Statistics 251, Autumn 2020 — Homework 9

Due date: 11:30am on Friday, December 4, 2020 on Gradescope.

Instructions: Please present your solutions in a legible, coherent manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Points may be deducted for unclear or messy solutions.

Collaboration and Academic Integrity: You are encouraged to collaborate on homework. However, you must write your solutions alone and **understand what you write**. When submitting your homework, list in the space below any sources you used (in print, online, or human) other than the textbook or the teaching staff.

1. [10pts] Let $X_1, \ldots, X_{20}$ be independent Poisson random variables with mean 1.
   a. Use Markov's inequality to obtain a bound on
   $$\mathbb{P}\left(\sum_1^{20} X_i > 15\right).$$
   b. Use the Central Limit Theorem to approximate
   $$\mathbb{P}\left(\sum_1^{20} X_i > 15\right).$$
2. [10pts] Civil engineers believe that W, the amount of weight (in units of 1000 pounds) that a certain span of a bridge can withstand without structural damage, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 pounds) of a car is a random variable with mean 3 and standard deviation 0.3. Approximately how many cars would have to be on the bridge span for the probability of structural damage to exceed 0.1?
3. [10pts] We use 100 components in a sequential fashion. That is, component 1 is initially put in use, and upon failure, it is replaced by component 2, which is itself replaced upon failure by component 3, and so on. If the lifetime of component i is exponentially distributed with mean $\frac{i+10}{10}$, $i = 1, \ldots, 100$, estimate the probability that the total life of all components will exceed 1200.

   Repeat this computation if the lifetime of component i is uniform over $(0, 20+i/5)$, $i = 1, \ldots, 100$.
4. [10pts] A die is continually rolled until the total sum of all rolls exceeds 300. Approximate the probability that at least 80 rolls are necessary.
5. [10pts] Explain why a gamma random variable with parameters (t, λ) has an approximately normal distribution when t is large.
6. [10pts] Find the moment generating function of the normal distribution with parameter (μ, σ^2).
7. [10pts] Consider random variables $X_n : [0,1] \to \{0,1\}$ defined as follows. Let $\Omega = [0,1]$ be the sample space. For $\omega \in [0,1]$, define for certain intervals A_n the random variables by
   $$X_n(\omega) = \begin{cases} 1 & \text{if } \omega \in A_n, \\ 0 & \text{otherwise.} \end{cases}$$
   The intervals $\{A_n\}$ are defined as follows. The first 10 intervals are defined by $A_1 = [0, 0.1]$, $A_2 = [0.1, 0.2]$, and so on, with each interval of width 0.1 beginning at the right boundary of the previous interval until we hit $A_{10} = [0.9, 1]$. The next intervals will repeat the pattern with half the width so that $A_{11} = [0, 0.05]$, $A_{12} = [0.05, 0.1]$ and so on. Prove that
   a. A_n converges to 0 in probability.
   b. A_n does not converge to 0 almost surely.
8. [10pts] The strong law of large numbers states that, with probability 1, the successive arithmetic averages of a sequence of independent and identically distributed random variables converge to their expectation μ. What do the successive geometric averages converge to? That is, what does the limit
   $$\lim_{n\to\infty} \left(\prod_{i=1}^n X_i\right)^{1/n}$$
   evaluate to?