Automorphism groups of some homogeneous directed graphs

Midsummer Combinatorial Workshop XXIV

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Automorphism group of the random graph

Theorem (Truss, 1985)

The automorphism group of the random graph is simple.

Theorem (Macpherson and Tent, 2011)

Let M be a free homogeneous structure such that $Aut(M)$ is transitive on M but is not equal to $Sym(M)$. Then $Aut(M)$ is a simple.

Theorem (Tent and Ziegler, 2012)

Let $\mathcal{M}$ be a countable homogeneous structure with a stationary independence relation. If $g \in Aut(\mathcal{M})$ moves almost maximally, then any element of $Aut(\mathcal{M})$ is the product of sixteen conjugates of g.

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Stationary independence relation (SIR)

Definition

Let $\mathcal{M}$ be a structure. $\mathop{\smile\!\!\!\!\!\!|}$ is a stationary independence relation (SIR) if the following is satisfied for any substructure $A, B, C, D \subseteq \mathcal{M}$:

- *Invariance: A, C independence over B depends only on the type of ABC*
- *Monotonicity: $A \mathop{\smile\!\!\!\!\!\!|}_B CD \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B C, A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$*
- *Transitivity: $A \mathop{\smile\!\!\!\!\!\!|}_B C, A \mathop{\smile\!\!\!\!\!\!|}_{BC} D \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B D$*
- *Symmetry: $A \mathop{\smile\!\!\!\!\!\!|}_B C \Rightarrow C \mathop{\smile\!\!\!\!\!\!|}_B A$*
- *Existence: If p is a type over B and C is a finite set, then p has a realisation that is independent from C over B*
- *Stationarity: If $\bar{a}$ and $\bar{a}'$ have the same type over B and are both independent from C over B, then $\bar{a}$ and $\bar{a}'$ have the same type over BC.*

SIR on the random graph

Theorem

Let R be the random graph. For any finite substructure $A, B, C \subseteq R$, define $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ if for any $a \in A \setminus B, c \in C \setminus B$, a, c are not connected. Then $\mathop{\smile\!\!\!\!\!\!\mid}$ is a SIR.

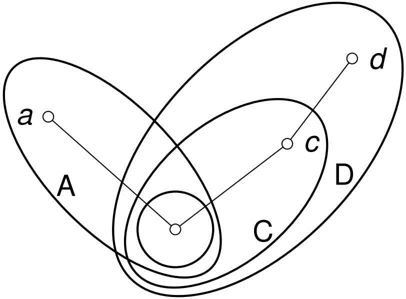
For example, let's check transitivity: $A \mathop{\smile\!\!\!\!\!\!\mid}_B C, A \mathop{\smile\!\!\!\!\!\!\mid}_{BC} D \Rightarrow A \mathop{\smile\!\!\!\!\!\!\mid}_B D$.

SIR on the random graph

Theorem

Let R be the random graph. For any finite substructure $A, B, C \subseteq R$, define $A \downfree_B C$ if for any $a \in A \setminus B, c \in C \setminus B$, a, c are not connected. Then $\downfree$ is a SIR.

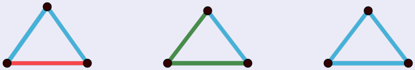
For example, let's check transitivity: $A \downfree_B C, A \downfree_{BC} D \Rightarrow A \downfree_B D$.



Cherlin's graphs

Example

Suppose we have three choices of colours-red, green and blue. We want to colour the edges of the countable complete graph using these colours randomly in a way such that the following triangles do not appear.



Denote the resulting graph by $\mathcal{M}$. We can find a SIR on $\mathcal{M}$ by 'putting an order' on the colours. For finite substructure $A, B, C \subseteq \mathcal{M}$, we can define $A \mathop{\smile\!\!\!\!\!\!|}_B C$ if for any $a \in A \setminus B, c \in C \setminus B$, (a, c) is coloured by red if doing so does not create a forbidden triangle. Otherwise, it is coloured by green. It can be shown that $\mathop{\smile\!\!\!\!\!\!|}$ is a SIR.

Theorem

$Aut(\mathcal{M})$ is simple.

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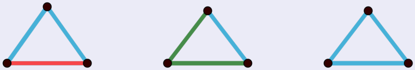

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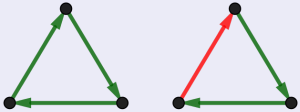
$Aut(\mathcal{M})$ is simple.

Cherlin's digraphs

Example

Suppose we have two choices of colours-red and green. We want to choose a colour and a direction randomly for every edge of the countable complete graph in a way such that the following triangles do not appear.



Denote the resulting graph by $\mathcal{M}$. For finite substructure $A, B, C \subseteq \mathcal{M}$.

Question

is there a SIR on $\mathcal{M}$? Is $Aut(\mathcal{M})$ simple?

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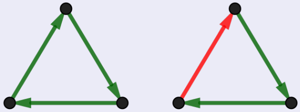

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Stationary weak independence relation (SWIR)

Definition

Let $\mathcal{M}$ be a homogeneous structure and suppose $A \mathop{\smile\!\!\!\!\!\!|}_B C$ is a ternary relation among finite substructure A, B, C of $\mathcal{M}$. We say that $\mathop{\smile\!\!\!\!\!\!|}$ is a stationary weak independence relation if the following axioms are statisfied:

- *Invariance: for any $g \in Aut(\mathcal{M})$, if $A \mathop{\smile\!\!\!\!\!\!|}_B C$, then $gA \mathop{\smile\!\!\!\!\!\!|}_{gB} gC$*
- *Monotonicity:* $A \mathop{\smile\!\!\!\!\!\!|}_B CD \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B C,\ A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$
  $AD \mathop{\smile\!\!\!\!\!\!|}_B C \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B C,\ D \mathop{\smile\!\!\!\!\!\!|}_{AB} D$
- *Transitivity:* $A \mathop{\smile\!\!\!\!\!\!|}_B C,\ A \mathop{\smile\!\!\!\!\!\!|}_{BC} D \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B D$
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Stationary weak independence relation (SWIR)

Definition (continued)

- *Existence: If p is a type over B and C is a finite set, then p has a realisation a such that $a \mathop{\smile\!\!\!\!\!\mid}_B C$.*
  If p is a type over B and C is a finite set, then p has a realisation a such that $C \mathop{\smile\!\!\!\!\!\mid}_B a$.
- *Stationarity: If a and a' have the same type over B and $a \mathop{\smile\!\!\!\!\!\mid}_B C$, $a' \mathop{\smile\!\!\!\!\!\mid}_B C$, then a and a' have the same type over BC.*
  If a and a' have the same type over B and $C \mathop{\smile\!\!\!\!\!\mid}_B a$, $C \mathop{\smile\!\!\!\!\!\mid}_B a'$, then a and a' have the same type over BC.

Theorem

Let $\mathcal{M}$ be a countable homogeneous structure with a stationary weak independence relation. If $g \in Aut(\mathcal{M})$ moves almost R-maximally and L-maximally, then any element of $Aut(\mathcal{M})$ is the product of sixteen conjugates of g.

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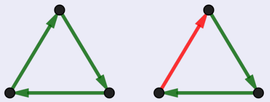

Denote the resulting graph by $\mathcal{M}$. For finite substructure $A, B, C \subseteq \mathcal{M}$. we can define $A \mathop{\smile\!\!\!\!\!\!|}_B C$ if for any $a \in A \setminus B, c \in C \setminus B$, (a, c) is coloured by red with direction $a \to c$ if doing so does not create a forbidden triangle. Otherwise, it is coloured by red with direction $c \to a$. It can be checked that $\mathop{\smile\!\!\!\!\!\!|}$ is a SWIR.

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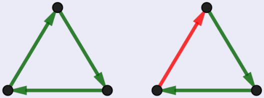

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$Aut(\mathcal{M})$ is simple.

Further applications

- dense linear order $(\mathbb{Q}, \leq)$
- the linearly ordered random graph
  (linearly ordered free homogeneous structure)

Reference

- *G. L. Cherlin. The Classification of Countable Homogeneous Directed Graphs and Countable Homogeneous n-Tournaments. American Mathematical Soc., 1998.*
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