Model Theoretic Approach to Homogeneous Structures and their Automorphism Groups

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Overview

- Amalgamation classes and semi-free amalgamation.
- Stationary independence relation on the structures.
- Automorphism groups of structures with the stationary independence relation

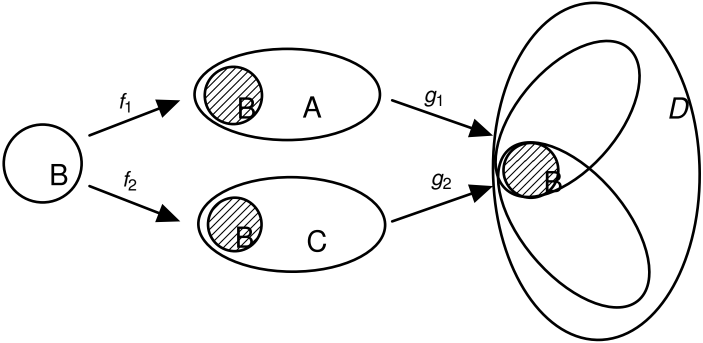
Amalgamation Class

Definition

Let $\mathcal{L}$ be a relational language, an amalgamation class is a class $\mathcal{C}$ of finite $\mathcal{L}$-structures satisfying the following three conditions:

1. Hereditary property: *For every $A \in \mathcal{C}$ and any substructure $B \subseteq A$ we have $B \in \mathcal{C}$;*
2. Joint embedding property: *For every $A, B \in \mathcal{C}$ there exists $C \in \mathcal{C}$ such that C contains both A and B as substructures;*
3. Amalgamation property: *For A, B, C and $f_1 : B \to A$, $f_2 : B \to C$ are embeddings, there is $D \in \mathcal{C}$ and embeddings $g_1 : A \to D, g_2 : C \to D$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*

Amalgamation Property



Examples

- the class of all finite graphs with free amalgamation

Definition

We say A, C are freely amalgamated over B, if for any $a \in g_1(A) \setminus g_1 f_1(B), c \in g_2(C) \setminus g_2 f_2(B)$, a, c are not in any relation.

- the class of all finite K_n-free graphs
- the class of all finite rational metric spaces

Fraïssé's Theorem

Definition

An $\mathcal{L}$-structure $\mathcal{M}$ is homogeneous if isomorphisms beteween finite substructures extend to automorphisms of $\mathcal{M}$.

Theorem (Fraïssé's Theorem)

There is a one-to-one correspondence between amalgamation classes and countable homogeneous structures, called the Fraïssé limit

$$\begin{tikzcd}
\boxed{\text{Countable Homogeneous structures } \mathcal{M}} \arrow[d, leftrightarrow, "\mathcal{M} \text{ embeds every structure in } \mathcal{C}", "\mathcal{C} = \{ \text{ all finite substructure of } \mathcal{M}\}"'] \\
\boxed{\text{Amalgamation classes } \mathcal{C}}
\end{tikzcd}$$

Examples

- the class of all finite graphs with free amalgamation
  Fraïssé limit: the random graph
- the class of all finite K_n-free graphs
  Fraïssé limit: the generic K_n-free graph
- the class of all finite rational metric spaces
  Fraïssé limit: the universal homogeneous rational metric space
  Its completion is the Urysohn space.

Semi-free amalgamation

Definition

$\mathcal{C}$ is a semi-free amalgamation class if there exists $\mathcal{L}' \subsetneq \mathcal{L}$ such that for any finite structures $A, B, C \in \mathcal{C}$ and embeddings $f_1 : B \to A, f_2 : B \to C$, there exist $D \in \mathcal{C}$ and embeddings $g_1 : A \to D, g_2 : C \to D$ such that $g_1 f_1(B) = g_2 f_2(B) = g_1(A) \cap g_2(C)$ and for any $a \in g_1(A) \setminus g_1 f_1(B), c \in g_2(C) \setminus g_2 f_2(B)$, a, c are in some relation $R \in \mathcal{L}'$.

Note this is a generalisation of the free amalgamation.

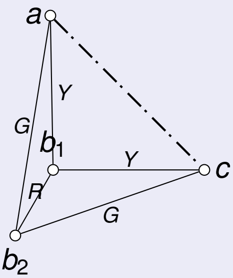
Semi-free amalgamation classes

Let $\mathcal{L}$ a relational language, semi-free amalgamation class can be given in the form of forbidden triangles:

- S a set of the forbidden triangles
- $Forb^c(S)$ the set of all finite completely labelled $\mathcal{L}$-structures that do not embed any triangle in S.

Example

Let $\mathcal{L} = \{R, G, Y\}$ and $S = \{RYY, GGY, YYY\}$.



Stationary Independence Relation

Definition

An n-type over B is a set $p(x_1, \ldots, x_n)$ of formulas in $\mathcal{L}(B)$ such that for every finite subset $p_0 \subseteq p$, there are some elements $b_1, \ldots, b_n \in \mathcal{M}$ such that $\mathcal{M} \models p_0(b_1, ..., b_n)$.
We say $(b_1, ..., b_n)$ is a realisation of p if $\mathcal{M} \models p(b_1, ..., b_n)$.

For a homogeneous structure $\mathcal{M}$, $\bar{a}, \bar{b}$ have the same type over B in $\mathcal{M}$ if and only if they are in the same $Aut(\mathcal{M})_B$-orbit.

Stationary Independence Relation

Definition

Let $\mathcal{M}$ be a structure. $\mathop{\smile\!\!\!\!\!\!|}$ is a stationary independence relation if the following is satisfied for any substructure $A, B, C, D \subseteq \mathcal{M}$:

- *Invariance: A, C independence over B depends only on the type of ABC*
- *Monotonicity: $A \mathop{\smile\!\!\!\!\!\!|}_B CD \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B C, A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$*
- *Transitivity: $A \mathop{\smile\!\!\!\!\!\!|}_B C, A \mathop{\smile\!\!\!\!\!\!|}_{BC} D \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B D$*
- *Symmetry: $A \mathop{\smile\!\!\!\!\!\!|}_B C \Rightarrow C \mathop{\smile\!\!\!\!\!\!|}_B A$*
- *Existence: If p is a type over B and C is a finite set, then p has a realisation that is independent from C over B*
- *Stationarity: If $\bar{a}$ and $\bar{a}'$ have the same type over B and are both independent from C over B, then $\bar{a}$ and $\bar{a}'$ have the same type over BC.*

Definition

$g \in Aut(\mathcal{M})$ moves almost maximally if for every 1-type over a finite set X has a realisation a such that $a \mathop{\smile\!\!\!\!\!\!\mid}_X ga$.

Theorem (Tent and Ziegler, 2012, Corollary 5.4)

Let $\mathcal{M}$ be a countable homogeneous structure with a stationary independence relation. If $g \in Aut(\mathcal{M})$ moves almost maximally, then any element of $Aut(\mathcal{M})$ is the product of sixteen conjugates of g.

- we want to find a stationary independence relation on the Fraïssé limit $\mathcal{M}$ of the semi-free amalgamation classes. The key is to find a 'unique' amalgam for every A, C over B.
- we want to find an automorphism of $\mathcal{M}$ that moves almost maximally.

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