Quaternionic Degenerate Eisenstein series on $\mathbf{U}(2, n)$

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Abstract

tba

1 Unitary groups of split rank 2

If some notation appearing in this section is not defined, then it should be defined in [HMY24].

Let E be an imaginary quadratic extension of $\mathbb{Q}$, and we denote the nonzero element in $\mathrm{Gal}(E/\mathbb{Q})$ by c or $x \mapsto \overline{x}$. Write the norm of $E/\mathbb{Q}$ as $|\,|$.

Define $\mathbf{G} = \mathbf{U}(V) = \mathbf{U}(2, n)$ to be the unitary group of a non-degenerate Hermitian space $(V, \langle\, , \,\rangle)$ over E with signature $(2, n)$. We write the $\mathbf{G}$-action on V as a right action.

Fix a pair of isotropic lines $(U, U^\vee)$ inside V such that $\langle U, U^\vee \rangle \neq 0$, and take V_0 to be the orthogonal complement of $U \oplus U^\vee$, which is a Hermitian space of signature $(1, n-1)$. Fix $b_1 \in U$ and $b_2 \in U^\vee$ such that $\langle b_1, b_2 \rangle = 1$.

One can define a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ as the stabilizer of U, and it has the following realization of Levi decomposition $\mathbf{P} = \mathbf{M}\mathbf{N}$:

— $\mathbf{M}$ is the stabilizer of U and $U^\vee$, and it is isomorphic to $\mathrm{Res}_{E/\mathbb{Q}}(\mathbf{G}_\mathrm{m}) \times \mathbf{U}(V_0)$. One can write the $\mathbf{M}$-action on $(u, w, u^\vee) \in U \oplus V_0 \oplus U^\vee$ explicitly:

$$(u, w, u^\vee).(z, h) = (z^{-1}u, wh, \overline{z}u^\vee).$$

— $\mathbf{N}$ is a Heisenberg, and it is isomorphic to the group

$$\left\{(v, \lambda) \in V_0 \times \mathrm{Res}_{E/\mathbb{Q}}\mathbf{G}_\mathrm{a} \,\middle|\, \overline{\lambda} = -\lambda\right\},$$

equipped with the following multiplication:

$$(v_1, \lambda_1) \cdot (v_2, \lambda_2) := \left(v_1 + v_2, \lambda_1 + \lambda_2 - \frac{\langle v_1, v_2 \rangle - \langle v_2, v_1 \rangle}{2}\right).$$

The action of the corresponding element $n(v, \lambda) \in \mathbf{N}$ on V is given as:

$$b_1 \mapsto b_1,\ b_2 \mapsto (-\frac{1}{2}\langle v, v \rangle + \lambda)b_1 + b_2 + v,\ w \in V_0 \mapsto -\langle w, v \rangle b_1 + w.$$

One has

$$(z,h)n(v,\lambda)(z^{-1},h^{-1}) = n(\bar{z}vh^{-1},|z|\lambda).$$

Denote the center of $\mathbf{N}$ by $\mathbf{N}_0$. We denote by ν the similitude character of $\mathbf{M}$: $\nu(z,h) = z \in \mathrm{Res}_{E/\mathbb{Q}}\mathbf{G}_{\mathrm{m}}$, so that $b_1(z,h) = \nu(z,h)^{-1}b_1$.

Fix an addictive character $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$, so that ψ_p has conductor $\mathbb{Z}_p$ and $\psi_\infty(x) = e^{2\pi ix}$. We have the following identification $V_0 \xrightarrow{\sim} \mathrm{Hom}([\mathbf{N}],\mathbb{S}^1)$: for any element $T \in V_0$, one associates a unitary character χ_T of $[\mathbf{N}]$ by

$$\chi_T : [\mathbf{N}] \to \mathbb{C}^\times,\ n(v,\lambda) \mapsto \psi(-\mathrm{Im}\langle T,v\rangle).$$

I am really confused with this identification in [HMY24]. This character χ_T seems not well-defined: the symplectic form $-\mathrm{Im}\langle T,v\rangle = \frac{\langle v,T\rangle - \langle T,v\rangle}{2i}$ involves $1/2i$, which may not lie in E. Maybe one should define χ_T to be $n(v,\lambda) \mapsto \psi(\mathrm{Re}\langle T,v\rangle)$.

For a finite place p of $\mathbb{Q}$, we define E_p to be $E_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} E$, which is isomorphic to

— $\mathbb{Q}_p \times \mathbb{Q}_p$, if p splits in E;

— a degree 2 unramified (*resp.* ramified) extension of $\mathbb{Q}_p$, if p is inert (*resp.* ramified).

2 Heisenberg Eisenstein series

Choose a section $f = f_{\ell,\infty} \otimes f_{fte}$ of $\mathrm{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}|\nu|^s$ (unnormalized) as follow:

— $f_{\ell,\infty}$ is the $\mathbb{V}_\ell = \left(\mathrm{Sym}^{2\ell} V_2^+ \otimes \det_{\mathbf{U}(2)}^{-\ell}\right) \boxtimes \mathbf{1}$-valued, K_∞-equivariant induced section whose restriction to $\mathbf{M}(\mathbb{R})$ is

$$f_{\ell,\infty}((z,h),s) = |z|^s[u_1^\ell][u_2^\ell],$$

where $[u_i^k] := u_i^k/k!$.

— f_{fte} is defined as

$$f_{fte}(g_f,\Phi_f,s) = \int_{\mathbf{GL}_1(\mathbb{A}_{E,f})} |t|^s\Phi_f(t\cdot b_1g_f)dt,$$

where $\Phi_f = \prod \Phi_p$ is a Schwartz-Bruhat function on $V(\mathbb{A}_f)$.

One defines the degenerate Heisenberg Eisenstein series:

$$E_\ell(g,\Phi_f,s) = \sum_{\gamma\in\mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})} f(\gamma g,s).$$

To prove the modularity of this Eisenstein series, one needs the following result:

Proposition 2.1. *The $\mathbb{V}_\ell$-valued section $f_{\ell,\infty}$ of $\mathrm{Ind}_{\mathbf{P}(\mathbb{R})}^{\mathbf{G}(\mathbb{R})}|\nu|^s$, as a function on $\mathbf{G}(\mathbb{R})$, is killed by the Schmid operators $\mathcal{D}_\ell^\pm$ if and only if $s = \ell + 1$.*

Proof. Write an element $(z,h) \in \mathbf{M}(\mathbb{R})$ as $m = (h,r,\theta)$ so that $z = re^{i\theta}$, $r \in \mathbb{R}_{>0}$ and $\theta \in [0,2\pi)$. Using this coordinates, we define a function F on $\mathbf{M}(\mathbb{R})$, sending (h,r,θ) to r^s. By [HMY24, Proposition 3.10], $f_{\ell,\infty}$ is killed by $\mathcal{D}_\ell^{\pm}$ if and only if

$$\begin{cases} (r\partial r - 2(\ell+1))F(m) = 0 \\ [u_2 \otimes \overline{v_k}]^+ F(m) = 0, \text{ if } 1 \le k < n \\ [u_2 \otimes \overline{v_k}]^- F(m) = 0, \text{ if } 1 \le k < n \end{cases}$$

It can be easily verified that these equations hold if and only if $s = \ell + 1$. □

As a consequence, the Eisenstein series $E_\ell(g, \Phi_f, s = \ell+1)$ is a quaternionic modular form of weight ℓ.

3 The Fourier expansion of $E_\ell(g, \Phi_f, s = \ell+1)$

3.1 Abstract Fourier expansion

In this subsection, we give the "abstract" Fourier expansion of $E_\ell(g, \Phi_f, s)$.

Lemma 3.1. *The right $\mathbf{P}(\mathbb{Q})$-space $\mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})$, the space of isotropic lines in $V(\mathbb{Q})$, has exactly 3 orbits of $\mathbf{P}(\mathbb{Q})$, represented respectively by $\mathbb{Q}b_1$, $\mathbb{Q}v_0$ and $\mathbb{Q}b_2$, where v_0 is an arbitrary non-zero isotropic vector in $V_0(\mathbb{Q})$.*

Proof. Directly by the explicit action given in §1. □

Set $\mathbf{G}(\mathbb{Q}) = \bigsqcup_{i=0}^{2} \mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})$, such that $w_0 = 1$, $b_1w_1 = v_0$ and $b_1w_2 = b_2$. Now we can write the degenerate Eisenstein series as

$$E_\ell(g, \Phi_f, s) = \sum_{i=0}^{2} E_{\ell,i}(g, \Phi_f, s),\ E_{\ell,i}(g, \Phi_f, s) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q})\backslash\mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})} f(\gamma g, s),$$

thus $E_{\ell,0}(g, \Phi_f, s) = f(g,s)$. From now on, when there is no confusion we will omit the ℓ and Φ_f in $E_\ell(g, \Phi_f, s)$, and write it as $E(g,s) = \sum_{i=0}^{2} E_i(g,s)$.

Lemma 3.2. *Assume that* $\mathrm{Re}(s) \gg 0$ *so that the sum defining $E(g,s)$ converges absolutely. Then one has the following expressions for the $E_i(g,s)$:*

(1) Let $\mathcal{L}_0$ be the set of non-zero isotropic lines ℓ in V_0 and for any $\ell \in \mathcal{L}_0$, select $\gamma(\ell) \in \mathbf{G}(\mathbb{Q})$ with $b_1\gamma(\ell) \in \ell$. Then

$$E_1(g,s) = \sum_{\ell \in \mathcal{L}_0} \sum_{\mu \in (\ell)^\perp \mathbf{N}_0(\mathbb{Q})\backslash\mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu g, s).$$

(2) One has

$$E_2(g,s) = \sum_{\mu \in \mathbf{N}(\mathbb{Q})} f(w_2\mu g, s).$$

For any $T \in V_0$, we set

$$E_i^T(g,s) = \int_{\mathbf{N}(F)\backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) E_i(ng,s) dn, \ i = 0,1,2.$$

Lemma 3.3. *(1) If T is anisotropic, then $E_1^T = 0$. If T is isotropic, define $\mathbf{N}_T = (\ell_T)^{\perp}\mathbf{N}_0 \subseteq \mathbf{N}$, then*

$$E_1^T(g,s) = \int_{\mathbf{N}_T(\mathbb{A})\backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell_T)ng,s) dn.$$

(2) For any $T \in V_0$, one has

$$E_2^T(g,s) = \int_{\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(w_2 ng,s) dn.$$

Proof. It suffices only to prove the $i = 1$ case. For any $\ell \in \mathcal{L}_0$, set

$$\mathbf{N}_\ell = \left\{ n(v,\lambda) \in \mathbf{N} \,\middle|\, v \in \ell^\perp \right\} \subseteq \mathbf{N}.$$

For any $T \in V_0$,

$$\begin{aligned} E_1^T(g,s) &= \sum_{\ell \in \mathcal{L}_0} \int_{[N]} \chi_T^{-1}(n) \left(\sum_{\mu \in \mathbf{N}_\ell(\mathbb{Q})\backslash \mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu ng,s) \right) dn \\ &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{Q})\backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell)ng,s) dn \\ &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A})\backslash \mathbf{N}(\mathbb{A})} \left(\int_{[\mathbf{N}_\ell]} \chi_T^{-1}(r) dr \right) \chi_T^{-1}(n) f(\gamma(\ell)ng,s) dn \\ &= \sum_{\ell \in \mathcal{L}_0, \chi_T|_{\mathbf{N}_\ell} \equiv 1} \int_{\mathbf{N}_\ell(\mathbb{A})\backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell)ng,s) dn. \end{aligned}$$

Then the lemma follows from the fact that $\chi_T|_{\mathbf{N}_\ell} \equiv 1$ if and only if $T \in \ell$, *i.e.* T is isotropic and $\ell = \ell_T$. □

3.2 Computation of constant term

I think we also need a refined version of [HMY24, Corollary 1.2], with a description on its constant term.

3.2.1 The $i = 0$-term

Lemma 3.4. *For $g \in \mathbf{P}(\mathbb{A})$,*

$$E_0(g,s) = f(g,s) = |\nu(g)|^s \zeta_E(s)[u_1^n][u_2^n].$$

Proof. For $g_f \in \mathbf{P}(\mathbb{A}_f)$, we have

$$\begin{aligned} f_{fte}(g_f, s) &= \int_{\mathbb{A}_{E,f}^\times} |t|^s \Phi_f(tb_1 g_f) dt \\ &= \int_{\mathbb{A}_{E,f}^\times} |t|^s \Phi_f(t\nu(g_f)^{-1} b_1) dt \\ &= |\nu(g_f)|^s \int_{\mathbb{A}_{E,f}^\times} |t|^s \Phi_f(tb_1) dt. \end{aligned}$$

Thus, the non-archimedean contribution is $|\nu(g_f)|^s \zeta_E(s)$. Combining with $f_{n,\infty}(g_\infty, s) = |\nu(g_\infty)|^s [u_1^n][u_2^n]$, we get the desired identity. $\square$

3.2.2 The $i = 1$-term

We fix a non-zero isotropic vector in V_0, such that $v_0 = b_1 \gamma_0$, and set $\ell_0 = Ev_0$. Define $\mathbf{P}_0$ be the stabilizer of ℓ_0 in $\mathbf{U}(V_0)$, which is a parabolic subgroup of $\mathbf{M}$. We denote the similitude character of $\mathbf{P}_0$ by λ, *i.e.* $v_0 g = \lambda(g)^{-1} v_0$ for any $g \in \mathbf{P}_0$. For $g \in \mathbf{P}_0(\mathbb{A})$, we have:

$$\begin{aligned} E_1^0(g, s) &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma(\ell) ng, s) dn \\ &= \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{Q})} \int_{\mathbf{N}_{\ell_0}(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma_0 \gamma ng, s) dn \end{aligned}$$

If we set $f_0(g, s) = \int_{\mathbf{N}_{\ell_0}(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma_0 ng, s) dn$, then for $\mathrm{Re}(s) \gg 0$,

$$E_1^0(g, s) = \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{Q})} f_0(\gamma g, s),$$

and it defines an Eisenstein series on $\mathbf{M}$.

At finite places, one has

$$\begin{aligned} \int_{\mathbf{N}_{\ell_0}(\mathbb{A}_{E,f}) \backslash \mathbf{N}(\mathbb{A}_{E,f})} f(\gamma_0 ng, s) dn &= \int_{x \in \mathbb{A}_f} \int_{t \in \mathbb{A}_f^\times} |t|^s \Phi_{fte}(t(v_0 + xb_1) g) dt dx \\ &= \int_{x \in \mathbb{A}_f} \int_{t \in \mathbb{A}_f^\times} |t|^s \Phi_{fte}(t\lambda(g)^{-1} v_0 + tx\nu(g)^{-1} b_1) dt dx \\ &= |\lambda(g)|_f^{s-1} |\nu(g)|_f \int_{x \in \mathbb{Q}_p} \int_{t \in \mathbb{Q}_p^\times} |t|^{s-1} \Phi_p(tv_0 + xb_1) dt dx \\ &= |\lambda(g)|_f^{s-1} |\nu(g)|_f \zeta(s-1). \end{aligned}$$

Now we switch to the archimedean place. Set $c_1 = v_0$ and c_2 another isotropic vector in V_0 with $\langle c_1, c_2 \rangle = 1$. We can take $u_2 = \frac{1}{\sqrt{2}}(c_1 + c_2)$ and $v_{n-1} = \frac{1}{\sqrt{2}}(c_1 - c_2)$. One picks the following representatives for $\mathbf{N}_{\ell_0}(\mathbb{R}) \backslash \mathbf{N}(\mathbb{R})$: $\{n(-xc_2, 0),\ x \in \mathbb{R}\}$, thus

$$b_1 \gamma_0 n(-xc_2, 0) g = \lambda(g)^{-1} c_1 + x\nu(g)^{-1} b_1.$$

The projection of this vector to V_2^+ is $\frac{1}{\sqrt{2}}(x\nu(g)^{-1}u_1 + \lambda(g)^{-1}u_2)$. So there exists an element $mk \in \mathbf{M}(\mathbb{R})K_\infty$ such that

— $\nu(m) = (|\lambda(g)|^{-1} + x^2|\nu(g)|^{-1})^{-1/2}$,

— the projection of $k \in K_\infty \simeq \mathrm{U}(V_2^+) \times \mathrm{U}(V_n^-)$ to $\mathrm{U}(V_2^+)$ is

$$(u_1, u_2) \mapsto (u_1, u_2) \begin{pmatrix} \frac{x\nu(g)^{-1}}{(|\lambda(g)|^{-1}+x^2|\nu(g)|^{-1})^{1/2}} & \frac{-\overline{\lambda(g)}^{-1}}{(|\lambda(g)|^{-1}+x^2|\nu(g)|^{-1})^{1/2}} \\ \frac{\lambda(g)^{-1}}{(|\lambda(g)|^{-1}+x^2|\nu(g)|^{-1})^{1/2}} & \frac{x\overline{\nu(g)}^{-1}}{(|\lambda(g)|^{-1}+x^2|\nu(g)|^{-1})^{1/2}} \end{pmatrix}$$

— and $mk \in \mathbf{P}(\mathbb{R})^{\nu=1}\gamma_0 n(-xc_2, 0)g$.

With these preparations, we can start to compute the archimedean integral:

$$\begin{aligned} &\int_{\mathbf{N}_{\ell_0}(\mathbb{R})\backslash\mathbf{N}(\mathbb{R})} f_{\ell,\infty}(\gamma_0 ng)dn \\ =&\int_{x\in\mathbb{R}} \frac{\left(x^2|\nu(g)|^{-1}u_1u_2 + x\left(-\nu(g)^{-1}\overline{\lambda(g)}^{-1}u_1^2 + \overline{\nu(g)}^{-1}\lambda(g)^{-1}u_2^2\right) - |\lambda(g)|^{-1}u_1u_2\right)^\ell}{(\ell!)^2(|\lambda(g)|^{-1} + x^2|\nu(g)|^{-1})^{s+\ell}} dx \end{aligned}$$

Set $\alpha = \nu(g)^{-1}$ and $\beta = \lambda(g)^{-1}$, then the above integral is written as:

$$\int_{\mathbb{R}} \frac{\left(\left(x^2|\alpha| - |\beta|\right)u_1u_2 - x\alpha\overline{\beta}u_1^2 + x\overline{\alpha}\beta u_2^2\right)^\ell}{(\ell)^2(x^2|\alpha| + |\beta|)^{s+\ell}} dx$$

3.3 Rank 1 Fourier coefficients

3.4 Rank 2 Fourier coefficients

One decomposes $E_2^T(1,s)$ as $\prod_v E_{2,v}^T(s)$.

3.4.1 Finite places

$$\begin{aligned} E_{2,p}^T(s) &= \int_{\mathbf{N}(\mathbb{Q}_p)} \chi_T^{-1}(n) f_p(w_2 n, s) dn \\ &= \int_{\mathbf{N}(\mathbb{Q}_p)} \int_{t\in E_p^\times} \chi_T^{-1}(n)|t|^s \Phi_p(t\cdot b_1 w_2 n) dt dn \\ &= \int_{v\in\mathbf{V}_0(\mathbb{Q}_p)} \int_{\substack{x\in E_p \\ \overline{x}=-x}} \int_{t\in E_p^\times} \chi_T^{-1}(v)|t|^s \Phi_p\left(t\left(\left(-\langle v,v\rangle/2 + x\right)b_1 + b_2 + v\right)\right) dt dx dv \end{aligned}$$

Take $\mathcal{O}_{E_p}$ to be $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_E \subseteq E_p$, $\mathfrak{p}$ to be its maximal ideal, and $\mathcal{V}$ an $\mathcal{O}_{E_p}$-lattice of $V_0(\mathbb{Q}_p) = V_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that $\mathcal{V} \otimes_{\mathcal{O}_{E_p}} (\mathcal{O}_{E,p}/\mathfrak{p})$ is a non-degenerate Hermitian space over $\mathcal{O}_{E_p}/\mathfrak{p}$. Assume that Φ_p is the characteristic function of the lattice $\mathcal{O}_{E_p}b_1 \oplus \mathcal{V} \oplus \mathcal{O}_{E_p}b_2$.

Write down a precise definition of $\mathcal{V}$?

If p splits in E: $E_p = \mathbb{Q}_p \times \mathbb{Q}_p$ and

$$
\begin{aligned}
E^T_{2,p}(s) &= \sum_{r_1 \geq 0, r_2 \geq 0} |p|^{(r_1+r_2)s} \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \left(\int_{\substack{x \in (p^{-r_1}, p^{-r_2})\mathcal{O} + \langle v,v\rangle/2 \\ x+\overline{x}=0}} dx \right) dv \\
&= \sum_{r_1 \geq 0, r_2 \geq 0} |p|^{(r_1+r_2)s} \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^{\max(r_1,r_2)}\langle v,v\rangle \in \mathbb{Z}_p) \cdot |p|^{-\min(r_1,r_2)} dv \\
&= \sum_{r_1 \geq 0, r_2 \geq 0} |p|^{(r_1+r_2)s - \min(r_1,r_2)} \left(\int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^{\max(r_1,r_2)}\langle v,v\rangle \in \mathbb{Z}_p) dv \right).
\end{aligned}
$$

If p is inert in E: E_p is an unramified quadratic field extension of $\mathbb{Q}_p$, and

$$
\begin{aligned}
E^T_{2,p}(s) &= \sum_{r \geq 0} |p|^{rs} \int_{v \in p^{-r}\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^r\langle v,v\rangle \in \mathbb{Z}_p) \left(\int_{x \in \langle v,v\rangle/2 + p^{-r}\mathcal{O}_{E_p}} \mathrm{Char}(x+\overline{x}=0) dx \right) dv \\
&= \sum_{r \geq 0} |p|^{r(s-1)} p^{-1} \left(\int_{v \in p^{-r}\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^r\langle v,v\rangle \in \mathbb{Z}_p) dv \right)
\end{aligned}
$$

If p is ramified in E: (Assume $p \neq 2$) Then E_p is a ramified quadratic field extension of $\mathbb{Q}_p$, and we choose an uniformizer ϖ of $\mathfrak{p} \subset \mathcal{O}_{E_p}$ such that $\varpi + \overline{\varpi} = 0$and $\varpi^2 \in p\mathbb{Z}_p^\times$. The integral for $E^T_{2,p}$ can be rewritten as:

$$
\begin{aligned}
E^T_{2,p}(s) &= \sum_{r \geq 0} |\varpi|^{rs} \int_{v \in \varpi^{-r}\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^{\lfloor r/2 \rfloor}\langle v,v\rangle \in \mathbb{Z}_p) \left(\int_{x \in \mathbb{Q}_p} \mathrm{Char}(x\varpi - \frac{\langle v,v\rangle}{2} \in \varpi^{-r}\mathcal{O}_{E_p}) dx \right) dv \\
&= \sum_{r \geq 0} |\varpi|^{rs} p^{\lceil r/2 \rceil} \left(\int_{v \in \varpi^{-r}\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^{\lfloor r/2 \rfloor}\langle v,v\rangle \in \mathbb{Z}_p) dv \right)
\end{aligned}
$$

3.4.2 Archimedean place

Remark 3.5. In this article, when we write $|z|$ for $z \in \mathbb{C}$, it means the *norm of z with respect to the extension* $\mathbb{C}/\mathbb{R}$, *i.e.* $|z| = z\overline{z}$ instead of the usual modulus $\sqrt{z\overline{z}}$. I admit that this is somehow strange and confusing, but I will fix this problem if this draft could become a paper...

We first analyze the function $f_{\ell,\infty}(w_2 n, s = \ell + 1)$:

Lemma 3.6. *For any $v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}$ and $x \in \mathbb{R}$, we set:*

$$
\alpha(v,x) = -\frac{\langle v,v\rangle}{2} + ix + 1, \ \beta(v) = \sqrt{2}\langle v, u_2\rangle,
$$

then we have

$$
f_{\ell,\infty}(w_2 n(v, ix), s) = \frac{(\alpha u_1 + \beta u_2)^\ell (-\overline{\beta} u_1 + \overline{\alpha} u_2)^\ell}{(|\alpha(v,x)| + |\beta(v)|)^{\ell+s} (\ell!)^2}.
$$

Proof. One has

$$b_1 w_2 n(v, ix) = (-\langle v, v\rangle/2 + ix)b_1 + b_2 + v.$$

Suppose that we can decompose $w_2 n(v, ix)$ as pk for some $p \in \mathbf{P}(\mathbb{R})$ and $k \in K_\infty$, then

$$b_1 w_2 n(v, ix)k^{-1} = b_1 p = \nu(p)^{-1} b_1. \tag{1}$$

Let k_+ be the factor of k in $\mathrm{U}(V_2^+)$, and $v = v_+ + v_- \in \mathbb{C}u_2 \oplus \mathrm{Span}_{\mathbb{C}}(v_1, \dots, v_{n-1})$. Taking the V_2^+ components of Equation (1), one gets:

$$\frac{1}{\sqrt{2}}(\alpha(v, x)u_1 + \beta(v)u_2) = \left[\frac{1}{\sqrt{2}}\left(-\frac{\langle v, v\rangle}{2} + ix + 1\right)u_1 + v_+\right]k_+^{-1} = \nu(p)^{-1}\frac{u_1}{\sqrt{2}}.$$

The norms of both sides give us the identity $|\alpha(v, x)| + |\beta(v)| = |\nu(p)|^{-1}$. One may assume that $\nu(p)^{-1} = \sqrt{|\alpha(v, x)| + |\beta(v)|}$, then in the basis of u_1, u_2, the element $k^+ \in \mathrm{U}(V_2^+)$ can be written as the Hermitian matrix

$$\frac{1}{\sqrt{|\alpha(v, x)| + |\beta(v)|}}\begin{pmatrix} \alpha(v, x) & -\overline{\beta(v)} \\ \beta(v) & \overline{\alpha(v, x)} \end{pmatrix}.$$

Plug k^+ and $\nu(p)$ into $f_{\ell,\infty}(w_2 n(v, ix), s) = f_{\ell,\infty}(pk, s) = |\nu(p)|^s [u_1^\ell][u_2^\ell].k^+$, and we get the desired value. $\square$

Let $I_0(T; \ell)$ be the coefficient of $[u_1^\ell][u_2^\ell]$ in $E_{2,\infty}^T(s = \ell + 1)$, which can be written as

$$\begin{aligned} I_0(T; \ell) &= \int_{v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}} \int_{x \in \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} \frac{\binom{\ell}{k}\binom{\ell}{k}(x^2 + A)^k(-B)^{\ell-k}}{(x^2 + A + B)^{2\ell+1}} dx dv \\ &= \int_{v \in V_0 \otimes \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \int_{x \in \mathbb{R}} \frac{(x^2 + A)^k}{(x^2 + A + B)^{2\ell+1}} dx dv \end{aligned}$$

where $A = |\alpha| - x^2 = (1 - \langle v, v\rangle/2)^2$, and $B = |\beta| = 2|\langle v, u_2\rangle|$.

Lemma 3.7. *For any real number C, D and two natural numbers $m < n$, we have*

$$\int_{\mathbb{R}} \frac{(x^2 + C)^m}{(x^2 + D)^n} dx = \frac{D^{m-n+1/2}}{(n-1)!} \sum_{k=0}^{m} \binom{m}{k} \left(\frac{C}{D}\right)^{m-k} \Gamma(k + 1/2)\Gamma(n - k - 1/2).$$

Proof. An exercise of calculus. $\square$

Now Lemma 3.7 tells us $I_0(T; \ell)$ is the Fourier transform of the function

$$F_{0,\ell}(v) := \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \frac{(A + B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{A}{A + B}\right)^{k-j} \Gamma(j + 1/2)\Gamma(2\ell + 1/2 - j)$$

Set $z = B/(A + B)$, then this function becomes

$$\frac{2^{-4\ell}}{(2\ell)!}\pi(A + B)^{-\ell-1/2} \sum_{k=0}^{\ell}\sum_{j=0}^{k} \binom{\ell}{k}^2 \binom{k}{j} \frac{(2j)!(4\ell - 2j)!}{j!(2\ell - j)!}(-z)^{\ell-k}(1 - z)^{k-j}$$

We write:

$$\sum_{k=0}^{\ell}\sum_{j=0}^{k} c_{j,k}(-z)^{\ell-k}(1-z)^{k-j} = \sum_{r=0}^{\ell}(-1)^r C(r) z^r,$$

where $c_{j,k} = \binom{\ell}{k}^2\binom{k}{j}\frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!}$. The term $(-z)^{\ell-k}(1-z)^{k-j}$ has a non-zero z^r term if and only if $\ell-k \le r \le \ell-j$, thus

$$(-1)^r C(r) = \sum_{j=0}^{\ell-r}\sum_{k=\ell-r}^{\ell} c_{j,k}\binom{k-j}{r-\ell+k}(-1)^{\ell-k+r-\ell+k} = (-1)^r \sum_{j=0}^{\ell-r}\sum_{k=\ell-r}^{\ell} c_{j,k}\binom{k-j}{r-\ell+k}.$$

To compute $C(r)$, we need the following lemma:

Lemma 3.8. *(1) For integers $0 \le a \le b$, one has*

$$\sum_{i=a}^{b}\binom{b}{i}\binom{b-a}{b-i} = \binom{2b-a}{b}.$$

(2) For any integer $0 \le r \le \ell$, one has

$$\sum_{i=0}^{\ell-r}\frac{\binom{2\ell}{i}\binom{\ell-r}{i}}{\binom{4\ell}{2i}} = 2^{2\ell-2r}\frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}.$$

Proof. The identity in (1) is obvious. For the identity in (2), the LHS can be rewritten as:

$$\sum_{i=0}^{\ell-r}\frac{(-\ell+r)_j(1/2)_j}{(-2\ell+1/2)_j j!} = {}_2F_1(-(\ell-r), 1/2; -2\ell+1/2; 1),$$

where $(x)_j$ is the (rising) Pochhammer symbol, and ${}_2F_1$ is the hypergeometric function. By Chu-Vandermond identity, this value of hypergeometric function is

$$\frac{(-2\ell)_{\ell-r}}{(-2\ell+1/2)_{\ell-r}} = 2^{\ell-r}\frac{\frac{(2\ell)!}{(\ell+r)!}}{\frac{(4\ell-1)!!}{(2\ell+2r-1)!!}} = 2^{2\ell-2r}\frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}.$$

□

Now we return to the value of $C(r)$:

$$
\begin{aligned}
C(r) &= \sum_{j=0}^{\ell-r}\sum_{k=\ell-r}^{\ell} \frac{(\ell!)^2(2j)!(4\ell-2j)!}{k!((\ell-k)!)^2(j!)^2(r-\ell+k)!(\ell-r-j)!(2\ell-j)!} \\
&= \sum_{j=0}^{\ell-r} \frac{\ell!(2j)!(4\ell-2j)!}{r!(j!)^2(\ell-r-j)!(2\ell-j)!} \sum_{k=\ell-r}^{\ell} \binom{\ell}{k}\binom{r}{\ell-k} \\
\text{(by (1) of Lemma 3.8)} &= \frac{\ell!}{r!}\cdot\frac{(\ell+r)!}{\ell!r!}\sum_{j=0}^{\ell-r}\frac{(2j)!(4\ell-2j)!}{(j!)^2(\ell-r-j)!(2\ell-j)!} \\
&= \frac{(\ell+r)!}{(r!)^2}\cdot\frac{(4\ell)!}{(\ell-r)!(2\ell)!}\sum_{j=0}^{\ell-r}\frac{\binom{2\ell}{j}\binom{\ell-r}{j}}{\binom{4\ell}{2j}} \\
\text{(by (2) of Lemma 3.8)} &= \frac{(\ell+r)!}{(r!)^2}\cdot\frac{(4\ell)!}{(\ell-r)!(2\ell)!}\cdot 2^{2\ell-2r}\frac{\frac{(2\ell+2r)!}{((\ell+r)!)^2}}{\frac{(4\ell)!}{((2\ell)!)^2}} \\
&= 2^{2\ell-2r}\frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!}.
\end{aligned}
$$

Putting the value of $C(r)$ into $F_{0,\ell}$, we have

$$
\begin{aligned}
F_{0,\ell}(v) =& \frac{2^{-4\ell}\pi}{(2\ell)!(A+B)^{\ell+1/2}}\sum_{r=0}^{\ell}(-1)^r 2^{2\ell-2r}\frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!} \\
=& \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\sum_{r=0}^{\ell}\frac{(-\ell)_r(\ell+1/2)_r}{1_r}\frac{z^r}{r!} \\
=& \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\cdot {}_2F_1(-\ell,\ell+1/2;1;z).
\end{aligned}
$$

We have shown the following result:

Proposition 3.9. *For $T\in V_0$ with $\langle T,T\rangle>0$,T the coefficient $I_0(T;\ell)$ of $[u_1^\ell][u_2^\ell]$ in $E_{2,\infty}^T(s=\ell+1)$ is the Fourier transform of the function:*

$$
F_{0,\ell}(v) = \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\cdot {}_2F_1(-\ell,\ell+1/2;1;\frac{B}{A+B}),
$$

where $A=\left(1-\frac{\langle v,v\rangle}{2}\right)^2$ and $B=2|\langle v,u_2\rangle|^2$.

Let's recall the following result for quadratic spaces by Pollack: I know there are notation problems again... just let it be like this for now

Proposition 3.10. *($v=0$ case of [PS22, Proposition 4.5.3]) Let $(V',(\,,\,))=V_2\oplus V_n$ be a non-degenerate quadratic space over $\mathbb{R}$ of signature $(2,n)$, and v_1,v_2 an orthonormal basis of V_2. Set*

$$
I_0(x;\ell)=\int_{V'}e^{i(\omega,x)}\mathrm{Char}(q(\omega)>0)q(\omega)^{\ell-n/2}K_0(\sqrt{2}|(\omega,v_1+\sqrt{-1}v_2)|)d\omega.
$$

This integral is absolutely convergent and we have

$$I_0(x;\ell) = (2\pi)^{(n+2)/2} 2^{\ell-1-n/2}\Gamma(\ell+1)\Gamma(\ell+1-n/2)F_4(\ell+1,\ell+1;\ell+1;1;-\|x_n\|^2/2;-\|x_2\|^2/2),$$

where $x = x_2 + x_n$, *and* $F_4(a,b;c;d;x;y)$ *is Appell's hypergeometric function.*

Now we look at how this result fits into our unitary group setting. One take V' to be $(V_0, \mathrm{Re}(\langle\,,\,\rangle))$, as a quadratic space of signature $(2, 2n-2)$, $v_1' = u_2$, $v_2' = iu_2$, then

$$\begin{aligned} I_0(x;\ell) &= \int_{T\in V_0,\, \langle T,T\rangle>0} e^{i\mathrm{Re}\langle T,x\rangle}\langle T,T\rangle^{\ell-n+1}K_0(\sqrt{2}|\langle T,u_2\rangle|)dT \\ &= \int_{T\in V_0,\, \langle T,T\rangle>0} e^{2\pi i\mathrm{Re}\langle T,x\rangle}(2\pi)^{2\ell-2n+2}\langle T,T\rangle^{\ell-n+1}K_0(2\sqrt{2}\pi|\langle T,u_2\rangle|)(2\pi)^{2n}dT \\ &= (2\pi)^{2\ell+2}\int_{T\in V_0,\, \langle T,T\rangle>0} e^{2\pi i\mathrm{Re}\langle T,x\rangle}\langle T,T\rangle^{\ell-n+1}K_0(2\sqrt{2}\pi|\langle T,u_2\rangle|)dT, \end{aligned}$$

here $|z| = \sqrt{z\bar{z}}$, and for any $\ell > n-1$ we have

$$I_0(x;\ell) = (2\pi)^n 2^{\ell-n}\ell!(\ell-n+1)!(A+B)^{-(\ell+1)/2}{}_2F_1(-\ell/2,(\ell+1)/2;1;B/(A+B)).$$

Replace ℓ by 2ℓ (now $\ell > \frac{n-1}{2}$), and we have

$$I_0(x;2\ell) = (2\pi)^n 2^{2\ell-n}(2\ell)!(2\ell-n+1)!(A+B)^{-\ell-1/2}\cdot {}_2F_1(-\ell,\ell+1/2;1;B/(A+B)).$$

Comparing this with Proposition 3.9, we get the following theorem:

Theorem 3.11. *For* $T \in V_0$ *with* $\langle T,T\rangle > 0$ *and* $\ell > \frac{n-1}{2}$, *we have*

$$I_0(T;\ell) = \frac{2^{-\ell-2}\pi^{4\ell-n+3}\langle T,T\rangle^{2\ell-n+1}}{\ell!(2\ell-n+1)!}\mathcal{W}_T(1).$$

References

[HMY24] Anton Hilado, Finn McGlade, and Pan Yan. *Fourier Coefficients and Algebraic Cusp Forms on* $\mathrm{U}(2,n)$. 2024. arXiv: 2404.17743 [math.NT]. URL: https://arxiv.org/abs/2404.17743.

[PS22] Aaron Pollack and Gordan Savin. "Modular forms on indefinite orthogonal groups of rank three". English. In: *J. Number Theory* 238 (2022), pp. 611–675. ISSN: 0022-314X. DOI: 10.1016/j.jnt.2021.09.011.