Mathematics UN1102 Name: ____________________
Section 1, Fall 2019
Midterm 1 UNI: ____________________
October 7, 2019
Time Limit: 75 Minutes

Instructions: This exam contains 7 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the back side of the problem sheets or the blank pages stapled to the end of the exam.

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Question	1	2	3	4	5	6	7	Total
Points	10	15	15	15	15	20	15	100
Score								

Formulas

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$$

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Problem 1 (10 points) Evaluate the definite integral

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx.$$

Answer: $\boxed{\frac{4}{3}}$

Substitute $u = x^3 + 1$ to obtain $du = 3x^2 dx$ and hence

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}} dx = \frac{1}{3}\int_1^9 \frac{du}{\sqrt{u}} = \frac{1}{3}\left[2\sqrt{u}\right]_1^9 = \frac{4}{3}.$$

Problem 2 (15 points) Evaluate the indefinite integral

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx.$$

Answer: $\boxed{\frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C}$

Substitute $x = \sin\theta$ so that $dx = \cos\theta d\theta$. We obtain

$$\int \frac{x^2}{\sqrt{1-x^2}}dx = \int \sin^2\theta d\theta = \int \frac{1-\cos 2\theta}{2}d\theta = \frac{1}{2}\theta - \frac{\sin(2\theta)}{4} + C = \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C.$$

Problem 3 (15 points) Evaluate the indefinite integral

$$\int x \ln x \; dx.$$

Answer: $\boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$

Applying integration by parts with $f(x) = \ln x$ and $g'(x) = x$, we have $g(x) = \frac{1}{2}x^2$ and $f'(x) = \frac{1}{x}$, so we obtain

$$\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C.$$

Problem 4 (15 points) Evaluate the indefinite integral

$$\int \frac{x+2}{x^3+2x^2+2x} dx.$$

Hint: It may help to first write a partial fraction decomposition of the integrand.

Answer: $\boxed{\ln x + \frac{1}{2}\ln|x^2+2x+2| + C}$

We factor the denominator as $x^3 + 2x^2 + 2x = x(x^2 + 2x + 2) = x((x + 1)^2 + 1)$. We now solve for

$$\frac{x+2}{x^3+2x^2+2x} = \frac{A}{x} + \frac{B(x+1)+C}{(x+1)^2+1}.$$

Multiplying both sides by $x^3 + 2x^2 + 2x$, we find that

$$x + 2 = A((x + 1)^2 + 1) + B(x^2 + x) + Cx = (A + B)x^2 + (2A + B + C)x + 2A.$$

Matching coefficients, this implies that $A = 1$, hence $B = -1$ and $C = 0$. We conclude that

$$\int \frac{x+2}{x^3+2x^2+2x} dx = \int \frac{1}{x} + \frac{x+1}{(x+1)^2+1} dx = \ln x + \frac{1}{2}\ln|x^2+2x+2| + C.$$

Problem 5 (15 points) Consider the improper integral

$$\int_0^2 e^x x^{-2}\, dx.$$

(a) (5 points) Write the definition of this improper integral as a limit.

Answer: $\boxed{\lim_{a\to 0^+} \int_a^2 e^x x^{-2} dx}$

(b) (10 points) Determine whether the improper integral converges or diverges and explain why. If the improper integral converges, compute its value.

Answer: $\boxed{\text{diverges}}$

Notice that $e^x x^{-2} \geq x^{-2} \geq 0$ and that $\int_0^2 x^{-2} dx$ diverges because $-2 \leq -1$. Therefore, the integral diverges by the comparison theorem.

Problem 6 (20 points) Let A be the area enclosed by the graph of $(x-2)^2 + y^2 \leq 1$.

(a) (10 points) Sketch A (Hint: it is a circle), and set up a definite integral to compute its area. You **do not** need to evaluate the integral.

Answer: $\boxed{\text{Picture omitted; area is given by } \int_1^3 2\sqrt{1-(x-2)^2}dx}$

(b) (10 points) Compute the volume of the solid of revolution obtained by rotating A about the y-axis. State whether you are using the method of disks/washers or the method of cylindrical shells.

Answer: $\boxed{4\pi^2}$

Using cylindrical shells, we have

$$V = \int_1^3 2\pi x \cdot 2\sqrt{1-(x-2)^2}dx = 4\pi \int_{-1}^1 (u+2)\sqrt{1-u^2}du = 4\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta + 2)\cos^2\theta d\theta$$

$$= 4\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\cos^2\theta + \cos(2\theta) + 1 d\theta = 4\pi \left[-\frac{1}{3}\cos^3\theta + \frac{1}{2}\sin(2\theta) + \theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\pi^2.$$

Using washers, we have

$$V = \int_{-1}^1 \pi\Big((2+\sqrt{1-y^2})^2 - (2-\sqrt{1-y^2})^2\Big)dy = \int_{-1}^1 8\pi\sqrt{1-y^2}dy$$

$$= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta = 4\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2\theta) + 1 d\theta = 4\pi \left[\frac{1}{2}\sin(2\theta) + \theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\pi^2.$$

Problem 7 (15 points) Compute the length of the graph of

$$f(x) = \frac{2}{3}x^{\frac{3}{2}}$$

between $x = 0$ and $x = 1$.

Answer: $\boxed{\frac{4\sqrt{2}}{3} - \frac{2}{3}}$

The arc length is given by

$$\int_0^1 \sqrt{1 + f'(x)^2} dx = \int_0^1 \sqrt{1+x} dx = \left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_0^1 = \frac{4\sqrt{2}}{3} - \frac{2}{3}.$$