FIBONACCI LATTICES

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Abstract. This paper gives an overview of the poset F and its two generalizations $\mathrm{Fib}(r)$ and $Z(r)$, collectively known as Fibonacci lattices. First defined by Stanley in [8] and [6], they are thus named because the Fibonacci numbers appear as the number of elements of each rank in $\mathrm{Fib}(1)$ and $Z(1)$. We begin with some basic properties of F and then introduce its generalizations $\mathrm{Fib}(r)$ and $Z(r)$. We discuss maximal chains and multichains of intervals in these lattices and present a correspondence between the properties of $\mathrm{Fib}(r)$ and $Z(r)$ found in [7]. More combinatorial proofs of these correspondences were given in [4] and [5], and we give the idea of these proofs here.

1. Introduction

In [8], Stanley first introduced a Fibonacci lattice F as an example of a lattice that serves as a Fibonacci analogue to the Young tableau. He defined $F = J(K)$ as the lattice of order ideals of the poset K of ordered pairs (m, n) with $m \in \mathbb{N}$, $n \in \{0, 1\}$ under the order relation $(m, n) < (m', n')$ if and only if $n = 0$ and $m < m'$. As shown in Figure 1, we can biject F to the set of words with letters in $\{1, 2\}$ corresponding to the number of elements in each diagonal slice of the order ideal. In this case, the word u covers v if and only if v is obtained

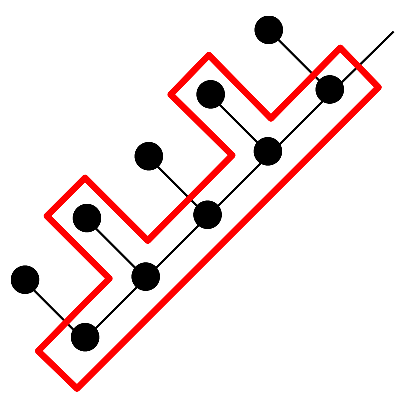

Figure 1. An element of F corresponding to the word 12121

by changing a 2 to a 1 in u or by removing a terminal 1 in u. The Fibonacci numbers appear as the number of elements of each rank in this lattice which shares certain properties with Young tableau, thereby motivating a type of Fibonacci tableau which we will discuss in the first part of this paper.

Stanley then introduced two new lattices $\mathrm{Fib}(r)$ and $Z(r)$ in [6] that generalize F by adding additional 1's to the set of allowed letters. These two lattices have the same underlying set, but their order relations are different generalizations of the condition on F. In the second half of the paper, we will discuss the surprising correspondence between them.

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2. The Fibonacci Lattice

As defined in the introduction, the Fibonacci lattice F is the finite distributive lattice of order ideals of a poset K whose Hasse diagram is shown in Figure 1. In this section, we follow the approach given by Stanley in [8] to discuss some of its properties. First note that its name is inspired by the following proposition.

Proposition 1. The Fibonacci lattice F contains f_k elements of rank k, where f_k is the k^{th} Fibonacci number (we take the convention $f_0 = f_1 = 1$).

First Proof. Let r_k be the number of elements of rank k. We have by observation that $r_0 = r_1 = 1$. Now, given an order ideal I of K with $|I| = k$, note that $I - \hat{0}$ is either an order ideal of K of cardinality $k-1$ or the disjoint union of a point and an order ideal of cardinality $k-2$. This defines a bijection, so we find $r_k = r_{k-1} + r_{k-2}$, completing the proof. $\square$

Second Proof. Viewing F instead as the set of words with letters in $\{1, 2\}$, we see that elements of rank k are in bijection with compositions of k with elements in $\{1, 2\}$. It is then clear that we have $r_k = r_{k-1}+r_{k-2}$, which completes the proof with our initial conditions. $\square$

For any distributive lattice $L = J(P)$, define $e(I)$ for $I \in L$ as the number of saturated chains from $\emptyset$ to I; equivalently, these are order preserving bijections $I \to k$. We now consider results relating to this measure for F.

Proposition 2. The number of saturated chains $e(I)$ satisfies

$$\sum_{|I|=k} e(I) = t_k \qquad \text{and} \qquad \sum_{|I|=k} e(I)^2 = k!,$$

where we are summing over all elements $I \in F$ of rank k and $t_k = \#\{w \in \mathfrak{S}_k \mid w^2 = 1\}$ satisfies $t_0 = t_1 = 1$ and $t_{k+1} = t_k + kt_{k-1}$.

Proof. Take $x_k = \sum_{|I|=k} e(I)$ and $y_k = \sum_{|I|=k} e(I)^2$. Let x be the element $(1, 1)$ in K. Note that the order-preserving bijections $\sigma : I \to k$ satisfy either (1) $x \in I$ or (2) $x \notin I$. In case (1), the ordering induced by σ with $\sigma(x)$ and $\sigma(\hat{0})$ removed forms an order-preserving bijection $I - \{\hat{0}, x\} \to k - 2$, which can be done in x_{k-2} ways; but $\sigma(x)$ has $k - 1$ possible values, so this gives a total of $(k-1)x_{k-2}$ such maps. In case (2), removing $\sigma(\hat{0})$ gives a map $I - \{\hat{0}\} \to k - 1$, which can be done in x_{k-1} ways. This means that $x_k = x_{k-1} + (k-1)x_{k-2}$, which is exactly the recursion for t_k; the proof follows from checking the initial conditions $x_0 = x_1 = 1$.

We now count pairs of order-preserving bijections $\sigma, \tau : I \to k$. Again we consider cases (1) and (2). In case (1), we can specify $\sigma(x)$ and $\tau(x)$ in a total of $(k - 1)^2$ ways to obtain $(k - 1)^2 y_{k-2}$. In case (2), we simply remove $\sigma(\hat{0})$ and $\tau(\hat{0})$ as before to obtain y_{k-1}. Thus we find the recursion $y_k = y_{k-1} + (k - 1)^2 y_{k-2}$, which finishes the proof upon checking that $y_0 = y_1 = 1$. $\square$

These values are identical to those obtained for a Young tableau (viewed as order ideals of the poset $\mathbb{N}^2$), suggesting that we can define an analogue using the Fibonacci lattice.

Definition 3. Define a *Fibonacci tableau* to be a pair (I, σ), where $I \in F$ is an element of the Fibonacci lattice, and $\sigma : I \to k$ is an order-preserving bijection. We call I the shape of the tableau, and we call k the size.

Note here that σ defines a labeling of the elements of the order ideal I. A Fibonacci tableau with shape shown in our first example is given below in Figure 2. Interpreted in

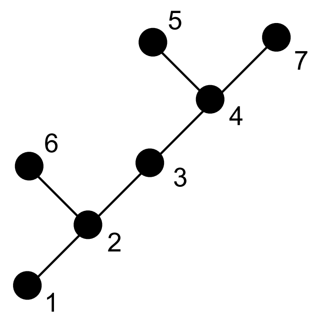

Figure 2. A Fibonacci tableau with shape 12121

this context, the results of Proposition 2 suggest that there should exist a correspondence between certain permutations and Fibonacci tableaux in the spirit of the one given for Young tableaux. Such a correspondence is given by the following proposition.

Proposition 4. Let F_k be the set of Fibonacci tableaux of size k. There exists a bijection $\{w \in \mathfrak{S}_k \mid w^2 = 1\} \simeq F_k$.

Proof. Given a tableau (I, σ), we construct $w \in \mathfrak{S}_k$ with $w^2 = 0$ to be the permutation with two-cycles given by the pairs $(\sigma(i,0), \sigma(i,1))$ for all i for which $(i,1) \in I$. For instance, for the permutation in Figure 2, we have $w = (1)(2\,6)(3)(4\,5)(7)$. Any order two permutation can be obtained in this way, so this establishes the desired bijection. $\square$

A similar bijection exists for $\mathfrak{S}_k \simeq \left\{ \Big((I,\sigma),(I,\tau)\Big) \Big| \, \sigma, \tau : I \to k \right\}$, where the latter is the set of pairs of Fibonacci tableaux of the same size and shape. For a pair of tableau (I, σ) and (I, τ), construct a permutation w as follows. Given $x = (m, n) \in I$, take $x' = (m, 1-n)$ and define w by

$$w(\sigma(x)) = \begin{cases} \tau(x) & x' \in I \\ \tau(x') & x' \notin I. \end{cases}$$

Note that w is a valid permutation since $\tau(x)$ appears only once in the image of w. These two bijections place Fibonacci tableaux in close relation to Young tableaux, and an investigation of such tableaux in more generality is given in [3].

3. Two Generalizations

In Section 2, we introduced the interpretation of the Fibonacci lattice F as words with letters in the set $\{1, 2\}$. In this section, we discuss two different generalizations of this lattice given by Stanley in [6]. Given some $r \in \mathbb{N}$, define the letter set $A_r = \{1_1, \ldots, 1_r, 2\}$, and let A_r^* be the free monoid generated by A_r. We now define the following two posets on the same underlying set A_r^*.

Definition 5. Define $\mathrm{Fib}(r)$ to be the poset with objects A_r^* and order relation such that u covers v if and only if v is obtained from u by either (1) deleting the terminal element if it is of the form 1_i or (2) changing a 2 in u to 1_i for some i.

Definition 6. Define $Z(r)$ to be the poset with objects A_r^* and order relation such that u covers v if and only if v is obtained from U by either (1) deleting the first letter that is not 2 or (2) changing one of the initial string of 2's in u to a 1_i for some i.

Note that Fib(1) is exactly the lattice F discussed in Section 2. Further, both Fib(r) and $Z(r)$ are graded with $\hat{0}$ with rank function ρ given by the sum of the elements as numbers, since each cover step reduces this sum by 1. Like Fib(1), then, $Z(1)$ also has the Fibonacci number f_k as the number of elements of rank k, providing justification for its name. Note further that Fib(r) is a upper-semimodular lattice, while $Z(r)$ is a modular lattice [6].

The lattice $Z(r)$ is one of the more prominent examples of an r-differential poset (defined below) as introduced by Stanley in [6] and is known as the Fibonacci r-differential poset. As such, it is a natural analogue to the more widely studied Young tableau, the prototypical differential poset, and we may expect it to have some similar properties.

Definition 7. For $r \in \mathbb{N}$, let a poset P be *r-differential* if it satisfies the following conditions:

1. P is locally finite and graded with $\hat{0}$
2. If $x \neq y$, then the number of elements in P covered by both x and y is equal to the number covering both x and y.
3. If $x \in P$ covers k elements in P, then x covers $k + r$ elements in P.

While $Z(r)$ is r-differential, Fib(r) is not; however, Fib(r) does have some nice properties resulting from its close relation to F. For example, we have the following proposition in the vein of our previous results for F.

Proposition 8. The square sum of the number of maximal chains to elements of rank k is given for Fib(r) by

$$\sum_{|x|=k} e(x)^2 = r^k k!.$$

Proof. Interpret $\sum_{|x|=k} e(x)^2$ as the number of pairs of saturated chains between $\hat{0}$ and elements of rank k. Any such pair of chains in Fib(r) can also be viewed in Fib(1) by ignoring the subscripts on the 1's. Let $o(x)$ and $t(x)$ be the number of 1's and 2's, respectively, in x. Then, note that each $I \in \mathrm{Fib}(1)$ is thus mapped to by $r^{o(x)}$ different elements of Fib(r), and each element $x' \in \mathrm{Fib}(r)$ that maps to I has $e(\mathrm{Fib}(r), x') = r^{t(x)} e(\mathrm{Fib}(1), x)$, since each time we convert 2 to 1_i, we introduce a new choice between the r possible values of i. Therefore, we find that

$$\sum_{|x|=k, x\in \mathrm{Fib}(r)} e(\mathrm{Fib}(r), x)^2 = \sum_{|x|=k, x\in \mathrm{Fib}(r)} r^{2t(x)} e(\mathrm{Fib}(1), x)^2 = \sum_{|x|=k, x\in \mathrm{Fib}(1)} r^{o(x)+2t(x)} e(\mathrm{Fib}(1), x)^2 = r^k k!,$$

since by Proposition 2, there are $k!$ such pairs in Fib(1). $\square$

Remarkably, that these two lattices share several properties relating to their linear extensions and more generally multichains, allowing us to transfer some calculations for Fib(r) to $Z(r)$. The result below, which we initially state without proof, establishes this correspondence.

Theorem 9. For the lattices Fib(r) and $Z(r)$, we have that the following properties hold:

1. The number of maximal chains $e(x)$ of $[\hat{0}, x]$ is equal in Fib(r) and $Z(r)$ for all $x \in A_r^*$.
2. The number of multichains $\hat{0} = x_0 \leq x_1 \leq \cdots \leq x_n = x$ of length n in $[\hat{0}, x]$ is equal in Fib(r) and $Z(r)$ for all $x \in A_r^*$.

Part 1 of this theorem and Proposition 8 together give us that $\sum_{|x|=k} e(Z(r),x)^2 = r^k k!$ by carrying over the result from $\mathrm{Fib}(r)$. The correspondence between $\mathrm{Fib}(r)$ and $Z(r)$ also satisfies the following further properties for the direct enumeration of elements and chains that follow from Theorem 9.

Corollary 10. For any x, the number of elements in $[\emptyset, x]$ is the same for $\mathrm{Fib}(r)$ and $Z(r)$.

Proof. This follows by taking 2 element multichains in Theorem 9. □

Corollary 11. For any x, the number of n-element chains in $[\emptyset, x]$ is the same for $\mathrm{Fib}(r)$ and $Z(r)$.

Proof. Let $L_n(P)$ be the number of multichains of length n in a poset P and let c_n be the number of n-element chains. Note that any n-element multichain is composed of a j-element chain for $j < n$ (chosen in c_j ways) and then $n-1-j$ elements equal to those in the chain (chosen in $\left(\!\binom{j-1}{n-j+1}\!\right) = \binom{n-2}{j-1}$ ways). Thus, we see that

$$L_n(P) = \sum_{j=1}^{n} c_j \binom{n-2}{j-1},$$

so the numbers $L_n(P)$ determine the numbers c_n uniquely and the result follows. □

Stanley proved Theorem 9 using largely algebraic methods in [6] and [7], respectively, and asked for more combinatorial bijective proofs, some of which have since been found. These proofs are fairly involved, and we choose to present them separately in the next section.

4. Two Bijections

In [4] and [5], Kremer and O'Hara gave bijective proofs of Theorem 9 part 1 and Corollary 10, answering Stanley's call for a combinatorial resolution to these problems. We present the idea of these proofs in the theorems below.

Theorem 12. For $x \in A_r^*$, the number $e(x)$ of maximal chains $\hat{0} = x_0 < x_1 < \cdots < x_n = x$ satisfies $e(\mathrm{Fib}(r), x) = e(Z(r), x)$.

Proof sketch. The bijection consists of the following three steps:

1. Map a maximal chain Y to a matrix $M(Y)$ by inserting 0's into the locations of removed elements and viewing the resulting words as column vectors of a matrix. An example is given below for the chain $Y = \emptyset < 1_1 < 1_1 1_2 < 1_1 2 < 1_1 2 1_2$ in $[\hat{0}, 1_1 2 1_2]_{\mathrm{Fib}(r)}$:
$$M(Y) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1_1 \\ 0 & 0 & 1_1 & 2 & 2 \\ 0 & 1_2 & 1_2 & 1_2 & 1_2 \end{bmatrix}.$$
2. Create a modified matrix $\bar{M}(Y)$ from $M(Y)$ by removing all elements below the first non-zero entry and the first 2 sequentially row by row beginning at the top. This process applied to the matrix above yields
$$\bar{M}(Y) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1_1 \\ 0 & 0 & 1_1 & 2 & \\ 0 & 1_2 & & & \end{bmatrix}.$$

Note that we can reconstruct $\bar{M}(Y)$ given $M(Y)$, since any removed entry must take a value equal to the entry on its left (since some other entry has already changed at that step in the chain).

3. Given a modified matrix $\bar{M}(Y)$ for a chain $Y \in [\hat{0}, x]_{Z(r)}$, we construct a modified matrix $\bar{M}(Z)$ for a chain $Z \in [\hat{0}, x]_{\mathrm{Fib}(r)}$. Note that each row is of the form $(0, 0, \ldots, 0, 1_i, 2, \ldots, 2)$; sequentially from the top of the matrix, we change the tth row to $(0, 1_i, 1_i, \ldots, 1_i, 2, \ldots, 2)$ (ignoring removed entries) and then apply the modification process to the tth row of the partially formed matrix. The resulting $\bar{M}(Z)$ for our example is then

$$\bar{M}(Z) = \begin{bmatrix} 0 & 1_1 & 1_1 & 1_1 & 1_1 \\ 0 & & 1_1 & 1_1 & 2 \\ 0 & & & 1_2 & \end{bmatrix}.$$

We now claim that the final map between modified matrices $\bar{M}(Y)$ and $\bar{M}(Z)$ is a bijection. Note that we can reverse this map by reversing its action on rows, so it suffices to show that each row in $\bar{M}(Y)$ is of the form $(0, \ldots, 0, 1_i, 2, \ldots, 2)$ and that each row in $\bar{M}(Z)$ is of the form $(0, 1_i, \ldots, 1_i, 2, \ldots, 2)$ for chains $Y \in [\hat{0}, x]_{Z(r)}$ and $Z \in [\hat{0}, x]_{\mathrm{Fib}(r)}$. Note that the cover relations for $Z(r)$ guarantee that an occurrence of 1_i that is not below the first occurrence of a non-zero entry in a row can only happen once, since once 1_i is inserted, all changes can only occur in rows above 1_i; this guarantees the desired form for the rows. A similar observation for $\mathrm{Fib}(r)$ completes the proof. □

Theorem 13. For any $x \in A_r^*$, the interval $[\hat{0}, x]$ has the same number of elements in both $\mathrm{Fib}(r)$ and $Z(r)$.

Proof sketch. Define the r-Boolean tree $T_{(n,r)}$ to be a rooted tree such that (1) the root is labeled $(n{+}1)_r$ and (2) a vertex labeled k_i has the $r(k{-}1)$ children labeled $1_1, 1_2, \ldots, 1_r, \ldots, (k-1)_1, (k-1)_2, \ldots, (k-1)_r$.

Define an r-colored subset of $[n]$ to be a subset of $[n]$ in which each element has a label in $[r]$. Then, take the r-Boolean poset $B_{(n,r)}$ to be the set of r-colored subsets of $[n]$ ordered by reverse inclusion. Observe that $T_{(n,r)}$ and $B_{(n,r)}$ each have $(r+1)^n$ elements; an easy induction on n suffices for $T_{(n,r)}$ and we have $|B_{(n,r)}| = \sum_{k=0}^{n} r^k \binom{n}{k}$.

Take $M_{\mathrm{Fib}(r)}(x)$ to be the sublattice of $[\hat{0}, x]_{\mathrm{Fib}(r)}$ containing $\hat{0}$ and x such that the lattice consists of all ways to build up x as a word from left to right; for instance, for $x = 1_2 2 1_1$, the rank 1, 2, and 3 elements would be $\{1_2\}$, $\{1_2 1_1, 1_2 1_2\}$, and $\{1_2 2\}$. Further, take $M_{Z(r)}(x)$ to be the sublattice of $[\hat{0}, x]_{Z(r)}$ such that the 2's in the word x are built up and then the 1's inserted right to left. For $x = 1_2 2 1_1$ again, the elements of rank 1, 2, and 3 are $\{1_1, 1_2\}$, $\{2\}$, and $\{2 1_2\}$. Note that $M_{\mathrm{Fib}(r)}(x)$ and $M_{Z(r)}(x)$ are sublattices of $[\hat{0}, x]$.

The idea of the proof is then as follows. We partition $[\hat{0}, x]_{\mathrm{Fib}(r)}$ into the disjoint sublattices $M_{\mathrm{Fib}(r)}(x)$ and a number of subposets isomorphic to $B_{(n,r)}$ for some n; $[\hat{0}, x]_{Z(r)}$ is similarly divided into $M_{Z(r)}(x)$ and an equal number of subposets isomorphic to $T_{(n,r)}$. The existence of bijections $M_{\mathrm{Fib}(r)}(x) \simeq M_{Z(r)}(x)$ and $T_{(n,r)} \simeq B_{(n,r)}$ then completes the proof. We omit the details of these bijections here for brevity's sake. □

5. Discussion

The Fibonacci lattices provide a variety of unexpected analogues to existing combinatorial structures. Viewed as tableaux, they can be placed in comparison with the more traditional

Young tableaux, while their connection with differential posets is made more clear by the correspondence between $\mathrm{Fib}(r)$ and $Z(r)$. The bijections given in Section 4 between maximal chains and intervals of these lattices characterize the transformation of the non-differential poset $\mathrm{Fib}(r)$ to the r-differential poset $Z(r)$. To some degree, then, they measure how close $\mathrm{Fib}(r)$ is to being r-differential and provide some intuition for how much information an r-differential poset carries.

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