EE 512 Stochastic Processes

Summary and Review: Part 1

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Course Info

- EE 512 Stochastic Processes, Spring 2022
- Instructor: Dr. Ben Reichardt, Associate Professor of Electrical and Computer Engineering (ECE) and Computer Science at USC
- Time & Location: TTH 12:30-13:50, OHE 132
- Topics: overview of probability, Poisson processes, renewal theory, discrete-time Markov chains, continuous-time Markov chains, martingales, random walks, Brownian motion, stochastic integration, stochastic differential equations and finance applications, simulation

  * Not a measure-theoretic stochastic processes course
- Textbook(s): There is no required textbook but the following are recommended for reference

  ◦ Ross, S. M. *Stochastic Processes*.

  ◦ Mikosch, T. *Elementary Stochastic Calculus with Finance in View*, World Scientific, 1998.

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Common Discrete Distributions

Table 7.1 Discrete Probability Distribution.

	Probability mass function, $p(x)$	Moment generating function, $M(t)$	Mean	Variance
Binomial with parameters n, p; $0 \leq p \leq 1$	$\binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, \ldots, n$	$(pe^t + 1 - p)^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$ $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \leq p \leq 1$	$p(1-p)^{x-1}$ $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative binomial with parameters r, p; $0 \leq p \leq 1$	$\binom{n-1}{r-1} p^r (1-p)^{n-r}$ $n = r, r+1, \ldots$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

Common Continuous Distributions

Table 7.2 Continuous Probability Distribution.

	Probability density function, $f(x)$	Moment generating function, $M(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters $(s, \lambda), \lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^s$	$\frac{s}{\lambda}$	$\frac{s}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2

cf. Sheldon Ross, *A First Course in Probability*, 9th

Conditioning and Independence

When we talk about events,

- Conditional Probability

$$\mathbb{P}\left(A|B\right) = \frac{\mathbb{P}\left(A \cap B\right)}{\mathbb{P}\left(B\right)} = \frac{\mathbb{P}\left(A\right)\mathbb{P}\left(B|A\right)}{\mathbb{P}\left(B\right)}$$

- Law of Total Probability

$$\mathbb{P}\left(A\right) = \sum_i \mathbb{P}\left(A \cap \Omega_i\right) = \mathbb{P}\left(\Omega_i\right)\mathbb{P}\left(A|\Omega_i\right)$$

- Law of Total Expectation (see Tower Rule later)

$$\mathbb{E}\left[A\right] = \sum_i \mathbb{P}\left(\Omega_i\right)\mathbb{E}\left[A|\Omega_i\right]$$

Conditioning and Independence

- Conditional PDF

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_y(y)}$$

- Conditional expectation

$$\mathbb{E}\left[X|Y=t\right] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|t)\, dx$$

$$\mathbb{E}\left[X|Y\leq t\right] = \int_{-\infty}^{t}\int_{-\infty}^{\infty} x\, f_{X|Y}(x,y) f_Y(y)\, dx\, dy$$

- Independence: If $X_1, \ldots, X_n$ are independent

$$\mathbb{E}\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n}\mathbb{E}\left[X_i\right]$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\mathrm{Var}\left(X_i\right)$$

Conditional Expectation & Conditional Variance

- Tower Rule (i.e. Total Expectation law applied to r.v.)

$$\mathbb{E}\left[\mathbb{E}\left[X|Y\right]\right] = \mathbb{E}\left[X\right]$$

- More properties:

$$\mathbb{E}\left[X\, g(Y)|Y\right] = g(Y)\mathbb{E}\left[X|Y\right]$$
$$\mathbb{E}\left[\mathbb{E}\left[X|Y,Z\right]|Y\right] = \mathbb{E}\left[X|Y\right]$$

- Conditional Variance

$$\begin{aligned}\mathrm{Var}\left(X|Y\right) &:= \mathbb{E}\left[(X-\mathbb{E}\left[X|Y\right])^2|Y\right] \\ &= \mathbb{E}\left[\mathrm{Var}\left(X|Y\right)\right] + \mathrm{Var}\left(\mathbb{E}\left[X|Y\right]\right)\end{aligned}$$

Important Inequalities

- Union Bound:

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

- Markov's inequality: If $X \geq 0$,

$$\mathbb{P}(X \geq k) \leq \frac{\mathbb{E}[X]}{k}$$

- Chebyshev's inequality: $\mathrm{Var}(X) < \infty$,

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq k) \leq \frac{\mathrm{Var}(X)}{k^2}$$

Important Inequalities

With the *moment generating function*, $M(t) = \mathbb{E}\left[e^{tX}\right]$, we have

- Chernoff Bound:
$$\mathbb{P}\left(X \geq a\right) \leq e^{-ta} M(t)$$
- Hoeffding's inequality (see Azuma's inequality later):
Let $X_1, \ldots, X_n$ be n independent random variables such that $a_i \leq X_i \leq b_i \; \forall \, i$ almost surely. Let $S_n = X_1 + \cdots + X_n$
$$\mathbb{P}\left(\left|S_n - \mathbb{E}\left[S_n\right]\right| \geq t\right) \leq 2 \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

Aside: Properties of MGF

- $\frac{d^n}{dt^n} M(t)\Big|_{t=0} = \mathbb{E}\left[X^n\right]$
- If X, Y are independent, $M_{X+Y}(t) = M_X(t) M_Y(t)$

- Continuity of Probability

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcup_{j=1}^{n} E_j\right) = \mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right)$$

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcap_{j=1}^{n} E_j\right) = \mathbb{P}\left(\bigcap_{j=1}^{\infty} E_j\right)$$

- Finitely/Infinitely Many Events Occur

∞ many events occur

$E_1, E_2, E_3, \dots$

{finitely many E_j occur} $\iff$ $\exists$ n sufficiently large that for all $j \geq n$ no E_j occurs

{infinitely many E_j occur} $\iff$ for all n, $\exists$ $j \geq n$ so E_j occurs

$$\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} E_j$$

Measure-theoretic Concepts & Consequences

Borel-Cantelli Lemma

If $\sum_{j=1}^{\infty} \mathbb{P}(E_j) < \infty$, then $\mathbb{P}(\text{infinitely many } E_j \text{ occur}) = 0$.

For *independent* events,

$$\sum_{j=1}^{\infty} \mathbb{P}(E_j) < \infty \iff \mathbb{P}(\text{infinitely many } E_j \text{ occur}) = 0$$

Limit Theorems

Strong Law of Large Numbers

Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean μ.

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n}(X_1 + \cdots + X_n) = \mu\right) = 1$$

(converges almost surely)

Limit Theorems

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(converges almost surely)

Central Limit Theorem

Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite mean μ and finite variance σ^2.

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \leq a\right) = \Phi(a)$$

(converges in distribution, see also Berry–Esseen Theorem for convergence rate)

Other Things to Note for Solving Problems

For problems about (finding) expectations,

- For any continuous *non-negative* random variable $X \geq 0$,

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\, dx$$

- Recurrence relation!
- Integration by parts
- Conditioning and Bayes rule
- Fubini Theorem: exchange the order of integration

  ...

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Poisson and Exponential Random Variables

- Interpretation: Let $X \sim \text{Poisson}(\lambda)$. X represents the number of events happened in unit time. Let $Y \sim \text{Exp}(\lambda)$. Y represents the length of time between 2 consecutive events.
- Poisson is the continuous-time analogue of binomial distribution, and exponential is the continuous-time analogue of geometric distribution.

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Binomial Converges to Poisson

Let $X_n \sim \text{Binomial}(n, p_n)$ If $np_n \to \lambda$ as $n \to \infty$,

$$\mathbb{P}\left(X_n = k\right) \to e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{as } n \to \infty$$

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- Summation of i.i.d. R.V.'s
  ◦ If $X_1 \sim \text{Poisson}(\lambda_1)$ and $X_2 \sim \text{Poisson}(\lambda_2)$ are independent, then $X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

  ◦ If $X_1, X_2 \sim \text{Exp}(\lambda)$ are i.i.d., then $X_1 + X_2 \sim \text{Gamma}(2,\lambda)$.

Poisson Process: Motivation

▷ Stochastic processes: think not just of random variables, but random variables ordered in time.

▷ Assumptions for a Poisson process

- Bernoulli process (i.i.d. Bernoulli trials) with infinitesimal length of time intervals
- at most 1 event can happen in an infinitesimal length of time (we disregard the "multiple events" cases that has small probability $o(\delta_t)$)

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▷ Counting process: $N(t) \in \mathbb{N} \cup \{0\}$ represents the number of events happened up to time t.

Independent and Stationary Increments

◦ A counting process has **independent increments** if the number of events that occur in *disjoint* time intervals are *independent*.

◦ A counting process has **stationary increments** if the distribution of the number of events occurred in any time interval depends *only* on the length of the interval (so independent of where the time interval is).

Poisson Process: Definitions

Poisson Process

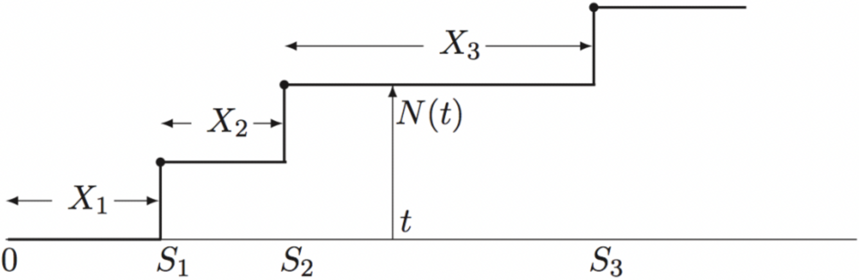
A **Poisson process** with rate λ is a counting process with
i) $N(0) = 0$
ii) independent increments
iii) the number of events in any time interval of length t, denoted as $N(t+s) - N(s)$, follows Poisson(λt)

▷ Random Variables of Interest:

- Count: the number of events occurred by t. $N(t) \sim \text{Poisson}(\lambda t)$
- Interarrival time: the time between 2 consecutive events $n-1$ and n. $X_n \sim \text{Exp}(\lambda)$
- Arrivial time: the time until event n occurred. $S_n = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \lambda)$ [1]

[1] a Gamma distribution whose first parameter is a natural number is called Erlang distribution.

Poisson Process: Definitions



Observe: $N(t) < n \iff S_n > t$

$N(t) \geq n \iff S_n \leq t$

$N(t) = n \iff S_n \leq t < S_{n+1}$

Memoryless Property

In Poisson processes, the **memoryless property** has its meanings in two folds:

◦ Independent increments, and thus i.i.d. interarrival times
* also holds for renewal processes, see next chapter

◦ The exponential r.v. is (the only continuous) r.v. that satisfies
$\forall\, s, t > 0:$

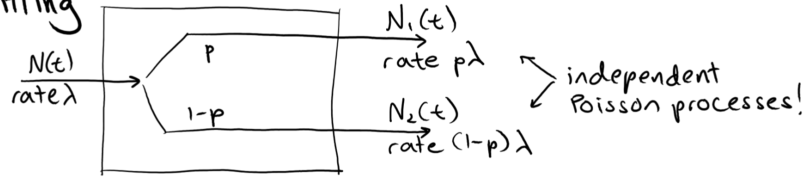
$$\mathbb{P}\left(X > s + t \middle| X > t\right) = \mathbb{P}\left(X > s\right)$$

Combining and Splitting Poisson Processes

Combining $\{N_1(t)\}$ rate λ_1 Poisson
$\{N_2(t)\}$ rate λ_2 Poisson $\Big\}$ independent

$\Rightarrow \{N_1(t)+N_2(t)\}$ is a Poisson process w/ rate $\lambda_1+\lambda_2$.

Splitting



Poisson Process: More Important Results

Minimum of 2 independent Exponential random variables

Let $X_1 \sim \text{Exp}(\lambda_1), X_2 \sim \text{Exp}(\lambda_2)$ be independent. Then
- ∘ $Z := \min\{X_1, X_2\} \sim \text{Exp}(\lambda_1 + \lambda_2)$
- ∘ $\mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
- ∘ Z is independent of the events $\{X_1 < X_2\}, \{X_1 > X_2\}$

Conditional Distribution of Arrival Times

For a Poisson process, *conditioned on* $N(t) = n$, the set of arrival times $\{S_1, \ldots, S_n\}$ and a set of n i.i.d. Uniform(0,t) variables $\{U_1, \ldots, U_n\}$ has the same distribution.

* An important theorem for simulation and problem solving

Non-homogeneous Poisson Processes

▷ For **non-homogeneous** Poisson processes with rate $\lambda(t)$,

$$\forall\, r \geq s \geq 0 : N(r) - N(s) \sim \text{Poisson}\left(\int_s^r \lambda(t)\, dt\right)$$

* It still has independent increments, but probably not stationary.

Example: M/G/∞ queuing system

Consider a queuing system that has infinite servers and the customer arrivals is a Poisson process with rate λ.

- $S_n :=$ the time that n-th customer arrives
- $T :=$ the service time that each server needs
- $Y(t) :=$ the number of customers in service at time t.

$$Y(t) \sim \text{Poisson}\left(\lambda \int_0^t \mathbb{P}\left(T > x\right)\, dx\right)$$

Example: M/G/∞ Queue

$N(t)$ = # customers arrived by time $t \sim \text{Poisson}(\lambda t)$.

$Y(t)$ still in service

$Z(t)$ have left

Condition on $N(t)=n$

$\Rightarrow \{S_1, S_2, \ldots, S_n\} \sim \{U_1, U_2, \ldots, U_n\}$

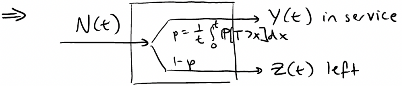
i.i.d. Uniform$(0,t)$

Consider one of these customers.

$$\mathbb{P}[\text{still in service at time } t] = \mathbb{P}[\underbrace{U}_{\text{arrival time}} + \underbrace{T}_{\text{service time}} > t]$$

$$= \frac{1}{t}\int_0^t \mathbb{P}[T > x]\,dx$$

This does not depend on n!



$$Y(t) \sim \text{Poisson}\left(\lambda t \cdot \frac{1}{t}\int_0^t \mathbb{P}[T>x]dx\right). \checkmark \quad \square$$

Compound Poisson Processes

$\triangleright$ A stochastic process $\{X(t) : t \geq 0\}$ is said to be a **compound Poisson process** if it can be represented by

$$X(t) = \sum_{t=0}^{N(t)} X_t$$

where $\{N(t) : t \geq 0\}$ is a Poisson process and $\{X_i\}_{i=1}^{\infty}$ are i.i.d. random variables that are *independent* of the process $\{N(t) : t \geq 0\}$.

- Tower rule: $\mathbb{E}[X(t)] = \mathbb{E}[N(t)]\,\mathbb{E}[X_i] = \lambda t \mathbb{E}[X_i]$ (see Wald's identity later)
- $\mathrm{Var}(X(t)) = \mathbb{E}[N(t)]\,\mathrm{Var}(X_i) + \mathrm{Var}(N(t))\,\mathbb{E}[X_i]^2 = \lambda t \mathbb{E}\left[X_i^2\right]$

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Renewal Process

A counting process which has i.i.d. interarrival times is called a **renewal process**.
∘ If the interarrival times after time t are *i.i.d.*, it is called **delayed renewal process**.

- Renewal processes have independent and stationary increments.
- When determining whether a stochastic process is a renewal process, be careful with

  ∘ Whether X_1 and $\{X_i, i \geq 2\}$ have the same distribution

  ∘ Whether $\{X_i, i \in \mathbb{N}\}$ are independent, i.e. whether there is some "internal" state of event n that X_{n+1} depends on

Inspection Paradox

Put it in words: When you want to check the interarrival time, you are more likely to fall into a long time interval and wait longer.

Inspection Paradox

Let s be the time of inspection. For any renewal process, $\forall\, t \geq 0$:

$$\mathbb{E}\left[X_{N(t)+1}\right] \geq \mathbb{E}\left[X_1\right]$$

$$\lim_{t\to\infty} \mathbb{E}\left[X_{N(t)+1}\right] = \frac{\mathbb{E}\left[X_1^2\right]}{\mathbb{E}\left[X_1\right]}$$

$$\lim_{t\to\infty} \mathbb{E}\left[S_{N(t)+1} - s\right] = \frac{\mathbb{E}\left[X_1^2\right]}{2\,\mathbb{E}\left[X_1\right]}$$

For a Poisson process with rate λ,

$$\mathbb{P}\left(X_{N(t)+1} > x\right) = e^{-\lambda x}(1 + \lambda \min\{x, t\})$$

Limit Theorems

We assume we know the distribution of X_i. Let $\mu := \mathbb{E}[X_i]$.

$\triangleright$ Can infinitely many events occur in a finite amount of time?

– No, by strong law of large numbers

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu\right) = 1$$

so as $n \to \infty$, $S_n \to \infty$ with probability 1.

$\triangleright$ Distribution of $N(t)$

- $\forall\, 0 \leq t < \infty : N(t) < \infty$ with probability 1
- as $n \to \infty$, $N(t) \to \infty$ with probability 1

Limit Theorems

Strong Law for Renewal Processes

$$\mathbb{P}\left(\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{\mu}\right)=1$$

Renewal Reward Theorem (weighted version of above)

For a renewal process, let $R_1, \ldots$ be random variables such that the pairs (X_n, R_n) are i.i.d. (i.e. R_n's are i.i.d. and R_n is independent of X_m for all $n \neq m$).

$$\mathbb{P}\left(\lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{N(t)} R_n=\frac{\mathbb{E}\left[R_n\right]}{\mathbb{E}\left[X_n\right]}\right)=1$$

Application: Proof of Inspection Paradox (waiting time)

Let $I(t) := \int_0^t (S_{N(s)+1} - s)\, ds$. Then $I(t) = \frac{1}{2}\left(X_1^2 + \cdots + X_{N(t)}^2\right) + \text{Rem}$

$$\frac{1}{2}\sum_{j=1}^{N(t)} X_j^2 \leq I(t) \leq \frac{1}{2}\sum_{j=1}^{N(t)+1} X_j^2$$

$$\frac{\frac{1}{2}\sum_{j=1}^{N(t)} X_j^2}{N(t)} \cdot \frac{N(t)}{t} \leq \frac{I(t)}{t} \leq \frac{\frac{1}{2}\sum_{j=1}^{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{t}$$

Take $t \to \infty$, by squeeze theorem we have

$$\lim_{t\to\infty} \int_0^t (S_{N(s)+1} - s)\, ds = \frac{\mathbb{E}\left[X_1^2\right]}{2\mathbb{E}\left[X_1\right]}$$

Limit Theorems

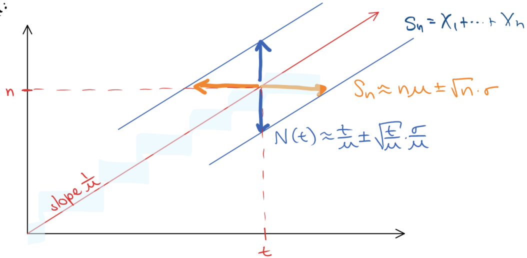
Theorem: Central Limit Theorem for $N(t)$

For a renewal process with $\mu = \mathbb{E}[X_j]$, $\sigma^2 = \mathrm{Var}(X_j) < \infty$,

$$\mathbb{P}\left[N(t) \leq \frac{t}{\mu} + k \cdot \sqrt{\frac{t}{\mu}} \cdot \frac{\sigma}{\mu}\right] \xrightarrow[t\to\infty]{} \mathbb{P}[\underbrace{Z}_{\text{Normal}(0,1)} \leq k]$$

"$N(t)$ tends to Gaussian with mean $\frac{t}{\mu}$ and standard deviation $\sqrt{\frac{t}{\mu}} \cdot \frac{\sigma}{\mu}$."

Idea:



Remark: We know that $\mathbb{E}[S_n] = n\mu$, $\mathrm{Var}(S_n) = n\sigma^2$.
We have also proved that, almost surely, $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$.
But we have **not** shown that $\mathbb{E}[N(t)] \to \frac{t}{\mu}$.

Wald's Identity

$\triangleright$ $N \in \{0, 1, \ldots\}$ is a **stopping time** for the sequence $\{X_1, \ldots\}$ if the event $\{N = n\}$ ONLY depends on $X_1, \ldots, X_n$.

* For a renewal process, the stopping time is $N(t) + 1$, not $N(t)$.

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* For a renewal process, the stopping time is $N(t) + 1$, not $N(t)$.

Wald's Identity

Let $X_1, \ldots, X_n$ be i.i.d. with $|\mathbb{E}[X_j]| < \infty$. Let N be a stopping time for the sequence with $\mathbb{E}[N] < \infty$.

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n\right] = \mathbb{E}[N]\,\mathbb{E}[X_j]$$

* A way to compute $\mathbb{E}[N]$
* Corollary: For a renewal process $\mathbb{E}\left[S_{N(t)+1}\right] = (\mathbb{E}[N(t)] + 1)\,\mathbb{E}[X]$

Elementary Renewal Theorem

Elementary Renewal Theorem

For a renewal process,

$$\lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} = \frac{1}{\mu}$$

* part of the proof uses the corollary of Wald's identity

* This theorem is about the asymptotic "slope" of the scant line between $(0,0)$ and $(t, \mathbb{E}[N(t)])$. We will see Blackwell's Theorem that shows the asymptotic "slope", or say derivative, *at* the point $(t, \mathbb{E}[N(t)])$

Blackwell's Theorem

$\triangleright$ We call the distribution of interarrival times X_j **lattice** if X_j can only takes on values that are *integer* multiples of d for some $d > 0$.

* If X_j only takes on rational values, then it is lattice.

Blackwell's Theorem (non-lattice version)

For a renewal process with *non-lattice* interarrival times,

$$\forall\, \delta > 0 : \lim_{t \to \infty} \mathbb{E}\left[N(t+\delta) - N(t)\right] = \frac{\delta}{\mu}$$

Blackwell's Theorem (lattice version)

For a renewal process with *non-lattice* interarrival times,

$$\lim_{n \to \infty} \mathbb{E}\left[\text{the number of events happened at } nd\right] = \frac{d}{\mu}$$

References

- Professor Ben Reichardt's Lecture Notes. USC Spring 2022.
- Ross, Sheldon M. *A First Course in Probability*. Boston: Pearson, 2019.
- Ross, Sheldon M., et al. *Stochastic Processes*. Vol. 2. New York: Wiley, 1996.