Mathematics UN1102
Section 1, Fall 2019
Final Exam Solutions
December 16, 2019
Time Limit: 170 Minutes

Name: ____________________

UNI: ____________________

Instructions: This exam contains 10 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. **Do not write on the back side of your test papers.**

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Question	1	2	3	4	5	6	7	8	9	10	Total
Points	8	12	10	10	10	10	10	10	10	10	100
Score											

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

$$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2 - x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\quad R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\quad R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $\quad R = 1$

Problem 1 (8 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (2 points) Suppose $f(x)$ is a continuous function on $[0, \infty)$. If the series $\sum_{n=1}^{\infty} f(n)$ converges, then the improper integral $\int_0^{\infty} f(x)dx$ converges.

Answer: $\boxed{\text{False.}}$ Consider $f(x) = \sin(\pi x)$, for which $f(n) = 0$.

(b) (2 points) Suppose $f(x)$ and $g(x)$ are continuous functions on $[0, \infty)$. If the improper integral $\int_0^{\infty} f(x)dx$ converges and $g(x) \leq f(x)$ for $x \in [0, \infty)$, then $\int_0^{\infty} g(x)dx$ converges.

Answer: $\boxed{\text{False.}}$ Consider $f(x) = \frac{1}{x^2+1}$ and $g(x) = -\frac{1}{x+1}$.

(c) (2 points) If $f(x)$ is equal to its Maclaurin series for all x and $g(x)$ is equal to its Maclaurin series for all x, then $f(x) + g(x)$ is equal to its Maclaurin series for all x.

Answer: $\boxed{\text{True.}}$ By the additive property of power series.

(d) (2 points) If $y_1(x)$ and $y_2(x)$ are solutions to $y' + 2xy = 0$ with $y_1(0) = y_2(0)$, then $y_1(x) = y_2(x)$.

Answer: $\boxed{\text{True.}}$ First order differential equations have solutions determined by a single initial condition.

Problem 2 (12 points) This problem is on two pages. Evaluate the following integrals.

(a) (4 points)

$$\int x^2 \sin(x^3) dx.$$

Answer: $\boxed{-\frac{1}{3}\cos(x^3) + C}$

Set $u = x^3$ so that $du = 3x^2 dx$, hence we get

$$\int x^2 \sin(x^3) dx = \frac{1}{3}\int \sin(u) du = -\frac{1}{3}\cos(u) + C = -\frac{1}{3}\cos(x^3) + C.$$

(b) (4 points)

$$\int \frac{3x^3 + x}{x^4 - 1} dx.$$

Answer: $\boxed{\frac{1}{2}\ln|x^2 + 1| + \ln|x - 1| + \ln|x + 1| + C}$

Find the partial fraction decomposition

$$\frac{3x^3 + x}{x^4 - 1} = \frac{x}{x^2 + 1} + \frac{1}{x - 1} + \frac{1}{x + 1}.$$

Integration then yields the claimed answer.

(c) (4 points)

$$\int e^x \sin(x) dx.$$

Answer: $\boxed{\int e^x \sin(x) dx = \frac{1}{2} e^x (\sin(x) - \cos(x))}$

By integration by parts, we have

$$\int e^x \sin(x) dx = -e^x \cos(x) + \int e^x \cos(x) dx = -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x) dx.$$

Solving yields $\int e^x \sin(x) dx = \frac{1}{2} e^x (\sin(x) - \cos(x))$.

Problem 3 (10 points) Consider the region A bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

(a) (2 points) Sketch the region A.

Answer: $\boxed{\text{See Wolfram Alpha for a sketch.}}$

(b) (3 points) Find the area of the region A.

Answer: $\boxed{\frac{1}{3}.}$

The area is given by

$$\int_0^1 (\sqrt{x} - x^2)dx = [\frac{2}{3}x^{3/2} - \frac{1}{3}x^3]_0^1 = \frac{1}{3}.$$

(c) (5 points) Find the volume of the solid of revolution obtained by rotating A about the y-axis.

Answer: $\boxed{\frac{3}{10}\pi}$

We use the cylinder method to obtain

$$V = \int_0^1 2\pi x(\sqrt{x} - x^2)dx = 2\pi[\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{4}x^4]_0^1 = \frac{3}{10}\pi.$$

Problem 4 (10 points) This problem is on two pages. Determine whether each of the following series is convergent or divergent. Justify your answer.

(a) (2 points)

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Answer: $\boxed{\text{Converges.}}$

Notice that for $n \geq 4$, we have

$$\frac{2^n}{n!} \leq 2\frac{4}{n(n-1)},$$

so the series converges by comparison.

(b) (2 points)

$$\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+1}.$$

Answer: $\boxed{\text{Converges.}}$ By comparison with $\sum_{n=1}^{\infty} \frac{1}{e^n}$.

(c) (3 points)

$$\sum_{n=1}^{\infty}(-1)^n \ln(n).$$

Answer: $\boxed{\text{Diverges.}}$ Because $\lim_{n\to\infty} \ln(n) = \infty$.

(d) (3 points)

$$\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2+1}.$$

Answer: $\boxed{\text{Converges.}}$ Because $\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2+1}$ converges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Problem 5 (10 points) Consider the power series

$$\sum_{n=1}^{\infty} \ln(n) \frac{(x+2)^n}{2^n}$$

(a) (5 points) Determine its radius of convergence.

Answer: $\boxed{R = 2}$

Applying the ratio test, we have

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} \frac{|x+2|}{2} = \frac{|x+2|}{2},$$

meaning that $L < 1$ if and only if $x \in (-4, 0)$, hence $R = 2$.

(b) (5 points) Determine its interval of convergence.

Answer: $\boxed{(-4, 0).}$

It suffices to check $x = -4$ and $x = 0$. At $x = -4$, we obtain $\sum_{n=1}^{\infty} (-1)^n \ln(n)$, which diverges. At $x = 0$, we obtain $\sum_{n=1}^{\infty} \ln(n)$, which diverges. So the interval is $(-4, 0)$.

Problem 6 (10 points)

(a) (6 points) Find the Maclaurin series of the function

$$f(x) = \frac{1}{8 + x^3}.$$

What is its interval of convergence?

Answer: $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{8^{n+1}} \text{ for } x \in (-2, 2)}$

We write

$$f(x) = \frac{1}{8} \frac{1}{1 + \frac{1}{8}x^3} = \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{8^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{8^{n+1}},$$

where we substitute the Maclaurin series for $\frac{1}{1-y}$ with $y = -\frac{x^3}{8}$. This converges for $y \in (-1, 1)$, which corresponds to $x \in (-2, 2)$.

(b) (4 points) Write down a series expression for

$$\int_0^1 f(x) dx.$$

Answer: $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1}{(3n+1)8^{n+1}}}$

We integrate term by term to get

$$\int_0^1 f(x) dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(3n+1)8^{n+1}}.$$

Problem 7 (10 points) Consider the curve given by $r = 1 + \sin(\theta)$ in polar coordinates for $\theta \in [0, 2\pi]$.

(a) (5 points) Determine the θ-values of the points where the curve has horizontal or vertical tangents.

Answer: $\boxed{\text{Horizontal tangents at } \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \text{ and vertical tangents at } \theta = \frac{\pi}{6}, \frac{5\pi}{6}.}$

The slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{r'(\theta)\sin(\theta) + r(\theta)\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\sin(\theta)} = \frac{\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta) + \cos(\theta)}{\cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) - \sin(\theta)}.$$

Notice that the numerator is zero when

$$\cos(\theta)(2\sin(\theta) + 1) = 0 \iff \theta = \frac{\pi}{2}, \frac{3\pi}{2} \text{ or } \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$$

The denominator is zero when

$$\begin{aligned}
\cos^2(\theta) - \sin^2(\theta) - \sin(\theta) = 0 &\iff 1 - 2\sin^2(\theta) - \sin(\theta) = 0 \\
&\iff (2\sin(\theta) - 1)(\sin(\theta) + 1) = 0 \\
&\iff \sin(\theta) = \frac{1}{2}, -1 \\
&\iff \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}.
\end{aligned}$$

We conclude that there are horizontal tangents at $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ and vertical tangents at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

(b) (5 points) Determine the area contained within the curve.

Answer: $\boxed{\frac{3\pi}{2}}$

The area is

$$\int_0^{2\pi} \frac{1}{2} r(\theta)^2 d\theta = \int_0^{2\pi} \frac{1}{2}(1 + 2\sin(\theta) + \sin^2(\theta)) d\theta = \frac{3\pi}{2}.$$

Problem 8 (10 points) Consider the differential equation

$$\frac{dy}{dx} = xy + 1.$$

(a) (5 points) Let $y = f(x)$ be the solution to the equation with initial condition $f(0) = 1$. Write an equation for the line tangent to $f(x)$ at $(0, 1)$.

Answer: $\boxed{y = x + 1}$

We have $f'(0) = 1$, so the equation is $y = x + 1$.

(b) (5 points) Find the approximation for $f(0.2)$ given by Euler's method with step size 0.1.

Answer: $\boxed{1.211}$

We have $f(0.1) \approx x_1 = f(0) + 0.1 \cdot 1 = 1.1$ and that

$$f(0.2) \approx x_2 = x_1 + 0.1 \cdot (0.1 \cdot 1.1 + 1) = 1.1 + 0.111 = 1.211.$$

Problem 9 (10 points) Solve the differential equation

$$\frac{dy}{dx} = (y^2 + 1)\sin(x)$$

with initial condition $y(0) = 1$.

Answer: $\boxed{y(x) = \tan\left(\frac{\pi}{4} + 1 - \cos(x)\right).}$

We obtain

$$\frac{1}{y^2 + 1}\frac{dy}{dx} = \sin(x),$$

which after integrating both sides yields

$$\arctan(y) = \int \frac{dy}{y^2 + 1} = \int \sin(x)dx = -\cos(x) + C,$$

meaning that

$$y(x) = \tan(C - \cos(x)).$$

Solving for the initial condition yields

$$1 = y(0) = \tan(C - 1),$$

hence $C = \frac{\pi}{4} + 1$.

Problem 10 (10 points) Find the general solution to the differential equation

$$xy' + y = x\sin(x).$$

Answer: $\boxed{y(x) = -\cos(x) + \frac{\sin(x)}{x} + \frac{C}{x}.}$

Dividing by x, we get the equation

$$y' + \frac{1}{x}y = \sin(x).$$

We obtain the integrating factor $I(x) = e^{\int \frac{1}{x}dx} = x$. Multiplying, we get

$$(xy)' = x\sin(x),$$

hence by integration by parts we have

$$xy = \int x\sin(x)dx = -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x) + C,$$

meaning the general solution is

$$y(x) = -\cos(x) + \frac{\sin(x)}{x} + \frac{C}{x}.$$