The wonder of the Rado graph

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March 21, 2018

Outline

- Rado's construction
- Probabilistic construction
- Number theoretic construction
- Homogeneity and universality
- Automorphism group of the Rado Graph
- Topological dynamics
- Ramsey theory

The Rado Graph

A graph G consists of a vertex set V together with a set of edges $E \subseteq V \times V$. We will only consider simple graphs, i.e. undirected graphs containing no loops or multiple edges.

An induced subgraph of G consists of a subset $V' \subseteq V$ and a subset $E' \subseteq E$ such that $E' = E \cap (V' \times V')$.

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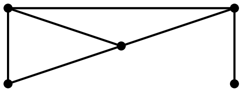

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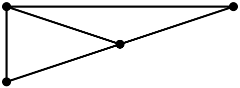

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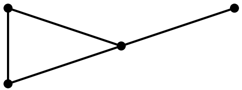

The Rado Graph: Rado's Construction

In 1964, Richard Rado constructed a countable universal graph R, i.e. every finite or countable graph occurs as an induced subgraph.

The vertex set of the Rado graph is the set $\mathbb{N}$ and given any two vertices $x, y \in \mathbb{N}$ such that $x < y$, we join x to y if when writing y in base 2, its x-th digit is 1.

$$\overset{\bullet}{0} \quad \overset{\bullet}{1} \quad \overset{\bullet}{2} \quad \overset{\bullet}{3} \quad \overset{\bullet}{4} \quad \overset{\bullet}{5} \quad \overset{\bullet}{6} \quad \overset{\bullet}{7} \quad \cdots\cdot$$

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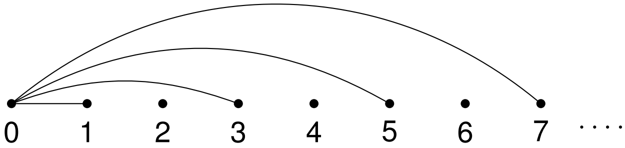

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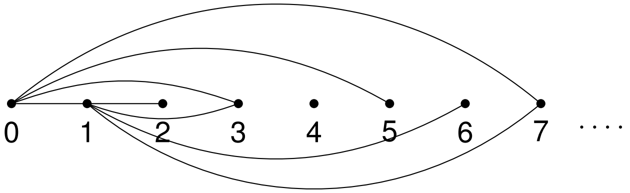

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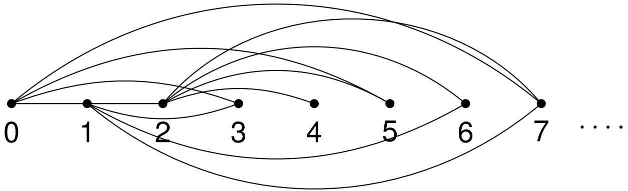

The Rado Graph: Probabilistic Construction

Erdős and Rényi, around the same time, showed the following:

Theorem

There is a countable graph R with the following property: if a graph G on a fixed countable vertex set is chosen by selecting edges independently at random with probability $\frac{1}{2}$, then the probability that G is isomorphic to R is 1.

The Rado Graph

Lemma

- *With probability 1, a countable random graph satisfy the following property:*
  ($\star$) given any finite subset U, V of the vertex set, there is a vertex x that is joined to every vertex in U and to no vertex in V.
- *Any two countable graphs satisfy ($\star$) are isomorphic.*
- *The Rado graph satisfy the above property .*

Definition

A structure M is $\aleph_0$-categorical if any countable structure satisfying the same first-order sentences as M is isomorphic to M.

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The Rado Graph: Number Theoretic Construction

Definition

Let p be an odd prime not dividing a, then a is a quadratic residue mod p if the congruence equation $x^2 \equiv a \pmod{p}$ has a solution.

Law of Quadratic Reciprocity

For primes p, q, if $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$, then p is a quadratic residue mod q if and only if q is a quadratic residue mod p.

So we can construct a countable graph whose vertices are all the primes congruent to 1 mod 4 such that two vertices p, q are joined if p is a quadratic residue mod q.

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The Rado Graph: Number Theoretic Construction

This graph satisfies $(\star)$, hence is isomorphic to the Rado graph

Proof.

Let $U = \{u_i\}$ and $V = \{v_j\}$ be finite sets of primes congruent to $1 \mod 4$. For each $u_i \in U$, let a_i be a quadratic residue mod u_i and for each v_j, let b_j be a quadratic non-residue mod v_j.

By the Chinese Reminder Theorem, the following system of simultaneous congruence have a solution mod $4 \prod_{u_i \in U} u_i \prod_{v_j \in V} v_j$:

$$
\begin{aligned}
x &\equiv a_i \pmod{u_i} \text{ for all } u_i \in U \\
x &\equiv b_j \pmod{v_j} \text{ for all } v_j \in V \\
x &\equiv 1 \pmod{4}
\end{aligned}
$$

By Dirichlet's Theorem, this congruence class contains a prime. $\square$

Symmetries

For a finite graph, the more symmetric a graph is, the smaller the probability of its occurrence is:

Graph				
Symmetries	6	2	2	6
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

But the Rado graph is highly symmetric:

Theorem

The Rado graph is homogeneous, i.e. every partial isomorphism can be extended to an automorphism.

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Universality

Theorem

The Rado graph is universal, i.e. every finite graph can be embedded in R

Theorem (Fraïssé's Theorem)

There is a one-to-one correspondence between amalgamation classes and countable homogeneous structures.

Countable Homogeneous structures M

$\mathcal{C} = \{$ all finite substructure of $M\}$ | M embeds every structure in $\mathcal{C}$

Amalgamation classes $\mathcal{C}$

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$$\begin{tikzcd} \boxed{\text{Countable Homogeneous structures } M} \arrow[d, leftrightarrow, "\mathcal{C} = \{\text{ all finite substructure of } M\}"', "M \text{ embeds every structure in } \mathcal{C}"] \\ \boxed{\text{Amalgamation classes } \mathcal{C}} \end{tikzcd}$$

Automorphism Group of R

- $Aut(R)$ is simple
- $Aut(R)$ has cardinality $2^{\aleph_0}$
- $Aut(R)$ has finitely many orbits on R^n for every n.

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For a countable first-order structure M, M is $\aleph_0$-categorical if and only if $Aut(M)$ has finitely many orbits on M^n for every n.

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Automorphism Group of R: the Topological Aspects

Let $X = \mathbb{N}$, there is a natural topology on $G \leq Sym(X)$ with basic open sets $S(\bar{a}, \bar{b}) = \{g \in G | g\bar{a} = \bar{b}\}$ where $\bar{a}, \bar{b} \in X^n$ for some n.

- Each basic open set is closed, so G is totally disconnected.
- There are countably many basic open sets, i.e. G is second countable.
- G is separable, i.e. it contains a countable dense subset.

Lemma

Any second-countable space is separable.

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- G is metrizable.
  Let $d(g_1, g_2) = \frac{1}{n}$ where n is the smallest integer such that $g_1 n \neq g_2 n$.
- G is completely metrizable.
  Let $d'(g_1, g_2) = d(g_1, g_2) + d(g_1^{-1}, g_2^{-1})$
- G is a Polish group, i.e. a topological group which is separable and completely metrizable.

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Topological Dynamics

- G is extremely amendable

Definition

Let G be a topological group.

- *A G-flow is a continuous action of G on a topological space X, usually assumed to be a compact Hausdorff space.*
- *A subflow is a invariant set under the action.*
- *A flow is minimal if it has no proper subflows.*
- *By Zorn's Lemma, every G-flow admits a minimal flow.*
- *There is also a universal minimal flow, i.e. a minimal flow that can be mapped to any minimal G-flow.*
- *G is extremely amenable if its universal minimal flow consists of a single point.*

Ramsey Class

Theorem (KPT Correspondence)

Let X be a countable set, and G a closed subgroup of $Sym(X)$. Then G is extremely amenable if and only if it is the automorphism group of a homogeneous structure whose age is a Ramsey class of ordered structure.

Definition

A class $\mathcal{C}$ of finite structures is a Ramsey class if, given any $n \in \mathbb{N}$ and a pair A, B of structures in $\mathcal{C}$, there exists a structure $C \in \mathcal{C}$ such that, if we colour the A-substructures of C with n colours, then there is a B-substructure of C, all of whose A-substructures have the same colour.

Reference

- Cameron, Peter J. "The random graph." The Mathematics of Paul Erdös II. Springer, Berlin, Heidelberg, 1997. 333-351.
- Kechris, Alexander S., Vladimir G. Pestov, and Stevo Todorcevic. "Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups." Geometric and Functional Analysis 15.1 (2005): 106-189.