

Detropticalized RSK correspondence

"Tropicalization": $ab \rightarrow a+b$, $\frac{a}{b} \rightarrow a-b$, $a+b \rightarrow \max\{a, b\}$.

① Row insertion. 1) (classical (tropical)). Alphabet $\{1, 2, \dots, n\}$. For a nondecreasing word w and $K \in \{1, 2, \dots, n\}$, if $K \geq$ all letters in w , attach it in the end. otherwise, find the leftmost letter greater than K and push it out with K . For nondecreasing words x, a perform insertions of letters of a one by one from left to right to get $x \xrightarrow{\begin{matrix} a \\ \downarrow b \\ \text{word formed by letters that were pushed out.} \end{matrix}} y \leftarrow \text{new word}$ $x = x_1 x_2 \dots x_n$, $y = y_1 y_2 \dots y_n$, $a = a_1 a_2 \dots a_n$, $b = b_1 b_2 \dots b_n$, $b_1 = 0$. Let $\mu_j = x_1 + \dots + x_j$, $v_j = y_1 + \dots + y_j$, $\nu_j = \max\{\mu_k + a_k \mid K \leq k \leq j\}$, $\vartheta_j = \max\{\mu_k, \nu_k \mid K \leq k \leq j\}$. So in general $\vartheta_1 = \mu_1 + a_1$, $\vartheta_2 = \max\{\nu_1, \mu_2 + a_2, \dots, \nu_j = \max\{\vartheta_{j-1}, \mu_j\} + a_j, \dots\}$, since $\vartheta_1 \leq \vartheta_2 \leq \dots \leq \vartheta_n$. Hence by induction on j we have $\vartheta_j = \max\{\mu_1 + a_1 + \dots + a_j, \mu_2 + a_2 + \dots + a_j, \dots, \mu_j + a_j\} = \max_{1 \leq k \leq j} \{x_1 + \dots + x_k + a_k + \dots + a_j\}$. Note $y_1 = \vartheta_1$, $y_j = \vartheta_j - \vartheta_{j-1}$, $2 \leq j \leq n$, $b_1 = 0$, $b_j = a_j + x_j - y_j = a_j + \mu_j - \mu_{j-1} - \vartheta_j + \vartheta_{j-1}$ for $2 \leq j \leq n$.

Example: $22345 \xrightarrow{1245} 122445$ $x = 1^0 2^2 3^1 4^1 5^1$, $a = 1^1 2^4 3^0 4^1 5^1 \Rightarrow$ from the table $\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 \\ \hline x & 0 & 2 & 1 & 1 & 1 \\ a & 1 & 1 & 0 & 1 & 1 \end{array}$ we get $\vartheta_1 = 1$, $\vartheta_2 = 3$, $\vartheta_3 = 3$, $\vartheta_4 = 5$, $\vartheta_5 = 6$ and so $y = 1^2 2^3 3^0 4^2 5^1$, $b = 1^0 2^1 3^1 4^0 5^1$. usually take $x_i, y_i, a_i, b_i \in \mathbb{R}_{>0}$

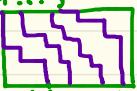
2) Detropicalized. $\xrightarrow{b} y$ $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$, $\mu_j = x_1 + \dots + x_j$, $\nu_j = y_1 + \dots + y_j$ ($1 \leq j \leq n$). $\vartheta_1 = \mu_1 + a_1$, $\vartheta_j = (\vartheta_{j-1} + a_j) + a_j$, $2 \leq j \leq n$. $y_1 = \vartheta_1$, $y_j = \frac{\vartheta_j}{\vartheta_{j-1}}$, $2 \leq j \leq n$ and $b_j = a_j + \frac{\vartheta_j}{\vartheta_{j-1}} = a_j + \frac{\mu_j - \mu_{j-1}}{\vartheta_{j-1} - \vartheta_j}$, $2 \leq j \leq n$. By induction on j we get $\vartheta_j = \mu_1 a_1 + \dots + a_j + \mu_2 a_2 + \dots + a_j + \dots + \mu_j a_j = x_1 a_1 a_2 \dots a_j + x_1 x_2 a_2 \dots a_j + \dots + x_1 \dots x_j a_j$. For $c = (c_1, \dots, c_s)$ with $c_1 \dots c_s \neq 0$ let $H(c) = (\text{diag}(c\bar{c}) - \Delta)^{-1}$ where $\bar{c} = (\frac{1}{c_1}, \dots, \frac{1}{c_s})$, $\Delta_{ij} = 1$ if $j = i+1$ for $1 \leq i \leq s-1$ and 0 otherwise. For $c = (c_1, \dots, c_n)$ let $H_n(c) = \begin{bmatrix} c_1 & & & \\ 0 & H(c) & & \\ & \ddots & \ddots & \\ & 0 & 0 & c_n \end{bmatrix}$. $H_1(c) = H(c)$ - upper triangular with $H(c)_{ij} = c_i \dots c_j$ for $1 \leq i \leq j \leq n$. Claim: $H_1(x) H_1(a) = H_2(b) H_1(y)$. Proof: We need

$$\left[\begin{smallmatrix} \frac{1}{a_1} - 1 & & & \\ \frac{1}{a_2} - 1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ \frac{1}{a_n} - 1 & & & \end{smallmatrix} \right] \left[\begin{smallmatrix} \frac{1}{x_1} - 1 & & & \\ \frac{1}{x_2} - 1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ \frac{1}{x_n} - 1 & & & \end{smallmatrix} \right] = \left[\begin{smallmatrix} \frac{1}{y_1} - 1 & & & \\ \frac{1}{y_2} - 1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ \frac{1}{y_n} - 1 & & & \end{smallmatrix} \right] \left[\begin{smallmatrix} 1 & 0 & & \\ \frac{1}{b_2} - 1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ \frac{1}{b_n} - 1 & & & \end{smallmatrix} \right] \text{ or in other words } \frac{1}{a_i x_1} = \frac{1}{y_1} \\ \frac{1}{a_1 x_2} = \frac{1}{y_2} \dots \frac{1}{a_j x_j} = \frac{1}{y_j} \text{ for } 2 \leq j \leq n, \text{ and}$$

$-\frac{1}{q_1} - \frac{1}{x_2} = -\frac{1}{b_2}, \quad -\frac{1}{q_j} - \frac{1}{x_{j+1}} = -\frac{1}{y_j} - \frac{1}{b_{j+1}}$ for $2 \leq j \leq n-1$, so it is enough to show $\frac{1}{q_1} + \frac{m_2}{m_2} = \frac{v_1 v_2}{q_2 m_2 v_1}$ and $\frac{1}{q_j} + \frac{m_j}{m_{j+1}} = \frac{v_{j-1}}{y_j} + \frac{m_j v_{j+1}}{q_{j+1} m_{j+1} v_j}$ for $2 \leq j \leq n-1$. First is equivalent to $\frac{v_2 - v_1}{m_2 a_2} = \frac{v_0}{m_1 a_1}$ (which are both 1) and other are equivalent to $\frac{v_{j+1} - v_j}{a_{j+1} m_{j+1}} = \frac{v_j - v_{j-1} a_j}{a_{j+1} m_{j+1}}$ (which are both 1). \square

② Multiple insertions: 1) **Classical (tropical)** x^1, \dots, x^m - nondecreasing words in $\mathbb{S}_1, \dots, \mathbb{S}_3$. $(x^1, x^2, \dots, x^m) \rightarrow$ semistandard Young tableau of shape λ with $\ell(\lambda) \leq m$ and filled with 1, 2, ..., n with $\#i = x_i^1 + \dots + x_i^m$, where $x_i^j = \begin{smallmatrix} x_1^j & x_2^j & \dots & x_n^j \\ 1 & 2 & \dots & n \end{smallmatrix}$. The resulting tableau has rows $y^{1,m}, y^{2,m}, \dots, y^{l,m}$ where $l = \min(m, n)$. Suppose $y^{j,m} = j^{p_1} \dots n^{p_l}$, $1 \leq j \leq l$.

Theorem: $p_j = T_j^i - T_{j-1}^i$, $p_i = T_i^j - T_{i-1}^j - T_i^{j-1} + T_{i-1}^{j-1}$ for $i > j$, where for $i \geq j$ $T_i^j = \max_{(Y_1, \dots, Y_j)} (x_{Y_1} + \dots + x_{Y_j})$ over j -tuples of nonintersecting paths $Y_k: (k, 1) \rightarrow (i-j+k, m)$. $1 \leq k \leq j$ in the $m \times n$ rectangle. $\sum_{i=1}^l p_i = 0$ $\begin{array}{|c|c|c|c|c|c|} \hline & 1 & 2 & \dots & l & \\ \hline & (i,j) \rightarrow X_i & & & & \\ \hline \end{array}$



2) **Detropicalized:** Use the same scheme as above for $x^1, \dots, x^m \in \mathbb{R}_{>0}^n$. $H_{K+1}(y^{K+1, K+1}) \dots H_1(y^{1, K+1}) H(x^K) = H_K(y^{K, K}) H_{K+1}(y^{K+1, K}) \dots H_1(y^{1, K}) \Rightarrow H(x^1) H(x^2) \dots H(x^m) = H_r(y^{1,m}) \dots H_2(y^{2,m}) H_1(y^{1,m})$ \star For $l = \min(m, n)$

Denote both sides by H . Observe that $\det H_{1, \dots, l} = \sum_{\substack{i_1, \dots, i_l \\ \text{down and to the right}}} x_{i_1} \dots x_{i_l}$ where the sum is over l -tuples of nonintersecting paths Y_1, \dots, Y_l in $m \times n$ rectangle

with $Y_k: (k, 1) \rightarrow (j_k, m)$ and $x_{i_r} = \prod_{(a,b) \in Y_r} x_a^b$ (true for $r=1$ + Gessel-Viennot).

On the other hand $\det H_{1, \dots, l} = \sum_{\substack{s_1, \dots, s_l \\ \text{up and to the right}}} y_{s_1} \dots y_{s_l}$ where the sum is over l -tuples of nonintersecting paths S_1, \dots, S_l (s_{i_r} up and to the right) in the trapezoid.

$m \leq n$ or triangle $m \geq n$, where $s_k: i_k$ -th point on the bottom boundary $\rightarrow j_k$ -th point on the top boundary

and $y_{s_k} = \prod_{j \in S_k} y_j^{b_j}$ where $y^{s, m} = (y_{s_1}^1, \dots, y_{s_l}^1)$ (true for $r=1$ + Gessel-Viennot).

Let $T_i^j = \det H_{i, j, \dots, i}$ for $1 \leq j \leq i \leq n$. If $j \leq m$, $T_i^j > 0$ from first expression. On the other hand $T_i^j = \prod_{b \in j, a \leq i} y_a^b \Rightarrow T_i^j = \frac{T_i^l}{T_{i-1}^{l-1}}$ for $1 \leq i \leq l$ and $y_i^j = \frac{T_i^l}{T_{i-1}^{l-1} T_{i-1}^{l-1}}$ $1 \leq j \leq i$, $1 \leq j \leq l$. In particular $\exists b$ it follows that $y^{s, m}$ are uniquely determined from \star , this is detropicalized version of the theorem above. Tropicalize it to get a proof of those theorems.

Example: $m=3, n=4, x^1=2234, x^2=1344, x^3=11224 \Rightarrow P = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}, x = \{x_i^j\}_{i,j}$, $1 \leq i \leq 4, 1 \leq j \leq 3$ is given by $\begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{bmatrix} \Rightarrow T = (T_{i,j}^j)_{i,j} = \begin{bmatrix} 3 & 5 & 5 & 7 \\ 7 & 9 & 12 \\ 9 & 13 \end{bmatrix} \Rightarrow P = (P_{i,j}^j)_{i,j} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ 3 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(3) Schützenberger involution 1) Classical (+tropical) For $k \in \{1, 2, \dots, n\}$ let $K^k = n - k + 1$ and for a word $w = w_1 \dots w_k$ let $w^* = w_k^* \dots w_1^*$. Then

Fact 1: involution $w \rightarrow w^*$ induces an involution $P \rightarrow P^*$ on the set of semistandard Young tableaux preserving the shape.

Theorem: Let P be a semistandard Young tableau, p_{ij}^k - number of i -s in the j -th row, \tilde{p}_{ij}^k - number of i -s in the j -th row of P^* ($1 \leq j \leq n$). Then $\tilde{p}_{ij}^k = b_{i-1}^{j-1} c_i^k$, $\tilde{p}_{ij}^j = b_i^j + b_{i-1}^{j-1} - b_{i-1}^{j-1} b_{i-1}^{j-1}$ ($i > j$), where for $1 \leq j \leq n$ $b_i^j = \max_{(x_1, \dots, x_j)} p_{x_1} + \dots + p_{x_j}$ over j -tuples of (down and to the left) nonintersecting paths in a triangle, $\gamma_k : (n-j+k, 1) \rightarrow (n-i+k, 1)$ $\forall 1 \leq k \leq j$

 Proof: tropicalize the detropicalized version of this result for mon. Observe that if $w = x^1 x^2 \dots x^m$, where $x^j = x_1^j x_2^j \dots x_n^j$, $1 \leq j \leq m$, $w^* = (x^1)^* \dots (x^2)^* \dots (x^m)^*$ where $(x^j)^* = x_n^j x_{n-1}^j \dots x_1^j$.

2) De tropicalized For $x = (x_1, x_2, \dots, x_n)$ let $x_{\#} = (x_n, \dots, x_2, x_1)$. Suppose $m \leq n$. If we have $H(x^1) H(x^2) \dots H(x^m) = H_m(u^m) H_{m-1}(u^{m-1}) \dots H_1(u^1)$ (i.e. $u^i = v^{s_i m}$), then Theorem: $H(x_{\#}) \dots H(x_{\#n}) H(x_{\#}^1) = H_m(v^m) H_{m-1}(v^{m-1}) \dots H_1(v^1)$ where $v^j = (v_1^j, \dots, v_n^j)$ and $v_{ij}^j = \frac{b_i^j}{c_i^j}$, $v_{ij}^j = b_i^j b_{i-1}^{j-1} / b_{i-1}^{j-1} b_i^j$ for $j < i$, where $b_i^j = \sum_{(x_1, \dots, x_j)} u_{x_1} \dots u_{x_j}$ over all j -tuples of nonintersecting (down and to the left) paths in a trapezoid

 Proof: let Y be $n \times n$ -matrix given by $y_{i,j} = \prod_{(a,b) \in Y} u_a^b$. On the other hand it is $H_m(v^m) H_{m-1}(v^{m-1}) \dots H_1(v^1)$ for some $v^j = (v_1^j, \dots, v_n^j)$. Denote both sides by H , then $\det H_{i_1, i_2, \dots, i_n} = \prod_{j=1}^n V_{i_j}^b$ and on the other hand it is $c_i^j > 0$ $1 \leq j \leq n, j \leq m \Rightarrow v_{ij}^j = \frac{b_i^j}{c_i^j}$, for $i > j$ $v_{ij}^j = b_i^j b_{i-1}^{j-1} / b_{i-1}^{j-1} b_i^j$ \square

(4) RSK correspondence 1) Classical (+tropical) Let $A = (a_{ij}^j) \in \text{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$ $1 \leq j \leq n, 1 \leq i \leq m$. Let $w_i^j = a_1^j \dots a_n^j$, $w_j = a_1^j \dots a_m^j$, $P = P(w^1 \dots w^m)$, $Q = P(w_1 \dots w_n)$, $A \rightarrow (P, Q)$ - bijection between $\text{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$ and pairs of semistandard Young tableaux of the same shape, Q filled with $\{1, \dots, m\}$, P filled with $\{1, \dots, n\}$, other variations $A \rightarrow (P^*, Q)$, $A \rightarrow (P, Q^*)$, $A \rightarrow (P^*, Q^*)$.

$p_j^i, q_j^i, \tilde{p}_j^i, \tilde{q}_j^i$ - number of j -s in the i -th row in P, Q, P^S, Q^S . Common shape
 $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r), \rho = \min\{\lambda_i, n\}, \lambda_i = p_j^i + \dots + p_n^i = q_j^i + \dots + q_m^i = \tilde{p}_j^i + \dots + \tilde{p}_m^i = \tilde{q}_j^i + \dots + \tilde{q}_m^i$
 $\text{RSK}^*: A \rightarrow \text{Sym}(A), Q^S(\text{Sym}(A))$, same as $A \rightarrow P(w^m \dots w^1), Q(w_n \dots w_1)$

Inverse Formula: Let $\text{RSK}^*: A \rightarrow (u, v)$ of shape $\lambda = (\lambda_1, \dots, \lambda_r), \rho = \min\{\lambda_i, n\}$.
Set $y_j^i = \sum_{k=1}^{i-1} u_{n-j+k}^k - \sum_{k=1}^i u_{n-j+k+1}^k, i < j, = \lambda_i - \sum_{k=1}^{i-1} u_k^k - \sum_{k=1}^i v_m^k, i = j, \sum_{k=1}^{i-1} v_{m-i+k}^k - \sum_{k=1}^i v_{m-i+k+1}^k$
 $i > j$ for $1 \leq i \leq m, 1 \leq j \leq n$. Set $T_j^i = \max_{(Y_1, \dots, Y_\nu)} y_{r_1} + \dots + y_{r_\nu}$ where $r = \min(i, j)$ and
the maximum is taken over ν -tuples of nonintersecting paths $Y_k: (1, n-i+k) \rightarrow$
 $(n-j+k, 1), k = 1, \dots, r$, $Y_\nu = \sum_{(a,b) \in Y_\nu} y_a^b$. Then $q_j^i = T_j^i + T_{j-1}^{i-1} - T_{j-1}^i, u_j^i | V_j^i$ in the i -th
row of U/V .

Proof: Tropicalize the detropicalized version.

2) Detropicalized Same scheme. Inverse Formula: Let $X = (x_{ij}^i), 1 \leq i \leq m, 1 \leq j \leq n, H(x^m) \cdots H(x^1) = H_p(u^i) \cdots H_q(u^i)$ for $u^i = (u_1^i, \dots, u_n^i)$, $H(x_n) \cdots H(x_1) = H_p(v^i) \cdots H_q(v^i)$ for $v^i = (v_1^i, \dots, v_m^i)$. Set $y_j^i = u_{n-j+i}^i \cdots u_{n-i+1}^i / u_{n-j+i+1}^i \cdots u_{n-i}^i$
 $i < j, = \lambda_i / u_1^i \cdots u_{n-i}^i v_m^i \cdots v_{n-j+1}^i, i = j, = v_{m-i+j}^i \cdots v_{m-i+1}^i / v_{m-i}^i \cdots v_{m-j+1}^i, i > j$. Here
 $\lambda_i = u_1^i \cdots u_n^i = v_1^i \cdots v_m^i$. Set $c_j^i = \sum_{(a,b) \in Y_\nu} y_{r_1}^a \cdots y_{r_\nu}^b$ where $r = \min(i, j)$ and the sum is
taken over ν -tuples of non intersecting paths $Y_k: (1, n-i+k) \rightarrow (n-j+k, 1), k = 1, \dots, r$,
 $Y_\nu = \prod_{(a,b) \in Y_\nu} y_a^b$, then $x_{ij}^i = \frac{b_j^i b_{j-1}^{i-1}}{c_{j-1}^i c_j^i}$.

Proof: Let $p_j^i = \sum_y x_{ij}^y$ where the sum is over all paths $(1, i) \rightarrow (j, 1)$
going up and to the right, $x_{ij}^y = \prod_{(a,b) \in Y} x_{ia}^b$. We want to show that for
 $1 \leq i_1 < \dots < i_r \leq m, 1 \leq j_1 < \dots < j_r \leq n$ set $P_{j_1, \dots, j_r} = \det Q_{i_1, i_2, \dots, i_r, j_1, \dots, j_r}$ where
 $Q = (q_{ij}^i)$ where $q_{ij}^i = \sum_y y_{r_1}^a \cdots y_{r_\nu}^b$ where the sum is over all paths $(1, i) \rightarrow (j, 1)$
going up and to the right, $y_{r_1}^a = \prod_{(a,b) \in Y} y_a^b$. If this is true, we can apply it to
 $i_1 = 1, i_2 = 2, \dots, i_r = r, j_1 = j-r+1, \dots, j_r = j$ for $r \leq j$ or to $i_1 = 1, i_2 = 2, \dots, i_r = r$ and
 $i_1 = i-r+1, \dots, i_r = i$ for $i \geq r$ to get $\prod_{(a,b) \in Y} x_{ia}^b = b_j^i \Rightarrow x_{ij}^i = \frac{b_j^i b_{j-1}^{i-1}}{b_{j-1}^i b_j^i}$. Let R be given
by $R_j^i = P_{i+1, \dots, i+r, j}$. We need to show $Q = R$. It is enough to show $\det Q_{j-i+1, \dots, j, i} =$
 $\det R_{j-i+1, \dots, j, i} \neq 0$ for $i \leq j$ and $\det Q_{i_1, \dots, i_r, j} = \det R_{i_1, \dots, i_r, j} \neq 0$. Indeed,
we can then show $q_{ij}^i = R_j^i$ by induction on $i+j$. If $i \leq j$, $\det Q_{j-i+1, \dots, j, i} =$
 $= \prod_{(a,b) \in Y} y_{r_1}^a \cdots y_{r_\nu}^b = \det(H_{i+1}(u^i) \cdots H_q(u^i))_{n-j+1, \dots, n-i+1} = \det(H_1 x^m) \cdots H(x^1))_{n-j+1, \dots, n-i+1} =$
 $\det R_{j-i+1, \dots, j, i} = \sum_{(a,b) \in Y} x_{ia}^a \cdots x_{ir}^r$ where the sum is over all ν -tuples
of nonintersecting paths in a $m \times n$ rectangle always going up and to the
right with $Y_k: (k, m) \rightarrow (n-j+k, 1)$. We need to show that this is equal to

$\det R_{i+1, \dots, j}^{1, \dots, i} = \det P_{n-j+i, \dots, n-j+i}^{m-i+1, \dots, m} = \sum_{S_1, \dots, S_i} x_{S_1} \cdots x_{S_i}$ where the sum is over all i -tuples of nonintersecting paths S_1, \dots, S_i in a $m \times n$ rectangle going up and to the right with $S_k: (1, n-j+k) \rightarrow (m-i+k, 1)$, $1 \leq k \leq i$. To show that these two sums are the same observe that they are both equal to $\prod_{\substack{k, l: \\ k \in [m], l \in [n], \\ m-k+l \geq i+1}} x_k^l \cdot \sum_{S_1, \dots, S_i} x_{S_1} \cdots x_{S_i}$ where the sum is over all i -tuples of nonintersecting paths in $m \times n$ rectangle going up and to the right with $S_k: (k, m-i+k) \rightarrow (n-j+k, 1)$ for $1 \leq k \leq i$.

In a similar fashion for $j \leq i$
 $\det Q_{i+1, \dots, j}^{i-j+1, \dots, i} = \prod_{\substack{k, l: \\ k \in [m], l \in [j], \\ m-k+l \geq i+1}} y_k^l = \prod_{\substack{k, l: \\ k \in [j], l \in [m-i+j] \\ c \leq k, d \leq l}} V_k^c = \det P_{n-j+1, \dots, n}^{m-i+1, \dots, m-i+j} =$

$= \det R_{1, \dots, j}$ and we are done \square

Reference: M. Noumi & Y. Yamada, Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions, in Representation Theory of Algebraic Groups and Quantum Groups, Adv. Stud. Pure Math. 40, 2004.