Automorphism groups of homogeneous structures with stationary weak independence relations

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For any unbounded isometry g of the Urysohn space the normal subgroup $\langle g \rangle^G$ is all of G. In fact, any element of G is the product of eight conjugates of g.

Theorem (Tent and Ziegler, 2012)

Let \mathcal{M} be a countable homogeneous structure with a stationary independence relation. If $g \in \operatorname{Aut}(\mathcal{M})$ moves almost maximally, then any element of $\operatorname{Aut}(\mathcal{M})$ is the product of sixteen conjugates of g.

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Let M be a free homogeneous structure such that Aut(M) is transitive on M but is not equal to Sym(M). Then Aut(M) is a simple.

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Background: Amalgamation classes

Definition

An \mathcal{L} -structure \mathcal{M} is homogeneous if isomorphisms between finite substructures extend to automorphisms of \mathcal{M} .

Definition

Let $\mathcal L$ be a relational language, an amalgamation class is a class $\mathcal C$ of finite $\mathcal L$ -structures satisfying the following three conditions:

- Hereditary property: For every $A \in C$ and any substructure $B \subseteq A$ we have $B \in C$;
- ② Joint embedding property: For every $A, B \in \mathcal{C}$ there exists $C \in \mathcal{C}$ such that C contains both A and B as substructures;
- ⓐ Amalgamation property: For A, B, C and $f_1: B \to A$, $f_2: B \to C$ are embeddings, there is D ∈ C and embeddings $g_1: A \to D$, $g_2: C \to D$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

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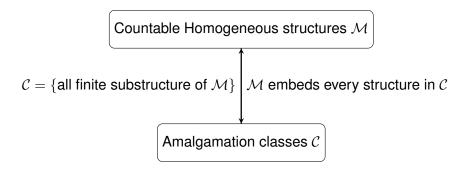
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Background: Fraïssé's Theorem

Theorem (Fraïssé's Theorem)

There is a one-to-one correspondence between amalgamation classes and countable homogeneous structures.



Stationary weak independence relation

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Definition

Let \mathcal{M} be a homogeneous structure and suppose $A \bigcup_B C$ is a ternary relation among finite substructure A, B, C of \mathcal{M} . We say that \bigcup is a stationary weak independence relation if the following axioms are statisfied:

- Invariance: for any $g \in Aut(M)$, if $A \bigcup_B C$, then $gA \bigcup_{aB} gC$
- Monotonicity: $A \downarrow_B CD \Rightarrow A \downarrow_B C$, $A \downarrow_{BC} D$ $AD \downarrow_B C \Rightarrow A \downarrow_B C$, $D \downarrow_{AB} D$
- Transitivity: $A \bigcup_{B} C$, $A \bigcup_{BC} D \Rightarrow A \bigcup_{B} D$ $A \bigcup_{B} C$, $D \bigcup_{AB} C \Rightarrow D \bigcup_{B} C$

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Stationary weak independence relation

Definition (continued)

- Existence: If p is a type over B and C is a finite set, then p has a realisation a such that a \(\sum_B C \).
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 - If p is a type over B and C is a finite set, then p has a realisation a such that $C \bigcup_B a$.
- Stationarity: If a and a' have the same type over B and a $\bigcup_B C$, a' $\bigcup_B C$, then a and a' have the same type over B and C $\bigcup_B a$,
 - If a and a' have the same type over B and $C \bigcup_B a$, $C \bigcup_B a'$, then a and a' have the same type over BC.

It is a stationary independence relation if in addition, it satisfies Symmetry: $A \bigcup_B C \Rightarrow C \bigcup_B A$

Examples

- the class of all finite graphs with free amalgamation Fraïssé limit: the random graph
- the class of all finite K_n -free graphs with free amalgamation Fraïssé limit: the universal K_n -free graph
- the class of all finite rational metric spaces
 Fraïssé limit: the universal homogeneous rational metric space
 Its completion is the Urysohn space.
- the class of all finite edge-coloured graphs without some forbidden triangles with semi-free amalgamation (Cherlin)
- the class of all finite linear orders Fraïssé limit: dense linear order (\mathbb{Q}, \leq)
- linearly ordered free amalgamation classes

Main result

Definition

 $g \in Aut(\mathcal{M})$ moves almost R-maximally if for every type over a finite set X has a realisation a such that a \bigcup_X ga.

 $g \in Aut(\mathcal{M})$ moves almost L-maximally if for every type over a finite set X has a realisation a such that $ga \bigcup_{Y} a$.

Theorem

Let \mathcal{M} be a countable homogeneous structure with a stationary weak independence relation. If $g \in Aut(\mathcal{M})$ such that g and g^{-1} moves almost R-maximally and L-maximally, then any element of $Aut(\mathcal{M})$ is the product of sixteen conjugates of g.

Applications

- \bullet (\mathbb{Q}, \leq)
- Cherlin's directed graphs determined by forbidden triangles



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Reference

- G. L. Cherlin. The Classification of Countable Homogeneous Directed Graphs and Countable Homogeneous n-Tournaments. American Mathematical Soc., 1998.
- K. Tent and M. Ziegler. "On the isometry group of the Urysohn space". In: Journal of the London Mathematical Society 87.1 (2013), pp. 289-303.