国家税务总局徐州市税务局稽查局

Convex Opt HW 10

1、 $f(x)=\max\{x_i\}$, $f^*(y)=\sup\limits_{x\in\mathbb{R}^n}\{y^Tx-\max\{x_i\}\}$

①若y有元素 $y_i<0$，则令 $x=\begin{bmatrix}0\\ \vdots\\ -t\\ \vdots\\ 0\end{bmatrix}=-t\cdot e_i$

则 $\lim\limits_{t\to\infty} y^Tx-\max\{x_i\}=\lim\limits_{t\to\infty}-t\cdot y_i=+\infty$, $y\notin \mathrm{dom}\, f^*$

②若 y有元素 $y_i>1$，令 $x=\begin{bmatrix}0\\ \vdots\\ t\\ \vdots\\ 0\end{bmatrix}=te_i$

$\lim\limits_{t\to\infty} y^Tx-\max\{x_i\}=\lim\limits_{t\to\infty}(y_i-1)t=+\infty$, $y\notin \mathrm{dom}\, f^*$

③若 $\sum\limits_{i=1}^n y_i\neq 1$，则令 $x=\begin{bmatrix}t\\ \vdots\\ t\end{bmatrix}$, $y^Tx-f(x)=t(\sum\limits_{i=1}^n y_i-1)$

则 $\lim\limits_{t\to+\infty} y^Tx-f(x)$ 与 $\lim\limits_{t\to-\infty} y^Tx-f(x)$ 中有一个为 $+\infty$

$\therefore$ $y\notin \mathrm{dom}\, f^*$.

$\therefore$ $\mathrm{dom}\, f^*=\{y\mid 0\le y_i\le 1, \sum\limits_{i=1}^n y_i=1\}$,

for $y\in$ dom f^*, $f^*(y)=\sup\limits_{x\in\mathbb{R}^n}\{y^Tx-\max\{x_i\}\}=\sup\limits_{x\in\mathbb{R}^n}\{\sum\limits_{i=1}^n y_ix_i-\max x_i\}\le 0$

且 $x=0$ 时，$y^Tx-\max\{x_i\}=0$

$\therefore f^*(y)=\begin{cases}0, & 0\le y_i\le 1, \sum\limits_{i=1}^n y_i=1\\ +\infty, & \text{otherwise}\end{cases}$

2、 $M_C(x)=\inf\{t>0\mid t^{-1}x\in C\}$

设 $t_1=M_C(x)=\inf\{t\mid t>0, \frac{x}{t}\in C\}$

$t_2=M_C(y)=\inf\{t\mid t>0, \frac{y}{t}\in C\}$

则 $\frac{x}{t_1}\in C$, $\frac{y}{t_2}\in C$, 且 C is convex

故 $\frac{\theta x+(1-\theta)y}{\theta t_1+(1-\theta)t_2}=\frac{\theta t_1}{\theta t_1+(1-\theta)t_2}\cdot\frac{x}{t_1}+\frac{(1-\theta)t_2}{\theta t_1+(1-\theta)t_2}\frac{y}{t_2}$ 为 $\frac{x}{t_1}$, $\frac{y}{t_2}$ 的凸组合

$\therefore \frac{\theta x+(1-\theta)y}{\theta t_1+(1-\theta)t_2}\in C$, 即 $\theta t_1+(1-\theta)t_2\ge\inf\{t\mid t>0, \frac{\theta x+(1-\theta)y}{t}\in C\}$

即：$\theta\cdot M_C(x)+(1-\theta)M_C(y)\ge M_C(\theta x+(1-\theta)y)\Rightarrow M_C(\cdot)$ is convex.

$M_C^*(y) = \sup_x \{y^T x - M_C(x)\}$

Lemma: $M_C(tx) = t \cdot M_C(x)$

proof: $M_C(tx) = \inf\{u>0 \mid \frac{tx}{u} \in C\}$

$= t \cdot \inf\{\frac{u}{t}>0 \mid \frac{x}{u/t} \in C\} = t \cdot \inf\{w>0 \mid \frac{x}{w} \in C\}$

$= t \cdot M_C(x)$

故若 $\exists x_0$ s.t. $y^T x_0 > M_C(x_0)$，则 $\lim_{t\to\infty} y^T(tx_0) - M_C(tx_0) = +\infty$

① $0 \in C$，则 $M_C(0)=0$，对于 $y^T x - M_C(x) \le 0, \forall x$ 的 y，有

$\sup_x\{y^T x - M_C(x)\} = y^T \cdot 0 - M_C(0) = 0$

$\therefore$ 此时 $f^*(y) = \begin{cases} 0, & \forall x_0, \text{s.t. } y^T x_0 \le M_C(x_0) \\ +\infty, & \exists x_0, \text{s.t. } y^T x_0 > M_C(x_0) \end{cases}$

② $0 \notin C$，则 $0 \notin \operatorname{dom} M_C$，$\operatorname{dom} M = \{x \mid \exists t>0, \text{s.t. } \frac{x}{t} \in C\}$ 为一个凸锥.

对于 $x \in \operatorname{dom} M_C$，易知：$\lim_{s\to 0} M_C(sx) = \lim_{s\to 0} s \cdot M_C(x) = 0$

故 $\sup_x\{y^T x - M_C(x)\} = \lim_{s\to 0} y^T(sx) - M_C(sx) = 0$

$\therefore f^*(y) = \begin{cases} 0, & \forall x_0, y^T x_0 \le M_C(x_0) \\ +\infty, & \exists x_0, y^T x_0 > M_C(x_0) \end{cases}$

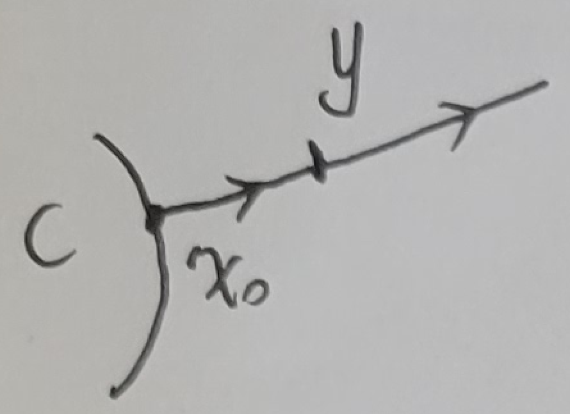
3. ① $f^*(y) = \sup\limits_{x} \{ y^Tx - \frac{1}{2}\|x\|^2 + \frac{1}{2}d_C^2(x) \}$

$y \in C$, 则 $y^Tx - \frac{1}{2}\|x\|^2 + \frac{1}{2}d_C^2(x) \le y^Tx - \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|y\|^2$

当 $x=y$ 时,等号成立

$\therefore f^*(y) = \frac{1}{2}\|y\|^2$

② $y \notin C$, 令 $x_0 = \arg\min\limits_{x\in C} \{ \|y-x\| \}$, $x = x_0 + t\cdot(y-x_0)$



则 $y^Tx - \frac{1}{2}\|x\|^2 + \frac{1}{2}d_C^2(x)$

$= y^Tx_0 + t\cdot y^T(y-x_0) - \frac{1}{2}\|x_0+t\cdot(y-x_0)\|^2 + \frac{1}{2}t^2\cdot\|y-x_0\|^2$

易知 $\lim\limits_{t\to\infty} y^Tx - \frac{1}{2}\|x\|^2 + \frac{1}{2}d_C^2(x) = +\infty$

$= t\cdot(y-x_0)^T\cdot(y-x_0) + y^Tx_0 - \frac{1}{2}\|x_0\|^2$

故不存在.

综上, $f^*(y) = \begin{cases} \frac{1}{2}\|y\|^2 & y\in C \\ +\infty, & y\notin C \end{cases}$

4. $f^*(y) = \sup\limits_x \{ y^T x - \sqrt{\alpha^2 + \|x\|^2} \}$

且已知：由范数对偶，$\|y\|^* = \sup\limits_x \{ y^T x - \|x\| \} = \begin{cases} 0, & \|y\|_* \le 1 \\ +\infty, & \|y\|_* > 1 \end{cases}$

① $\|y\|_* \le 1$

则 $y^T x \le \|x\| \le \sqrt{\alpha^2 + \|x\|^2}$ $\therefore \sup\limits_x \{ y^T x - \sqrt{\alpha^2 + \|x\|^2} \} \le 0$

而 $\exists$ x_0 s.t. $y^T x_0 = \|x_0\|$

$\therefore \lim\limits_{t\to\infty} y^T(t x_0) - \sqrt{\alpha^2 + \|t x\|^2} = \lim\limits_{t\to\infty} \|t x_0\| - \sqrt{\alpha^2 + \|t x_0\|^2} = 0$

$\therefore f^*(y) = 0$

② $\|y\|_* > 1$，设 $y = y_0(1+\varepsilon)$，$\|y_0\|_* = 1$，$\varepsilon > 0$

取 x_0 s.t. $y_0^T x_0 = \|x_0\|$

则 $\lim\limits_{t\to\infty} y^T(t x_0) - \sqrt{\alpha^2 + \|t x_0\|^2} = \lim\limits_{t\to\infty} \|t x_0\| - \sqrt{\alpha^2 + \|t x_0\|^2} + \lim\limits_{t\to\infty} \varepsilon t \|x_0\|$

$= +\infty$

综上，$f^*(y) = \begin{cases} 0, & \|y\|_* \le 1 \\ +\infty, & \|y\|_* > 1 \end{cases}$

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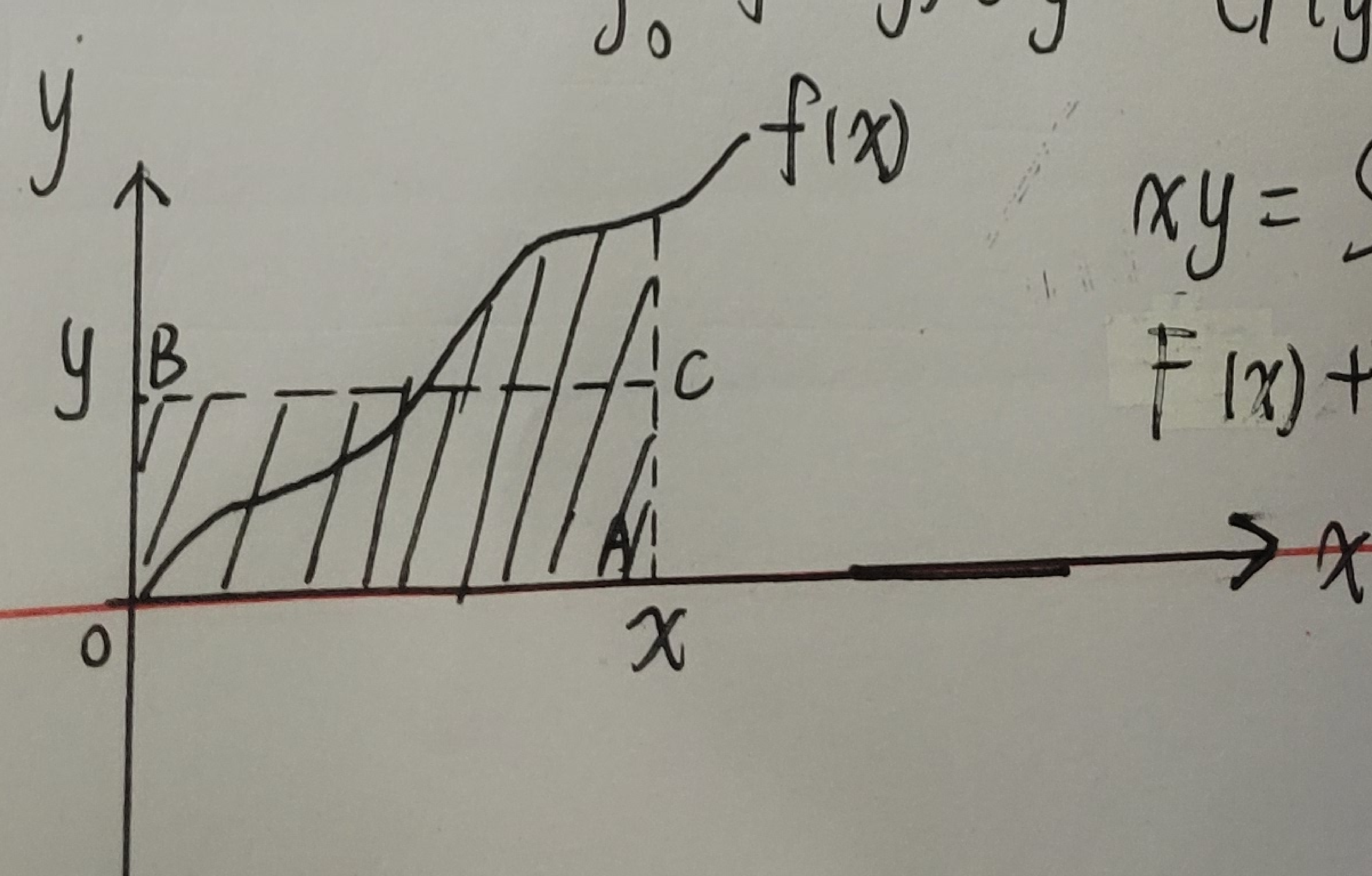
5. $F^*(y) = \sup\limits_x \{ yx - \int_0^x f(a)\,da \}$

$\frac{d}{dx}\left(yx - \int_0^x f(a)\,da\right) = y - f(x)$ $\begin{cases} \le 0, & \text{when } x \ge f^{-1}(y) \\ \ge 0, & \text{when } x \le f^{-1}(y) \end{cases}$

$\therefore F^*(y) = y f^{-1}(y) - \int_0^{f^{-1}(y)} f(a)\,da$

$= y f^{-1}(y) - \left(a f(a) \Big|_0^{f^{-1}(y)} - \int_0^{a=f^{-1}(y)} a\, df(a) \right)$

$= \int_0^y f^{-1}(y)\,dy = G(y)$，that is, F & G are conjugates



$xy = S_{\text{长方形 OABC}}$

$F(x) + G(y) = S_{\text{阴影}}$

易知 $xy \le F(x) + G(y)$

6.

$B_{f(\cdot)}(x,y) = f(x) - f(y) - \langle \nabla f(y), x-y \rangle$，查阅文献后认为应有：
f is convex

且 $h^*(\nabla h(y)) = \sup\limits_{x} \{ x^T \nabla h(y) - h(x) \} = y^T \nabla h(y) - h(y)$ ①

故 $B_{h^*}(\nabla h(x), \nabla h(y))$

$= h^*(\nabla h(x)) - h^*(\nabla h(y)) - \langle \nabla h(x) - \nabla h(y), \nabla h^*(\nabla h(y)) \rangle$

由①两边求梯度，$\underbrace{\nabla h^*(\nabla h(y))}_{\nabla^2 h(y)} = \nabla^2 h(y) \cdot y \Rightarrow \nabla h^*(\nabla h(y)) = y$

$\therefore B_{h^*}(\nabla h(x), \nabla h(y)) = x^T \nabla h(x) - h(x) - y^T \nabla h(y) + h(y) - (y)^T(\nabla h(x) - \nabla h(y))$

$= x^T \nabla h(x) - h(x) + h(y) - y^T \nabla h(x)$

$= B_h(y, x)$，得证.

7. $(f \square g)(x) = \inf\limits_{y+z=x} f(y) + g(x)$

$\therefore (f \square g)^*(y)$

$= \sup\limits_{x} \{ y^T x - (f \square g)(x) \} = \sup\limits_{x} \{ y^T x - \inf\limits_{z+w=x} (f(z) + g(w)) \}$

$= \sup\limits_{z,w} \{ [y^T z - f(z)] + [y^T w - g(w)] \}$ 这一步看作"两步优化"，为同时取极值和分步取极值。

$= \sup\limits_{z} \{ y^T z - f(z) \} + \sup\limits_{w} \{ y^T w - g(w) \} = f^* + g^*$