Problem Set 6 Solutions

Note: Thanks to Kevin Lee for some of the solutions.

1. For a walk that reaches level $2r$ after $2n$ steps and that reaches $2r$ for the first time after $2k$ steps, divide the walk into the the portion from steps 0 to $2k$ and the portion from steps $2k$ to $2n$. The latter portion must be a return to 0 walk, which happens with probability u_{2n-2k}. The former portion is a positive walk of $2k$ steps that has been inverted; these are in bijection with return to zero walks of length $2k$, hence this event occurs with probability u_{2k}. Because the two separate portions of the walk occur independently, we can multiply these probabilities to obtain that $\mathbb{P}(T_{2n}^{2r} = 2k) = u_{2k}u_{2n-2k}$, as desired.

2. Given a walk that returns to 0 exactly once, split it into the portion that reaches 0 and the portion thereafter, which is a non-zero walk. Using the bijection described in class, map the ending non-zero walk to a walk that returns to zero and preserve the initial portion of the walk to produce a walk that that returns to zero. Similarly, for a walk that returns to 0 exactly once, keep the first portion the same, and map the remainder of the walk back using the inverse of the bijection given in class. This gives a bijection between the desired sets, hence they have the same cardinality.

3. As established in lecture, when $p = \frac{1}{2}$, the probability that when you start with k you will go broke is $p_k = 1 - \frac{k}{N}$. We wish to find $1 - p_k = \frac{k}{N}$. Now, since k and N will scale similarly no matter how much the tickets cost, we find that $p_k = \frac{2}{3}$ regardless of the price of the ticket.

On the other hand, if $p = \frac{1}{3}$, we find that $p_k = \frac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^N - 1}$. Note that $\frac{q}{p} = 2$. If tickets cost \$1, then $k = 8$ and $N = 12$ implies that $1 - p_k = \frac{17}{273}$. If tickets are \$2, then $1 - p_k = \frac{5}{21}$. If tickets are \$4, then $1 - p_k = \frac{3}{7}$.

4. Note that either Hillary or Obama obtained the last vote. There is a probability of $\frac{n}{m+n}$ that Obama received the last vote and a probability of $\frac{m}{m+n}$ that Hillary received the last vote. We thus find the recurrence relation

$$p(m,n) = \frac{n}{m+n} p(m, n-1) + \frac{m}{m+n} p(m-1, n).$$

Now, we claim that $p(m,n) = \frac{m-n}{m+n}$ satisfies this recurrence. Plugging in, we get that

$$\frac{n}{m+n}\left(\frac{m-n+1}{m+n-1}\right) + \frac{m}{m+n}\left(\frac{m-n-1}{m+n-1}\right) = \frac{m^2 - n^2 + n - m}{(m+n)(m+n-1)} = \frac{(m+n-1)(m-n)}{(m+n)(m+n-1)} = \frac{m-n}{m+n} = p(m,n)$$

This is exactly what the ballot theorem gives us, since Hillary is $m - n$ ahead of Obama with $m + n$ votes, and we never want $m - n$ to reach 0.

5. The balance after the first day must be \$1 afterwards, if the balance drops to $-\$1$, then the reflection principle tells us that the number of paths that do that is the same as the number of paths that go from \$1 to $-\$3$ within $2n - 2$ days. There are $\binom{2n-2}{n+1}$ ways of this happening. This is compared to the number of paths that go between the two end points, which is $\binom{2n-2}{n-1}$. Thus, the probability of this happening is

$$\begin{aligned}\frac{1}{2^{2n}}\left(\binom{2n-2}{n-1} - \binom{2n-2}{n+1}\right) &= \frac{1}{2^{2n}}\left(\frac{(2n-2)!}{(n-1)!(n-1)!} - \frac{(2n-2)!}{(n+1)!(n-3)!}\right) \\ &= \frac{1}{2^{2n}}\left(\frac{(n(n+1) - (n-2)(n-1))(2n-2)!}{(n+1)!(n-1)!}\right) = \left(\frac{1}{2^{2n}}\right)\frac{(4n-2)(2n-2)!}{(n+1)!(n-1)!} \\ &= \left(\frac{1}{2^{2n}}\right)\frac{2n(2n-1)(2n-2)!}{(n+1)!(n-1)!n} = \left(\frac{1}{2^{2n}}\right)\frac{2n!}{(n+1)(n!)^2}.\end{aligned}$$

6. (a) There are $N_{2n}(0,0)$ paths that never return to the origin. Further, each of these walks ends on an even coordinate, since there an even number $(2n)$ of steps. Therefore, whatever the next step is, it is impossible to reach the origin because there are an odd number of steps. Therefore, each valid path of the first $2n$ steps has 2 choices for the last step, for a total of $2N_{2n}(0,0)$ paths as desired. For a three vote election, the four possibilities that don't result in a tie are

$$HHH, HHB, BBH, BBB.$$

Similarly, for a five vote election the twelve possibilities are

$$HHHHH, HHHHB, HHHBH, HHHBB, HHBHH, HHBHB, BBHBH,$$
$$BBHBB, BBBHH, BBBHB, BBBBH, BBBBB.$$

(b) For each non-negative path of length $2n+1$, we can add two steps in either direction to the end of the path. Neither of these paths become negative, since our original path was non-negative at an odd coordinate. Therefore, the total number of non-negative paths of length $2n+1$ is half the number of length $2n+2$. Therefore, we see that the number of non-negative paths of length $2n+1$ is $\frac{1}{2}N_{2n+2}(0,0)$. For a three vote election, the three possibilities are

$$HHH, HHB, HBH.$$

For a five vote election, the ten possibilities are

$$HHHHH, HHHHB, HHHBH, HHHBB, HHBHH, HHBHB, HHBBH, HBHHH, HBHHB, HBHBH.$$