Statistics 251, Autumn 2020 — Homework 6

Due date: 11:30am on Monday, November 9, 2020 on Gradescope.

Instructions: Please present your solutions in a legible, coherent manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Points may be deducted for unclear or messy solutions.

Collaboration and Academic Integrity: You are encouraged to collaborate on homework. However, you must write your solutions alone and **understand what you write**. When submitting your homework, list in the space below any sources you used (in print, online, or human) other than the textbook or the teaching staff.

1. [10pts] Alice figures that the total number of thousands of miles that an auto can be driven before it would need to be junked is an exponential random variable with parameter 20. Bob has a used car that he claims has been driven only 10000 miles. If Alice purchases the car, what is the probability that she would get at least 20000 additional miles out of it? What if the lifetime mileage of the car is not exponentially distributed, but rather is (in thousands of miles) uniformly distributed over $(0, 40)$?
2. [10pts] Let $X_1, \cdots, X_n$ be independent exponential random variables having a common parameter λ. Determine the distribution of $\min(X_1, \cdots, X_n)$.
3. [10pts] Given $X \sim \text{Gamma}(\alpha, \lambda)$, calculate $\mathbb{E}(X^2)$ and then $\text{Var}(X)$.
4. [10pts] If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$.
5. [10pts] Let X be a continuous random variable having cumulative distribution function F. Define the random variable Y by $Y = F(X)$. Show that Y is uniformly distributed over $(0, 1)$.
6. [10pts] The joint probability density function of X and Y is given by

   $$f(x, y) = c(y^2 - x^2)e^{-y} \qquad -y \le x \le y, 0 < y < \infty$$

   (a) Find c,
   (b) Find the marginal densities of X and Y,
   (c) Find $\mathbb{E}[X]$.
7. [10pts] Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of
   (a) X_1, X_2;
   (b) X_1, X_2, X_3.
8. [10pts] The joint density of X and Y is given by

   $$f(x, y) = \begin{cases} xe^{-x-y} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

   Are X and Y independent? If, instead, f were given by

   $$f(x, y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   would X and Y be independent?