年 月 日

Convex Opt HW9

1、(a) 令 $g(\vec{y}) = \log(\sum_{i=1}^{n} e^{y_i})$, $g(\vec{y})$ is convex

故 $\log(\sum_{i=1}^{m} \exp\{a_i^T x + b_i\}) = g(A\vec{x}+b)$

易知 $g(Ax+b)$ is convex (用定义).

$\therefore f(x) = -\log(-g(Ax+b))$, $-\log(x)$ is non-increasing and convex

$-g(Ax+b)$ is concave.

故 $f(x)$ is convex on dom f.

(b) $h(x, u) = x^T x/u$ is convex on $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$

$\therefore$ $v - x^T x/u$ is concave on $\mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \cup \{uv > x^T x\}$

$f(x, u, v) = -\sqrt{u \cdot (v - x^T x/u)} = g(u, v - x^T x/u)$

$g = -\sqrt{\cdot}$ is convex and non-increasing in all arguments

Besides, u is concave, $v - x^T x/u$ is concave

$\therefore$ f is convex on dom f.

(c) $f(x, u, v) = -\log u - \log(v - x^T x/u)$

由(b), $v - x^T x/u$ is concave, $-\log(y)$ is non-increasing and convex

$\therefore$ $-\log(v - x^T x/u)$ is convex, $-\log(u)$ is convex

$\therefore$ f is convex on dom f.

(d)

$f(x, t) = -t^{(p-1)/p}(t - \|x\|_p^p/t^{p-1})^{1/p}$

令 $g(u, v) = -u^{1/p} v^{1-1/p}$, g is convex on $\mathbb{R}^2_+$

且 g 关于 u, v 都递减 non-increasing

$f(x, t) = g(t - \|x\|_p^p/t^{p-1}, t)$

$\because$ $\|x\|_p^p/t^{p-1}$ is convex $\therefore$ $t - \|x\|_p^p/t^{p-1}$ is concave, t is concave

$\therefore$ f is convex on dom f

(e) $f(x,t) = -\log(t^p - \|x\|_p^p)$

$$= -(p-1)\log t - \log(t - \|x\|_p^p / t^{p-1})$$

由(d)，$t - \|x\|_p^p / t^{p-1}$ is concave, $-\log(\cdot)$ is convex & non-increasing

∴ $-\log(t - \|x\|_p^p / t^{p-1})$ is convex on its domain

而 $-(p-1)\log t$ is also convex on its domain

∴ $f(x,t)$ is convex on $\mathrm{dom} f$.

2. (a) $f^*(y) = \sup\limits_x \; y^T x - f(x)$

① 若 y 有元素 $y_i < 0$，则令 $x = \begin{bmatrix} 0 \\ \vdots \\ -t \\ \vdots \\ 0 \end{bmatrix}$ → 第 i 个 $= -te_i$

有：$\lim\limits_{t\to+\infty} y^T x - f(x) = \lim\limits_{t\to+\infty} -t y_i = +\infty$

∴ $y \succeq 0$

② 若 y 有元素 $y_i > 1$，

则令 $x = te_i$.

则 $y^T x - f(x) = (y_i - 1)t$，$\lim\limits_{t\to+\infty} y^T x - f(x) = +\infty$

∴ $y \preceq 1$

③ 若 $\sum\limits_{i=1}^n y_i \neq r$

则令 $x = \begin{bmatrix} t \\ \vdots \\ t \end{bmatrix}$. $y^T x - f(x) = t(\sum\limits_{i=1}^n y_i - r)$

$t \in (-\infty, +\infty)$，则 $\lim\limits_{t\to-\infty} y^T x - f(x) = +\infty$ 或 $\lim\limits_{t\to+\infty} y^T x - f(x) = +\infty$

∴ $\sum\limits_{i=1}^n y_i = r$

而当 y 满足 $\forall i$，$0 \le y_i \le 1$，$\sum\limits_{i=1}^n y_i = r$ 时，

$$y^T x - f(x) = y^T x - \sup_{\substack{0 \le y_i \le 1 \\ \sum_{i=1}^n y_i = r}} y^T x \le 0$$

而当 $x = 0$ 时，$y^T x - f(x) = 0$

故 $f^*(y) = \begin{cases} 0, & \begin{cases} \forall i, 0 \le y_i \le 1 \\ \sum_{i=1}^n y_i = r \end{cases} \\ +\infty, & \text{otherwise} \end{cases}$，$\mathrm{dom} f^* = \left\{ y \,\middle|\, \begin{array}{l} 0 \le y_i \le 1, \forall i \\ \sum_{i=1}^n y_i = r \end{array} \right\}$

年 月 日

(b) 令 x_i 表示: $y=a_i x+b_i$ 与 $y=a_{i+1}x+b_{i+1}$ 交点.

① 若 $y>a_m$, 则 显然 $yx-f(x)$ 在 $\mathbb{R}$ 上递增

$\lim_{x\to+\infty} yx-f(x) = \lim_{x\to+\infty}(y-a_m)x = +\infty$, 故不在 $\mathrm{dom}\, f^*$ 上.

② 若 $y<a_1$, 同理, $\lim_{x\to-\infty} yx-f(x)=+\infty$, 不在 $\mathrm{dom}\, f^*$ 上.

③ 若 $a_i \le y \le a_{i+1}$, 有: $x_i = \frac{b_i-b_{i+1}}{a_{i+1}-a_i}$

当 $x<x_i$ 时, $yx-f(x)$ 在每一段均 non-increasing

$x>x_i$ 时, $yx-f(x)$ 在每一段均 non-decreasing

$\therefore\ f^*(y) = yx_i - f(x_i) = y\frac{b_i-b_{i+1}}{a_{i+1}-a_i} - \frac{a_{i+1}b_i-a_ib_{i+1}}{a_{i+1}-a_i}$

综上, $f^*(y)=\begin{cases}\frac{b_i-b_{i+1}}{a_{i+1}-a_i}y-\frac{a_{i+1}b_i-a_ib_{i+1}}{a_{i+1}-a_i}, & a_i\le y\le a_{i+1}\\ +\infty, & \text{otherwise}\end{cases}$, $\mathrm{dom}\, f^* = \{y \mid a_1\le y\le a_n\}$

(c) [i] $p>1$. $f^*(y) = \sup_{x>0}(yx-x^p)$

$\frac{d}{dx}(yx-x^p) = y-px^{p-1}$, $\frac{d^2}{dx^2}(yx-x^p) = -p(p-1)x^{p-2}<0$

$\therefore$ $y\le 0$ 时, $\sup_{x>0}(yx-x^p)=0$; $y>0$ 时, $\sup_{x>0}(yx-x^p) = (p^{-\frac{1}{p-1}}-1)y^{\frac{p}{p-1}}$

$f^*(y)=\begin{cases}(p^{-\frac{1}{p-1}}-1)y^{\frac{p}{p-1}}, & y>0\\ 0, & y\le 0\end{cases}$. $\mathrm{dom}\, f^* = \mathbb{R}$

[ii] $p<0$ $f^*(y) = \sup_{x>0} yx-x^p$

$y>0$ 时, $\lim_{x\to+\infty} yx-x^p=+\infty$ $\therefore$ 不属于 $\mathrm{dom}\, f^*$

$y\le 0$ 时, $\frac{d}{dx}(yx-x^p) = y-px^{p-1}$, $\frac{d^2}{dx^2}(\cdots) = -p(p-1)x^{p-2}<0$

$\therefore\ f^*(y)=\begin{cases}+\infty, & y>0\\ (p^{-\frac{1}{p-1}}-1)y^{\frac{p}{p-1}}, & y\le 0\end{cases}$. $\mathrm{dom}\, f^* = (-\infty, 0]$

(d) $f^*(y) = \sup\limits_{x>0} \; y^T x + (\prod x_i)^{\frac{1}{n}}$

① 若 $\exists i$, s.t. $y_i > 0$, 则取 $x = e_i \cdot t + \sum\limits_{j\neq i} e_j = \begin{bmatrix} 1 \\ \vdots \\ t \\ \vdots \\ 1 \end{bmatrix}$

$\lim\limits_{t\to\infty} y^T x + (\prod x_i)^{\frac{1}{n}} = \lim\limits_{t\to\infty} t y_i + t^{\frac{1}{n}} - \sum\limits_{j\neq i} y_i = +\infty$

故不属于 dom f^*

② 若 $y_i \le 0$

(i) 若 $\left(\prod\limits_{i=1}^{n} |y_i|\right)^{\frac{1}{n}} \geq \frac{1}{n}$

则由基本不等式, $y^T x + (\prod x_i)^{\frac{1}{n}}$

$= \sqrt[n]{x_1 \cdots x_n} - \frac{|y_1| x_1 + \cdots + |y_n| x_n}{n} \le \sqrt[n]{x_1 \cdots x_n} - \sqrt[n]{x_1 \cdots x_n} = 0$

且 $\lim\limits_{x\to 0} y^T x + (\prod x_i)^{\frac{1}{n}} = 0$, 故 $f^*(y) = 0$

(ii) 若 $\left(\prod\limits_{i=1}^{n} |y_i|\right)^{\frac{1}{n}} < 1/n$

取 $x_i = -\frac{t}{y_i}$, 有: $x^T y - f(x) = t\cdot(-n) - t\cdot\left(\prod\left(-\frac{1}{y_i}\right)\right)^{\frac{1}{n}}$

$= t\left(\left(\prod -\frac{1}{y}\right)^{\frac{1}{n}} - n\right)$, ~~其中~~ $\left(\prod\limits_{i=1}^{n} -\frac{1}{y_i}\right) > n$

$\therefore \lim\limits_{t\to\infty} x^T y - f(x) = +\infty$, 故不属于 dom f^*

综上, $f^*(y) = \begin{cases} 0, & \left(\prod\limits_{i=1}^{n} |y_i|\right)^{\frac{1}{n}} \geq \frac{1}{n},\ y_i \le 0,\ \forall i \\ +\infty, & \text{otherwise} \end{cases}$

$\text{dom} f^* = \left\{ \vec{y} \;\middle|\; \vec{y} \le 0,\ \left(\prod\limits_{i=1}^{n} |y_i|\right)^{\frac{1}{n}} \geq \frac{1}{n} \right\}$

(e) 设 $y = \begin{bmatrix} \vec{y_1} \\ y_2 \end{bmatrix}\begin{matrix} \}n \\ \}1 \end{matrix}$

$f^*(y) = \sup\limits_{\text{dom} f} \; y_1^T x + y_2 t + \log(t^2 - x^T x)$

~~若 $\|y_1\|_2 + y_2 \geq 0$, 则令 $\vec{x} = u \cdot \vec{y_1}$, $t = u \cdot y_2$~~

~~$\lim\limits_{u\to\infty} y_1^T x + y_2 t + \log(t^2 - x^T x) = \lim\limits_{u\to\infty} u(\|y_1\|_2 + y_2^2)$~~

年 月 日

若 $\|y_1\|_2 + y_2 \geq 0$，则令 $x = s \cdot y_1$，$t = s\|y_1\|_2 + \varepsilon \cdot s$，$\varepsilon > 0$

则 $y_1^T x + y_2 t = (y_1^T y_1 + y_2\|y_1\| + \varepsilon)s > \varepsilon \cdot s$

$\therefore \lim_{s\to\infty} y_1^T x + y_2 t + \log(t^2 - x^T x) = +\infty$，不成立.

故 $\|y_1\|_2 < -y_2$.

而 $\frac{\partial}{\partial x}\log(t^2 - x^T x) = \frac{-2}{t^2 - x^T x}\vec{x}$，$\frac{\partial}{\partial t}\log(t^2 - x^T x) = \frac{2}{t^2 - x^T x}t$

令 $y_1 = \frac{-2}{t^2 - x^T x}\vec{x}$，$y_2 = \frac{2}{t^2 - x^T x}t \Rightarrow \vec{x} = \frac{2\vec{y}_1}{y_2 - y_1^T y_1}$　$t = \frac{-2y_2}{y_2 - y_1^T y_1}$

有：

$$f^*(\vec{y}_1, y_2) = y_1^T x + y_2 t + \log(t^2 - x^T x)$$

$$= -2 + \log\frac{4}{y_2^2 - y_1^T y_1}$$，$\mathrm{dom}\, f^* = \{\vec{y} \mid y_2^2 - y_1^T y_1 > 0\}$

$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

3. (a) $g^*(y) = \sup_{x\in \mathrm{dom} f} (y+c)^T x + f(x) + d$

$$= d + \sup_{x\in\mathrm{dom} f} (y+c)^T x + f(x) = d + f^*(y+c)$$

$(\mathrm{dom}\, g^* = \{y \mid y+c \in \mathrm{dom}\, f^*\})$

(b) 设 $g(x,t) = t f(x/t)$，$\mathrm{dom}\, g = \{(x,t) \mid x/t \in \mathrm{dom} f, t>0\}$

$$g^*(y,s) = \sup_{\substack{\cancel{(y,s)} \in \mathrm{dom}\, g \\ (x,t)}} y^T x + st - t f(x/t) = \sup_t \left\{ t \cdot \left(\sup_{x/t \in \mathrm{dom} f} \left(y^T\left(\frac{x}{t}\right) - f\left(\frac{x}{t}\right)\right) + s \right) \right\}$$

$$= \sup_{t>0} t \cdot f^*(y) + st = \begin{cases} 0, & f^*(y) + s \leq 0 \\ +\infty, & f^*(y) + s > 0 \end{cases}$$

(c) $g(x) = \inf_z f(x,z)$

则 $g^*(y) = \sup_{x\in\mathrm{dom} g} y^T x - g^*(x) = \sup_{(x,z)\in\mathrm{dom} f} y^T x - f(x,z) = f^*(y,0)$

记 $f(x,z) = \begin{cases} +\infty, & Az+b \neq x \\ h(z), & Az+b = x \end{cases}$，则 $f^*(y,w) = \inf_{Az+b} (y^T x - w^T z - h(z)) = b^T y + h^*(A^T y - w)$

$\therefore g^*(y) = f^*(y,0) = b^T y + h^*(A^T y)$