Problem Set 8 Solutions

Note: Thanks to Kevin Lee for some of the solutions.

1. (a) Notice that

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{\pi}\left(1-x^2-y^2\right)\,dy = \frac{2}{\pi}\left(2\sqrt{1-x^2}\left(1-x^2\right) - \frac{2\left(1-x^2\right)^{\frac{3}{2}}}{3}\right) = \boxed{\frac{8}{3\pi}\left(1-x^2\right)^{\frac{3}{2}}}$$

where $x \in [-1,1]$. Now, we have that

$$F_X(x) = \int_{-1}^{x} f_X(t)\,dt = \frac{8}{3\pi}\int_{\sin^{-1}(-1)}^{\sin^{-1}(x)} \left(1-\sin^2 u\right)^{\frac{3}{2}}\cos u\,du = \frac{8}{3\pi}\int_{-\pi/2}^{\sin^{-1}(x)} \cos^4 u\,du.$$

Now, to find a better formula for $\cos^4 u$, note that we have $\cos 2u = 2\cos^2 u - 1$ and $\cos 4u = 2\cos^2 2u - 1 = 8\cos^4 u - 8\cos^2 u + 2 - 1$. Thus, we can solve a set of linear equations to get that $\cos^4 u = \frac{\cos 4u}{8} + \frac{\cos 2u}{2} + \frac{3}{8}$. Our integral therefore evaluates to

$$F_X(x) = \frac{8}{3\pi}\left(\frac{3\sin^{-1}x}{8} + \frac{\sin\left(2\sin^{-1}x\right)}{4} + \frac{\sin\left(4\sin^{-1}x\right)}{32} - \frac{-3\pi}{16}\right) = \boxed{\frac{1}{2} + \frac{\sin^{-1}x}{\pi} + x\sqrt{1-x^2}\left(\frac{5}{3\pi} - \frac{2}{3\pi}x^2\right)}.$$

Here we've used the fact that the terms with sin become $\sin 2u = 2\sin u\cos u$ and $\sin 4u = 4\sin u\cos u\left(1-2\sin^2 u\right)$, and if $u = \sin^{-1}x$, we get $2x\sqrt{1-x}$ and $4x\sqrt{1-x^2}\left(1-2x^2\right)$ respectively.

(b) We may compute:

$$f_{Y|X}(y,x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{2}{\pi}\left(1-x^2-y^2\right)}{\frac{8}{3\pi}\left(1-x^2\right)^{\frac{3}{2}}} = \boxed{\frac{3}{4}\left(1-x^2-y^2\right)\left(1-x^2\right)^{-\frac{3}{2}}}.$$

(c) We have the reverse transformation $x = x$ and $y = \pm\sqrt{r^2-x^2}$. The Jacobian is therefore $\frac{2r}{\sqrt{r^2-x^2}}$, where we add a factor of 2 to account for the fact that y is a doubly-valued function of x. So we find

$$f_{X,R}(x,r) = \frac{4r\left(1-r^2\right)}{\pi\sqrt{r^2-x^2}}$$

and

$$f_{R|X}(r,x) = \frac{4r\left(1-r^2\right)}{\pi\sqrt{r^2-x^2}} \cdot \frac{3\pi}{8\left(1-x^2\right)^{\frac{3}{2}}} = \boxed{\frac{3}{2}r\left(1-r^2\right)\left(r^2-x^2\right)^{-1}\left(1-x^2\right)^{-\frac{3}{2}}}.$$

2. First, we get that $f_{X,Y}(x,y) = f_X(x)\,f_Y(y) = \frac{\lambda^5}{2}xy^2e^{-\lambda(x+y)}$. Since we have that $X = UV$ and $Y = U(1-V)$, the Jacobian works out to be $\frac{\partial X}{\partial U}\frac{\partial Y}{\partial V} - \frac{\partial X}{\partial V}\frac{\partial Y}{\partial U} = -UV - U(1-V) = -U$. Thus, we have that

$$f_{U,V}(u,v) = \boxed{\frac{\lambda^5}{2}u^4 v\left(1-v\right)^2 e^{-\lambda u}}.$$

Note that since $X, Y > 0$, we have the bounds $U > 0$ and $0 < V < 1$ which are not affected by each other. Thus, we find

$$f_U(u) = \int_0^1 f_{U,V}(u,v)\,dv = \frac{\lambda^5}{2}u^4e^{-\lambda u}\int_0^1 v\left(1-v\right)^2 dv = \boxed{\frac{\lambda^5}{24}u^4e^{-\lambda u}}.$$

We also get that $f_V(v) = \int_0^\infty f_{U,V}(u,v)\,du = \frac{\lambda^5}{2} v(1-v)^2 \int_0^\infty u^4 e^{-\lambda u} du$. Repeatedly evaluating the last integral by parts gives us that

$$f_V(v) = \frac{24\lambda^5}{2} v(1-v)^2 \frac{1}{\lambda^5} = \boxed{12v(1-v)^2}.$$

Thus, we verify that $f_U(u) f_V(v) = f_{U,V}(u,v)$, so the variables are independent.

3. Consider a fixed R. The line AR divides the triangle into two smaller triangles. Either P and Q are in the same smaller triangle, which happens with probability r^2 and $(1-r)^2$ where r is the position of R on BC, or one is on each side. For the former case, we can simply reduce down to the expected area of the triangle made with two random points and a vertex. From the course notes this is $\frac{4}{27}r$ or $\frac{4}{27}(1-r)$. Otherwise, let P and Q be vectors with the origin at R. The area is simply the cross product, or $|P_x Q_y - Q_x P_y|$. Based on the properties of expectation, and the fact that P and Q are chosen independently, this means that the expected area is $\mathbb{E}(P_x)\mathbb{E}(Q_y) - \mathbb{E}(Q_x)\mathbb{E}(P_y)$. But this is just the area of the triangle formed by the centroids of triangles ABR and ARC and R. To compute this area, we extend the lines from R to the centroids until they touch AB and AC respectively. The area of that quadrilateral is half the area of the original triangle, because the two lines are medians and cut the area of the smaller triangles in half. Then, the area of the top triangle is a quarter of the total area. Thus, we are left with $\frac{1}{4}$ of the original area. Since the centroids are located $\frac{2}{3}$ of the way above the line, we have a smaller triangle which has an area of $\frac{4}{9}$ the original. Therefore, this triangle has area $\frac{1}{9}$ of the original. If we integrate over possible values of r, this gives us the final formula $\int_0^1 r^2\left(\frac{4}{27}r\right) + (1-r)^2\left(\frac{4}{27}(1-r)\right) + \left(1 - r^2 - \left(1-r^2\right)\right)\left(\frac{1}{9}\right) dr$. Simplifying a bit gives us $2\int_0^1 \frac{4}{27}r^3 dr + \frac{1}{9}\int_0^1 2r - 2r^2 dr = \frac{2}{27} + \frac{1}{9} - \frac{2}{27} = \frac{1}{9}$. Thus, we get that $\mathbb{E}|PQR| = \frac{1}{9}\mathbb{E}|ABC|$ as expected.

4. **(a)** Note that X/Y is simply the reciprocal of the slope of a randomly distributed point in a 1×1 box. Thus, if the slope is between $\frac{1}{a}$ and $\frac{1}{b}$, as long as $1 < a < b$ the points that satisfy the inequality $a < x/y < b$ are in a triangle with an area of $\frac{1}{2}\left(\frac{1}{a} - \frac{1}{b}\right) = \boxed{\frac{b-a}{2ab}}$. Note that when $a, b < 1$, we can just interpret everything as its reciprocal to get $\frac{1}{2}(b-a)$.

(b) For $k = 0$, we need $X/Y \in \left(0, \frac{1}{2}\right)$. Using the above formula gives us $\boxed{\frac{1}{4}}$. For $k = 1$, $X/Y \in \left(\frac{1}{2}, \frac{3}{2}\right)$. This is $\frac{1}{4} + \frac{1}{6} = \boxed{\frac{5}{12}}$. Otherwise, $X/Y \in \left(\frac{2k-1}{2}, \frac{2k+1}{2}\right)$ gives us $\boxed{\frac{2}{4k^2-1}}$ from **(a)**.

(c) Note that because of the way that we originally derived the formula, $\frac{2}{4k^2-1} = \frac{1}{2k-1} - \frac{1}{2k+1}$. We wish to sum these terms, except when $k = 1$, where we want $\frac{5}{12}$ instead. So the total sum for odd k is $\frac{\pi}{4} - 1 + \frac{1}{3} + \frac{5}{12} = \frac{\pi}{4} - \frac{1}{4}$. Since we want the probability that S is even, which is the complement of this event, we obtain $\boxed{\frac{5-\pi}{4}}$.