EE 512 Stochastic Processes

Summary and Review: Part 2

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Notations

- $p_{i,j}$: (one-step) transition probability from state i to state j
- $P^n_{i,j}$: the probability of starting from state i and transit to state j after n steps
$$P^n_{i,j} := \mathbb{P}\left(X_{n+m} = j \,|\, X_m = i\right)$$
- $f^n_{i,j}$: the probability that starting in state i, the first transition into state j occurs at time n
$$\begin{aligned} f^0_{i,j} &:= 0 \\ f^n_{i,j} &:= \mathbb{P}\left(X_n = j, X_k \neq j \;\; \forall k = 1, \ldots, n-1 \,|\, X_0 = i\right) \end{aligned}$$
- $f_{i,j} := \sum_{n=0}^{\infty} f^n_{i,j}$
  The probability of ever making a transition to state j given that the process starts from state i
- Hitting time $T_j := \min\{n \in \mathbb{N} : X_n = j\}$: the first time that the system is at state j
- $\mu_{i,j} = \mathbb{E}_i[T_j] := \mathbb{E}[T_j \,|\, X_0 = i]$: the expected time to transit to state j given that the initial state is i

Markov Chains: Concepts

Consider a (discrete-time) stochastic process $\{X_n, n = 0, 1, \dots\}$ that takes on a *finite* or *countable* set of values. If $X_n = i$, the process is said to be in state i at time n.

- **Markov property**: Transition probabilities only depend on current state but not the history

$$\mathbb{P}\left(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\right) = \mathbb{P}\left(X_{n+1} = j | X_n = i\right)$$

If the given stochastic process satisfies Markov property, then it is a Markov chain.

* We denote the transition probability as $p_{i,j} := \mathbb{P}\left(X_{n+1} = j | X_n = i\right)$. It satisfies $\forall\, i : \sum_{j=0}^{\infty} p_{i,j} = 1$ (must transit into some state).

Markov Chains: Concepts

- Transition matrix

$$P := \begin{bmatrix} p_{0,0} & p_{0,1} & \cdots \\ p_{1,0} & p_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

- Distribution
  ◦ At each time, we have a probability distribution $\mathbb{P}(X_t) := \pi^{(t)}$ of which state the random variable is at. It is represented by a row vector

$$\begin{bmatrix} \pi_0^{(t)} & \pi_1^{(t)} & \dots \end{bmatrix}$$

  ◦ We perform a 1-timestep update of the state by multiplying the transition matrix to the vector of current distribution.

- Stationary distribution and Equilibrium
  If the distribution satisfies $\pi P = \pi$, it is the **stationary distribution** and the Markov chain reaches equilibrium. Essentially, π is the eigenvector of P associated with eigenvalue 1.

Stationary Distribution

- Does the stationary distribution always exist? When is it unique?
  – No, see *irreducible, aperiodic* Markov chains later.
  – No, transition matrix can have multiple eigenvectors associated with eigenvalue 1. Note that if π_1 and π_2 are two stationary distributions, then $\forall\, \alpha \in [0, 1] : \alpha\pi_1 + (1 - \alpha)\pi_2$ is also a stationary distribution.

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- How to find the stationary distribution
  – Apply "cuts" to the graph, i.e. the outflow at the cut is the same of the inflow, to get a set of equations that often have some recursive structure. Then use the fact that the total probability mass of a distribution is 1 (normalization condition) to solve for the stationary distribution.

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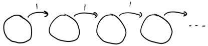
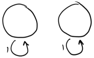
- Does a Markov chain necessarily converge to the stationary distribution?
  – No, see an example later

Bad examples:

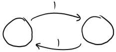
1) 

No stationary distribution: state diverges to ∞
"transient"

2) 

$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(p, 1-p)$ is stationary for any
- the graph is disconnected

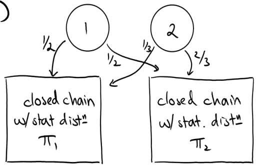
3) 

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$(\frac{1}{2}, \frac{1}{2})$ is the unique stationary distribution,
but the system does not converge

It just oscillates with period 2:

$$X_n = \begin{cases} X_0 & \text{, if } n \text{ is even} \\ 1 - X_0 & \text{, if } n \text{ is odd} \end{cases}$$

4) 

- multiple stationary distributions
- state converges to stationarity, but doesn't forget where it started

$X_0 \in$ chain 1 $\Rightarrow (\pi_1, 0)$

$X_0 \in$ chain 2 $\Rightarrow (0, \pi_2)$

$X_0 = 1 \Rightarrow (\frac{1}{2}\pi_1, \frac{1}{2}\pi_2)$

$X_0 = 2 \Rightarrow (\frac{1}{3}\pi_1, \frac{2}{3}\pi_2)$

Irreducibility and Periodicity

- Accessibility: State i is accessible to j if $\exists\, n \in \mathbb{N} : P^n_{i,j} > 0$ (The probability of getting from i to j in n steps in not zero). Denote this as $i \to j$. Additionally, if j is also accessible to i, we say i and j **communicate**, denoted as $i \leftrightarrow j$.

  * Communication is an equivalence relation. It defines equivalent classes.

- Irredicibility: A Markov chain is **irreducible** if $\forall\, i, j : i \leftrightarrow j$.

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- Irredicibility: A Markov chain is **irreducible** if $\forall\, i, j : i \leftrightarrow j$.
- Recurrence: state j is **recurrent** if $f_{j,j} = \sum_{n=0}^{\infty} f^n_{j,j} = 1$, i.e. given that the system starts on state j, it will come back with probability 1.

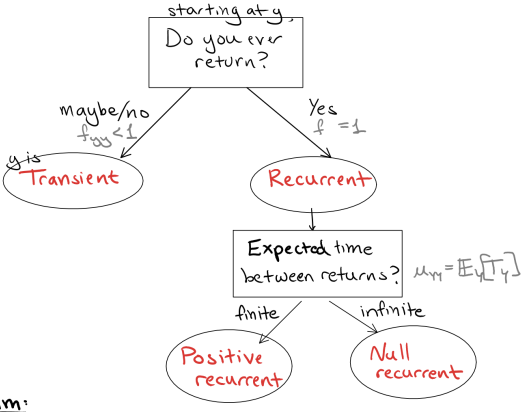
  ◦ If a state is NOT recurrent, then it is **transient**

  ◦ a recurrent state i is **null recurrent** if there exists some j that communicates with i and

$$\lim_{n\to\infty} P^n_{i,j} = 0$$

  Otherwise, it is **positive recurrent**.

Transient and recurrent states



Claim:

- If y is transient, then $\forall x$,

$$\mathbb{P}_x[N_y < \infty] = 1$$

$$\mathbb{E}_x[N_y] = \frac{f_{xy}}{1 - f_{yy}}$$

- If y is recurrent, then

$$\mathbb{P}_y[N_y = \infty] = 1$$

you return to y infinitely many times!

Also, $\mathbb{P}_x[N_y = \infty] = \mathbb{P}_x[T_y < \infty] = f_{xy}$.

Irreducibility and Periodicity

- Periodicity: The **period** of state i is $d(i) := \gcd(\{n \mid P^n_{i,i} > 0\})$.
  * **Aperiodic** means the period is 1.
  * Any self-loop makes the state aperiodic.
  * If $i \leftrightarrow j$, then $d(i) = d(j)$.
  * A Markov chain is called **aperiodic** if all its states have period 1.
  * If it is impossible to get back to state i, the period is defined as ∞.

For Chinese readers, I recommend this post on the concepts we introduced and the limit theorems [1]

[1] https://zhuanlan.zhihu.com/p/389201529

Irreducibility and Periodicity

Irreducible Aperiodic Markov Chain

An irreducible aperiodic Markov chain satisfies either one of the following:

1) states are all transient or all null recurrent, i.e. $\forall j : \mu_{j,j} = \infty$ or equivalently

$$\forall i,j : \lim_{n\to\infty} P^n_{i,j} = \frac{1}{\mu_{j,j}} = 0$$

In this case, there is no stationary distribution.

2) states are all positive recurrent, i.e. $\forall j : \mu_{j,j} < \infty$ or equivalently

$$\forall i,j : \lim_{n\to\infty} P^n_{i,j} = \frac{1}{\mu_{j,j}} > 0$$

In this case, $\left(\frac{1}{\mu_{1,1}}, \frac{1}{\mu_{2,2}}, \dots\right)$ is the unique stationary distribution (**ergodic theorem**).

* If a irreducible, aperiodic Markov chain has *finitely* many states, then it is ergodic and falls into the second case.

Renewal Theories Applied to Markov Chains

Consider a delayed renewal process that starts from state i and the event is defined as arriving at state j.

Elementary Renewal Theorem and Blackwell's Theorem (lattice)

If $i \leftrightarrow j$, then

i) $\mathbb{P}\left(\lim_{n\to\infty} \frac{N_j(n)}{n} = \frac{1}{\mu_{j,j}} \,|\, X_0 = i\right) = 1$

ii) $\lim_{n\to\infty} \frac{\mathbb{E}\left[N_j(n) \,|\, X_0 = i\right]}{n} = \frac{1}{\mu_{j,j}}$

iii) If j is aperiodic, $\lim_{n\to\infty} P^n_{i,j} = \frac{1}{\mu_{j,j}}$

iv) If period$(j) = d$, $\lim_{n\to\infty} P^{nd}_{j,j} = \frac{d}{\mu_{j,j}}$

Reversible Markov Chains

Consider an irreducible Markov chain $\{X_n, n \geq 0\}$ on the finite state space $\mathcal{X}$. We may wonder when it is the case that the Markov chain "looks the same" regardless of whether we run it forwards or backwards in time.

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Given $N \in \mathbb{N}$, we define the **reversed chain** $Y_n := X_{N-n}$ for $n = 0, 1, \ldots, N$ with transition probability matrix P.

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Reversed Chain

If the irreducible Markov chain $\{X_n, n \geq 0\}$ is started from the stationary distribution π, then the reversed chain $\{Y_n, n = 0, \ldots, N\}$ is an irreducible Markov chain with transition probabilities $\hat{P}(x, y) = \frac{\pi(y)P(y,x)}{\pi(x)}$ for all $x, y \in \mathcal{X}$. The stationary distribution for the reversed chain is also π.

Reversible Markov Chains: Detailed Balance

Now we can answer the question when the reversed chain looks the same as the original chain, i.e. $\hat{P}(x, y) = P(x, y) \, \forall \, x, y \in \mathcal{X}$.

Reversible Markov chains

A Markov chain is **reversible** if $\forall \, x, y \in \mathcal{X} : \pi(x)P(x, y) = \pi(y)P(y, x)$. This condition is called **detailed balance**.

Detailed balance condition implies that the current distribution is stationary. In fact, it is stronger than the conditions required for a stationary distribution [2].

[2] https://en.wikipedia.org/wiki/Detailed_balance

Application: Metropolis Algorithm

Suppose we want to sample from a complicated (discrete) probability distribution p. We may not know the probabilities p_j exactly but we know $a_j = c \cdot p_j \; \forall j$. For instance, there are so many states that it is difficult to calculate the normalization factor $A = \sum_{j=1}^{m} a_j$.

We can still sample a sequence of i.i.d. random variable that follows the desired distribution by making it the stationary distribution of a certain Markov chain.

Metropolis Algorithm

Let Q be an irreducible transition matrix on states $1, \ldots, m$ such that $p_{i,j} = p_{j,i} \, \forall \, i, j$. Now define a Markov chain $\{X_n, n \geq 0\}$.
1) If $X_n = i$, then propose a random variable with distribution q_i (the i-th row of Q).
2) If the random variable takes on value j, then we set $X_{n+1} = j$ with probability $\min\{1, \frac{a_j}{a_i}\}$ and set $X_{n+1} = i$ otherwise.

Mixing Time for Finite Chains

We wonder how fast the distribution of a Markov chain converges to the stationary distribution.

Mixing Time

For a Markov chain with transition matrix P and stationary distribution π, the **mixing time** is the smallest t such that

$$\forall x : d_{TV}(\pi, xP^t) < \frac{1}{2e}$$

where the **total variation difference** between two probability distributions is defined as $d_{TV}(\pi_1, \pi_2) := \frac{1}{2} \int_{-\infty}^{\infty} |\pi_1(x) - \pi_2(x)| \, dx$

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Example: sample a random graph coloring

Let the maximum degree be Δ. If the number of available colors $q > 4\Delta$, then the mixing time is $\mathcal{O}(n \log n)$, where n is the number of vertices.

Approach: Construct a Markov chain on colorings, with uniform stationary distribution. We run the chain until it converges.

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Definition

If we generalize from discrete time to continuous time, the Markov property becomes

$$\mathbb{P}\left(X(t+s)=j | X(s)=i, X(u)=x(u), 0 \leq u < s\right) = \mathbb{P}\left(X(t+s)=j | X(s)=i\right)$$

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By this property, we can find that the time that the system stays in state i, denoted as τ_i, follows an exponential distribution since

$$\mathbb{P}\left(\tau_i>s+t|\tau_i>s\right)=\mathbb{P}\left(\tau_i>t\right)$$

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Now we can define a **continuous time Markov chain**. It is a stochastic process having the properties that each time it enters state i:
i) the amount of time it spends in the state before making a transition into a different state follows $\mathrm{Exp}(v_i)$; and
ii) when the process leaves state i, it will enter the next state j with some probability $p_{i,j}$, where $\sum_{j\neq i} p_{i,j}=1$

Transition Rate

$\triangleright$ Idea: We regard the 2 phases of Poisson "clock" and transition as the *combined* process of the independent Poisson processes *split* from the Poisson "clock".

Namely, waiting at state i for time that is exponentially distributed with rate v_i and transit to state j with probability $p_{i,j}$ is EQUIVALENT to in the next δ time, transition to state j with probability $\delta v_i p_{i,j}$ for all $j \neq i$ or stay in state i with probability $1 - \delta v_i$

Time Evolution of Continuous-time Markov Chains

Chapman-Kolmogorov Equations

Let $P(t) = (P_{i,j}(t))$ be the transition matrix where $P_{i,j}(t) := \mathbb{P}\left(X(t) = j \,|\, X(0) = i\right)$ Then

$$P(s+t) = P(s)P(t)$$

Examples: Queues

Examples: Birth-Death Process in Continuous Time

Examples: Tandem Queues

References

- Professor Ben Reichardt's Lecture Notes. USC Spring 2022.
- Ross, Sheldon M., et al. *Stochastic Processes*. Vol. 2. New York: Wiley, 1996.