Automorphism groups of homogeneous structures with stationary weak independence relations

Yibei Li

Imperial College London

SEEMOD, University of Cambridge

Theorem (Tent and Ziegler, 2012)

For any unbounded isometry g of the Urysohn space the normal subgroup $\langle g \rangle^G$ is all of G. In fact, any element of G is the product of eight conjugates of g.

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Let $\mathcal{M}$ be a countable homogeneous structure with a stationary independence relation. If $g \in Aut(\mathcal{M})$ moves almost maximally, then any element of $Aut(\mathcal{M})$ is the product of sixteen conjugates of g.

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Let M be a free homogeneous structure such that $Aut(M)$ is transitive on M but is not equal to $Sym(M)$. Then $Aut(M)$ is a simple.

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Background: Amalgamation classes

Definition

An $\mathcal{L}$-structure $\mathcal{M}$ is homogeneous if isomorphisms beteween finite substructures extend to automorphisms of $\mathcal{M}$.

Definition

Let $\mathcal{L}$ be a relational language, an amalgamation class is a class $\mathcal{C}$ of finite $\mathcal{L}$-structures satisfying the following three conditions:

1. Hereditary property: *For every $A \in \mathcal{C}$ and any substructure $B \subseteq A$ we have $B \in \mathcal{C}$;*
2. Joint embedding property: *For every $A, B \in \mathcal{C}$ there exists $C \in \mathcal{C}$ such that C contains both A and B as substructures;*
3. Amalgamation property: *For A, B, C and $f_1 : B \to A$, $f_2 : B \to C$ are embeddings, there is $D \in \mathcal{C}$ and embeddings $g_1 : A \to D$, $g_2 : C \to D$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*

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Background: Fraïssé's Theorem

Theorem (Fraïssé's Theorem)

There is a one-to-one correspondence between amalgamation classes and countable homogeneous structures.

$$\begin{tikzcd} \boxed{\text{Countable Homogeneous structures } \mathcal{M}} \arrow[d, <->, "\mathcal{C} = \{\text{all finite substructure of } \mathcal{M}\}"', "\mathcal{M} \text{ embeds every structure in } \mathcal{C}"] \\ \boxed{\text{Amalgamation classes } \mathcal{C}} \end{tikzcd}$$

Stationary weak independence relation

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Definition

Let $\mathcal{M}$ be a homogeneous structure and suppose $A \mathop{\smile\hspace{-0.9em}|}_B C$ is a ternary relation among finite substructure A, B, C of $\mathcal{M}$. We say that $\mathop{\smile\hspace{-0.9em}|}$ is a stationary weak independence relation if the following axioms are statisfied:

(i) *Invariance: for any $g \in Aut(\mathcal{M})$, if $A \mathop{\smile\hspace{-0.9em}|}_B C$, then $gA \mathop{\smile\hspace{-0.9em}|}_{gB} gC$*

(ii) *Monotonicity:* $A \mathop{\smile\hspace{-0.9em}|}_B CD \Rightarrow A \mathop{\smile\hspace{-0.9em}|}_B C, A \mathop{\smile\hspace{-0.9em}|}_{BC} D$
$$AD \mathop{\smile\hspace{-0.9em}|}_B C \Rightarrow A \mathop{\smile\hspace{-0.9em}|}_B C, D \mathop{\smile\hspace{-0.9em}|}_{AB} D$$

(iii) *Transitivity:* $A \mathop{\smile\hspace{-0.9em}|}_B C, A \mathop{\smile\hspace{-0.9em}|}_{BC} D \Rightarrow A \mathop{\smile\hspace{-0.9em}|}_B D$
$$A \mathop{\smile\hspace{-0.9em}|}_B C, D \mathop{\smile\hspace{-0.9em}|}_{AB} C \Rightarrow D \mathop{\smile\hspace{-0.9em}|}_B C$$

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  $A \mathop{\smile\!\!\!\!\!|}_B C,\ D \mathop{\smile\!\!\!\!\!|}_{AB} C \Rightarrow D \mathop{\smile\!\!\!\!\!|}_B C$

Stationary weak independence relation

Definition (continued)

1. *Existence: If p is a type over B and C is a finite set, then p has a realisation a such that $a \mathop{\smile\!\!\!\!\!\!|}_B C$.*
   If p is a type over B and C is a finite set, then p has a realisation a such that $C \mathop{\smile\!\!\!\!\!\!|}_B a$.
2. *Stationarity: If a and a' have the same type over B and $a \mathop{\smile\!\!\!\!\!\!|}_B C$, $a' \mathop{\smile\!\!\!\!\!\!|}_B C$, then a and a' have the same type over BC.*
   If a and a' have the same type over B and $C \mathop{\smile\!\!\!\!\!\!|}_B a$, $C \mathop{\smile\!\!\!\!\!\!|}_B a'$, then a and a' have the same type over BC.

It is a stationary independence relation if in addition, it satisfies

Symmetry: $A \mathop{\smile\!\!\!\!\!\!|}_B C \Rightarrow C \mathop{\smile\!\!\!\!\!\!|}_B A$

Examples

- the class of all finite graphs with free amalgamation
  Fraïssé limit: the random graph
- the class of all finite K_n-free graphs with free amalgamation
  Fraïssé limit: the universal K_n-free graph
- the class of all finite rational metric spaces
  Fraïssé limit: the universal homogeneous rational metric space
  Its completion is the Urysohn space.
- the class of all finite edge-coloured graphs without some forbidden triangles with semi-free amalgamation (Cherlin)
- the class of all finite linear orders
  Fraïssé limit: dense linear order $(\mathbb{Q}, \leq)$
- linearly ordered free amalgamation classes

Main result

Definition

$g \in Aut(\mathcal{M})$ *moves almost R-maximally if for every type over a finite set* X *has a realisation* a *such that* $a \mathop{\smile\!\!\!\!\!\!\mid}_X ga$.

$g \in Aut(\mathcal{M})$ *moves almost L-maximally if for every type over a finite set* X *has a realisation* a *such that* $ga \mathop{\smile\!\!\!\!\!\!\mid}_X a$.

Theorem

Let $\mathcal{M}$ *be a countable homogeneous structure with a stationary weak independence relation. If* $g \in Aut(\mathcal{M})$ *such that* g *and* g^{-1} *moves almost R-maximally and L-maximally, then any element of* $Aut(\mathcal{M})$ *is the product of sixteen conjugates of* g.

Applications

- $(\mathbb{Q}, \leq)$
- *Cherlin's directed graphs determined by forbidden triangles*

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Reference

- *G. L. Cherlin. The Classification of Countable Homogeneous Directed Graphs and Countable Homogeneous n-Tournaments. American Mathematical Soc., 1998.*
- *K. Tent and M. Ziegler. "On the isometry group of the Urysohn space". In: Journal of the London Mathematical Society 87.1 (2013), pp. 289-303.*