A NEW INTEGRAL FORMULA FOR HECKMAN-OPDAM HYPERGEOMETRIC FUNCTIONS

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Abstract. We provide formulas for the multivariate Bessel functions and Heckman-Opdam hypergeometric functions as representation-valued integrals over dressing orbits. Our expression is the quasi-classical limit of the realization of Macdonald polynomials as traces of intertwiners of quantum groups given by Etingof and Kirillov Jr. in [EK94]. Integration over the Liouville tori of the Gelfand-Tsetlin integrable system and adjunction for higher Calogero-Moser Hamiltonians relates our expression to the integral realization over Gelfand-Tsetlin polytopes which appeared in the recent work [BG13] of Borodin and Gorin on the β-Jacobi corners ensemble.

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Date: June 14, 2014.

1. Introduction

1.1. **Heckman-Opdam hypergeometric functions.** Fix a complex number k and a positive integer N. The rational and trigonometric Calogero-Moser integrable systems in the variables $\{\lambda_i\}_{1\leq i\leq N}$ are the quantum integrable systems with quadratic Hamiltonians

$$L_{p_2}(k) = \sum_i \partial_i^2 - 2k(k+1)\sum_{i<j}\frac{1}{(\lambda_i-\lambda_j)^2} \text{ and}$$
$$L_{p_2}^{\text{trig}}(k) = \sum_i \partial_i^2 - k(k+1)\sum_{i<j}\frac{1}{2\sinh^2\left(\frac{\lambda_i-\lambda_j}{2}\right)}.$$

They are completely integrable systems, meaning that $L_{p_2}(k)$ and $L_{p_2}^{\text{trig}}(k)$ fit into families $L_p(k)$ and $L_p^{\text{trig}}(k)$ of commuting Hamiltonians defined for each symmetric polynomial p. It was shown in [HO87, Hec87, Opd88a, Opd88b] that $L_p^{\text{trig}}(k-1)$ have a family of holomorphic joint eigenfunctions $\Phi_k(\Lambda, s)$ indexed by a parameter $s = (s_1, \ldots, s_N)$ so that

$$L_p^{\text{trig}}(k-1)\Phi_k(\Lambda, s) = p(s)\Phi_k(\Lambda, s). \tag{1}$$

The functions $\Phi_k(\Lambda, s)$ are known as the Heckman-Opdam hypergeometric functions. The corresponding rational degenerations are a family of joint eigenfunctions $\phi_k(\lambda, s)$ of $L_p(k-1)$ so that

$$L_p(k-1)\phi_k(\lambda, s) = p(s)\phi_k(\lambda, s)$$

and are known as multivariate Bessel functions.

1.2. **Poisson-Lie group structure on $\mathfrak{su}(N)$ and $SU(N)$.** The Lie algebra $\mathfrak{sl}_N(\mathbb{C})$ has real Iwasawa decomposition $\mathfrak{sl}_N = \mathfrak{su}_N \oplus \mathfrak{sb}_N$ with $\mathfrak{sb}_N \simeq \mathfrak{su}_N^*$. We identify $\mathfrak{su}_N^*$ with $\mathfrak{p}_N$, the trivial Lie algebra of $N\times N$ Hermitian matrices by the map $x \mapsto \frac{1}{2}(x + x^*)$. Equip $\mathfrak{p}_N$ with the Kirillov-Kostant-Souriau Poisson structure, and denote the coadjoint orbit of $\lambda \in \mathfrak{p}_N$ by $\mathcal{O}_\lambda$. Denote the symplectic form and Liouville measure on $\mathcal{O}_\lambda$ by ω_λ and $d\mu_\lambda$, respectively.

In the corresponding Iwasawa decomposition $SL(N) = SU(N)SB(N)$ for the group, give $SU(N)$ the Lu-Weinstein Poisson-Lie structure (see [LW90]) so that $SB(N)$ is the dual Poisson-Lie group to $SU(N)$. Identify $SB(N)$ with the Poisson manifold P_N^+ of $N\times N$ positive definite Hermitian matrices via $\text{sym}(b) = (b^*b)^{1/2}$ so that sym intertwines the dressing and conjugation actions of $SU(N)$ on $SB(N)$ and P_N^+. For $\Lambda = e^\lambda \in P_N^+$, denote by $\mathcal{O}_\Lambda$, ω_Λ, and $d\mu_\Lambda$ the dressing orbit containing Λ, its symplectic form, and its Liouville measure.

1.3. **The main results.** Restrict now to the case of positive integer k. View $W_{k-1} := \text{Sym}^{(k-1)N}(\mathbb{C}^N)$ as the $U(N)$-representation on degree $(k-1)N$ homogeneous polynomials in $x_1, \ldots, x_N$, and let $f_{k-1} : \mathcal{O}_\lambda \to W_{k-1}$ and $F_{k-1} : \mathcal{O}_\Lambda \to W_{k-1}$ denote the unique $U(N)$-equivariant maps such that $f_{k-1}(\lambda) = F_{k-1}(\Lambda) = w_{k-1} := (x_1\cdots x_N)^{k-1}$. Our main results are Theorems 3.1 and 4.2, which realize the multivariate Bessel functions and Heckman-Opdam hypergeometric functions as representation-valued integrals over coadjoint and dressing orbits under the identification of $W_{k-1}[0] \simeq \mathbb{C}\cdot w_{k-1}$ with $\mathbb{C}$.

Theorem 3.1. The multivariate Bessel function $\phi_k(\lambda, s)$ admits the integral representation

$$\phi_k(\lambda, s) = \frac{1}{\prod_{i<j}(s_i - s_j)^{k-1}}\int_{X\in\mathcal{O}_\lambda} f_{k-1}(X)e^{\sum_{l=1}^N s_l X_{ll}}d\mu_\lambda.$$

Theorem 4.2. The Heckman-Opdam hypergeometric function $\Phi_k(\Lambda, s)$ admits the integral representation

$$\Phi_k(\Lambda, s) = \frac{1}{\prod_{a=1}^{k-1}\prod_{i<j}(s_i - s_j - a)}\int_{X\in\mathcal{O}_\Lambda} F_{k-1}(X)\prod_{l=1}^N\left(\frac{\det(X_l)}{\det(X_{l-1})}\right)^{s_l}d\mu_\Lambda,$$

where X_l is the principal $l\times l$ submatrix of X.

Remark. The integral of Theorem 3.1 may be viewed as a generalization of its $k = 1$ case, which is the HCIZ integral of [HC57a, HC57b, IZ80].

1.4. **Existing integral formulas and connection to β-Jacobi corners ensemble.** The Heckman-Opdam functions appeared in the work [BG13] of Borodin-Gorin on the β-Jacobi corners ensemble, where they were obtained as a certain scaling limit of the Macdonald polynomials $P_\lambda(x;q,t)$.

Theorem 1.1 ([BG13, Proposition 6.2])**.** For any positive real $k>0$, the Heckman-Opdam hypergeometric function with parameter k is the following scaling limit of Macdonald polynomials

$$\Phi_k(\lambda,s)=\lim_{\varepsilon\to 0}\varepsilon^{kN(N-1)/2}P_{\lfloor\varepsilon^{-1}\lambda\rfloor}(e^{\varepsilon s};q,q^k)\text{ with }q=e^{\varepsilon}.$$

For $\lambda_1\geq\cdots\geq\lambda_N\in\mathbb{R}^N$, define the Gelfand-Tsetlin polytope to be

$$\mathrm{GT}_\lambda:=\{(\mu_i^l)_{1\leq i\leq l,1\leq l<N}\mid \mu_i^{l+1}\geq\mu_i^l\geq\mu_{i+1}^{l+1}\},$$

where we take $\mu_i^N=\lambda_i$. A point $\{\mu_i^l\}$ in GT_λ is called a Gelfand-Tsetlin pattern. By applying the scaling of Theorem 1.1 to the combinatorial formula for Macdonald polynomials, Borodin-Gorin established the following integral formula over Gelfand-Tsetlin patterns for the Heckman-Opdam hypergeometric function.

Theorem 1.2 ([BG13, Definition 6.1 and Proposition 6.3])**.** For any positive real $k>0$, the Heckman-Opdam hypergeometric function is given by

$$\Phi_k(\lambda,s)=\frac{1}{\Gamma(k)^{N(N-1)/2}}\int_{\mu\in\mathrm{GT}_\lambda}e^{\left(\sum_{l=1}^N s_l\left(\sum_{i=1}^l\mu_i^l-\sum_{i=1}^{l-1}\mu_i^{l-1}\right)\right)}$$
$$\prod_{l=1}^{N-1}\frac{\prod_{i=1}^l\prod_{j=1}^{l+1}|e^{-\mu_i^l}-e^{-\mu_j^{l+1}}|^{k-1}}{\prod_{i<j}(e^{-\mu_i^l}-e^{-\mu_j^l})^{k-1}\prod_{i<j}(e^{-\mu_i^{l+1}}-e^{-\mu_j^{l+1}})^{k-1}}\prod_{l=1}^{N-1}e^{(k-1)\sum_{i=1}^l\mu_i^l}\prod_i d\mu_i^l.$$

Remark. The main result of [KK96, Theorem 6.3] gives for each Weyl chamber a contour integral formula for a solution to the hypergeometric system (1) holomorphic in that Weyl chamber. These formulas have the same integrand as the integral of Theorem 1.2 but contours which are different for each Weyl chamber.

1.5. **Connection to intertwiners of quantum group representations.** We discuss briefly the motivation for the formula of Theorem 4.2. For a dominant integral weight λ, let L_λ denote the corresponding highest weight irreducible representation of $U_q(\mathfrak{gl}_N)$. Let $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha$ be half the sum of the positive roots. In [EK94], it was shown that there exists a unique intertwiner $\varphi_\lambda: L_{\lambda+(k-1)\rho}\to L_{\lambda+(k-1)\rho}\otimes W_{k-1}$ of $U_q(\mathfrak{gl}_N)$-representations such that the highest weight vector $v_\lambda\in L_\lambda$ is mapped to

$$\varphi_\lambda(v_\lambda)=v_\lambda\otimes w_{k-1}+(\text{lower order terms}),$$

where the lower order terms lie only in spaces of weight less than λ in the L_λ tensor factor. They expressed Macdonald polynomials in terms of these intertwiners in the following theorem.

Theorem 1.3 ([EK94, Theorem 1])**.** The Macdonald polynomial at $t=q^k$ is given by

$$P_\lambda(x;q,q^k)=\frac{\mathrm{Tr}|_{L_{\lambda+(k-1)\rho}}(\varphi_\lambda x^h)}{\mathrm{Tr}|_{L_{(k-1)\rho}}(\varphi_0 x^h)}.\tag{2}$$

Applying the limit transition of [BG13] to one side of (2) yields the Heckman-Opdam hypergeometric functions. Applying it to the other side yields a quasi-classical limit, under which we have the degenerations

$$\mathrm{Tr}|_{L_{\lambda+(k-1)\rho}}(-)\mapsto\int_{\mathcal{O}_\Lambda}-\,d\mu_\Lambda\qquad\text{and}\qquad\varphi_\lambda\mapsto f_{k-1}.$$

Together, these degenerations provide the form of Theorem 4.2.

1.6. **Outline of method and organization.** We outline our approach. The Gelfand-Tsetlin action on the dressing orbit defines a classical integrable system whose moment map is the logarithmic Gelfand-Tsetlin map GT of [FR96, AM07]. Integration over the Liouville tori reduces the integral of Theorem 4.2 to an integral with respect to the Duistermaat-Heckman measure $\mathrm{GT}_*(d\mu_\Lambda)$ on GT_λ, which is the Lebesgue measure. This yields an integral expression for $\Phi_k(\Lambda,s)$ over GT_λ. The new integrand differs from that of Theorem 1.2, but we show equality of the integrals by applying adjunction for higher Calogero-Moser Hamiltonians.

The remainder of this paper is organized as follows. In Section 2, we give the geometric setup for our integral formulas. In Section 3, we prove Theorem 3.1 in the rational setting, establishing in particular the key Proposition 3.4. In Section 4, we use Proposition 3.4 to prove Theorem 4.2 in the trigonometric setting. In Section 5, we provide proofs for some technical lemmas whose proofs were deferred.

1.7. **Acknowledgements.** The author thanks his Ph.D. advisor P. Etingof for suggesting the problem and providing the idea for the proof of Proposition 3.4 and A. Borodin for helpful discussions. Y. S. was supported by a NSF graduate research fellowship (NSF Grant #1122374).

2. Geometric setup

2.1. **Notations.** For sets of variables $\{x_i\}$ and $\{y_i\}$, we denote the Vandermonde determinant by $\Delta(x) = \prod_{i<j}(x_i - x_j)$, and the product of differences by $\Delta(x, y) = \prod_{i,j}(x_i - y_j)$.

2.2. **Gelfand-Tsetlin coordinates.** Define the *Gelfand-Tsetlin map* $\mathrm{gt} : \mathcal{O}_\lambda \to \mathrm{GT}_\lambda$ by

$$\mathrm{gt}(X) = \{\lambda_i(X_l)\}_{1\le i\le l, 1\le l<N},$$

where X_l is the principal $l \times l$ submatrix of X, and $\lambda_1(X_l) \geq \ldots \geq \lambda_l(X_l)$ are its eigenvalues. Define the *logarithmic Gelfand-Tsetlin map* $\mathrm{GT} : \mathcal{O}_\Lambda \to \mathrm{GT}_\lambda$ by

$$\mathrm{GT}(X) = \{\log(\lambda_i(X_l))\}_{1\le i\le l, 1\le l<N}.$$

By a theorem of Ginzburg and Weinstein (see [GW92]), the Poisson structures we have described on $\mathfrak{sb}_N$ and $SB(N)$ make them isomorphic as Poisson manifolds. By [AM07], there exists a Ginzburg-Weinstein isomorphism $\mathfrak{sb}_N \to SB(N)$ which intertwines the logarithmic and ordinary Gelfand-Tsetlin maps. In particular, this map restricts to a symplectomorphism $\mathcal{O}_\lambda \to \mathcal{O}_\Lambda$.

2.3. **Gelfand-Tsetlin integrable system.** Let $T := T_1 \times \cdots \times T_{N-1}$ be a torus of dimension $\frac{N(N-1)}{2}$, where $\dim T_l = l$. For $t_l \in T_l$ and X in $\mathcal{O}_\lambda$ or $\mathcal{O}_\Lambda$ whose principal $l \times l$ submatrix X_l is diagonalized by $X_l = U_l \Lambda_l U_l^*$, the *Gelfand-Tsetlin* action of t_l on X is defined as

$$t_l \cdot X = \mathrm{Ad}_{\overline{U_l t_l U_l^*}}(X),$$

where for $Y_l \in U(l)$, the matrix $\overline{Y_l} \in SU(N)$ is defined to be the square block matrix

$$\overline{Y_l} = \left(\begin{array}{ccc|c} & & & 0 \\ & Y_l & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & cI_{N-l} \end{array}\right),$$

where c is chosen so that $\overline{Y_l} \in SU(N)$. The actions of T_l preserve $l \times l$ principal submatrices and pairwise commute, giving actions of T on $\mathcal{O}_\lambda$ and $\mathcal{O}_\Lambda$. These actions are Hamiltonian with moment maps gt and GT, respectively, and the corresponding classical integrable system is known as the Gelfand-Tsetlin integrable system (see [AM07, GS83, FR96] for more about this integrable system).

We may use the Gelfand-Tsetlin action to write any X_0 in $\mathrm{gt}^{-1}(\mu)$ or $\mathrm{GT}^{-1}(\mu)$ in a special form. Write X_0 as either $u_N \lambda u_N^*$ or $u_N \Lambda u_N^*$ for some unitary matrix u_N and decompose u_N as

$$u_N = \overline{u}_1(\overline{u}_1^*\overline{u}_2)\cdots(\overline{u}_{N-1}^* u_N)$$

for $u_m \in U(m)$ and $v_m := \overline{u}_{m-1}^* u_m$ satisfying either

$$(v_m \mu^m v_m^*)_{m-1} = \mu^{m-1} \qquad \text{or} \qquad (v_m e^{\mu^m} v_m^*)_{m-1} = e^{\mu^{m-1}},$$

where $(M)_{m-1}$ denotes the principal $(m-1) \times (m-1)$ submatrix of a matrix M. Lemma 2.1 gives a compatibility property between this decomposition and the Gelfand-Tsetlin action.

Lemma 2.1. For any $l \leq m$ and $t_m \in T_m$, we have

$$t_m \cdot \mathrm{ad}_{\overline{v}_l\cdots\overline{v}_N}(\lambda) = \mathrm{ad}_{\overline{v}_l\cdots\overline{v}_m}(t_m \cdot \mathrm{ad}_{\overline{v}_{m+1}\cdots\overline{v}_N}(\lambda)), \text{ and}$$
$$t_m \cdot \mathrm{ad}_{\overline{v}_l\cdots\overline{v}_N}(\Lambda) = \mathrm{ad}_{\overline{v}_l\cdots\overline{v}_m}(t_m \cdot \mathrm{ad}_{\overline{v}_{m+1}\cdots\overline{v}_N}(\Lambda)).$$

Proof. By construction, the principal $m \times m$ submatrix of $\mathrm{ad}_{\overline{v}_{m+1}\cdots\overline{v}_N}(\lambda)$ is diagonal, implying that

$$t_m \cdot \mathrm{ad}_{\overline{v}_l\cdots\overline{v}_N}(\lambda) = \mathrm{ad}_{\mathrm{ad}_{\overline{v}_l\cdots\overline{v}_m}(t_m)}(\mathrm{ad}_{\overline{v}_l\cdots\overline{v}_N}(\lambda)) = \mathrm{ad}_{\overline{v}_l\cdots\overline{v}_m}(t_m \cdot \mathrm{ad}_{\overline{v}_{m+1}\cdots\overline{v}_N}(\lambda)).$$

An analogous proof yields the lemma for Λ in place of λ. $\square$

2.4. **Duistermaat-Heckman measures.** The pushforwards $\mathrm{gt}_*(d\mu_\lambda)$ and $\mathrm{GT}_*(d\mu_\Lambda)$ of the Liouville measures on $\mathcal{O}_\lambda$ and $\mathcal{O}_\Lambda$ to GT_λ are called Duistermaat-Heckman measures. Because the Ginzburg-Weinstein isomorphism intertwines the two Gelfand-Tsetlin maps, the two Duistermaat-Heckman measures on GT_λ coincide. It is known (see [GN50, Bar01, AB04, Section 5.6]) that the Duistermaat-Heckman measure for the coadjoint orbit $\mathcal{O}_\lambda$ is proportional to the Lebesgue measure on the Gelfand-Tsetlin polytope. To compute the normalization constant, we recall Harish-Chandra's formula (see [Kir99, Theorem 3, Section 3])

$$\int_{\mathcal{O}_\lambda} e^{(b,x)} d\mu_\lambda = \frac{\sum_{w\in W}(-1)^w e^{(w\lambda,x)}}{\prod_{i<j}(x_i - x_j)},$$

which upon taking $x \to 0$ (via $x = \varepsilon \cdot \rho$ and $\varepsilon \to 0$) shows that

$$\mathrm{Vol}(\mathcal{O}_\lambda) = \frac{\prod_{i<j}(\lambda_i - \lambda_j)}{(N-1)!\cdots 1!}.$$

On the other hand, it is known (see [Ols13, Corollary 3.2]) that $\mathrm{Vol}(\mathrm{GT}_\lambda) = \frac{\prod_{i<j}(\lambda_i-\lambda_j)}{(N-1)!\cdots 1!}$, meaning that $\mathrm{gt}_*(d\mu_\lambda) = 1_{\mathrm{GT}_\lambda} \cdot dx$. This discussion establishes the following Proposition 2.2.

Proposition 2.2. The Duistermaat-Heckman measures $\mathrm{gt}_*(d\mu_\lambda) = \mathrm{GT}_*(d\mu_\Lambda)$ are equal to the Lebesgue measure dx on the Gelfand-Tsetlin polytope. Explicitly, we have

$$\mathrm{gt}_*(d\mu_\lambda) = \mathrm{GT}_*(d\mu_\Lambda) = 1_{\mathrm{GT}_\lambda} dx.$$

3. The rational case

3.1. **Multivariate Bessel functions.** Recall that the multivariate Bessel functions are the family of joint eigenfunctions $\phi_k(\lambda, s)$ of the rational Calogero-Moser Hamiltonians $L_p(k-1)$ so that

$$L_p(k-1)\phi_k(\lambda, s) = p(s)\phi_k(\lambda, s). \tag{3}$$

They are degenerations of Heckman-Opdam hypergeometric functions and admit the integral representation

$$\phi_k(\lambda, s) = \frac{1}{\Gamma(k)^{N(N-1)/2}} \int_{\mu\in\mathrm{GT}_\lambda} e^{\sum_{l=1}^N s_l(\sum_i \mu_i^l - \sum_i \mu_i^{l-1})} \prod_{l=1}^{N-1} \frac{|\Delta(\mu^l, \mu^{l+1})|^{k-1}}{\Delta(\mu^l)^{k-1}\Delta(\mu^{l+1})^{k-1}} \prod_{i=1}^{l} d\mu_i^l, \tag{4}$$

which is the rational limit of Theorem 1.2. This integral identity implies the shift identity

$$e^{c\sum_i \lambda_i}\phi_k(\lambda, s) = \phi_k(\lambda, s_1 + c, \ldots, s_N + c). \tag{5}$$

3.2. **Statement of the result.** Let $W_{k-1} = \mathrm{Sym}^{(k-1)N}(\mathbb{C}^N)$, and let $w_{k-1} = (x_1 \cdots x_N)^{k-1} \in W_{k-1}$. Let $f_{k-1} : \mathcal{O}_\lambda \to W_{k-1}$ be the unique $U(N)$-equivariant map so that $f_{k-1}(\lambda) = w_{k-1}$. Define the representation-valued integral

$$\psi_k(\lambda, s) = \int_{X\in\mathcal{O}_\lambda} f_{k-1}(X) e^{\sum_{l=1}^N s_l X_{ll}} d\mu_\lambda$$

over the coadjoint orbit $\mathcal{O}_\lambda$. The integrand and Liouville measure are invariant under the action of the maximal torus of $U(N)$, so $\psi_k(\lambda, s)$ lies in $W_{k-1}[0] = \mathbb{C} \cdot w_{k-1}$. We interpret the integrals $\psi_k(\lambda, s)$ as complex-valued functions by identifying $\mathbb{C} \cdot w_{k-1}$ with $\mathbb{C}$. Our first result relates these integrals to the multivariate Bessel functions.

Theorem 3.1. The multivariate Bessel function $\phi_k(\lambda, s)$ admits the integral representation

$$\phi_k(\lambda, s) = \frac{1}{\prod_{i<j}(s_i - s_j)^{k-1}} \int_{X\in\mathcal{O}_\lambda} f_{k-1}(X) e^{\sum_{l=1}^N s_l X_{ll}} d\mu_\lambda.$$

3.3. **Adjoints of rational Calogero-Moser operators.** The rational Dunkl operators in variables μ_i are

$$D_i(k) = \partial_i - k \sum_{j \neq i} \frac{1}{\mu_i - \mu_j}(1 - s_{ij}).$$

Let m denote the restriction of a differential-difference operator to its differential part. For a symmetric polynomial p, define $\overline{L}_p(k) := m(p(D_i(k)))$. Recall that

$$\overline{L}_p(k) = \Delta(\mu)^k \circ L_p(k) \circ \Delta(\mu)^{-k}$$

is a conjugate of the rational Calogero-Moser Hamiltonian corresponding to p. Define $D_i(k)^\dagger := -D_i(k)$ to be the formal adjoint for $D_i(k)$ with respect to the inner product $\langle f, g\rangle = \int f(\mu)\overline{g}(\mu)\Delta(\mu)^{-2k} d\mu$. We characterize the adjoint of $\overline{L}_p(k)$ in terms of $D_i(k)^\dagger$ by Proposition 3.2. For multi-indices $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$, write $\beta \leq \alpha$ if $\alpha_i \leq \beta_i$ for all i.

Proposition 3.2. Let A be a rectangular domain. Let $p = \sum_\alpha c_\alpha \mu^\alpha$ be a symmetric function and f and g be symmetric functions on A. If for each non-zero monomial μ^α appearing in p, $\partial_\mu^\beta f$ vanishes on the boundary of A for any $\beta \leq \alpha$, then we have the adjunction relation

$$\int_A (\overline{L}_p(k) f(\mu))\, \overline{g}(\mu)\, \Delta(\mu)^{-2k} d\mu = \int_A f(\mu)\, m(p(D_i(k)^\dagger)(\overline{g}(\mu))\, \Delta(\mu)^{-2k} d\mu.$$

Proof. If A is replaced by $\mathbb{R}^N$, the statement holds because the adjoint and formal adjoint of $D_i(k)$ coincide. The adjoint of $\overline{L}_p(k)$ as a differential operator does not depend on the domain. Therefore, integration by parts shows the two sides of the desired relation differ by the sum of several terms, each of which contains a factor which is the evaluation of $\partial_\mu^\beta(f)$ on a point of the boundary of A for some $\beta \leq \alpha$ with μ^α appearing in p. These terms vanish, giving the lemma. □

3.4. **A matrix element computation.** Recall that sequences $\{\lambda_i\}_{1 \leq i \leq n}$ and $\{\mu_i\}_{1 \leq i \leq n-1}$ *interlace* if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n,$$

which we denote by $\mu \prec \lambda$. Define the real matrix $u(\mu, \lambda)$ by

$$u(\mu,\lambda)_{ij} = \begin{cases} \left(\frac{\prod_l (\mu_l - \lambda_j)}{\prod_{l \neq j}(\lambda_l - \lambda_j)}\right)^{1/2} & i = n \\ (\lambda_j - \mu_i)^{-1} \left(\frac{\prod_l (\mu_l - \lambda_j)}{\prod_{l \neq j}(\lambda_l - \lambda_j)}\right)^{1/2} \left(-\frac{\prod_l (\lambda_l - \mu_i)}{\prod_{l \neq i}(\mu_l - \mu_i)}\right)^{1/2} & i < n, \end{cases}$$

where each square root is applied to a non-negative real number because $\mu \prec \lambda$, and we take the non-negative branch. The following lemma, whose proof is given in Section 5.1, shows that $u(\mu, \lambda)$ conjugates a diagonal matrix to a matrix with diagonal principal submatrix.

Lemma 3.3. The matrix $u(\mu, \lambda)$ is unitary, and the $(n-1) \times (n-1)$ principal submatrix of

$$u(\mu, \lambda)\, \mathrm{diag}(\lambda_1, \ldots, \lambda_n)\, u(\mu, \lambda)^*$$

is $\mathrm{diag}(\mu_1, \ldots, \mu_{n-1})$.

We would like to understand a specific matrix element of $u(\mu, \lambda)$ in $\mathrm{Sym}^{(k-1)N}\mathbb{C}^N$. For this, we first compute an auxiliary quantity. Let $Z_k(\mu, \lambda)$ denote the coefficient of $(x_1 \cdots x_l)^k$ in the polynomial

$$\frac{1}{(l-n+1)!} \prod_{j=1}^{l} \left(\sum_{i=1}^{n-1} \frac{x_i}{\mu_i - \lambda_j} + x_n + \cdots + x_l \right)^k.$$

By Proposition 3.4, we may express $Z_k(\mu, \lambda)$ via a conjugated Calogero-Moser Hamiltonian; we defer the proof to Section 5.2. The computation of the desired matrix element of $u(\mu, \lambda)$ is an easy consequence.

Proposition 3.4. We may write

$$Z_k(\mu, \lambda) = k!^{-(n-1)} \Delta(\mu, \lambda)^{-k} \overline{L}_{\mu_{n-1} \cdots \mu_1}(k)^k \Delta(\mu, \lambda)^k.$$

Lemma 3.5. The coefficient of $(x_1 \cdots x_{n-1})^{k-1}$ in $u(\mu, \lambda) \cdot (x_1 \cdots x_{n-1})^{k-1}$ is

$$(-1)^{(k-1)n(n-1)/2}(k-1)!^{-(n-1)} \Delta(\mu)^{1-k} \Delta(\lambda)^{1-k} (\overline{L}_{\mu_1 \cdots \mu_{n-1}}(k-1)^\dagger)^{k-1} \Delta(\mu, \lambda)^{k-1}.$$

Proof. By Lemma 3.3, the desired coefficient is given by

$$(-1)^{(k-1)(n+2)(n-1)/2}\frac{\Delta(\mu,\lambda)^{k-1}}{\Delta(\mu)^{k-1}\Delta(\lambda)^{k-1}}Z_{k-1}(\mu,\lambda),$$

which by Proposition 3.4 is equal to

$$(-1)^{(k-1)(n+2)(n-1)/2}(k-1)!^{-(n-1)}\Delta(\mu)^{1-k}\Delta(\lambda)^{1-k}\overline{L}_{\mu_{n-1}\cdots\mu_1}(k-1)^{k-1}\Delta(\mu,\lambda)^{k-1}.$$

To recover the desired form, it remains to notice that

$$(-1)^{n-1}\overline{L}_{\mu_{n-1}\cdots\mu_1}(k-1)^{k-1} = (\overline{L}_{\mu_{n-1}\cdots\mu_1}(k-1)^{k-1})^\dagger. \qquad \square$$

3.5. **Proof of Theorem 3.1.** Integrating over Liouville tori of the Gelfand-Tsetlin integrable system on $\mathcal{O}_\lambda$ yields the expression

$$\psi_k(\lambda,s) = \int_{\mu\in\mathrm{GT}_\lambda}\int_{t\in T, X_0\in \mathrm{gt}^{-1}(\mu)} f_{k-1}(t\cdot X_0)dt\; e^{\sum_{l=1}^N s_l(\sum_i \mu_i^l - \sum_i \mu_i^{l-1})}\mathrm{gt}_*(d\mu_\lambda),$$

where dt is an invariant probability measure on T and μ_i^l are the Gelfand-Tsetlin coordinates. Recall that $\mathrm{gt}_*(d\mu_\lambda)$ is equal to Lebesgue measure on GT_λ by Proposition 2.2. Adopting the notations of Section 2.3, by repeated application of Lemma 2.1 we have for $t = t_1\cdots t_{N-1}$ in the Gelfand-Tsetlin torus that

$$t\cdot X_0 = \mathrm{ad}(\overline{v}_1)t_1\cdot\mathrm{ad}(\overline{v}_2)\cdots t_{N-1}\cdot\mathrm{ad}(\overline{v}_N)\cdot\lambda.$$

On the other hand, if $w\in W_{k-1}$ lies in $\mathbb{C}[x_1,\ldots,x_l](x_{l+1}\cdots x_N)^{k-1}$, then

$$\int_{T_l} t_l\cdot w\, dt_l = \{\text{coefficient of } (x_1\cdots x_N)^{k-1} \text{ in } w\}.$$

Together, these imply that

$$\int_{t\in T, X_0\in\mathrm{gt}^{-1}(\mu)} f_{k-1}(t\cdot X_0)\,dt = \prod_{m=1}^{N-1} W_m,$$

where W_m denotes the coefficient of $(x_1\cdots x_m)^{k-1}$ in $v_m\cdot(x_1\cdots x_m)^{k-1}$. Recall that v_m was chosen so that $(v_m\,\mathrm{diag}(\mu^{m+1})v_m^*)_m = \mathrm{diag}(\mu^m)$, meaning by Lemma 3.5 that

$$\begin{aligned} W_m &= \frac{\Delta(\mu^m,\mu^{m+1})^{k-1}}{\Delta(\mu^m)^{k-1}\Delta(\mu^{m+1})^{k-1}}Z_{k-1}(\mu^m,\mu^{m+1}) \\ &= \frac{(-1)^{(k-1)m(m+1)/2}(k-1)!^{-m}}{\Delta(\mu^m)^{k-1}\Delta(\mu^{m+1})^{k-1}}(\overline{L}_{\mu_1\cdots\mu_m}(k-1)^\dagger)^{k-1}\Delta(\mu^m,\mu^{m+1})^{k-1}. \end{aligned}$$

Substituting in this result, inducting on N, applying the integral formula (4), and applying (5) in $N-1$ variables with $c = -s_N$, we obtain

$$\begin{aligned} \psi_k(\lambda,s) &= \int_{\mu\in\mathrm{GT}_\lambda}\prod_{m=1}^{N-1}W_m e^{\sum_{l=1}^N s_l(\sum_i\mu^l - \sum_i \mu^{l-1})}\prod_l d\mu_i^l \\ &= \Gamma(k)^{-N}\int_{\mu\prec\lambda} Z_{N-1}e^{s_N(\sum_i\lambda - \sum_i\mu)}\prod_{1\le i<j\le N-1}(s_i-s_j)^{k-1}\phi_k(\mu,s)\prod_i d\mu_i \\ &= e^{s_N\sum_i\lambda_i}\Gamma(k)^{-N}\prod_{1\le i<j\le N-1}(s_i-s_j)^{k-1}\int_{\mu\prec\lambda}W_{N-1}e^{-s_N\sum_i\mu_i}\phi_k(\mu,s_1,\ldots,s_{N-1})\prod_i d\mu_i \\ &= (-1)^{(k-1)N(N-1)/2}e^{s_N\sum_i\lambda_i}\Gamma(k)^{-N} \\ &\quad \prod_{1\le i<j\le N-1}(s_i-s_j)^{k-1}\int_{\mu\prec\lambda}\frac{(\overline{L}_{\mu_1\cdots\mu_{N-1}}(k-1)^\dagger)^{k-1}\Delta(\mu,\lambda)^{k-1}}{\Delta(\mu)^{k-1}\Delta(\lambda)^{k-1}}\phi_k(\mu,s_1-s_N,\ldots,s_{N-1}-s_N)\prod_i d\mu_i. \end{aligned}$$

Applying Proposition 3.2, (5), and (3) to the last expression yields the desired expression

$$\begin{aligned}\psi_k(\lambda,s) &= \frac{e^{s_N\sum_i \lambda_i}}{\Delta(\lambda)^{k-1}}(-1)^{(k-1)N(N-1)/2}\Gamma(k)^{-N}\prod_{1\le i\le j\le N-1}(s_i-s_j)^{k-1}\\
&\quad\int_{\mu\prec\lambda}\frac{\Delta(\mu,\lambda)^{k-1}}{\Delta(\mu)^{2(k-1)}}(\overline{L}_{\mu_1\cdots\mu_{N-1}}(k-1))^{k-1}\Delta(\mu)^{k-1}\phi_k(\mu,s_1-s_N,\dots,s_{N-1}-s_N)\prod_i d\mu_i\\
&= e^{s_N\sum_i\lambda_i}(-1)^{(k-1)N(N-1)/2}\Gamma(k)^{-N}\prod_{1\le i\le j\le N}(s_i-s_j)^{k-1}\int_{\mu\prec\lambda}\frac{\Delta(\mu,\lambda)^{k-1}\phi_k(\mu,s_1-s_N,\dots,s_{N-1}-s_N)}{\Delta(\mu)^{k-1}\Delta(\lambda)^{k-1}}\prod_i d\mu_i\\
&= e^{s_N\sum_i\lambda_i}\prod_{1\le i\le j\le N}(s_i-s_j)^{k-1}\phi_k(\lambda,s_1-s_N,\dots,s_{N-1}-s_N,0)\\
&= \prod_{1\le i\le j\le N}(s_i-s_j)^{k-1}\phi_k(\lambda,s).\end{aligned}$$

4. The trigonometric case

4.1. **Heckman-Opdam hypergeometric functions.** Recall that the Heckman-Opdam hypergeometric functions $\Phi_k(\Lambda,s)$ are the family of joint eigenfunctions $\Phi_k(\Lambda,s)$ of $L_p^{\text{trig}}(k-1)$ so that

$$L_p^{\text{trig}}(k-1)\Phi_k(\Lambda,s)=p(s)\Phi_k(\Lambda,s).$$

It was shown in [BG13] that the Heckman-Opdam hypergeometric functions may be represented as the integral over Gelfand-Tsetlin patterns given in Theorem 1.2. We rewrite this integral for our purposes as

$$\begin{aligned}\Phi_k(\Lambda,s) &= \frac{1}{\Gamma(k)^{N(N-1)/2}}\int_{\mu\in\mathrm{GT}_\lambda}e^{\left(\sum_{l=1}^N s_l\left(\sum_{i=1}^l\mu_i^l-\sum_{i=1}^{l-1}\mu_i^{l-1}\right)\right)}\\
&\quad\prod_{l=1}^{N-1}\frac{\prod_{i=1}^l\prod_{j=1}^{l+1}|e^{\mu_i^l}-e^{\mu_j^{l+1}}|^{k-1}}{\prod_{i<j}(e^{\mu_i^l}-e^{\mu_j^l})^{k-1}\prod_{i<j}(e^{\mu_i^{l+1}}-e^{\mu_j^{l+1}})^{k-1}}\prod_{l=1}^{N-1}e^{-(k-1)\sum_{i=1}^l\mu_i^l}\prod_i d\mu_i^l.\end{aligned}$$

As in the rational setting, this integral representation implies a shift identity

$$e^{c\sum_i\lambda_i}\Phi_k(\Lambda,s)=\Phi_k(\Lambda,s_1+c,\dots,s_N+c). \tag{6}$$

Define the conjugated Calogero-Moser Hamiltonian as

$$\overline{L}_p^{\text{trig}}(k)=e^{-\frac{(n-1)k}{2}\sum_i\mu_i}\Delta(e^\mu)^k\circ L_p^{\text{trig}}(k)\circ e^{\frac{(n-1)k}{2}\sum_i\mu_i}\Delta(e^\mu)^{-k}. \tag{7}$$

Lemma 4.1 shows how it acts on an appropriately normalized Heckman-Opdam hypergeometric function.

Lemma 4.1. For any symmetric polynomial p, we have

$$\Delta(e^\mu)^{1-k}\overline{L}_p^{\text{trig}}(k-1)\Delta(e^\mu)^{k-1}\Phi_k(\mu,s)=p\left(s_1+\frac{(N-2)(k-1)}{2},\dots,s_{N-1}+\frac{(N-2)(k-1)}{2}\right)\Phi_k(\mu,s).$$

Proof. Using (7) and the shift identity (6) for $\Phi_k(\mu,s)$, we compute

$$\begin{aligned}&\Delta(e^\mu)^{1-k}\overline{L}_p^{\text{trig}}(k-1)\Delta(e^\mu)^{k-1}\Phi_k(\mu,s)\\
&\quad= e^{-\frac{(N-2)(k-1)}{2}\sum_i\mu_i}L_p^{\text{trig}}(k-1)e^{\frac{(N-2)(k-1)}{2}\sum_i\mu_i}\Phi_k(\mu,s)\\
&\quad= e^{-\frac{(N-2)(k-1)}{2}\sum_i\mu_i}L_p^{\text{trig}}(k-1)\Phi_k\left(\mu,s_1+\frac{(N-2)(k-1)}{2},\dots,s_{N-1}+\frac{(N-2)(k-1)}{2}\right)\\
&\quad= p\left(s_1+\frac{(N-2)(k-1)}{2},\dots,s_{N-1}+\frac{(N-2)(k-1)}{2}\right)\Phi_k(\mu,s).\end{aligned}$$

□

4.2. **Statement of the result.** Let $F_{k-1} : \mathcal{O}_\Lambda \to W_{k-1}$ be the unique $U(N)$-equivariant map so that $F_{k-1}(\Lambda) = w_{k-1}$. Define the representation-valued integral

$$\Psi_k(\Lambda, s) = \int_{X \in \mathcal{O}_\Lambda} F_{k-1}(X) \prod_{l=1}^{N} \left(\frac{\det(X_l)}{\det(X_{l-1})} \right)^{s_l} d\mu_\Lambda,$$

where X_l denotes the principal $l \times l$ submatrix of X. As in the rational case, the integrand and Liouville measure in the definition of $\Psi_k(\Lambda, s)$ are invariant under the action of the maximal torus of $U(N)$, so $\Psi_k(\Lambda, s)$ lies in $W_{k-1}[0] = \mathbb{C} \cdot w_{k-1}$. We will again interpret it as a complex-valued function via the identification of $\mathbb{C} \cdot w_{k-1}$ with $\mathbb{C}$. Our result in the trigonometric setting uses these integrals to express the Heckman-Opdam hypergeometric functions.

Theorem 4.2. The Heckman-Opdam hypergeometric function $\Phi_k(\Lambda, s)$ admits the integral representation

$$\Phi_k(\Lambda, s) = \frac{1}{\prod_{a=1}^{k-1} \prod_{i<j} (s_i - s_j - a)} \int_{X \in \mathcal{O}_\Lambda} F_{k-1}(X) \prod_{l=1}^{N} \left(\frac{\det(X_l)}{\det(X_{l-1})} \right)^{s_l} d\mu_\Lambda.$$

4.3. **Adjoints of trigonometric Calogero-Moser operators.** The trigonometric Dunkl operators in variables μ_i are defined by

$$T_{\mu_i}(k) = \partial_i - k \sum_{\alpha > 0} (\alpha, \mu_i) \frac{1}{1 - e^{-\alpha}} (1 - s_\alpha) + k(\rho, \mu_i).$$

For a symmetric polynomial p, $m(p(T_i(k))) = \overline{L}_p^{\text{trig}}(k)$ is the conjugate (7) of the trigonometric Calogero-Moser Hamiltonian corresponding to p.

Remark. Our sign convention for $T_{\mu_i}(k)$ is opposite from [Hec97] for consistency with the rational case.

We require also the following result on adjoints of $T_{\mu_i}(k)$. By [Opd88a, Lemma 7.8], the formal adjoint of $T_{\mu_i}(k)$ with respect to the inner product

$$\langle f, g \rangle_k = \int f(\mu) \overline{g}(\mu) \Delta(e^\mu)^{-2k} d\mu$$

is given by

$$\begin{aligned} T_{\mu_i}(k)^\dagger &= -\partial_i + k \sum_{j<i} \frac{e^{\mu_i}}{e^{\mu_i} - e^{\mu_j}} (1 - s_{ij}) - k \sum_{j>i} \frac{e^{\mu_j}}{e^{\mu_j} - e^{\mu_i}} (1 - s_{ij}) + k \left(\frac{n}{2} - i \right) \\ &= -\partial_i + k \sum_{j \neq i} \frac{e^{\mu_i}}{e^{\mu_i} - e^{\mu_j}} (1 - s_{ij}) + k \sum_{j>i} s_{ij} - k \frac{n-2}{2} \\ &= -T_{\mu_i}(k) - k \sum_{j<i} s_{ij} + k \sum_{j>i} s_{ij}. \end{aligned} \tag{8}$$

We may again characterize the adjoint of $\overline{L}_p^{\text{trig}}(k)$ in terms of its formal adjoint by Proposition 4.3.

Proposition 4.3. Let A be a rectangular domain. Let $p = \sum_\alpha c_\alpha \mu^\alpha$ be a symmetric function and f and g be symmetric functions on A. If for each non-zero monomial μ^α appearing in p, $\partial_\mu^\beta f$ vanishes on the boundary of A for any $\beta \leq \alpha$, then we have the adjunction relation

$$\int_A (\overline{L}_p^{\text{trig}}(k) f(\mu)) \, \overline{g}(\mu) \, \Delta(e^\mu)^{-2k} d\mu = \int_A f(\mu) \, m(p(T_i(k)^\dagger))(\overline{g}(\mu)) \, \Delta(e^\mu)^{-2k} d\mu.$$

Proof. The proof is the same as for Proposition 3.2. □

4.4. **Matrix elements in the trigonometric case.** Take $l \geq n-1$ and consider variables $\lambda_1, \ldots, \lambda_l$ and $\mu_1, \ldots, \mu_{n-1}$. Recall that $Z_k(e^\mu, e^\lambda)$ denotes the coefficient of $(x_1 \cdots x_l)^k$ in the polynomial

$$\frac{1}{(l-n+1)!} \prod_{j=1}^{l} \left(\sum_{i=1}^{n-1} \frac{x_i}{e^{\mu_i} - e^{\lambda_j}} + x_n + \cdots + x_l \right)^k.$$

We express $Z_k(e^\mu, e^\lambda)$ via trigonometric Calogero-Moser Hamiltonians in Proposition 4.4.

Proposition 4.4. We have the identity

$$Z_k(e^\mu, e^\lambda) = (-1)^{n-1} k!^{-(n-1)} \Delta(e^\mu, e^\lambda)^{-k} \left(e^{-\sum_i \mu_i} \overline{L}^{\text{trig}}_{\prod_{i=1}^{n-1}(\mu_i - k\frac{n-2}{2})}(k)^\dagger\right)^k \Delta(e^\mu, e^\lambda)^k.$$

Proof. We use the result in the rational case. By Proposition 3.4 and (7), it suffices to check that

$$e^{-\sum_i \mu_i}(-1)^{n-1}\Big(T_{\mu_1}(k)^\dagger - k\frac{n-2}{2}\Big)\cdots\Big(T_{\mu_{n-1}}(k)^\dagger - k\frac{n-2}{2}\Big) = D_{e^{\mu_1}}(k)\cdots D_{e^{\mu_{n-1}}}(k)$$

on $\mathbb{C}[e^{\mu_i}]^{S_{n-1}}$. We may rewrite $T_{\mu_i}(k)$ in the form

$$T_{\mu_i}(k) = \partial_{\mu_i} - k\sum_{j\neq i}\frac{e^{\mu_i}}{e^{\mu_i} - e^{\mu_j}}(1 - s_{ij}) - k\sum_{j<i} s_{ij} + k\frac{n-2}{2} = e^{\mu_i} D_{e^{\mu_i}}(k) - k\sum_{j<i} s_{ij} + k\frac{n-2}{2}, \tag{9}$$

where $D_{e^{\mu_i}}(k)$ is the rational Dunkl operator in the exponential variables e^{μ_i}. By (9), we see that

$$D_{e^{\mu_i}}(k) = e^{-\mu_i}\Big(T_{\mu_i}(k) - k\sum_{j<i} s_{ij} + k\frac{n-2}{2}\Big).$$

Further, we may check that $T_{\mu_i}(k)e^{-\mu_j} = e^{-\mu_j}(T_{\mu_i}(k) - ks_{ij})$, so shifting each $e^{-\mu_i}$ term to the beginning of the expression, we see by (9) and (8) that

$$\begin{aligned} D_{e^{\mu_{n-1}}}(k)\cdots D_{e^{\mu_1}}(k) &= e^{-\sum_i \mu_i}\prod_{i=1}^{n-1}\Big(T_{\mu_i}(k) - k\sum_{j<i} s_{ij} + k\sum_{j>i} s_{ij} + k\frac{n-2}{2}\Big) \\ &= e^{-\sum_i \mu_i}(-1)^{n-1}\prod_{i=1}^{n-1}\Big(T_{\mu_i}(k)^\dagger - k\frac{n-2}{2}\Big). \end{aligned}$$

□

4.5. **Proof of Theorem 4.2.** We again compute $\Psi_k(\Lambda, s)$ by integrating over the Liouville tori given by the Gelfand-Tsetlin coordinates. We may write

$$\Psi_k(\Lambda, s) = \int_{\mu\in \mathrm{GT}_\lambda}\int_{t\in T, X_0\in \mathrm{GT}^{-1}(\mu)} F_{k-1}(t\cdot X_0)dt\, e^{\sum_{l=1}^N s_l(\sum_i \mu_i^l - \sum_i \mu_i^{l-1})}\mathrm{GT}_*(d\mu_\Lambda), \tag{10}$$

where dt is the invariant probability measure on the torus, and μ_i^l are the logarithmic Gelfand-Tsetlin coordinates. As in the rational case, by Lemma 2.1, we have

$$\int_{t\in T, X_0\in\mathrm{GT}^{-1}(\mu)} F_{k-1}(t\cdot X_0)dt = \prod_{m=1}^{N-1} W_m,$$

where W_m denotes the coefficient of $(x_1\cdots x_m)^{k-1}$ in $v_m\cdot(x_1\cdots x_m)^{k-1}$. Notice that $(v_m \operatorname{diag}(e^{\mu^{m+1}})v_m^*)_m = \operatorname{diag}(e^{\mu^m})$. By Lemma 3.3, we have

$$W_m = (-1)^{(k-1)(m+3)m/2}\frac{\Delta(e^{\mu^m}, e^{\mu^{m+1}})^{k-1}}{\Delta(e^{\mu^m})^{k-1}\Delta(e^{\mu^{m+1}})^{k-1}} Z_{k-1}(e^{\mu^m}, e^{\mu^{m+1}}).$$

Noting that $\mathrm{GT}_*(d\mu_\Lambda) = 1_{\mathrm{GT}_\lambda}\cdot dx$ by Proposition 2.2 and inducting on N, we transform (10) to

$$\begin{aligned} \Psi_k(\Lambda, s) &= \int_{\mu\in\mathrm{GT}_\lambda}\prod_{m=1}^{N-1} W_m e^{\sum_{l=1}^N s_l(\sum_i \mu_i^l - \sum_i \mu_i^{l-1})}\prod_i d\mu_i^l \\ &= (-1)^{(k-1)(N+2)(N-1)/2}\int_{\mu\prec\lambda}\frac{\Delta(e^\mu, e^\lambda)^{k-1} Z_{k-1}(e^\mu, e^\lambda)}{\Delta(e^\mu)^{k-1}\Delta(e^\lambda)^{k-1}} e^{s_N(\sum_i\lambda_i - \sum_i \mu_i)} \\ &\qquad \prod_{a=1}^{k-1}\prod_{1\leq i<j\leq N-1}(s_i - s_j - a)\Phi_k(\mu, s)\prod_i d\mu_i \\ &= (-1)^{(k-1)(N+2)(N-1)/2}\prod_{a=1}^{k-1}\prod_{1\leq i<j\leq N-1}(s_i - s_j - a)e^{s_N\sum_i\lambda_i} \\ &\qquad \int_{\mu\prec\lambda}\frac{\Delta(e^\mu, e^\lambda)^{k-1}Z_{k-1}(e^\mu, e^\lambda)}{\Delta(e^\mu)^{k-1}\Delta(e^\lambda)^{k-1}}\Phi_k(\mu, s')\prod_i d\mu_i, \end{aligned}$$

where $s' = \Big(s_1 - s_N, \ldots, s_{N-1} - s_N\Big)$ and the last equality follows from the $c = -s_N$ case of (6). By Lemma 4.1 and (6), we see that

$$\Big(\Delta(e^\mu)^{1-k}\overline{L}^{\text{trig}}_{\prod_{i=1}^{N-1}(\mu_i - \frac{(N-2)(k-1)}{2})}(k-1)\Delta(e^\mu)^{k-1}e^{-\sum_i \mu_i}\Big)^{k-1}\Phi_k(\mu, s') = e^{-(k-1)\sum_i \mu_i}\prod_{a=1}^{k-1}\prod_i (s_i - s_N - a)\Phi_k(\mu, s'),$$

so by expressing $Z_{k-1}(e^\mu, e^\lambda)$ using Proposition 4.4 and applying Proposition 4.3 and the shift identity, we obtain the desired expression

$$\begin{aligned}\Psi_k(\Lambda, s) &= (-1)^{(k-1)N(N-1)/2}\prod_{a=1}^{k-1}\prod_{1\le i<j\le N}(s_i - s_j - a)e^{s_N\sum_i \lambda_i - (k-1)\sum_i \mu_i}\Gamma(k)^{-(N-1)}\int_{\mu\prec\lambda}\frac{\Delta(e^\mu, e^\lambda)^{k-1}\Phi_k(\mu, s')}{\Delta(e^\lambda)^{k-1}\Delta(e^\mu)^{k-1}}d\mu \\ &= \prod_{a=1}^{k-1}\prod_{1\le i<j\le N}(s_i - s_j - a)\Gamma(k)^{-(N-1)}\int_{\mu\prec\lambda}e^{s_N(\sum_i \lambda_i - \sum_i \mu_i)}\frac{\Delta(e^\mu, e^\lambda)^{k-1}}{\Delta(e^\lambda)^{k-1}\Delta(e^\mu)^{k-1}}e^{-(k-1)\sum_i \mu_i}\Phi_k(\mu, s)d\mu \\ &= \prod_{a=1}^{k-1}\prod_{1\le i<j\le N}(s_i - s_j - a)\Phi_k(\lambda, s).\end{aligned}$$

5. Proofs of some technical lemmas

5.1. **Proof of Lemma 3.3.** We verify the statement by direct computation. Write $u = u(\mu, \lambda)$ and $\lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Define the non-negative real numbers $x_1, \ldots, x_{n-1}$ by

$$x_i^2 = -\frac{\prod_j(\lambda_j - \mu_i)}{\prod_{j\ne i}(\mu_j - \mu_i)},$$

where we note that the right side of the definition is non-negative because λ and μ interlace. Define $y = \sum_i \lambda_i - \sum_i \mu_i$. For $i < n$, our definition of u implies that

$$u_{ij} = \frac{x_i}{\lambda_j - \mu_i}u_{nj}. \tag{11}$$

We first claim that $u\lambda = \mu' u$ for the matrix

$$\mu' = \left(\begin{array}{ccccc|c} \mu_1 & & & & & x_1 \\ & \mu_2 & & & & x_2 \\ & & \ddots & & & \vdots \\ & & & \mu_{n-2} & & x_{n-2} \\ & & & & \mu_{n-1} & x_{n-1} \\ \hline x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} & y \end{array}\right).$$

For $i < n$, this holds for each element of row i by the equality

$$\lambda_j u_{ij} = \mu_j u_{ij} + x_i u_{nj}$$

implied by (11). For row n, we must check that

$$\lambda_j u_{nj} = \sum_{i=1}^{n-1} x_i u_{ij} + y u_{nj} = \left(y + \sum_{i=1}^{n-1}\frac{x_i^2}{\lambda_j - \mu_i}\right)u_{nj},$$

for which it suffices to check that

$$\sum_{i=1}^{n-1}\frac{\prod_{l\ne j}(\lambda_l - \mu_i)}{\prod_{l\ne i}(\mu_l - \mu_i)} = \sum_{i\ne j}\lambda_i - \sum_i \mu_i. \tag{12}$$

The left side of (12) is a symmetric rational function in the μ_i which may be expressed as a quotient

$$\frac{P(\mu)}{\prod_{i<j}(\mu_i - \mu_j)},$$

whose numerator $P(\mu)$ has degree at most $\frac{n(n-1)}{2} + 1$ in the μ-variables. Therefore, $P(\mu)$ is antisymmetric, meaning the quotient is symmetric of degree at most 1. In particular, it takes the form $C_1 + C_2\sum_i \mu_i$ for

C_1 and C_2 constant in μ. Noting that the coefficient of $\mu_1^{n-1}\mu_2^{n-3}\mu_3^{n-2}\cdots\mu_{n-2}$ in $P(\mu)$ is -1 shows that $C_2 = -1$. Finally, C_1 is a polynomial of degree 1 in λ, so it is given by

$$C_1 = \sum_i \frac{\mu_i^{n-2}(-1)^{n-2}\sum_{l\neq j}\lambda_l}{\prod_{l\neq i}(\mu_l - \mu_i)} = \sum_i \frac{\mu_i^{n-2}}{\prod_{l\neq i}(\mu_i - \mu_l)} \cdot \left(\sum_{l\neq j}\lambda_l\right) = \sum_{i\neq j}\lambda_i,$$

where the last equality follows by noting that $\sum_i \frac{\mu_i^{n-2}}{\prod_{l\neq i}(\mu_i-\mu_l)}$ is symmetric of degree 0 in μ and a rational function whose denominator is $\prod_{i<j}(\mu_i - \mu_j)$ and whose numerator contains $\mu_1^{n-2}\mu_2^{n-3}\cdots\mu_{n-2}$ with coefficient 1. This establishes (12).

It remains to check that u is unitary. For this, we check that the columns of u are orthonormal. Choose any $1 \leq a < b \leq n$. We have that

$$\sum_i u_{ia}u_{ib} = \left(\sum_i \frac{x_i^2}{(\lambda_a - \mu_i)(\lambda_b - \mu_i)} + 1\right) u_{na}u_{nb} = \left(1 - \sum_i \frac{\prod_{j\neq a,b}(\lambda_j - \mu_i)}{\prod_{j\neq i}(\mu_j - \mu_i)}\right) u_{na}u_{nb}.$$

Observe that $\sum_i \frac{\prod_{j\neq a,b}(\lambda_j-\mu_i)}{\prod_{j\neq i}(\mu_j-\mu_i)}$ is symmetric in the μ_i and may be expressed as a rational function with denominator $\prod_{i<j}(\mu_i - \mu_j)$ and numerator of degree at most $\frac{n(n-1)}{2}$ in μ. Further, the coefficient of $\mu_1^{n-2}\mu_2^{n-3}\cdots\mu_{n-2}$ in the numerator is 1, so we conclude that

$$1 - \sum_i \frac{\prod_{j\neq a,b}(\lambda_j - \mu_i)}{\prod_{j\neq i}(\mu_j - \mu_i)} = 0, \tag{13}$$

hence $\sum_i u_{ia}u_{ib} = 0$. It remains only to show that

$$1 = \sum_i u_{ia}^2 = \left(1 + \sum_i \frac{x_i^2}{(\lambda_a - \mu_i)^2}\right) u_{na}^2,$$

for which we must check that

$$\frac{\prod_{l\neq a}(\lambda_l - \lambda_a)}{\prod_l(\mu_l - \lambda_a)} = 1 - \sum_i \frac{\prod_{j\neq a}(\lambda_j - \mu_i)}{(\lambda_a - \mu_i)\prod_{j\neq i}(\mu_j - \mu_i)},$$

which is equivalent to

$$\prod_{l\neq a}(\lambda_l - \lambda_a) = \prod_l(\mu_l - \lambda_a)\left(1 - \sum_i \frac{\prod_{j\neq a}(\lambda_j - \mu_i)}{(\lambda_a - \mu_i)\prod_{j\neq i}(\mu_j - \mu_i)}\right). \tag{14}$$

View both sides of (14) as polynomials in λ_a. If $\lambda_a = \lambda_b$ for $b \neq a$, the right side becomes

$$1 - \sum_i \frac{\prod_{j\neq a,b}(\lambda_j - \mu_i)}{\prod_{j\neq i}(\mu_j - \mu_i)} = 0$$

by (13). Therefore, both sides of (14) are polynomials in λ_a of the same degree with the same roots and the same leading coefficient $(-1)^{n-1}$, so they are equal, completing the proof.

Remark. The expressions above for x_i^2 and y appeared previously in [Ner03].

5.2. **Proof of Proposition 3.4.** Before beginning the proof, we outline our approach. We first obtain an alternate expression for $Z_1(\mu, \lambda)$ in Lemma 5.1. We then observe that $Z_k(\mu, \lambda)$ is a constant multiple of $Z_1(\mu', \lambda')$ for sets of variables μ' and λ' which contain k duplicate copies of each value of μ and λ. Relating Calogero-Moser Hamiltonians at different values of k in Lemma 5.2 leads to the result.

Lemma 5.1. For any $\kappa \in \mathbb{C}$, we have

$$\Delta(\mu, \lambda)^{-\kappa} D_{\mu_{n-1}}(\kappa)\cdots D_{\mu_1}(\kappa)\Delta(\mu, \lambda)^{\kappa} = \kappa^{n-1} Z_1(\mu, \lambda).$$

Proof. We first claim that

$$\Delta(\mu, \lambda)^{-\kappa} D_{\mu_a}(\kappa)\cdots D_{\mu_1}(\kappa)\Delta(\mu, \lambda)^{\kappa} = \kappa^a \sum_{\substack{\sigma:\{1,\ldots,a\}\\ \to\{1,\ldots,l\}\\ \sigma(i)\neq\sigma(j)}} \prod_{i=1}^a (\mu_i - \lambda_{\sigma(i)})^{-1}. \tag{15}$$

Taking $a = n-1$ in (15) and expanding the product in the definition of $Z_1(\mu, \lambda)$ then completes the proof. We prove (15) by induction on a. The base case $a = 1$ holds because $D_{\mu_1}(\kappa)$ acts by ∂_1 on the symmetric function $\Delta(\mu, \lambda)^\kappa$ in μ. For the induction step, note that $D_{\mu_a}(\kappa) \cdots D_{\mu_1}(\kappa)\Delta(\mu, \lambda)^\kappa$ is symmetric in $\mu_{a+1}, \dots \mu_{n-1}$ by the inductive hypothesis. Applying $D_{\mu_{a+1}}$, we see that

$$\begin{aligned}
&\Delta(\mu, \lambda)^\kappa D_{\mu_{a+1}}(\kappa)(D_{\mu_a}(\kappa) \cdots D_{\mu_1}(\kappa)\Delta(\mu, \lambda)^\kappa) \\
&= \kappa^{a+1} \sum_{j=1}^{l} (\mu_{a+1} - \lambda_j)^{-1} \sum_{\substack{\sigma:\{1,\dots,a\} \\ \to\{1,\dots,l\} \\ \sigma(i)\neq\sigma(j)}} \prod_{i=1}^{a} (\mu_i - \lambda_{\sigma(i)})^{-1} - \kappa^{a+1} \sum_{\substack{\sigma:\{1,\dots,a\} \\ \to\{1,\dots,l\} \\ \sigma(i)\neq\sigma(j)}} \prod_{i=1}^{a} (\mu_i - \lambda_{\sigma(i)})^{-1} \sum_{i=1}^{a} (\mu_{a+1} - \lambda_{\sigma(i)})^{-1} \\
&= \kappa^{a+1} \sum_{\substack{\sigma:\{1,\dots,a+1\} \\ \to\{1,\dots,l\} \\ \sigma(i)\neq\sigma(j)}} \prod_{i=1}^{a+1} (\mu_i - \lambda_{\sigma(i)})^{-1},
\end{aligned}$$

where we repeatedly make use of the identity

$$\frac{1}{\mu_{a+1} - \mu_i}\Big((\mu_{a+1} - \lambda_j) - (\mu_i - \lambda_j)\Big) = 1. \qquad \square$$

Proof of Proposition 3.4. Replace l by kl and apply Lemma 5.1 with $\kappa = \frac{1}{k}$, k copies of each λ_j, and $k(n-1)$ different variables $\mu_1^1, \dots, \mu_1^k, \dots, \mu_{n-1}^1, \dots, \mu_{n-1}^k$. We obtain

$$\Delta(\{\mu_i^j\}, \{\lambda_i\})^{-1} D_{\mu_{n-1}^k}(1/k) \cdots D_{\mu_1^1}(1/k)\Delta(\{\mu_i^j\}, \{\lambda_i\}) = k^{-(n-1)k} Z_1(\{\mu_i^j\}, \{\lambda_i^j\}). \tag{16}$$

Now, make the specialization $\mu_1^1 = \cdots = \mu_1^k = \mu_1, \dots, \mu_{n-1}^1 = \cdots = \mu_{n-1}^k = \mu_{n-1}$. We first claim that

$$Z_1(\{\mu_i^j\}, \{\lambda_i^j\}) = k!^{n-1} Z_k(\{\mu_i\}, \{\lambda_i\})$$

under this specialization. Indeed, we see that

$$\begin{aligned}
Z_1(\{\mu_i^j\}, \{\lambda_i^j\}) &= \sum_{\substack{\sigma:\{1,\dots,(n-1)\}\times\{1,\dots,k\} \\ \to\{1,\dots,l\}\times\{1,\dots,k\} \\ \sigma(i_1,j_1)\neq\sigma(i_2,j_2)}} \prod_{i,j} (\mu_i^j - \lambda_{\sigma(i,j)_1}^{\sigma(i,j)_2})^{-1} \\
&= \sum_{\substack{\sigma^1,\dots,\sigma^{n-1}\subset\{1,\dots,l\}\times\{1,\dots,k\} \\ |\sigma^i|=k \\ \sigma^i\cap\sigma^j=\emptyset}} k!^{n-1} \prod_i \prod_{(j,p)\in\sigma^i} (\mu_i - \lambda_j^p)^{-1} \\
&= k!^{n-1} \sum_{\substack{\sigma_1^1,\dots,\sigma_l^1,\dots,\sigma_1^{n-1},\dots,\sigma_l^{n-1} \\ \sum_j \sigma_j^i = k \\ \sum_i \sigma_j^i \leq k}} \prod_i \prod_j \binom{k}{\sigma_j^1, \dots, \sigma_j^{n-1}} (\mu_i - \lambda_j)^{-\sigma_j^i},
\end{aligned}$$

which is a direct expansion of $Z_k(\{\mu_i\}, \{\lambda_i\})$. The conclusion will now follow from Lemma 5.2, which describes what occurs under specialization to the other side of Lemma 5.1. Indeed, applying Lemma 5.2 for $p(y) = y_1^1 \cdots y_{n-1}^k$ to (16), we see that

$$\begin{aligned}
Z_k(\{\mu_i\}, \{\lambda_i\}) &= k!^{-(n-1)} k^{(n-1)k} k^{-(n-1)k} \Delta(\{\mu_i\}, \{\lambda_i\})^{-k} D_{\mu_{n-1}}(k)^k \cdots D_{\mu_1}(k)^k \Delta(\{\mu_i\}, \{\lambda_i\})^k \\
&= k!^{-(n-1)} \Delta(\{\mu_i\}, \{\lambda_i\})^{-k} D_{\mu_{n-1}}(k)^k \cdots D_{\mu_1}(k)^k \Delta(\{\mu_i\}, \{\lambda_i\})^k. \qquad \square
\end{aligned}$$

Lemma 5.2. Let $p \in \mathbb{C}[y_1^1, \dots, y_{n-1}^k]^{S_{k(n-1)}}$ be a symmetric polynomial. Then the map $\mathrm{Res}_k : \mathbb{C}[\mu_i^j] \to \mathbb{C}[\mu_i]$ given by $\mu_i^j \mapsto \mu_i$ satisfies

$$\begin{aligned}
&\mathrm{Res}_k \circ p(D_{\mu_1^1}(1/k), \dots, D_{\mu_{n-1}^k}(1/k)) \\
&= p\Big(\frac{1}{k} D_{\mu_1}(k), \dots, \frac{1}{k} D_{\mu_1}(k), \dots, \frac{1}{k} D_{\mu_{n-1}}(k), \dots, \frac{1}{k} D_{\mu_{n-1}}(k)\Big) \circ \mathrm{Res}_k.
\end{aligned}$$

Proof. Let $H_{1/k,(n-1)k}$ and $H_{k,(n-1)}$ denote the rational Cherednik algebras of $S_{(n-1)k}$ and S_{n-1}, respectively. Within $H_{1/k,(n-1)k}$ and $H_{k,(n-1)}$, denote the power sums $p_a(x) = \sum_{i,j}(x_i^j)^a$ and $p'_a(x) = \sum_i x_i^a$, and define $p_a(y), p'_a(y)$ similarly. Write $\Theta_{1/k,(n-1)k} : H_{1/k,(n-1)k} \to \mathrm{End}(\mathbb{C}[\mu_i^j])$ and $\Theta_{k,n-1} : H_{k,n-1} \to \mathrm{End}(\mathbb{C}[\mu_i])$ for the Dunkl embeddings induced by $\Theta_{1/k,(n-1)k}(x_i^j) = \mu_i^j$, $\Theta_{1/k,(n-1)k}(y_i^j) = D_{\mu_i^j}(1/k)$, $\Theta_{k,n-1}(x_i) = kx_i$, and $\Theta_{k,n-1}(y_i) = \frac{1}{k}D_{\mu_i}(k)$. In this language, we wish to show that

$$\mathrm{Res}_k \circ \Theta_{1/k,(n-1)k}(p_a(y)) = \Theta_{k,n-1}(p'_a(y)) \circ \mathrm{Res}_k. \tag{17}$$

Suppose first that the statement held for $p_2(y)$. Then, we have for any a that

$$\mathrm{Res}_k \circ \Theta_{1/k,(n-1)k}(\mathrm{ad}^a_{p_2(y)} p_a(x)) = \Theta_{k,(n-1)}(\mathrm{ad}^a_{p'_2(y)} p'_a(x)) \circ \mathrm{Res}_k \tag{18}$$

Recall that for $h = \frac{1}{2}\sum_{i,j}(x_{i,j}y_{i,j} + y_{i,j}x_{i,j})$ and $h' = \frac{1}{2}\sum_i(x_iy_i + y_ix_i)$, the triples

$$(f, e, h) = \left(\frac{1}{2}p_2(y), -\frac{1}{2}p_2(x), h\right) \qquad \text{and} \qquad (f', e', h') = \left(\frac{1}{2}p'_2(y), -\frac{1}{2}p'_2(x), h'\right)$$

are copies of $\mathfrak{sl}_2$ inside $H_{1/k,(n-1)k}$ and $H_{k,n-1}$ corresponding to the $SL_2(\mathbb{C})$-actions given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x_i = ax_i + by_i, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} y_i = cx_i + dy_i,$$

and similar formulas for x_i^j, y_i^j. In particular, $p_a(x)$ and $p'_a(x)$ are highest weight vectors of weight a for these representations, meaning that $\mathrm{ad}^a_{p_2(y)/2} p_a(x)$ and $\mathrm{ad}^a_{p'_2(y)/2} p'_a(x)$ are given by the same fixed constant multiple of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p_a(x) = p_a(y) \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p'_a(x) = p'_a(y),$$

respectively. Combining with (18) and canceling common constant factors yields the desired relation (17).

It remains to check the statement for $p_2(y)$ directly. Observe that

$$\mathrm{Res}_k \circ \sum_j \partial_{\mu_i^j} = \partial_{\mu_i} \circ \mathrm{Res}_k,$$

which implies that

$$\mathrm{Res}_k\left(\sum_{j_1,j_2} \frac{\partial_{\mu_{i_1}^{j_1}} - \partial_{\mu_{i_2}^{j_2}}}{\mu_{i_1}^{j_1} - \mu_{i_2}^{j_2}} f\right) = k\frac{\partial_{\mu_{i_1}} - \partial_{\mu_{i_2}}}{\mu_{i_1} - \mu_{i_2}} \mathrm{Res}_k(f). \tag{19}$$

Secondly, for any partition τ with at most k parts, let $m_\tau(\mu_i^j)$ be the monomial symmetric function in $\mu_i^1, \ldots, \mu_i^k$. Then we see that

$$\begin{aligned}
&\mathrm{Res}_k\left(\Big(\sum_j \partial^2_{\mu_i^j} - \frac{2}{k}\sum_{j_1<j_2} \frac{\partial_{\mu_i^{j_1}} - \partial_{\mu_i^{j_2}}}{\mu_i^{j_1} - \mu_i^{j_2}}\Big) m_\tau(\mu_i^j)\right) \\
&\quad = \left(\sum_j \tau_j(\tau_j - 1) - \frac{2}{k}\sum_{j_1<j_2}\frac{1}{2}\Big(\tau_{j_1}(\tau_{j_1} - 1 - \tau_{j_2}) + \tau_{j_2}(\tau_{j_2} - 1 - \tau_{j_1})\Big)\right) k!\, \mu_i^{|\tau|-2} \\
&\quad = \left(\frac{1}{k}\sum_i \tau_i(\tau_i - 1) + \frac{2}{k}\sum_{j_1<j_2} \tau_{j_1}\tau_{j_2}\right)(\mu_i^j)^{-2}\mathrm{Res}_k(\mu_\lambda(\mu_i^j)) \\
&\quad = \frac{1}{k}|\tau|(|\tau| - 1)(\mu_i^j)^{-2}\mathrm{Res}_k(m_\tau(\mu_i^j)) \\
&\quad = \frac{1}{k}\partial^2_{\mu_i}\mathrm{Res}_k(m_\tau(\mu_i^j)).
\end{aligned} \tag{20}$$

Combining (19) and (20), the statement for $p_2(y)$ follows by computing

$$\begin{aligned}
\mathrm{Res}_k \circ \overline{L}_{p_2}(1/k) &= \mathrm{Res}_k \circ \left(\sum_{i,j} \partial^2_{\mu_i^j} - \frac{2}{k} \sum_{(i_1,j_1)<(i_2,j_2)} \frac{\partial_{\mu_{i_1}^{j_1}} - \partial_{\mu_{i_2}^{j_2}}}{\mu_{i_1}^{j_1} - \mu_{i_2}^{j_2}} \right) \\
&= \mathrm{Res}_k \circ \left(\sum_i \left(\sum_j \partial^2_{\mu_i^j} - \frac{2}{k} \sum_{j_1<j_2} \frac{\partial_{\mu_i^{j_1}} - \partial_{\mu_i^{j_2}}}{\mu_i^{j_1} - \mu_i^{j_2}} \right) - \frac{2}{k} \sum_{i_1 \neq i_2} \sum_{j_1, j_2} \frac{\partial_{\mu_{i_1}^{j_1}} - \partial_{\mu_{i_2}^{j_2}}}{\mu_{i_1}^{j_1} - \mu_{i_2}^{j_2}} \right) \\
&= \frac{1}{k} \left(\sum_i \partial^2_{\mu_i} - 2k \sum_{i_1 \neq i_2} \frac{\partial_{\mu_{i_1}} - \partial_{\mu_{i_2}}}{\mu_{i_1} - \mu_{i_2}} \right) \circ \mathrm{Res}_k \\
&= \frac{1}{k} \overline{L}_{p_2}(k) \circ \mathrm{Res}_k. \qquad \square
\end{aligned}$$

Remark. Lemma 5.2 may also be extracted from [CEE09, Proposition 9.5(ii)] relating representations of the rational Cherednik algebras $H_{1/k}(S_{(n-1)k})$ and $H_k(S_{n-1})$. We include the direct proof here to keep the exposition self-contained.

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