

# Rationality of the Fourier expansion of quaternionic Heisenberg Eisenstein series on $\mathbf{U}(2, n)$

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**Abstract**

tba

## 1 Introduction

## 2 Preliminaries on the unitary groups of split rank 2

Fix an imaginary quadratic field extension  $E$  of  $\mathbb{Q}$ , as a subfield of  $\mathbb{C}$ . Denote the nonzero element in  $\text{Gal}(E/\mathbb{Q})$  by  $x \mapsto \bar{x}$ , and for any  $\mathbb{Q}$ -algebra  $R$  we extend this involution to  $E \otimes_{\mathbb{Q}} R$  and define the following two maps:

- $\text{Tr}_{E \otimes_{\mathbb{Q}} R} : E \otimes_{\mathbb{Q}} R \rightarrow R, x \mapsto x + \bar{x}$ , and
- $| \cdot |_{E \otimes_{\mathbb{Q}} R} : E \otimes_{\mathbb{Q}} R \rightarrow R, x \mapsto |x|_{E \otimes_{\mathbb{Q}} R} = x\bar{x}$ .

Denote by  $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$  (*resp.*  $\mathbb{A}_E$ ) the adèle ring of  $\mathbb{Q}$  (*resp.*  $E$ ). Fix an additive  $\psi = \psi_{\mathbb{Q}}$  on  $\mathbb{Q} \backslash \mathbb{A}$  satisfying that for every finite prime  $p$ ,  $\psi_p$  is an additive character of  $\mathbb{Q}_p$  with conductor  $\mathbb{Z}_p$ , and at the archimedean place,  $\psi_{\infty}(x) = e^{2\pi i x}$ . Set  $\psi_E$  to be the additive character of  $E \backslash \mathbb{A}_E$  that sends  $x \in E$  to  $\psi(\text{Tr}_{\mathbb{A}_E}(x))$ .

### 2.1 Hermitian spaces

For any natural number  $n \geq 2$ , let  $(\mathbf{V}, \langle, \rangle)$  be a non-degenerate Hermitian space over  $E$  with signature  $(2, n)$ , where the Hermitian form is conjugate-linear in the second variable. For any  $\mathbb{Q}$ -algebra  $R$ , we denote by  $\mathbf{V}(R)$  the Hermitian module  $\mathbf{V} \otimes_{\mathbb{Q}} R$  over  $E \otimes_{\mathbb{Q}} R$ .

Fix a pair of isotropic vectors  $(b_1, b_2)$  in  $\mathbf{V}$  such that  $\langle b_1, b_2 \rangle = 1$ , and take  $\mathbf{V}_0$  to be the orthogonal complement of  $Eb_1 \oplus Eb_2$  in  $\mathbf{V}$ , which is a non-degenerate Hermitian space of signature  $(1, n-1)$ . We fix again a pair of isotropic vectors  $(c_1, c_2)$  in  $\mathbf{V}_0$  with  $\langle c_1, c_2 \rangle = 1$ ,

then  $\mathbf{V}$  is the orthogonal direct sum of  $Eb_1 \oplus Eb_2 \oplus Ec_1 \oplus Ec_2$  and a Hermitian space  $\mathbf{V}_{00}$  with signature  $(0, n-2)$ .

Take  $u_1 = \frac{1}{\sqrt{2}}(b_1 + b_2)$ ,  $u_2 = \frac{1}{\sqrt{2}}(c_1 + c_2)$ ,  $v_n = \frac{1}{\sqrt{2}}(b_1 - b_2)$  and  $v_{n-1} = \frac{1}{\sqrt{2}}(c_1 - c_2)$ , and they form an orthogonal basis of  $\mathbb{C}b_1 \oplus \mathbb{C}b_2 \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \subseteq V := \mathbf{V}(\mathbb{R})$ . Fix a basis  $\{v_1, \dots, v_{n-2}\}$  of  $V_{00} := \mathbf{V}_{00}(\mathbb{R})$  such that  $\langle v_i, v_j \rangle = -\delta_{ij}$  for any  $1 \leq i, j \leq n-2$ . We have obtained a basis of  $V$  with the following properties:

- $\langle u_i, u_j \rangle = \delta_{ij}$ , for any  $1 \leq i, j \leq 2$ ;
- $\langle v_i, v_j \rangle = -\delta_{ij}$ , for any  $1 \leq i, j \leq n$ ;
- $\langle u_i, v_j \rangle = 0$  for any  $1 \leq i \leq 2$  and  $1 \leq j \leq n$ ;
- $\{u_2, v_1, \dots, v_{n-1}\}$  is a basis for  $V_0 := \mathbf{V}_0(\mathbb{R})$ .

Set  $V_2^+ = \mathbb{C}u_1 \oplus \mathbb{C}u_2$  and  $V_n^- = \mathbb{C}v_1 \oplus \dots \oplus \mathbb{C}v_n$ , and we have a decomposition  $V_2^+ \oplus V_n^-$  of  $V$  into definite subspaces.

## 2.2 Unitary group $\mathbf{U}(2, n)$ and its Heisenberg parabolic subgroup

Define  $\mathbf{G} = \mathbf{U}(\mathbf{V}) = \mathbf{U}(2, n)$  to be the unitary group of the Hermitian space  $(\mathbf{V}, \langle, \rangle)$ , and we write the  $\mathbf{G}$ -action on  $\mathbf{V}$  as a right action.

The Heisenberg parabolic subgroup  $\mathbf{P}$  of  $\mathbf{G}$  is defined as the stabilizer of the isotropic line spanned by  $b_1$ , and it admits the following Levi decomposition  $\mathbf{P} = \mathbf{M}\mathbf{N}$ :

- The Levi factor  $\mathbf{M}$  is chosen to be the stabilizer in  $\mathbf{P}$  of the isotropic line  $Eb_2$ . By its action on  $Eb_1 \oplus \mathbf{V}_0 \oplus Eb_2$ , one identifies this subgroup with  $\text{Res}_{E/\mathbb{Q}}(\mathbf{G}_m) \times \mathbf{U}(\mathbf{V}_0)$ , so that  $(z, h) \in \text{Res}_{E/\mathbb{Q}}(\mathbf{G}_m) \times \mathbf{U}(\mathbf{V}_0) \simeq \mathbf{M}$  acts on  $\mathbf{V}$  by

$$(xb_1 + w + yb_2) \cdot (z, h) = z^{-1}xb_1 + wh + \bar{z}yb_2, \text{ for any } x, y \in \mathbf{G}_a, w \in \mathbf{V}_0.$$

- The unipotent radical  $\mathbf{N}$  is a two-step nilpotent group, and it is isomorphic to the group

$$\{(v, \lambda) \in \mathbf{V}_0 \times \text{Res}_{E/\mathbb{Q}}\mathbf{G}_a \mid \bar{\lambda} = -\lambda\},$$

equipped with the following multiplication:

$$(v_1, \lambda_1) \cdot (v_2, \lambda_2) := \left( v_1 + v_2, \lambda_1 + \lambda_2 - \frac{\langle v_1, v_2 \rangle - \langle v_2, v_1 \rangle}{2} \right).$$

The action of the corresponding element  $n(v, \lambda) \in \mathbf{N}$  on  $\mathbf{V}$  is given as:

$$b_1 \mapsto b_1, b_2 \mapsto \left(-\frac{1}{2}\langle v, v \rangle + \lambda\right)b_1 + b_2 + v, w \in \mathbf{V}_0 \mapsto -\langle w, v \rangle b_1 + w.$$

The center of  $\mathbf{N}$  is  $\mathbf{Z} := \{n(0, \lambda) \mid \bar{\lambda} = -\lambda\}$ , which has dimension 1.

The conjugation action of  $\mathbf{M}$  on  $\mathbf{N}$  is given by

$$(z, h)n(v, \lambda)(z^{-1}, h^{-1}) = n(\bar{z}vh^{-1}, |z|\lambda).$$

Denote by  $\nu$  the similitude character of  $\mathbf{P}$ :

$$\nu((z, h)n(v, \lambda)) = z \in \text{Res}_{E/\mathbb{Q}} \mathbf{G}_m, \text{ for any } (z, h) \in \mathbf{M}, n(v, \lambda) \in \mathbf{N},$$

then we have  $b_1 p = \nu(p)^{-1} b_1$  for any  $p \in \mathbf{P}$ .

Using the unitary character  $\psi_E : E \setminus \mathbb{A}_E \rightarrow \mathbb{S}^1 \subset \mathbb{C}^\times$ , we identify  $\mathbf{V}_0$  with  $\text{Hom}([\mathbf{N}], \mathbb{S}^1)$  by sending  $T \in \mathbf{V}_0$  to the unitary character  $\chi_T(n(v, \lambda)) = \psi_E(\langle T, v \rangle) = \psi(\text{Tr}_{\mathbb{A}_E} \langle T, v \rangle)$ . Its archimedean analogue is

$$V_0 \xrightarrow{\cong} \text{Hom}(\mathbf{N}(\mathbb{R}), \mathbb{S}^1), T \mapsto \left( \chi_{T, \infty} : v \mapsto e^{2\pi i(\langle T, v \rangle + \langle v, T \rangle)} \right).$$

### 3 Quaternionic Heisenberg Eisenstein series

#### 3.1 Quaternionic modular forms on $\mathbf{U}(2, n)$

We first recall from [HMY24, §3] the theory of quaternionic modular forms on the unitary group  $\mathbf{G} = \mathbf{U}(2, n)$ .

In §2.1, we have a decomposition  $V = V_2^+ \oplus V_n^-$  of  $V = \mathbf{V}(\mathbb{R})$  into definite subspaces. The group  $K_\infty = \mathbf{U}(V_2^+) \times \mathbf{U}(V_n^-) \simeq \mathbf{U}(2) \times \mathbf{U}(n)$  is a maximal compact subgroup of  $\mathbf{U}(2, n) := \mathbf{G}(\mathbb{R})$ . For any integer  $\ell \geq 1$ , consider the following irreducible representation of  $K_\infty = \mathbf{U}(V_2^+) \times \mathbf{U}(V_n^-)$ :

$$\mathbb{V}_\ell := \left( \text{Sym}^{2\ell} V_2^+ \otimes \det_{\mathbf{U}(V_2^+)}^{-\ell} \right) \boxtimes \mathbf{1}$$

We fix a basis  $\{[u_1^{\ell-v}][u_2^{+v}], -\ell \leq v \leq \ell\}$  of  $\mathbb{V}_\ell$ , where  $[u_i^j] := u_i^j/j!$ .

When  $\ell \geq n$ , Gross and Wallach [GW96] construct an irreducible unitary discrete series representation  $\Pi_\ell$  of  $\mathbf{G}(\mathbb{R})$ , which contains  $\mathbb{V}_\ell$  as its minimal  $K_\infty$ -type with multiplicity 1, and two Schmid operators  $\mathcal{D}_\ell^\pm$  associated with  $\Pi_\ell$  are constructed in [HMY24, §3.1].

**Definition 3.1.** [HMY24, Definition 3.1] A *weight  $\ell$  quaternionic modular form* on  $\mathbf{G} = \mathbf{U}(2, n)$  is a smooth function  $F : \mathbf{G}(\mathbb{A}) \rightarrow \mathbb{V}_\ell$  of moderate growth such that:

- (1) For any  $g \in \mathbf{G}(\mathbb{A})$  and  $\gamma \in \mathbf{G}(\mathbb{Q})$ ,  $F(\gamma g) = F(g)$ .
- (2) For every  $k \in K_\infty$  and  $g \in \mathbf{G}(\mathbb{A})$ ,  $F(gk) = F(g) \cdot k$ .
- (3) The functions  $\mathcal{D}_\ell^\pm \left( F|_{\mathbf{G}(\mathbb{R})} \right)$  vanish identically on  $\mathbf{G}(\mathbb{R})$ .

Let  $F$  be a weight  $\ell$  quaternionic modular form on  $\mathbf{G}$ . If  $T \in \mathbf{V}_0$ , then the  $T$ -th Fourier coefficient of  $F$  along  $\mathbf{N}$  is

$$F_{\mathbf{N}, T} : \mathbf{G}(\mathbb{A}) \rightarrow \mathbb{V}_\ell, g \mapsto \int_{[\mathbf{N}]} F(ng) \overline{\chi_T(n)} dn.$$

We have the following result on the Fourier expansion of  $F$ :

**Theorem 3.2.** [HMY24, Corollary 3.7] Let  $F : \mathbf{G}(\mathbb{A}) \rightarrow \mathbb{V}_\ell$  be a weight  $\ell$  quaternionic modular form, and  $F_{\mathbf{Z}}$  (resp.  $F_{\mathbf{N}}$ ) the constant term of  $F$  along  $\mathbf{Z}$  (resp.  $\mathbf{N}$ ). There exist locally constant functions

$$\{a_T(F, \cdot) : \mathbf{G}(\mathbb{A}_f) \rightarrow \mathbb{C}\}_{T \in \mathbf{V}_0, \langle T, T \rangle \geq 0}$$

such that for any  $g_f \in \mathbf{G}(\mathbb{A}_f)$  and  $g_\infty \in \mathbf{G}(\mathbb{R})$  one has:

$$F_{\mathbf{Z}}(g_f g_\infty) = F_{\mathbf{N}}(g_f g_\infty) + \sum_{\substack{T \in \mathbf{V}_0 \\ \langle T, T \rangle \geq 0}} a_T(F, g_f) \mathcal{W}_T(g_\infty), \quad (3.1)$$

where  $\mathcal{W}_T$  is the  $\mathbb{V}_\ell$ -valued function on  $\mathbf{G}(\mathbb{R})$  given by the formula

$$\mathcal{W}_T(nmk) = \chi_{T, \infty}(n) \sum_{v=-\ell}^{\ell} |\nu(m)|^{2\ell+2} \left( \frac{|\beta_T(m)|}{\beta_T(m)} \right)^v K_v(|\beta_T(m)|) [u_1^{\ell-v}] [u_2^{\ell+v}] \cdot k, \quad (3.2)$$

for any  $n \in \mathbf{N}(\mathbb{R})$ ,  $m \in \mathbf{M}(\mathbb{R})$  and  $k \in K_\infty$ . Here  $K_v$  denotes the K-Bessel function  $K_v(x) = \frac{1}{2} \int_0^\infty t^{v-1} e^{-x(t+t^{-1})} dt$  and  $\beta_T : \mathbf{M}(\mathbb{R}) \rightarrow \mathbb{C}$  is defined by mapping  $(z, h)$  to  $2\sqrt{2}\pi \langle u_2, zT \cdot h \rangle$ .

### 3.2 Heisenberg Eisenstein series

For  $\ell \geq n$ , the quaternionic discrete series  $\Pi_\ell$  can be embedded into the (unnormalized) degenerate principal series  $\text{Ind}_{\mathbf{P}(\mathbb{R})}^{\mathbf{G}(\mathbb{R})} |\nu|_{\mathbb{C}}^{\ell+1}$  [Wal03, Theorem 9] (by our notation  $|z|_{\mathbb{C}} = z\bar{z} = |z|^2$ ), thus we now construct an Eisenstein series on  $\mathbf{G}$  from some section of  $\text{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})} |\nu|_{\mathbb{A}_E}^s$ ,  $s \in \mathbb{C}$ .

Suppose that  $\Phi_f = \otimes_{p < \infty} \Phi_p$  is a Schwartz-Bruhat function on  $\mathbf{V}(\mathbb{A}_f)$ . We choose a section  $f_\ell(g_f g_\infty, \Phi_f, s) = f_{f_{te}}(g_f, \Phi_f, s) f_{\ell, \infty}(g_\infty, s)$  of  $\text{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})} |\nu|_{\mathbb{A}_E}^s$  as follows:

- The archimedean part  $f_{\ell, \infty}$  is the  $\mathbb{V}_\ell$ -valued,  $K_\infty$ -equivariant induced section whose restriction to  $\mathbf{M}(\mathbb{R})$  satisfies  $f_{\ell, \infty}((z, h), s) = |z|_{\mathbb{C}}^s [u_1^\ell] [u_2^\ell] = |z|^{2s} [u_1^\ell] [u_2^\ell]$  for any  $(z, h) \in \mathbf{M}(\mathbb{R})$ .
- The finite part  $f_{f_{te}}$  is defined by  $f_{f_{te}}(g_f, \Phi_f, s) = \int_{\text{GL}_1(\mathbb{A}_{E,f})} |t|_{\mathbb{A}_{E,f}}^s \Phi_f(t \cdot b_1 g_f) dt$ .

We set the (Heisenberg) Eisenstein series

$$E_\ell(g, \Phi_f, s) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{Q})} f_\ell(\gamma g, \Phi_f, s).$$

**Proposition 3.3.** For  $\ell \geq n$ , the Eisenstein series  $E_\ell(g, \Phi_f, s = \ell + 1)$  is a weight  $\ell$  quaternionic modular form on  $\mathbf{G} = \mathbf{U}(2, n)$ .

*Proof.* The only condition in Definition 3.1 that is not obvious is the third one. It suffices to show that the restriction of  $f_{\ell, \infty}$  to  $\mathbf{M}(\mathbb{R})$  is killed by the Schmid operators  $\mathcal{D}_\ell^\pm$ .

Write an element  $(z, h) \in \mathbf{M}(\mathbb{R})$  as  $m = (h, r, \theta)$  so that  $z = re^{i\theta}$ ,  $r \in \mathbb{R}_{>0}$  and  $\theta \in [0, 2\pi)$ . Using this coordinates, we define a function  $F(h, r, \theta) = r^{2s}$  on  $\mathbf{M}(\mathbb{R})$  so that  $f_{\ell, \infty}(m) = F(m) [u_1^\ell] [u_2^\ell]$ . By [HMY24, Proposition 3.10],  $f_{\ell, \infty}$  is killed by  $\mathcal{D}_\ell^\pm$  if and only if  $(r\partial r - 2(\ell + 1))F \equiv 0$  on  $\mathbf{M}(\mathbb{R})$ , which holds when  $s = \ell + 1$ .  $\square$

PROGRESS MARK

### 3.3 Abstract Fourier expansion

In this subsection, we give the “abstract” Fourier expansion of  $E_\ell(g, \Phi_f, s)$ .

**Lemma 3.4.** *The right  $\mathbf{P}(\mathbb{Q})$ -space  $\mathbf{P}(\mathbb{Q}) \setminus \mathbf{G}(\mathbb{Q})$ , the space of isotropic lines in  $V(\mathbb{Q})$ , has exactly 3 orbits of  $\mathbf{P}(\mathbb{Q})$ , represented respectively by  $\mathbf{Q}b_1$ ,  $\mathbf{Q}v_0$  and  $\mathbf{Q}b_2$ , where  $v_0$  is an arbitrary non-zero isotropic vector in  $V_0(\mathbb{Q})$ .*

*Proof.* Directly by the explicit action given in §2. □

Set  $\mathbf{G}(\mathbb{Q}) = \bigsqcup_{i=0}^2 \mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})$ , such that  $w_0 = 1$ ,  $b_1w_1 = v_0$  and  $b_1w_2 = b_2$ . Now we can write the degenerate Eisenstein series as

$$E_\ell(g, \Phi_f, s) = \sum_{i=0}^2 E_{\ell,i}(g, \Phi_f, s), \quad E_{\ell,i}(g, \Phi_f, s) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q}) \setminus \mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})} f(\gamma g, s),$$

thus  $E_{\ell,0}(g, \Phi_f, s) = f(g, s)$ . From now on, when there is no confusion we will omit the  $\ell$  and  $\Phi_f$  in  $E_\ell(g, \Phi_f, s)$ , and write it as  $E(g, s) = \sum_{i=0}^2 E_i(g, s)$ .

**Lemma 3.5.** *Assume that  $\operatorname{Re}(s) \gg 0$  so that the sum defining  $E(g, s)$  converges absolutely. Then one has the following expressions for the  $E_i(g, s)$ :*

- (1) *Let  $\mathcal{L}_0$  be the set of non-zero isotropic lines  $\ell$  in  $V_0$  and for any  $\ell \in \mathcal{L}_0$ , select  $\gamma(\ell) \in \mathbf{G}(\mathbb{Q})$  with  $b_1\gamma(\ell) \in \ell$ . Then*

$$E_1(g, s) = \sum_{\ell \in \mathcal{L}_0} \sum_{\mu \in (\ell)^\perp \mathbf{N}_0(\mathbb{Q}) \setminus \mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu g, s).$$

- (2) *One has*

$$E_2(g, s) = \sum_{\mu \in \mathbf{N}(\mathbb{Q})} f(w_2\mu g, s).$$

For any  $T \in V_0$ , we set

$$E_i^T(g, s) = \int_{\mathbf{N}(F) \setminus \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) E_i(ng, s) dn, \quad i = 0, 1, 2.$$

**Lemma 3.6.** (1) *If  $T$  is anisotropic, then  $E_1^T = 0$ . If  $T$  is isotropic, define  $\mathbf{N}_T = (\ell_T)^\perp \mathbf{N}_0 \subseteq \mathbf{N}$ , then*

$$E_1^T(g, s) = \int_{\mathbf{N}_T(\mathbb{A}) \setminus \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell_T)ng, s) dn.$$

- (2) *For any  $T \in V_0$ , one has*

$$E_2^T(g, s) = \int_{\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(w_2ng, s) dn.$$

*Proof.* It suffices only to prove the  $i = 1$  case. For any  $\ell \in \mathcal{L}_0$ , set

$$\mathbf{N}_\ell = \left\{ n(v, \lambda) \in \mathbf{N} \mid v \in \ell^\perp \right\} \subseteq \mathbf{N}.$$

For any  $T \in V_0$ ,

$$\begin{aligned} E_1^T(g, s) &= \sum_{\ell \in \mathcal{L}_0} \int_{[N]} \chi_T^{-1}(n) \left( \sum_{\mu \in \mathbf{N}_\ell(\mathbb{Q}) \setminus \mathbf{N}(\mathbb{Q})} f(\gamma(\ell) \mu n g, s) \right) dn \\ &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{Q}) \setminus \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell) n g, s) dn \\ &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \setminus \mathbf{N}(\mathbb{A})} \left( \int_{[N_\ell]} \chi_T^{-1}(r) dr \right) \chi_T^{-1}(n) f(\gamma(\ell) n g, s) dn \\ &= \sum_{\ell \in \mathcal{L}_0, \chi_T|_{\mathbf{N}_\ell} \equiv 1} \int_{\mathbf{N}_\ell(\mathbb{A}) \setminus \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell) n g, s) dn. \end{aligned}$$

Then the lemma follows from the fact that  $\chi_T|_{\mathbf{N}_\ell} \equiv 1$  if and only if  $T \in \ell$ , i.e.  $T$  is isotropic and  $\ell = \ell_T$ .  $\square$

## 4 Rank 2 Fourier coefficients of $E_\ell$

## 5 Rank 1 Fourier coefficients of $E_\ell$

## 6 The constant term of $E_\ell$

## 7 The Fourier expansion of $E_\ell(g, \Phi_f, s = \ell + 1)$

### 7.1 Computation of constant term

The constant term of  $E_\ell(g, \Phi_f, s = \ell + 1)$  consists of three parts:

- $E_{\ell,0}(g, \Phi_f, s) = f(g, s),$
- $E_1^0(g, s) = \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \setminus \mathbf{N}(\mathbb{A})} f(\gamma(\ell) n g, s) dn,$
- and  $E_2^0(g, s) = \int_{\mathbf{N}(\mathbb{A})} f(w_2 n g) dn.$

#### 7.1.1 The $i = 0$ -term

**Lemma 7.1.** For  $g \in \mathbf{P}(\mathbb{A})$ ,

$$E_0(g, s) = f(g, s) = |\nu(g)|_E^s \zeta_E(s) [u_1^n] [u_2^n].$$

*Proof.* For  $g_f \in \mathbf{P}(\mathbb{A}_f)$ , we have

$$\begin{aligned} f_{fte}(g_f, s) &= \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(tb_1 g_f) dt \\ &= \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(t\nu(g_f)^{-1} b_1) dt \\ &= |\nu(g_f)|_E^s \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(tb_1) dt. \end{aligned}$$

Thus, the non-archimedean contribution is  $|\nu(g_f)|_E^s \zeta_E(s)$ . Combining with  $f_{n,\infty}(g_\infty, s) = |\nu(g_\infty)|_E^s [u_1^n][u_2^n]$ , we get the desired identity.  $\square$

### 7.1.2 The $i = 1$ -term

We fix a non-zero isotropic vector in  $V_0$ , such that  $v_0 = b_1 \gamma_0$ , and set  $\ell_0 = Ev_0$ . Define  $\mathbf{P}_0$  be the stabilizer of  $\ell_0$  in  $\mathbf{U}(V_0)$ , which is a parabolic subgroup of  $\mathbf{M}$ . We denote the similitude character of  $\mathbf{P}_0$  by  $\lambda$ , i.e.  $v_0 g = \lambda(g)^{-1} v_0$  for any  $g \in \mathbf{P}_0$ . For  $g \in \mathbf{P}_0(\mathbb{A})$ , we have:

$$\begin{aligned} E_1^0(g, s) &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma(\ell)ng, s) dn \\ &= \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{Q})} \int_{\mathbf{N}_{\ell_0}(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma_0 \gamma ng, s) dn \end{aligned}$$

If we set  $f_0(g, s) = \int_{\mathbf{N}_{\ell_0}(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma_0 ng, s) dn$ , then for  $\text{Re}(s) \gg 0$ ,

$$E_1^0(g, s) = \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{Q})} f_0(\gamma g, s),$$

and it defines an Eisenstein series on  $\mathbf{M}$ . Now we want to determine this section  $f_0(g, s) \in \text{Ind}_{\mathbf{P}_0(\mathbb{A})}^{\mathbf{M}(\mathbb{A})} |\lambda|^s$  (still unnormalized).

At finite places, one has

$$\begin{aligned} \int_{\mathbf{N}_{\ell_0}(\mathbb{A}_{E,f}) \backslash \mathbf{N}(\mathbb{A}_{E,f})} f(\gamma_0 ng, s) dn &= \int_{x \in \mathbb{A}_{E,f}} \int_{t \in \mathbb{A}_{E,f}^\times} |t|_E^s \Phi_{fte}(t(v_0 + xb_1)g) dt dx \\ &= \int_{x \in \mathbb{A}_{E,f}} \int_{t \in \mathbb{A}_{E,f}^\times} |t|_E^s \Phi_{fte}(t\lambda(g)^{-1} v_0 + tx\nu(g)^{-1} b_1) dt dx \\ &= |\lambda(g)|_f^{s-1} |\nu(g)|_f \int_{x \in \mathbb{A}_{E,f}} \int_{t \in \mathbb{A}_{f,E}^\times} |t|_E^{s-1} \Phi_{fte}(tv_0 + xb_1) dt dx \\ &= |\lambda(g)|_f^{s-1} |\nu(g)|_f \zeta_E(s-1). \end{aligned}$$

Now we switch to the archimedean place. Set  $c_1 = v_0$  and  $c_2$  another isotropic vector in  $V_0$  with  $\langle c_1, c_2 \rangle = 1$ . We can take  $u_2 = \frac{1}{\sqrt{2}}(c_1 + c_2)$  and  $v_{n-1} = \frac{1}{\sqrt{2}}(c_1 - c_2)$ . We identify  $\mathbf{N}_{\ell_0}(\mathbb{R}) \backslash \mathbf{N}(\mathbb{R})$  with  $\mathbb{C}$  via the map  $n(v, \lambda) \mapsto -\langle c_1, v \rangle$ , and for any  $z \in \mathbb{C}$  we choose an element  $n(z) \in \mathbf{N}(\mathbb{R})$  under this identification.

**Lemma 7.2.** When  $\ell \geq 1$ , we have  $f_{0,\infty} \equiv 0$ .

*Proof.* Similarly to Lemma 7.3, for any  $g \in \mathbf{P}_0(\mathbb{R})$  one has

$$f_{\ell,\infty}(\gamma_0 n(z)g) = \frac{(z\nu(g)^{-1}u_1 + \lambda(g)^{-1}u_2)^\ell (-\overline{\lambda(g)}^{-1}u_1 + \overline{z\nu(g)^{-1}u_2})^\ell}{(|\lambda(g)|^{-2} + |z|^2|\nu(g)|^{-2})^{2\ell+1}(\ell!)^2}.$$

From now on we write  $\lambda(g)$  (resp.  $\nu(g)$ ) as  $\lambda$  (resp.  $\nu$ ).

The coefficient of  $[u_1^{\ell-v}][u_2^{\ell+v}]$  in  $f_{\ell,\infty}(\gamma_0 n(z)g)$  is

$$\begin{aligned} & \frac{(\ell-v)!(\ell+v)!}{(\ell!)^2(|\lambda|^{-2} + |z|^2|\nu|^{-2})^{2\ell+1}} \sum_{\substack{0 \leq i, j \leq \ell \\ i+j=\ell-v}} \binom{\ell}{i} \binom{\ell}{j} z^i \nu^{-i} \lambda^{-(\ell-i)} (-\overline{\lambda})^{-j} (\overline{z\nu^{-1}})^{\ell-j} \\ &= \frac{(\ell-v)!(\ell+v)!|\nu|^{-2\ell} \nu^v \lambda^{-v} |z|^{2\ell} z^{-v}}{(\ell!)^2(|\lambda|^{-2} + |z|^2|\nu|^{-2})^{2\ell+1}} \sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)} (-1)^j \binom{\ell}{j} \binom{\ell}{j+v} |\nu|^{2j} |\lambda|^{-2j} |z|^{-2j}, \end{aligned}$$

and the corresponding coefficient in  $f_{0,\infty}(g, s)$  is the integral of this function over  $z \in \mathbb{C}$ . Set  $w = z\lambda/\nu$ , then we have  $dw = |\lambda/\nu|^2 dz$  and this function equals

$$\frac{(\ell-v)!(\ell+v)!|\lambda|^{2\ell+2} w^{-v} |w|^{2\ell}}{(\ell!)^2(1 + |w|^2)^{2\ell+1}} \sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)} (-1)^j \binom{\ell}{j} \binom{\ell}{j+v} |w|^{-2j}.$$

Hence the coefficient of  $[u_1^{\ell-v}][u_2^{\ell+v}]$  in  $f_{0,\infty}(g, s = \ell + 1)$  equals

$$\frac{(\ell-v)!(\ell+v)!}{(\ell!)^2} |\lambda|^{2\ell} |\nu|^2 \sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)} (-1)^j \binom{\ell}{j} \binom{\ell}{j+v} \left( \int_{\mathbb{C}} \frac{w^{-v} |w|^{2\ell-2j}}{(1 + |w|^2)^{2\ell+1}} dw \right) \quad (7.1)$$

Notice that the inner integral is nonzero only when  $v = 0$ , and we have

$$\int_{\mathbb{C}} \frac{|w|^{2\ell-2j}}{(1 + |w|^2)^{2\ell+1}} dw = \pi \frac{(\ell-j)!(\ell+j-1)!}{(2\ell)!}, \quad j = 0, 1, \dots, \ell.$$

So we only need to consider the coefficient of  $[u_1^\ell][u_2^\ell]$ , which is equal to

$$\begin{aligned} & \pi((2\ell)!)^{-1} |\lambda|^{2\ell} |\nu|^2 \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j}^2 (\ell-j)!(\ell+j-1)! \\ &= \frac{\pi(\ell!)^2 |\lambda|^{2\ell} |\nu|^2}{\ell(2\ell)!} {}_2F_1(\ell, -\ell; 1; 1), \end{aligned}$$

which is 0 when  $\ell \leq 1$ . □

### 7.1.3 The $i = 2$ -term

This term vanishes when we have the condition  $\ell > n + 1$  (by analyzing the singularity of  $K_v(x)$  at  $x = 0$ ).



## 7.2 Rank 1 Fourier coefficients

For rank 1 Fourier coefficients, one needs to calculate

$$E_1^T(g, s) = \int_{\mathbf{N}_T(\mathbb{A}) \setminus \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell_T)ng, s) dn$$

for any nonzero isotropic vector  $T$ .

### 7.2.1 Non-archimedean components

For any prime  $p$ , we identify  $\mathbf{N}_T(\mathbb{Q}_p) \setminus \mathbf{N}(\mathbb{Q}_p)$  with  $E_p$  via the map  $n(v, \lambda) \mapsto -\langle T, v \rangle$ , thus

$$\begin{aligned} E_{1,p}^T(1, s) &= \int_{x \in E_p} \int_{t \in E_p^\times} \psi_E(x) |t|_E^s \Phi_p(tx b_1 + tT) dt dx \\ &= \int_{x \in E_p} \int_{t \in E_p^\times} \psi_E(x/t) |t|_E^{s-1} \Phi_p(x b_1 + tT) dt dx \end{aligned}$$

**7.2.1.1 Split case** When  $p$  is split in  $E$ , we write  $T$  as  $(T_1, T_2)$  with  $T_i \in \mathbb{Z}_p^n, i = 1, 2$ . Let  $\alpha = v_p(T_1)$  and  $\beta = v_p(T_2)$ , then we have:

$$\begin{aligned} E_{1,p}^T &:= E_{1,p}^T(1, s = \ell + 1) \\ &= \int_{(x_1, x_2) \in \mathbb{Q}_p^2} \int_{(t_1, t_2) \in (\mathbb{Q}_p \times \mathbb{Q}_p)^\times} \psi_p(x_1/t_1 + x_2/t_2) |t_1 t_2|_p^{s-1} \Phi_p((x_1, x_2)b_1 + (t_1, t_2)T) dt_1 dt_2 dx_1 dx_2 \\ &= \sum_{\substack{r_1 \geq -\alpha \\ r_2 \geq -\beta}} p^{-\ell(r_1+r_2)} \int_{(x_1, x_2) \in \mathbb{Z}_p} \psi_p(x_1/p^{r_1}) \psi_p(x_2/p^{r_2}) dx_1 dx_2 \\ &= \sum_{\substack{r_1 \geq -\alpha \\ r_2 \geq -\beta}} p^{-\ell(r_1+r_2)} \left( \frac{1}{p^{r_1}} \sum_{x_1 \in \mathbb{Z}/p^{r_1}\mathbb{Z}} \psi_p(x_1/p^{r_1}) \right) \left( \frac{1}{p^{r_2}} \sum_{x_2 \in \mathbb{Z}/p^{r_2}\mathbb{Z}} \psi_p(x_2/p^{r_2}) \right) \\ &= (1 + p^\ell + p^{2\ell} + \dots + p^{\alpha\ell}) (1 + p^\ell + p^{2\ell} + \dots + p^{\beta\ell}). \end{aligned}$$

**7.2.1.2 Inert case** When  $p$  is inert in  $E$  and  $p \neq 2$ , we have:

$$\begin{aligned} E_{1,p}^T &= \sum_{r \geq -v_p(T)} p^{-2r\ell} \int_{x \in \mathcal{O}_{E_p}} \psi_E(x/p^r) dx \\ &= 1 + p^{2\ell} + p^{4\ell} + \dots + p^{2\ell v_p(T)}. \end{aligned}$$

**7.2.1.3 Ramified case**

### 7.2.2 Archimedean components

Again, we identify  $\mathbf{N}_T(\mathbb{R}) \setminus \mathbf{N}(\mathbb{R})$  with  $\mathbb{C}$  via the map  $n(v, \lambda) \mapsto -\langle T, v \rangle$ , and for any  $z \in \mathbb{C}$  we choose an element  $n(z) \in \mathbf{N}(\mathbb{R})$  corresponding to  $z$  under this identification.

**Lemma 7.3.** *For any  $z \in \mathbb{C}$ , we have*

$$f_{\ell, \infty}(\gamma(\ell_T)n(z), s = \ell + 1) = \frac{(zu_1 + \beta u_2)^\ell (-\bar{\beta}u_1 + \bar{z}u_2)^\ell}{(|z|^2 + |\beta|^2)^{2\ell+1}(\ell!)^2},$$

where  $\beta = \beta(T) := \sqrt{2}\langle T, u_2 \rangle$ .

*Proof.* Suppose that  $\gamma(\ell_T)n(z) = pk$  for  $p \in \mathbf{P}(\mathbb{R})$  and  $k = (k_+, k_-) \in K_\infty = \mathbf{U}(V_2^+) \times \mathbf{U}(V_n^-)$ , then one has

$$T + zb_1 = b_1\gamma(\ell_T)n(z) = b_1pk = \nu(p)^{-1}b_1k.$$

Taking the projection to  $V_2^+$  of both sides, we obtain that

$$zu_1 + \sqrt{2}\langle T, u_2 \rangle u_2 = \nu(p)^{-1}u_1k_+.$$

One can take  $\nu(p)$  to be  $(|z|^2 + |\beta|^2)^{-1/2}$ , then

$$k_+ = \nu(p) \begin{pmatrix} z & -\bar{\beta} \\ \beta & \bar{z} \end{pmatrix} \in \mathbf{U}(V_2^+),$$

which gives us the desired value of  $f_{\ell, \infty}$ . □

Using Lemma 7.3, the coefficient of  $[u_1^\ell][u_2^\ell]$  in  $E_{1, \infty}^T := E_{1, \infty}^T(1, s = \ell + 1)$  is equal to

$$\int_{z \in \mathbb{C}} e^{2\pi i \operatorname{Re}(z)} \frac{\sum_{k=0}^{\ell} \binom{\ell}{k}^2 |z|^{2k} (-|\beta|^2)^{\ell-k}}{(|z|^2 + |\beta|^2)^{2\ell+1}} dz. \quad (7.2)$$

We set  $B = |\beta|^2$  and  $z = x + iy$ , where  $x, y \in \mathbb{R}$ , then the integral in Equation (7.2) equals

$$\begin{aligned} & \int_{\mathbb{R}^2} e^{2\pi i x} \sum_{k=0}^{\ell} \binom{\ell}{k}^2 (-B)^{\ell-k} \frac{(x^2 + y^2)^k}{(x^2 + y^2 + B)^{2\ell+1}} dx dy \\ &= \int_{\mathbb{R}} e^{2\pi i x} \sum_{k=0}^{\ell} \binom{\ell}{k}^2 (-B)^{\ell-k} \left( \int_{\mathbb{R}} \frac{(y^2 + x^2)}{(y^2 + x^2 + B)^{2\ell+1}} dy \right) dx \end{aligned}$$

By Lemma 7.9, the inner integral over  $y \in \mathbb{R}$  equals

$$\frac{(x^2 + B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^k \binom{k}{j} \left( \frac{x^2}{x^2 + B} \right)^{k-j} \Gamma(j + 1/2) \Gamma(2\ell - j + 1/2).$$

Similarly to our calculation in §7.4, we have

$$E_{1,\infty}^T = 2^{-2\ell} \pi \sum_{k=0}^{\ell} \frac{(-B)^k 2^{-2k} (2\ell + 2k)!}{(k!)^2 (\ell + k)! (\ell - k)!} \int_{\mathbb{R}} \frac{e^{2\pi i x} dx}{(x^2 + B)^{\ell+k+1/2}}$$

Using Basset's integral, one has

$$\int_{\mathbb{R}} \frac{e^{iwt} dt}{(t^2 + z^2)^{n+1/2}} = \frac{2\sqrt{\pi}}{\Gamma(n+1/2)} \left( \frac{|w|}{2|z|} \right)^n K_n(|wz|), \quad w, z \in \mathbb{R}.$$

Plugging this into  $E_{1,\infty}^T$ , we obtain:

$$\begin{aligned} E_{1,T}^{\infty} &= 2^{-2\ell} \pi \sum_{k=0}^{\ell} \frac{(-B)^k 2^{-2k} (2\ell + 2k)!}{(k!)^2 (\ell + k)! (\ell - k)!} \cdot \frac{2^{\ell+k+1} (\ell + k)! B^{-(\ell+k)/2} (2\pi)^{\ell+k} K_{\ell+k}(2\pi\sqrt{B})}{(2\ell + 2k)!} \\ &= 2\pi B^{-\ell} \sum_{k=0}^{\ell} \frac{(-1)^k}{(k!)^2 (\ell - k)!} (\pi\sqrt{B})^{\ell+k} K_{\ell+k}(2\pi\sqrt{B}). \end{aligned}$$

**Lemma 7.4.** *For any  $C > 0$ , we have*

$$S_{\ell} := \sum_{k=0}^{\ell} \frac{(-1)^k C^{\ell+k} K_{\ell+k}(2C)}{(k!)^2 (\ell - k)!} = \frac{(-1)^{\ell} C^{2\ell}}{(\ell!)^2} K_0(2C).$$

*Proof.* Using the recurrence relation for K-Bessel functions:

$$K_{n+1}(z) = K_{n-1}(z) + \frac{2n}{z} K_n(z),$$

we can write  $-C^2 S_{\ell}$  as

$$\begin{aligned} -C^2 S_{\ell} &= \sum_{k=0}^{\ell} (-1)^{k+1} \frac{C^{\ell+k+2}}{(k!)^2 (\ell - k)!} K_{\ell+k}(2C) \\ &= \sum_{j=1}^{\ell+1} (-1)^j \frac{C^{\ell+1+j}}{((j-1)!)^2 (\ell - j + 1)!} \left( K_{\ell+1+j}(2C) - \frac{\ell+j}{C} K_{\ell+j}(2C) \right) \\ &= \sum_{j=1}^{\ell+1} \frac{(-1)^j C^{\ell+1+j} K_{\ell+1+j}(2C)}{((j-1)!)^2 (\ell - j + 1)!} + \sum_{j=0}^{\ell} \frac{(-1)^j (\ell + j + 1) C^{\ell+1+j} K_{\ell+j+1}(2C)}{(j!)^2 (\ell - j)!} \\ &= \frac{(\ell + 1) C^{\ell+1} K_{\ell+1}(2C)}{\ell!} + \sum_{j=1}^{\ell} \frac{(-1)^j C^{\ell+1+j} K_{\ell+1+j}(2C)}{(j!)^2 (\ell + 1 - j)!} (\ell + 1)^2 + (-1)^{\ell+1} \frac{C^{2\ell+2} K_{2\ell+2}(2C)}{(\ell!)^2} \\ &= (\ell + 1)^2 S_{\ell+1}. \end{aligned}$$

Since  $S_0 = K_0(2C)$  and  $S_{\ell+1} = -\frac{C^2}{(\ell+1)^2} S_{\ell}$ , we have

$$S_{\ell} = (-1)^{\ell} \frac{C^{2\ell}}{(\ell!)^2} K_0(2C). \quad \square$$

By this lemma, one has

$$E_{1,T}^{\infty} = 2\pi B^{-\ell} \cdot \frac{(-1)^{\ell} (\pi\sqrt{B})^{2\ell}}{(\ell!)^2} K_0(2\pi\sqrt{B}) = \frac{2\pi^{2\ell+1}}{(\ell!)^2} K_0(2\pi\sqrt{B}).$$

### 7.3 Rank 2 Fourier coefficients: non-archimedean components

One decomposes  $E_2^T(1, s)$  as  $\prod_v E_{2,v}^T(s)$ .

$$\begin{aligned} E_{2,p}^T(s) &= \int_{\mathbf{N}(\mathbb{Q}_p)} \chi_T^{-1}(n) f_p(w_2 n, s) dn \\ &= \int_{\mathbf{N}(\mathbb{Q}_p)} \int_{t \in E_p^\times} \chi_T^{-1}(n) |t|^s \Phi_p(t \cdot b_1 w_2 n) dt dn \\ &= \int_{v \in V_0(\mathbb{Q}_p)} \int_{\substack{x \in E_p \\ \bar{x} = -x}} \int_{t \in E_p^\times} \chi_T^{-1}(v) |t|^s \Phi_p(t((- \langle v, v \rangle / 2 + x) b_1 + b_2 + v)) dt dx dv \end{aligned}$$

Take  $\mathcal{O}_{E_p}$  to be  $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_E \subseteq E_p$ ,  $\mathfrak{p}$  to be its maximal ideal, and  $\mathcal{V}$  an  $\mathcal{O}_{E_p}$ -lattice of  $V_0(\mathbb{Q}_p) = V_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$  such that  $\mathcal{V} \otimes_{\mathcal{O}_{E_p}} (\mathcal{O}_{E_p} / \mathfrak{p})$  is a non-degenerate Hermitian space over  $\mathcal{O}_{E_p} / \mathfrak{p}$ . Assume that  $\Phi_p$  is the characteristic function of the lattice  $\mathcal{O}_{E_p} b_1 \oplus \mathcal{V} \oplus \mathcal{O}_{E_p} b_2$ .

**Write down a precise definition of  $\mathcal{V}$ ?**

**Lemma 7.5.** (1) If  $p$  splits in  $E$ , then

$$E_{2,p}^T(s) = \sum_{r_1, r_2 \geq 0} p^{-(r_1+r_2)s + \min(r_1, r_2)} \left( \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\max(r_1, r_2)} \langle v, v \rangle \in \mathbb{Z}_p) dv \right).$$

(2) If  $p$  is inert in  $E$ , then

$$E_{2,p}^T(s) = \sum_{r \geq 0} p^{-2rs+r} \left( \int_{v \in p^{-r}\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^r \langle v, v \rangle / 2 \in \mathbb{Z}_p) dv \right).$$

(3) If  $p$  is ramified in  $E$ , then

*Proof.* (1) If  $p$  splits in  $E$ , then  $E_p = \mathbb{Q}_p \times \mathbb{Q}_p$ :

$$E_{2,p}^T(s) = \sum_{r_1 \geq 0, r_2 \geq 0} |p|^{(r_1+r_2)s} \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \left( \int_{\substack{x \in (p^{-r_1}, p^{-r_2})\mathcal{O} + \langle v, v \rangle / 2 \\ x + \bar{x} = 0}} dx \right) dv$$

If we write  $x = (y, -y) \in E_p$ , then  $x \in (p^{-r_1}, p^{-r_2})\mathcal{O} + \langle v, v \rangle / 2$  is equivalent to

$$y \in p^{-r_1}\mathbb{Z}_p + \langle v, v \rangle / 2, -y \in p^{-r_2}\mathbb{Z}_p + \langle v, v \rangle / 2.$$

There are such  $x$  only if  $\langle v, v \rangle \in p^{-\max(r_1, r_2)}\mathbb{Z}_p$ . If  $r_1 \geq r_2$ , then  $y \in p^{-r_2}\tilde{y} - \langle v, v \rangle / 2$  for some  $\tilde{y} \in \mathbb{Z}_p$ . Since the measure on the line  $\{x = (y, -y) \in E_p\}$  is the normalized Haar measure  $dy$  on  $\mathbb{Q}_p$ , one has  $dy = p^{r_2}d\tilde{y}$ , and

$$\int_{x=(y, -y) \in p^{-r_1}\mathbb{Z}_p \times p^{-r_2}\mathbb{Z}_p + \langle v, v \rangle / 2} dx = \int_{\tilde{y} \in \mathbb{Z}_p} p^{r_2} d\tilde{y} = p^{r_2} = p^{\min(r_1, r_2)}.$$

Hence

$$E_{2,p}^T(s) = \sum_{r_1, r_2 \geq 0} p^{-(r_1+r_2)s+\min(r_1, r_2)} \left( \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\max(r_1, r_2)} \langle v, v \rangle \in \mathbb{Z}_p) dv \right).$$

(2) If  $p$  is inert in  $E$ , then  $E_p$  is an unramified quadratic field extension of  $\mathbb{Q}_p$ , and

$$E_{2,p}^T(s) = \sum_{r \geq 0} |p|_{E_p}^{rs} \int_{v \in p^{-r}\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^r \langle v, v \rangle \in \mathbb{Z}_p) \left( \int_{x \in \langle v, v \rangle / 2 + p^{-r}\mathcal{O}_{E_p}} \text{Char}(x + \bar{x} = 0) dx \right) dv$$

Choose a unit  $u \in \mathcal{O}_{E_p}^\times$  with  $u + \bar{u} = 0$ , then the line  $\{x \in E_p \mid x + \bar{x} = 0\}$  can be written as  $\{yu, y \in \mathbb{Q}_p\}$ , with the normalized Haar measure  $dy$ . There exist elements  $yu \in \langle v, v \rangle / 2 + p^{-r}\mathcal{O}_{E_p}$  if and only if  $\langle v, v \rangle / 2 \in p^{-r}\mathbb{Z}_p$ . When we have  $\langle v, v \rangle / 2 \in p^{-r}\mathbb{Z}_p$ , any element  $yu \in \langle v, v \rangle / 2 + p^{-r}\mathcal{O}_{E_p}$  is of the form

$$yu = \langle v, v \rangle / 2 + p^{-r} (-p^r \langle v, v \rangle / 2 + \tilde{y}u), \tilde{y} \in \mathbb{Z}_p,$$

thus one has

$$\begin{aligned} \int_{x \in \langle v, v \rangle / 2 + p^{-r}\mathcal{O}_{E_p}} \text{Char}(x + \bar{x} = 0) dx &= \text{Char}(p^r \langle v, v \rangle / 2 \in \mathbb{Z}_p) |p|_{\mathbb{Q}_p}^{-r} \int_{\mathbb{Z}_p} d\tilde{y} \\ &= p^r \text{Char}(p^r \langle v, v \rangle / 2 \in \mathbb{Z}_p), \end{aligned}$$

which gives us the desired identity.  $\square$

**If  $p$  is ramified in  $E$ :** (Assume  $p \neq 2$ ) Then  $E_p$  is a ramified quadratic field extension of  $\mathbb{Q}_p$ , and we choose an uniformizer  $\omega$  of  $\mathfrak{p} \subset \mathcal{O}_{E_p}$  such that  $\omega + \bar{\omega} = 0$  and  $\omega^2 \in p\mathbb{Z}_p^\times$ . The integral for  $E_{2,p}^T$  can be rewritten as:

$$\begin{aligned} E_{2,p}^T(s) &= \sum_{r \geq 0} |\omega|^{rs} \int_{v \in \omega^{-r}\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\lfloor r/2 \rfloor} \langle v, v \rangle \in \mathbb{Z}_p) \left( \int_{x \in \mathbb{Q}_p} \text{Char}(x\omega - \frac{\langle v, v \rangle}{2} \in \omega^{-r}\mathcal{O}_{E_p}) dx \right) dv \\ &= \sum_{r \geq 0} |\omega|^{rs} p^{\lfloor r/2 \rfloor} \left( \int_{v \in \omega^{-r}\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\lfloor r/2 \rfloor} \langle v, v \rangle \in \mathbb{Z}_p) dv \right) \end{aligned}$$

### 7.3.1 Split case

When  $p$  splits in  $E$ , then one can view the Hermitian  $\mathbb{Q}_p \times \mathbb{Q}_p$ -space  $V_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$  as the direct sum of two copies of quadratic  $\mathbb{Q}_p$ -space  $(\mathbb{Q}_p^n, ((x_i), (y_i)) = \sum x_i y_i)$ , and one has

$$2\text{Re}\langle (x_1, y_1), (x_2, y_2) \rangle = (x_1, y_2) + (x_2, y_1).$$

Denote by  $L$  the lattice  $\mathbb{Z}_p^n$  in  $\mathbb{Q}_p^n$ . We set

$$\begin{aligned} S(r_1, r_2) &:= \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\max(r_1, r_2)} \langle v, v \rangle \in \mathbb{Z}_p) dv \\ &= \int_{\substack{x \in p^{-r_1}L \\ y \in p^{-r_2}L}} \psi_p^{-1}((x, T_2) + (y, T_1)) \text{Char}(p^{\max(r_1, r_2)}(x, y) \in \mathbb{Z}_p) dx dy, \end{aligned}$$

for any integers  $r_1, r_2 \geq 0$ .

**Proposition 7.6.** *Let  $p$  be a prime split in  $E$ , and  $r_1 \leq r_2$  be two natural numbers. For a vector  $T = (T_1, T_2) \in \mathcal{V}$  with  $\alpha = v_p(T_1), \beta = v_p(T_2), v_p(\langle T, T \rangle) = \gamma$  and  $B = B(r_1, r_2) := \min(r_1, \beta, \alpha + r_1 - r_2)$ , one has*

$$S(r_1, r_2) = p^{r_2 n - r_1} \left( -\text{Char}(\gamma < r_2 + B) \cdot p^{r_1 - 1 + (n-1)(\gamma - r_2 + 1)} + \sum_{k=0}^{\min(B, \gamma - r_2)} p^{kn} \phi(p^{r_1 - k}) \right),$$

which is nonzero only if  $r_2 \leq \min(\gamma + 1, r_2 + \alpha)$ . In particular, if  $\langle T, T \rangle \in \mathcal{O}_{E_p}^\times$ , then the nonzero terms are

$$S(0, 0) = 1 \text{ and } S(1, 1) = -p^{n-1}.$$

*Proof.* The integral  $S(r_1, r_2)$  can be rewritten as an exponential sum:

$$S(r_1, r_2) = \sum_{\substack{x \in L/p^{r_1}L \\ y \in L/p^{r_2}L}} \psi_p \left( \frac{(x, T_2)}{p^{r_1}} + \frac{(y, T_1)}{p^{r_2}} \right) \text{Char}((x, y) \in p^{r_1} \mathbb{Z}_p).$$

The characteristic function has the following expression:

$$\text{Char}((x, y) \in p^{r_1} \mathbb{Z}_p) = p^{-r_1} \sum_{u \in \mathbb{Z}/p^{r_1} \mathbb{Z}} \psi_p \left( \frac{u(x, y)}{p^{r_1}} \right),$$

thus

$$S(r_1, r_2) = p^{-r_1} \sum_{\substack{y \in L/p^{r_2}L \\ u \in \mathbb{Z}/p^{r_1} \mathbb{Z}}} \psi_p^{-1} \left( \frac{(y, T_1)}{p^{r_2}} \right) \left( \sum_{x \in L/p^{r_1}L} \psi_p^{-1} \left( \frac{(x, T_2 - uy)}{p^{r_1}} \right) \right).$$

The inner sum equals  $p^{r_1 n}$  when  $T_2 - uy \in p^{r_1} L$ , otherwise it equals 0. Hence one has

$$S(r_1, r_2) = p^{r_1(n-1)} \sum_{\substack{y \in L/p^{r_2}L \\ u \in \mathbb{Z}/p^{r_1} \mathbb{Z}}} \psi_p^{-1} \left( \frac{(y, T_1)}{p^{r_2}} \right) \text{Char}(T_2 - uy \in p^{r_1} L).$$

The equation  $T_2 = uy \bmod p^{r_1} L$  has solution if and only if  $k := v_p(u) \leq v_p(T_2) = \beta$ , and in this case,  $y = u^{-1} T_2 + p^{r_1 - k} z$  for  $z \in L/p^{r_2 - r_1 + k} L$ . Plugging this in the exponential sum and replacing  $u$  by  $p^k u$ , we obtain that

$$\begin{aligned} S(r_1, r_2) &= p^{r_1(n-1)} \sum_{k=0}^{\min(r_1, \beta)} \sum_{u \in (\mathbb{Z}/p^{r_1-k} \mathbb{Z})^\times} \sum_{z \in L/p^{r_2-r_1+k} L} \psi_p^{-1} \left( \frac{(T_1, u^{-1} p^{-k} T_2 + p^{r_1-k} z)}{p^{r_2}} \right) \\ &= p^{r_1(n-1)} \sum_{k=0}^{\min(r_1, \beta)} \sum_{u \in (\mathbb{Z}/p^{r_1-k} \mathbb{Z})^\times} \psi_p^{-1} \left( \frac{u^{-1} (T_1, T_2)}{p^{r_2+k}} \right) \left( \sum_{z \in L/p^{r_2-r_1+k} L} \psi_p^{-1} \left( \frac{(z, T_1)}{p^{r_2-r_1+k}} \right) \right) \\ &= p^{r_1(n-1)} \sum_{k=0}^{\min(r_1, \beta)} p^{(r_2-r_1+k)n} \text{Char}(\alpha \geq r_2 - r_1 + k) \left( \sum_{u \in (\mathbb{Z}/p^{r_1-k} \mathbb{Z})^\times} \psi_p^{-1} \left( \frac{u^{-1}}{p^{r_2+k-\gamma}} \right) \right) \end{aligned}$$

For the inner sum, we have

$$\sum_{u \in (\mathbb{Z}/p^{r_1-k}\mathbb{Z})^\times} \psi_p^{-1} \left( \frac{u^{-1}}{p^{r_2+k-\gamma}} \right) = \begin{cases} \phi(p^{r_1-k}) = |(\mathbb{Z}/p^{r_1-k})^\times| & , \text{ if } r_2 + k - \gamma \leq 0 \\ -p^{r_1-k-1} & , \text{ if } r_2 + k - \gamma = 1 \\ 0 & , \text{ if } r_2 + k - \gamma > 1 \end{cases}$$

Combining all these together, we get the following formula: let  $B = B(r_1, r_2, \alpha, \beta)$  be  $\min(r_1, \beta, \alpha + r_1 - r_2)$ , then

$$S(r_1, r_2) = p^{r_2 n - r_1} \left( -\text{Char}(\gamma < r_2 + B) \cdot p^{r_1 - 1 + (n-1)(\gamma - r_2 + 1)} + \sum_{k=0}^{\min(B, \gamma - r_2)} p^{kn} \phi(p^{r_1 - k}) \right).$$

This is nonzero only if  $r_1 \leq r_2 \leq \min(\gamma + 1, r_1 + \alpha)$ , which only hold for finitely many terms. Particularly, if  $T = (T_1, T_2)$  is unramified, then  $\alpha = \beta = \gamma = 0$ , thus  $r_1 = r_2 \leq 1$ . In this case we have

$$S(0, 0) = 1, S(1, 1) = -p^{n-1}.$$

□

## 7.4 Rank 2 Fourier coefficients: archimedean components

*Remark 7.7.* In this article, when we write  $|z|$  for  $z \in \mathbb{C}$ , it means the *norm* of  $z$  with respect to the extension  $\mathbb{C}/\mathbb{R}$ , i.e.  $|z| = z\bar{z}$  instead of the usual modulus  $\sqrt{z\bar{z}}$ . I admit that this is somehow strange and confusing, but I will fix this problem if this draft could become a paper...

We first analyze the function  $f_{\ell, \infty}(w_2 n, s = \ell + 1)$ :

**Lemma 7.8.** *For any  $v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}$  and  $x \in \mathbb{R}$ , we set:*

$$\alpha(v, x) = -\frac{\langle v, v \rangle}{2} + ix + 1, \beta(v) = \sqrt{2} \langle v, u_2 \rangle,$$

then we have

$$f_{\ell, \infty}(w_2 n(v, ix), s) = \frac{(\alpha u_1 + \beta u_2)^\ell (-\bar{\beta} u_1 + \bar{\alpha} u_2)^\ell}{(|\alpha(v, x)|^2 + |\beta(v)|^2)^{\ell+s} (\ell!)^2}.$$

*Proof.* One has

$$b_1 w_2 n(v, ix) = (-\langle v, v \rangle / 2 + ix) b_1 + b_2 + v.$$

Suppose that we can decompose  $w_2 n(v, ix)$  as  $pk$  for some  $p \in \mathbf{P}(\mathbb{R})$  and  $k \in K_\infty$ , then

$$b_1 w_2 n(v, ix) k^{-1} = b_1 p = v(p)^{-1} b_1. \quad (7.3)$$

Let  $k_+$  be the factor of  $k$  in  $U(V_2^+)$ , and  $v = v_+ + v_- \in \mathbb{C} u_2 \oplus \text{Span}_{\mathbb{C}}(v_1, \dots, v_{n-1})$ . Taking the  $V_2^+$  components of Equation (7.3), one gets:

$$\frac{1}{\sqrt{2}} (\alpha(v, x) u_1 + \beta(v) u_2) = \left[ \frac{1}{\sqrt{2}} \left( -\frac{\langle v, v \rangle}{2} + ix + 1 \right) u_1 + v_+ \right] k_+^{-1} = v(p)^{-1} \frac{u_1}{\sqrt{2}}.$$

The norms of both sides give us the identity  $|\alpha(v, x)|^2 + |\beta(v)|^2 = |\nu(p)|^{-2}$ . One may assume that  $\nu(p)^{-1} = \sqrt{|\alpha(v, x)|^2 + |\beta(v)|^2}$ , then in the basis of  $u_1, u_2$ , the element  $k^+ \in U(V_2^+)$  can be written as the Hermitian matrix

$$\frac{1}{\sqrt{|\alpha(v, x)|^2 + |\beta(v)|^2}} \begin{pmatrix} \alpha(v, x) & -\overline{\beta(v)} \\ \beta(v) & \overline{\alpha(v, x)} \end{pmatrix}.$$

Plug  $k^+$  and  $\nu(p)$  into  $f_{\ell, \infty}(w_2 n(v, ix), s) = f_{\ell, \infty}(pk, s) = |\nu(p)|^{2s} [u_1^\ell] [u_2^\ell] \cdot k^+$ , and we get the desired value.  $\square$

Let  $I_0(T; \ell)$  be the coefficient of  $[u_1^\ell] [u_2^\ell]$  in  $E_{2, \infty}^T(s = \ell + 1)$ , which can be written as

$$\begin{aligned} I_0(T; \ell) &= \int_{v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}} \int_{x \in \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} \frac{\binom{\ell}{k} \binom{\ell}{k} (x^2 + A)^k (-B)^{\ell-k}}{(x^2 + A + B)^{2\ell+1}} dx dv \\ &= \int_{v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \int_{x \in \mathbb{R}} \frac{(x^2 + A)^k}{(x^2 + A + B)^{2\ell+1}} dx dv \end{aligned}$$

where  $A = |\alpha|^2 - x^2 = (1 - \langle v, v \rangle / 2)^2$ , and  $B = |\beta|^2 = 2|\langle v, u_2 \rangle|^2$ .

**Lemma 7.9.** *For any real number  $C, D$  and two natural numbers  $m < n$ , we have*

$$\int_{\mathbb{R}} \frac{(x^2 + C)^m}{(x^2 + D)^n} dx = \frac{D^{m-n+1/2}}{(n-1)!} \sum_{k=0}^m \binom{m}{k} \left(\frac{C}{D}\right)^{m-k} \Gamma(k+1/2) \Gamma(n-k-1/2).$$

*Proof.* An exercise of calculus.  $\square$

Now Lemma 7.9 tells us  $I_0(T; \ell)$  is the Fourier transform of the function

$$F_{0, \ell}(v) := \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \frac{(A+B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^k \binom{k}{j} \left(\frac{A}{A+B}\right)^{k-j} \Gamma(j+1/2) \Gamma(2\ell+1/2-j)$$

Set  $z = B/(A+B)$ , then this function becomes

$$\frac{2^{-4\ell}}{(2\ell)!} \pi (A+B)^{-\ell-1/2} \sum_{k=0}^{\ell} \sum_{j=0}^k \binom{\ell}{k}^2 \binom{k}{j} \frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!} (-z)^{\ell-k} (1-z)^{k-j}$$

We write:

$$\sum_{k=0}^{\ell} \sum_{j=0}^k c_{j,k} (-z)^{\ell-k} (1-z)^{k-j} = \sum_{r=0}^{\ell} (-1)^r C(r) z^r,$$

where  $c_{j,k} = \binom{\ell}{k}^2 \binom{k}{j} \frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!}$ . The term  $(-z)^{\ell-k} (1-z)^{k-j}$  has a non-zero  $z^r$  term if and only if  $\ell - k \leq r \leq \ell - j$ , thus

$$(-1)^r C(r) = \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} c_{j,k} \binom{k-j}{r-\ell+k} (-1)^{\ell-k+r-\ell+k} = (-1)^r \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} c_{j,k} \binom{k-j}{r-\ell+k}.$$

To compute  $C(r)$ , we need the following lemma:



**Lemma 7.10.** (1) For integers  $0 \leq a \leq b$ , one has

$$\sum_{i=a}^b \binom{b}{i} \binom{b-a}{b-i} = \binom{2b-a}{b}.$$

(2) For any integer  $0 \leq r \leq \ell$ , one has

$$\sum_{i=0}^{\ell-r} \frac{\binom{2\ell}{i} \binom{\ell-r}{i}}{\binom{4\ell}{2i}} = 2^{2\ell-2r} \frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}.$$

*Proof.* The identity in (1) is obvious. For the identity in (2), the LHS can be rewritten as:

$$\sum_{i=0}^{\ell-r} \frac{(-\ell+r)_i (1/2)_i}{(-2\ell+1/2)_{i+j} j!} = {}_2F_1(-(\ell-r), 1/2; -2\ell+1/2; 1),$$

where  $(x)_j$  is the (rising) Pochhammer symbol, and  ${}_2F_1$  is the hypergeometric function. By Chu-Vandermonid identity, this value of hypergeometric function is

$$\frac{(-2\ell)_{\ell-r}}{(-2\ell+1/2)_{\ell-r}} = 2^{\ell-r} \frac{\frac{(2\ell)!}{(\ell+r)!}}{\frac{(4\ell-1)!!}{(2\ell+2r-1)!!}} = 2^{2\ell-2r} \frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}. \quad \square$$

Now we return to the value of  $C(r)$ :

$$\begin{aligned} C(r) &= \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} \frac{(\ell!)^2 (2j)! (4\ell-2j)!}{k! ((\ell-k)!)^2 (j!)^2 (r-\ell+k)! (\ell-r-j)! (2\ell-j)!} \\ &= \sum_{j=0}^{\ell-r} \frac{\ell! (2j)! (4\ell-2j)!}{r! (j!)^2 (\ell-r-j)! (2\ell-j)!} \sum_{k=\ell-r}^{\ell} \binom{\ell}{k} \binom{r}{\ell-k} \\ (\text{by (1) of Lemma 7.10}) &= \frac{\ell!}{r!} \cdot \frac{(\ell+r)!}{\ell! r!} \sum_{j=0}^{\ell-r} \frac{(2j)! (4\ell-2j)!}{(j!)^2 (\ell-r-j)! (2\ell-j)!} \\ &= \frac{(\ell+r)!}{(r!)^2} \cdot \frac{(4\ell)!}{(\ell-r)! (2\ell)!} \sum_{j=0}^{\ell-r} \frac{\binom{2\ell}{j} \binom{\ell-r}{j}}{\binom{4\ell}{2j}} \\ (\text{by (2) of Lemma 7.10}) &= \frac{(\ell+r)!}{(r!)^2} \cdot \frac{(4\ell)!}{(\ell-r)! (2\ell)!} \cdot 2^{2\ell-2r} \frac{\frac{(2\ell+2r)!}{((\ell+r)!)^2}}{\frac{(4\ell)!}{((2\ell)!)^2}} \\ &= 2^{2\ell-2r} \frac{(2\ell)! (2\ell+2r)!}{(r!)^2 (\ell+r)! (\ell-r)!}. \end{aligned}$$

Putting the value of  $C(r)$  into  $F_{0,\ell}$ , we have

$$\begin{aligned} F_{0,\ell}(v) &= \frac{2^{-4\ell}\pi}{(2\ell)!(A+B)^{\ell+1/2}} \sum_{r=0}^{\ell} (-1)^r 2^{2\ell-2r} \frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!} z^r \\ &= \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}} \sum_{r=0}^{\ell} \frac{(-\ell)_r(\ell+1/2)_r}{1_r} \frac{z^r}{r!} \\ &= \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}} \cdot {}_2F_1(-\ell, \ell+1/2; 1; z). \end{aligned}$$

We have shown the following result:

**Proposition 7.11.** For  $T \in V_0$  with  $\langle T, T \rangle > 0$ ,  $T$  the coefficient  $I_0(T; \ell)$  of  $[u_1^\ell][u_2^\ell]$  in  $E_{2,\infty}^T(s = \ell + 1)$  is the Fourier transform of the function:

$$F_{0,\ell}(v) = \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}} \cdot {}_2F_1(-\ell, \ell+1/2; 1; \frac{B}{A+B}),$$

where  $A = \left(1 - \frac{\langle v, v \rangle}{2}\right)^2$  and  $B = 2|\langle v, u_2 \rangle|^2$ .

Let's recall the following result for quadratic spaces by Pollack: **I know there are notation problems again... just let it be like this for now**

**Proposition 7.12.** ( $v = 0$  case of [PS22, Proposition 4.5.3]) Let  $(V', (\cdot, \cdot)) = V_2 \oplus V_n$  be a non-degenerate quadratic space over  $\mathbb{R}$  of signature  $(2, n)$ , and  $v_1, v_2$  an orthonormal basis of  $V_2$ . Set

$$I_0(x; \ell) = \int_{V'} e^{i(\omega, x)} \text{Char}(q(\omega) > 0) q(\omega)^{\ell-n/2} K_0(\sqrt{2}|(\omega, v_1 + \sqrt{-1}v_2)|) d\omega.$$

This integral is absolutely convergent and we have

$$I_0(x; \ell) = (2\pi)^{(n+2)/2} 2^{\ell-1-n/2} \Gamma(\ell+1) \Gamma(\ell+1-n/2) F_4(\ell+1, \ell+1; \ell+1; 1; -\|x_n\|^2/2; -\|x_2\|^2/2),$$

where  $x = x_2 + x_n$ , and  $F_4(a, b; c; d; x; y)$  is Appell's hypergeometric function.

Now we look at how this result fits into our unitary group setting. One take  $V'$  to be  $(V_0, \text{Re}(\langle \cdot, \cdot \rangle))$ , as a quadratic space of signature  $(2, 2n-2)$ ,  $v'_1 = u_2$ ,  $v'_2 = iu_2$ , then

$$\begin{aligned} I_0(x; \ell) &= \int_{T \in V_0, \langle T, T \rangle > 0} e^{i\text{Re}\langle T, x \rangle} \langle T, T \rangle^{\ell-n+1} K_0(\sqrt{2}|\langle T, u_2 \rangle|) dT \\ &= \int_{T \in V_0, \langle T, T \rangle > 0} e^{2\pi i \text{Re}\langle T, x \rangle} (2\pi)^{2\ell-2n+2} \langle T, T \rangle^{\ell-n+1} K_0(2\sqrt{2}\pi|\langle T, u_2 \rangle|) (2\pi)^{2n} dT \\ &= (2\pi)^{2\ell+2} \int_{T \in V_0, \langle T, T \rangle > 0} e^{2\pi i \text{Re}\langle T, x \rangle} \langle T, T \rangle^{\ell-n+1} K_0(2\sqrt{2}\pi|\langle T, u_2 \rangle|) dT, \end{aligned}$$

here  $|z| = \sqrt{z\bar{z}}$ , and for any  $\ell > n-1$  we have

$$I_0(x; \ell) = (2\pi)^n 2^{\ell-n} \ell! (\ell-n+1)! (A+B)^{-(\ell+1)/2} {}_2F_1(-\ell/2, (\ell+1)/2; 1; B/(A+B)).$$

Replace  $\ell$  by  $2\ell$  (now  $\ell > \frac{n-1}{2}$ ), and we have

$$I_0(x; 2\ell) = (2\pi)^n 2^{2\ell-n} (2\ell)! (2\ell - n + 1)! (A + B)^{-\ell-1/2} \cdot {}_2F_1(-\ell, \ell + 1/2; 1; B/(A + B)).$$

Comparing this with Proposition 7.11, we get the following theorem:

**Theorem 7.13.** *For  $T \in V_0$  with  $\langle T, T \rangle > 0$  and  $\ell > \frac{n-1}{2}$ , we have*

$$I_0(T; \ell) = \frac{2^{-\ell-2} \pi^{4\ell-n+3} \langle T, T \rangle^{2\ell-n+1}}{\ell! (2\ell - n + 1)!} \mathcal{W}_T(1).$$

## 8 Our progress

If there is no mistake in the previous calculations, that means

$$\begin{aligned} & E_\ell(g, \Phi_f, s = \ell + 1) \\ &= f_\ell(g, s = \ell + 1) + \pi^{2\ell+1} \left( \sum_{\substack{T \neq 0 \in \mathcal{V} \\ \langle T, T \rangle = 0}} \square \cdot \sigma_E(T) \mathcal{W}_T(g) + \sum_{\substack{T \in \mathcal{V} \\ \langle T, T \rangle > 0}} \square \cdot \langle T, T \rangle^{2\ell-n+1} \mathcal{W}_T(g) \right), \end{aligned}$$

where  $f_\ell(g, s = \ell + 1) = \zeta_E(\ell + 1) |\nu(g)|_E^{\ell+1} [u_1^n][u_2^n]$  and  $\square$  means certain rational factor.

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