Problem Set 7 Solutions

Note: Thanks to Kevin Lee for some of the solutions.

1. (a) Integrating the density function over the domain and transforming to polar coordinates, we find that

$$\int_A f(x,y)dxdy = C\int_0^{2\pi}\int_0^1 \left(1-(r\cos\theta)^2-(r\sin\theta)^2\right) rdrd\theta = 2C\pi\int_0^1 \left(1-r^2\right) rdr = 2C\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}C = 1.$$

Thus we find $C = 2/\pi$.
(b) Note that R represents the distance of the dart from the bullseye. We can use the same integration formula as before, but integrating r only until the desired radius to get the distribution function. This gives us $F_R(x) = \frac{2}{\pi}2\pi\int_0^x r - r^3\, dr = 4\left(\frac{x^2}{2}-\frac{x^4}{4}\right) = 2x^2 - x^4$. The density function is $f_R(x) = F_R'(x) = 4x - 4x^3$. (Note that both these values are only for the range $[0,1]$; outside this range they are 0.)

2. First solution: We have $\mathbb{E}\left(X^4\right) = \int_{-\infty}^{\infty} x^4 f_X(x)\,dx = \int_{-\infty}^{\infty}\frac{x^4}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}dx$. If we let $G(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{-k\frac{x^2}{2}}dx$, then note that

$$G(x)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(xy)^4 e^{-k\frac{x^2+y^2}{2}}dxdy = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} r^8 2^{-4}\sin^4(2\theta)\,e^{-k\frac{r^2}{2}}r\,drd\theta$$
$$= \frac{1}{2^5\pi}\left(\int_0^{2\pi}\sin^4(2\theta)\,d\theta\right)\left(\int_0^{\infty} r^9 e^{-k\frac{r^2}{2}}dr\right) = \frac{3}{2^7}\int_0^{\infty} r^9 e^{-k\frac{r^2}{2}}dr = \frac{3}{2^7}\frac{384}{k^5} = \frac{9}{k^5},$$

where the last integral is by repeated integration by parts. Thus, we find $\mathbb{E}\left(X^4\right) = \frac{1}{\sigma}G\left(\frac{1}{\sigma^2}\right) = 3\sigma^4$.
Second solution: Recall from class that $G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k\frac{x^2}{2}}dx = k^{-1/2}$. Then, we may compute

$$G''(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{-x^2}{2}\right)^2 e^{-k\frac{x^2}{2}}dx = \frac{3}{4}k^{-5/2},$$

which means that $\mathbb{E}(X^4) = \frac{1}{\sigma}(4G''(1/\sigma^2)) = 3\sigma^4$.

3. First, we calculate the variance. Notice that

$$\mathbb{E}\left(X^2\right) = \int_{-\infty}^{\infty}\frac{x^2}{2}\lambda e^{-\lambda|x|}dx = \int_0^{\infty} x^2\lambda e^{-\lambda x}dx = \int_0^{\infty}\frac{u^2}{\lambda^2}\lambda e^{-u}\frac{1}{\lambda}du = \frac{\Gamma(3)}{\lambda^2} = \frac{2}{\lambda^2},$$

where we've made the substitution $u = \lambda x$. Thus, the variance is $\mathbb{E}\left(X^2\right) - \mathbb{E}(X)^2 = \frac{2}{\lambda^2}$ and $\sigma^2 = \frac{2}{\lambda^2} \implies \lambda = \sqrt{2}\sigma^{-1}$. We can then calculate

$$\mathbb{E}\left(X^4\right) = \int_{-\infty}^{\infty}\frac{x^4}{2}\lambda e^{-\lambda|x|}dx = \int_0^{\infty} x^4\lambda e^{-\lambda x}dx = \frac{\Gamma(5)}{\lambda^4} = \frac{24}{\lambda^4}$$

using the same trick. Substituting our value for λ gives us $6\sigma^4$.

4. We instead compute the probability that all of the angles are less than $x\pi$. This means that we can never travel more than x along the circle (which we will assume has circumference 1) before we hit the next point. Picking an arbitrary starting point (at location 0), the next point P_1 must be within x after and the previous point P_2 must be within x before. Thus, if we have a square of the possible positions of P_1 and P_2, we get an $x \times x$ square. However, we also have the additional constraints that P_2 comes after P_1 and that they are within x of each other. This is represented by two diagonal lines with intercepts 0 and x respectively. However since P_1 and P_2 are symmetric, we can mirror the entire valid region along the diagonal. Thus, we simply need to find the area between the diagonals with intercepts $\pm x$ in the two $x \times x$ squares. Clearly if $x > \frac{1}{2}$, then the squares actually overlap. Thus, the total area is actually everything except the $(1-x)\times(1-x)$ squares in the opposite corners and the $(1-x)\times(1-x)$ square partitioned off by the diagonals. Otherwise, if $x < \frac{1}{3}$ then the area is 0, and if $\frac{1}{3} \le x \le \frac{1}{2}$ then the two triangles compose a $(3x-1)\times(3x-1)$ square. Since this is the complement probability, subtracting from 1 gives us the desired function $b(x) = \begin{cases} 1-(3x-1)^2 & \frac{1}{3} \le x \le \frac{1}{2} \\ 3(1-x)^2 & \frac{1}{2} < x \le 1 \end{cases}$.

To compute the density, we differentiate $1 - b(x)$ to get $f_X(x) = \begin{cases} 18x - 6 & \frac{1}{3} \le x \le \frac{1}{2} \\ 6 - 6x & \frac{1}{2} < x \le 1 \end{cases}$. The expected value is given by the formula $\int_{1/3}^{1/2} 18x^2 - 6x\, dx + \int_{1/2}^{1} 6x - 6x^2\, dx = \frac{1}{9} + \frac{1}{2} = \frac{11}{18}$. Of course, each of these values is for our adjusted degrees. In terms of actual radians, the density function is actually $f_X(x) = \begin{cases} 18\frac{x}{\pi} - 6 & \frac{\pi}{3} \le x \le \frac{\pi}{2} \\ 6 - 6\frac{x}{\pi} & \frac{\pi}{2} \le x \le \pi \end{cases}$ and the expected value is $\frac{11\pi}{18}$.

5. **(a)** Recall that $f_{Y|X}(y,x) = \frac{f(x,y)}{f_X(x)}$. The marginal distribution function is then

$$f_X(x) = \int_x^\infty f(x,y)\,dy = \int_x^\infty \lambda^2 e^{-\lambda y} dy = \lambda e^{-\lambda x}.$$

Thus, $f_{Y|X}(y,x) = \lambda e^{-\lambda(y-x)}$. The expectation is then

$$\int_x^\infty y\lambda e^{-\lambda(y-x)} dy = \int_0^\infty (y+x)\,\lambda e^{-\lambda y} dy = \frac{1}{\lambda} + x$$

because the latter part is just the PDF of an exponential distribution.
(b) Now, we have

$$f_X = \int_0^\infty x e^{-x(y+1)} dy = -e^{-x(y+1)}|_0^\infty = e^{-x}.$$

Thus, $f_{Y|X}(y,x) = xe^{-xy}$. The expectation is

$$\int_0^\infty yxe^{-xy} dy = -ye^{-xy}|_0^\infty + \int_0^\infty e^{-xy} dy = \frac{1}{x}$$

via integration by parts.

6. **(a)** Since X and Y are uniformly distributed, we see that $f(x,y) = C$ for some constant C. Integrating, we find that $\int_A f(x,y)dxdy = \int_0^1 \int_0^y C dydx = C/2 = 1$. So we see that $f(x,y) = C = 2$.
(b) Note that $\mathbb{P}(X > 1/2 \cap Y > 1/2) = 0$, but $\mathbb{P}(X > 1/2), \mathbb{P}(Y > 1/2) \neq 0$.
(c) As noted in the course notes, the inverse transformation is $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. This gives a Jacobian of $\frac{1}{2}$. Thus, the density function is $\frac{1}{2}\int_{-\infty}^\infty f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) dv = \frac{1}{2}\int_{-u}^u 2\,dv = 2u$. Now, notice that $\mathbb{P}(U < .1), \mathbb{P}(V > .1) > 0$, but $\mathbb{P}(U < .1 \cap V > .1) = 0$, so U and V are not independent.
(d) For a given v, u must be in the range $(|v|, 1)$. Thus, we have that $f_{U|V}(u,v) = \frac{1}{1-|v|}$ on the range $(|v|, 1)$ and 0 elsewhere.
(e) The inverse transformation is $x = \frac{u}{v+1}$ and $y = \frac{uv}{v+1}$. This gives us a Jacobian of $\frac{u}{(v+1)^2}$. We can simply integrate $\int_{-\infty}^\infty \frac{u}{(v+1)^2} f\left(\frac{u}{v+1}, \frac{uv}{v+1}\right) dv = \int_0^\infty \frac{2u}{(v+1)^2} dv = 2u\int_1^\infty \frac{1}{v^2} = 2u$. For independence, we may compute $F_{U,V}(u,v) = \int_0^v \int_0^u f_{u,v}(u,v)dudv = \int_0^v \frac{dv}{(v+1)^2}u^2 = \frac{u^2 v}{v+1}$, $F_U(u) = \int_0^u \int_0^\infty f_{U,V}(u,v)dvdu = \int_0^u 2udu = u^2$, and $F_V(v) = \int_0^v \int_0^1 f_{U,V}(u,v)dudv = \int_0^v \frac{dv}{(v+1)^2} = \frac{v}{v+1}$, so we see that $F_{U,V}(u,v) = F_U(u)F_V(v)$.