Laguerre and Jacobi analogues of the Warren process

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I. Dyson Brownian motion and Warren process

II. Laguerre and Jacobi Warren processes

III. Existence of solutions to SDER's

IV. Intertwining diffusions à la Pal-Shkolnikov

Dyson Brownian motion

Let $X_n(t)$ be a standard Brownian motion in $n \times n$ Hermitian matrices with eigenvalues

$$\lambda^n(t) = (\lambda_1^n(t) \leq \cdots \leq \lambda_n^n(t)).$$

Theorem (Dyson 1962)

The $\lambda^n(t)$ are Markovian and solve the SDE

$$d\lambda_i^n(t) = dB_i^n(t) + \sum_{j \neq i} \frac{1}{\lambda_i^n(t) - \lambda_j^n(t)} dt, \qquad 1 \leq i \leq n,$$

where $B_i^n(t)$ are independent standard real Brownian motions.

- Called Dyson Brownian motion
- Indep. BM's made non-intersecting via Doob h-transform

GUE corners process

Definition

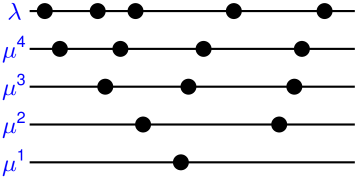
The **GUE corners** distribution is the joint distribution of eigenvalues

$$\lambda^k(t) = (\lambda_1^k(t) \leq \cdots \leq \lambda_k^k(t))$$

of $k \times k$ principal submatrices of $X_n(t)$ for $1 \leq k \leq n$.

- If $\mu^n = \lambda$, then $\{\mu_i^k\}$ lies in the Gelfand-Tsetlin polytope

$$\mathbb{GT}(\lambda) := \{\mu_i^k \mid \mu_i^k \leq \mu_i^{k-1} \leq \mu_{i+1}^k, \mu^n = \lambda\}.$$



- Distribution of $\{\lambda^k(t)\}$ is **Gibbs**, i.e. uniform on $\mathbb{GT}(\lambda^n(t))$. Note: $\mathbb{GT}(\lambda)$ has volume $\frac{\Delta(\lambda)}{(n-1)!\cdots 1!}$.

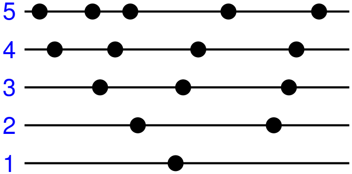
Warren's process (I)

Consider the SDE's with reflection

$$d\mu_i^k(t) = dB_i^k(t) + \frac{1}{2}dL_i^{k,+}(t) - \frac{1}{2}dL_i^{k,-}(t), 1 \le i \le k, 1 \le k \le n$$

in the domain $\mathbb{GT}_n := \{\mu_i^k \mid \mu_{i-1}^{k-1} \le \mu_i^k \le \mu_i^{k-1}\}$, where

- $B_i^k(t)$ are standard real Brownian motions;
- $L_i^{k,-}(t)$ is local time of $\mu_i^k(t) - \mu_i^{k-1}(t)$ at 0;
- $L_i^{k,+}(t)$ is local time of $\mu_i^k(t) - \mu_{i-1}^{k-1}(t)$ at 0.



"BM's at level k interlace with and reflect off BM's at level $k-1$."

Warren's process (II)

Theorem (Warren 2007)

(a) There is a unique weak solution $\{\mu_i^k(t)\}$ to

$$d\mu_i^k(t) = dB_i^k(t) + \frac{1}{2} dL_i^{k,+}(t) - \frac{1}{2} dL_i^{k,-}(t)$$

when started at 0 with entrance law

$$(2\pi)^{-n} t^{-n^2/2} \Delta(\mu^n) \prod_{i=1}^{n} e^{-(\mu_i^n)^2/2t} \prod_{k=1}^{n} \prod_{i=1}^{k} d\mu_i^k.$$

Follows by transforming triangular array of Brownian motions via a **deterministic** Skorokhod map.

Warren's process (III)

Theorem (Warren 2007)

(b) The projection of $\{\mu_i^k(t)\}$ to level k is Markovian and has the law of Dyson Brownian motion with entrance law

$$(2\pi)^{-k/2} t^{-k^2/2} \Delta(\mu^k)^2 \prod_{i=1}^{k} e^{-(\mu_i^k)^2/2t} d\mu_i^k.$$

(c) The fixed time distribution of $\{\mu_i^k(t)\}$ is the GUE corners distribution.

Note: **Is not** joint evolution of eigenvalues of principal submatrices of $X_n(t)$. (Adler-Nordenstam-van Moerbeke 2014)

Warren's process (IV)

Several proofs:

- Warren 2007: Explicit computation of semigroups
- Pal-Shkolnikov 2013: Stochastic process approach using intertwining diffusions

Generalizations:

- Brownian particles: [FF], [MOW], [WW]
- General β: Gorin-Shkolnikov

This talk: generalize to processes coming from Laguerre and Jacobi random matrix ensembles:

- Existence is more complicated.
- For projection and Gibbs properties, follow [PS].

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Laguerre eigenvalue process

Dynamic model for Wishart ensemble:

- Let $X_n(t)$ be $n \times p$ matrix of complex Brownian motions.
- Define $\Sigma_n(t) = X_n(t)^* X_n(t)$ with non-zero eigenvalues

$$0 \leq l_1^n(t) \leq \cdots \leq l_n^n(t).$$

Theorem (Konig-O'Connell 2001)

The process $\{l_i^n(t)\}$ is Markov and solves

$$dl_i^n = 2\sqrt{l_i^n}\,dB_i + 2(p-n+1)dt + \sum_{j \neq i} \frac{4 l_i^n}{l_i^n - l_j^n} dt.$$

Corresponds to n indep. dim. $2(p-n+1)$ squared Bessel processes conditioned never to intersect via Doob h-transform.

Laguerre corners process

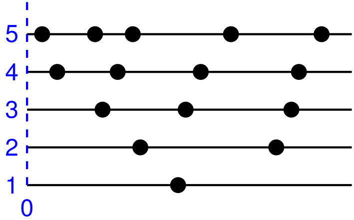
Static multilevel Wishart model:

- Consider $\Sigma_n = X_n^* X_n$ with non-zero eigenvalues

$$0 \leq l_1^n \leq \cdots \leq l_n^n$$

for changing n.

Eigenvalues of Σ_n and Σ_{n-1} **interlace** ($l_{i-1}^n \leq l_{i-1}^{n-1} \leq l_i^n$):



Eigenvalues have Gibbs measure

$$\mathbb{P}(\{l^k\} \in B \mid l^p = \lambda) \propto \frac{\mathrm{vol}(B)}{\Delta(\lambda)}.$$

Laguerre Warren process (I)

Consider the SDE with (oblique) reflection

$$dl_i^k(t) = 2\sqrt{l_i^k(t)}dB_i^k(t) + 2(p-k+1)dt + dL_i^{k,+}(t) - dL_i^{k,-}(t)$$

in the domain

$$\mathbb{GT}_{p,p} := \{l_i^n \mid 0 \leq l_{i-1}^{n-1} \leq l_i^n \leq l_i^{n-1}\}_{1\leq i\leq n, 1\leq n\leq p},$$

where

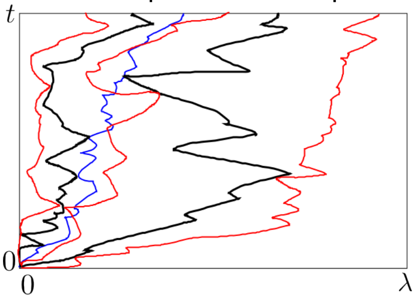
- $2\sqrt{l_i^k(t)}dB_i^k(t) + d\,dt$ – squared Bessel process of dim. d;
- $L_i^{k,+}$: local time at zero of $l_i^k - l_{i-1}^{k-1}$;
- $L_i^{k,-}$: local time at zero of $l_i^{k-1} - l_i^k$.

Laguerre Warren process (II)

Laguerre Warren SDER:

$$dl_i^k(t) = 2\sqrt{l_i^k(t)}dB_i^k(t) + 2(p-k+1)dt + dL_i^{k,+}(t) - dL_i^{k,-}(t)$$

Intuition: squared Bessel processes reflecting off each other



- L1 (blue) evolves as free squared Bessel process
- L2 (black) reflects off L1
- L3 (red) reflects off L2
- ...

Note: Dimension of squared Bessel process is level-dependent.

Laguerre Warren process (III)

Theorem (S.)

For any Gibbs initial condition, the Laguerre Warren SDER admits a unique strong solution with:

1. Its projection to level n is Markov and coincides in law with the Laguerre eigenvalues process of rank p and level n.
2. Its fixed time distribution at any $t > 0$ is Gibbs.
3. It may be started from $l_i^n(0) = 0$ with entrance law

$$\Delta(l^p) \prod_{i=1}^{p} e^{-\frac{l_i^p}{2t}} \prod_{n=1}^{p} \prod_{i=1}^{n} dl_i^n .$$

- Generalization of Warren's process with squared Bessel
- Fixed time distribution is Laguerre corners process

Left edge of the Laguerre Warren process

Consider the projection to the left edge:

$$l_1^1(t) \geq l_1^2(t) \geq \cdots \geq l_1^p(t) \geq 0.$$

Evolution is Markovian and satisfies

$$dl_1^k(t) = 2\sqrt{l_1^k(t)}dB_1^k(t) + 2(p-k+1)dt - dL_i^{k,-}(t),$$

where $L_i^{k,-}$ is local time at zero of $l_1^{k-1}(t) - l_1^k(t)$.

Smallest eigenvalue of $p \times p$ sample covariance matrix $\stackrel{d}{=} \lambda_1^p(t)$

- Particle system has **local interactions**, but
- Produces hard edge of RMT!

Jacobi eigenvalues process

Fix parameters (p, q) and $n \leq p, q$, and let $N = p + q$. Let

$$0 \leq \mu_1^n(t) \leq \cdots \leq \mu_n^n(t) \leq 1$$

be the singular values of the top left $n \times p$ submatrix of a Brownian motion on the space of unitary $N \times N$ matrices.

Theorem (Doumerc 2005)

The singular values $\{\mu_i^n(t)\}$ solve

$$\begin{aligned} d\mu_i^n(t) = {} & 2\sqrt{\mu_i^n(t)(1-\mu_i^n(t))}dB_i^n(t) + 2(p-n+1)dt \\ & + 2(p+q-2n+2)\mu_i^n(t)dt + \sum_{j\neq i} \frac{4\mu_i^n(1-\mu_i^n(t))}{\mu_i^n(t)-\mu_j^n(t)}dt \end{aligned}$$

and have invariant measure proportional to

$$\Delta(\mu^n)^2 \prod_{i=1}^{n} (\mu_i^n)^{p-n}(1-\mu_i^n)^{q-n} d\mu_i^n.$$

Jacobi Warren process (I)

Consider the SDE with (oblique) reflection

$$dj_i^n(t) = 2\sqrt{j_i^n(t)(1 - j_i^n(t))}dB_i^n(t) + 2(p - n + 1)dt$$
$$+ 2(p + q - 2n + 2)j_i^n(t)dt + \frac{1}{2}dL_i^{n,+}(t) - \frac{1}{2}dL_i^{n,-}(t),$$

where

- Univariate Jacobi process corresponds to

$$2\sqrt{j_i^n(t)(1 - j_i^n(t))}dB_i^n(t) + \Big(a + (a + b)j_i^n(t)\Big)dt.$$

- $L_i^{n,+}(t)$ is local time at 0 of $j_i^n(t) - j_{i-1}^{n-1}(t)$;
- $L_i^{n,-}(t)$ is local time at 0 of $j_i^{n-1}(t) - j_i^n(t)$.

Jacobi Warren process (II)

Theorem (S.)

For any Gibbs initial condition, the Jacobi Warren SDER admits a unique strong solution with:

1. Its projection to level n is Markov and has the law of Jacobi eigenvalues process with parameters (p, q) and level n.
2. Its fixed time distribution at any $t > 0$ is Gibbs.
3. It may be started with invariant measure $(m = \min\{p, q\})$:

$$\Delta(j^m) \prod_{i=1}^{m} (j_i^m)^{p-m} (1 - j_i^m)^{q-m} \prod_{n=1}^{m} \prod_{i=1}^{n} dj_i^n.$$

- Fixed time distribution $=$ Jacobi corners ensemble of RMT.

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Characteristics of Laguerre / Jacobi Warren SDER's

Laguerre Warren SDER:

$$dl_i^k(t) = 2\sqrt{l_i^k(t)}dB_i^k(t) + 2(p-k+1)dt + dL_i^{k,+}(t) - dL_i^{k,-}(t)$$

with domain in Gelfand-Tsetlin cone

$$\mathbb{GT}_{p,p} := \{l_i^n \mid 0 \le l_{i-1}^{n-1} \le l_i^n \le l_i^{n-1}\}_{1\le i\le n, 1\le n\le p}.$$

Special features:

- Diffusion term $2\sqrt{l_i^k(t)}dB_i^k(t)$ is **singular**.
- Domain $\mathbb{GT}_{p,p}$ is polyhedral cone with **singular** boundary.
- Reflection off boundary is **oblique**.

Existence for SDER's in 1-D

Reduce to 1-D setting with time-dep boundaries $L_t < U_t$:

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt + d\Phi_t - d\Psi_t.$$

A strong solution is a triple (X_t, Φ_t, Ψ_t) so that:

- $X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \Phi_t - \Psi_t$ and $X_0 = x_0$;
- $L_t \le X_t \le U_t$ for all t;
- Φ_t, Ψ_t non-decreasing, bd variation, and $\Phi_0 = \Psi_0 = 0$;
- $\int_0^\infty 1_{X_t \neq L_t} d\Phi_t = \int_0^\infty 1_{X_t \neq U_t} d\Psi_t = 0$.

Theorem (Slominski-Wojciechowski 2013)

If b Lipschitz, σ locally Lipschitz, and

$$|\sigma(x) - \sigma(x')|^2 \le \rho(|x - x'|)$$
$$|\sigma(x)|^2 + |b(x)|^2 \le K(1 + |x|^2)$$

for some K and $\rho : \mathbb{R}_+ \to \mathbb{R}$ so that $\int_{0+} \frac{1}{\rho(s)} ds = \infty$, we have strong existence and uniqueness.

Existence for Laguerre and Jacobi Warren process

Construct the process level-by-level:

- Does not require a Gibbs initial condition.
- By 1-D criterion, suffices to exclude simult. collisions.
- Change coordinates and apply Girsanov to reduce to reflected BM's with specified reflection and covariance.
- Triple collisions for RBM studied by Sarantsev, Bruggeman-Sarantsev.

Theorem (S.-Sarantsev)

If the initial conditions $\{l_i^n(0)\}$ for the Laguerre or Jacobi Warren SDE's have no collisions, they admit a unique strong solution with no simultaneous collisions.

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The Pal-Shkolnikov framework (I)

Let X and Y be diffusion processes with generators

$$\mathcal{A}^X := \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{m} b_i(x)\partial_{x_i}$$

$$\mathcal{A}^Y := \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}(y)\partial_{y_i}\partial_{y_j} + \sum_{i=1}^{n} \gamma_i(y)\partial_{y_i}$$

on domains $\mathcal{X}$ and $\mathcal{Y}$. Let $D \subset \mathcal{X} \times \mathcal{Y}$ have polyhedral closure, and set $D(y) := \{x \mid (x, y) \in D\}$.
Let L be a stochastic transition operator mapping $C_0(\mathcal{X})$ to $C_0(\mathcal{Y})$ by the Gibbs restriction

$$(Lf)(y) := \int_{D(y)} f(x)\Lambda(y, x)dx$$

for a link function $\Lambda : D \to \mathbb{R}_{\geq 0}$.

The Pal-Shkolnikov framework (II)

Definition (Pal-Shkolnikov 2015)

A process $Z = (Z_1, Z_2)$ is an **intertwining of diffusions** X and Y with link operator L if:

(i) $Z_1 \stackrel{d}{=} X$ and $Z_2 \stackrel{d}{=} Y$, where $\stackrel{d}{=}$ denotes equality in law, and

$$\mathbb{E}[f(Z_1(0)) \mid Z_2(0) = y] = (Lf)(y),$$

for all bounded Borel measurable functions f on $D(y)$.

(ii) The transition semigroups P_t and Q_t of Z_1 and Z_2 are intertwined, meaning that $Q_t L = L P_t$ for all $t \geq 0$.

(iii) The process Z_1 is Markovian with respect to the joint filtration generated by (Z_1, Z_2).

(iv) For any $s \geq 0$, conditional on $Z_2(s)$, the random variable $Z_1(s)$ is independent of $\{Z_2(u), 0 \leq u \leq s\}$ and is conditionally distributed according to L.

A criterion for intertwining diffusions (I)

Consider the SDER on domain D for $Z = (Z_1, Z_2)$ given by

$$
\begin{aligned}
dZ_1(t) &= \sigma_X(Z_1(t))dB_X(t) + b(Z_1(t))dt + d\Phi_1(t) \\
dZ_2(t) &= \sigma_Y(Z_2(t))dB_Y(t) + \gamma(Z_2(t))dt \\
&\qquad\qquad + \langle \rho(Z_2(t)), \nabla_{z_2}[\log \Lambda(Z_2(t), Z_1(t))]\rangle dt + d\Phi(t)
\end{aligned}
$$

where $\partial D(y)$ is a polytope with η the outward unit normal and Ψ^i the y_i-derivative of parametrized boundary points, and

- $a(x) = \sigma_X(x)\sigma_X(x)^T$;
- $\rho(y) = \sigma_Y(y)\sigma_Y(y)^T$;
- Φ_1 implements reflection of Z_1 off $\partial\mathcal{X}$;
- Φ implements reflection of Z_2 off $\partial D(y)$ in direction

$$\sum_{i,j=1}^{n} \rho_{ij}\langle \Psi^i, \eta\rangle \partial_{y_j}.$$

A criterion for intertwining diffusions (II)

Generator for the Feller diffusion Z is

$$\mathcal{A}^Z := \mathcal{A}^X + \mathcal{A}^Y + \sum_{i,j=1}^{n} \rho_{ij}(y)\partial_{y_i}[\log \Lambda(y,x)]\partial_{y_j}$$

with domain containing

$$\mathcal{D}(\mathcal{A}^Z) := \{f \in C_c^2(D) \mid \langle u, \nabla f(z)\rangle = 0 \text{ for } u \in \widetilde{U}(z),\, z \in \partial D\},$$

where $\widetilde{U}(z)$ is the set of reflection directions at z.

Definition

The process Z is **regular** if $\mathcal{D}(\mathcal{A}^Z)$ is a core for $\mathcal{A}^Z$.

A criterion for intertwining diffusions (III)

Consider condition

$$\int_{D(y)} \Lambda \mathcal{A}^X(f)dx = \int_{D(y)} \mathcal{A}^Y(\Lambda) f dx + \sum_{j=1}^{n} \gamma_j \int_{\partial D(y)} \Lambda f \langle \Psi^j, \eta \rangle d\theta(x)$$

$$(\star) \quad + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \int_{\partial D(y)} \Big(\mathrm{div}_{\partial x}(\Lambda f \Psi^j) + 2\partial_{y_j}(\Lambda f) \Big) \langle \Psi^i, \eta \rangle d\theta(x).$$

Theorem (S.)

Suppose D, L, X, Y satisfy technical assumptions and that any $f \in \mathcal{D}(\mathcal{A}^Z)$ satisfies $(\star)$. If a weak solution Z to the SDER is a regular Feller diffusion with generator $\mathcal{A}^Z$, then for Gibbs initial conditions, Z is an intertwining of X and Y with link L.

- Generalizes result of Pal-Shkolnikov for $a_{ij} = \rho_{ij} = \delta_{ij}$.

A criterion for intertwining diffusions (IV)

Condition $(\star)$ is implied by:

- $\mathcal{A}^Y(\Lambda) = (\mathcal{A}^X)^*(\Lambda)$;
- on $\partial D(y)$:

$$\Lambda\langle b, \eta\rangle - \frac{1}{2}\Lambda\langle \mathrm{div}_x a, \eta\rangle - \langle\langle a, \eta\rangle, \nabla_x \Lambda\rangle$$
$$= \sum_j \Lambda\gamma_j\langle \Psi^j, \eta\rangle + \sum_{i,j=1}^n \rho_{ij}\partial_{y_j}(\Lambda)\langle \Psi^i, \eta\rangle;$$

- on $\partial D(y)$:

$$\langle a, \eta\rangle = \sum_{i,j=1}^n \rho_{ij}\Psi^j\langle \Psi^i, \eta\rangle.$$

Recall: η is unit normal to $\partial D(y)$

Ψ^i is y_i-derivative of point on $\partial D(y)$.

A criterion for intertwining diffusions (V)

Key point of the proof:

- Check Gibbs property:

$$\mathbb{E}\left[\int_{D(y)} \Lambda(Y(t),x)f(x,Y(t))dx \mid Y(0)=y\right]$$
$$= \int_{D(y)} \Lambda(y,x)\mathbb{E}[f(Z_1(t),Z_2(t)) \mid Z(0)=(x,y)]dx.$$

- Infinitesimally reduce to:

$$\mathcal{A}^Y \int_{D(y)} \Lambda f\, dx = \int_{D(y)} \Lambda\, \mathcal{A}^Z(f)dx.$$

Summary

This talk:

1. Define Laguerre and Jacobi Warren processes as solutions to certain reflected SDE's.
2. Show these processes preserve Gibbs measures and project to Laguerre and Jacobi eigenvalues processes.
3. Prove general criterion for existence of an intertwining diffusion generalizing that of Pal-Shkolnikov.

References:

- Y. S. (with an appendix by A. Sarantsev), Laguerre and Jacobi analogues of the Warren process, 2017.