Quaternionic Degenerate Eisenstein series on $\mathbf{U}(2,n)$

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Abstract

tba

1 Unitary groups of split rank 2

If some notation appearing in this section is not defined, then it should be defined in [HMY24].

Let E be an imaginary quadratic extension of $\mathbb{Q}$, and we denote the nonzero element in $\mathrm{Gal}(E/\mathbb{Q})$ by c or $x \mapsto \overline{x}$. Write the norm of $E/\mathbb{Q}$ as $|\,|$.

Define $\mathbf{G} = \mathbf{U}(V) = \mathbf{U}(2,n)$ to be the unitary group of a non-degenerate Hermitian space $(V, \langle\ ,\ \rangle)$ over E with signature $(2,n)$. We write the $\mathbf{G}$-action on V as a right action.

Fix a pair of isotropic lines $(U, U^\vee)$ inside V such that $\langle U, U^\vee \rangle \neq 0$, and take V_0 to be the orthogonal complement of $U \oplus U^\vee$, which is a Hermitian space of signature $(1, n-1)$. Fix $b_1 \in U$ and $b_2 \in U^\vee$ such that $\langle b_1, b_2 \rangle = 1$.

One can define a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ as the stabilizer of U, and it has the following realization of Levi decomposition $\mathbf{P} = \mathbf{MN}$:

— $\mathbf{M}$ is the stabilizer of U and $U^\vee$, and it is isomorphic to $\mathrm{Res}_{E/\mathbb{Q}}(\mathbf{G}_\mathrm{m}) \times \mathbf{U}(V_0)$. One can write the $\mathbf{M}$-action on $(u, w, u^\vee) \in U \oplus V_0 \oplus U^\vee$ explicitly:

$$(u, w, u^\vee).(z, h) = (z^{-1}u, wh, \overline{z}u^\vee).$$

— $\mathbf{N}$ is a Heisenberg, and it is isomorphic to the group

$$\left\{(v, \lambda) \in V_0 \times \mathrm{Res}_{E/\mathbb{Q}}\mathbf{G}_\mathrm{a} \,\middle|\, \overline{\lambda} = -\lambda\right\},$$

equipped with the following multiplication:

$$(v_1, \lambda_1) \cdot (v_2, \lambda_2) := \left(v_1 + v_2, \lambda_1 + \lambda_2 - \frac{\langle v_1, v_2 \rangle - \langle v_2, v_1 \rangle}{2}\right).$$

The action of the corresponding element $n(v, \lambda) \in \mathbf{N}$ on V is given as:

$$b_1 \mapsto b_1,\ b_2 \mapsto (-\frac{1}{2}\langle v, v \rangle + \lambda)b_1 + b_2 + v,\ w \in V_0 \mapsto -\langle w, v \rangle b_1 + w.$$

One has

$$(z,h)n(v,\lambda)(z^{-1},h^{-1}) = n(\bar{z}vh^{-1}, |z|\lambda).$$

Denote the center of $\mathbf{N}$ by $\mathbf{N}_0$. We denote by ν the similitude character of $\mathbf{M}$: $\nu(z,h) = z \in \mathrm{Res}_{E/\mathbb{Q}}\mathbf{G}_{\mathrm{m}}$, so that $b_1(z,h) = \nu(z,h)^{-1}b_1$.

Fix an addictive character $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$, so that ψ_p has conductor $\mathbb{Z}_p$ and $\psi_\infty(x) = e^{2\pi i x}$. We have the following identification $V_0 \xrightarrow{\sim} \mathrm{Hom}([\mathbf{N}], \mathbb{S}^1)$: for any element $T \in V_0$, one associates a unitary character χ_T of $[\mathbf{N}]$ by

$$\chi_T : [\mathbf{N}] \to \mathbb{C}^\times,\ n(v,\lambda) \mapsto \psi(-\mathrm{Im}\langle T, v\rangle).$$

I am really confused with this identification in [HMY24]. This character χ_T seems not well-defined: the symplectic form $-\mathrm{Im}\langle T, v\rangle = \frac{\langle v,T\rangle - \langle T,v\rangle}{2i}$ involves $1/2i$, which may not lie in E. Maybe one should define χ_T to be $n(v,\lambda) \mapsto \psi(\mathrm{Re}\langle T, v\rangle)$.

For a finite place p of $\mathbb{Q}$, we define E_p to be $E_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} E$, which is isomorphic to

— $\mathbb{Q}_p \times \mathbb{Q}_p$, if p splits in E;

— a degree 2 unramified (*resp.* ramified) extension of $\mathbb{Q}_p$, if p is inert (*resp.* ramified).

2 Heisenberg Eisenstein series

Choose a section $f = f_{\ell,\infty} \otimes f_{fte}$ of $\mathrm{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}|\nu|^s$ (unnormalized) as follow:

— $f_{\ell,\infty}$ is the $\mathbb{V}_\ell = \left(\mathrm{Sym}^{2\ell} V_2^+ \otimes \det_{\mathbf{U}(2)}^{-\ell}\right) \boxtimes \mathbf{1}$-valued, K_∞-equivariant induced section whose restriction to $\mathbf{M}(\mathbb{R})$ is

$$f_{\ell,\infty}((z,h),s) = |z|^s[u_1^\ell][u_2^\ell],$$

where $[u_i^k] := u_i^k/k!$.

— f_{fte} is defined as

$$f_{fte}(g_f, \Phi_f, s) = \int_{\mathbf{GL}_1(\mathbb{A}_{E,f})} |t|^s \Phi_f(t \cdot b_1 g_f) dt,$$

where $\Phi_f = \prod \Phi_p$ is a Schwartz-Bruhat function on $V(\mathbb{A}_f)$.

One defines the degenerate Heisenberg Eisenstein series:

$$E_\ell(g, \Phi_f, s) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})} f(\gamma g, s).$$

To prove the modularity of this Eisenstein series, one needs the following result:

Proposition 2.1. *The $\mathbb{V}_\ell$-valued section $f_{\ell,\infty}$ of $\mathrm{Ind}_{\mathbf{P}(\mathbb{R})}^{\mathbf{G}(\mathbb{R})}|\nu|^s$, as a function on $\mathbf{G}(\mathbb{R})$, is killed by the Schmid operators $\mathcal{D}_\ell^\pm$ if and only if $s = \ell + 1$.*

Proof. Write an element $(z,h) \in \mathbf{M}(\mathbb{R})$ as $m = (h,r,\theta)$ so that $z = re^{i\theta}$, $r \in \mathbb{R}_{>0}$ and $\theta \in [0,2\pi)$. Using this coordinates, we define a function F on $\mathbf{M}(\mathbb{R})$, sending (h,r,θ) to r^s. By [HMY24, Proposition 3.10], $f_{\ell,\infty}$ is killed by $\mathcal{D}_\ell^\pm$ if and only if

$$\begin{cases} (r\partial r - 2(\ell+1))F(m) = 0 \\ [u_2 \otimes \overline{v_k}]^+ F(m) = 0, \text{ if } 1 \le k < n \\ [u_2 \otimes \overline{v_k}]^- F(m) = 0, \text{ if } 1 \le k < n \end{cases}$$

It can be easily verified that these equations hold if and only if $s = \ell + 1$. □

As a consequence, the Eisenstein series $E_\ell(g, \Phi_f, s = \ell+1)$ is a quaternionic modular form of weight ℓ.

3 The Fourier expansion of $E_\ell(g, \Phi_f, s = \ell+1)$

3.1 Abstract Fourier expansion

In this subsection, we give the "abstract" Fourier expansion of $E_\ell(g, \Phi_f, s)$.

Lemma 3.1. *The right $\mathbf{P}(\mathbb{Q})$-space $\mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})$, the space of isotropic lines in $V(\mathbb{Q})$, has exactly 3 orbits of $\mathbf{P}(\mathbb{Q})$, represented respectively by $\mathbb{Q}b_1$, $\mathbb{Q}v_0$ and $\mathbb{Q}b_2$, where v_0 is an arbitrary non-zero isotropic vector in $V_0(\mathbb{Q})$.*

Proof. Directly by the explicit action given in §1. □

Set $\mathbf{G}(\mathbb{Q}) = \bigsqcup_{i=0}^2 \mathbf{P}(\mathbb{Q}) w_i \mathbf{P}(\mathbb{Q})$, such that $w_0 = 1$, $b_1 w_1 = v_0$ and $b_1 w_2 = b_2$. Now we can write the degenerate Eisenstein series as

$$E_\ell(g, \Phi_f, s) = \sum_{i=0}^{2} E_{\ell,i}(g, \Phi_f, s),\ E_{\ell,i}(g, \Phi_f, s) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q})\backslash \mathbf{P}(\mathbb{Q}) w_i \mathbf{P}(\mathbb{Q})} f(\gamma g, s),$$

thus $E_{\ell,0}(g, \Phi_f, s) = f(g,s)$. From now on, when there is no confusion we will omit the ℓ and Φ_f in $E_\ell(g, \Phi_f, s)$, and write it as $E(g,s) = \sum_{i=0}^2 E_i(g,s)$.

Lemma 3.2. *Assume that* $\mathrm{Re}(s) \gg 0$ *so that the sum defining $E(g,s)$ converges absolutely. Then one has the following expressions for the $E_i(g,s)$:*

(1) Let $\mathcal{L}_0$ be the set of non-zero isotropic lines ℓ in V_0 and for any $\ell \in \mathcal{L}_0$, select $\gamma(\ell) \in \mathbf{G}(\mathbb{Q})$ with $b_1\gamma(\ell) \in \ell$. Then

$$E_1(g,s) = \sum_{\ell \in \mathcal{L}_0} \sum_{\mu \in (\ell)^\perp \mathbf{N}_0(\mathbb{Q}) \backslash \mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu g, s).$$

(2) One has

$$E_2(g,s) = \sum_{\mu \in \mathbf{N}(\mathbb{Q})} f(w_2 \mu g, s).$$

For any $T \in V_0$, we set

$$E_i^T(g,s) = \int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) E_i(ng,s)dn,\ i = 0,1,2.$$

Lemma 3.3. *(1) If T is anisotropic, then $E_1^T = 0$. If T is isotropic, define $\mathbf{N}_T = (\ell_T)^{\perp}\mathbf{N}_0 \subseteq \mathbf{N}$, then*

$$E_1^T(g,s) = \int_{\mathbf{N}_T(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell_T)ng,s)dn.$$

(2) For any $T \in V_0$, one has

$$E_2^T(g,s) = \int_{\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(w_2 ng,s)dn.$$

Proof. It suffices only to prove the $i = 1$ case. For any $\ell \in \mathcal{L}_0$, set

$$\mathbf{N}_\ell = \left\{ n(v,\lambda) \in \mathbf{N} \,\middle|\, v \in \ell^{\perp} \right\} \subseteq \mathbf{N}.$$

For any $T \in V_0$,

$$\begin{aligned}
E_1^T(g,s) &= \sum_{\ell \in \mathcal{L}_0} \int_{[N]} \chi_T^{-1}(n) \left(\sum_{\mu \in \mathbf{N}_\ell(\mathbb{Q})\backslash\mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu ng,s) \right) dn \\
&= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{Q})\backslash\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell)ng,s)dn \\
&= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})} \left(\int_{[\mathbf{N}_\ell]} \chi_T^{-1}(r)dr \right) \chi_T^{-1}(n) f(\gamma(\ell)ng,s)dn \\
&= \sum_{\ell \in \mathcal{L}_0,\, \chi_T|_{\mathbf{N}_\ell} \equiv 1} \int_{\mathbf{N}_\ell(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell)ng,s)dn.
\end{aligned}$$

Then the lemma follows from the fact that $\chi_T|_{\mathbf{N}_\ell} \equiv 1$ if and only if $T \in \ell$, *i.e.* T is isotropic and $\ell = \ell_T$. □

3.2 Computation of constant term

The constant term of $E_\ell(g,\Phi_f,s = \ell+1)$ consists of three parts:

— $E_{\ell,0}(g,\Phi_f,s) = f(g,s)$,

— $E_1^0(g,s) = \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})} f(\gamma(\ell)ng,s)dn$,

— and $E_2^0(g,s) = \int_{\mathbf{N}(\mathbb{A})} f(w_2 ng)dn$.

3.2.1 The $i = 0$-term

Lemma 3.4. *For $g \in \mathbf{P}(\mathbb{A})$,*

$$E_0(g,s) = f(g,s) = |\nu(g)|_E^s \zeta_E(s)[u_1^n][u_2^n].$$

Proof. For $g_f \in \mathbf{P}(\mathbb{A}_f)$, we have

$$\begin{aligned} f_{fte}(g_f,s) &= \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(tb_1 g_f)dt \\ &= \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(t\nu(g_f)^{-1}b_1)dt \\ &= |\nu(g_f)|_E^s \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(tb_1)dt. \end{aligned}$$

Thus, the non-archimedean contribution is $|\nu(g_f)|_E^s \zeta_E(s)$. Combining with $f_{n,\infty}(g_\infty, s) = |\nu(g_\infty)|_E^s [u_1^n][u_2^n]$, we get the desired identity. □

3.2.2 The $i = 1$-term

We fix a non-zero isotropic vector in V_0, such that $v_0 = b_1\gamma_0$, and set $\ell_0 = Ev_0$. Define $\mathbf{P}_0$ be the stabilizer of ℓ_0 in $\mathbf{U}(V_0)$, which is a parabolic subgroup of $\mathbf{M}$. We denote the similitude character of $\mathbf{P}_0$ by λ, *i.e.* $v_0 g = \lambda(g)^{-1}v_0$ for any $g \in \mathbf{P}_0$. For $g \in \mathbf{P}_0(\mathbb{A})$, we have:

$$\begin{aligned} E_1^0(g,s) &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A})\backslash \mathbf{N}(\mathbb{A})} f(\gamma(\ell)ng,s)dn \\ &= \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q})\backslash \mathbf{M}(\mathbb{Q})} \int_{\mathbf{N}_{\ell_0}(\mathbb{A})\backslash \mathbf{N}(\mathbb{A})} f(\gamma_0\gamma ng,s)dn \end{aligned}$$

If we set $f_0(g,s) = \int_{\mathbf{N}_{\ell_0}(\mathbb{A})\backslash \mathbf{N}(\mathbb{A})} f(\gamma_0 ng,s)dn$, then for $\mathrm{Re}(s) \gg 0$,

$$E_1^0(g,s) = \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q})\backslash \mathbf{M}(\mathbb{Q})} f_0(\gamma g,s),$$

and it defines an Eisenstein series on $\mathbf{M}$. Now we want to determine this section $f_0(g,s) \in \mathrm{Ind}_{\mathbf{P}_0(\mathbb{A})}^{\mathbf{M}(\mathbb{A})}|\lambda|^s$ (still unnormalized).

At finite places, one has

$$\begin{aligned} \int_{\mathbf{N}_{\ell_0}(\mathbb{A}_{E,f})\backslash \mathbf{N}(\mathbb{A}_{E,f})} f(\gamma_0 ng,s)dn &= \int_{x\in\mathbb{A}_{E,f}} \int_{t\in\mathbb{A}_{E,f}^\times} |t|^s \Phi_{fte}(t(v_0 + xb_1)g)dtdx \\ &= \int_{x\in\mathbb{A}_{E,f}} \int_{t\in\mathbb{A}_{E,f}^\times} |t|^s \Phi_{fte}(t\lambda(g)^{-1}v_0 + tx\nu(g)^{-1}b_1)dtdx \\ &= |\lambda(g)|_f^{s-1}|\nu(g)|_f \int_{x\in\mathbb{A}_{E,f}} \int_{t\in\mathbb{A}_{f,E}^\times} |t|^{s-1}\Phi_{fte}(tv_0 + xb_1)dtdx \\ &= |\lambda(g)|_f^{s-1}|\nu(g)|_f \zeta_E(s-1). \end{aligned}$$

Now we switch to the archimedean place. Set $c_1 = v_0$ and c_2 another isotropic vector in V_0 with $\langle c_1, c_2 \rangle = 1$. We can take $u_2 = \frac{1}{\sqrt{2}}(c_1 + c_2)$ and $v_{n-1} = \frac{1}{\sqrt{2}}(c_1 - c_2)$.

Lemma 3.5. *We have*

$$f_{0,\infty}(g,s) =$$

Proof. One picks the following representatives for $\mathbf{N}_{\ell_0}(\mathbb{R})\backslash\mathbf{N}(\mathbb{R})$: $\{n(-xc_2, 0),\ x \in \mathbb{R}\}$, thus $b_1\gamma_0 n(-xc_2,0)g = \lambda(g)^{-1}c_1 + x\nu(g)^{-1}b_1$. Suppose that $\gamma_0 n(-xc_2,0)g = pk$ for some $p \in \mathbf{P}(\mathbb{R})$ and $k = (k^+, k^-) \in K_\infty$, then one has

$$\lambda(p)^{-1}b_1 k = \lambda(g)^{-1}c_1 + x\nu(g)^{-1}b_1.$$

Projecting both sides to V_2^+, one has

$$\lambda(p)^{-1}u_1 k^+ = x\nu(g)^{-1}u_1 + \lambda(g)^{-1}u_2.$$

So one can take p such that

$$\lambda(p)^{-1} = \sqrt{|\lambda(g)|^{-2} + x^2|\nu(g)|^{-2}}$$

and

$$k^+ = \lambda(p)\begin{pmatrix} x\nu(g)^{-1} & -\overline{\lambda(g)}^{-1} \\ \lambda(g)^{-1} & x\overline{\nu(g)}^{-1} \end{pmatrix}.$$

Hence we have

$$\begin{aligned} f_{\ell,\infty}(\gamma_0 n(-xc_2,0)g) &= f_{\ell,\infty}(pk) \\ &= |\nu(p)|^s([u_1^\ell][u_2^\ell])k \\ &= \frac{(x\nu(g)^{-1}u_1 + \lambda(g)^{-1}u_2)^\ell(-\overline{\lambda(g)}^{-1}u_1 + x\overline{\nu(g)}^{-1}u_2)^\ell}{(|\lambda(g)|^{-2} + x^2|\nu(g)|^{-2})^{s+\ell}(\ell!)^2} \end{aligned}$$

The coefficient of $[u_1^{\ell-v}][u_2^{\ell+v}]$ in $f_{\ell,\infty}(\gamma_0 n(-xc_2,0)g)$ is

$$\begin{aligned} &\frac{(\ell-v)!(\ell+v)!}{(\ell!)^2(|\lambda|^{-2} + x^2|\nu|^{-2})^{2\ell+1}} \sum_{\substack{0\le i,j\le \ell \\ i+j=\ell-v}} \binom{\ell}{i}\binom{\ell}{j} x^i \nu^{-i}\lambda^{-(\ell-i)}(-\overline{\lambda})^{-j}(x\overline{\nu}^{-1})^{\ell-j} \\ =&\frac{(\ell-v)!(\ell+v)!|\nu|^{-2\ell}\nu^v\lambda^{-v}}{(\ell!)^2(|\lambda|^{-2} + x^2|\nu|^{-2})^{2\ell+1}} \sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)} (-1)^j \binom{\ell}{j}\binom{\ell}{j+v} |\nu|^{2j}|\lambda|^{-2j}x^{2\ell-v-2j} \end{aligned}$$

Using the fact that

$$\int_{x\in\mathbb{R}} \frac{x^{2m}}{(|\lambda|^{-2} + x^2|\nu|^{-2})^{2\ell+1}} = \pi|\lambda|^{4\ell-2m+1}|\nu|^{2m+1}2^{-4\ell}\frac{(4\ell-2m)!(2m)!}{(2\ell-m)!m!(2\ell)!},$$

we can integrate $f_{\ell,\infty}(\gamma_0 n(-xc_2,0)g)$ over $x\in\mathbb{R}$, and its $[u_1^{\ell-v}][u_2^{\ell+v}]$ coefficient is 0 when v is odd, otherwise, it is

$$\frac{\pi|\lambda|^{2\ell+1}|\nu|}{(2\ell)!(\ell!)^2 2^{4\ell}}\cdot(\ell-v)!(\ell+v)!(\nu/\lambda)^v|\nu/\lambda|^{-v}\sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)}(-1)^j\binom{\ell}{j}\binom{\ell}{j+v}\frac{(2\ell+v+2j)!(2\ell-v-2j)!}{(\ell+v/2+j)!(\ell-v/2-j)!}.$$

The inner sum is symmetric under $v\mapsto -v$, and when v is non-negative, then it equals

$$\begin{aligned}&\sum_{j=0}^{\ell-v}(-1)^j\binom{\ell}{j}\binom{\ell}{j+v}\frac{(2\ell+v+2j)!(2\ell-v-2j)!}{(\ell+v/2+j)!(\ell-v/2-j)!}\\ =&\frac{2^{4\ell}}{\pi}\Gamma(\ell+v/2+1/2)\Gamma(\ell-v/2+1/2){}_3F_2(-\ell,-\ell+v,\ell+v/2+1;v+1,-\ell+v/2+1/2;1).\end{aligned}$$

So we have

$$f_{0,\infty}(s)=\frac{|\lambda(g)|^{2\ell+1}|\nu(g)|}{(2\ell)!}\sum_{\substack{-\ell\le v\le\ell\\ v\text{ is even}}}\binom{\ell+|v|}{\ell}\Gamma(\ell+v/2+1)\Gamma(\ell-v/2+1)\left(\frac{\nu(g)\overline{\lambda(g)}}{\overline{\nu(g)}\lambda(g)}\right)^{v/2}{}_3F_2(-\ell,-\ell+v,\ell+v/2+1;v+1,-\ell+v/2+1/2;1).$$

□

There must be some mistake in this computation. I need to check it.

3.3 Rank 1 Fourier coefficients

For rank 1 Fourier coefficients, one needs to calculate

$$E_1^T(g,s)=\int_{\mathbf{N}_T(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})}\chi_T^{-1}(n)f(\gamma(\ell_T)ng,s)dn$$

for any nonzero isotropic vector T.

3.3.1 Non-archimedean components

For any prime p, we identify $\mathbf{N}_T(\mathbb{Q}_p)\backslash\mathbf{N}(\mathbb{Q}_p)$ with E_p via the map $n(v,\lambda)\mapsto -\langle T,v\rangle$, thus

$$\begin{aligned}E_{1,p}^T(1,s)&=\int_{x\in E_p}\int_{t\in E_p^\times}\psi_E(x)|t|_E^s\Phi_p(txb_1+tT)dtdx\\&=\int_{x\in E_p}\int_{t\in E_p^\times}\psi_E(x/t)|t|_E^{s-1}\Phi_p(xb_1+tT)dtdx\end{aligned}$$

3.3.2 Archimedean components

Again, we identify $\mathbf{N}_T(\mathbb{R})\backslash\mathbf{N}(\mathbb{R})$ with $\mathbb{C}$ via the map $n(v,\lambda)\mapsto -\langle T,v\rangle$, and for any $z\in\mathbb{C}$ we choose an element $n(z)\in\mathbf{N}(\mathbb{R})$ corresponding to z under this identification.

Lemma 3.6. *For any $z \in \mathbb{C}$, we have*

$$f_{\ell,\infty}(\gamma(\ell_T)n(z), s = \ell+1) = \frac{(zu_1 + \beta u_2)^\ell (-\overline{\beta} u_1 + \bar{z} u_2)^\ell}{(|z|^2 + |\beta|^2)^{2\ell+1}(\ell!)^2},$$

where $\beta = \beta(T) := \sqrt{2}\langle T, u_2\rangle$.

Proof. Suppose that $\gamma(\ell_T)n(z) = pk$ for $p \in \mathbf{P}(\mathbb{R})$ and $k = (k_+, k_-) \in K_\infty = \mathrm{U}(V_2^+) \times \mathrm{U}(V_n^-)$, then one has

$$T + zb_1 = b_1\gamma(\ell_T)n(z) = b_1 pk = \nu(p)^{-1} b_1 k.$$

Taking the projection to V_2^+ of both sides, we obtain that

$$zu_1 + \sqrt{2}\langle T, u_2\rangle u_2 = \nu(p)^{-1} u_1 k_+.$$

One can take $\nu(p)$ to be $(|z|^2 + |\beta|^2)^{-1/2}$, then

$$k_+ = \nu(p)\begin{pmatrix} z & -\overline{\beta} \\ \beta & \bar{z} \end{pmatrix} \in \mathrm{U}(V_2^+),$$

which gives us the desired value of $f_{\ell,\infty}$. □

Using Lemma 3.6, the coefficient of $[u_1^\ell][u_2^\ell]$ in $E_{1,\infty}^T := E_{1,\infty}^T(1, s = \ell+1)$ is equal to

$$\int_{z\in\mathbb{C}} e^{2\pi i \mathrm{Re}(z)} \frac{\sum_{k=0}^{\ell} \binom{\ell}{k}^2 |z|^{2k}(-|\beta|^2)^{\ell-k}}{(|z|^2 + |\beta|^2)^{2\ell+1}} dz. \tag{1}$$

We set $B = |\beta|^2$ and $z = x + iy$, where $x, y \in \mathbb{R}$, then the integral in Equation (1) equals

$$\begin{aligned} &\int_{\mathbb{R}^2} e^{2\pi i x} \sum_{k=0}^{\ell} \binom{\ell}{k}^2 (-B)^{\ell-k} \frac{(x^2+y^2)^k}{(x^2+y^2+B)^{2\ell+1}} dx dy \\ =& \int_{\mathbb{R}} e^{2\pi i x} \sum_{k=0}^{\ell} \binom{\ell}{k}^2 (-B)^{\ell-k} \left(\int_{\mathbb{R}} \frac{(y^2+x^2)}{(y^2+x^2+B)^{2\ell+1}} dy \right) dx \end{aligned}$$

By Lemma 3.12, the inner integral over $y \in \mathbb{R}$ equals

$$\frac{(x^2+B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{x^2}{x^2+B}\right)^{k-j} \Gamma(j+1/2)\Gamma(2\ell-j+1/2).$$

Similarly to our calculation in §3.5, we have

$$E_{1,\infty}^T = 2^{-2\ell}\pi \sum_{k=0}^{\ell} \frac{(-B)^k 2^{-2k}(2\ell+2k)!}{(k!)^2(\ell+k)!(\ell-k)!} \int_{\mathbb{R}} \frac{e^{2\pi i x} dx}{(x^2+B)^{\ell+k+1/2}}$$

Using Basset's integral, one has

$$\int_{\mathbb{R}} \frac{e^{iwt}dt}{(t^2+z^2)^{n+1/2}} = \frac{2\sqrt{\pi}}{\Gamma(n+1/2)} \left(\frac{|w|}{2|z|}\right)^n K_n(|wz|),\ w, z \in \mathbb{R}.$$

Plugging this into $E^T_{1,\infty}$, we obtain:

$$\begin{aligned} E^{\infty}_{1,T} &= 2^{-2\ell}\pi \sum_{k=0}^{\ell} \frac{(-B)^k 2^{-2k}(2\ell+2k)!}{(k!)^2(\ell+k)!(\ell-k)!} \cdot \frac{2^{l+k+1}(\ell+k)! B^{-(\ell+k)/2}(2\pi)^{\ell+k} K_{\ell+k}(2\pi\sqrt{B})}{(2\ell+2k)!} \\ &= 2\pi B^{-\ell} \sum_{k=0}^{\ell} \frac{(-1)^k}{(k!)^2(\ell-k)!}(\pi\sqrt{B})^{\ell+k} K_{\ell+k}(2\pi\sqrt{B}). \end{aligned}$$

Lemma 3.7. *For any $C > 0$, we have*

$$S_\ell := \sum_{k=0}^{\ell} \frac{(-1)^k C^{\ell+k} K_{\ell+k}(2C)}{(k!)^2(\ell-k)!} = \frac{(-1)^\ell C^{2\ell}}{(\ell!)^2} K_0(2C).$$

Proof. Using the recurrence relation for K-Bessel functions:

$$K_{n+1}(z) = K_{n-1}(z) + \frac{2n}{z} K_n(z),$$

we can write $-C^2 S_\ell$ as

$$\begin{aligned} -C^2 S_\ell &= \sum_{k=0}^{\ell} (-1)^{k+1} \frac{C^{\ell+k+2}}{(k!)^2(\ell-k)!} K_{\ell+k}(2C) \\ &= \sum_{j=1}^{\ell+1} (-1)^j \frac{C^{\ell+1+j}}{((j-1)!)^2(\ell-j+1)!} \left(K_{\ell+1+j}(2C) - \frac{\ell+j}{C} K_{\ell+j}(2C) \right) \\ &= \sum_{j=1}^{\ell+1} \frac{(-1)^j C^{\ell+1+j} K_{\ell+1+j}(2C)}{((j-1)!)^2(\ell-j+1)!} + \sum_{j=0}^{\ell} \frac{(-1)^j(\ell+j+1)C^{\ell+1+j} K_{\ell+j+1}(2C)}{(j!)^2(\ell-j)!} \\ &= \frac{(\ell+1)C^{\ell+1} K_{\ell+1}(2C)}{\ell!} + \sum_{j=1}^{\ell} \frac{(-1)^j C^{\ell+1+j} K_{\ell+1+j}(2C)}{(j!)^2(\ell+1-j)!}(\ell+1)^2 + (-1)^{\ell+1} \frac{C^{2\ell+2} K_{2\ell+2}(2C)}{(\ell!)^2} \\ &= (\ell+1)^2 S_{\ell+1}. \end{aligned}$$

Since $S_0 = K_0(2C)$ and $S_{\ell+1} = -\frac{C^2}{(\ell+1)^2} S_\ell$, we have

$$S_\ell = (-1)^\ell \frac{C^{2\ell}}{(\ell!)^2} K_0(2C). \qquad \square$$

By this lemma, one has

$$E^{\infty}_{1,T} = 2\pi B^{-\ell} \cdot \frac{(-1)^\ell (\pi\sqrt{B})^{2\ell}}{(\ell!)^2} K_0(2\pi\sqrt{B}) = \frac{2\pi^{2\ell+1}}{(\ell!)^2} K_0(2\pi\sqrt{B}).$$

3.4 Rank 2 Fourier coefficients: non-archimedean components

One decomposes $E_2^T(1,s)$ as $\prod_v E_{2,v}^T(s)$.

$$\begin{aligned}
E_{2,p}^T(s) &= \int_{\mathbf{N}(\mathbb{Q}_p)} \chi_T^{-1}(n) f_p(w_2 n, s) dn \\
&= \int_{\mathbf{N}(\mathbb{Q}_p)} \int_{t\in E_p^\times} \chi_T^{-1}(n)|t|^s \Phi_p(t\cdot b_1 w_2 n) dt dn \\
&= \int_{v\in \mathbf{V}_0(\mathbb{Q}_p)} \int_{\substack{x\in E_p \\ \bar{x}=-x}} \int_{t\in E_p^\times} \chi_T^{-1}(v)|t|^s \Phi_p\left(t\left(\left(-\langle v,v\rangle/2+x\right)b_1+b_2+v\right)\right) dt dx dv
\end{aligned}$$

Take $\mathcal{O}_{E_p}$ to be $\mathbb{Z}_p\otimes_{\mathbb{Z}}\mathcal{O}_E\subseteq E_p$, $\mathfrak{p}$ to be its maximal ideal, and $\mathcal{V}$ an $\mathcal{O}_{E_p}$-lattice of $V_0(\mathbb{Q}_p)=V_0\otimes_{\mathbb{Q}}\mathbb{Q}_p$ such that $\mathcal{V}\otimes_{\mathcal{O}_{E_p}}(\mathcal{O}_{E,p}/\mathfrak{p})$ is a non-degenerate Hermitian space over $\mathcal{O}_{E_p}/\mathfrak{p}$. Assume that Φ_p is the characteristic function of the lattice $\mathcal{O}_{E_p}b_1\oplus\mathcal{V}\oplus\mathcal{O}_{E_p}b_2$.

Write down a precise definition of $\mathcal{V}$?

Lemma 3.8. *(1) If p splits in E, then*

$$E_{2,p}^T(s)=\sum_{r_1,r_2\geq 0} p^{-(r_1+r_2)s+\min(r_1,r_2)}\left(\int_{v\in(p^{-r_1},p^{-r_2})\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^{\max(r_1,r_2)}\langle v,v\rangle\in\mathbb{Z}_p)dv\right).$$

(2) If p is inert in E, then

$$E_{2,p}^T(s)=\sum_{r\geq 0}p^{-2rs+r}\left(\int_{v\in p^{-r}\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}\left(p^r\langle v,v\rangle/2\in\mathbb{Z}_p\right)dv\right).$$

(3) If p is ramified in E, then

Proof. (1) If p splits in E, then $E_p=\mathbb{Q}_p\times\mathbb{Q}_p$:

$$E_{2,p}^T(s)=\sum_{r_1\geq 0, r_2\geq 0}|p|^{(r_1+r_2)s}\int_{v\in(p^{-r_1},p^{-r_2})\mathcal{V}}\chi_T^{-1}(v)\left(\int_{\substack{x\in(p^{-r_1},p^{-r_2})\mathcal{O}+\langle v,v\rangle/2\\ x+\bar{x}=0}}dx\right)dv$$

If we write $x=(y,-y)\in E_p$, then $x\in(p^{-r_1},p^{-r_2})\mathcal{O}+\langle v,v\rangle/2$ is equivalent to

$$y\in p^{-r_1}\mathbb{Z}_p+\langle v,v\rangle/2,\ -y\in p^{-r_2}\mathbb{Z}_p+\langle v,v\rangle/2.$$

There are such x only if $\langle v,v\rangle\in p^{-\max(r_1,r_2)}\mathbb{Z}_p$, If $r_1\geq r_2$, then $y\in p^{-r_2}\widetilde{y}-\langle v,v\rangle/2$ for some $\widetilde{y}\in\mathbb{Z}_p$. Since the measure on the line $\{x=(y,-y)\in E_p\}$ is the normalized Haar measure dy on $\mathbb{Q}_p$, one has $dy=p^{r_2}d\widetilde{y}$, and

$$\int_{x=(y,-y)\in p^{-r_1}\mathbb{Z}_p\times p^{-r_2}\mathbb{Z}_p+\langle v,v\rangle/2}dx=\int_{\widetilde{y}\in\mathbb{Z}_p}p^{r_2}d\widetilde{y}=p^{r_2}=p^{\min(r_1,r_2)}.$$

Hence

$$E_{2,p}^{T}(s)=\sum_{r_1,r_2\geq 0}p^{-(r_1+r_2)s+\min(r_1,r_2)}\left(\int_{v\in(p^{-r_1},p^{-r_2})\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^{\max(r_1,r_2)}\langle v,v\rangle\in\mathbb{Z}_p)dv\right).$$

(2) If p is inert in E, then E_p is an unramified quadratic field extension of $\mathbb{Q}_p$, and

$$E_{2,p}^{T}(s)=\sum_{r\geq 0}|p|_{E_p}^{rs}\int_{v\in p^{-r}\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^r\langle v,v\rangle\in\mathbb{Z}_p)\left(\int_{x\in\langle v,v\rangle/2+p^{-r}\mathcal{O}_{E_p}}\mathrm{Char}(x+\overline{x}=0)dx\right)dv$$

Choose a unit $u\in\mathcal{O}_{E_p}^{\times}$ with $u+\overline{u}=0$, then the line $\left\{x\in E_p\,\middle|\,x+\overline{x}=0\right\}$ can be written as $\{yu,\ y\in\mathbb{Q}_p\}$, with the normalized Haar measure dy. There exist elements $yu\in\langle v,v\rangle/2+p^{-r}\mathcal{O}_{E_p}$ if and only if $\langle v,v\rangle/2\in p^{-r}\mathbb{Z}_p$. When we have $\langle v,v\rangle/2\in p^{-r}\mathbb{Z}_p$, any element $yu\in\langle v,v\rangle/2+p^{-r}\mathcal{O}_{E_p}$ is of the form

$$yu=\langle v,v\rangle/2+p^{-r}\left(-p^r\langle v,v\rangle/2+\widetilde{y}u\right),\ \widetilde{y}\in\mathbb{Z}_p,$$

thus one has

$$\begin{aligned}\int_{x\in\langle v,v\rangle/2+p^{-r}\mathcal{O}_{E_p}}\mathrm{Char}(x+\overline{x}=0)dx&=\mathrm{Char}(p^r\langle v,v\rangle/2\in\mathbb{Z}_p)|p|_{\mathbb{Q}_p}^{-r}\int_{\mathbb{Z}_p}d\widetilde{y}\\&=p^r\mathrm{Char}(p^r\langle v,v\rangle/2\in\mathbb{Z}_p),\end{aligned}$$

which gives us the desired identity. □

If p is ramified in E: (Assume $p\neq 2$) Then E_p is a ramified quadratic field extension of $\mathbb{Q}_p$, and we choose an uniformizer ϖ of $\mathfrak{p}\subset\mathcal{O}_{E_p}$ such that $\varpi+\overline{\varpi}=0$ and $\varpi^2\in p\mathbb{Z}_p^{\times}$. The integral for $E_{2,p}^{T}$ can be rewritten as:

$$\begin{aligned}E_{2,p}^{T}(s)&=\sum_{r\geq 0}|\varpi|^{rs}\int_{v\in\varpi^{-r}\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^{\lfloor r/2\rfloor}\langle v,v\rangle\in\mathbb{Z}_p)\left(\int_{x\in\mathbb{Q}_p}\mathrm{Char}(x\varpi-\frac{\langle v,v\rangle}{2}\in\varpi^{-r}\mathcal{O}_{E_p})dx\right)dv\\&=\sum_{r\geq 0}|\varpi|^{rs}p^{\lceil r/2\rceil}\left(\int_{v\in\varpi^{-r}\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^{\lfloor r/2\rfloor}\langle v,v\rangle\in\mathbb{Z}_p)dv\right)\end{aligned}$$

3.4.1 Split case

When p splits in E, then one can view the Hermitian $\mathbb{Q}_p\times\mathbb{Q}_p$-space $V_0\otimes_{\mathbb{Q}}\mathbb{Q}_p$ as the direct sum of two copies of quadratic $\mathbb{Q}_p$-space $\left(\mathbb{Q}_p^n,((x_i),(y_i))=\sum x_iy_i\right)$, and one has

$$2\mathrm{Re}\langle(x_1,y_1),(x_2,y_2)\rangle=(x_1,y_2)+(x_2,y_1).$$

Denote by L the lattice $\mathbb{Z}_p^n$ in $\mathbb{Q}_p^n$. We set

$$\begin{aligned}S(r_1,r_2)&:=\int_{v\in(p^{-r_1},p^{-r_2})\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^{\max(r_1,r_2)}\langle v,v\rangle\in\mathbb{Z}_p)dv\\&=\int_{\substack{x\in p^{-r_1}L\\ y\in p^{-r_2}L}}\psi_p^{-1}\left((x,T_2)+(y,T_1)\right)\mathrm{Char}(p^{\max(r_1,r_2)}(x,y)\in\mathbb{Z}_p)dxdy,\end{aligned}$$

for any integers $r_1,r_2\geq 0$.

Proposition 3.9. *Let p be a prime split in E, and $r_1 \le r_2$ be two natural numbers. For a vector $T = (T_1, T_2) \in \mathcal{V}$ with $\alpha = v_p(T_1), \beta = v_p(T_2), v_p(\langle T, T\rangle) = \gamma$ and $B = B(r_1, r_2) := \min(r_1, \beta, \alpha + r_1 - r_2)$, one has*

$$S(r_1, r_2) = p^{r_2 n - r_1}\left(-\mathrm{Char}(\gamma < r_2 + B)\cdot p^{r_1 - 1 + (n-1)(\gamma - r_2 + 1)} + \sum_{k=0}^{\min(B, \gamma - r_2)} p^{kn}\phi(p^{r_1 - k})\right),$$

which is nonzero only if $r_2 \le \min(\gamma + 1, r_2 + \alpha)$. In particular, if $\langle T, T\rangle \in \mathcal{O}_{E_p}^{\times}$, then the nonzero terms are

$$S(0,0) = 1 \text{ and } S(1,1) = -p^{n-1}.$$

Proof. The integral $S(r_1, r_2)$ can be rewritten as an exponential sum:

$$S(r_1, r_2) = \sum_{\substack{x \in L/p^{r_1}L \\ y \in L/p^{r_2}L}} \psi_p\left(\frac{(x, T_2)}{p^{r_1}} + \frac{(y, T_1)}{p^{r_2}}\right)\mathrm{Char}\left((x, y) \in p^{r_1}\mathbb{Z}_p\right).$$

The characteristic function has the following expression:

$$\mathrm{Char}\left((x, y) \in p^{r_1}\mathbb{Z}_p\right) = p^{-r_1}\sum_{u \in \mathbb{Z}/p^{r_1}\mathbb{Z}} \psi_p\left(\frac{u(x, y)}{p^{r_1}}\right),$$

thus

$$S(r_1, r_2) = p^{-r_1}\sum_{\substack{y \in L/p^{r_2}L \\ u \in \mathbb{Z}/p^{r_1}\mathbb{Z}}} \psi_p^{-1}\left(\frac{(y, T_1)}{p^{r_2}}\right)\left(\sum_{x \in L/p^{r_1}L} \psi_p^{-1}\left(\frac{(x, T_2 - uy)}{p^{r_1}}\right)\right).$$

The inner sum equals $p^{r_1 n}$ when $T_2 - uy \in p^{r_1}L$, otherwise it equals 0. Hence one has

$$S(r_1, r_2) = p^{r_1(n-1)}\sum_{\substack{y \in L/p^{r_2}L \\ u \in \mathbb{Z}/p^{r_1}\mathbb{Z}}} \psi_p^{-1}\left(\frac{(y, T_1)}{p^{r_2}}\right)\mathrm{Char}(T_2 - uy \in p^{r_1}L).$$

The equation $T_2 = uy \bmod p^{r_1}L$ has solution if and only if $k := v_p(u) \le v_p(T_2) = \beta$, and in this case, $y = u^{-1}T_2 + p^{r_1 - k}z$ for $z \in L/p^{r_2 - r_1 + k}L$. Plugging this in the exponential sum and replacing u by $p^k u$, we obtain that

$$\begin{aligned}
S(r_1, r_2) &= p^{r_1(n-1)}\sum_{k=0}^{\min(r_1, \beta)}\sum_{u \in (\mathbb{Z}/p^{r_1 - k}\mathbb{Z})^{\times}}\sum_{z \in L/p^{r_2 - r_1 + k}L} \psi_p^{-1}\left(\frac{(T_1, u^{-1}p^{-k}T_2 + p^{r_1 - k}z)}{p^{r_2}}\right) \\
&= p^{r_1(n-1)}\sum_{k=0}^{\min(r_1, \beta)}\sum_{u \in (\mathbb{Z}/p^{r_1 - k}\mathbb{Z})^{\times}} \psi_p^{-1}\left(\frac{u^{-1}(T_1, T_2)}{p^{r_2 + k}}\right)\left(\sum_{z \in L/p^{r_2 - r_1 + k}L} \psi_p^{-1}\left(\frac{(z, T_1)}{p^{r_2 - r_1 + k}}\right)\right) \\
&= p^{r_1(n-1)}\sum_{k=0}^{\min(r_1, \beta)} p^{(r_2 - r_1 + k)n}\mathrm{Char}(\alpha \ge r_2 - r_1 + k)\left(\sum_{u \in (\mathbb{Z}/p^{r_1 - k}\mathbb{Z})^{\times}} \psi_p^{-1}\left(\frac{u^{-1}}{p^{r_2 + k - \gamma}}\right)\right)
\end{aligned}$$

For the inner sum, we have

$$\sum_{u\in(\mathbb{Z}/p^{r_1-k}\mathbb{Z})^\times} \psi_p^{-1}\left(\frac{u^{-1}}{p^{r_2+k-\gamma}}\right) = \begin{cases} \phi(p^{r_1-k}) = |(\mathbb{Z}/p^{r_1-k})^\times| & \text{, if } r_2+k-\gamma \le 0 \\ -p^{r_1-k-1} & \text{, if } r_2+k-\gamma = 1 \\ 0 & \text{, if } r_2+k-\gamma > 1 \end{cases}$$

Combining all these together, we get the following formula: let $B = B(r_1, r_2, \alpha, \beta)$ be $\min(r_1, \beta, \alpha + r_1 - r_2)$, then

$$S(r_1, r_2) = p^{r_2 n - r_1}\left(-\mathrm{Char}(\gamma < r_2 + B)\cdot p^{r_1-1+(n-1)(\gamma-r_2+1)} + \sum_{k=0}^{\min(B,\gamma-r_2)} p^{kn}\phi(p^{r_1-k})\right).$$

This is nonzero only if $r_1 \le r_2 \le \min(\gamma+1, r_1+\alpha)$, which only hold for finitely many terms. Particularly, if $T = (T_1, T_2)$ is unramified, then $\alpha = \beta = \gamma = 0$, thus $r_1 = r_2 \le 1$. In this case we have

$$S(0,0) = 1,\ S(1,1) = -p^{n-1}.$$

□

3.5 Rank 2 Fourier coefficients: archimedean components

Remark 3.10. In this article, when we write $|z|$ for $z \in \mathbb{C}$, it means the *norm of z with respect to the extension $\mathbb{C}/\mathbb{R}$, i.e.* $|z| = z\bar{z}$ instead of the usual modulus $\sqrt{z\bar{z}}$. I admit that this is somehow strange and confusing, but I will fix this problem if this draft could become a paper...

We first analyze the function $f_{\ell,\infty}(w_2 n, s = \ell+1)$:

Lemma 3.11. *For any $v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}$ and $x \in \mathbb{R}$, we set:*

$$\alpha(v,x) = -\frac{\langle v,v\rangle}{2} + ix + 1,\ \beta(v) = \sqrt{2}\langle v, u_2\rangle,$$

then we have

$$f_{\ell,\infty}(w_2 n(v,ix), s) = \frac{(\alpha u_1 + \beta u_2)^\ell(-\bar{\beta}u_1 + \bar{\alpha}u_2)^\ell}{(|\alpha(v,x)| + |\beta(v)|)^{\ell+s}(\ell!)^2}.$$

Proof. One has

$$b_1 w_2 n(v,ix) = (-\langle v,v\rangle/2 + ix)b_1 + b_2 + v.$$

Suppose that we can decompose $w_2 n(v,ix)$ as pk for some $p \in \mathbf{P}(\mathbb{R})$ and $k \in K_\infty$, then

$$b_1 w_2 n(v,ix)k^{-1} = b_1 p = \nu(p)^{-1}b_1. \tag{2}$$

Let k_+ be the factor of k in $\mathrm{U}(V_2^+)$, and $v = v_+ + v_- \in \mathbb{C}u_2 \oplus \mathrm{Span}_{\mathbb{C}}(v_1, \ldots, v_{n-1})$. Taking the V_2^+ components of Equation (2), one gets:

$$\frac{1}{\sqrt{2}}(\alpha(v,x)u_1 + \beta(v)u_2) = \left[\frac{1}{\sqrt{2}}\left(-\frac{\langle v,v\rangle}{2} + ix + 1\right)u_1 + v_+\right]k_+^{-1} = \nu(p)^{-1}\frac{u_1}{\sqrt{2}}.$$

The norms of both sides give us the identity $|\alpha(v,x)| + |\beta(v)| = |\nu(p)|^{-1}$. One may assume that $\nu(p)^{-1} = \sqrt{|\alpha(v,x)| + |\beta(v)|}$, then in the basis of u_1, u_2, the element $k^+ \in \mathrm{U}(V_2^+)$ can be written as the Hermitian matrix

$$\frac{1}{\sqrt{|\alpha(v,x)| + |\beta(v)|}} \begin{pmatrix} \alpha(v,x) & -\overline{\beta(v)} \\ \beta(v) & \overline{\alpha(v,x)} \end{pmatrix}.$$

Plug k^+ and $\nu(p)$ into $f_{\ell,\infty}(w_2 n(v,ix),s) = f_{\ell,\infty}(pk,s) = |\nu(p)|^s [u_1^\ell][u_2^\ell].k^+$, and we get the desired value. $\square$

Let $I_0(T;\ell)$ be the coefficient of $[u_1^\ell][u_2^\ell]$ in $E_{2,\infty}^T(s = \ell+1)$, which can be written as

$$\begin{aligned} I_0(T;\ell) &= \int_{v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}} \int_{x \in \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} \frac{\binom{\ell}{k}\binom{\ell}{k}(x^2+A)^k(-B)^{\ell-k}}{(x^2+A+B)^{2\ell+1}} dxdv \\ &= \int_{v \in V_0 \otimes \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \int_{x \in \mathbb{R}} \frac{(x^2+A)^k}{(x^2+A+B)^{2\ell+1}} dxdv \end{aligned}$$

where $A = |\alpha| - x^2 = (1 - \langle v,v \rangle/2)^2$, and $B = |\beta| = 2|\langle v, u_2 \rangle|$.

Lemma 3.12. *For any real number C, D and two natural numbers $m < n$, we have*

$$\int_{\mathbb{R}} \frac{(x^2+C)^m}{(x^2+D)^n} dx = \frac{D^{m-n+1/2}}{(n-1)!} \sum_{k=0}^{m} \binom{m}{k} \left(\frac{C}{D}\right)^{m-k} \Gamma(k+1/2)\Gamma(n-k-1/2).$$

Proof. An exercise of calculus. $\square$

Now Lemma 3.12 tells us $I_0(T;\ell)$ is the Fourier transform of the function

$$F_{0,\ell}(v) := \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \frac{(A+B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{A}{A+B}\right)^{k-j} \Gamma(j+1/2)\Gamma(2\ell+1/2-j)$$

Set $z = B/(A+B)$, then this function becomes

$$\frac{2^{-4\ell}}{(2\ell)!} \pi (A+B)^{-\ell-1/2} \sum_{k=0}^{\ell} \sum_{j=0}^{k} \binom{\ell}{k}^2 \binom{k}{j} \frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!} (-z)^{\ell-k} (1-z)^{k-j}$$

We write:

$$\sum_{k=0}^{\ell} \sum_{j=0}^{k} c_{j,k} (-z)^{\ell-k} (1-z)^{k-j} = \sum_{r=0}^{\ell} (-1)^r C(r) z^r,$$

where $c_{j,k} = \binom{\ell}{k}^2 \binom{k}{j} \frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!}$. The term $(-z)^{\ell-k}(1-z)^{k-j}$ has a non-zero z^r term if and only if $\ell - k \leq r \leq \ell - j$, thus

$$(-1)^r C(r) = \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} c_{j,k} \binom{k-j}{r-\ell+k} (-1)^{\ell-k+r-\ell+k} = (-1)^r \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} c_{j,k} \binom{k-j}{r-\ell+k}.$$

To compute $C(r)$, we need the following lemma:

Lemma 3.13. *(1) For integers $0 \le a \le b$, one has*

$$\sum_{i=a}^{b} \binom{b}{i}\binom{b-a}{b-i} = \binom{2b-a}{b}.$$

(2) For any integer $0 \le r \le \ell$, one has

$$\sum_{i=0}^{\ell-r} \frac{\binom{2\ell}{i}\binom{\ell-r}{i}}{\binom{4\ell}{2i}} = 2^{2\ell-2r}\frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}.$$

Proof. The identity in (1) is obvious. For the identity in (2), the LHS can be rewritten as:

$$\sum_{i=0}^{\ell-r} \frac{(-\ell+r)_j(1/2)_j}{(-2\ell+1/2)_j j!} = {}_2F_1(-(\ell-r), 1/2; -2\ell+1/2; 1),$$

where $(x)_j$ is the (rising) Pochhammer symbol, and ${}_2F_1$ is the hypergeometric function. By Chu-Vandermond identity, this value of hypergeometric function is

$$\frac{(-2\ell)_{\ell-r}}{(-2\ell+1/2)_{\ell-r}} = 2^{\ell-r}\frac{\frac{(2\ell)!}{(\ell+r)!}}{\frac{(4\ell-1)!!}{(2\ell+2r-1)!!}} = 2^{2\ell-2r}\frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}. \qquad \square$$

Now we return to the value of $C(r)$:

$$\begin{aligned}
C(r) &= \sum_{j=0}^{\ell-r}\sum_{k=\ell-r}^{\ell} \frac{(\ell!)^2(2j)!(4\ell-2j)!}{k!((\ell-k)!)^2(j!)^2(r-\ell+k)!(\ell-r-j)!(2\ell-j)!} \\
&= \sum_{j=0}^{\ell-r} \frac{\ell!(2j)!(4\ell-2j)!}{r!(j!)^2(\ell-r-j)!(2\ell-j)!} \sum_{k=\ell-r}^{\ell} \binom{\ell}{k}\binom{r}{\ell-k} \\
\text{(by (1) of Lemma 3.13)} &= \frac{\ell!}{r!}\cdot\frac{(\ell+r)!}{\ell!r!}\sum_{j=0}^{\ell-r}\frac{(2j)!(4\ell-2j)!}{(j!)^2(\ell-r-j)!(2\ell-j)!} \\
&= \frac{(\ell+r)!}{(r!)^2}\cdot\frac{(4\ell)!}{(\ell-r)!(2\ell)!}\sum_{j=0}^{\ell-r}\frac{\binom{2\ell}{j}\binom{\ell-r}{j}}{\binom{4\ell}{2j}} \\
\text{(by (2) of Lemma 3.13)} &= \frac{(\ell+r)!}{(r!)^2}\cdot\frac{(4\ell)!}{(\ell-r)!(2\ell)!}\cdot 2^{2\ell-2r}\frac{\frac{(2\ell+2r)!}{((\ell+r)!)^2}}{\frac{(4\ell)!}{((2\ell)!)^2}} \\
&= 2^{2\ell-2r}\frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!}.
\end{aligned}$$

Putting the value of $C(r)$ into $F_{0,\ell}$, we have

$$\begin{aligned}F_{0,\ell}(v) =&\frac{2^{-4\ell}\pi}{(2\ell)!(A+B)^{\ell+1/2}}\sum_{r=0}^{\ell}(-1)^r 2^{2\ell-2r}\frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!}z^r\\ =&\frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\sum_{r=0}^{\ell}\frac{(-\ell)_r(\ell+1/2)_r}{1_r}\frac{z^r}{r!}\\ =&\frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\cdot {}_2F_1(-\ell,\ell+1/2;1;z).\end{aligned}$$

We have shown the following result:

Proposition 3.14. *For $T \in V_0$ with $\langle T,T\rangle > 0$,T the coefficient $I_0(T;\ell)$ of $[u_1^\ell][u_2^\ell]$ in $E^T_{2,\infty}(s = \ell+1)$ is the Fourier transform of the function:*

$$F_{0,\ell}(v) = \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\cdot {}_2F_1(-\ell,\ell+1/2;1;\frac{B}{A+B}),$$

where $A = \left(1-\frac{\langle v,v\rangle}{2}\right)^2$ and $B = 2|\langle v,u_2\rangle|^2$.

Let's recall the following result for quadratic spaces by Pollack: I know there are notation problems again... just let it be like this for now

Proposition 3.15. *($v = 0$ case of [PS22, Proposition 4.5.3]) Let $(V', (\,,\,)) = V_2 \oplus V_n$ be a non-degenerate quadratic space over $\mathbb{R}$ of signature $(2,n)$, and v_1, v_2 an orthonormal basis of V_2. Set*

$$I_0(x;\ell) = \int_{V'} e^{i(\omega,x)}\mathrm{Char}(q(\omega)>0)q(\omega)^{\ell-n/2}K_0(\sqrt{2}|(\omega, v_1+\sqrt{-1}v_2)|)d\omega.$$

This integral is absolutely convergent and we have

$$I_0(x;\ell) = (2\pi)^{(n+2)/2}2^{\ell-1-n/2}\Gamma(\ell+1)\Gamma(\ell+1-n/2)F_4(\ell+1,\ell+1;\ell+1;1;-\|x_n\|^2/2;-\|x_2\|^2/2),$$

where $x = x_2 + x_n$, and $F_4(a,b;c;d;x;y)$ is Appell's hypergeometric function.

Now we look at how this result fits into our unitary group setting. One take V' to be $(V_0, \mathrm{Re}(\langle\,,\,\rangle))$, as a quadratic space of signature $(2, 2n-2)$, $v_1' = u_2$, $v_2' = iu_2$, then

$$\begin{aligned}I_0(x;\ell) &= \int_{T\in V_0,\,\langle T,T\rangle>0} e^{i\mathrm{Re}\langle T,x\rangle}\langle T,T\rangle^{\ell-n+1}K_0(\sqrt{2}|\langle T,u_2\rangle|)dT\\ &= \int_{T\in V_0,\,\langle T,T\rangle>0} e^{2\pi i\mathrm{Re}\langle T,x\rangle}(2\pi)^{2\ell-2n+2}\langle T,T\rangle^{\ell-n+1}K_0(2\sqrt{2}\pi|\langle T,u_2\rangle|)(2\pi)^{2n}dT\\ &= (2\pi)^{2\ell+2}\int_{T\in V_0,\,\langle T,T\rangle>0} e^{2\pi i\mathrm{Re}\langle T,x\rangle}\langle T,T\rangle^{\ell-n+1}K_0(2\sqrt{2}\pi|\langle T,u_2\rangle|)dT,\end{aligned}$$

here $|z| = \sqrt{z\bar{z}}$, and for any $\ell > n-1$ we have

$$I_0(x;\ell) = (2\pi)^n 2^{\ell-n}\ell!(\ell-n+1)!(A+B)^{-(\ell+1)/2}{}_2F_1(-\ell/2,(\ell+1)/2;1;B/(A+B)).$$

Replace ℓ by 2ℓ (now $\ell > \frac{n-1}{2}$), and we have

$$I_0(x;2\ell) = (2\pi)^n 2^{2\ell-n}(2\ell)!(2\ell-n+1)!(A+B)^{-\ell-1/2} \cdot {}_2F_1(-\ell,\ell+1/2;1;B/(A+B)).$$

Comparing this with Proposition 3.14, we get the following theorem:

Theorem 3.16. *For $T \in V_0$ with $\langle T,T\rangle > 0$ and $\ell > \frac{n-1}{2}$, we have*

$$I_0(T;\ell) = \frac{2^{-\ell-2}\pi^{4\ell-n+3}\langle T,T\rangle^{2\ell-n+1}}{\ell!(2\ell-n+1)!}\mathcal{W}_T(1).$$

References

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