Quaternionic Degenerate Eisenstein series on $\mathbf{U}(2, n)$

October 3, 2025

Abstract

tba

1 Unitary groups of split rank 2

If some notation appearing in this section is not defined, then it should be defined in [HMY24].

Let E be an imaginary quadratic extension of $\mathbb{Q}$, and we denote the nonzero element in $\mathrm{Gal}(E/\mathbb{Q})$ by c or $x \mapsto \bar{x}$. Write the norm of $E/\mathbb{Q}$ as $|\,|$.

Define $\mathbf{G} = \mathbf{U}(V) = \mathbf{U}(2, n)$ to be the unitary group of a non-degenerate Hermitian space $(V, \langle\,,\,\rangle)$ over E with signature $(2, n)$. We write the $\mathbf{G}$-action on V as a right action.

Fix a pair of isotropic lines $(U, U^\vee)$ inside V such that $\langle U, U^\vee \rangle \neq 0$, and take V_0 to be the orthogonal complement of $U \oplus U^\vee$, which is a Hermitian space of signature $(1, n-1)$. Fix $b_1 \in U$ and $b_2 \in U^\vee$ such that $\langle b_1, b_2 \rangle = 1$.

One can define a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ as the stabilizer of U, and it has the following realization of Levi decomposition $\mathbf{P} = \mathbf{MN}$:

— $\mathbf{M}$ is the stabilizer of U and $U^\vee$, and it is isomorphic to $\mathrm{Res}_{E/\mathbb{Q}}(\mathbf{G_m}) \times \mathbf{U}(V_0)$. One can write the $\mathbf{M}$-action on $(u, w, u^\vee) \in U \oplus V_0 \oplus U^\vee$ explicitly:

$$(u, w, u^\vee).(z, h) = (z^{-1}u, wh, \bar{z}u^\vee).$$

— $\mathbf{N}$ is a Heisenberg, and it is isomorphic to the group

$$\left\{ (v, \lambda) \in V_0 \times \mathrm{Res}_{E/\mathbb{Q}}\mathbf{G_a} \,\middle|\, \bar{\lambda} = -\lambda \right\},$$

equipped with the following multiplication:

$$(v_1, \lambda_1) \cdot (v_2, \lambda_2) := \left(v_1 + v_2, \lambda_1 + \lambda_2 - \frac{\langle v_1, v_2 \rangle - \langle v_2, v_1 \rangle}{2} \right).$$

The action of the corresponding element $n(v, \lambda) \in \mathbf{N}$ on V is given as:

$$b_1 \mapsto b_1,\ b_2 \mapsto (-\frac{1}{2}\langle v, v \rangle + \lambda)b_1 + b_2 + v,\ w \in V_0 \mapsto -\langle w, v \rangle b_1 + w.$$

One has
$$(z,h)n(v,\lambda)(z^{-1},h^{-1}) = n(\bar{z}vh^{-1}, |z|\lambda).$$

Denote the center of $\mathbf{N}$ by $\mathbf{N}_0$. We denote by ν the similitude character of $\mathbf{M}$: $\nu(z,h) = z \in \mathrm{Res}_{E/\mathbb{Q}}\mathbf{G_m}$, so that $b_1(z,h) = \nu(z,h)^{-1}b_1$.

Fix an addictive character $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$, so that ψ_p has conductor $\mathbb{Z}_p$ and $\psi_\infty(x) = e^{2\pi i x}$. We have the following identification $V_0 \xrightarrow{\sim} \mathrm{Hom}([\mathbf{N}],\mathbb{S}^1)$: for any element $T \in V_0$, one associates a unitary character χ_T of $[\mathbf{N}]$ by

$$\chi_T : [\mathbf{N}] \to \mathbb{C}^\times, \; n(v,\lambda) \mapsto \psi(-\mathrm{Im}\langle T,v\rangle).$$

I am really confused with this identification in [HMY24]. This character χ_T seems not well-defined: the symplectic form $-\mathrm{Im}\langle T,v\rangle = \frac{\langle v,T\rangle - \langle T,v\rangle}{2i}$ involves $1/2i$, which may not lie in E. Maybe one should define χ_T to be $n(v,\lambda) \mapsto \psi(\mathrm{Re}\langle T,v\rangle)$.

For a finite place p of $\mathbb{Q}$, we define E_p to be $E_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} E$, which is isomorphic to

— $\mathbb{Q}_p \times \mathbb{Q}_p$, if p splits in E;

— a degree 2 unramified (*resp.* ramified) extension of $\mathbb{Q}_p$, if p is inert (*resp.* ramified).

2 Heisenberg Eisenstein series

Choose a section $f = f_{\ell,\infty} \otimes f_{fte}$ of $\mathrm{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}|\nu|^s$ (unnormalized) as follow:

— $f_{\ell,\infty}$ is the $\mathbb{V}_\ell = \left(\mathrm{Sym}^{2\ell}V_2^+ \otimes \det_{\mathbf{U}(2)}^{-\ell}\right) \boxtimes \mathbf{1}$-valued, K_∞-equivariant induced section whose restriction to $\mathbf{M}(\mathbb{R})$ is

$$f_{\ell,\infty}((z,h),s) = |z|^s[u_1^\ell][u_2^\ell],$$

where $[u_i^k] := u_i^k/k!$.

— f_{fte} is defined as

$$f_{fte}(g_f,\Phi_f,s) = \int_{\mathbf{GL}_1(\mathbb{A}_{E,f})} |t|^s \Phi_f(t \cdot b_1 g_f)dt,$$

where $\Phi_f = \prod \Phi_p$ is a Schwartz-Bruhat function on $V(\mathbb{A}_f)$.

One defines the degenerate Heisenberg Eisenstein series:

$$E_\ell(g,\Phi_f,s) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})} f(\gamma g, s).$$

To prove the modularity of this Eisenstein series, one needs the following result:

Proposition 2.1. *The $\mathbb{V}_\ell$-valued section $f_{\ell,\infty}$ of $\mathrm{Ind}_{\mathbf{P}(\mathbb{R})}^{\mathbf{G}(\mathbb{R})}|\nu|^s$, as a function on $\mathbf{G}(\mathbb{R})$, is killed by the Schmid operators $\mathcal{D}_\ell^\pm$ if and only if $s = \ell + 1$.*

Proof. Write an element $(z, h) \in \mathbf{M}(\mathbb{R})$ as $m = (h, r, \theta)$ so that $z = re^{i\theta}$, $r \in \mathbb{R}_{>0}$ and $\theta \in [0, 2\pi)$. Using this coordinates, we define a function F on $\mathbf{M}(\mathbb{R})$, sending (h, r, θ) to r^s. By [HMY24, Proposition 3.10], $f_{\ell, \infty}$ is killed by $\mathcal{D}_\ell^\pm$ if and only if

$$\begin{cases} (r\partial r - 2(\ell + 1))F(m) = 0 \\ [u_2 \otimes \overline{v_k}]^+ F(m) = 0, \text{ if } 1 \leq k < n \\ [u_2 \otimes \overline{v_k}]^- F(m) = 0, \text{ if } 1 \leq k < n \end{cases}$$

It can be easily verified that these equations hold if and only if $s = \ell + 1$. $\qquad \square$

As a consequence, the Eisenstein series $E_\ell(g, \Phi_f, s = \ell + 1)$ is a quaternionic modular form of weight ℓ.

3 The Fourier expansion of $E_\ell(g, \Phi_f, s = \ell + 1)$

3.1 Abstract Fourier expansion

In this subsection, we give the "abstract" Fourier expansion of $E_\ell(g, \Phi_f, s)$.

Lemma 3.1. *The right $\mathbf{P}(\mathbb{Q})$-space $\mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})$, the space of isotropic lines in $V(\mathbb{Q})$, has exactly 3 orbits of $\mathbf{P}(\mathbb{Q})$, represented respectively by $\mathbb{Q}b_1$, $\mathbb{Q}v_0$ and $\mathbb{Q}b_2$, where v_0 is an arbitrary non-zero isotropic vector in $V_0(\mathbb{Q})$.*

Proof. Directly by the explicit action given in §1. $\qquad \square$

Set $\mathbf{G}(\mathbb{Q}) = \bigsqcup_{i=0}^2 \mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})$, such that $w_0 = 1$, $b_1 w_1 = v_0$ and $b_1 w_2 = b_2$. Now we can write the degenerate Eisenstein series as

$$E_\ell(g, \Phi_f, s) = \sum_{i=0}^2 E_{\ell,i}(g, \Phi_f, s), \quad E_{\ell,i}(g, \Phi_f, s) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q})\backslash\mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})} f(\gamma g, s),$$

thus $E_{\ell,0}(g, \Phi_f, s) = f(g, s)$. From now on, when there is no confusion we will omit the ℓ and Φ_f in $E_\ell(g, \Phi_f, s)$, and write it as $E(g, s) = \sum_{i=0}^2 E_i(g, s)$.

Lemma 3.2. *Assume that $\mathrm{Re}(s) \gg 0$ so that the sum defining $E(g, s)$ converges absolutely. Then one has the following expressions for the $E_i(g, s)$:*

(1) Let $\mathcal{L}_0$ be the set of non-zero isotropic lines ℓ in V_0 and for any $\ell \in \mathcal{L}_0$, select $\gamma(\ell) \in \mathbf{G}(\mathbb{Q})$ with $b_1\gamma(\ell) \in \ell$. Then

$$E_1(g, s) = \sum_{\ell \in \mathcal{L}_0} \sum_{\mu \in (\ell)^\perp \mathbf{N}_0(\mathbb{Q})\backslash\mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu g, s).$$

(2) One has

$$E_2(g, s) = \sum_{\mu \in \mathbf{N}(\mathbb{Q})} f(w_2\mu g, s).$$

3

For any $T \in V_0$, we set

$$E_i^T(g,s) = \int_{\mathbf{N}(F) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) E_i(ng, s) dn, \ i = 0, 1, 2.$$

Lemma 3.3. *(1) If T is anisotropic, then $E_1^T = 0$. If T is isotropic, define $\mathbf{N}_T = (\ell_T)^{\perp} \mathbf{N}_0 \subseteq \mathbf{N}$, then*

$$E_1^T(g,s) = \int_{\mathbf{N}_T(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell_T) ng, s) dn.$$

(2) For any $T \in V_0$, one has

$$E_2^T(g,s) = \int_{\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(w_2 ng, s) dn.$$

Proof. It suffices only to prove the $i = 1$ case. For any $\ell \in \mathcal{L}_0$, set

$$\mathbf{N}_\ell = \left\{ n(v, \lambda) \in \mathbf{N} \,\middle|\, v \in \ell^{\perp} \right\} \subseteq \mathbf{N}.$$

For any $T \in V_0$,

$$
\begin{aligned}
E_1^T(g,s) &= \sum_{\ell \in \mathcal{L}_0} \int_{[N]} \chi_T^{-1}(n) \left(\sum_{\mu \in \mathbf{N}_\ell(\mathbb{Q}) \backslash \mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu ng, s) \right) dn \\
&= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{Q}) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell) ng, s) dn \\
&= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} \left(\int_{[\mathbf{N}_\ell]} \chi_T^{-1}(r) dr \right) \chi_T^{-1}(n) f(\gamma(\ell) ng, s) dn \\
&= \sum_{\ell \in \mathcal{L}_0, \chi_T|_{\mathbf{N}_\ell} \equiv 1} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell) ng, s) dn.
\end{aligned}
$$

Then the lemma follows from the fact that $\chi_T|_{\mathbf{N}_\ell} \equiv 1$ if and only if $T \in \ell$, *i.e.* T is isotropic and $\ell = \ell_T$. $\square$

3.2 Computation of constant term

I think we also need a refined version of [HMY24, Corollary 1.2], with a description on its constant term.

3.2.1 The $i = 0$-term

Lemma 3.4. *For $g \in \mathbf{P}(\mathbb{A})$,*

$$E_0(g,s) = f(g,s) = |v(g)|^s \zeta_E(s) [u_1^n][u_2^n].$$

Proof. For $g_f \in \mathbf{P}(\mathbb{A}_f)$, we have

$$
\begin{aligned}
f_{fte}(g_f, s) &= \int_{\mathbb{A}_{E,f}^\times} |t|^s \Phi_f(t b_1 g_f) dt \\
&= \int_{\mathbb{A}_{E,f}^\times} |t|^s \Phi_f(t \nu(g_f)^{-1} b_1) dt \\
&= |\nu(g_f)|^s \int_{\mathbb{A}_{E,f}^\times} |t|^s \Phi_f(t b_1) dt.
\end{aligned}
$$

Thus, the non-archimedean contribution is $|\nu(g_f)|^s \zeta_E(s)$. Combining with $f_{n,\infty}(g_\infty, s) = |\nu(g_\infty)|^s [u_1^n][u_2^n]$, we get the desired identity. $\qquad\square$

3.2.2 The $i = 1$-term

We fix a non-zero isotropic vector in V_0, such that $v_0 = b_1 \gamma_0$, and set $\ell_0 = E v_0$. Define $\mathbf{P}_0$ be the stabilizer of ℓ_0 in $\mathbf{U}(V_0)$, which is a parabolic subgroup of $\mathbf{M}$. We denote the similitude character of $\mathbf{P}_0$ by λ, *i.e.* $v_0 g = \lambda(g)^{-1} v_0$ for any $g \in \mathbf{P}_0$. For $g \in \mathbf{P}_0(\mathbb{A})$, we have:

$$
\begin{aligned}
E_1^0(g, s) &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma(\ell) n g, s) dn \\
&= \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{Q})} \int_{\mathbf{N}_{\ell_0}(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma_0 \gamma n g, s) dn
\end{aligned}
$$

If we set $f_0(g, s) = \int_{\mathbf{N}_{\ell_0}(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma_0 n g, s) dn$, then for $\mathrm{Re}(s) \gg 0$,

$$
E_1^0(g, s) = \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{Q})} f_0(\gamma g, s),
$$

and it defines an Eisenstein series on $\mathbf{M}$. Now we want to determine this section $f_0(g, s) \in \mathrm{Ind}_{\mathbf{P}_0(\mathbb{A})}^{\mathbf{M}(\mathbb{A})} |\lambda|^s$ (still unnormalized).

At finite places, one has

$$
\begin{aligned}
\int_{\mathbf{N}_{\ell_0}(\mathbb{A}_{E,f}) \backslash \mathbf{N}(\mathbb{A}_{E,f})} f(\gamma_0 n g, s) dn &= \int_{x \in \mathbb{A}_{E,f}} \int_{t \in \mathbb{A}_{E,f}^\times} |t|^s \Phi_{fte}(t(v_0 + x b_1) g) dt dx \\
&= \int_{x \in \mathbb{A}_{E,f}} \int_{t \in \mathbb{A}_{E,f}^\times} |t|^s \Phi_{fte}(t \lambda(g)^{-1} v_0 + t x \nu(g)^{-1} b_1) dt dx \\
&= |\lambda(g)|_f^{s-1} |\nu(g)|_f \int_{x \in \mathbb{A}_{E,f}} \int_{t \in \mathbb{A}_{f,E}^\times} |t|^{s-1} \Phi_{fte}(t v_0 + x b_1) dt dx \\
&= |\lambda(g)|_f^{s-1} |\nu(g)|_f \zeta_E(s-1).
\end{aligned}
$$

Now we switch to the archimedean place. Set $c_1 = v_0$ and c_2 another isotropic vector in V_0 with $\langle c_1, c_2 \rangle = 1$. We can take $u_2 = \frac{1}{\sqrt{2}}(c_1 + c_2)$ and $v_{n-1} = \frac{1}{\sqrt{2}}(c_1 - c_2)$.

5

Lemma 3.5. *We have*
$$f_{0,\infty}(g,s) =$$

Proof. One picks the following representatives for $\mathbf{N}_{\ell_0}(\mathbb{R})\backslash\mathbf{N}(\mathbb{R})$: $\{n(-xc_2,0),\ x \in \mathbb{R}\}$, thus $b_1\gamma_0 n(-xc_2,0)g = \lambda(g)^{-1}c_1 + xv(g)^{-1}b_1$. Suppose that $\gamma_0 n(-xc_2,0)g = pk$ for some $p \in \mathbf{P}(\mathbb{R})$ and $k = (k^+,k^-) \in K_\infty$, then one has

$$\lambda(p)^{-1}b_1 k = \lambda(g)^{-1}c_1 + xv(g)^{-1}b_1.$$

Projecting both sides to V_2^+, one has

$$\lambda(p)^{-1}u_1 k^+ = xv(g)^{-1}u_1 + \lambda(g)^{-1}u_2.$$

So one can take p such that

$$\lambda(p)^{-1} = \sqrt{|\lambda(g)|^{-2} + x^2|v(g)|^{-2}}$$

and

$$k^+ = \lambda(p) \begin{pmatrix} xv(g)^{-1} & -\overline{\lambda(g)}^{-1} \\ \lambda(g)^{-1} & \overline{xv(g)}^{-1} \end{pmatrix}.$$

Hence we have

$$
\begin{aligned}
f_{\ell,\infty}(\gamma_0 n(-xc_2,0)g) &= f_{\ell,\infty}(pk) \\
&= |v(p)|^s([u_1^\ell][u_2^\ell])k \\
&= \frac{(xv(g)^{-1}u_1 + \lambda(g)^{-1}u_2)^\ell(-\overline{\lambda(g)}^{-1}u_1 + \overline{xv(g)}^{-1}u_2)^\ell}{(|\lambda(g)|^{-2} + x^2|v(g)|^{-2})^{s+\ell}(\ell!)^2}
\end{aligned}
$$

The coefficient of $[u_1^{\ell-v}][u_2^{\ell+v}]$ in $f_{\ell,\infty}(\gamma_0 n(-xc_2,0)g)$ is

$$
\frac{(\ell-v)!(\ell+v)!}{(\ell!)^2(|\lambda|^{-2} + x^2|v|^{-2})^{2\ell+1}} \sum_{\substack{0 \le i,j \le \ell \\ i+j=\ell-v}} \binom{\ell}{i}\binom{\ell}{j} x^i v^{-i}\lambda^{-(\ell-i)}(-\overline{\lambda})^{-j}(x\overline{v}^{-1})^{\ell-j}
$$

$$
= \frac{(\ell-v)!(\ell+v)!|v|^{-2\ell}v^v\lambda^{-v}}{(\ell!)^2(|\lambda|^{-2} + x^2|v|^{-2})^{2\ell+1}} \sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)} (-1)^j \binom{\ell}{j}\binom{\ell}{j+v} |v|^{2j}|\lambda|^{-2j}x^{2\ell-v-2j}
$$

Using the fact that

$$
\int_{x\in\mathbb{R}} \frac{x^{2m}}{(|\lambda|^{-2} + x^2|v|^{-2})^{2\ell+1}} = \pi|\lambda|^{4\ell-2m+1}|v|^{2m+1}2^{-4\ell}\frac{(4\ell-2m)!(2m)!}{(2\ell-m)!m!(2\ell)!},
$$

we can integrate $f_{\ell,\infty}(\gamma_0 n(-xc_2, 0)g)$ over $x \in \mathbb{R}$, and its $[u_1^{\ell-v}][u_2^{\ell+v}]$ coefficient is 0 when v is odd, otherwise, it is

$$\frac{\pi|\lambda|^{2\ell+1}|v|}{(2\ell)!(\ell!)^2 2^{4\ell}} \cdot (\ell-v)!(\ell+v)!(v/\lambda)^v|v/\lambda|^{-v} \sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)} (-1)^j \binom{\ell}{j}\binom{\ell}{j+v} \frac{(2\ell+v+2j)!(2\ell-v-2j)!}{(\ell+v/2+j)!(\ell-v/2-j)!}.$$

The inner sum is symmetric under $v \mapsto -v$, and when v is non-negative, then it equals

$$\sum_{j=0}^{\ell-v}(-1)^j \binom{\ell}{j}\binom{\ell}{j+v}\frac{(2\ell+v+2j)!(2\ell-v-2j)!}{(\ell+v/2+j)!(\ell-v/2-j)!}$$
$$=\frac{2^{4\ell}}{\pi}\Gamma(\ell+v/2+1/2)\Gamma(\ell-v/2+1/2)\,_3F_2(-\ell,-\ell+v,\ell+v/2+1;v+1,-\ell+v/2+1/2;1).$$

So we have

$$f_{0,\infty}(s)=\frac{|\lambda(g)|^{2\ell+1}|v(g)|}{(2\ell)!}\sum_{\substack{-\ell\leq v\leq\ell\\v\text{ is even}}}\binom{\ell+|v|}{\ell}\Gamma(\ell+v/2+1)\Gamma(\ell-v/2+1)\left(\frac{v(g)\overline{\lambda(g)}}{\overline{v(g)}\lambda(g)}\right)^{v/2}\,_3F_2(-\ell,-\ell+v,\ell+v/2+1;v+1,-\ell+v/2+1/2;1).$$

$\square$

There must be some mistake in this computation. I need to check it.

3.3 Rank 1 Fourier coefficients

3.4 Rank 2 Fourier coefficients

One decomposes $E_2^T(1,s)$ as $\prod_v E_{2,v}^T(s)$.

3.4.1 Finite places

$$E_{2,p}^T(s) = \int_{\mathbf{N}(\mathbb{Q}_p)} \chi_T^{-1}(n) f_p(w_2 n, s) dn$$
$$= \int_{\mathbf{N}(\mathbb{Q}_p)}\int_{t\in E_p^\times} \chi_T^{-1}(n)|t|^s \Phi_p(t\cdot b_1 w_2 n)dtdn$$
$$= \int_{v\in\mathbf{V}_0(\mathbb{Q}_p)}\int_{\substack{x\in E_p\\\overline{x}=-x}}\int_{t\in E_p^\times} \chi_T^{-1}(v)|t|^s\Phi_p\left(t\left((-\langle v,v\rangle/2+x)b_1+b_2+v\right)\right)dtdxdv$$

Take $\mathcal{O}_{E_p}$ to be $\mathbb{Z}_p\otimes_{\mathbb{Z}}\mathcal{O}_E\subseteq E_p$, $\mathfrak{p}$ to be its maximal ideal, and $\mathcal{V}$ an $\mathcal{O}_{E_p}$-lattice of $V_0(\mathbb{Q}_p)=V_0\otimes_{\mathbb{Q}}\mathbb{Q}_p$ such that $\mathcal{V}\otimes_{\mathcal{O}_{E_p}}(\mathcal{O}_{E,p}/\mathfrak{p})$ is a non-degenerate Hermitian space over $\mathcal{O}_{E_p}/\mathfrak{p}$. Assume that Φ_p is the characteristic function of the lattice $\mathcal{O}_{E_p}b_1\oplus\mathcal{V}\oplus\mathcal{O}_{E_p}b_2$.

Write down a precise definition of $\mathcal{V}$?

Lemma 3.6. *(1) If p splits in E, then*

$$E_{2,p}^T(s) = \sum_{r_1,r_2\geq 0} p^{-(r_1+r_2)s+\min(r_1,r_2)}\left(\int_{v\in(p^{-r_1},p^{-r_2})\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^{\max(r_1,r_2)}\langle v,v\rangle\in\mathbb{Z}_p)dv\right).$$

7

(2) If p is inert in E, then

$$E_{2,p}^T(s) = \sum_{r \geq 0} p^{-2rs+r} \left(\int_{v \in p^{-r}\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}\left(p^r \langle v, v \rangle /2 \in \mathbb{Z}_p\right) dv \right).$$

(3) If p is ramified in E, then

Proof. (1) If p splits in E, then $E_p = \mathbb{Q}_p \times \mathbb{Q}_p$:

$$E_{2,p}^T(s) = \sum_{r_1 \geq 0, r_2 \geq 0} |p|^{(r_1+r_2)s} \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \left(\int_{\substack{x \in (p^{-r_1}, p^{-r_2})\mathcal{O} + \langle v, v \rangle /2 \\ x + \bar{x} = 0}} dx \right) dv$$

If we write $x = (y, -y) \in E_p$, then $x \in (p^{-r_1}, p^{-r_2})\mathcal{O} + \langle v, v \rangle /2$ is equivalent to

$$y \in p^{-r_1}\mathbb{Z}_p + \langle v, v \rangle /2, \quad -y \in p^{-r_2}\mathbb{Z}_p + \langle v, v \rangle /2.$$

There are such x only if $\langle v, v \rangle \in p^{-\max(r_1, r_2)}\mathbb{Z}_p$, If $r_1 \geq r_2$, then $y \in p^{-r_2}\tilde{y} - \langle v, v \rangle /2$ for some $\tilde{y} \in \mathbb{Z}_p$. Since the measure on the line $\{x = (y, -y) \in E_p\}$ is the normalized Haar measure dy on $\mathbb{Q}_p$, one has $dy = p^{r_2}d\tilde{y}$, and

$$\int_{x=(y,-y)\in p^{-r_1}\mathbb{Z}_p \times p^{-r_2}\mathbb{Z}_p + \langle v,v \rangle /2} dx = \int_{\tilde{y} \in \mathbb{Z}_p} p^{r_2}d\tilde{y} = p^{r_2} = p^{\min(r_1, r_2)}.$$

Hence

$$E_{2,p}^T(s) = \sum_{r_1, r_2 \geq 0} p^{-(r_1+r_2)s + \min(r_1, r_2)} \left(\int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^{\max(r_1, r_2)} \langle v, v \rangle \in \mathbb{Z}_p) dv \right).$$

(2) If p is inert in E, then E_p is an unramified quadratic field extension of $\mathbb{Q}_p$, and

$$E_{2,p}^T(s) = \sum_{r \geq 0} |p|_{E_p}^{rs} \int_{v \in p^{-r}\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^r \langle v, v \rangle \in \mathbb{Z}_p) \left(\int_{x \in \langle v,v \rangle /2 + p^{-r}\mathcal{O}_{E_p}} \mathrm{Char}(x + \bar{x} = 0) dx \right) dv$$

Choose a unit $u \in \mathcal{O}_{E_p}^\times$ with $u + \bar{u} = 0$, then the line $\{x \in E_p \,|\, x + \bar{x} = 0\}$ can be written as $\{yu, y \in \mathbb{Q}_p\}$, with the normalized Haar measure dy. There exist elements $yu \in \langle v, v \rangle /2 + p^{-r}\mathcal{O}_{E_p}$ if and only if $\langle v, v \rangle /2 \in p^{-r}\mathbb{Z}_p$. When we have $\langle v, v \rangle /2 \in p^{-r}\mathbb{Z}_p$, any element $yu \in \langle v, v \rangle /2 + p^{-r}\mathcal{O}_{E_p}$ is of the form

$$yu = \langle v, v \rangle /2 + p^{-r}\left(-p^r \langle v, v \rangle /2 + \tilde{y}u\right), \tilde{y} \in \mathbb{Z}_p,$$

thus one has

$$\int_{x \in \langle v,v \rangle /2 + p^{-r}\mathcal{O}_{E_p}} \mathrm{Char}(x + \bar{x} = 0) dx = \mathrm{Char}(p^r \langle v, v \rangle /2 \in \mathbb{Z}_p) |p|_{\mathbb{Q}_p}^{-r} \int_{\mathbb{Z}_p} d\tilde{y}$$

$$= p^r \mathrm{Char}(p^r \langle v, v \rangle /2 \in \mathbb{Z}_p),$$

which gives us the desired identity. □

8

If p is ramified in E: (Assume $p \neq 2$) Then E_p is a ramified quadratic field extension of $\mathbb{Q}_p$, and we choose an uniformizer ϖ of $\mathfrak{p} \subset \mathcal{O}_{E_p}$ such that $\varpi + \overline{\varpi} = 0$ and $\varpi^2 \in p\mathbb{Z}_p^\times$. The integral for $E_{2,p}^T$ can be rewritten as:

$$E_{2,p}^T(s) = \sum_{r \geq 0} |\varpi|^{rs} \int_{v \in \varpi^{-r}\mathcal{V}} \chi_T^{-1}(v)\mathrm{Char}(p^{\lfloor r/2 \rfloor}\langle v, v\rangle \in \mathbb{Z}_p)\left(\int_{x \in \mathbb{Q}_p} \mathrm{Char}(x\varpi - \frac{\langle v, v\rangle}{2} \in \varpi^{-r}\mathcal{O}_{E_p})dx \right) dv$$

$$= \sum_{r \geq 0} |\varpi|^{rs} p^{\lceil r/2 \rceil} \left(\int_{v \in \varpi^{-r}\mathcal{V}} \chi_T^{-1}(v)\mathrm{Char}(p^{\lfloor r/2 \rfloor}\langle v, v\rangle \in \mathbb{Z}_p)dv \right)$$

3.4.2 Archimedean place

Remark 3.7. In this article, when we write $|z|$ for $z \in \mathbb{C}$, it means the *norm of z with respect to the extension $\mathbb{C}/\mathbb{R}$, i.e.* $|z| = z\overline{z}$ instead of the usual modulus $\sqrt{z\overline{z}}$. I admit that this is somehow strange and confusing, but I will fix this problem if this draft could become a paper...

We first analyze the function $f_{\ell,\infty}(w_2n, s = \ell + 1)$:

Lemma 3.8. *For any $v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}$ and $x \in \mathbb{R}$, we set:*

$$\alpha(v, x) = -\frac{\langle v, v\rangle}{2} + ix + 1, \quad \beta(v) = \sqrt{2}\langle v, u_2\rangle,$$

then we have

$$f_{\ell,\infty}(w_2n(v, ix), s) = \frac{(\alpha u_1 + \beta u_2)^\ell (-\overline{\beta}u_1 + \overline{\alpha}u_2)^\ell}{(|\alpha(v, x)| + |\beta(v)|)^{\ell+s}(\ell!)^2}.$$

Proof. One has

$$b_1 w_2 n(v, ix) = (-\langle v, v\rangle/2 + ix)b_1 + b_2 + v.$$

Suppose that we can decompose $w_2 n(v, ix)$ as pk for some $p \in \mathbf{P}(\mathbb{R})$ and $k \in K_\infty$, then

$$b_1 w_2 n(v, ix)k^{-1} = b_1 p = \nu(p)^{-1}b_1. \tag{1}$$

Let k_+ be the factor of k in $\mathrm{U}(V_2^+)$, and $v = v_+ + v_- \in \mathbb{C}u_2 \oplus \mathrm{Span}_{\mathbb{C}}(v_1, \ldots, v_{n-1})$. Taking the V_2^+ components of Equation (1), one gets:

$$\frac{1}{\sqrt{2}}(\alpha(v, x)u_1 + \beta(v)u_2) = \left[\frac{1}{\sqrt{2}}\left(-\frac{\langle v, v\rangle}{2} + ix + 1 \right)u_1 + v_+ \right]k_+^{-1} = \nu(p)^{-1}\frac{u_1}{\sqrt{2}}.$$

The norms of both sides give us the identity $|\alpha(v, x)| + |\beta(v)| = |\nu(p)|^{-1}$. One may assume that $\nu(p)^{-1} = \sqrt{|\alpha(v, x)| + |\beta(v)|}$, then in the basis of u_1, u_2, the element $k^+ \in \mathrm{U}(V_2^+)$ can be written as the Hermitian matrix

$$\frac{1}{\sqrt{|\alpha(v, x)| + |\beta(v)|}}\begin{pmatrix} \alpha(v, x) & -\overline{\beta(v)} \\ \beta(v) & \overline{\alpha(v, x)} \end{pmatrix}.$$

Plug k^+ and $\nu(p)$ into $f_{\ell,\infty}(w_2n(v, ix), s) = f_{\ell,\infty}(pk, s) = |\nu(p)|^s[u_1^\ell][u_2^\ell].k^+$, and we get the desired value. $\qquad\square$

Let $I_0(T; \ell)$ be the coefficient of $[u_1^\ell][u_2^\ell]$ in $E_{2,\infty}^T(s = \ell + 1)$, which can be written as

$$I_0(T; \ell) = \int_{v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}} \int_{x \in \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} \frac{\binom{\ell}{k}\binom{\ell}{k}(x^2 + A)^k(-B)^{\ell-k}}{(x^2 + A + B)^{2\ell+1}} dx dv$$

$$= \int_{v \in V_0 \otimes \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \int_{x \in \mathbb{R}} \frac{(x^2 + A)^k}{(x^2 + A + B)^{2\ell+1}} dx dv$$

where $A = |\alpha| - x^2 = (1 - \langle v, v \rangle / 2)^2$, and $B = |\beta| = 2|\langle v, u_2 \rangle|$.

Lemma 3.9. *For any real number C, D and two natural numbers $m < n$, we have*

$$\int_{\mathbb{R}} \frac{(x^2 + C)^m}{(x^2 + D)^n} dx = \frac{D^{m-n+1/2}}{(n-1)!} \sum_{k=0}^{m} \binom{m}{k} \left(\frac{C}{D}\right)^{m-k} \Gamma(k + 1/2)\Gamma(n - k - 1/2).$$

Proof. An exercise of calculus. □

Now Lemma 3.9 tells us $I_0(T; \ell)$ is the Fourier transform of the function

$$F_{0,\ell}(v) := \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \frac{(A + B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{A}{A+B}\right)^{k-j} \Gamma(j + 1/2)\Gamma(2\ell + 1/2 - j)$$

Set $z = B/(A + B)$, then this function becomes

$$\frac{2^{-4\ell}}{(2\ell)!} \pi (A + B)^{-\ell-1/2} \sum_{k=0}^{\ell} \sum_{j=0}^{k} \binom{\ell}{k}^2 \binom{k}{j} \frac{(2j)!(4\ell - 2j)!}{j!(2\ell - j)!} (-z)^{\ell-k}(1 - z)^{k-j}$$

We write:

$$\sum_{k=0}^{\ell} \sum_{j=0}^{k} c_{j,k} (-z)^{\ell-k}(1 - z)^{k-j} = \sum_{r=0}^{\ell} (-1)^r C(r) z^r,$$

where $c_{j,k} = \binom{\ell}{k}^2 \binom{k}{j} \frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!}$. The term $(-z)^{\ell-k}(1 - z)^{k-j}$ has a non-zero z^r term if and only if $\ell - k \leq r \leq \ell - j$, thus

$$(-1)^r C(r) = \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} c_{j,k} \binom{k-j}{r-\ell+k} (-1)^{\ell-k+r-\ell+k} = (-1)^r \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} c_{j,k} \binom{k-j}{r-\ell+k}.$$

To compute $C(r)$, we need the following lemma:

Lemma 3.10. *(1) For integers $0 \leq a \leq b$, one has*

$$\sum_{i=a}^{b} \binom{b}{i} \binom{b-a}{b-i} = \binom{2b-a}{b}.$$

(2) *For any integer $0 \leq r \leq \ell$, one has*

$$\sum_{i=0}^{\ell-r} \frac{\binom{2\ell}{i}\binom{\ell-r}{i}}{\binom{4\ell}{2i}} = 2^{2\ell-2r}\frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}.$$

Proof. The identity in (1) is obvious. For the identity in (2), the LHS can be rewritten as:

$$\sum_{i=0}^{\ell-r} \frac{(-\ell+r)_j(1/2)_j}{(-2\ell+1/2)_j j!} = {}_2F_1(-(\ell-r),1/2;-2\ell+1/2;1),$$

where $(x)_j$ is the (rising) Pochhammer symbol, and ${}_2F_1$ is the hypergeometric function. By Chu-Vandermond identity, this value of hypergeometric function is

$$\frac{(-2\ell)_{\ell-r}}{(-2\ell+1/2)_{\ell-r}} = 2^{\ell-r}\frac{\frac{(2\ell)!}{(\ell+r)!}}{\frac{(4\ell-1)!!}{(2\ell+2r-1)!!}} = 2^{2\ell-2r}\frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}. \qquad \square$$

Now we return to the value of $C(r)$:

$$C(r) = \sum_{j=0}^{\ell-r}\sum_{k=\ell-r}^{\ell} \frac{(\ell!)^2(2j)!(4\ell-2j)!}{k!((\ell-k)!)^2(j!)^2(r-\ell+k)!(\ell-r-j)!(2\ell-j)!}$$

$$= \sum_{j=0}^{\ell-r} \frac{\ell!(2j)!(4\ell-2j)!}{r!(j!)^2(\ell-r-j)!(2\ell-j)!} \sum_{k=\ell-r}^{\ell}\binom{\ell}{k}\binom{r}{\ell-k}$$

$$\text{(by (1) of Lemma 3.10)} = \frac{\ell!}{r!}\cdot\frac{(\ell+r)!}{\ell!r!}\sum_{j=0}^{\ell-r}\frac{(2j)!(4\ell-2j)!}{(j!)^2(\ell-r-j)!(2\ell-j)!}$$

$$= \frac{(\ell+r)!}{(r!)^2}\cdot\frac{(4\ell)!}{(\ell-r)!(2\ell)!}\sum_{j=0}^{\ell-r}\frac{\binom{2\ell}{j}\binom{\ell-r}{j}}{\binom{4\ell}{2j}}$$

$$\text{(by (2) of Lemma 3.10)} = \frac{(\ell+r)!}{(r!)^2}\cdot\frac{(4\ell)!}{(\ell-r)!(2\ell)!}\cdot 2^{2\ell-2r}\frac{\frac{(2\ell+2r)!}{((\ell+r)!)^2}}{\frac{(4\ell)!}{((2\ell)!)^2}}$$

$$= 2^{2\ell-2r}\frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!}.$$

Putting the value of $C(r)$ into $F_{0,\ell}$, we have

$$F_{0,\ell}(v) = \frac{2^{-4\ell}\pi}{(2\ell)!(A+B)^{\ell+1/2}}\sum_{r=0}^{\ell}(-1)^r 2^{2\ell-2r}\frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!}$$

$$= \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\sum_{r=0}^{\ell}\frac{(-\ell)_r(\ell+1/2)_r}{1_r}\frac{z^r}{r!}$$

$$= \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\cdot {}_2F_1(-\ell,\ell+1/2;1;z).$$

We have shown the following result:

Proposition 3.11. *For $T \in V_0$ with $\langle T, T \rangle > 0$, T the coefficient $I_0(T; \ell)$ of $[u_1^\ell][u_2^\ell]$ in $E_{2,\infty}^T(s = \ell + 1)$ is the Fourier transform of the function:*

$$F_{0,\ell}(v) = \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}} \cdot {}_2F_1(-\ell, \ell + 1/2; 1; \frac{B}{A+B}),$$

where $A = \left(1 - \frac{\langle v,v \rangle}{2}\right)^2$ and $B = 2|\langle v, u_2 \rangle|^2$.

Let's recall the following result for quadratic spaces by Pollack: I know there are notation problems again... just let it be like this for now

Proposition 3.12. *($v = 0$ case of [PS22, Proposition 4.5.3]) Let $(V', (\,,\,)) = V_2 \oplus V_n$ be a non-degenerate quadratic space over $\mathbb{R}$ of signature $(2, n)$, and v_1, v_2 an orthonormal basis of V_2. Set*

$$I_0(x; \ell) = \int_{V'} e^{i(\omega, x)} \mathrm{Char}(q(\omega) > 0) q(\omega)^{\ell - n/2} K_0(\sqrt{2}|(\omega, v_1 + \sqrt{-1}v_2)|) d\omega.$$

This integral is absolutely convergent and we have

$$I_0(x; \ell) = (2\pi)^{(n+2)/2} 2^{\ell-1-n/2} \Gamma(\ell+1)\Gamma(\ell+1-n/2) F_4(\ell+1, \ell+1; \ell+1; 1; -\|x_n\|^2/2; -\|x_2\|^2/2),$$

where $x = x_2 + x_n$, and $F_4(a, b; c; d; x; y)$ is Appell's hypergeometric function.

Now we look at how this result fits into our unitary group setting. One take V' to be $(V_0, \mathrm{Re}(\langle\,,\,\rangle))$, as a quadratic space of signature $(2, 2n-2)$, $v_1' = u_2$, $v_2' = iu_2$, then

$$\begin{aligned}
I_0(x; \ell) &= \int_{T \in V_0, \langle T, T \rangle > 0} e^{i\mathrm{Re}\langle T, x \rangle} \langle T, T \rangle^{\ell-n+1} K_0(\sqrt{2}|\langle T, u_2 \rangle|) dT \\
&= \int_{T \in V_0, \langle T, T \rangle > 0} e^{2\pi i \mathrm{Re}\langle T, x \rangle} (2\pi)^{2\ell-2n+2} \langle T, T \rangle^{\ell-n+1} K_0(2\sqrt{2}\pi|\langle T, u_2 \rangle|) (2\pi)^{2n} dT \\
&= (2\pi)^{2\ell+2} \int_{T \in V_0, \langle T, T \rangle > 0} e^{2\pi i \mathrm{Re}\langle T, x \rangle} \langle T, T \rangle^{\ell-n+1} K_0(2\sqrt{2}\pi|\langle T, u_2 \rangle|) dT,
\end{aligned}$$

here $|z| = \sqrt{z\bar{z}}$, and for any $\ell > n - 1$ we have

$$I_0(x; \ell) = (2\pi)^n 2^{\ell-n} \ell! (\ell-n+1)! (A+B)^{-(\ell+1)/2} {}_2F_1(-\ell/2, (\ell+1)/2; 1; B/(A+B)).$$

Replace ℓ by 2ℓ (now $\ell > \frac{n-1}{2}$), and we have

$$I_0(x; 2\ell) = (2\pi)^n 2^{2\ell-n} (2\ell)! (2\ell-n+1)! (A+B)^{-\ell-1/2} \cdot {}_2F_1(-\ell, \ell+1/2; 1; B/(A+B)).$$

Comparing this with Proposition 3.11, we get the following theorem:

Theorem 3.13. *For $T \in V_0$ with $\langle T, T \rangle > 0$ and $\ell > \frac{n-1}{2}$, we have*

$$I_0(T; \ell) = \frac{2^{-\ell-2}\pi^{4\ell-n+3}\langle T, T \rangle^{2\ell-n+1}}{\ell!(2\ell-n+1)!} \mathcal{W}_T(1).$$

References

[HMY24] Anton Hilado, Finn McGlade, and Pan Yan. *Fourier Coefficients and Algebraic Cusp Forms on* $\mathrm{U}(2, n)$. 2024. arXiv: 2404.17743 [math.NT]. URL: https://arxiv.org/abs/2404.17743.

[PS22] Aaron Pollack and Gordan Savin. "Modular forms on indefinite orthogonal groups of rank three". English. In: *J. Number Theory* 238 (2022), pp. 611–675. ISSN: 0022-314X. DOI: 10.1016/j.jnt.2021.09.011.