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Convex Opt HW6 2100011025 王奕博

1. 对于 K^o 与 $\tilde{K}^o$ (interiors of K and $\tilde{K}$), 易知 K^o 与 $\tilde{K}^o$ 仍是凸集

故对 K^o 与 $\tilde{K}^o$ 应用 separating hyperplane theorem, 有:

$\exists\, y, b$, s.t. $\forall x \in K^o$, $y^T x > b$; $\forall x \in \tilde{K}^o$, $y^T x < b$.

而 $\exists \{x_n\} \in K^o$, s.t. $\lim_{n\to\infty} \vec{x_n} = 0$. 故: $\lim_{n\to\infty} y^T x_n \geq \lim_{n\to\infty} b$, $b \leq 0$

同理, $b \geq 0$. 故 $b = 0$

K is a cone, ∵ $\forall x \in \partial K$, $\exists \{x_n\} \in K^o$, s.t. $\lim_{n\to\infty} x_n = x$. 故 $y^T x \geq 0$

∴ $\exists y$, s.t. $\forall x \in K$, $y^T x \geq 0$

$\forall x \in \tilde{K}$, $y^T x \leq 0$ ∴ $\exists y$ s.t. $y \in K^*$, $-y \in \tilde{K}^*$

2. 若 $\nexists x$, s.t. $x > 0$ 且 $Ax = b$

则令 $C_1 = \{x \mid x > 0\}$, $C_2 = \{x \mid Ax = b\}$

即: 两者无交点. 易知 C_1, C_2 均为 convex set,

故~~不~~存在 a, 使得: $\begin{cases} \forall x \in C_1, & a^T x \geq d \\ \forall x \in C_2, & a^T x \leq d \end{cases}$ ①

~~其~~等价为

条件 ① 等价于: $\begin{cases} a \geq 0 \\ d \leq 0 \\ a^T x \leq d,\ \forall x \in C_2 \end{cases}$ ②

而 C_2 是仿射集, 即 $\{a^T x \mid x \in C_2\}$ 也是仿射集, 且是一维.

一维仿射集只有: $\emptyset$, $\{a\}$, $\mathbb{R}$, 故 $\{a^T x \mid x \in C_2\}$ 等于 $\mathbb{R}$ 或 $\{a^T x_0\}$. 而 $a^T x$ 有界, 故 $a^T x$ 在 $x \in C_2$ 时为常数.

故 ② 又等价于:

$\begin{cases} \vec{a} \geq 0 \\ a^T x = d \leq 0,\ \forall x \in C_2 \end{cases}$, 令 $B = A^-$, 即伪逆 即 $A = P \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} Q$, P, Q 可逆

则 $B = Q^{-1} \begin{bmatrix} 0 & \\ & I \end{bmatrix} P^{-1}$.

有: $N(A) = R(B)$, $C_2 = x_0 + R(B) = \{x_0 + B\xi \mid \forall \xi\}$, 而 $a^T B \xi = d$, $\forall \xi$, 只有当:

$a^T B = 0$, 即 $a \in N(B^T)$ 时成立

$a \in N(B^T) = N(Q^{T^{-1}} \begin{bmatrix} 0 & \\ & I \end{bmatrix}^T P^{T^{-1}})$

而 $A^T = P^T \begin{bmatrix} I & \\ & 0 \end{bmatrix}^T Q^T$

故 $N(B^T) = R(A^T)$ $\therefore a \in R(A^T)$

故 $\exists \lambda$, s.t. $A^T\lambda = a$，则 $a^T x = \lambda^T A x = \lambda^T b \le 0$

即 ② 等价于 $\exists \lambda$, s.t. $\begin{cases} A^T\lambda \geqslant 0 \\ A^T\lambda \neq 0 \\ \lambda^T b \le 0 \end{cases}$，故得证.

3. $C = \{ x \in \mathbb{R}^n \mid |x_i| \le 1, \forall i = 1, \cdots, n \}$

故 C 可以等价表示为:

$\{ x \in \mathbb{R}^n \mid e_i^T (x - e_i) \le 0, -e_i^T (x + e_i) \le 0, \forall i = 1, \cdots, n \}$

$= \{ x \in \mathbb{R}^n \mid e_i^T x \le 1, e_i^T x \geqslant -1 \}_{i=1,\cdots,n}$

故其 supporting hyperplane 为: $e_i^T x = 1$, $e_i^T x = -1$, $\forall i = 1, \cdots, n$

4. 若 f 不为常数, i.e., $\exists x_0, \Delta x$, s.t. $f(x_0 + \Delta x) \neq f(x_0)$, 不妨令 $f(x_0 + \Delta x) > f(x_0)$

则 $\because$ f is convex，$\therefore$ $f(x_0 + 2\Delta x) + f(x_0) \geqslant 2 f(x_0 + \Delta x)$

即，若令 $\Delta f = f(x_0 + \Delta x) - f(x_0)$ $(\Delta f > 0)$，则 $f(x_0 + 2\Delta x) - f(x_0) \geqslant 2\Delta f$

同理， $f(x_0 + n\Delta x) \geqslant n \Delta f + f(x_0)$，$\Delta f > 0$

故 $\lim_{n \to \infty} f(x_n + n\Delta x) = \infty$，与有界矛盾

故 $f = \text{Const}$ on $\mathbb{R}^n$

5. (a) $f''(x) = e^x > 0$ on $\mathbb{R}$

故 f is convex. 而 $\forall \mu > 0$, $e^x - 1 - \frac{1}{2}\mu x^2$ 不 convex. 故不 strongly convex

(b) $\nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$，不正定也不负定, 故不 convex 也不 concave

(c) $\nabla^2 f = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}$. 由 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} > 0 \Leftrightarrow \begin{cases} a > 0 \\ d > 0 \\ ad - bc > 0 \end{cases}$

可判定 $\nabla^2 f > 0$, 其 convex

而 取 $x_1 = x_2 = x$，有 $\lambda_1 = \frac{3}{x^4}$，$\lambda_2 = \frac{1}{x^4}$. $\lim_{x\to+\infty} \lambda_{1,2} = 0$

故 不存在 $\mu > 0$, s.t. $\nabla^2 f \succeq \mu I$ 故不 strongly convex.

(d)

$$\nabla^2 f = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}.$$

并不正定、不负定.
故 f 不 convex 也不 concave

(e) $\nabla^2 f = \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}$. $\lambda_1 = 0$, $\lambda_2 = \frac{2}{x_2} + \frac{2x_1^2}{x_2^3} > 0$

故 $\nabla^2 f \succeq 0$，并 convex. 而 $\lambda_1 = 0$, 故不存在 $\mu > 0$, s.t. $\nabla^2 f - \mu I \succeq 0$

故 不 strongly convex

(f) $\nabla^2 f = \begin{bmatrix} (\alpha-1)\alpha x_1^{\alpha-2} x_2^{1-\alpha} & (1-\alpha)\alpha x_1^{\alpha-1} x_2^{-\alpha} \\ (1-\alpha)\alpha x_1^{\alpha-1} x_2^{-\alpha} & -(1-\alpha)\alpha x_1^{\alpha} x_2^{-\alpha-1} \end{bmatrix}$.

$\lambda_1 = 0$，$\lambda_2 = \alpha(1-\alpha)(x_1^{\alpha-2} x_2^{\alpha} + x_1^{\alpha} x_2^{-\alpha-1}) < 0$

故 $\nabla^2 f \leq 0$，而 $\lambda_1 = 0$，故 不 strongly convex.

故 concave 故不存在 $\mu > 0$, s.t. $\nabla^2 f - \mu I \succeq 0$.

6. 令 $\frac{a_1}{a_3} = \theta \in (0,1)$，原式等价为：

$\langle \nabla f(x_3), \theta x_1 + (1-\theta) x_2 - x_3 \rangle \leq \theta f(x_1) + (1-\theta) f(x_2) - f(x_3)$

而 $\langle \nabla f(x_3), y - x_3 \rangle \leq f(y) - f(x_3)$

取 $y = \theta x_1 + (1-\theta) x_2$

$LHS \leq f(\theta x_1 + (1-\theta) x_2) - f(x_3)$

而 $f(x)$ is convex 故 $f(\theta x_1 + (1-\theta) x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$

故得证。