Rationality of the Fourier expansion of quaternionic Heisenberg Eisenstein series on $\mathbf{U}(2,n)$

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Abstract

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1 Introduction

2 Preliminaries on the unitary groups of split rank 2

Fix an imaginary quadratic field extension E of $\mathbb{Q}$, as a subfield of $\mathbb{C}$. Denote the nonzero element in $\mathrm{Gal}(E/\mathbb{Q})$ by $x \mapsto \overline{x}$, and for any $\mathbb{Q}$-algebra R we extend this involution to $E \otimes_{\mathbb{Q}} R$ and define $\mathrm{Tr}_{E\otimes_{\mathbb{Q}} R} : E \otimes_{\mathbb{Q}} R \to R$ to be the trace map $x \mapsto x + \overline{x}$.

For any place v of $\mathbb{Q}$, let $|\,|_{\mathbb{Q}_v}$ be the normalized absolute value on the local field $\mathbb{Q}_v$. Set $E_v := E \otimes_{\mathbb{Q}} \mathbb{Q}_v$, and denote by $|\,|_{E_v}$ the absolute value on E_v defined by $x \mapsto |x\overline{x}|_{\mathbb{Q}_v}$.

Denote by $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ (*resp.* $\mathbb{A}_E$) the adèle ring of $\mathbb{Q}$ (*resp.* E). Fix an additive $\psi = \psi_{\mathbb{Q}}$ on $\mathbb{Q}\backslash\mathbb{A}$ satisfying that for every finite prime p, ψ_p is an additive character of $\mathbb{Q}_p$ with conductor $\mathbb{Z}_p$, and at the archimedean place, $\psi_\infty(x) = e^{2\pi i x}$. Set ψ_E to be the additive character of $E\backslash\mathbb{A}_E$ that sends $x \in E$ to $\psi(\mathrm{Tr}_{\mathbb{A}_E}(x))$.

2.1 Hermitian spaces

For any natural number $n \geq 2$, let $(\mathbf{V}, \langle\, , \rangle)$ be a non-degenerate Hermitian space over E with signature $(2,n)$, where the Hermitian form is conjugate-linear in the second variable. For any $\mathbb{Q}$-algebra R, we denote by $\mathbf{V}(R)$ the Hermitian module $\mathbf{V} \otimes_{\mathbb{Q}} R$ over $E \otimes_{\mathbb{Q}} R$.

Fix a pair of isotropic vectors (b_1, b_2) in $\mathbf{V}$ such that $\langle b_1, b_2 \rangle = 1$, and take $\mathbf{V}_0$ to be the orthogonal complement of $Eb_1 \oplus Eb_2$ in $\mathbf{V}$, which is a non-degenerate Hermitian space of signature $(1, n-1)$. We fix again a pair of isotropic vectors (c_1, c_2) in $\mathbf{V}_0$ with $\langle c_1, c_2 \rangle = 1$, then $\mathbf{V}$ is the orthogonal direct sum of $Eb_1 \oplus Eb_2 \oplus Ec_1 \oplus Ec_2$ and a Hermitian space $\mathbf{V}_{00}$ with signature $(0, n-2)$.

Take $u_1 = \frac{1}{\sqrt{2}}(b_1 + b_2)$, $u_2 = \frac{1}{\sqrt{2}}(c_1 + c_2)$, $v_n = \frac{1}{\sqrt{2}}(b_1 - b_2)$ and $v_{n-1} = \frac{1}{\sqrt{2}}(c_1 - c_2)$, and they form an orthogonal basis of $\mathbb{C}b_1 \oplus \mathbb{C}b_2 \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \subseteq V := \mathbf{V}(\mathbb{R})$. Fix a basis $\{v_1, \ldots, v_{n-2}\}$ of $V_{00} := \mathbf{V}_{00}(\mathbb{R})$ such that $\langle v_i, v_j \rangle = -\delta_{ij}$ for any $1 \le i, j \le n-2$. We have obtained a basis of V with the following properties:

- $\langle u_i, u_j \rangle = \delta_{ij}$, for any $1 \le i, j \le 2$;
- $\langle v_i, v_j \rangle = -\delta_{ij}$, for any $1 \le i, j \le n$;
- $\langle u_i, v_j \rangle = 0$ for any $1 \le i \le 2$ and $1 \le j \le n$;
- $\{u_2, v_1, \ldots, v_{n-1}\}$ is a basis for $V_0 := \mathbf{V}_0(\mathbb{R})$.

Set $V_2^+ = \mathbb{C}u_1 \oplus \mathbb{C}u_2$ and $V_n^- = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n$, and we have a decomposition $V_2^+ \oplus V_n^-$ of V into definite subspaces.

2.2 Unitary group $\mathbf{U}(2, n)$ and its Heisenberg parabolic subgroup

Define $\mathbf{G} = \mathbf{U}(\mathbf{V}) = \mathbf{U}(2, n)$ to be the unitary group of the Hermitian space $(\mathbf{V}, \langle\, , \rangle)$, and we write the $\mathbf{G}$-action on $\mathbf{V}$ as a right action.

The Heisenberg parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is defined as the stabilizer of the isotropic line spanned by b_1, and it admits the following Levi decomposition $\mathbf{P} = \mathbf{MN}$:

- The Levi factor $\mathbf{M}$ is chosen to be the stabilizer in $\mathbf{P}$ of the isotropic line Eb_2. By its action on $Eb_1 \oplus \mathbf{V}_0 \oplus Eb_2$, one identifies this subgroup with $\mathrm{Res}_{E/\mathbb{Q}}(\mathbf{G}_\mathrm{m}) \times \mathbf{U}(\mathbf{V}_0)$, so that $(z, h) \in \mathrm{Res}_{E/\mathbb{Q}}(\mathbf{G}_\mathrm{m}) \times \mathbf{U}(\mathbf{V}_0) \simeq \mathbf{M}$ acts on $\mathbf{V}$ by
$$(xb_1 + w + yb_2) \cdot (z, h) = z^{-1}xb_1 + wh + \bar{z}yb_2, \text{ for any } x, y \in \mathbf{G}_\mathrm{a},\ w \in \mathbf{V}_0.$$

- The unipotent radical $\mathbf{N}$ is a two-step nilpotent group, and it is isomorphic to the group
$$\left\{(v, \lambda) \in \mathbf{V}_0 \times \mathrm{Res}_{E/\mathbb{Q}}\mathbf{G}_\mathrm{a} \,\middle|\, \overline{\lambda} = -\lambda\right\},$$
equipped with the following multiplication:
$$(v_1, \lambda_1) \cdot (v_2, \lambda_2) := \left(v_1 + v_2, \lambda_1 + \lambda_2 - \frac{\langle v_1, v_2 \rangle - \langle v_2, v_1 \rangle}{2}\right).$$
The action of the corresponding element $n(v, \lambda) \in \mathbf{N}$ on $\mathbf{V}$ is given as:
$$b_1 \mapsto b_1,\ b_2 \mapsto (-\frac{1}{2}\langle v, v \rangle + \lambda)b_1 + b_2 + v,\ w \in \mathbf{V}_0 \mapsto -\langle w, v \rangle b_1 + w.$$
The center of $\mathbf{N}$ is $\mathbf{Z} := \left\{n(0, \lambda) \,\middle|\, \overline{\lambda} = -\lambda\right\}$, which has dimension 1.

The conjugation action of $\mathbf{M}$ on $\mathbf{N}$ is given by
$$(z, h)n(v, \lambda)(z^{-1}, h^{-1}) = n(\bar{z}vh^{-1}, |z|\lambda).$$

Denote by ν the similitude character of $\mathbf{P}$:

$$\nu\left((z,h)n(v,\lambda)\right) = z \in \mathrm{Res}_{E/\mathbb{Q}}\mathbf{G}_{\mathrm{m}},\ \text{for any } (z,h) \in \mathbf{M},\ n(v,\lambda) \in \mathbf{N},$$

then we have $b_1 p = \nu(p)^{-1} b_1$ for any $p \in \mathbf{P}$.

Using the unitary character $\psi_E : E\backslash\mathbb{A}_E \to \mathbb{S}^1 \subset \mathbb{C}^\times$, we identify $\mathbf{V}_0$ with $\mathrm{Hom}([\mathbf{N}], \mathbb{S}^1)$ by sending $T \in \mathbf{V}_0$ to the unitary character $\chi_T(n(v,\lambda)) = \psi_E(\langle T, v\rangle) = \psi(\mathrm{Tr}_{\mathbb{A}_E}\langle T, v\rangle)$. Its archimedean analogue is

$$V_0 \xrightarrow{\simeq} \mathrm{Hom}(\mathbf{N}(\mathbb{R}), \mathbb{S}^1),\ T \mapsto \left(\chi_{T,\infty} : v \mapsto e^{2\pi i(\langle T,v\rangle + \langle v,T\rangle)}\right).$$

3 Quaternionic Heisenberg Eisenstein series

3.1 Quaternionic modular forms on $\mathbf{U}(2,n)$

We first recall from [HMY24, §3] the theory of quaternionic modular forms on the unitary group $\mathbf{G} = \mathbf{U}(2,n)$.

In §2.1, we have a decomposition $V = V_2^+ \oplus V_n^-$ of $V = \mathbf{V}(\mathbb{R})$ into definite subspaces. The group $K_\infty = \mathrm{U}(V_2^+) \times \mathrm{U}(V_n^-) \simeq \mathrm{U}(2) \times \mathrm{U}(n)$ is a maximal compact subgroup of $\mathrm{U}(2,n) := \mathbf{G}(\mathbb{R})$. For any integer $\ell \geq 1$, consider the following irreducible representation of $K_\infty = \mathrm{U}(V_2^+) \times \mathrm{U}(V_n^-)$:

$$\mathbb{V}_\ell := \left(\mathrm{Sym}^{2\ell} V_2^+ \otimes \det{}_{\mathbf{U}(V_2^+)}^{-\ell}\right) \boxtimes \mathbf{1}$$

We fix a basis $\{[u_1^{\ell-v}][u_2^{\ell+v}],\ -\ell \leq v \leq \ell\}$ of $\mathbb{V}_\ell$, where $[u_i^j] := u_i^j/j!$.

When $\ell \geq n$, Gross and Wallach [GW96] construct an irreducible unitary discrete series representation Π_ℓ of $\mathbf{G}(\mathbb{R})$, which contains $\mathbf{V}_\ell$ as its minimal K_∞-type with multiplicity 1, and two Schmid operators $\mathcal{D}_\ell^\pm$ associated with Π_ℓ are constructed in [HMY24, §3.1].

Definition 3.1. [HMY24, Definition 3.1] A *weight ℓ quaternionic modular form on* $\mathbf{G} = \mathbf{U}(2,n)$ is a smooth function $F : \mathbf{G}(\mathbb{A}) \to \mathbb{V}_\ell$ of moderate growth such that:

(1) For any $g \in \mathbf{G}(\mathbb{A})$ and $\gamma \in \mathbf{G}(\mathbb{Q})$, $F(\gamma g) = F(g)$.

(2) For every $k \in K_\infty$ and $g \in \mathbf{G}(\mathbb{A})$, $F(gk) = F(g) \cdot k$.

(3) The functions $\mathcal{D}_\ell^\pm\left(F|_{\mathbf{G}(\mathbb{R})}\right)$ vanish identically on $\mathbf{G}(\mathbb{R})$.

Let F be a weight ℓ quaternionic modular form on $\mathbf{G}$. If $T \in \mathbf{V}_0$, then the *T-th Fourier coefficient of F along* $\mathbf{N}$ is

$$F_{\mathbf{N},T} : \mathbf{G}(\mathbb{A}) \to \mathbb{V}_\ell,\ g \mapsto \int_{[\mathbf{N}]} F(ng)\overline{\chi_T(n)}dn.$$

We have the following result on the Fourier expansion of F:

Theorem 3.2. *[HMY24, Corollary 3.7] Let $F : \mathbf{G}(\mathbb{A}) \to \mathbb{V}_\ell$ be a weight ℓ quaternionic modular form, and $F_{\mathbf{Z}}$* (resp. *$F_{\mathbf{N}}$) the constant term of F along $\mathbf{Z}$* (resp. $\mathbf{N}$). *There exist locally constant functions*

$$\{a_T(F,\cdot) : \mathbf{G}(\mathbb{A}_f) \to \mathbb{C}\}_{T\in \mathbf{V}_0,\, \langle T,T\rangle \geq 0}$$

such that for any $g_f \in \mathbf{G}(\mathbb{A}_f)$ and $g_\infty \in \mathbf{G}(\mathbb{R})$ one has:

$$F_{\mathbf{Z}}(g_f g_\infty) = F_{\mathbf{N}}(g_f g_\infty) + \sum_{\substack{T\in \mathbf{V}_0 \\ \langle T,T\rangle \geq 0}} a_T(F, g_f)\mathcal{W}_T(g_\infty), \tag{3.1}$$

where $\mathcal{W}_T$ is the $\mathbb{V}_\ell$-valued function on $\mathbf{G}(\mathbb{R})$ given by the formula

$$\mathcal{W}_T\,(nmk) = \chi_{T,\infty}(n) \sum_{v=-\ell}^{\ell} |\nu(m)|^{2\ell+2} \left(\frac{|\beta_T(m)|}{\beta_T(m)}\right)^{v} K_v\left(|\beta_T(m)|\right) [u_1^{\ell-v}][u_2^{\ell+v}] \cdot k, \tag{3.2}$$

for any $n \in \mathbf{N}(\mathbb{R}), m \in \mathbf{M}(\mathbb{R})$ and $k \in K_\infty$. Here K_v denotes the K-Bessel function $K_v(x) = \frac{1}{2}\int_0^\infty t^{v-1}e^{-x(t+t^{-1})}dt$ and $\beta_T : \mathbf{M}(\mathbb{R}) \to \mathbb{C}$ is defined by mapping (z,h) to $4\sqrt{2}\pi\langle u_2, zT\cdot h\rangle$.

Remark 3.3. Notice that in [HMY24], the function $\beta_T(z,h)$ is defined as $2\sqrt{2}\pi\langle u_2, zT\cdot h\rangle$. This is because their additive character ψ_E is chosen to be $\psi_E(x) = \psi(\frac{1}{2}\operatorname{Tr}_{\mathbb{A}_E}(x))$.

3.2 Heisenberg Eisenstein series

For $\ell \geq n$, the quaternionic discrete series Π_ℓ can be embedded into the (unnormalized) degenerate principal series $\operatorname{Ind}_{\mathbf{P}(\mathbb{R})}^{\mathbf{G}(\mathbb{R})}|\nu|_{\mathbb{C}}^{\ell+1}$ [Wal03, Theorem 9] (by our notation $|z|_{\mathbb{C}} = z\bar{z} = |z|^2$), thus we now construct an Eisenstein series on $\mathbf{G}$ from some section of $\operatorname{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}|\nu|_{\mathbb{A}_E}^{s}$, $s \in \mathbb{C}$.

Suppose that $\Phi_f = \otimes_{p<\infty}\Phi_p$ is a Schwartz-Bruhat function on $\mathbf{V}(\mathbb{A}_f)$. We choose a section $f_\ell(g_f g_\infty, \Phi_f, s) = f_{fte}(g_f, \Phi_f, s) f_{\ell,\infty}(g_\infty, s)$ of $\operatorname{Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})}|\nu|^s$ as follows:

- The archimedean part $f_{\ell,\infty}$ is the $\mathbb{V}_\ell$-valued, K_∞-equivariant induced section whose restriction to $\mathbf{M}(\mathbb{R})$ satisfies $f_{\ell,\infty}((z,h),s) = |z|_{\mathbb{C}}^{s}[u_1^\ell][u_2^\ell] = |z|^{2s}[u_1^\ell][u_2^\ell]$ for any $(z,h) \in \mathbf{M}(\mathbb{R})$.
- The finite part f_{fte} is defined by $f_{fte}(g_f, \Phi_f, s) = \int_{\mathbf{GL}_1(\mathbb{A}_{E,f})} |t|_{\mathbb{A}_{E,f}}^{s}\Phi_f(t\cdot b_1 g_f)dt$.

We set the (Heisenberg) Eisenstein series

$$E_\ell(g, \Phi_f, s) = \sum_{\gamma\in \mathbf{P}(\mathbb{Q})\backslash \mathbf{G}(\mathbb{Q})} f_\ell(\gamma g, \Phi_f, s).$$

Proposition 3.4. *For $\ell \geq n$, the Eisenstein series $E_\ell(g, \Phi_f, s = \ell+1)$ is a weight ℓ quaternionic modular form on $\mathbf{G} = \mathbf{U}(2,n)$.*

Proof. The only condition in Definition 3.1 that is not obvious is the third one. It suffices to show that the restriction of $f_{\ell,\infty}$ to $\mathbf{M}(\mathbb{R})$ is killed by the Schmid operators $\mathcal{D}_\ell^\pm$.

Write an element $(z,h) \in \mathbf{M}(\mathbb{R})$ as $m = (h,r,\theta)$ so that $z = re^{i\theta}$, $r \in \mathbb{R}_{>0}$ and $\theta \in [0,2\pi)$. Using this coordinates, we define a function $F(h,r,\theta) = r^{2s}$ on $\mathbf{M}(\mathbb{R})$ so that $f_{\ell,\infty}(m) = F(m)[u_1^\ell][u_2^\ell]$. By [HMY24, Proposition 3.10], $f_{\ell,\infty}$ is killed by $\mathcal{D}_\ell^\pm$ if and only if $(r\partial r - 2(\ell+1))F \equiv 0$ on $\mathbf{M}(\mathbb{R})$, which holds when $s = \ell+1$. □

PROGRESS MARK

3.3 Abstract Fourier expansion

In this subsection, we give the "abstract" Fourier expansion of $E_\ell(g,\Phi_f,s)$.

Lemma 3.5. *The right $\mathbf{P}(\mathbb{Q})$-space $\mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})$, the space of isotropic lines in $V(\mathbb{Q})$, has exactly 3 orbits of $\mathbf{P}(\mathbb{Q})$, represented respectively by $\mathbb{Q}b_1$, $\mathbb{Q}v_0$ and $\mathbb{Q}b_2$, where v_0 is an arbitrary non-zero isotropic vector in $V_0(\mathbb{Q})$.*

Proof. Directly by the explicit action given in §2. □

Set $\mathbf{G}(\mathbb{Q}) = \bigsqcup_{i=0}^2 \mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})$, such that $w_0 = 1$, $b_1w_1 = v_0$ and $b_1w_2 = b_2$. Now we can write the degenerate Eisenstein series as

$$E_\ell(g,\Phi_f,s) = \sum_{i=0}^{2} E_{\ell,i}(g,\Phi_f,s),\ E_{\ell,i}(g,\Phi_f,s) = \sum_{\gamma\in\mathbf{P}(\mathbb{Q})\backslash\mathbf{P}(\mathbb{Q})w_i\mathbf{P}(\mathbb{Q})} f(\gamma g,s),$$

thus $E_{\ell,0}(g,\Phi_f,s) = f(g,s)$. From now on, when there is no confusion we will omit the ℓ and Φ_f in $E_\ell(g,\Phi_f,s)$, and write it as $E(g,s) = \sum_{i=0}^2 E_i(g,s)$.

Lemma 3.6. *Assume that $\mathrm{Re}(s) \gg 0$ so that the sum defining $E(g,s)$ converges absolutely. Then one has the following expressions for the $E_i(g,s)$:*

(1) Let $\mathcal{L}_0$ be the set of non-zero isotropic lines ℓ in V_0 and for any $\ell \in \mathcal{L}_0$, select $\gamma(\ell) \in \mathbf{G}(\mathbb{Q})$ with $b_1\gamma(\ell) \in \ell$. Then

$$E_1(g,s) = \sum_{\ell\in\mathcal{L}_0}\ \sum_{\mu\in(\ell)^\perp\mathbf{N}_0(\mathbb{Q})\backslash\mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu g,s).$$

(2) One has

$$E_2(g,s) = \sum_{\mu\in\mathbf{N}(\mathbb{Q})} f(w_2\mu g,s).$$

For any $T \in V_0$, we set

$$E_i^T(g,s) = \int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n)E_i(ng,s)dn,\ i = 0,1,2.$$

Lemma 3.7. *(1) If T is anisotropic, then $E_1^T = 0$. If T is isotropic, define $\mathbf{N}_T = (\ell_T)^\perp\mathbf{N}_0 \subseteq \mathbf{N}$, then*

$$E_1^T(g,s) = \int_{\mathbf{N}_T(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n)f(\gamma(\ell_T)ng,s)dn.$$

(2) For any $T \in V_0$, one has

$$E_2^T(g,s) = \int_{\mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(w_2 n g, s) dn.$$

Proof. It suffices only to prove the $i = 1$ case. For any $\ell \in \mathcal{L}_0$, set

$$\mathbf{N}_\ell = \left\{ n(v,\lambda) \in \mathbf{N} \,\middle|\, v \in \ell^\perp \right\} \subseteq \mathbf{N}.$$

For any $T \in V_0$,

$$\begin{aligned} E_1^T(g,s) &= \sum_{\ell \in \mathcal{L}_0} \int_{[N]} \chi_T^{-1}(n) \left(\sum_{\mu \in \mathbf{N}_\ell(\mathbb{Q}) \backslash \mathbf{N}(\mathbb{Q})} f(\gamma(\ell)\mu n g, s) \right) dn \\ &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{Q}) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell) n g, s) dn \\ &= \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} \left(\int_{[\mathbf{N}_\ell]} \chi_T^{-1}(r) dr \right) \chi_T^{-1}(n) f(\gamma(\ell) n g, s) dn \\ &= \sum_{\ell \in \mathcal{L}_0, \chi_T|_{\mathbf{N}_\ell} \equiv 1} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} \chi_T^{-1}(n) f(\gamma(\ell) n g, s) dn. \end{aligned}$$

Then the lemma follows from the fact that $\chi_T|_{\mathbf{N}_\ell} \equiv 1$ if and only if $T \in \ell$, *i.e.* T is isotropic and $\ell = \ell_T$. □

4 Rank 2 Fourier coefficients of E_ℓ

4.1 Rank 2 Fourier coefficients: non-archimedean components

One decomposes $E_2^T(1,s)$ as $\prod_v E_{2,v}^T(s)$.

$$\begin{aligned} E_{2,p}^T(s) &= \int_{\mathbf{N}(\mathbb{Q}_p)} \chi_T^{-1}(n) f_p(w_2 n, s) dn \\ &= \int_{\mathbf{N}(\mathbb{Q}_p)} \int_{t \in E_p^\times} \chi_T^{-1}(n) |t|^s \Phi_p(t \cdot b_1 w_2 n) dt dn \\ &= \int_{v \in \mathbf{V}_0(\mathbb{Q}_p)} \int_{\substack{x \in E_p \\ \overline{x} = -x}} \int_{t \in E_p^\times} \chi_T^{-1}(v) |t|^s \Phi_p\left(t\left(\left(-\langle v,v\rangle/2 + x\right) b_1 + b_2 + v\right)\right) dt dx dv \end{aligned}$$

Take $\mathcal{O}_{E_p}$ to be $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_E \subseteq E_p$, $\mathfrak{p}$ to be its maximal ideal, and $\mathcal{V}$ an $\mathcal{O}_{E_p}$-lattice of $V_0(\mathbb{Q}_p) = V_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that $\mathcal{V} \otimes_{\mathcal{O}_{E_p}} (\mathcal{O}_{E,p}/\mathfrak{p})$ is a non-degenerate Hermitian space over $\mathcal{O}_{E_p}/\mathfrak{p}$. Assume that Φ_p is the characteristic function of the lattice $\mathcal{O}_{E_p} b_1 \oplus \mathcal{V} \oplus \mathcal{O}_{E_p} b_2$.

Write down a precise definition of $\mathcal{V}$?

Lemma 4.1. *(1) If p splits in E, then*

$$E_{2,p}^{T}(s)=\sum_{r_1,r_2\geq 0}p^{-(r_1+r_2)s+\min(r_1,r_2)}\left(\int_{v\in(p^{-r_1},p^{-r_2})\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^{\max(r_1,r_2)}\langle v,v\rangle\in\mathbb{Z}_p)dv\right).$$

(2) If p is inert in E, then

$$E_{2,p}^{T}(s)=\sum_{r\geq 0}p^{-2rs+r}\left(\int_{v\in p^{-r}\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}\left(p^r\langle v,v\rangle/2\in\mathbb{Z}_p\right)dv\right).$$

(3) If p is ramified in E, then

Proof. (1) If p splits in E, then $E_p=\mathbb{Q}_p\times\mathbb{Q}_p$:

$$E_{2,p}^{T}(s)=\sum_{r_1\geq 0,r_2\geq 0}|p|^{(r_1+r_2)s}\int_{v\in(p^{-r_1},p^{-r_2})\mathcal{V}}\chi_T^{-1}(v)\left(\int_{\substack{x\in(p^{-r_1},p^{-r_2})\mathcal{O}+\langle v,v\rangle/2\\ x+\overline{x}=0}}dx\right)dv$$

If we write $x=(y,-y)\in E_p$, then $x\in(p^{-r_1},p^{-r_2})\mathcal{O}+\langle v,v\rangle/2$ is equivalent to

$$y\in p^{-r_1}\mathbb{Z}_p+\langle v,v\rangle/2,\ -y\in p^{-r_2}\mathbb{Z}_p+\langle v,v\rangle/2.$$

There are such x only if $\langle v,v\rangle\in p^{-\max(r_1,r_2)}\mathbb{Z}_p$, If $r_1\geq r_2$, then $y\in p^{-r_2}\widetilde{y}-\langle v,v\rangle/2$ for some $\widetilde{y}\in\mathbb{Z}_p$. Since the measure on the line $\{x=(y,-y)\in E_p\}$ is the normalized Haar measure dy on $\mathbb{Q}_p$, one has $dy=p^{r_2}d\widetilde{y}$, and

$$\int_{x=(y,-y)\in p^{-r_1}\mathbb{Z}_p\times p^{-r_2}\mathbb{Z}_p+\langle v,v\rangle/2}dx=\int_{\widetilde{y}\in\mathbb{Z}_p}p^{r_2}d\widetilde{y}=p^{r_2}=p^{\min(r_1,r_2)}.$$

Hence

$$E_{2,p}^{T}(s)=\sum_{r_1,r_2\geq 0}p^{-(r_1+r_2)s+\min(r_1,r_2)}\left(\int_{v\in(p^{-r_1},p^{-r_2})\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^{\max(r_1,r_2)}\langle v,v\rangle\in\mathbb{Z}_p)dv\right).$$

(2) If p is inert in E, then E_p is an unramified quadratic field extension of $\mathbb{Q}_p$, and

$$E_{2,p}^{T}(s)=\sum_{r\geq 0}|p|_{E_p}^{rs}\int_{v\in p^{-r}\mathcal{V}}\chi_T^{-1}(v)\mathrm{Char}(p^r\langle v,v\rangle\in\mathbb{Z}_p)\left(\int_{x\in\langle v,v\rangle/2+p^{-r}\mathcal{O}_{E_p}}\mathrm{Char}(x+\overline{x}=0)dx\right)dv$$

Choose a unit $u\in\mathcal{O}_{E_p}^{\times}$ with $u+\overline{u}=0$, then the line $\left\{x\in E_p\,\middle|\,x+\overline{x}=0\right\}$ can be written as $\{yu,\ y\in\mathbb{Q}_p\}$, with the normalized Haar measure dy. There exist elements $yu\in\langle v,v\rangle/2+p^{-r}\mathcal{O}_{E_p}$ if and only if $\langle v,v\rangle/2\in p^{-r}\mathbb{Z}_p$. When we have $\langle v,v\rangle/2\in p^{-r}\mathbb{Z}_p$, any element $yu\in\langle v,v\rangle/2+p^{-r}\mathcal{O}_{E_p}$ is of the form

$$yu=\langle v,v\rangle/2+p^{-r}\left(-p^r\langle v,v\rangle/2+\widetilde{y}u\right),\ \widetilde{y}\in\mathbb{Z}_p,$$

thus one has

$$\begin{aligned}\int_{x\in\langle v,v\rangle/2+p^{-r}\mathcal{O}_{E_p}}\mathrm{Char}(x+\overline{x}=0)dx&=\mathrm{Char}(p^r\langle v,v\rangle/2\in\mathbb{Z}_p)|p|_{\mathbb{Q}_p}^{-r}\int_{\mathbb{Z}_p}d\widetilde{y}\\&=p^r\mathrm{Char}(p^r\langle v,v\rangle/2\in\mathbb{Z}_p),\end{aligned}$$

which gives us the desired identity. □

If p is ramified in E: (Assume $p \neq 2$) E_p is a ramified quadratic field extension of $\mathbb{Q}_p$, and we choose an uniformizer ϖ of $\mathfrak{p} \subset \mathcal{O}_{E_p}$ such that $\varpi + \overline{\varpi} = 0$ and $\varpi^2 = p$. Then the integral $E^T_{2,p}(s)$ equals

$$\sum_{r \geq 0} p^{-rs} \int_{v \in \varpi^{-r}\mathcal{V}} \chi_T^{-1}(v) \int_{x \in \mathbb{Q}_p} \mathrm{Char}(x\varpi - \frac{\langle v, v\rangle}{2} \in \varpi^{-r}\mathcal{O}_{E_p}) dx dv.$$

Suppose that $x\varpi - \langle v, v\rangle/2 = \varpi^{-r}(y + \varpi z)$ for $y, z \in Z_p$.

- If r is even, one has $x\varpi - \langle v, v\rangle/2 = (p^{-r/2}z)\varpi + p^{-r/2}y$, which implies that $\langle v, v\rangle \in p^{-r/2}\mathbb{Z}_p$ and $x \in p^{-r/2}\mathbb{Z}_p$.
- If r is odd, one has $x\varpi - \langle v, v\rangle/2 = (p^{-(r+1)/2}y)\varpi + p^{-(r-1)/2}z$, which implies that $\langle v, v\rangle \in p^{-(r-1)/2}\mathbb{Z}_p$ and $x \in p^{-(r+1)/2}\mathbb{Z}_p$.

Hence

$$E^T_{2,p}(s) = \sum_{r \geq 0} p^{-rs} p^{\lceil r/2 \rceil} \int_{v \in \varpi^{-r}\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^{\lfloor r/2 \rfloor}\langle v, v\rangle \in \mathbb{Z}_p) dv.$$

4.1.1 Split case

When p splits in E, then one can view the Hermitian $\mathbb{Q}_p \times \mathbb{Q}_p$-space $V_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ as the direct sum of two copies of quadratic $\mathbb{Q}_p$-space $\left(\mathbb{Q}_p^n, ((x_i), (y_i)) = \sum x_i y_i\right)$, and one has

$$2\mathrm{Re}\langle (x_1, y_1), (x_2, y_2)\rangle = (x_1, y_2) + (x_2, y_1).$$

Denote by L the lattice $\mathbb{Z}_p^n$ in $\mathbb{Q}_p^n$. We set

$$\begin{aligned} S(r_1, r_2) &:= \int_{v \in (p^{-r_1}, p^{-r_2})\mathcal{V}} \chi_T^{-1}(v) \mathrm{Char}(p^{\max(r_1, r_2)}\langle v, v\rangle \in \mathbb{Z}_p) dv \\ &= \int_{\substack{x \in p^{-r_1}L \\ y \in p^{-r_2}L}} \psi_p^{-1}\left((x, T_2) + (y, T_1)\right) \mathrm{Char}(p^{\max(r_1, r_2)}(x, y) \in \mathbb{Z}_p) dx dy, \end{aligned}$$

for any integers $r_1, r_2 \geq 0$.

Proposition 4.2. *Let p be a prime split in E, and $r_1 \leq r_2$ be two natural numbers. For a vector $T = (T_1, T_2) \in \mathcal{V}$ with $\alpha = v_p(T_1), \beta = v_p(T_2), v_p(\langle T, T\rangle) = \gamma$ and $B = B(r_1, r_2) := \min(r_1, \beta, \alpha + r_1 - r_2)$, one has*

$$S(r_1, r_2) = p^{r_2 n - r_1}\left(-\mathrm{Char}(\gamma < r_2 + B) \cdot p^{r_1 - 1 + (n-1)(\gamma - r_2 + 1)} + \sum_{k=0}^{\min(B, \gamma - r_2)} p^{kn}\phi(p^{r_1 - k})\right),$$

which is nonzero only if $r_2 \leq \min(\gamma + 1, r_2 + \alpha)$. In particular, if $\langle T, T\rangle \in \mathcal{O}_{E_p}^\times$, then the nonzero terms are

$$S(0, 0) = 1 \text{ and } S(1, 1) = -p^{n-1}.$$

Proof. The integral $S(r_1, r_2)$ can be rewritten as an exponential sum:

$$S(r_1, r_2) = \sum_{\substack{x \in L/p^{r_1}L \\ y \in L/p^{r_2}L}} \psi_p \left(\frac{(x, T_2)}{p^{r_1}} + \frac{(y, T_1)}{p^{r_2}} \right) \mathrm{Char}\left((x, y) \in p^{r_1}\mathbb{Z}_p \right).$$

The characteristic function has the following expression:

$$\mathrm{Char}\left((x, y) \in p^{r_1}\mathbb{Z}_p \right) = p^{-r_1} \sum_{u \in \mathbb{Z}/p^{r_1}\mathbb{Z}} \psi_p \left(\frac{u(x, y)}{p^{r_1}} \right),$$

thus

$$S(r_1, r_2) = p^{-r_1} \sum_{\substack{y \in L/p^{r_2}L \\ u \in \mathbb{Z}/p^{r_1}\mathbb{Z}}} \psi_p^{-1} \left(\frac{(y, T_1)}{p^{r_2}} \right) \left(\sum_{x \in L/p^{r_1}L} \psi_p^{-1} \left(\frac{(x, T_2 - uy)}{p^{r_1}} \right) \right).$$

The inner sum equals $p^{r_1 n}$ when $T_2 - uy \in p^{r_1}L$, otherwise it equals 0. Hence one has

$$S(r_1, r_2) = p^{r_1(n-1)} \sum_{\substack{y \in L/p^{r_2}L \\ u \in \mathbb{Z}/p^{r_1}\mathbb{Z}}} \psi_p^{-1} \left(\frac{(y, T_1)}{p^{r_2}} \right) \mathrm{Char}(T_2 - uy \in p^{r_1}L).$$

The equation $T_2 = uy \bmod p^{r_1}L$ has solution if and only if $k := v_p(u) \leq v_p(T_2) = \beta$, and in this case, $y = u^{-1}T_2 + p^{r_1-k}z$ for $z \in L/p^{r_2-r_1+k}L$. Plugging this in the exponential sum and replacing u by $p^k u$, we obtain that

$$\begin{aligned}
S(r_1, r_2) &= p^{r_1(n-1)} \sum_{k=0}^{\min(r_1, \beta)} \sum_{u \in (\mathbb{Z}/p^{r_1-k}\mathbb{Z})^\times} \sum_{z \in L/p^{r_2-r_1+k}L} \psi_p^{-1} \left(\frac{(T_1, u^{-1}p^{-k}T_2 + p^{r_1-k}z)}{p^{r_2}} \right) \\
&= p^{r_1(n-1)} \sum_{k=0}^{\min(r_1, \beta)} \sum_{u \in (\mathbb{Z}/p^{r_1-k}\mathbb{Z})^\times} \psi_p^{-1} \left(\frac{u^{-1}(T_1, T_2)}{p^{r_2+k}} \right) \left(\sum_{z \in L/p^{r_2-r_1+k}L} \psi_p^{-1} \left(\frac{(z, T_1)}{p^{r_2-r_1+k}} \right) \right) \\
&= p^{r_1(n-1)} \sum_{k=0}^{\min(r_1, \beta)} p^{(r_2-r_1+k)n} \mathrm{Char}(\alpha \geq r_2 - r_1 + k) \left(\sum_{u \in (\mathbb{Z}/p^{r_1-k}\mathbb{Z})^\times} \psi_p^{-1} \left(\frac{u^{-1}}{p^{r_2+k-\gamma}} \right) \right)
\end{aligned}$$

For the inner sum, we have

$$\sum_{u \in (\mathbb{Z}/p^{r_1-k}\mathbb{Z})^\times} \psi_p^{-1} \left(\frac{u^{-1}}{p^{r_2+k-\gamma}} \right) = \begin{cases} \phi(p^{r_1-k}) = |(\mathbb{Z}/p^{r_1-k})^\times| & \text{, if } r_2 + k - \gamma \leq 0 \\ -p^{r_1-k-1} & \text{, if } r_2 + k - \gamma = 1 \\ 0 & \text{, if } r_2 + k - \gamma > 1 \end{cases}$$

Combining all these together, we get the following formula: let $B = B(r_1, r_2, \alpha, \beta)$ be $\min(r_1, \beta, \alpha + r_1 - r_2)$, then

$$S(r_1, r_2) = p^{r_2 n - r_1} \left(-\mathrm{Char}(\gamma < r_2 + B) \cdot p^{r_1 - 1 + (n-1)(\gamma - r_2 + 1)} + \sum_{k=0}^{\min(B, \gamma - r_2)} p^{kn} \phi(p^{r_1-k}) \right).$$

This is nonzero only if $r_1 \le r_2 \le \min(\gamma+1, r_1+\alpha)$, which only hold for finitely many terms. Particularly, if $T = (T_1, T_2)$ is unramified, then $\alpha = \beta = \gamma = 0$, thus $r_1 = r_2 \le 1$. In this case we have

$$S(0,0) = 1,\ S(1,1) = -p^{n-1}.$$

□

4.2 Rank 2 Fourier coefficients: archimedean components

We first analyze the function $f_{\ell,\infty}(w_2 n, s = \ell+1)$ on $n \in \mathbf{N}(\mathbb{R})$.

Lemma 4.3. *For any $v \in V_0 = \mathbf{V}_0(\mathbb{R})$ and $x \in \mathbb{R}$, set:*

$$\alpha(v,x) = -\frac{\langle v, v\rangle}{2} + ix + 1, \text{ and } \beta(v) = \sqrt{2}\langle v, u_2\rangle.$$

Then we have

$$f_{\ell,\infty}(w_2 n(v, ix), s) = \frac{(\alpha u_1 + \beta u_2)^\ell (-\overline{\beta} u_1 + \overline{\alpha} u_2)^\ell}{(|\alpha(v,x)|^2 + |\beta(v)|^2)^{\ell+s} (\ell!)^2}.$$

Proof. From the explicit action of $\mathbf{P}$ on $\mathbf{V}$, one has

$$b_1 w_2 n(v, ix) = (-\langle v, v\rangle/2 + ix) b_1 + b_2 + v.$$

Under the Iwasawa decomposition $\mathbf{G}(\mathbb{R}) = \mathbf{P}(\mathbb{R}) K_\infty$, we write $w_2 n(v, ix)$ as pk for some $p \in \mathbf{P}(\mathbb{R})$ and $k \in K_\infty$, then

$$b_1 w_2 n(v, ix) = \nu(p)^{-1}(b_1 k). \tag{4.1}$$

Let k_+ be the factor of k in $\mathrm{U}(V_2^+)$, and $v = v_+ + v_- \in \mathbb{C}u_2 \oplus \mathrm{Span}_{\mathbb{C}}(v_1, \ldots, v_{n-1})$. Taking the V_2^+ components of both sides of (4.1), we obtain:

$$\frac{1}{\sqrt{2}}(\alpha(v,x) u_1 + \beta(v) u_2) = \nu(p)^{-1} \frac{u_1 k_+}{\sqrt{2}}.$$

The norms of both sides give us the identity $|\alpha(v,x)|^2 + |\beta(v)|^2 = |\nu(p)|^{-2}$. One may assume that $\nu(p)^{-1} = \sqrt{|\alpha(v,x)|^2 + |\beta(v)|^2}$, then in the basis of u_1, u_2, the element $k^+ \in \mathrm{U}(V_2^+)$ can be written as the Hermitian matrix

$$\frac{1}{\sqrt{|\alpha(v,x)|^2 + |\beta(v)|^2}} \begin{pmatrix} \alpha(v,x) & -\overline{\beta(v)} \\ \beta(v) & \overline{\alpha(v,x)} \end{pmatrix}.$$

Plug k^+ and $\nu(p)$ into $f_{\ell,\infty}(w_2 n(v, ix), s) = f_{\ell,\infty}(pk, s) = |\nu(p)|^{2s}[u_1^\ell][u_2^\ell] \cdot k^+$, and we get the desired value. □

Using Lemma 4.3, one can write the archimedean coefficient $E^T_{2,\infty} := E^T_{2,\infty}(1, \ell+1)$ as a concrete integral over $v \in V_0$ and $x \in \mathbb{R}$. In particular, we have the following formula for the coefficient of $[u_1^\ell][u_2^\ell]$ of $E^T_{2,\infty}$, denoted by $I_0(T;\ell)$:

$$\begin{aligned} I_0(T;\ell) &= \int_{v\in V_0}\int_{x\in\mathbb{R}} \chi_T^{-1}(v) \sum_{k=0}^{\ell} \frac{\binom{\ell}{k}\binom{\ell}{k}(x^2+A)^k(-B)^{\ell-k}}{(x^2+A+B)^{2\ell+1}} dxdv \\ &= \int_{v\in V_0} \chi_T^{-1}(v) \sum_{k=0}^{\ell}(-B)^{\ell-k}\binom{\ell}{k}^2 \int_{x\in\mathbb{R}} \frac{(x^2+A)^k}{(x^2+A+B)^{2\ell+1}} dxdv \end{aligned}$$

where $A = A(v) := |\alpha|^2 - x^2 = \left(1 - \frac{\langle v,v\rangle}{2}\right)^2$, and $B = B(v) := |\beta|^2 = 2|\langle v, u_2\rangle|^2$.

By [HMY24, Theorem 3.5], for a non-zero $T \in V_0$, when $\langle T, T\rangle < 0$, the $\mathbb{V}_\ell$-valued function $E^T_{2,\infty}(g, \ell+1)$ on $\mathbf{G}(\mathbb{R})$ is identically zero; when $\langle T, T\rangle \geq 0$, $E^T_{2,\infty}(g, \ell+1)$ must be a multiple of the generalized Whittaker function $\mathcal{W}_T(g)$ given by (3.2). To exactly know this multiple, it suffices to compare the coefficients of $[u_1^\ell][u_2^\ell]$ in $E^T_{2,\infty}$ and $\mathcal{W}_T(1)$ for any element T with $\langle T, T\rangle \geq 0$, *i.e.* compare $I_0(T;\ell)$ and $K_0(4\sqrt{2}\pi|\langle u_2, T\rangle|) = K_0(4\pi\sqrt{B(T)})$.

4.2.1 Fourier transforms

Equipped with the symmetric form $(v, w) := \mathrm{Tr}_{\mathbb{C}}\langle v, w\rangle = 2\,\mathrm{Re}\langle v, w\rangle$, we view V_0 as a quadratic space with signature $(2, 2n-2)$, and define the Fourier transform of a Schwartz function f on V_0 to be

$$\widehat{f}(T) = \int_{v\in V_0} f(v) e^{-2\pi i (T,v)} dv,\ T \in V_0.$$

We use the following lemma to write $I_0(T;\ell)$ as a Fourier transform on V_0:

Lemma 4.4. *For any real number C, D and two natural numbers $m < n$, we have*

$$\int_{\mathbb{R}} \frac{(x^2+C)^m}{(x^2+D)^n} dx = \frac{D^{m-n+1/2}}{(n-1)!} \sum_{k=0}^{m} \binom{m}{k} \left(\frac{C}{D}\right)^{m-k} \Gamma(k+1/2)\Gamma(n-k-1/2).$$

Proof. An exercise of calculus. □

Applying Lemma 4.4 to $I_0(T;\ell)$, we obtain that $I_0(T;\ell) = \widehat{F_{0,\ell}}(T)$, where

$$F_{0,\ell}(v) := \sum_{k=0}^{\ell} (-B)^{\ell-k} \binom{\ell}{k}^2 \frac{(A+B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{A}{A+B}\right)^{k-j} \frac{\pi\cdot(2j)!(4\ell-2j)!}{2^{4\ell}\cdot j!(2\ell-j)!}.$$

Now we can reduce the problem to the comparison of $F_{0,\ell}(T)$ and the inverse Fourier transform of (the product of certain function with) $K_0(4\pi\sqrt{B(T)})$. Recall the following result for quadratic spaces by Pollack:

Proposition 4.5. *($v = 0$ case of [PS22, Proposition 4.5.3]) Let $(V', (\,,\,)) = V_2 \oplus V_m$ be a non-degenerate quadratic space over $\mathbb{R}$ of signature $(2, m)$, and v_1', v_2' an orthonormal basis of V_2. For $x = x_2 + x_m \in V_2 \oplus V_m$, define $\|x\|^2 := (x_2, x_2) - (x_m, x_m)$. For any integer $\ell > m/2$ and $x = x_2 + x_m \in V_2 \oplus V_m$ with $\|x_2\| + \|x_m\| < \sqrt{2}$, we have*

$$\int_{V'} e^{i(\omega,x)}\mathrm{Char}((\omega,\omega) > 0)\left(\frac{(\omega,\omega)}{2}\right)^{\ell - m/2} K_0(\sqrt{2}|(\omega, v_1' + \sqrt{-1}v_2')|)d\omega$$
$$=2^{\ell}\pi^{\frac{m+2}{2}}\ell! \cdot \Gamma(\ell - m/2 + 1)\tau(\sqrt{2}x_n, \sqrt{2}x_2)^{-\ell-1}{}_2F_1\left(-\frac{\ell}{2}, \frac{\ell+1}{2}; 1; \frac{2\|x_2\|^2}{\tau(\sqrt{2}x_n, \sqrt{2}x_2)^2}\right),$$

where $\tau(\sqrt{2}x_n, \sqrt{2}x_2)^2 := \left(1 + \frac{\|x_n\|^2 - \|x_2\|^2}{2}\right)^2 + 2\|x_2\|^2$. Here

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{z^n}{n!},\ |z| < 1$$

is the hypergeometric function, where $(q)_n$ is the (rising) Pochhammer symbol.

Take V' to be $(V_0, \frac{1}{2}(,) = \mathrm{Re}\langle\,,\,\rangle)$, as a quadratic space of signature $(2, 2n-2)$, and $v_1' = u_2$, $v_2' = iu_2$. Now we rewrite the objects appearing in Proposition 4.5 in terms of our unitary notations:

- The K-Bessel function appearing in the integral is $K_0(\sqrt{2}|\langle\omega, u_2\rangle|) = K_0(\sqrt{B(\omega)})$.
- The norm $\|x_2\|^2$ equals $|\langle x, u_2\rangle|^2 = B(x)/2$.
- The function $\tau(\sqrt{2}x_n, \sqrt{2}x_2)^2$ equals $(1 - \frac{\langle x,x\rangle}{2})^2 + 2|\langle x, u_2\rangle|^2 = A(x) + B(x)$.

Hence Proposition 4.5 can be rewritten as:

$$2^{-\ell+n-1}\int_{V_0} e^{i(\omega,x)/2}\mathrm{Char}(\langle\omega,\omega\rangle > 0)\langle w, w\rangle^{\ell-n+1}K_0(\sqrt{B(\omega)})d\omega$$
$$=2^{\ell}\pi^n\ell!(\ell - n + 1)!(A(x) + B(x))^{-\frac{\ell+1}{2}}{}_2F_1\left(-\frac{\ell}{2}, \frac{\ell+1}{2}; 1; \frac{B(x)}{A(x) + B(x)}\right).$$

We replace the variable ω in the integral by $4\pi\omega$ and replace ℓ by 2ℓ with $\ell > (n-1)/2$, then we obtain that the inverse Fourier transform of

$$\mathrm{Char}(\langle\omega,\omega\rangle > 0)\langle\omega,\omega\rangle^{2\ell-n+1}K_0(4\pi\sqrt{B(\omega)})$$

is

$$\pi^{n-4\ell-2} \cdot 2^{-4\ell-n-1}(2\ell)!(2\ell - n + 1)!(A(x) + B(x))^{-\ell-1/2}{}_2F_1\left(-\ell, \ell + 1/2; 1; \frac{B(x)}{A(x) + B(x)}\right). \tag{4.2}$$

4.2.2 Write $F_{0,\ell}$ in terms of hypergeometric functions

Now we want to write the function $F_{0,\ell}$ in a form close to (4.2).
Set $z(v) = B(v)/(A(v)+B(v))$, then we have:

$$F_{0,\ell}(v) = \frac{2^{-4\ell}}{(2\ell)!}\pi(A+B)^{-\ell-1/2}\sum_{k=0}^{\ell}\sum_{j=0}^{k}\binom{\ell}{k}^2\binom{k}{j}\frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!}(-z)^{\ell-k}(1-z)^{k-j}$$

Proposition 4.6. *For any integers $0 \le k \le \ell$ and $0 \le j \le k$, set*

$$c_{j,k} = \binom{\ell}{k}^2\binom{k}{j}\frac{(2j)!(4\ell-2j)!}{j!(2\ell-j)!}.$$

Then

$$\sum_{k=0}^{\ell}\sum_{j=0}^{k} c_{j,k}(-z)^{\ell-k}(1-z)^{k-j} = \sum_{r=0}^{\ell}(-1)^r 2^{2\ell-2r}\frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!}z^r. \tag{4.3}$$

Proof. We write:

$$\sum_{k=0}^{\ell}\sum_{j=0}^{k} c_{j,k}(-z)^{\ell-k}(1-z)^{k-j} = \sum_{r=0}^{\ell}(-1)^r C(r) z^r,$$

The term $(-z)^{\ell-k}(1-z)^{k-j}$ has a non-zero z^r term if and only if $\ell-k \le r \le \ell-j$, thus

$$(-1)^r C(r) = \sum_{j=0}^{\ell-r}\sum_{k=\ell-r}^{\ell} c_{j,k}\binom{k-j}{r-\ell+k}(-1)^{\ell-k+r-\ell+k} = (-1)^r\sum_{j=0}^{\ell-r}\sum_{k=\ell-r}^{\ell} c_{j,k}\binom{k-j}{r-\ell+k}.$$

To compute $C(r)$, we need the following lemma:

Lemma 4.7. *(1) For integers $0 \le a \le b$, one has*

$$\sum_{i=a}^{b}\binom{b}{i}\binom{b-a}{b-i} = \binom{2b-a}{b}.$$

(2) For any integer $0 \le r \le \ell$, one has

$$\sum_{i=0}^{\ell-r}\frac{\binom{2\ell}{i}\binom{\ell-r}{i}}{\binom{4\ell}{2i}} = 2^{2\ell-2r}\frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}.$$

Proof of Lemma 4.7. The identity in (1) is obvious. For the identity in (2), the left-hand side can be rewritten as:

$$\sum_{i=0}^{\ell-r}\frac{(-\ell+r)_j(1/2)_j}{(-2\ell+1/2)_j j!} = {}_2F_1(-(\ell-r),1/2;-2\ell+1/2;1),$$

By Chu-Vandermond identity, this value of hypergeometric function equals

$$\frac{(-2\ell)_{\ell-r}}{(-2\ell+1/2)_{\ell-r}} = 2^{\ell-r}\frac{\frac{(2\ell)!}{(\ell+r)!}}{\frac{(4\ell-1)!!}{(2\ell+2r-1)!!}} = 2^{2\ell-2r}\frac{\binom{2\ell+2r}{\ell+r}}{\binom{4\ell}{2\ell}}.$$ □

Now we return to the calculation of $C(r)$:

$$
\begin{aligned}
C(r) &= \sum_{j=0}^{\ell-r}\sum_{k=\ell-r}^{\ell} \frac{(\ell!)^2(2j)!(4\ell-2j)!}{k!((\ell-k)!)^2(j!)^2(r-\ell+k)!(\ell-r-j)!(2\ell-j)!} \\
&= \frac{\ell!}{r!}\sum_{j=0}^{\ell-r}\frac{(2j)!(4\ell-2j)!}{(j!)^2(\ell-r-j)!(2\ell-j)!}\sum_{k=\ell-r}^{\ell}\binom{\ell}{k}\binom{r}{\ell-k} \\
\text{(by (1) of Lemma 4.7)} &= \frac{\ell!}{r!}\cdot\frac{(\ell+r)!}{\ell!r!}\sum_{j=0}^{\ell-r}\frac{(2j)!(4\ell-2j)!}{(j!)^2(\ell-r-j)!(2\ell-j)!} \\
&= \frac{(\ell+r)!}{(r!)^2}\cdot\frac{(4\ell)!}{(\ell-r)!(2\ell)!}\sum_{j=0}^{\ell-r}\frac{\binom{2\ell}{j}\binom{\ell-r}{j}}{\binom{4\ell}{2j}} \\
\text{(by (2) of Lemma 4.7)} &= \frac{(\ell+r)!}{(r!)^2}\cdot\frac{(4\ell)!}{(\ell-r)!(2\ell)!}\cdot 2^{2\ell-2r}\frac{\frac{(2\ell+2r)!}{((\ell+r)!)^2}}{\frac{(4\ell)!}{((2\ell)!)^2}} \\
&= 2^{2\ell-2r}\frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!}.
\end{aligned}
$$

$\square$

Applying Proposition 4.6 to $F_{0,\ell}$, we have

$$
\begin{aligned}
F_{0,\ell}(v) =& \frac{2^{-4\ell}\pi}{(2\ell)!(A+B)^{\ell+1/2}}\sum_{k=0}^{\ell}\sum_{j=0}^{k} c_{j,k}(-z)^{\ell-k}(1-z)^{k-j} \\
=& \frac{2^{-4\ell}\pi}{(2\ell)!(A+B)^{\ell+1/2}}\sum_{r=0}^{\ell}(-1)^r 2^{2\ell-2r}\frac{(2\ell)!(2\ell+2r)!}{(r!)^2(\ell+r)!(\ell-r)!}z^r \\
=& \frac{2^{-2\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\sum_{r=0}^{\ell}\frac{(-\ell)_r(\ell+1/2)_r}{1_r}\frac{z^r}{r!} \\
=& \frac{2^{-2\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\cdot {}_2F_1(-\ell,\ell+1/2;1;z).
\end{aligned}
$$

We have shown the following result:

Proposition 4.8. *For any $T \in V_0$, the coefficient $I_0(T;\ell)$ of $[u_1^\ell][u_2^\ell]$ in $E_{2,\infty}^T$ satisfies*

$$I_0(T;\ell) = \widehat{F_{0,\ell}}(T),$$

where

$$F_{0,\ell}(v) = \frac{2^{-2\ell}(2\ell)!\pi}{(\ell!)^2(A+B)^{\ell+1/2}}\cdot {}_2F_1(-\ell,\ell+1/2;1;\frac{B}{A+B}),\ v \in V_0,$$

and $A(v) = \left(1-\frac{\langle v,v\rangle}{2}\right)^2$, $B(v) = 2|\langle v,u_2\rangle|^2$.

Comparing Proposition 4.8 with (4.2), we obtain that the function

$$\pi^{n-4\ell-3}2^{-2\ell-n-1}(\ell!)^2(2\ell-n+1)!F_{0,\ell}(v)$$

is equal to the inverse Fourier transform of

$$\mathrm{Char}(\langle \omega, \omega \rangle > 0) \langle \omega, \omega \rangle^{2\ell-n+1} K_0(4\pi \sqrt{B(\omega)}),$$

which directly implies the following theorem:

Theorem 4.9. *For any $T \in V_0$ and $\ell > \frac{n-1}{2}$, we have*

$$I_0(T;\ell) = \mathrm{Char}(\langle T, T \rangle > 0) \frac{2^{2\ell+n+1}\pi^{4\ell-n+3}\langle T, T \rangle^{2\ell-n+1}}{(\ell!)^2(2\ell-n+1)!} \mathcal{W}_T(1).$$

In particular, $E^T_{2,\infty}(g, \ell+1)$ is identically zero when $\langle T, T \rangle \leq 0$, and equals

$$\frac{2^{2\ell+n+1}\pi^{4\ell-n+3}\langle T, T \rangle^{2\ell-n+1}}{(\ell!)^2(2\ell-n+1)!} \mathcal{W}_T(g)$$

when $\langle T, T \rangle > 0$.

5 Rank 1 Fourier coefficients of E_ℓ

6 The constant term of E_ℓ

7 The Fourier expansion of $E_\ell(g, \Phi_f, s = \ell + 1)$

7.1 Computation of constant term

The constant term of $E_\ell(g, \Phi_f, s = \ell+1)$ consists of three parts:

- $E_{\ell,0}(g, \Phi_f, s) = f(g, s)$,
- $E_1^0(g, s) = \sum_{\ell \in \mathcal{L}_0} \int_{\mathbf{N}_\ell(\mathbb{A}) \backslash \mathbf{N}(\mathbb{A})} f(\gamma(\ell) n g, s) dn$,
- and $E_2^0(g, s) = \int_{\mathbf{N}(\mathbb{A})} f(w_2 n g) dn$.

7.1.1 The $i = 0$-term

Lemma 7.1. *For $g \in \mathbf{P}(\mathbb{A})$,*

$$E_0(g, s) = f(g, s) = |\nu(g)|_E^s \zeta_E(s) [u_1^n][u_2^n].$$

Proof. For $g_f \in \mathbf{P}(\mathbb{A}_f)$, we have

$$\begin{aligned} f_{fte}(g_f, s) &= \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(t b_1 g_f) dt \\ &= \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(t \nu(g_f)^{-1} b_1) dt \\ &= |\nu(g_f)|_E^s \int_{\mathbb{A}_{E,f}^\times} |t|_E^s \Phi_f(t b_1) dt. \end{aligned}$$

Thus, the non-archimedean contribution is $|\nu(g_f)|_E^s\zeta_E(s)$. Combining with $f_{n,\infty}(g_\infty, s) = |\nu(g_\infty)|_E^s[u_1^n][u_2^n]$, we get the desired identity. $\square$

7.1.2 The $i = 1$-term

We fix a non-zero isotropic vector in V_0, such that $v_0 = b_1\gamma_0$, and set $\ell_0 = Ev_0$. Define $\mathbf{P}_0$ be the stabilizer of ℓ_0 in $\mathbf{U}(V_0)$, which is a parabolic subgroup of $\mathbf{M}$. We denote the similitude character of $\mathbf{P}_0$ by λ, *i.e.* $v_0 g = \lambda(g)^{-1}v_0$ for any $g \in \mathbf{P}_0$. For $g \in \mathbf{P}_0(\mathbb{A})$, we have:

$$\begin{aligned}
E_1^0(g,s) &= \sum_{\ell\in\mathcal{L}_0}\int_{\mathbf{N}_\ell(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})} f(\gamma(\ell)ng,s)dn \\
&= \sum_{\gamma\in\mathbf{P}_0(\mathbb{Q})\backslash\mathbf{M}(\mathbb{Q})}\int_{\mathbf{N}_{\ell_0}(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})} f(\gamma_0\gamma ng,s)dn
\end{aligned}$$

If we set $f_0(g,s) = \int_{\mathbf{N}_{\ell_0}(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})} f(\gamma_0 ng,s)dn$, then for $\mathrm{Re}(s) \gg 0$,

$$E_1^0(g,s) = \sum_{\gamma\in\mathbf{P}_0(\mathbb{Q})\backslash\mathbf{M}(\mathbb{Q})} f_0(\gamma g,s),$$

and it defines an Eisenstein series on $\mathbf{M}$. Now we want to determine this section $f_0(g,s) \in \mathrm{Ind}_{\mathbf{P}_0(\mathbb{A})}^{\mathbf{M}(\mathbb{A})}|\lambda|^s$ (still unnormalized).

At finite places, one has

$$\begin{aligned}
\int_{\mathbf{N}_{\ell_0}(\mathbb{A}_{E,f})\backslash\mathbf{N}(\mathbb{A}_{E,f})} f(\gamma_0 ng,s)dn &= \int_{x\in\mathbb{A}_{E,f}}\int_{t\in\mathbb{A}_{E,f}^\times}|t|^s\Phi_{fte}(t(v_0+xb_1)g)dtdx \\
&= \int_{x\in\mathbb{A}_{E,f}}\int_{t\in\mathbb{A}_{E,f}^\times}|t|^s\Phi_{fte}(t\lambda(g)^{-1}v_0+tx\nu(g)^{-1}b_1)dtdx \\
&= |\lambda(g)|_f^{s-1}|\nu(g)|_f\int_{x\in\mathbb{A}_{E,f}}\int_{t\in\mathbb{A}_{f,E}^\times}|t|^{s-1}\Phi_{fte}(tv_0+xb_1)dtdx \\
&= |\lambda(g)|_f^{s-1}|\nu(g)|_f\zeta_E(s-1).
\end{aligned}$$

Now we switch to the archimedean place. Set $c_1 = v_0$ and c_2 another isotropic vector in V_0 with $\langle c_1, c_2\rangle = 1$. We can take $u_2 = \frac{1}{\sqrt{2}}(c_1+c_2)$ and $v_{n-1} = \frac{1}{\sqrt{2}}(c_1-c_2)$. We identify $\mathbf{N}_{\ell_0}(\mathbb{R})\backslash\mathbf{N}(\mathbb{R})$ with $\mathbb{C}$ via the map $n(v,\lambda) \mapsto -\langle c_1, v\rangle$, and for any $z \in \mathbb{C}$ we choose an element $n(z) \in \mathbf{N}(\mathbb{R})$ under this identification.

Lemma 7.2. *When $\ell \geq 1$, we have $f_{0,\infty} \equiv 0$.*

Proof. Similarly to Lemma 7.3, for any $g \in \mathbf{P}_0(\mathbb{R})$ one has

$$f_{\ell,\infty}(\gamma_0 n(z)g) = \frac{(z\nu(g)^{-1}u_1+\lambda(g)^{-1}u_2)^\ell(-\overline{\lambda(g)}^{-1}u_1+\overline{z\nu(g)^{-1}}u_2)^\ell}{(|\lambda(g)|^{-2}+|z|^2|\nu(g)|^{-2})^{2\ell+1}(\ell!)^2}.$$

From now on we write $\lambda(g)$ (*resp.* $\nu(g)$) as λ (*resp.* ν).

The coefficient of $[u_1^{\ell-v}][u_2^{\ell+v}]$ in $f_{\ell,\infty}(\gamma_0 n(z)g)$ is

$$\frac{(\ell-v)!(\ell+v)!}{(\ell!)^2(|\lambda|^{-2}+|z|^2|\nu|^{-2})^{2\ell+1}}\sum_{\substack{0\le i,j\le\ell\\ i+j=\ell-v}}\binom{\ell}{i}\binom{\ell}{j}z^i\nu^{-i}\lambda^{-(\ell-i)}(-\overline{\lambda})^{-j}(\overline{z\nu^{-1}})^{\ell-j}$$

$$=\frac{(\ell-v)!(\ell+v)!|\nu|^{-2\ell}\nu^v\lambda^{-v}|z|^{2\ell}z^{-v}}{(\ell!)^2(|\lambda|^{-2}+|z|^2|\nu|^{-2})^{2\ell+1}}\sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)}(-1)^j\binom{\ell}{j}\binom{\ell}{j+v}|\nu|^{2j}|\lambda|^{-2j}|z|^{-2j},$$

and the corresponding coefficient in $f_{0,\infty}(g,s)$ is the integral of this function over $z\in\mathbb{C}$. Set $w=z\lambda/\nu$, then we have $dw=|\lambda/\nu|^2dz$ and this function equals

$$\frac{(\ell-v)!(\ell+v)!|\lambda|^{2\ell+2}w^{-v}|w|^{2\ell}}{(\ell!)^2(1+|w|^2)^{2\ell+1}}\sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)}(-1)^j\binom{\ell}{j}\binom{\ell}{j+v}|w|^{-2j}.$$

Hence the coefficient of $[u_1^{\ell-v}][u_2^{\ell+v}]$ in $f_{0,\infty}(g,s=\ell+1)$ equals

$$\frac{(\ell-v)!(\ell+v)!}{(\ell!)^2}|\lambda|^{2\ell}|\nu|^2\sum_{j=\max(0,-v)}^{\min(\ell,\ell-v)}(-1)^j\binom{\ell}{j}\binom{\ell}{j+v}\left(\int_{\mathbb{C}}\frac{w^{-v}|w|^{2\ell-2j}}{(1+|w|^2)^{2\ell+1}}dw\right) \tag{7.1}$$

Notice that the inner integral is nonzero only when $v=0$, and we have

$$\int_{\mathbb{C}}\frac{|w|^{2\ell-2j}}{(1+|w|^2)^{2\ell+1}}dw=\pi\frac{(\ell-j)!(\ell+j-1)!}{(2\ell)!},\ j=0,1,\dots,\ell.$$

So we only need to consider the coefficient of $[u_1^{\ell}][u_2^{\ell}]$, which is equal to

$$\pi((2\ell)!)^{-1}|\lambda|^{2\ell}|\nu|^2\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}^2(\ell-j)!(\ell+j-1)!$$

$$=\frac{\pi(\ell!)^2|\lambda|^{2\ell}|\nu|^2}{\ell(2\ell)!}{}_2F_1(\ell,-\ell;1;1),$$

which is 0 when $\ell\le 1$. $\square$

7.1.3 The $i=2$-term

This term vanishes when we have the condition $\ell>n+1$ (by analyzing the singularity of $K_v(x)$ at $x=0$).

7.2 Rank 1 Fourier coefficients

For rank 1 Fourier coefficients, one needs to calculate

$$E_1^T(g,s)=\int_{\mathbf{N}_T(\mathbb{A})\backslash\mathbf{N}(\mathbb{A})}\chi_T^{-1}(n)f(\gamma(\ell_T)ng,s)dn$$

for any nonzero isotropic vector T.

7.2.1 Non-archimedean components

For any prime p, we identify $\mathbf{N}_T(\mathbb{Q}_p)\backslash\mathbf{N}(\mathbb{Q}_p)$ with E_p via the map $n(v,\lambda)\mapsto -\langle T,v\rangle$, thus

$$\begin{aligned} E_{1,p}^T(1,s) &= \int_{x\in E_p}\int_{t\in E_p^\times}\psi_E(x)|t|_E^s\Phi_p(txb_1+tT)dtdx \\ &= \int_{x\in E_p}\int_{t\in E_p^\times}\psi_E(x/t)|t|_E^{s-1}\Phi_p(xb_1+tT)dtdx \end{aligned}$$

7.2.1.1 Split case When p is split in E, we write T as (T_1,T_2) with $T_i\in\mathbb{Z}_p^n, i=1,2$. Let $\alpha=v_p(T_1)$ and $\beta=v_p(T_2)$, then we have:

$$\begin{aligned} E_{1,p}^T :=& E_{1,p}^T(1,s=\ell+1) \\ =& \int_{(x_1,x_2)\in\mathbb{Q}_p^2}\int_{(t_1,t_2)\in(\mathbb{Q}_p\times\mathbb{Q}_p)^\times}\psi_p\left(x_1/t_1+x_2/t_2\right)|t_1t_2|_p^{s-1}\Phi_p\left((x_1,x_2)b_1+(t_1,t_2)T\right)dt_1dt_2dx_1dx_2 \\ =& \sum_{\substack{r_1\geq-\alpha\\ r_2\geq-\beta}} p^{-\ell(r_1+r_2)}\int_{(x_1,x_2)\in\mathbb{Z}_p}\psi_p(x_1/p^{r_1})\psi_p(x_2/p^{r_2})dx_1dx_2 \\ =& \sum_{\substack{r_1\geq-\alpha\\ r_2\geq-\beta}} p^{-\ell(r_1+r_2)}\left(\frac{1}{p^{r_1}}\sum_{x_1\in\mathbb{Z}/p^{r_1}\mathbb{Z}}\psi_p(x_1/p^{r_1})\right)\left(\frac{1}{p^{r_1}}\sum_{x_2\in\mathbb{Z}/p^{r_2}\mathbb{Z}}\psi_p(x_2/p^{r_2})\right) \\ =& (1+p^\ell+p^{2\ell}+\cdots+p^{\alpha\ell})(1+p^\ell+p^{2\ell}+\cdots+p^{\beta\ell}). \end{aligned}$$

7.2.1.2 Inert case When p is inert in E and $p\neq 2$, we have:

$$\begin{aligned} E_{1,p}^T =& \sum_{r\geq -v_p(T)} p^{-2r\ell}\int_{x\in\mathcal{O}_{E_p}}\psi_E(x/p^r)dx \\ =& 1+p^{2\ell}+p^{4\ell}+\cdots+p^{2\ell v_p(T)}. \end{aligned}$$

7.2.1.3 Ramified case

7.2.2 Archimedean components

Again, we identify $\mathbf{N}_T(\mathbb{R})\backslash\mathbf{N}(\mathbb{R})$ with $\mathbb{C}$ via the map $n(v,\lambda)\mapsto -\langle T,v\rangle$, and for any $z\in\mathbb{C}$ we choose an element $n(z)\in\mathbf{N}(\mathbb{R})$ corresponding to z under this identification.

Lemma 7.3. *For any $z\in\mathbb{C}$, we have*

$$f_{\ell,\infty}(\gamma(\ell_T)n(z),s=\ell+1)=\frac{(zu_1+\beta u_2)^\ell(-\overline{\beta}u_1+\overline{z}u_2)^\ell}{(|z|^2+|\beta|^2)^{2\ell+1}(\ell!)^2},$$

where $\beta=\beta(T):=\sqrt{2}\langle T,u_2\rangle$.

Proof. Suppose that $\gamma(\ell_T)n(z) = pk$ for $p \in \mathbf{P}(\mathbb{R})$ and $k = (k_+, k_-) \in K_\infty = \mathrm{U}(V_2^+) \times \mathrm{U}(V_n^-)$, then one has

$$T + zb_1 = b_1\gamma(\ell_T)n(z) = b_1pk = \nu(p)^{-1}b_1k.$$

Taking the projection to V_2^+ of both sides, we obtain that

$$zu_1 + \sqrt{2}\langle T, u_2\rangle u_2 = \nu(p)^{-1}u_1k_+.$$

One can take $\nu(p)$ to be $(|z|^2 + |\beta|^2)^{-1/2}$, then

$$k_+ = \nu(p)\begin{pmatrix} z & -\overline{\beta} \\ \beta & \overline{z} \end{pmatrix} \in \mathrm{U}(V_2^+),$$

which gives us the desired value of $f_{\ell,\infty}$. $\square$

Using Lemma 7.3, the coefficient of $[u_1^\ell][u_2^\ell]$ in $E_{1,\infty}^T := E_{1,\infty}^T(1, s = \ell + 1)$ is equal to

$$\int_{z\in\mathbb{C}} e^{2\pi i \mathrm{Re}(z)} \frac{\sum_{k=0}^{\ell} \binom{\ell}{k}^2 |z|^{2k}(-|\beta|^2)^{\ell-k}}{(|z|^2 + |\beta|^2)^{2\ell+1}} dz. \tag{7.2}$$

We set $B = |\beta|^2$ and $z = x + iy$, where $x, y \in \mathbb{R}$, then the integral in (7.2) equals

$$\begin{aligned}
&\int_{\mathbb{R}^2} e^{2\pi i x} \sum_{k=0}^{\ell} \binom{\ell}{k}^2 (-B)^{\ell-k} \frac{(x^2+y^2)^k}{(x^2+y^2+B)^{2\ell+1}} dxdy \\
&= \int_{\mathbb{R}} e^{2\pi i x} \sum_{k=0}^{\ell} \binom{\ell}{k}^2 (-B)^{\ell-k} \left(\int_{\mathbb{R}} \frac{(y^2+x^2)}{(y^2+x^2+B)^{2\ell+1}} dy \right) dx
\end{aligned}$$

By Lemma 4.4, the inner integral over $y \in \mathbb{R}$ equals

$$\frac{(x^2+B)^{k-2\ell-1/2}}{(2\ell)!} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{x^2}{x^2+B}\right)^{k-j} \Gamma(j+1/2)\Gamma(2\ell-j+1/2).$$

Similarly to our calculation in §4.2, we have

$$E_{1,\infty}^T = 2^{-2\ell}\pi \sum_{k=0}^{\ell} \frac{(-B)^k 2^{-2k}(2\ell+2k)!}{(k!)^2(\ell+k)!(\ell-k)!} \int_{\mathbb{R}} \frac{e^{2\pi i x}dx}{(x^2+B)^{\ell+k+1/2}}$$

Using Basset's integral, one has

$$\int_{\mathbb{R}} \frac{e^{iwt}dt}{(t^2+z^2)^{n+1/2}} = \frac{2\sqrt{\pi}}{\Gamma(n+1/2)} \left(\frac{|w|}{2|z|}\right)^n K_n(|wz|),\ w, z \in \mathbb{R}.$$

Plugging this into $E_{1,\infty}^T$, we obtain:

$$\begin{aligned}
E_{1,T}^\infty &= 2^{-2\ell}\pi \sum_{k=0}^{\ell} \frac{(-B)^k 2^{-2k}(2\ell+2k)!}{(k!)^2(\ell+k)!(\ell-k)!} \cdot \frac{2^{l+k+1}(\ell+k)!B^{-(\ell+k)/2}(2\pi)^{\ell+k}K_{\ell+k}(2\pi\sqrt{B})}{(2\ell+2k)!} \\
&= 2\pi B^{-\ell} \sum_{k=0}^{\ell} \frac{(-1)^k}{(k!)^2(\ell-k)!} (\pi\sqrt{B})^{\ell+k} K_{\ell+k}(2\pi\sqrt{B}).
\end{aligned}$$

Lemma 7.4. *For any $C > 0$, we have*

$$S_\ell := \sum_{k=0}^{\ell} \frac{(-1)^k C^{\ell+k} K_{\ell+k}(2C)}{(k!)^2(\ell-k)!} = \frac{(-1)^\ell C^{2\ell}}{(\ell!)^2} K_0(2C).$$

Proof. Using the recurrence relation for K-Bessel functions:

$$K_{n+1}(z) = K_{n-1}(z) + \frac{2n}{z} K_n(z),$$

we can write $-C^2 S_\ell$ as

$$\begin{aligned}
-C^2 S_\ell &= \sum_{k=0}^{\ell} (-1)^{k+1} \frac{C^{\ell+k+2}}{(k!)^2(\ell-k)!} K_{\ell+k}(2C) \\
&= \sum_{j=1}^{\ell+1} (-1)^j \frac{C^{\ell+1+j}}{((j-1)!)^2(\ell-j+1)!} \left(K_{\ell+1+j}(2C) - \frac{\ell+j}{C} K_{\ell+j}(2C) \right) \\
&= \sum_{j=1}^{\ell+1} \frac{(-1)^j C^{\ell+1+j} K_{\ell+1+j}(2C)}{((j-1)!)^2(\ell-j+1)!} + \sum_{j=0}^{\ell} \frac{(-1)^j(\ell+j+1)C^{\ell+1+j}K_{\ell+j+1}(2C)}{(j!)^2(\ell-j)!} \\
&= \frac{(\ell+1)C^{\ell+1}K_{\ell+1}(2C)}{\ell!} + \sum_{j=1}^{\ell} \frac{(-1)^j C^{\ell+1+j} K_{\ell+1+j}(2C)}{(j!)^2(\ell+1-j)!}(\ell+1)^2 + (-1)^{\ell+1}\frac{C^{2\ell+2}K_{2\ell+2}(2C)}{(\ell!)^2} \\
&= (\ell+1)^2 S_{\ell+1}.
\end{aligned}$$

Since $S_0 = K_0(2C)$ and $S_{\ell+1} = -\frac{C^2}{(\ell+1)^2} S_\ell$, we have

$$S_\ell = (-1)^\ell \frac{C^{2\ell}}{(\ell!)^2} K_0(2C). \qquad \square$$

By this lemma, one has

$$E_{1,T}^\infty = 2\pi B^{-\ell} \cdot \frac{(-1)^\ell(\pi\sqrt{B})^{2\ell}}{(\ell!)^2} K_0(2\pi\sqrt{B}) = \frac{2\pi^{2\ell+1}}{(\ell!)^2} K_0(2\pi\sqrt{B}).$$

8 Our progress

If there is no mistake in the previous calculations, that means

$$\begin{aligned}
&E_\ell(g, \Phi_f, s = \ell+1) \\
=& f_\ell(g, s = \ell+1) + \pi^{2\ell+1} \left(\sum_{\substack{T \neq 0 \in \mathcal{V} \\ \langle T,T \rangle = 0}} \square \cdot \sigma_E(T) \mathcal{W}_T(g) + \sum_{\substack{T \in \mathcal{V} \\ \langle T,T \rangle > 0}} \square \cdot \langle T,T \rangle^{2\ell-n+1} \mathcal{W}_T(g) \right),
\end{aligned}$$

where $f_\ell(g, s = \ell+1) = \zeta_E(\ell+1)|\nu(g)|_E^{\ell+1}[u_1^n][u_2^n]$ and $\square$ means certain rational factor.

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