

Quaternionic Degenerate Eisenstein series on $\mathbf{U}(2, n)$

October 21, 2025

Abstract

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1 Unitary groups of split rank 2

If some notation appearing in this section is not defined, then it should be defined in [HMY24].

Let E be an imaginary quadratic extension of \mathbb{Q} , and we denote the nonzero element in $\text{Gal}(E/\mathbb{Q})$ by c or $x \mapsto \bar{x}$. Write the norm of E/\mathbb{Q} as $||$.

Define $\mathbf{G} = \mathbf{U}(V) = \mathbf{U}(2, n)$ to be the unitary group of a non-degenerate Hermitian space $(V, \langle \cdot, \cdot \rangle)$ over E with signature $(2, n)$. We write the \mathbf{G} -action on V as a right action.

Fix a pair of isotropic lines (U, U^\vee) inside V such that $\langle U, U^\vee \rangle \neq 0$, and take V_0 to be the orthogonal complement of $U \oplus U^\vee$, which is a Hermitian space of signature $(1, n - 1)$. Fix $b_1 \in U$ and $b_2 \in U^\vee$ such that $\langle b_1, b_2 \rangle = 1$.

One can define a parabolic subgroup \mathbf{P} of \mathbf{G} as the stabilizer of U , and it has the following realization of Levi decomposition $\mathbf{P} = \mathbf{M}\mathbf{N}$:

- \mathbf{M} is the stabilizer of U and U^\vee , and it is isomorphic to $\text{Res}_{E/\mathbb{Q}}(\mathbf{G}_m) \times \mathbf{U}(V_0)$. One can write the \mathbf{M} -action on $(u, w, u^\vee) \in U \oplus V_0 \oplus U^\vee$ explicitly:

$$(u, w, u^\vee). (z, h) = (z^{-1}u, wh, \bar{z}u^\vee).$$

- \mathbf{N} is a Heisenberg, and it is isomorphic to the group

$$\{(v, \lambda) \in V_0 \times \text{Res}_{E/\mathbb{Q}}\mathbf{G}_a \mid \bar{\lambda} = -\lambda\},$$

equipped with the following multiplication:

$$(v_1, \lambda_1) \cdot (v_2, \lambda_2) := \left(v_1 + v_2, \lambda_1 + \lambda_2 - \frac{\langle v_1, v_2 \rangle - \langle v_2, v_1 \rangle}{2} \right).$$

The action of the corresponding element $n(v, \lambda) \in \mathbf{N}$ on V is given as:

$$b_1 \mapsto b_1, b_2 \mapsto \left(-\frac{1}{2}\langle v, v \rangle + \lambda\right)b_1 + b_2 + v, w \in V_0 \mapsto -\langle w, v \rangle b_1 + w.$$

One has

$$(z, h)n(v, \lambda)(z^{-1}, h^{-1}) = n(\bar{z}vh^{-1}, |z|\lambda).$$

Denote the center of \mathbf{N} by \mathbf{N}_0 . We denote by ν the similitude character of \mathbf{M} : $\nu(z, h) = z \in \text{Res}_{E/\mathbb{Q}}\mathbf{G}_m$, so that $b_1(z, h) = \nu(z, h)^{-1}b_1$.

Fix an additive character $\psi : \mathbb{Q}\backslash\mathbb{A} \rightarrow \mathbb{C}^\times$, so that ψ_p has conductor \mathbb{Z}_p and $\psi_\infty(x) = e^{2\pi ix}$. We have the following identification $V_0 \xrightarrow{\sim} \text{Hom}([\mathbf{N}], \mathbb{S}^1)$: for any element $T \in V_0$, one associates a unitary character χ_T of $[\mathbf{N}]$ by

$$\chi_T : [\mathbf{N}] \rightarrow \mathbb{C}^\times, n(v, \lambda) \mapsto \psi(-\text{Im}\langle T, v \rangle).$$

I am really confused with this identification in [HMY24]. This character χ_T seems not well-defined: the symplectic form $-\text{Im}\langle T, v \rangle = \frac{\langle v, T \rangle - \langle T, v \rangle}{2i}$ involves $1/2i$, which may not lie in E . Maybe one should define χ_T to be $n(v, \lambda) \mapsto \psi(\text{Re}\langle T, v \rangle)$.

For a finite place p of \mathbb{Q} , we define E_p to be $E_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} E$, which is isomorphic to

- $\mathbb{Q}_p \times \mathbb{Q}_p$, if p splits in E ;
- a degree 2 unramified (*resp.* ramified) extension of \mathbb{Q}_p , if p is inert (*resp.* ramified).

2 Heisenberg Eisenstein series

Choose a section $f = f_{\ell, \infty} \otimes f_{fte}$ of $\text{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})} |\nu|^s$ (unnormalized) as follow:

- $f_{\ell, \infty}$ is the $\mathbb{V}_\ell = (\text{Sym}^{2\ell} V_2^+ \otimes \det_{\mathbf{U}(2)}^{-\ell}) \boxtimes \mathbf{1}$ -valued, K_∞ -equivariant induced section whose restriction to $\mathbf{M}(\mathbb{R})$ is

$$f_{\ell, \infty}((z, h), s) = |z|^s [u_1^\ell] [u_2^\ell],$$

where $[u_i^k] := u_i^k / k!$.

- f_{fte} is defined as

$$f_{fte}(g_f, \Phi_f, s) = \int_{\mathbf{GL}_1(\mathbb{A}_{E,f})} |t|^s \Phi_f(t \cdot b_1 g_f) dt,$$

where $\Phi_f = \prod \Phi_p$ is a Schwartz-Bruhat function on $V(\mathbb{A}_f)$.

One defines the degenerate Heisenberg Eisenstein series:

$$E_\ell(g, \Phi_f, s) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{Q})} f(\gamma g, s).$$

To prove the modularity of this Eisenstein series, one needs the following result:

Proposition 2.1. *The \mathbb{V}_ℓ -valued section $f_{\ell, \infty}$ of $\text{Ind}_{\mathbf{P}(\mathbb{R})}^{\mathbf{G}(\mathbb{R})} |\nu|^s$, as a function on $\mathbf{G}(\mathbb{R})$, is killed by the Schmid operators \mathcal{D}_ℓ^\pm if and only if $s = \ell + 1$.*

Proof. Write an element $(z, h) \in \mathbf{M}(\mathbb{R})$ as $m = (h, r, \theta)$ so that $z = re^{i\theta}$, $r \in \mathbb{R}_{>0}$ and $\theta \in [0, 2\pi)$. Using this coordinates, we define a function F on $\mathbf{M}(\mathbb{R})$, sending (h, r, θ) to r^s . By [HMY24, Proposition 3.10], $f_{\ell, \infty}$ is killed by \mathcal{D}_ℓ^\pm if and only if

$$\begin{cases} (r\partial_r - 2(\ell + 1))F(m) = 0 \\ [u_2 \otimes \bar{v}_k]^+ F(m) = 0, \text{ if } 1 \leq k < n \\ [u_2 \otimes \bar{v}_k]^- F(m) = 0, \text{ if } 1 \leq k < n \end{cases}$$

It can be easily verified that these equations hold if and only if $s = \ell + 1$. \square

As a consequence, the Eisenstein series $E_\ell(g, \Phi_f, s = \ell + 1)$ is a quaternionic modular form of weight ℓ .

3 The Fourier expansion of $E_\ell(g, \Phi_f, s = \ell + 1)$

3.1 Abstract Fourier expansion

In this subsection, we give the “abstract” Fourier expansion of $E_\ell(g, \Phi_f, s)$.

Lemma 3.1. *The right $\mathbf{P}(\mathbb{Q})$ -space $\mathbf{P}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{Q})$, the space of isotropic lines in $V(\mathbb{Q})$, has exactly 3 orbits of $\mathbf{P}(\mathbb{Q})$, represented respectively by $\mathbf{Q}b_1$, $\mathbf{Q}v_0$ and $\mathbf{Q}b_2$, where v_0 is an arbitrary non-zero isotropic vector in $V_0(\mathbb{Q})$.*

Proof. Directly by the explicit action given in §1. \square

Set $\mathbf{G}(\mathbb{Q}) = \bigsqcup_{i=0}^2 \mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})$, such that $w_0 = 1$, $b_1w_1 = v_0$ and $b_1w_2 = b_2$. Now we can write the degenerate Eisenstein series as

$$E_\ell(g, \Phi_f, s) = \sum_{i=0}^2 E_{\ell,i}(g, \Phi_f, s), \quad E_{\ell,i}(g, \Phi_f, s) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q}) \backslash \mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})} f(\gamma g, s),$$

thus $E_{\ell,0}(g, \Phi_f, s) = f(g, s)$. From now on, when there is no confusion we will omit the ℓ and Φ_f in $E_\ell(g, \Phi_f, s)$, and write it as $E(g, s) = \sum_{i=0}^2 E_i(g, s)$.

Lemma 3.2. *Assume that $\operatorname{Re}(s) \gg 0$ so that the sum defining $E(g, s)$ converges absolutely. Then one has the following expressions for the $E_i(g, s)$:*

- (1) *Let \mathcal{L}_0 be the set of non-zero isotropic lines ℓ in V_0 and for any $\ell \in \mathcal{L}_0$, select $\gamma(\ell) \in \mathbf{G}(\mathbb{Q})$ with $b_1\gamma(\ell) \in \ell$. Then*

$$E_1(g, s) = \sum_{\ell \in \mathcal{L}_0} \sum_{\mu \in (\ell)^\perp \cap \mathbf{N}_0(\mathbb{Q}) \backslash \mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu g, s).$$

- (2) *One has*

$$E_2(g, s) = \sum_{\mu \in \mathbf{N}(\mathbb{Q})} f(w_2\mu g, s).$$

For any $T \in V_0$, we set

$$E_i^T(g, s) = \int_{\mathbf{N}(F) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) E_i(ng, s) dn, i = 0, 1, 2.$$

Lemma 3.3. (1) If T is anisotropic, then $E_1^T = 0$. If T is isotropic, define $\mathbf{N}_T = (\ell_T)^\perp \mathbf{N}_0 \subseteq \mathbf{N}$, then

$$E_1^T(g, s) = \int_{\mathbf{N}_T(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell_T)ng, s) dn.$$

(2) For any $T \in V_0$, one has

$$E_2^T(g, s) = \int_{\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(w_2 ng, s) dn.$$

Proof. It suffices only to prove the $i = 1$ case. For any $\ell \in \mathcal{L}_0$, set

$$\mathbf{N}_\ell = \left\{ n(v, \lambda) \in \mathbf{N} \mid v \in \ell^\perp \right\} \subseteq \mathbf{N}.$$

For any $T \in V_0$,

$$\begin{aligned} E_1^T(g, s) &= \sum_{\ell \in \mathcal{L}_0} \int_{[N]} \chi_T^{-1}(n) \left(\sum_{\mu \in \mathbf{N}_\ell(\mathbb{Q}) \backslash \mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu ng, s) \right) dn \\ &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{Q}) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell)ng, s) dn \\ &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} \left(\int_{[\mathbf{N}_\ell]} \chi_T^{-1}(r) dr \right) \chi_T^{-1}(n) f(\gamma(\ell)ng, s) dn \\ &= \sum_{\ell \in \mathcal{L}_0, \chi_T|_{\mathbf{N}_\ell} \equiv 1} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell)ng, s) dn. \end{aligned}$$

Then the lemma follows from the fact that $\chi_T|_{\mathbf{N}_\ell} \equiv 1$ if and only if $T \in \ell$, i.e. T is isotropic and $\ell = \ell_T$. \square

3.2 Computation of constant term

The constant term of $E_\ell(g, \Phi_f, s = \ell + 1)$ consists of three parts:

- $E_{\ell,0}(g, \Phi_f, s) = f(g, s)$,
- $E_1^0(g, s) = \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma(\ell)ng, s) dn$,
- and $E_2^0(g, s) = \int_{\mathbf{N}(\mathbb{A})} f(w_2 ng) dn$.

3.2.1 The $i = 0$ -term

Lemma 3.4. For $g \in \mathbf{P}(\mathbb{A})$,

$$E_0(g, s) = f(g, s) = |\nu(g)|_E^s \zeta_E(s) [u_1^n] [u_2^n].$$

Proof. For $g_f \in \mathbf{P}(\mathbb{A}_f)$, we have

$$\begin{aligned} f_{fte}(g_f, s) &= \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(t b_1 g_f) dt \\ &= \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(t \nu(g_f)^{-1} b_1) dt \\ &= |\nu(g_f)|_E^s \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(t b_1) dt. \end{aligned}$$

Thus, the non-archimedean contribution is $|\nu(g_f)|_E^s \zeta_E(s)$. Combining with $f_{n,\infty}(g_\infty, s) = |\nu(g_\infty)|_E^s [u_1^n] [u_2^n]$, we get the desired identity. \square

3.2.2 The $i = 1$ -term

We fix a non-zero isotropic vector in V_0 , such that $v_0 = b_1 \gamma_0$, and set $\ell_0 = E v_0$. Define \mathbf{P}_0 be the stabilizer of ℓ_0 in $\mathbf{U}(V_0)$, which is a parabolic subgroup of \mathbf{M} . We denote the similitude character of \mathbf{P}_0 by λ , i.e. $v_0 g = \lambda(g)^{-1} v_0$ for any $g \in \mathbf{P}_0$. For $g \in \mathbf{P}_0(\mathbb{A})$, we have:

$$\begin{aligned} E_1^0(g, s) &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma(\ell) n g, s) dn \\ &= \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{Q})} \int_{\mathbf{N}_{\ell_0}(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma_0 \gamma n g, s) dn \end{aligned}$$

If we set $f_0(g, s) = \int_{\mathbf{N}_{\ell_0}(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma_0 n g, s) dn$, then for $\text{Re}(s) \gg 0$,

$$E_1^0(g, s) = \sum_{\gamma \in \mathbf{P}_0(\mathbb{Q}) \backslash \mathbf{M}(\mathbb{Q})} f_0(\gamma g, s),$$

and it defines an Eisenstein series on \mathbf{M} . Now we want to determine this section $f_0(g, s) \in \text{Ind}_{\mathbf{P}_0(\mathbb{A})}^{\mathbf{M}(\mathbb{A})} |\lambda|^s$ (still unnormalized).

At finite places, one has

$$\begin{aligned} \int_{\mathbf{N}_{\ell_0}(\mathbb{A}_{E,f}) \backslash \mathbf{N}(\mathbb{A}_{E,f})} f(\gamma_0 n g, s) dn &= \int_{x \in \mathbb{A}_{E,f}} \int_{t \in \mathbb{A}_{E,f}^\times} |t|^s \Phi_{fte}(t(v_0 + x b_1) g) dt dx \\ &= \int_{x \in \mathbb{A}_{E,f}} \int_{t \in \mathbb{A}_{E,f}^\times} |t|^s \Phi_{fte}(t \lambda(g)^{-1} v_0 + t x \nu(g)^{-1} b_1) dt dx \\ &= |\lambda(g)|_f^{s-1} |\nu(g)|_f \int_{x \in \mathbb{A}_{E,f}} \int_{t \in \mathbb{A}_{f,E}^\times} |t|^{s-1} \Phi_{fte}(t v_0 + x b_1) dt dx \\ &= |\lambda(g)|_f^{s-1} |\nu(g)|_f \zeta_E(s-1). \end{aligned}$$

Now we switch to the archimedean place. Set $c_1 = v_0$ and c_2 another isotropic vector in V_0 with $\langle c_1, c_2 \rangle = 1$. We can take $u_2 = \frac{1}{\sqrt{2}}(c_1 + c_2)$ and $v_{n-1} = \frac{1}{\sqrt{2}}(c_1 - c_2)$. We identify $\mathbf{N}_{\ell_0}(\mathbb{R}) \backslash \mathbf{N}(\mathbb{R})$ with \mathbb{C} via the map $n(v, \lambda) \mapsto -\langle c_1, v \rangle$, and for any $z \in \mathbb{C}$ we choose an element $n(z) \in \mathbf{N}(\mathbb{R})$ under this identification.

Lemma 3.5. *When $\ell \geq 1$, we have $f_{0,\infty} \equiv 0$.*

Proof. Similarly to Lemma 3.6, for any $g \in \mathbf{P}_0(\mathbb{R})$ one has

$$f_{\ell,\infty}(\gamma_0 n(z)g) = \frac{(z\nu(g)^{-1}u_1 + \lambda(g)^{-1}u_2)^\ell (-\bar{\lambda}(g)^{-1}u_1 + \bar{z}\nu(g)^{-1}u_2)^\ell}{(|\lambda(g)|^{-2} + |z|^2|\nu(g)|^{-2})^{2\ell+1}(\ell!)^2}.$$

From now on we write $\lambda(g)$ (resp. $\nu(g)$) as λ (resp. ν).

The coefficient of $[u_1^{\ell-v}][u_2^{\ell+v}]$ in $f_{\ell,\infty}(\gamma_0 n(z)g)$ is

$$\begin{aligned} & \frac{(\ell-v)!(\ell+v)!}{(\ell!)^2(|\lambda|^{-2} + |z|^2|\nu|^{-2})^{2\ell+1}} \sum_{\substack{0 \leq i,j \leq \ell \\ i+j=\ell-v}} \binom{\ell}{i} \binom{\ell}{j} z^i \nu^{-i} \lambda^{-(\ell-i)} (-\bar{\lambda})^{-j} (\bar{z}\nu^{-1})^{\ell-j} \\ &= \frac{(\ell-v)!(\ell+v)! |\nu|^{-2\ell} \nu^v \lambda^{-v} |z|^{2\ell} z^{-v}}{(\ell!)^2(|\lambda|^{-2} + |z|^2|\nu|^{-2})^{2\ell+1}} \sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)} (-1)^j \binom{\ell}{j} \binom{\ell}{j+v} |\nu|^{2j} |\lambda|^{-2j} |z|^{-2j}, \end{aligned}$$

and the corresponding coefficient in $f_{0,\infty}(g, s)$ is the integral of this function over $z \in \mathbb{C}$. Set $w = z\lambda/\nu$, then we have $dw = |\lambda/\nu|^2 dz$ and this function equals

$$\frac{(\ell-v)!(\ell+v)! |\lambda|^{2\ell+2} w^{-v} |w|^{2\ell}}{(\ell!)^2 (1+|w|^2)^{2\ell+1}} \sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)} (-1)^j \binom{\ell}{j} \binom{\ell}{j+v} |w|^{-2j}.$$

Hence the coefficient of $[u_1^{\ell-v}][u_2^{\ell+v}]$ in $f_{0,\infty}(g, s = \ell+1)$ equals

$$\frac{(\ell-v)!(\ell+v)! |\lambda|^{2\ell} |\nu|^2}{(\ell!)^2} \sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)} (-1)^j \binom{\ell}{j} \binom{\ell}{j+v} \left(\int_{\mathbb{C}} \frac{w^{-v} |w|^{2\ell-2j}}{(1+|w|^2)^{2\ell+1}} dw \right) \quad (1)$$

Notice that the inner integral is nonzero only when $v = 0$, and we have

$$\int_{\mathbb{C}} \frac{|w|^{2\ell-2j}}{(1+|w|^2)^{2\ell+1}} dw = \pi \frac{(\ell-j)!(\ell+j-1)!}{(2\ell)!}, \quad j = 0, 1, \dots, \ell.$$

So we only need to consider the coefficient of $[u_1^\ell][u_2^\ell]$, which is equal to

$$\begin{aligned} & \pi((2\ell)!)^{-1} |\lambda|^{2\ell} |\nu|^2 \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j}^2 (\ell-j)!(\ell+j-1)! \\ &= \frac{\pi(\ell!)^2 |\lambda|^{2\ell} |\nu|^2}{\ell(2\ell)!} {}_2F_1(\ell, -\ell; 1; 1), \end{aligned}$$

which is 0 when $\ell \leq 1$. \square

3.2.3 The $i = 2$ -term

This term vanishes when we have the condition $\ell > n + 1$ (by analyzing the singularity of $K_v(x)$ at $x = 0$).

3.3 Rank 1 Fourier coefficients

For rank 1 Fourier coefficients, one needs to calculate

$$E_1^T(g, s) = \int_{\mathbf{N}_T(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell_T)ng, s) dn$$

for any nonzero isotropic vector T .

3.3.1 Non-archimedean components

For any prime p , we identify $\mathbf{N}_T(\mathbb{Q}_p) \backslash \mathbf{N}(\mathbb{Q}_p)$ with E_p via the map $n(v, \lambda) \mapsto -\langle T, v \rangle$, thus

$$\begin{aligned} E_{1,p}^T(1, s) &= \int_{x \in E_p} \int_{t \in E_p^\times} \psi_E(x) |t|_E^s \Phi_p(tx b_1 + tT) dt dx \\ &= \int_{x \in E_p} \int_{t \in E_p^\times} \psi_E(x/t) |t|_E^{s-1} \Phi_p(x b_1 + tT) dt dx \end{aligned}$$

3.3.1.1 Split case When p is split in E , we write T as (T_1, T_2) with $T_i \in \mathbb{Z}_p^n, i = 1, 2$. Let $\alpha = v_p(T_1)$ and $\beta = v_p(T_2)$, then we have:

$$\begin{aligned} E_{1,p}^T &:= E_{1,p}^T(1, s = \ell + 1) \\ &= \int_{(x_1, x_2) \in \mathbb{Q}_p^2} \int_{(t_1, t_2) \in (\mathbb{Q}_p \times \mathbb{Q}_p)^\times} \psi_p(x_1/t_1 + x_2/t_2) |t_1 t_2|_p^{s-1} \Phi_p((x_1, x_2)b_1 + (t_1, t_2)T) dt_1 dt_2 dx_1 dx_2 \\ &= \sum_{\substack{r_1 \geq -\alpha \\ r_2 \geq -\beta}} p^{-\ell(r_1+r_2)} \int_{(x_1, x_2) \in \mathbb{Z}_p^2} \psi_p(x_1/p^{r_1}) \psi_p(x_2/p^{r_2}) dx_1 dx_2 \\ &= \sum_{\substack{r_1 \geq -\alpha \\ r_2 \geq -\beta}} p^{-\ell(r_1+r_2)} \left(\frac{1}{p^{r_1}} \sum_{x_1 \in \mathbb{Z}/p^{r_1}\mathbb{Z}} \psi_p(x_1/p^{r_1}) \right) \left(\frac{1}{p^{r_2}} \sum_{x_2 \in \mathbb{Z}/p^{r_2}\mathbb{Z}} \psi_p(x_2/p^{r_2}) \right) \\ &= (1 + p^\ell + p^{2\ell} + \cdots + p^{\alpha\ell})(1 + p^\ell + p^{2\ell} + \cdots + p^{\beta\ell}). \end{aligned}$$

3.3.1.2 Inert case When p is inert in E and $p \neq 2$, we have:

$$\begin{aligned} E_{1,p}^T &= \sum_{r \geq -v_p(T)} p^{-2r\ell} \int_{x \in \mathcal{O}_{E_p}} \psi_E(x/p^r) dx \\ &= 1 + p^{2\ell} + p^{4\ell} + \cdots + p^{2\ell v_p(T)}. \end{aligned}$$

3.3.1.3 Ramified case

3.3.2 Archimedean components

Again, we identify $\mathbf{N}_T(\mathbb{R}) \setminus \mathbf{N}(\mathbb{R})$ with \mathbb{C} via the map $n(v, \lambda) \mapsto -\langle T, v \rangle$, and for any $z \in \mathbb{C}$ we choose an element $n(z) \in \mathbf{N}(\mathbb{R})$ corresponding to z under this identification.

Lemma 3.6. *For any $z \in \mathbb{C}$, we have*

$$f_{\ell, \infty}(\gamma(\ell_T)n(z), s = \ell + 1) = \frac{(zu_1 + \beta u_2)^\ell (-\bar{\beta}u_1 + \bar{z}u_2)^\ell}{(|z|^2 + |\beta|^2)^{2\ell+1} (\ell!)^2},$$

where $\beta = \beta(T) := \sqrt{2}\langle T, u_2 \rangle$.

Proof. Suppose that $\gamma(\ell_T)n(z) = pk$ for $p \in \mathbf{P}(\mathbb{R})$ and $k = (k_+, k_-) \in K_\infty = \mathbf{U}(V_2^+) \times \mathbf{U}(V_n^-)$, then one has

$$T + z b_1 = b_1 \gamma(\ell_T)n(z) = b_1 p k = \nu(p)^{-1} b_1 k.$$

Taking the projection to V_2^+ of both sides, we obtain that

$$zu_1 + \sqrt{2}\langle T, u_2 \rangle u_2 = \nu(p)^{-1} u_1 k_+.$$

One can take $\nu(p)$ to be $(|z|^2 + |\beta|^2)^{-1/2}$, then

$$k_+ = \nu(p) \begin{pmatrix} z & -\bar{\beta} \\ \beta & \bar{z} \end{pmatrix} \in \mathbf{U}(V_2^+),$$

which gives us the desired value of $f_{\ell, \infty}$. \square

Using Lemma 3.6, the coefficient of $[u_1^\ell][u_2^\ell]$ in $E_{1, \infty}^T := E_{1, \infty}^T(1, s = \ell + 1)$ is equal to

$$\int_{z \in \mathbb{C}} e^{2\pi i \operatorname{Re}(z)} \frac{\sum_{k=0}^{\ell} \binom{\ell}{k}^2 |z|^{2k} (-|\beta|^2)^{\ell-k}}{(|z|^2 + |\beta|^2)^{2\ell+1}} dz. \quad (2)$$

We set $B = |\beta|^2$ and $z = x + iy$, where $x, y \in \mathbb{R}$, then the integral in Equation (2) equals

$$\begin{aligned} & \int_{\mathbb{R}^2} e^{2\pi ix} \sum_{k=0}^{\ell} \binom{\ell}{k}^2 (-B)^{\ell-k} \frac{(x^2 + y^2)^k}{(x^2 + y^2 + B)^{2\ell+1}} dx dy \\ &= \int_{\mathbb{R}} e^{2\pi ix} \sum_{k=0}^{\ell} \binom{\ell}{k}^2 (-B)^{\ell-k} \left(\int_{\mathbb{R}} \frac{(y^2 + x^2)}{(y^2 + x^2 + B)^{2\ell+1}} dy \right) dx \end{aligned}$$

By Lemma 3.12, the inner integral over $y \in \mathbb{R}$ equals

$$\frac{(x^2 + B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^k \binom{k}{j} \left(\frac{x^2}{x^2 + B} \right)^{k-j} \Gamma(j + 1/2) \Gamma(2\ell - j + 1/2).$$

Similarly to our calculation in §3.5, we have

$$E_{1,\infty}^T = 2^{-2\ell} \pi \sum_{k=0}^{\ell} \frac{(-B)^k 2^{-2k} (2\ell + 2k)!}{(k!)^2 (\ell+k)! (\ell-k)!} \int_{\mathbb{R}} \frac{e^{2\pi i x} dx}{(x^2 + B)^{\ell+k+1/2}}$$

Using Bassett's integral, one has

$$\int_{\mathbb{R}} \frac{e^{iwt} dt}{(t^2 + z^2)^{n+1/2}} = \frac{2\sqrt{\pi}}{\Gamma(n+1/2)} \left(\frac{|w|}{2|z|} \right)^n K_n(|wz|), w, z \in \mathbb{R}.$$

Plugging this into $E_{1,\infty}^T$, we obtain:

$$\begin{aligned} E_{1,T}^{\infty} &= 2^{-2\ell} \pi \sum_{k=0}^{\ell} \frac{(-B)^k 2^{-2k} (2\ell + 2k)!}{(k!)^2 (\ell+k)! (\ell-k)!} \cdot \frac{2^{\ell+k+1} (\ell+k)! B^{-(\ell+k)/2} (2\pi)^{\ell+k} K_{\ell+k}(2\pi\sqrt{B})}{(2\ell+2k)!} \\ &= 2\pi B^{-\ell} \sum_{k=0}^{\ell} \frac{(-1)^k}{(k!)^2 (\ell-k)!} (\pi\sqrt{B})^{\ell+k} K_{\ell+k}(2\pi\sqrt{B}). \end{aligned}$$

Lemma 3.7. *For any $C > 0$, we have*

$$S_{\ell} := \sum_{k=0}^{\ell} \frac{(-1)^k C^{\ell+k} K_{\ell+k}(2C)}{(k!)^2 (\ell-k)!} = \frac{(-1)^{\ell} C^{2\ell}}{(\ell!)^2} K_0(2C).$$

Proof. Using the recurrence relation for K-Bessel functions:

$$K_{n+1}(z) = K_{n-1}(z) + \frac{2n}{z} K_n(z),$$

we can write $-C^2 S_{\ell}$ as

$$\begin{aligned} -C^2 S_{\ell} &= \sum_{k=0}^{\ell} (-1)^{k+1} \frac{C^{\ell+k+2}}{(k!)^2 (\ell-k)!} K_{\ell+k}(2C) \\ &= \sum_{j=1}^{\ell+1} (-1)^j \frac{C^{\ell+1+j}}{((j-1)!)^2 (\ell-j+1)!} \left(K_{\ell+1+j}(2C) - \frac{\ell+j}{C} K_{\ell+j}(2C) \right) \\ &= \sum_{j=1}^{\ell+1} \frac{(-1)^j C^{\ell+1+j} K_{\ell+1+j}(2C)}{((j-1)!)^2 (\ell-j+1)!} + \sum_{j=0}^{\ell} \frac{(-1)^j (\ell+j+1) C^{\ell+1+j} K_{\ell+j+1}(2C)}{(j!)^2 (\ell-j)!} \\ &= \frac{(\ell+1) C^{\ell+1} K_{\ell+1}(2C)}{\ell!} + \sum_{j=1}^{\ell} \frac{(-1)^j C^{\ell+1+j} K_{\ell+1+j}(2C)}{(j!)^2 (\ell+1-j)!} (\ell+1)^2 + (-1)^{\ell+1} \frac{C^{2\ell+2} K_{2\ell+2}(2C)}{(\ell!)^2} \\ &= (\ell+1)^2 S_{\ell+1}. \end{aligned}$$

Since $S_0 = K_0(2C)$ and $S_{\ell+1} = -\frac{C^2}{(\ell+1)^2} S_{\ell}$, we have

$$S_{\ell} = (-1)^{\ell} \frac{C^{2\ell}}{(\ell!)^2} K_0(2C). \quad \square$$

By this lemma, one has

$$E_{1,T}^{\infty} = 2\pi B^{-\ell} \cdot \frac{(-1)^{\ell} (\pi\sqrt{B})^{2\ell}}{(\ell!)^2} K_0(2\pi\sqrt{B}) = \frac{2\pi^{2\ell+1}}{(\ell!)^2} K_0(2\pi\sqrt{B}).$$

3.4 Rank 2 Fourier coefficients: non-archimedean components

One decomposes $E_2^T(1, s)$ as $\prod_v E_{2,v}^T(s)$.

$$\begin{aligned} E_{2,p}^T(s) &= \int_{\mathbf{N}(\mathbb{Q}_p)} \chi_T^{-1}(n) f_p(w_2 n, s) dn \\ &= \int_{\mathbf{N}(\mathbb{Q}_p)} \int_{t \in E_p^\times} \chi_T^{-1}(n) |t|^s \Phi_p(t \cdot b_1 w_2 n) dt dn \\ &= \int_{v \in \mathbf{V}_0(\mathbb{Q}_p)} \int_{\substack{x \in E_p \\ \bar{x} = -x}} \int_{t \in E_p^\times} \chi_T^{-1}(v) |t|^s \Phi_p(t ((-\langle v, v \rangle / 2 + x) b_1 + b_2 + v)) dt dx dv \end{aligned}$$

Take \mathcal{O}_{E_p} to be $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_E \subseteq E_p$, \mathfrak{p} to be its maximal ideal, and \mathcal{V} an \mathcal{O}_{E_p} -lattice of $V_0(\mathbb{Q}_p) = V_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that $\mathcal{V} \otimes_{\mathcal{O}_{E_p}} (\mathcal{O}_{E,p}/\mathfrak{p})$ is a non-degenerate Hermitian space over $\mathcal{O}_{E_p}/\mathfrak{p}$. Assume that Φ_p is the characteristic function of the lattice $\mathcal{O}_{E_p} b_1 \oplus \mathcal{V} \oplus \mathcal{O}_{E_p} b_2$.

Write down a precise definition of \mathcal{V} ?

Lemma 3.8. (1) If p splits in E , then

$$E_{2,p}^T(s) = \sum_{r_1, r_2 \geq 0} p^{-(r_1+r_2)s + \min(r_1, r_2)} \left(\int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\max(r_1, r_2)} \langle v, v \rangle \in \mathbb{Z}_p) dv \right).$$

(2) If p is inert in E , then

$$E_{2,p}^T(s) = \sum_{r \geq 0} p^{-2rs+r} \left(\int_{v \in p^{-r}\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^r \langle v, v \rangle / 2 \in \mathbb{Z}_p) dv \right).$$

(3) If p is ramified in E , then

Proof. (1) If p splits in E , then $E_p = \mathbb{Q}_p \times \mathbb{Q}_p$:

$$E_{2,p}^T(s) = \sum_{r_1 \geq 0, r_2 \geq 0} |p|^{(r_1+r_2)s} \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \left(\int_{\substack{x \in (p^{-r_1}, p^{-r_2})\mathcal{O} + \langle v, v \rangle / 2 \\ x + \bar{x} = 0}} dx \right) dv$$

If we write $x = (y, -y) \in E_p$, then $x \in (p^{-r_1}, p^{-r_2})\mathcal{O} + \langle v, v \rangle / 2$ is equivalent to

$$y \in p^{-r_1}\mathbb{Z}_p + \langle v, v \rangle / 2, -y \in p^{-r_2}\mathbb{Z}_p + \langle v, v \rangle / 2.$$

There are such x only if $\langle v, v \rangle \in p^{-\max(r_1, r_2)}\mathbb{Z}_p$. If $r_1 \geq r_2$, then $y \in p^{-r_2}\tilde{y} - \langle v, v \rangle / 2$ for some $\tilde{y} \in \mathbb{Z}_p$. Since the measure on the line $\{x = (y, -y) \in E_p\}$ is the normalized Haar measure dy on \mathbb{Q}_p , one has $dy = p^{r_2}d\tilde{y}$, and

$$\int_{x=(y,-y) \in p^{-r_1}\mathbb{Z}_p \times p^{-r_2}\mathbb{Z}_p + \langle v, v \rangle / 2} dx = \int_{\tilde{y} \in \mathbb{Z}_p} p^{r_2} d\tilde{y} = p^{r_2} = p^{\min(r_1, r_2)}.$$

Hence

$$E_{2,p}^T(s) = \sum_{r_1, r_2 \geq 0} p^{-(r_1+r_2)s + \min(r_1, r_2)} \left(\int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\max(r_1, r_2)} \langle v, v \rangle \in \mathbb{Z}_p) dv \right).$$

(2) If p is inert in E , then E_p is an unramified quadratic field extension of \mathbb{Q}_p , and

$$E_{2,p}^T(s) = \sum_{r \geq 0} |p|_{E_p}^{rs} \int_{v \in p^{-r}\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^r \langle v, v \rangle \in \mathbb{Z}_p) \left(\int_{x \in \langle v, v \rangle / 2 + p^{-r}\mathcal{O}_{E_p}} \text{Char}(x + \bar{x} = 0) dx \right) dv$$

Choose a unit $u \in \mathcal{O}_{E_p}^\times$ with $u + \bar{u} = 0$, then the line $\{x \in E_p \mid x + \bar{x} = 0\}$ can be written as $\{yu, y \in \mathbb{Q}_p\}$, with the normalized Haar measure dy . There exist elements $yu \in \langle v, v \rangle / 2 + p^{-r}\mathcal{O}_{E_p}$ if and only if $\langle v, v \rangle / 2 \in p^{-r}\mathbb{Z}_p$. When we have $\langle v, v \rangle / 2 \in p^{-r}\mathbb{Z}_p$, any element $yu \in \langle v, v \rangle / 2 + p^{-r}\mathcal{O}_{E_p}$ is of the form

$$yu = \langle v, v \rangle / 2 + p^{-r}(-p^r \langle v, v \rangle / 2 + \tilde{y}u), \quad \tilde{y} \in \mathbb{Z}_p,$$

thus one has

$$\begin{aligned} \int_{x \in \langle v, v \rangle / 2 + p^{-r}\mathcal{O}_{E_p}} \text{Char}(x + \bar{x} = 0) dx &= \text{Char}(p^r \langle v, v \rangle / 2 \in \mathbb{Z}_p) |p|_{\mathbb{Q}_p}^{-r} \int_{\mathbb{Z}_p} d\tilde{y} \\ &= p^r \text{Char}(p^r \langle v, v \rangle / 2 \in \mathbb{Z}_p), \end{aligned}$$

which gives us the desired identity. \square

If p is ramified in E : (Assume $p \neq 2$) Then E_p is a ramified quadratic field extension of \mathbb{Q}_p , and we choose an uniformizer ω of $\mathfrak{p} \subset \mathcal{O}_{E_p}$ such that $\omega + \bar{\omega} = 0$ and $\omega^2 \in p\mathbb{Z}_p^\times$. The integral for $E_{2,p}^T$ can be rewritten as:

$$\begin{aligned} E_{2,p}^T(s) &= \sum_{r \geq 0} |\omega|^{rs} \int_{v \in \omega^{-r}\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\lfloor r/2 \rfloor} \langle v, v \rangle \in \mathbb{Z}_p) \left(\int_{x \in \mathbb{Q}_p} \text{Char}(x\omega - \frac{\langle v, v \rangle}{2} \in \omega^{-r}\mathcal{O}_{E_p}) dx \right) dv \\ &= \sum_{r \geq 0} |\omega|^{rs} p^{\lceil r/2 \rceil} \left(\int_{v \in \omega^{-r}\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\lfloor r/2 \rfloor} \langle v, v \rangle \in \mathbb{Z}_p) dv \right) \end{aligned}$$

3.4.1 Split case

When p splits in E , then one can view the Hermitian $\mathbb{Q}_p \times \mathbb{Q}_p$ -space $V_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ as the direct sum of two copies of quadratic \mathbb{Q}_p -space $(\mathbb{Q}_p^n, ((x_i), (y_i)) = \sum x_i y_i)$, and one has

$$2\text{Re}\langle (x_1, y_1), (x_2, y_2) \rangle = (x_1, y_2) + (x_2, y_1).$$

Denote by L the lattice \mathbb{Z}_p^n in \mathbb{Q}_p^n . We set

$$\begin{aligned} S(r_1, r_2) &:= \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \text{Char}(p^{\max(r_1, r_2)} \langle v, v \rangle \in \mathbb{Z}_p) dv \\ &= \int_{\substack{x \in p^{-r_1}L \\ y \in p^{-r_2}L}} \psi_p^{-1}((x, T_2) + (y, T_1)) \text{Char}(p^{\max(r_1, r_2)}(x, y) \in \mathbb{Z}_p) dx dy, \end{aligned}$$

for any integers $r_1, r_2 \geq 0$.

Proposition 3.9. Let p be a prime split in E , and $r_1 \leq r_2$ be two natural numbers. For a vector $T = (T_1, T_2) \in \mathcal{V}$ with $\alpha = v_p(T_1), \beta = v_p(T_2), v_p(\langle T, T \rangle) = \gamma$ and $B = B(r_1, r_2) := \min(r_1, \beta, \alpha + r_1 - r_2)$, one has

$$S(r_1, r_2) = p^{r_2 n - r_1} \left(-\text{Char}(\gamma < r_2 + B) \cdot p^{r_1 - 1 + (n-1)(\gamma - r_2 + 1)} + \sum_{k=0}^{\min(B, \gamma - r_2)} p^{kn} \phi(p^{r_1 - k}) \right),$$

which is nonzero only if $r_2 \leq \min(\gamma + 1, r_2 + \alpha)$. In particular, if $\langle T, T \rangle \in \mathcal{O}_{E_p}^\times$, then the nonzero terms are

$$S(0, 0) = 1 \text{ and } S(1, 1) = -p^{n-1}.$$

Proof. The integral $S(r_1, r_2)$ can be rewritten as an exponential sum:

$$S(r_1, r_2) = \sum_{\substack{x \in L/p^{r_1}L \\ y \in L/p^{r_2}L}} \psi_p \left(\frac{(x, T_2)}{p^{r_1}} + \frac{(y, T_1)}{p^{r_2}} \right) \text{Char}((x, y) \in p^{r_1} \mathbb{Z}_p).$$

The characteristic function has the following expression:

$$\text{Char}((x, y) \in p^{r_1} \mathbb{Z}_p) = p^{-r_1} \sum_{u \in \mathbb{Z}/p^{r_1} \mathbb{Z}} \psi_p \left(\frac{u(x, y)}{p^{r_1}} \right),$$

thus

$$S(r_1, r_2) = p^{-r_1} \sum_{\substack{y \in L/p^{r_2}L \\ u \in \mathbb{Z}/p^{r_1} \mathbb{Z}}} \psi_p^{-1} \left(\frac{(y, T_1)}{p^{r_2}} \right) \left(\sum_{x \in L/p^{r_1}L} \psi_p^{-1} \left(\frac{(x, T_2 - uy)}{p^{r_1}} \right) \right).$$

The inner sum equals $p^{r_1 n}$ when $T_2 - uy \in p^{r_1} L$, otherwise it equals 0. Hence one has

$$S(r_1, r_2) = p^{r_1(n-1)} \sum_{\substack{y \in L/p^{r_2}L \\ u \in \mathbb{Z}/p^{r_1} \mathbb{Z}}} \psi_p^{-1} \left(\frac{(y, T_1)}{p^{r_2}} \right) \text{Char}(T_2 - uy \in p^{r_1} L).$$

The equation $T_2 = uy \pmod{p^{r_1} L}$ has solution if and only if $k := v_p(u) \leq v_p(T_2) = \beta$, and in this case, $y = u^{-1} T_2 + p^{r_1 - k} z$ for $z \in L/p^{r_2 - r_1 + k} L$. Plugging this in the exponential sum and replacing u by $p^k u$, we obtain that

$$\begin{aligned} S(r_1, r_2) &= p^{r_1(n-1)} \sum_{k=0}^{\min(r_1, \beta)} \sum_{u \in (\mathbb{Z}/p^{r_1-k} \mathbb{Z})^\times} \sum_{z \in L/p^{r_2-r_1+k} L} \psi_p^{-1} \left(\frac{(T_1, u^{-1} p^{-k} T_2 + p^{r_1-k} z)}{p^{r_2}} \right) \\ &= p^{r_1(n-1)} \sum_{k=0}^{\min(r_1, \beta)} \sum_{u \in (\mathbb{Z}/p^{r_1-k} \mathbb{Z})^\times} \psi_p^{-1} \left(\frac{u^{-1} (T_1, T_2)}{p^{r_2+k}} \right) \left(\sum_{z \in L/p^{r_2-r_1+k} L} \psi_p^{-1} \left(\frac{(z, T_1)}{p^{r_2-r_1+k}} \right) \right) \\ &= p^{r_1(n-1)} \sum_{k=0}^{\min(r_1, \beta)} p^{(r_2-r_1+k)n} \text{Char}(\alpha \geq r_2 - r_1 + k) \left(\sum_{u \in (\mathbb{Z}/p^{r_1-k} \mathbb{Z})^\times} \psi_p^{-1} \left(\frac{u^{-1}}{p^{r_2+k-\gamma}} \right) \right) \end{aligned}$$

For the inner sum, we have

$$\sum_{u \in (\mathbb{Z}/p^{r_1-k}\mathbb{Z})^\times} \psi_p^{-1} \left(\frac{u^{-1}}{p^{r_2+k-\gamma}} \right) = \begin{cases} \phi(p^{r_1-k}) = |(\mathbb{Z}/p^{r_1-k})^\times| & , \text{ if } r_2 + k - \gamma \leq 0 \\ -p^{r_1-k-1} & , \text{ if } r_2 + k - \gamma = 1 \\ 0 & , \text{ if } r_2 + k - \gamma > 1 \end{cases}$$

Combining all these together, we get the following formula: let $B = B(r_1, r_2, \alpha, \beta)$ be $\min(r_1, \beta, \alpha + r_1 - r_2)$, then

$$S(r_1, r_2) = p^{r_2 n - r_1} \left(-\text{Char}(\gamma < r_2 + B) \cdot p^{r_1 - 1 + (n-1)(\gamma - r_2 + 1)} + \sum_{k=0}^{\min(B, \gamma - r_2)} p^{kn} \phi(p^{r_1-k}) \right).$$

This is nonzero only if $r_1 \leq r_2 \leq \min(\gamma + 1, r_1 + \alpha)$, which only hold for finitely many terms. Particularly, if $T = (T_1, T_2)$ is unramified, then $\alpha = \beta = \gamma = 0$, thus $r_1 = r_2 \leq 1$. In this case we have

$$S(0, 0) = 1, S(1, 1) = -p^{n-1}.$$

□

3.5 Rank 2 Fourier coefficients: archimedean components

Remark 3.10. In this article, when we write $|z|$ for $z \in \mathbb{C}$, it means the *norm of z with respect to the extension \mathbb{C}/\mathbb{R}* , i.e. $|z| = z\bar{z}$ instead of the usual modulus $\sqrt{z\bar{z}}$. I admit that this is somehow strange and confusing, but I will fix this problem if this draft could become a paper...

We first analyze the function $f_{\ell, \infty}(w_2 n, s = \ell + 1)$:

Lemma 3.11. *For any $v \in V_0 \otimes_{\mathbb{Q}} \mathbb{R}$ and $x \in \mathbb{R}$, we set:*

$$\alpha(v, x) = -\frac{\langle v, v \rangle}{2} + ix + 1, \beta(v) = \sqrt{2} \langle v, u_2 \rangle,$$

then we have

$$f_{\ell, \infty}(w_2 n(v, ix), s) = \frac{(\alpha u_1 + \beta u_2)^\ell (-\bar{\beta} u_1 + \bar{\alpha} u_2)^\ell}{(|\alpha(v, x)| + |\beta(v)|)^{\ell+s} (\ell!)^2}.$$

Proof. One has

$$b_1 w_2 n(v, ix) = (-\langle v, v \rangle / 2 + ix) b_1 + b_2 + v.$$

Suppose that we can decompose $w_2 n(v, ix)$ as $p k$ for some $p \in \mathbf{P}(\mathbb{R})$ and $k \in K_\infty$, then

$$b_1 w_2 n(v, ix) k^{-1} = b_1 p = v(p)^{-1} b_1. \quad (3)$$

Let k_+ be the factor of k in $U(V_2^+)$, and $v = v_+ + v_- \in \mathbb{C} u_2 \oplus \text{Span}_{\mathbb{C}}(v_1, \dots, v_{n-1})$. Taking the V_2^+ components of Equation (3), one gets:

$$\frac{1}{\sqrt{2}} (\alpha(v, x) u_1 + \beta(v) u_2) = \left[\frac{1}{\sqrt{2}} \left(-\frac{\langle v, v \rangle}{2} + ix + 1 \right) u_1 + v_+ \right] k_+^{-1} = v(p)^{-1} \frac{u_1}{\sqrt{2}}.$$

The norms of both sides give us the identity $|\alpha(v, x)| + |\beta(v)| = |\nu(p)|^{-1}$. One may assume that $\nu(p)^{-1} = \sqrt{|\alpha(v, x)| + |\beta(v)|}$, then in the basis of u_1, u_2 , the element $k^+ \in U(V_2^+)$ can be written as the Hermitian matrix

$$\frac{1}{\sqrt{|\alpha(v, x)| + |\beta(v)|}} \begin{pmatrix} \alpha(v, x) & -\overline{\beta(v)} \\ \beta(v) & \overline{\alpha(v, x)} \end{pmatrix}.$$

Plug k^+ and $\nu(p)$ into $f_{\ell, \infty}(w_2 n(v, ix), s) = f_{\ell, \infty}(pk, s) = |\nu(p)|^s [u_1^\ell][u_2^\ell].k^+$, and we get the desired value. \square

Let $I_0(T; \ell)$ be the coefficient of $[u_1^\ell][u_2^\ell]$ in $E_{2, \infty}^T(s = \ell + 1)$, which can be written as

$$\begin{aligned} I_0(T; \ell) &= \int_{v \in V_0 \otimes Q \mathbb{R}} \int_{x \in \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} \frac{\binom{\ell}{k} \binom{\ell}{k} (x^2 + A)^k (-B)^{\ell-k}}{(x^2 + A + B)^{2\ell+1}} dx dv \\ &= \int_{v \in V_0 \otimes \mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \int_{x \in \mathbb{R}} \frac{(x^2 + A)^k}{(x^2 + A + B)^{2\ell+1}} dx dv \end{aligned}$$

where $A = |\alpha| - x^2 = (1 - \langle v, v \rangle / 2)^2$, and $B = |\beta| = 2|\langle v, u_2 \rangle|$.

Lemma 3.12. *For any real number C, D and two natural numbers $m < n$, we have*

$$\int_{\mathbb{R}} \frac{(x^2 + C)^m}{(x^2 + D)^n} dx = \frac{D^{m-n+1/2}}{(n-1)!} \sum_{k=0}^m \binom{m}{k} \left(\frac{C}{D}\right)^{m-k} \Gamma(k+1/2) \Gamma(n-k-1/2).$$

Proof. An exercise of calculus. \square

Now Lemma 3.12 tells us $I_0(T; \ell)$ is the Fourier transform of the function

$$F_{0,\ell}(v) := \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \frac{(A+B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^k \binom{k}{j} \left(\frac{A}{A+B}\right)^{k-j} \Gamma(j+1/2) \Gamma(2\ell+1/2-j)$$

Set $z = B/(A+B)$, then this function becomes

$$\frac{2^{-4\ell}}{(2\ell)!} \pi (A+B)^{-\ell-1/2} \sum_{k=0}^{\ell} \sum_{j=0}^k \binom{\ell}{k}^2 \binom{k}{j} \frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!} (-z)^{\ell-k} (1-z)^{k-j}$$

We write:

$$\sum_{k=0}^{\ell} \sum_{j=0}^k c_{j,k} (-z)^{\ell-k} (1-z)^{k-j} = \sum_{r=0}^{\ell} (-1)^r C(r) z^r,$$

where $c_{j,k} = \binom{\ell}{k}^2 \binom{k}{j} \frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!}$. The term $(-z)^{\ell-k} (1-z)^{k-j}$ has a non-zero z^r term if and only if $\ell - k \leq r \leq \ell - j$, thus

$$(-1)^r C(r) = \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} c_{j,k} \binom{k-j}{r-\ell+k} (-1)^{\ell-k+r-\ell+k} = (-1)^r \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} c_{j,k} \binom{k-j}{r-\ell+k}.$$

To compute $C(r)$, we need the following lemma:

Lemma 3.13. (1) For integers $0 \leq a \leq b$, one has

$$\sum_{i=a}^b \binom{b}{i} \binom{b-a}{b-i} = \binom{2b-a}{b}.$$

(2) For any integer $0 \leq r \leq \ell$, one has

$$\sum_{i=0}^{\ell-r} \frac{\binom{2\ell}{i} \binom{\ell-r}{i}}{\binom{4\ell}{2i}} = 2^{2\ell-2r} \frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}.$$

Proof. The identity in (1) is obvious. For the identity in (2), the LHS can be rewritten as:

$$\sum_{i=0}^{\ell-r} \frac{(-\ell+r)_j (1/2)_j}{(-2\ell+1/2)_{j!}} = {}_2F_1(-(\ell-r), 1/2; -2\ell+1/2; 1),$$

where $(x)_j$ is the (rising) Pochhammer symbol, and ${}_2F_1$ is the hypergeometric function. By Chu-Vandermonde identity, this value of hypergeometric function is

$$\frac{(-2\ell)_{\ell-r}}{(-2\ell+1/2)_{\ell-r}} = 2^{\ell-r} \frac{\frac{(2\ell)!}{(\ell+r)!}}{\frac{(4\ell-1)!!}{(2\ell+2r-1)!!}} = 2^{2\ell-2r} \frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}. \quad \square$$

Now we return to the value of $C(r)$:

$$\begin{aligned} C(r) &= \sum_{j=0}^{\ell-r} \sum_{k=\ell-r}^{\ell} \frac{(\ell!)^2 (2j)! (4\ell-2j)!}{k! ((\ell-k)!)^2 (j!)^2 (r-\ell+k)! (\ell-r-j)! (2\ell-j)!} \\ &= \sum_{j=0}^{\ell-r} \frac{\ell! (2j)! (4\ell-2j)!}{r! (j!)^2 (\ell-r-j)! (2\ell-j)!} \sum_{k=\ell-r}^{\ell} \binom{\ell}{k} \binom{r}{\ell-k} \\ (\text{by (1) of Lemma 3.13}) &= \frac{\ell!}{r!} \cdot \frac{(\ell+r)!}{\ell! r!} \sum_{j=0}^{\ell-r} \frac{(2j)! (4\ell-2j)!}{(j!)^2 (\ell-r-j)! (2\ell-j)!} \\ &= \frac{(\ell+r)!}{(r!)^2} \cdot \frac{(4\ell)!}{(\ell-r)! (2\ell)!} \sum_{j=0}^{\ell-r} \frac{\binom{2\ell}{j} \binom{\ell-r}{j}}{\binom{4\ell}{2j}} \\ (\text{by (2) of Lemma 3.13}) &= \frac{(\ell+r)!}{(r!)^2} \cdot \frac{(4\ell)!}{(\ell-r)! (2\ell)!} \cdot 2^{2\ell-2r} \frac{\frac{(2\ell+2r)!}{((\ell+r)!)^2}}{\frac{(4\ell)!}{((2\ell)!)^2}} \\ &= 2^{2\ell-2r} \frac{(2\ell)! (2\ell+2r)!}{(r!)^2 (\ell+r)! (\ell-r)!}. \end{aligned}$$

Putting the value of $C(r)$ into $F_{0,\ell}$, we have

$$\begin{aligned} F_{0,\ell}(v) &= \frac{2^{-4\ell}\pi}{(2\ell)!(A+B)^{\ell+1/2}} \sum_{r=0}^{\ell} (-1)^r 2^{2\ell-2r} \frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!} z^r \\ &= \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}} \sum_{r=0}^{\ell} \frac{(-\ell)_r (\ell+1/2)_r}{1_r} \frac{z^r}{r!} \\ &= \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}} \cdot {}_2F_1(-\ell, \ell+1/2; 1; z). \end{aligned}$$

We have shown the following result:

Proposition 3.14. For $T \in V_0$ with $\langle T, T \rangle > 0$, the coefficient $I_0(T; \ell)$ of $[u_1^\ell][u_2^\ell]$ in $E_{2,\infty}^T(s = \ell + 1)$ is the Fourier transform of the function:

$$F_{0,\ell}(v) = \frac{2^{-3\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}} \cdot {}_2F_1(-\ell, \ell+1/2; 1; \frac{B}{A+B}),$$

where $A = \left(1 - \frac{\langle v, v \rangle}{2}\right)^2$ and $B = 2|\langle v, u_2 \rangle|^2$.

Let's recall the following result for quadratic spaces by Pollack: I know there are notation problems again... just let it be like this for now

Proposition 3.15. ($v = 0$ case of [PS22, Proposition 4.5.3]) Let $(V', (\cdot, \cdot)) = V_2 \oplus V_n$ be a non-degenerate quadratic space over \mathbb{R} of signature $(2, n)$, and v_1, v_2 an orthonormal basis of V_2 . Set

$$I_0(x; \ell) = \int_{V'} e^{i(\omega, x)} \text{Char}(q(\omega) > 0) q(\omega)^{\ell-n/2} K_0(\sqrt{2}|\langle \omega, v_1 + \sqrt{-1}v_2 \rangle|) d\omega.$$

This integral is absolutely convergent and we have

$$I_0(x; \ell) = (2\pi)^{(n+2)/2} 2^{\ell-1-n/2} \Gamma(\ell+1) \Gamma(\ell+1-n/2) F_4(\ell+1, \ell+1; \ell+1; -\|x_n\|^2/2; -\|x_2\|^2/2),$$

where $x = x_2 + x_n$, and $F_4(a, b; c; d; x; y)$ is Appell's hypergeometric function.

Now we look at how this result fits into our unitary group setting. One take V' to be $(V_0, \text{Re}(\langle \cdot, \cdot \rangle))$, as a quadratic space of signature $(2, 2n-2)$, $v'_1 = u_2$, $v'_2 = iu_2$, then

$$\begin{aligned} I_0(x; \ell) &= \int_{T \in V_0, \langle T, T \rangle > 0} e^{i\text{Re}\langle T, x \rangle} \langle T, T \rangle^{\ell-n+1} K_0(\sqrt{2}|\langle T, u_2 \rangle|) dT \\ &= \int_{T \in V_0, \langle T, T \rangle > 0} e^{2\pi i \text{Re}\langle T, x \rangle} (2\pi)^{2\ell-2n+2} \langle T, T \rangle^{\ell-n+1} K_0(2\sqrt{2}\pi|\langle T, u_2 \rangle|) (2\pi)^{2n} dT \\ &= (2\pi)^{2\ell+2} \int_{T \in V_0, \langle T, T \rangle > 0} e^{2\pi i \text{Re}\langle T, x \rangle} \langle T, T \rangle^{\ell-n+1} K_0(2\sqrt{2}\pi|\langle T, u_2 \rangle|) dT, \end{aligned}$$

here $|z| = \sqrt{z\bar{z}}$, and for any $\ell > n-1$ we have

$$I_0(x; \ell) = (2\pi)^n 2^{\ell-n} \ell! (\ell-n+1)! (A+B)^{-(\ell+1)/2} {}_2F_1(-\ell/2, (\ell+1)/2; 1; B/(A+B)).$$

Replace ℓ by 2ℓ (now $\ell > \frac{n-1}{2}$), and we have

$$I_0(x; 2\ell) = (2\pi)^n 2^{2\ell-n} (2\ell)! (2\ell - n + 1)! (A + B)^{-\ell-1/2} \cdot {}_2F_1(-\ell, \ell + 1/2; 1; B/(A + B)).$$

Comparing this with Proposition 3.14, we get the following theorem:

Theorem 3.16. *For $T \in V_0$ with $\langle T, T \rangle > 0$ and $\ell > \frac{n-1}{2}$, we have*

$$I_0(T; \ell) = \frac{2^{-\ell-2} \pi^{4\ell-n+3} \langle T, T \rangle^{2\ell-n+1}}{\ell! (2\ell - n + 1)!} \mathcal{W}_T(1).$$

4 Our progress

If there is no mistake in the previous calculations, that means

$$\begin{aligned} E_\ell(g, \Phi_f, s = \ell + 1) \\ = f_\ell(g, s = \ell + 1) + \pi^{2\ell+1} \left(\sum_{\substack{T \neq 0 \in \mathcal{V} \\ \langle T, T \rangle = 0}} \square \cdot \sigma_E(T) \mathcal{W}_T(g) + \sum_{\substack{T \in \mathcal{V} \\ \langle T, T \rangle > 0}} \square \cdot \langle T, T \rangle^{2\ell-n+1} \mathcal{W}_T(g) \right), \end{aligned}$$

where $f_\ell(g, s = \ell + 1) = \zeta_E(\ell + 1) |\nu(g)|_E^{\ell+1} [u_1^n] [u_2^n]$ and \square means certain rational factor.

References

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