1 Traffic properties

A **trajectory** is a function, $x(t)$, giving a vehicle's location at every time. The trajectory can never "bend backwards" or have a vertical segment, because that would imply the vehicle has two locations at the same time. At every time, there should only be one value of $x(t)$.

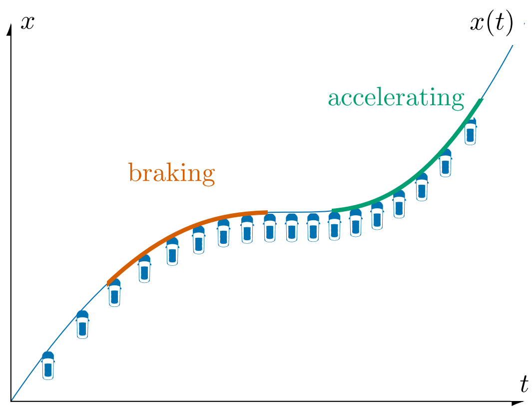


Figure 1: a vehicle trajectory,$x(t)$

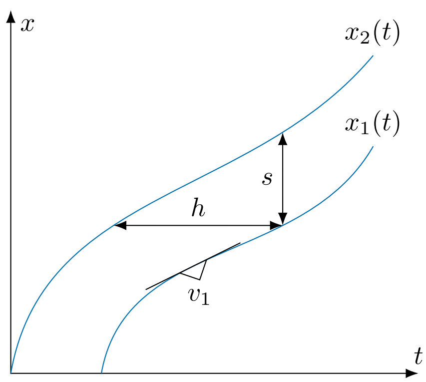


Figure 2: microscopic traffic properties

Trajectories help us visualize **microscopic traffic properties**. Consider the two vehicles with trajectories x_1 and x_2 in Fig. 2. For these two vehicles, we define *spacing* and *headway* to be:

- **spacing**, s
  the distance measured from nose-to-nose or tail-to-tail between two vehicles
- **headway**, h
  the time between when two veh's pass a point

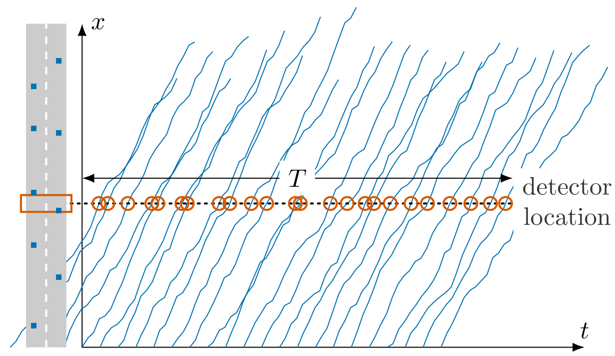
The first derivative of $x(t)$ is its speed, and its second derivative is its acceleration:

$$v(t) = \dot{x}(t) \tag{1}$$
$$a(t) = \ddot{x}(t) \tag{2}$$

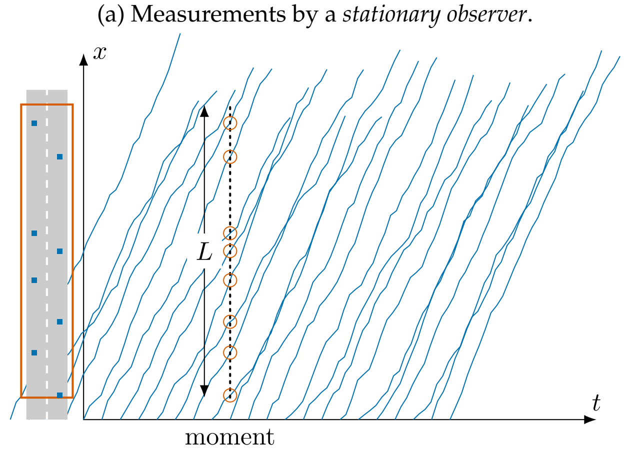
To talk about macroscopic traffic flow properties, we speak of two observers:

(i) a **stationary observer (SO)** who counts vehicles pass over time, at a fixed location. See Fig. 3a.

(ii) an **overhead observer (OO)** who counts vehicles along a length of road, at a fixed moment in time. See Fig. 3b.



(a) Measurements by a *stationary observer*.



(b) Measurements by an *overhead observer*

Figure 3: stationary vs overhead observers

These observers figure in several definitions:

- **average spacing** $\bar{s}$
  the average spacing between vehicles observed by an overhead observer
- **average headway** $\bar{h}$
  the average headway between vehicles observed by a stationary observer
- **density** $k = M/L \approx 1/\bar{s}$
  where M is the number of vehicles an overhead observer counts on a segment of road with length L
- **flow** $q = N/T \approx 1/\bar{h}$
  where N is the # of veh's a stationary observers counts pass during time T

- **time-mean speed** $\overline{v}_t$
  the average speed measured by a stationary observer. If an SO counts $i = 1, 2, \ldots, N$ vehicles, then time-mean speed is

$$\overline{v}_t = \frac{1}{N} \sum_{i=1}^{N} v_i. \tag{3}$$

- **space-mean speed** $\overline{v}_s$
  the average speed measured by an overhead observer. If an OO counts $j = 1, 2, \ldots, M$ vehicles, then the space-mean speed is

$$\overline{v}_s = \frac{1}{M} \sum_{k=1}^{M} v_j. \tag{4}$$

If all vehicles on a road have the same speed, then $\overline{v}_s = \overline{v}_t$. If there is a distribution of speeds, then $\overline{v}_s < \overline{v}_t$.

2 Stationary + homogeneous traffic

When traffic is stationary and homogeneous (i.e., when it measurements are the same at any time or any place), then two relationships hold.

First, a stationary observer can estimate space-mean speed by the following formula:

$$\overline{v}_s = \frac{N}{\sum_{i=1}^{N} \frac{1}{v_i}}, \tag{5}$$

where $i = 1, 2, \ldots, N$ are the indices of the N vehicles that the SO counts. In other words, space-mean speed becomes the *harmonic mean* of speeds measured.

Second, there is a relationship called the **fundamental identity**:

$$q = k\overline{v}_s. \tag{6}$$

This is useful. If you know two variables (e.g., flow and density) then you can get the third one.

3 Fundamental Diagram

In classical traffic theory, we presume a homogeneous section of road exhibits a function $q(k)$ giving flow as a function of density.

We usually plot the Fundamental Diagram in k/q space, as in Figure 4. A **traffic state** is a point in k/q space: a combination of a density, k, and a flow, q. For example, in Figure 4, S is a state on a road's fundamental diagram. By the relationship $q = kv$, each state is also associated with a traffic speed $v = q/k$. Visually, this is the slope of a line from the origin to the state, such as the slope v_S.

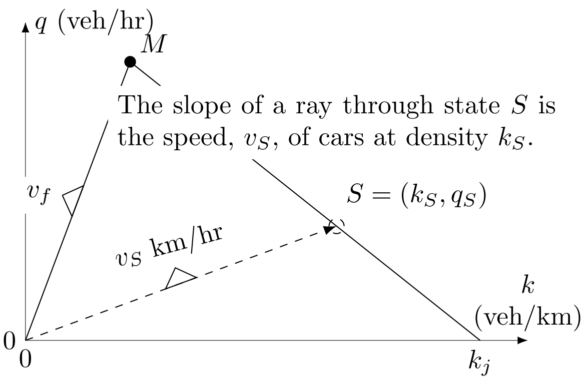


Figure 4: traffic speed as slope in k/q space

Fundamental digrams generally exhibit four properties, captured by the following variables:

- **capacity**, q_M (veh/hr)
  the maximum possible flow
- **critical density**, k_M (veh/hr)
  the density where the maximum flow is achieved
- **jam density**, k_j (veh/km)
  the density at which vehicles cannot move
- **free-flow speed**, v_f (km/hr)
  the maximum speed traffic can achieve, when there is no congestion to slow it down.

In this class, we generally work with *triangular* fundamental diagrams. Figure 5 illustrates. The triangular fundamental diagram can be written as a function:

$$q(k) = \min\left(v_f \cdot k, q_M \cdot \frac{k_j - k}{k_j - k_M}\right)$$

Traffic states on the *left side* of the fundamental diagram are called **uncongested** states. States on the right side are called **queued** or **congested** states. A **queue** is a block of traffic in a queued state, such as a line of vehicles waiting at a stop light or trying to pass through a tollbooth.

If you need to find the density in the *congested* regime which corresponds to a certain flow, you can use the formula:

$$k(q) = k_j - \frac{q}{q_M}(k_j - k_M). \tag{7}$$

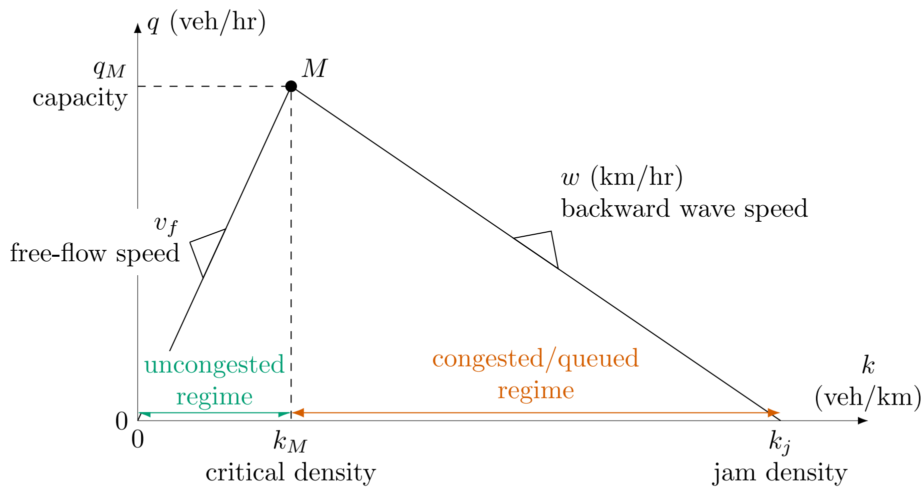


Figure 5: triangular fundamental diagram

4 Traffic Dynamics

The fundamental diagram allows us to make predictions about how traffic dynamics unfold.

Interfaces

Suppose that a road has one segment where the vehicles are in state A, and next to it is another segment where vehicles are in state B. See Figure 6. The boundary between them is called an **interface**.

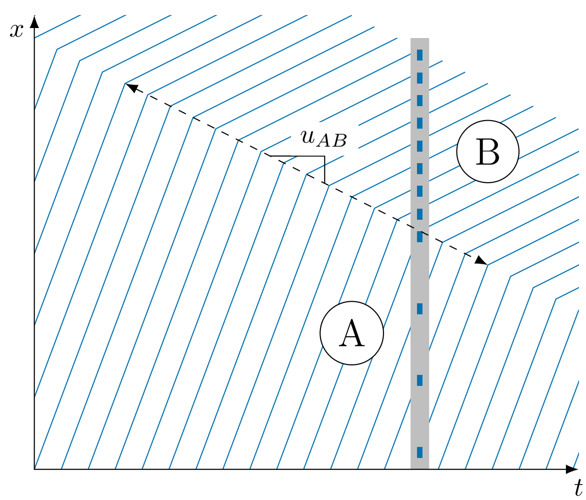


Figure 6: The **trajectory of an interface** in t/x space. Figure shows the road at a time $t = \hat{t}$. State A has a lower density than B. The interface moves upstream with velocity u_{AB}.

Often, as in Figure 6, the interface moves, either upstream (against traffic) or downstream (with traffic). In Fig. 6, the interface is moving upstream. The velocity of an interface between states A and B is found by the formula

$$u_{AB} = \frac{q_A - q_B}{k_A - k_B} \quad \text{(km/hr)}, \tag{8}$$

where k_A, q_A is the density/flow of state A, and k_B, q_B is the density/flow of state B. If you look at things in k/q space, then u_{AB} is equal to the slope of a line between the states A and B on the road's Fundamental Diagram. See Figure 7.

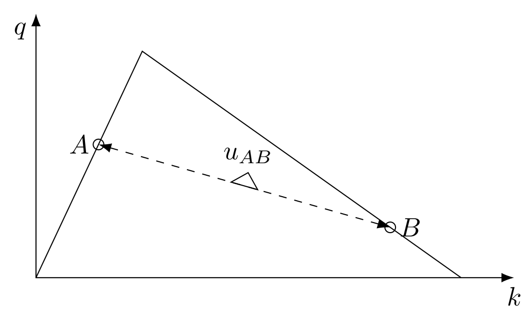


Figure 7: interface velocity in k/q space

Bottlenecks

A key class of problem involves **bottlenecks**: restrictions that limit a road's flow below its capacity. Consider this scenario, similar to your homework. Traffic on a road arrives from upstream at a constant rate q_U (veh/hr). This traffic moves freely in the state U shown in Figure 9. This is the road (0) in Fig. 8.

At some point, a bottleneck opens on the road with a capacity of μ, which is less than q_U. Hence a queue

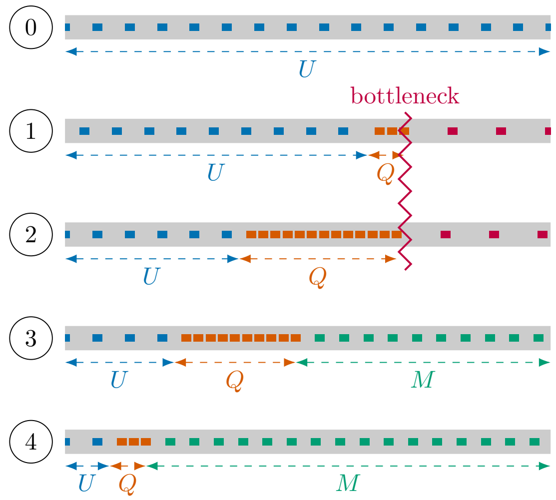


Figure 8: traffic states along the road

begins to form at the bottleneck, like the road 1. The flow in this queue will be equal to the capacity of the bottleneck. The traffic in the queue will exhibit state $Q = (k_Q, \mu)$, as shown in Figure 9: Q **is the congested state with the same flow as the bottleneck capacity**.

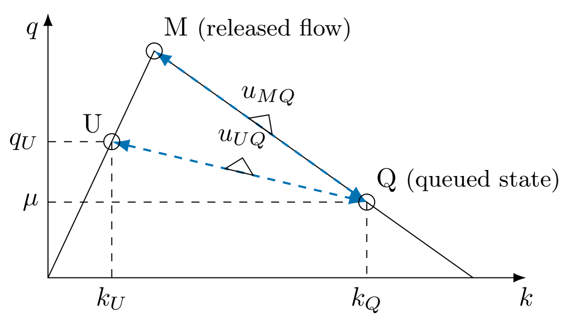


Figure 9: bottleneck states and interfaces

We have then, an interface between states U and Q. This interface moves with velocity u_{UQ}. Note that u_{UQ} is negative (the slope is downward), so that means the interface moves *upstream*: that is, the queue grows upstream along the road. *Note that does not mean the vehicles move backwards.* Thus, at time 2 of Fig. 8 the queue is longer.

Suppose next that the bottleneck is removed. Now, the released vehicles begin to move out of the bottleneck as fast as they can. This *released flow* is in state M…the state where the fundamental diagram has its capacity. See time 3 in Figure 8. So we have a *new* interface at the *front* of the queue: between states M and Q. This interface has speed u_{MQ}.

Note u_{MQ} also slopes downwards (so the new interface moves upstream, too). But it slopes more steeply than u_{UQ} does, so the front of the queue in this situation moves upstream *faster* than the back of the queue does. Hence the queue shrinks, and so in time 4 the queue has become small. Eventually, the front will "catch" the back, whereupon the queue dissipates.

5 Cumulative counts

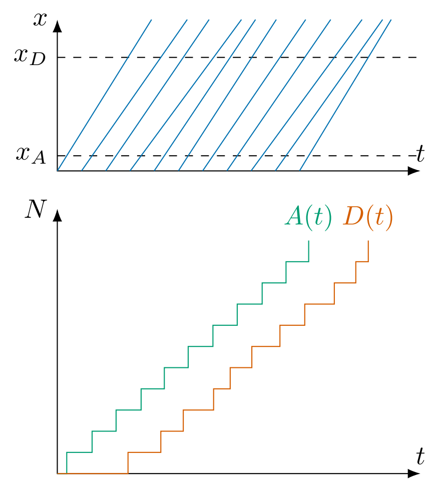


Figure 10: Detectors at x_A and x_D and associated cumulative counts

Figure 10 shows a road segment between two detectors: one upstream (at x_A) and one downstream (at x_D). $A(t)$ counts cumulative **arrivals** (vehicles crossing x_A) and $D(t)$ counts cumulative **departures** (vehicles crossing x_D).

If there are many vehicles, $A(t)$ and $D(t)$ become smooth lines like in Figure 11. In this figure…

- $M(t)$ is the vertical distance between the two curves at time t: the number of vehicles between x_A and x_D
- $w(n)$ is the travel time of vehicle n.
- The slopes of $A(t)$ and $D(t)$ are the flows of vehicles passing the upstream and downstream detectors, respectively.

To analyze congestion, we add a third curve $V(t)$ like in Figure 12, which shows what happens when there is a bottleneck with capacity μ between x_A and x_D.

- w_f (hr) is the **free-flow travel time** between x_A and x_D.
- $V(t)$ is the **virtual departure curve**: the cumu-

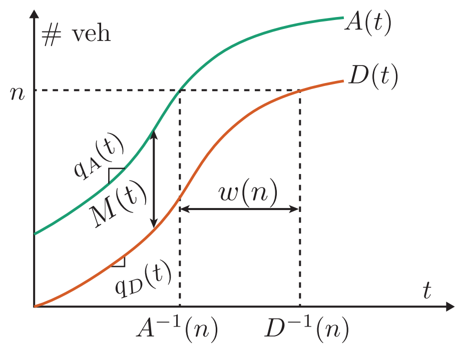


Figure 11: Cumulative arrivals and departures

lative number of vehicles that *would* pass x_D if vehicles moved at free-flow speed. It is just $A(t)$ shifted right by w_f.

– $d(n) = w(n) - w_f$ is the **delay** for vehicle n: the amount of extra travel time over the free-flow travel time.

– The area between V and D is **total delay**: the sum of $d(n)$ across all the vehicles.

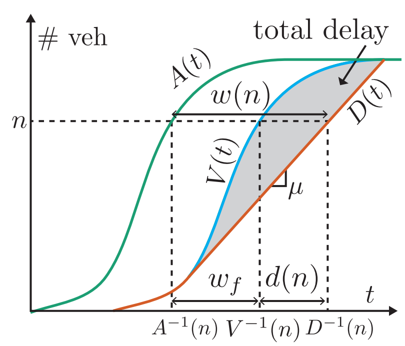


Figure 12: Virtual arrival curve

6 Vertical queues

When queues are very short, we can pretend that they have no length, and that $A(t) = V(t)$. The vertical queue problems we look at in this class involve bottlenecks with fixed capacity, μ. In this case, the vertical distance between $A(t)$ and $D(t)$ is the number of vehicles in queue at time t.

For such situations, we are given information about arrival flows, and we use it to construct $A(t)$. Then we use $A(t)$ to construct $D(t)$ using some rules:

I. The number of vehicles in queue is $M(t) = A(t) - D(t)$: the # of vehicles which have arrived at but not departed the bottleneck.

II. $D'(t) = \mu$ if $A(t) > D(t)$. That is, the bottleneck serves vehicle at capacity if there is a queue.

III. $D'(t) = \min[\mu, A'(t)]$ if $A(t) = D(t)$. If there is no queue (i.e., if $A(t) = D(t)$), then the bottleneck serves vehicles at a rate which is the minimum of the arrival rate, $A'(t)$, and the capacity, μ.

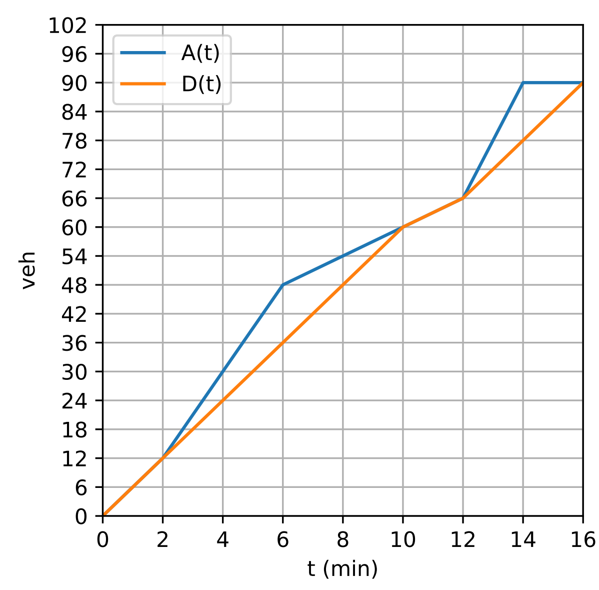
The **wait time** or **delay** or **time spent queueing** of a vehicle n is the horizontal distance between D and A for vehicle n.

The **total/aggregate delay/wait time** is the area between $A(t)$ and $D(t)$.

Example

– A parking gate has capacity $\mu = 6$ veh/min

– Vehicles arrive at the following rates, which are $A'(t)$ (the slope of $A(t)$)
  - 6 veh/min for 2 min
  - 9 veh/min for 4 min
  - 3 veh/min for 6 min
  - 12 veh/min for 2 min
  - 0 veh/min after



To solve this, we use the $A'(t)$ info to draw $A(t)$. We see at first that $A(t) = D(t)$. Then at $t = 2$, $A' > \mu$, so a queue starts to form, because vehicles are arriving at the gate faster than the gate can handle them. At $t = 6$, the arrival rate falls, but the queue is still there so $D'(t) = \mu$ still. The queue dies at $t = 10$, when $A(t) = D(t)$ again. So from $t = 10$ to $t = 12$, $D'(t) = A'(t)$ again. Then from $t = 12$ to $t = 14$,

$A'(t) > \mu$ again, and the queue builds. At $t = 14$, no more vehicles arive, but it takes 2 more minutes for the queue to die off.

7 Stop spacing

Each time a bus stops, it loses some time to braking, acceleration and dwell time. Suppose a bus' cruising speed is v_b (km/hr), that it brakes at a rate a_B (km/hr^2) and accelerates at rate a_A. Suppose the bus spends at least T_{dwell} stopped due to opening/closing doors etc. In this case, the bus' trajectory around a stop looks like in Fig. 13. τ is the **stop time** or **stop delay**: the time the bus loses each time it stops.

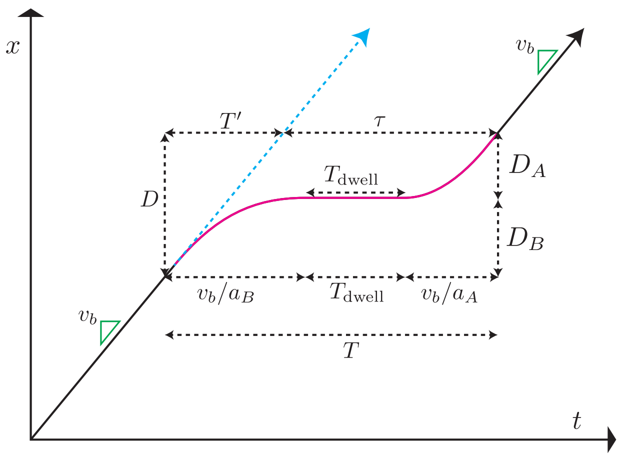


Figure 13: bus trajectory around a stop

Stop spacing affects travel times. Consider a bus route with stop spacing s (km/stop). The bus takes

$$\left(\frac{1}{v_b} + \frac{\tau}{s}\right) \quad \text{(hr/km)}$$

to go one kilometer. This is the bus' **unit travel time**. So if a person travels x (km) on the bus, they spend

$$x \cdot \left(\frac{1}{v_b} + \frac{\tau}{s}\right) \quad \text{(hr)} \tag{9}$$

on the bus.

If origins and destinations are uniformly distributed over the route, then the average access (walking) distance to/from stops is $s/2$. Suppose the **access speed** is v_w (km/hr). The average **access time** is $1/2sv_w$.

If the route has a headway H, and passengers arrive randomly at stops, then the average wait time is $H/2$. Thus, the average travel time of a trip traveling x (km) on the bus is

$$T = \frac{H}{2} + \frac{s}{2v_w} + x\left(\frac{1}{v_b} + \frac{\tau}{s}\right) \quad \text{(hr)}. \tag{10}$$

8 Fleet sizes

Consider an isolated bus route. Suppose it takes W (hr) for one bus to complete the route. (*You may have to calculate W based on other information.*) W is the route's **cycle time**. We are interested in the **minimum fleet size** required to serve the route while meeting two constraints.

The **first** constraint is a **maximum headway**: **how many buses do you need to have a headway no larger than $H_{\max}$ (hr/bus)?**

$$B \geq W/H_{\max} \quad \text{(bus)}. \tag{11}$$

So you calculate $W/H_{\max}$ and then choose the next largest integer. Since $1/H_{max}$ is a frequency, the right-hand side of this equation is Little's Law.

The **second** constraint is a **minimum capacity**: **how many buses do you need to ensure all the passengers can fit?** Suppose that each bus has a capacity of $P_{\max}$ passengers, demand on the route is Q pax/hr. Suppose also that each passenger spends W_{pax} on the bus. Then you need

$$\underbrace{B \cdot P_{\max}}_{\text{total capacity}} \geq \underbrace{Q \cdot W_{\text{pax}}}_{\text{pax on bus at once}} \tag{12}$$

Again, this is Little's Law. The right-hand side is the total number of passengers on the bus at once.

We have two minima (one for each constraint), so the minimum fleet size required is the larger of the two minima.