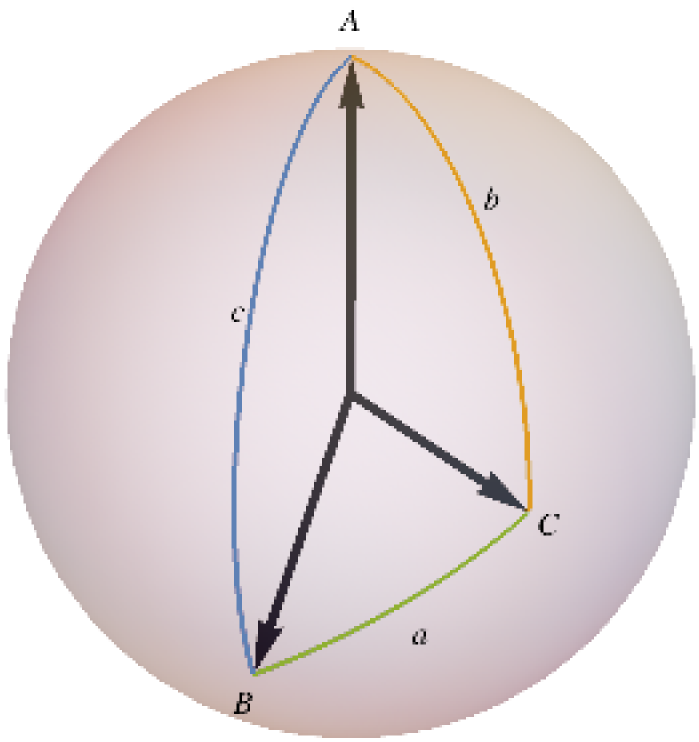
Spherical Trigonometry—Laws of Cosines and Sines

Students use vectors to to derive the spherical law of cosines. From there, they use the polar triangle to obtain the second law of cosines. Arithmetic leads to the law of sines. Comparisons are made to Euclidean laws of sines and cosines. Finally, the spherical triangle area formula is deduced.



Given a spherical triangle $\triangle ABC$, we can rotate the sphere so that A is the north pole. As is clear from the diagram above, the angle $\angle A$ determines along which great circles sides b and c lie, and the angles b and c then determine the locations of points C and B. So the triangle is determines by the lengths of two sides and the measure of the included angle. That is, there is a spherical SAS theorem.

Let's calculate a from b, c, and A. As the diagram suggests, use *vectors* to represent the points on the sphere. The nifty reason to do this is that dot products use cosines. That is, $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos(\theta)$ where θ is the angle between the vectors. As a bonus, the vectors from the center of the sphere to points A, B, and C are radii of a unit sphere, so their lengths are all one. This means that $\cos(a) = \mathbf{B} \cdot \mathbf{C}$.

So let's figure out the vectors $\mathbf{B}$ and $\mathbf{C}$ from the origin to the points B and C respectively. First, let's rotate the sphere along the axis through A until B lies in the xz-plane and its x-coordinate is positive. We can use simple trig to find the coordinates of $\mathbf{B}$ now. It is $(\sin(c), 0, \cos(c))$.

Vector $\mathbf{C}$ is a little harder to figure out, because it is not in any kind of nice position. To see where it is, start at the north pole, head south by angle c, and then rotate around from the xz-plane by amount A. By heading south, we end up at a point whose z-coordinate is $\cos(b)$ and which is $\sin(b)$ away from the axis. When we rotate it, the z-coordinate doesn't change. The x- and y- coordinates are the coordinates of a point on a circle of radius $\sin(b)$ rotated by A from the x-axis, so they are $(\sin(b)\cos(A), \sin b \sin(A))$. Thus, our coordinates for vector $\mathbf{C}$ are $(\sin(b)\cos(A), \sin(b)\sin(A), \cos(b))$.

Now we're almost done, because we just need to take the dot product of this with $\mathbf{B}$ to find $\cos(a)$. But dot products are easy—just multiply the coordinates and add. So we obtain $\cos(a) = \cos(A)\sin(b)\sin(c) + \cos(b)\cos(c)$.

Theorem: (Spherical law of cosines) $\cos(a) = \cos(A)\sin(b)\sin(c) + \cos(b)\cos(c)$. (And similarly for the other sides, of course.)

We can apply this theorem to the polar triangle. Recall that in the polar triangle, there is a relationship between the sides and angles of the original triangle: $a' = \pi - A$, $A' = \pi - a$, and so forth. So in the polar triangle, $\cos(a') = \cos(A')\sin(b')\sin(c') + \cos(b')\cos(c')$. We replace a' with $\pi - A$ and so forth. Since $\sin(\pi - x) = \sin(x)$ and $\cos(\pi - x) = -\cos(x)$, we obtain

Theorem: (Supplemental law of cosines) $\cos(A) = \cos(a)\sin(B)\sin(C) - \cos(B)\cos(C)$.

This last theorem tell us that if we know two angles, B and C, and the lengh of the side between them, we can compute the remaining angle. We can also solve it for $\cos(a)$:

$$\cos(a) = \frac{\cos(A) + \cos(B)\cos(C)}{\sin(B)\sin(C)}$$

which allows us to solve for the sides of a triangle knowing just the angles!

Of course, the original law of since could be solved for $\cos(A)$ which allows us to solve for the angles knowing the sides. So we now know that SAS, ASA, SSS, and AAA are all congruence theorems for spherical triangles.

What if you know two sides and a *non-included* angle? Or two angles and a *non-included* side? Those are homework questions!

Note that there is no AA *similarity* theorem in spherical geometry. Knowing two angles isn't enough to figure out the third. For instance, there are a wider variaty of triangles with two right angles—just pick any to non-antipodal points on the equator together with the north pole. And as we have just seen, AAA is actually a *congruence* theorem.

We can use the laws of cosines to figure out a law of sines for spherical trig. Let's just brute force it:

$$\cos(a) = \frac{\cos(A) + \cos(B)\cos(C)}{\sin(B)\sin(C)}$$
$$\cos^2(a) = \frac{\cos^2(A) + 2\cos(A)\cos(B)\cos(C) + \cos^2(B)\cos^2(C)}{\sin^2(B)\sin^2(C)}$$
$$\sin^2(a) = 1 - \cos^2(a) = 1 - \frac{\cos^2(A) + 2\cos(A)\cos(B)\cos(C) + \cos^2(B)\cos^2(C)}{\sin^2(B)\sin^2(C)}$$
$$= \frac{\sin^2(B)\sin^2(C) - \cos^2(A) - 2\cos(A)\cos(B)\cos(C) - \cos^2(B)\cos^2(C)}{\sin^2(B)\sin^2(C)}$$

Now the $\sin^2(B)\sin^2(C)$ can be replaced by $(1 - \cos^2(B))(1 - \cos^2(C)) = 1 - \cos^2(B) - \cos^2(C) + \cos^2(C)\cos^2(C)$. The last term of this cancels with the $-\cos^2(B)\cos^2(C)$ at the end of the numerator, so the numerator turns into $1 - \cos^2(A) - cos^2(B) - cos^2(C) - 2\cos(A)\cos(B)\cos(C)$. Dividing the whole equation by $sin^2(A)$ gives

$$\frac{\sin^2(a)}{\sin^2(A)} = \frac{1 - \cos^2(A) - \cos^2(B) - \cos^2(C) - 2\cos(A)\cos(B)\cos(C)}{\sin^2(A)\sin^2(B)\sin^2(C)}$$

. Finally, notice that the right-hand side is symmetric in A, B, and C, so it also equals $\sin^2(b)/\sin^2(B)$ and $\sin^2(c)/\sin^2(C)$. Also, none of our angles are allowed to be more than $180°$, so all the sines are positive anyway, and we can take square roots to obtain

Theorem: (Spherical law of sines) $\dfrac{\sin(a)}{\sin(A)} = \dfrac{\sin(b)}{\sin(B)} = \dfrac{\sin(c)}{\sin(C)}$.

Now how to these laws compare with the analogous laws from plane trigonometry? The key lies in understanding that if the radius of a sphere is very large, the surface looks flat. Or, since we are keeping the radius fixed, if the sides of the triangle are very small, then the surface looks flat.

But when a is small, $\sin(a) \approx a$. So as long as the sides are small, we can replace the sines in the numerator of the law of sines with just the side length—and we get the plane law of sines!

What about the laws of cosines? Well, when a is small, $\cos(a) \approx 1 - a^2/2$. So the spherical law of cosines is approsimately

$$1 - \frac{a^2}{2} = (1 - \frac{b^2}{2})(1 - \frac{c^2}{2}) + bc\cos(A)$$

(remember, A needn't be small, just the sides!). If we multiply this out and simplify, we get $a^2 = b^2 + c^2 - 2bc\cos(A) - b^2c^2/2$. Since b and c are small, the last term is very small and can be ignored–leaving the plane law of cosines!

What happens to the supplemental law of cosines when you make the small-side approximation? Do the homework to find out!

One last thing. How can we find the area of a spherical triangle?

First, if you have two great circles, they divide the sphere into four sections, called *lunes*. If you start at the vertex of an angle, and follow the geodesics they will meet again at the antipode, creating a lune. Note that the area of a lune is obviously just a multiple of the angle at which the geodesics meet. Since the area of the whole sphere is 4π, if the lune's angle is α, then the area of the lune is 2α.

Now consider the triangle $\triangle ABC$. The great circles that are the sides of this triangle actually split the sphere into eight spherical triangles. If the antipode of A is A' and so forth, these triangles are $\triangle ABC$, $\triangle ABC'$, $\triangle AB'C$, $\triangle AB'C'$, and four more with A' instead of A. Two more things to note: opposite triangles, like $\triangle A'BC'$ and $\triangle AB'C$ are congruent, so have the same area, and if you replace one point with its antipode, the two triangles make up a lune. For example, together triangles $\triangle ABC$ and $\triangle A'BC$ make up a lune with angle A. So together these two triangles have area $2\pi A$.

Next, notice that the four triangles $\triangle ABC$, $\triangle A'BC$, $\triangle AB'C$, and $\triangle A'B'C$ cover the area of one hemisphere—they are the four triangles situate around the point C and all lie on one side of great circle c. So their combined area is 2π. Now $\triangle ABC$ and $\triangle A'BC$ make up a lune with area $2A$. $\triangle ABC$ and $\triangle AB'C$ make up a lune with area $2B$, while triangles ABC and ABC' make up a lune with area $2C$. But $\triangle ABC'$ and $\triangle A'B'C$ have the same area. Adding these three lunes together gives $3\triangle ABC + \triangle AB'C + \triangle A'BC + \triangle A'B'C = 2(A + B + C)$. As we noted at the beginning of this paragraph, the four trianles with a C in them make up a hemisphere. Substituting, we find that $2\triangle ABC + 2\pi = 2(A + B + C)$. Solving for the area of $\triangle ABC$ yields

Theorem: The area of spherical triangle $\triangle ABC$ is $A + B + C - \pi$.

This quantity, $A + B + C - \pi$ is called the *excess*. It's amazing that the area has such a nice formula, depending only on the angles of the triangle!