Problem Set 9 Solutions

Note: Thanks to Kevin Lee for some of the solutions.

1. (a) For the "random endpoints" method, we can fix the first endpoint. If the second endpoint is θ radians away from the first, then consider the triangle formed by the chord and the center of the circle. If the point is chosen randomly along the chord, its expected distance squared is given by $\frac{1}{2b}\int_{-b}^{b} x^2 + r^2 dx = r^2 + \frac{1}{3}b^2$, where b is half the length of the chord and r is the height of the triangle (distance of the chord to the center). By trigonometry, we get that $b = \sin\frac{\theta}{2}$ and $h = \left|\cos\frac{\theta}{2}\right|$. Integrating over the values of θ, the expected value is

$$E = \frac{1}{\pi}2\int_0^{\pi/2} \cos^2\theta + \frac{1}{3}\sin^2\theta \, d\theta = \frac{2}{\pi}\left(\frac{\pi}{4} + \frac{1}{3}\frac{\pi}{4}\right) = \boxed{\frac{2}{3}}.$$

(b) For the "random radius" method, we can consider the same triangle as before. However, this time if we pick a specific radius r, then the base works out to be $\sqrt{1-r^2}$. Thus the expectation is given by

$$E = \int_0^1 r^2 + \frac{1}{3}\left(\sqrt{1-r^2}\right)^2 dr = \int_0^1 \frac{2}{3}r^2 + \frac{1}{3}dr = \frac{2}{9} + \frac{1}{3} = \boxed{\frac{5}{9}}.$$

(c) For the "random midpoint" method, we can use the exact same method as before, except now r is not uniformly distributed. Instead, it is distributed with density function $f_R(r) = 2r$. Thus, we now have

$$E = \int_0^1 \frac{2}{3}r\left(2r^2+1\right)dr = \frac{2}{3}\left(\frac{1}{2} + \frac{1}{2}\right) = \boxed{\frac{2}{3}}.$$

2. Let the chosen points have coordinates (X_1, Y_1) and (X_2, Y_2). Notice that X_1, Y_1, X_2, Y_2 are all independent random variables and that the area is $A = |X_1 - Y_1| \cdot |X_2 - Y_2|$. Hence, we find by independence that

$$\mathbb{E}[A] = \mathbb{E}[|X_1 - Y_1|] \cdot \mathbb{E}[|X_2 - Y_2|] = (1/3)^2 = 1/9,$$

where we know that $\mathbb{E}[|X_1 - Y_1|] = \mathbb{E}[|X_2 - Y_2|] = 1/3$ by Point 2 on Lecture 19.

3. Let D be the value of a single digit and S be their sum.
(a) By the definitions of generating functions, we find

$$G_D(s) = \frac{1}{10}\left(1 + s + s^2 + s^3 + \cdots + s^9\right) \text{ and } G_S(s) = G_D(s)^4 = \frac{1}{10^4}\left(1 + s + s^2 + s^3 + \cdots + s^9\right)^4.$$

(b) We wish to find the s^{25} term in the expansion of $G_S(s)$. We may first write

$$G_S(s) = \frac{1}{10^4}\frac{\left(1-s^{10}\right)^4}{(1-s)^4} = \frac{1}{10^4}\left(1 - \binom{4}{1}s^{10} + \binom{4}{2}s^{20} + \cdots\right)\left(1 + \binom{4}{3}s + \binom{5}{3}s^2 + \cdots\right)$$

by the negative binomial expansion. Picking out the pairs of terms in the product that will multiply to a term of s^{25}, we find the final coefficient to be $\frac{1}{10^4}\left(\binom{4}{2}\binom{8}{3} - \binom{4}{1}\binom{18}{3} + \binom{28}{3}\right) = \boxed{\frac{348}{10000}}$.

4. (a) Applying the definitions of generating functions, we compute

$$G_A(s) = \left(\frac{1}{6}\left(1 + s + s^2 + s^3 + s^4 + s^5\right)\right)^3 \text{ and } G_B(s) = \left(\frac{1}{4}\left(1 + s + s^2 + s^3\right)\right)^5.$$

(b) We again apply the negative binomial expansion. Notice that

$$G_A(s) = \frac{1}{6^3}\left(1 - \binom{3}{1}s^6 + \cdots\right)\left(1 + \binom{3}{2}s + \binom{4}{2}s^2 + \cdots\right),$$

so the coefficient of s^6 is $\frac{1}{6^3}\left(\binom{8}{2} - \binom{3}{1}\right) = \boxed{\frac{25}{216}}$. Similarly, we have that

$$G_B(s) = \frac{1}{4^5}\left(1 - \binom{5}{1}s^4 + \cdots\right)\left(1 + \binom{5}{4}s + \binom{6}{4}s^2 + \cdots\right),$$

so the coefficient of s^6 is $\frac{1}{4^5}\left(\binom{10}{4} - \binom{5}{1}\binom{6}{4}\right) = \boxed{\frac{135}{1024}}$. Thus, a score of 6 is more probable on Test B.

5. We first find the generating functions of T_1, T_2, T_3 via their mass functions, obtaining $G_{T_1}(s) = s$ and

$$G_{T_2}(s) = \frac{2}{3}s + \frac{2}{9}s^2 + \frac{2}{27}s^3 + \cdots = \sum_{k=1}^{\infty}\frac{2}{3^k}s^k = \frac{\frac{2}{3}s}{1 - \frac{1}{3}s} \text{ and } G_{T_3}(s) = \frac{1}{3}s + \frac{2}{9}s^2 + \frac{4}{27}s^3 + \cdots = \sum_{k=1}^{\infty}\frac{2^{k-1}}{3^k}s^k = \frac{\frac{1}{3}s}{1 - \frac{2}{3}s}$$

Since $T = T_1 + T_2 + T_3$, recall that we have $G_T(s) = G_{T_1}(s)\,G_{T_2}(s)\,G_{T_3}(s)$, so we have the partial fractions

$$G_T(s) = \frac{2}{9}s^3\frac{1}{\left(1 - \frac{1}{3}s\right)\left(1 - \frac{2}{3}s\right)} = \frac{2}{9}s^3\left(\frac{2}{1 - \frac{2}{3}s} - \frac{1}{1 - \frac{1}{3}s}\right).$$

Expanding into two geometric series and taking the coefficient gives us the probability mass function of T to be

$$f_T(n) = \boxed{\frac{2}{9}\left(2\left(\frac{2}{3}\right)^{n-3} - \left(\frac{1}{3}\right)^{n-3}\right) \text{ for } n \geq 3}.$$

6. (a) Applying the definition of generating functions, we see that

$$G_X(s) = \frac{1}{4}\left(1 + s + s^2 + s^3\right) \text{ and } G_Y(s) = G_X(s)^3 = \frac{1}{4^3}\left(1 + s + s^2 + s^3\right)^3.$$

(b) Suppose we have $G_U(s) = \sum p_i s^i = p_0 + p_1 s + p_2 s^2 + \cdots$ for some random variable U. Then note that $G_U'(s) = \sum p_i i s^i = p_1 + 2p_2 s + 3p_3 s^2 + \cdots$. Setting $s = 1$, this means that $G_U'(1) = \sum ip_i = p_1 + 2p_2 + 3p_3 + \cdots = \mathbb{E}[U]$. Taking $U = Y$, we may then compute

$$G_Y'(s) = \frac{3}{4^3}\left(1 + 2s + 3s^2\right)\left(1 + s + s^2 + s^3\right)^2 \implies \mathbb{E}[Y] = G_Y'(1) = \boxed{\frac{9}{2}}.$$

(c) We apply the negative binomial expansion, obtaining

$$G_Y(s) = \frac{1}{4^3}\frac{\left(1 - s^4\right)^3}{(1 - s)^3} = \frac{1}{4^3}\left(1 - \binom{3}{1}s^4 + \cdots\right)\left(1 + \binom{3}{2}s + \binom{4}{2}s^2 + \cdots\right),$$

so the coefficient of s^4 is $\frac{1}{4^3}\left(\binom{6}{2} - \binom{3}{1}\right) = \frac{12}{64} = \boxed{\frac{3}{16}}$.

(d) Note that $\left(1 + s + s^2 + s^3\right) = (1 + s)\left(1 + s^2\right)$, meaning that we have the alternate factorization $G_Y(s) = \frac{1}{4^3}\left(1 + s + s^2 + s^3\right)^3 = \frac{1}{4^3}(1 + s)^3\left(1 + s^2\right)^3$. Letting each of these groups correspond to one of the octahedral die, we find that two die, X_1 with numbers $\{0, 1, 1, 1, 2, 2, 2, 3\}$ and X_2 with numbers $\{0, 2, 2, 2, 4, 4, 4, 6\}$, satisfy the desired property. We may check that

$$\mathbb{P}(Y = 4) = \mathbb{P}(X_2 = 4)\,\mathbb{P}(X_1 = 0) + \mathbb{P}(X_2 = 2)\,\mathbb{P}(X_1 = 2) = \frac{3}{8}\frac{1}{8} + \frac{3}{8}\frac{3}{8} = \frac{12}{64} = \boxed{\frac{3}{16}},$$

which is the same as before.