linear combinations of the constant coordinates of the boundary, which will imply the conclusion.

We will be dealing with slightly more general tinkertoys and configurations in this section. In particular, we wish to consider configurations of sub-tinkertoys τ of the GL_n tinkertoys τ_n. We will call these *τ-honeycombs*; however, whenever the word honeycomb is used without qualification, we will still take it to mean τ_n-honeycomb.

3.1. **Degeneracies of Honeycombs.** The following lemma demonstrates that non-degenerate vertices of τ-honeycombs are in some sense generic. In particular, they cannot be vertices of the configuration cone of τ.

Lemma 1. Let h_τ be a τ-honeycomb. If τ contains an undirected loop passing through no degenerate vertices and containing each vertex at most once, then there is an open line segment contained in the configuration cone of τ that passes through h_τ and has the same image under the boundary map `BDRY` as h_τ.

Proof. This statement is very intuitive; the existence of the line segment follows from "expansion" and "contraction" of the loop. To formalize this, notice that the loop is the boundary of some (possibly concave) polygon with only 120° and 240° angles and that each vertex of the loop is a member of exactly one edge that is not itself part of the loop. Then, for any x in a sufficiently small interval $(-\varepsilon, \varepsilon)$, we may change the length of each edge adjoining a 120° angle by x and the length of each other edge by $-x$. It is easy to check that this defines a valid configuration with the same boundary conditions, since expanding an interior hexagon does not change the constant coordinates of the boundary. □

We are therefore interested in classifying degenerate honeycombs, as they form an extremal case. We say that a degeneracy is *simple* if there is only one degenerate edge at the relevant vertex. Our goal will be to show that any point in $\texttt{BDRY}(\tau_n)$ has a preimage in $\texttt{HONEY}(\tau_n)$ containing only simple degeneracies of a certain type. If a degeneracy at a vertex v of some honeycomb h is simple, then the four remaining edges adjoining v form two collinear pairs. This gives the following operation.

Definition 6. Let v be the vertex of a simple degeneracy in the configuration h of a tinkertoy τ. Then, define the operation of *eliding* the vertex v to output the pair $e_v(\tau, h) = (\tau_v, h_v)$, where τ_v is the tinkertoy with v removed and the corresponding collinear pairs of edges joined, and h_v is the configuration induced from h.

This process is illustrated in Figure 3.

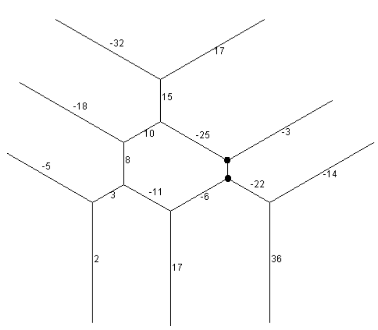

FIGURE 3. A honeycomb becoming simply degenerate, simply degenerate at vertex v, and after eliding vertex v.

Of course, there may be degeneracies outside of simple ones, and, using some rather technical arguments, we may classify all degeneracies at a vertex into five cases via the following theorem whose proof we omit for the sake of brevity.

Theorem 8. If v is a vertex in a τ_n-honeycomb, then the degeneracy of v falls into one of the following five classes. Here, the variable next to an edge denotes the multiplicity of that edge.

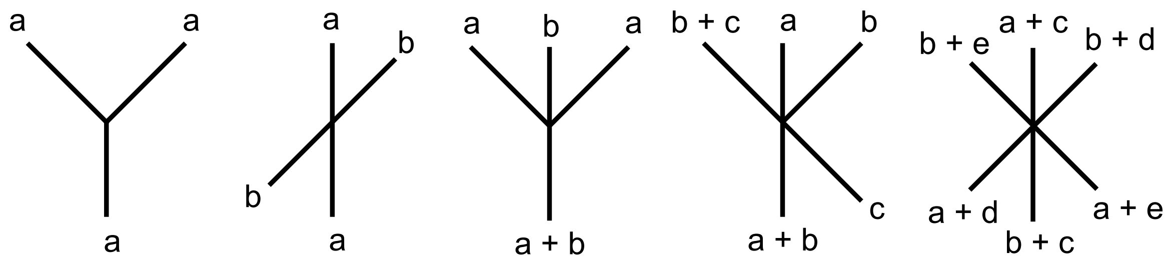

FIGURE 4. The possible types of degeneracy at a vertex.

3.2. **Largest Lifts.** By Lemma 1, a generic element (λ, μ, ν) in $\texttt{BDRY}(\tau_n)$ should have a rather large space of preimages in $\texttt{HONEY}(\tau_n)$, as for any honeycomb h with $\texttt{BDRY}(h) = (\lambda, \mu, \nu)$, we may expand or contract each simple loop containing only non-degenerate vertices. We wish to select a preimage with only simple degeneracies, which we will do by constructing a so-called "largest lift" map $\texttt{BDRY}(\tau_n) \to \texttt{HONEY}(\tau_n)$.

Notice that we have a projection map of cones $\texttt{BDRY} : \texttt{HONEY}(\tau_n) \to \texttt{BDRY}(\tau_n)$ that is induced by a linear mapping of their ambient vector spaces. We will first claim that this map is proper; that is, that the preimage of each point is compact.

Lemma 2. The map $\texttt{BDRY} : \texttt{HONEY}(\tau_n) \to \texttt{BDRY}(\tau_n)$ is proper.

Proof. Pick any $(\lambda, \mu, \nu) \in \texttt{BDRY}(\tau_n)$. Recall that Theorem 5 gave a homeomorphism $\texttt{HONEY}(\tau_n) \simeq \texttt{BZ}(n)$ that mapped $\texttt{BDRY}^{-1}(\lambda, \mu, \nu)$ to the space of quasi-BZ patterns of type (n, l, m, r) for $l_i = \lambda_i - \lambda_{i+1}$, $m_i = \mu_i - \mu_{i+1}$ and $r_i = \nu_i - \nu_{i+1}$. But for a fixed type (n, l, m, r), each entry of a quasi-BZ pattern has bounded magnitude, meaning that this forms a closed and bounded, hence compact subset of $\texttt{BZ}(n)$. Therefore, we see that $\texttt{BDRY}^{-1}(\lambda, \mu, \nu)$ is compact as needed. $\square$

Therefore, a generic linear functional ξ on the ambient vector space of $\texttt{HONEY}(\tau_n)$ attains a unique maximum on $\texttt{BDRY}^{-1}(\lambda, \mu, \nu)$ for generic (λ, μ, ν), that is, for (λ, μ, ν) whose preimage only contains points that lie in the interior of $\texttt{HONEY}(\tau_n)$ or its facets. We will now construct a special functional $\texttt{wperim} : \texttt{HONEY}(\tau_n) \to \mathbb{R}$, the *weighted perimeter* of a honeycomb h.

To do so, we first construct an auxiliary weighting function w on the regions of τ_n so that (1) $w(\alpha) = 0$ for each unbounded boundary region and (2) for any bounded region α surrounded by regions $\alpha_1, \ldots, \alpha_6$, $w(\alpha) > \frac{1}{6}\sum_{i=1}^{6} w(\alpha_i)$. It's clear that such a w always exists. For instance, we may take any strictly concave down function ϕ of x, y, z. Then, for C large enough, assigning to each interior hexagon the value of $\phi + C$ at the center of the hexagon defines a valid w by Jensen's inequality.

Then, define $\texttt{wperim} : \texttt{HONEY}(\tau_n) \to \mathbb{R}$ by

$$\texttt{wperim}(h) = \sum_{\alpha \in \tau_n} w(\alpha)\text{perimeter}(\alpha, h),$$

where the sum is over all hexagons α in τ_n and $\text{perimeter}(\alpha, h)$ denotes the perimeter of α in the honeycomb h. Since these perimeters depend linearly on the coordinates of the vertices, $\texttt{wperim}$ is valid linear functional. Notice here that a valid w could be constructed from any strictly concave down function, hence, we may choose w to make $\texttt{wperim}$ generic.

For a given choice of $\texttt{wperim}$, we may therefore define the *largest lift map* $\texttt{LIFT} : \texttt{BDRY}(\tau_n) \to \texttt{HONEY}(\tau_n)$ by

$$\texttt{LIFT}(\lambda, \mu, \nu) = \underset{h \in \texttt{BDRY}^{-1}(\lambda, \mu, \nu)}{\text{argmax}} \texttt{wperim}(h).$$

We now wish to show that the image of $\texttt{LIFT}$ contains honeycombs with simple degeneracies only. First, we have the following lemma on non-degenerate hexagons.

Lemma 3. Let h be a honeycomb and $\alpha \in h$ an interior hexagon which may be expanded. Then, h does not maximize $\texttt{wperim}(h)$.

Proof. Expand α by some length ε, increasing its perimeter by 6ε and decreasing the perimeter of each of its bounded neighbors $\alpha_1, \dots, \alpha_6$ by ϵ. Call the new honeycomb formed h'. The net change in weighted perimeter is therefore given by

$$\texttt{wperim}(h') - \texttt{wperim}(h) > 6\varepsilon \cdot w(\alpha) - \sum_{i=1}^{6} \varepsilon \cdot w(\alpha_i) > 0$$

by the definition of w. This gives the result. □

Therefore, by Lemma 1 any h in the image of $\texttt{LIFT}$ contains no loops containing only non-degenerate vertices, as we could then apply Lemma 3 to all hexagons contained in the loop simultaneously to obtain some h' with greater weighted perimeter. We now wish to apply Lemma 3 to eliminate the other types of degeneracies listed in Theorem 8. We will proceed via the following claim.

Lemma 4. For each of the types of degeneracy listed in Theorem 8, we may remove the degeneracy by simultaneously expanding some regions of τ_n (which may be boundary regions).

Proof sketch. The general strategy is to examine the hexagons in the tinkertoy in the defining configuration that degenerate into the given vertex, apply an inflation to the defining configuration, and examine the effect on the given configuration. The specific details vary, but for each of the specific cases in Theorem 8, it is clear which hexagons to inflate. For the sake of brevity, however, we will omit the details. □

Lemma 4 will allow us to better characterize the image of $\texttt{LIFT}$ in the following key result.

Theorem 9. Let $(\lambda, \mu, \nu) \in \texttt{BDRY}(\tau_n)$ be a point such that λ, μ, ν have no repeated coordinates. Then, $\texttt{LIFT}(\lambda, \mu, \nu)$ has only simple degeneracies; moreover, after eliding these degeneracies, the graph of the underlying tinkertoy contains no cycles.

Proof sketch. We first show that $h = \texttt{LIFT}(\lambda, \mu, \nu)$ contains only simple degeneracies. It is enough to check that none of the cases in Theorem 8 occur. The general strategy will be to check that the occurrence of any of these cases will allow us to apply Lemma 4 while inflating at least one interior hexagon, which will mean by Lemma 3 that h would not maximize $\texttt{wperim}(h)$. This proof again proceeds via a rather technical analysis of each case in Theorem 8, so we will omit the details of this step.

We now need to show that a τ-honeycomb $h = \texttt{LIFT}(\lambda, \mu, \nu)$ with no degeneracies has no cycles. Suppose for contradiction that there was some cycle in h. Then, by Lemma 1, we could find an open line segment in the configuration cone of τ that was contained in $\texttt{BDRY}^{-1}(\lambda, \mu, \nu)$, meaning that h is not a vertex of the polytope $\texttt{BDRY}^{-1}(\lambda, \mu, \nu)$. But this is a contradiction, since we may take our generic choice of the functional $\texttt{wperim}$ to make h a vertex of the polytope. □

3.3. **Integer Honeycombs.** Using Theorem 9, we may now characterize the coordinates of the vertices of $\texttt{LIFT}(\lambda, \mu, \nu)$. In particular, we wish to look at the image of $\texttt{LIFT}$ on integer boundary coefficients. For these, we have the following theorem.

Theorem 10. The map $\texttt{LIFT} : \texttt{BDRY}(\tau_n) \to \texttt{HONEY}(\tau_n)$ is piecewise $\mathbb{Z}$-linear. In particular, it sends integer boundary conditions to lattice honeycombs.

Proof. Since the fiber $\texttt{BDRY}^{-1}(\lambda,\mu,\nu)$ of any set of boundary coefficients (λ,μ,ν) is a polytope, it's clear that $\texttt{LIFT}$ is a priori piecewise linear. It is now enough by continuity to show that $\texttt{LIFT}$ is $\mathbb{Z}$-linear on the dense open set in $\texttt{BDRY}(\tau_n)$ of boundary conditions (λ,μ,ν) with no repeated entries. Pick such a point $(\lambda,\mu,\nu)\in\texttt{BDRY}(\tau_n)$ and notice by Theorem 9 that its largest lift $h=\texttt{LIFT}(\lambda,\mu,\nu)$ can contain no cycles or other degeneracies after eliding all simple degeneracies. Denote the resulting tinkertoy and honeycomb by τ and h_τ, respectively.

We wish to show that h is a lattice honeycomb with coordinates $\mathbb{Z}$-linear functions of (λ,μ,ν). First, it is enough to show this for h_τ, since the vertices of h are $\mathbb{Z}$-linearly dependent on those of h_τ (as they are formed by the intersections of edges in h_τ). Since τ is acyclic by Theorem 9, the graph formed by discarding all infinite edges from τ is a tree, so we may find a vertex v of degree 1 in this graph. But h_τ had no degeneracies, meaning that in τ, v must have been connected to two semi-infinite edges; the constant coordinates on these edges then uniquely determine the coordinates of v (as an evident $\mathbb{Z}$-linear combination). This then gives us the constant coordinate of the finite edge containing v, which allows us to inductively find the coordinates of all vertices of τ, completing the proof.

Notice that the fact that τ was acyclic was essential to this proof, as it allows us to relate the coordinates of interior vertices to the constant coordinates of the boundary. □

We are finally ready to give a proof of Theorem 3.

Proof of Theorem 3. If $(N\lambda,N\mu,N\nu)\in T'_n$, we have that $c_{N\lambda\,N\mu\,N\nu}\neq 0$, so by Corollary 7, there exists a honeycomb h such that $\texttt{BDRY}(h)=(N\lambda,N\mu,N\nu)$. Now, consider the honeycomb $h'=\frac{1}{N}h$. By the linearity of $\texttt{BDRY}$, we see that $\texttt{BDRY}(h')=(\lambda,\mu,\nu)$, meaning that $(\lambda,\mu,\nu)\in\texttt{BDRY}(\tau_n)$. By Theorem 10, the largest lift $\texttt{LIFT}(\lambda,\mu,\nu)$ is a lattice honeycomb, hence another application of Corollary 7 shows that $c_{\lambda\mu\nu}\neq 0$, as needed. □

4. Implications

Having now outlined the proof of Theorems 2 and 3, we have shown that the description of the semigroups T_n and T'_n by the inequalities in [6] is complete. In this section, we discuss the connections between these inequalities and the seemingly unrelated problem of understanding the spectra of sums of Hermitian matrices. We will follow mainly the exposition of [5] and [8].

Given a Hermitian matrix A, we define its *spectrum* $\alpha=(\alpha_1\geq\cdots\geq\alpha_n)$ to be a list of its eigenvalues in decreasing order. Now, recall that the sum of Hermitian matrices is hermitian, so we may consider the following question, which relates the linear condition of being Hermitian with the non-linear determination of eigenvalues.

Question. For which triples of spectra (α,β,γ) can we find Hermitian matrices A,B,C such that $A+B=C$ with spectra α,β,γ?

Computing the trace of a matrix A as the sum of its eigenvalues implies the obvious constraint

$$\sum_i \alpha_i+\sum_i \beta_i=\mathrm{Tr}(A)+\mathrm{Tr}(B)=\mathrm{Tr}(C)=\sum_i \gamma_i.$$

Several other constraints on the triples (α,β,γ) are known, all taking the form

$$\sum_{i\in I}\alpha_i+\sum_{j\in J}\beta_j\geq\sum_{k\in K}\gamma_k$$

for subsets $I,J,K\subset\{1,\ldots,n\}$ with common cardinality $|I|=|J|=|K|<n$. Denote this constraint by $(\texttt{IJK})$. The problem, then, is to find conditions on the subsets I, J, and K such that the constraints $(\texttt{IJK})$ are necessary and sufficient conditions on the spectra (α,β,γ).

This was resolved in some sense by Klyachko in [6] as follows. Associate to a subset $I\subset\{1,2,\ldots,n\}$ of cardinality p a Young diagram σ_I in the box $p\times n-p$ via the bijection between such subsets and lattice walks between the two corners of the box. Let s_I denote the Schubert cycle corresponding to σ_I in the Grassmannian $\mathrm{Gr}(p, n-p)$. We then have the following theorem.

Theorem 11. For subsets $I, J, K \subset \{1, 2, \ldots, n\}$ such that s_K is a component of $s_I \cdot s_J$, we have condition (`IJK`) . Together with the restriction on trace, the conditions (`IJK`) for these values of I, J, K form a complete answer to the question.

Since we have the decomposition $s_I \cdot s_J = \sum_K c_{\sigma_I \sigma_J}^{\sigma_K} s_K$ of Schubert cycles, already we see that the Littlewood-Richardson coefficients seem to be related to this problem; notice, however, that they parametrize the constraints here. Using the techniques of intersection theory, Klyachko is also able to deduce the following remarkable theorem directly addressing the structure of the Littlewood-Richardson coefficients.

Theorem 12. Let α, β, γ be integer spectra. Then, the following conditions are equivalent.

(1) There exists a positive integer N such that $c_{N\alpha\, N\beta}^{N\gamma} \neq 0$.
(2) The trace condition and condition (`IJK`) hold for (α, β, γ) for all K so that s_K is in the decomposition of $s_I \cdot s_J$.

Notice that by the Saturation Conjecture, condition (1) in Theorem 12 can be changed to simply read $c_{\alpha\beta}^{\gamma} \neq 0$. We conclude with the following interpretation of Theorem 12. Notice that determining whether s_K is in the Schubert cycle decomposition of $s_I \cdot s_J$ requires knowledge of a Littlewood-Richardson coefficient of order smaller than the orders of α, β, and γ. Using the implication (2) implies (1) in Theorem 12, then, we may determine the non-zero values of larger Littlewood-Richardson coefficients, giving in theory a recursive algorithm for determining the semigroup T_n. In passing, Theorem 11 also provides a complete answer to our question.

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