国家税务总局徐州市税务局稽查局

Convex Optimization HW2 王奕博 2100011025

1. a. interior : $\emptyset$ closure : $\emptyset$ boundary : $\emptyset$

b. interior : $\mathbb{R}^n$ closure : $\mathbb{R}^n$ boundary : $\emptyset$

c. interior: $(0,1) \cup (2,3) \cup (4,5)$ closure : $[0,1] \cup [2,3] \cup [4,5]$

boundary : $\{0, 1, 2, 3, 4, 5\}$

d. interior : $\{(x,y)^T \mid x>0, y>0\}$ closure : $\{(x,y)^T \mid x\geq 0, y\geq 0\}$

boundary : $\{(x,y)^T \mid (x=0, y\geq 0)$或$(x\geq 0, y=0)\}$

e. 由于 $C_5^c = \cdots \cup (-1,0) \cup (0,1) \cup \cdots$ 为开集

故 $(C_5^c)^o = C_5^c$, closure $=((C_5^c)^o)^c = C_5 = \{k \mid k\in\mathbb{Z}\}$

interior $= \emptyset$. boundary $= \bar{C} / C^o = C_5 = \{k \mid k\in\mathbb{Z}$

f. 其为离散点列, 故 interior $= \emptyset$

而 C_6 不为闭 (因为 $\lim\limits_{k\to\infty} \frac{1}{k} = 0$, 0不在 C_6 内), 而 $C_6 \cup \{0\}$ 为闭

又因为显然 $C_6 \cup \{0\}$ 为含 C_6 的最小闭集, 故:

closure $= \{0\} \cup \{\frac{1}{k} \mid k\in\mathbb{Z}\}$

boundary = closure / interior $= \{0\} \cup \{\frac{1}{k} \mid k\in\mathbb{Z}\}$

g. 其为离散点列, 故 interior $= \emptyset$

而 $\sin k, k\in\mathbb{Z}$ 在 $[-1,1]$ 上稠密, 故可知:

closure $= \{\begin{bmatrix} 1/k \\ \sin k \end{bmatrix}, k\in\mathbb{Z}\} \cup \{\begin{bmatrix} 0 \\ y \end{bmatrix}, y\in[-1,1]\}$

boundary = closure = ↺

4. (b) 若 $x \in (A\cap B)^\circ$，则 $\exists\,\varepsilon$, s.t. $B_\varepsilon(x) \subseteq A$ 且 $B_\varepsilon(x) \subseteq B$

故 $x \in A^\circ$ 且 $x \in B^\circ$，$x \in A^\circ \cap B^\circ$

若 $x \in A^\circ$ 且 $x \in B^\circ$，则 $\exists\,\varepsilon_1$ s.t. $B_{\varepsilon_1}(x) \subseteq A$，$\exists\,\varepsilon_2$ s.t. $B_{\varepsilon_2}(x) \subseteq B$

则 $\exists\,\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, s.t. $B_\varepsilon(x) \subseteq A\cap B$ $\therefore x \in (A\cap B)^\circ$

$\therefore (A\cap B)^\circ = A^\circ \cap B^\circ$

若 $x \in A^\circ$ 或 $x \in B^\circ$，则 $\{\exists\,\varepsilon_1$ s.t. $B_{\varepsilon_1}(x) \subseteq A\}$ 或 $\{\exists\,\varepsilon_2$ s.t. $B_{\varepsilon_2} \subseteq A\}$ 成立

则 $x \in A^\circ \cup B^\circ$ 一定成立.

$\therefore$ 故：$(A\cup B)^\circ \supseteq A^\circ \cup B^\circ$

令 $A = [0,1]$, $B = [1,2]$. 则 $A^\circ \cup B^\circ = (0,1)\cup(1,2)$, $(A\cup B)^\circ = (0,2) \neq A^\circ \cup B^\circ$.

(a) 利用 (b) 的结论,

$$\overline{A\cup B} = [[[A\cup B]^c]^\circ]^c = [[A^c \cap B^c]^\circ]^c \xlongequal{\text{上问结论}} [[A^c]^\circ \cap [B^c]^\circ]^c$$

$$= [[A^c]^\circ]^c \cup [[B^c]^\circ]^c = \bar{A} \cup \bar{B}.$$ 得证.

$$\overline{A\cap B} = [[[A\cap B]^c]^\circ]^c = [[A^c \cup B^c]^\circ]^c$$

而 $[A^c \cup B^c]^\circ \supseteq [A^c]^\circ \cup [B^c]^\circ$

故 $[[A^c \cup B^c]^\circ]^c \subseteq [[A^c]^\circ \cup [B^c]^\circ]^c = [[A^c]^\circ]^c \cap [[B^c]^\circ]^c$

$= \bar{A} \cap \bar{B}$. 得证.

令 $A = [0,1]$, $B = [1,2]$

则 $A\cap B = \emptyset$, $\overline{A\cap B} = \emptyset$

$\bar{A} = [0,1]$, $\bar{B} = [1,2]$, $\bar{A} \cap \bar{B} = \{1\} \neq \overline{A\cap B}$

2、"$\Rightarrow$" $x \in \partial C \Rightarrow x \in \bar{C}$ 且 $x \notin C^\circ$

$x \notin C^\circ$ 可知，$\forall \varepsilon > 0, \exists z \notin C$, s.t. $\|z - x\|_2 \le \varepsilon$

$x \in \bar{C}$ 可知，$x \notin (C^c)^\circ$. 故 $\forall \varepsilon > 0, \exists y \notin C^c$, s.t $\|y - x\|_2 \le \varepsilon$

而 $y \notin C^c \Leftrightarrow y \in C$. 即证.

"$\Leftarrow$"

$\forall \varepsilon > 0, \exists z \notin C$, s.t. $\|z - x\|_2 \le \varepsilon$ 可知，$x \notin C^\circ$

$\forall \varepsilon > 0, \exists y \in C$, 即 $y \notin C^c$, s.t. $\|y - x\|_2 \le \varepsilon$, 可知，$x \notin (C^c)^\circ$

故 $x \in ((C^c)^\circ)^c$ 且 $x \notin C^\circ$

即 $x \in ((C^c)^\circ)^c / C^\circ = \partial C$

3、以下使用两个分析学结论:

Lemma① C is closed $\Leftrightarrow$ $\mathbb{R}^n / C$ is open; vice versa.（开闭互补，经典结论）

Lemma② C is open $\Leftrightarrow$ $C = C^\circ$（应是定义）

则：[1] C is closed $\Leftrightarrow$ $\partial C \subseteq C$

Proof: "$\Rightarrow$": C is closed $\Rightarrow C^c$ is open $\Rightarrow (C^c)^\circ = C^c$

$\Rightarrow C = ((C^c)^\circ)^c$, 而 $C^\circ \subseteq C$ 显然

故 $\partial C = ((C^c)^\circ)^c / C^\circ \subseteq C$.

"$\Leftarrow$": $\partial C \subseteq C \Rightarrow ((C^c)^\circ)^c / C^\circ \subseteq C$, 而 $C^\circ \subseteq C$ 显然

$\Rightarrow ((C^c)^\circ)^c \subseteq C$

$\Rightarrow C^c \subseteq (C^c)^\circ$（利用 $A \subseteq B \Leftrightarrow \mathbb{R}^n/A \supseteq \mathbb{R}^n/B$）

而 $(C^c)^\circ \subseteq C^c$ 显然，$\therefore C^c = (C^c)^\circ$, 故 C^c 为开

$\therefore C$ 为闭.

[2] C is open $\Leftrightarrow$ $C \cup \partial C = \phi$

Proof: 注意到 C is open $\Leftrightarrow C^c$ is closed, $C \cup \partial C = \phi \Leftrightarrow \partial C \subseteq C^c$

而 Lemma 3: $\partial C = \partial C^c$. 证明: $\partial C = [[C^c]^\circ]^c / C^\circ$, $\partial C^c = [C^\circ]^c / [C^c]^\circ$

而 $A^c / B^c = B/A$, 故 $\partial C^c = [[C^c]^\circ]^c / C^\circ = \partial C$. 得证.
$(A \subseteq B)$

故 $C \cup \partial C = \phi \Leftrightarrow \partial C^c \subseteq C^c$ 而上面已证 C^c is closed $\Leftrightarrow \partial C^c \subseteq C^c$

故: C is open $\Leftrightarrow C \cup \partial C = \phi$

5、令 $x_n = 4e^{-1.5^n}$

则 $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n^{1.5}} = \lim\limits_{n\to\infty} \exp(\ln x_{n+1} - 1.5\ln x_n)$

$= \lim\limits_{n\to\infty} \exp\{\ln 4 - 1.5^{n+1} - 1.5(\ln 4 - 1.5^n)\}$

$= \lim\limits_{n\to\infty} \exp\{-0.5\ln 4\} = 4^{-0.5} = 0.5$

且 $\lim\limits_{n\to\infty} x_n = 0$，故得证.

(p is rate of convergence)

6. a. $\lim\limits_{n\to\infty} x_n = 0$，且 $\lim\limits_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = 0.5$，故 $p=1$，rate constant $= 0.5$

b. $\lim\limits_{n\to\infty} x_n = 1$，且 $\lim\limits_{n\to\infty} \left|\frac{x_{n+1}-1}{x_n-1}\right| = 0.01$，故 $p=1$，rate constant $= 0.01$

c. $\lim\limits_{n\to\infty} x_n = 0$，且 $\lim\limits_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^2} = \frac{2^{-2^{k+1}}}{2^{-2^k\cdot 2}} = 1$，故 $p=2$，rate constant $=1$

d. $\lim\limits_{n\to\infty} x_n = 0$，$\lim\limits_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim\limits_{n\to\infty} 3^{-2n-1} = 0$ 故 $p=1$，rate constant $= 0$

(Superlinear)

e. $\lim\limits_{n\to\infty} x_n = 0$，但 $\lim\limits_{n\to\infty} \frac{|x_{n+1}-1|}{|x_n-1|^r}$ 不存在，故不存在p与rate constant

7. (a) $\frac{d\|\vec{x}\|_p^p}{d\vec{x}} = \frac{d(x_1^p + \dots + x_n^p)}{d\vec{x}} = \begin{bmatrix} p x_1^{p-1} \\ \vdots \\ p x_n^{p-1} \end{bmatrix}$. 故 $\frac{df}{d\vec{x}} = \frac{d\|x\|_p}{d\|x\|_p^p} \cdot \frac{d\|x\|_p^p}{dx}$

$= \frac{1}{p\|x\|_p^{p-1}} \begin{bmatrix} p x_1^{p-1} \\ \vdots \\ p x_n^{p-1} \end{bmatrix}$

$= \frac{1}{\|x\|_p^{p-1}} \begin{bmatrix} x_1^{p-1} \\ \vdots \\ x_n^{p-1} \end{bmatrix}$.

(b) $\frac{df}{dx} = (a^Tx)\frac{d\, b^Tx}{dx} + (b^Tx)\cdot\frac{d\, a^Tx}{dx}$

$= (a^Tx)\vec{b} + (b^Tx)\vec{a}$.

(c) 设 $Ax = y$,

$\frac{df}{dy} = \frac{d}{dy}\frac{1}{2}\|y-b\|_2^2 = \vec{y} - \vec{b}$，即 $f(y+\Delta y) \approx f(y) + \langle \frac{df}{dy}, \Delta y \rangle$

$\therefore f(x+\Delta x) \approx f(x) + \langle Ax-b, A\Delta x \rangle$

$= f(x) + (Ax-b)^T \cdot A\Delta x = f(x) + \langle A^T(Ax-b), \Delta x \rangle \quad \therefore \frac{df}{dx} = A^T(Ax-b)$

d. 设 $\tilde{f}(y) = u^T g(y) = u_1 g(y_1) + \cdots + u_n g(y_n)$

$\therefore \frac{d\tilde{f}}{dy} = \begin{bmatrix} u_1 g_0' \\ \vdots \\ u_n g_0' \end{bmatrix}$

而由上问,易知: $\frac{df}{dx} = R^T \frac{d\tilde{f}}{dy} = R^T \begin{bmatrix} u_1 g_0'(y_1) \\ \vdots \\ u_n g_0'(y_n) \end{bmatrix}$

e. 在一阶小量意义下,考虑:

$f(X+\Delta X) = \|(X^T+\Delta X^T)A(X+\Delta X)\|_F^2 \overset{\text{抛弃二阶小量}}{\approx} \|X^TAX + \Delta X^TAX + X^TA\Delta X\|_F^2$

$= \sum^m \sum^n [(X^TAX)_{ij} + (\Delta X^TAX)_{ij} + (X^TA\Delta X)]^2$

$\approx \sum^m \sum^n (X^TAX)_{ij}^2 + 2(X^TAX)_{ij}(\Delta X^TAX + X^TA\Delta X)_{ij}$ → 抛弃二阶小量

$= \|X^TAX\|_F^2 + 2\sum\sum (X^TAX)_{ij}(\Delta X^TAX + X^TA\Delta X)_{ij}$ $\quad \sum\sum A_{ij}B_{ij} = tr(A^TB)$

$= \|X^TAX\|_F^2 + 2tr((X^TAX)^T(\Delta X^TAX + X^TA\Delta X))$

$= \|X^TAX\|_F^2 + 2tr(X^TAX \cdot \Delta X^T \cdot AX) + 2tr(X^TAX \cdot X^TA\Delta X)$

$= \|X^TAX\|_F^2 + tr(2X^TA\Delta X \cdot X^TAX + 2X^TAXX^TA\Delta X)$

$= \|X^TAX\|_F^2 + tr(4(AXX^TAX)^T\Delta X)$ → 利用 $tr(AB) = tr(BA)$

故: $\frac{\partial f}{\partial X} = 4AXX^TAX$.

f. 依然在一阶小量下考虑,部分过程与上面相同,

$f(X+\Delta X) \approx \|diag(X^TAX) + 2diag(X^TA\Delta X)\|_F^2$

$= \sum_{i=1}^m [(X^TAX)_{ii} + 2(X^TA\Delta X)_{ii}]^2 \approx \sum_{i=1}^m (X^TAX)_{ii}^2 + 4(X^TAX)_{ii}\cdot(X^TA\Delta X)_{ii}$

$= \|diag(X^TAX)\|_F^2 + 4tr(diag(X^TAX)\cdot diag(X^TA\Delta X))$

而易知, $tr(diag(A)\cdot diag(B)) = tr(diag(A)\cdot B)$

故 $f(X+\Delta X) \approx f(X) + tr(4diag(X^TAX)\cdot X^TA\Delta X)$, 由此, $\frac{df}{dX} = 4AXdiag(X^TAX)$

g. $f(X+\Delta X) = tr(A(X+\Delta X)B) = tr(AXB) + tr(A\Delta XB)$

$\xlongequal{tr(AB)=tr(BA)} tr(AXB) + tr(BA\Delta X)$, 故 $\frac{df}{dX} = (BA)^T = A^TB^T$