Convex Opt 最 优 化 HW11 王奕博 2100011025

1. (a) $\mathrm{Prox}_c(x) = \inf\limits_{w}\{f(w) + \frac{1}{2c}\|w-x\|^2\}$, $c>0$

$= \inf\limits_{w}\{\|w\|_2 + \frac{1}{2c}\|w-x\|_2^2\}$

direct computation:

取最小值时, $0 \in \partial[\|w\|_2 + \frac{1}{2c}\|w-x\|_2^2]$

设 $w^* = \mathrm{argmin}\{\cdots\}$, ① $w^* \neq 0$, 则有: $0 = \frac{\vec{w}}{\|w\|_2} + \frac{1}{c}(\vec{w}-\vec{x})$

$\therefore \vec{x} = \vec{w}\left(\frac{c}{\|w\|_2}+1\right)$, $\|x\|_2 = \left\|\frac{\vec{w}}{\|w\|_2}\right\|(c+\|w\|_2)$
$= c + \|w\|_2$

$\|w\|_2 + \frac{1}{2c}\|w-x\|_2^2$
$= \|w\|_2 + \frac{1}{2c}\cdot c^2 = \frac{c}{2} + \|w\|_2 = (c+\|w\|_2) - \frac{c}{2}$
$= \|x\|_2 - \frac{c}{2}$

② $w^* = 0$, 有: $0 \in \partial[\|w\|_2 + \frac{1}{2c}\|w-x\|_2^2] \Rightarrow \partial\|w\|_2\big|_{w=0} \ni \frac{\vec{x}}{c}$

则 $\frac{\vec{x}}{c} \in \{g \mid \|g\|_2 \le 1\}$, $\mathrm{Env}_c(x) = \frac{\|x\|_2^2}{2}$

$$\therefore \mathrm{Prox}_c(x) = \begin{cases} 0, & \|x\|_2 \le c \\ \vec{x} - \frac{c\vec{x}}{\|x\|_2}, & \|x\|_2 > c. \end{cases}$$

Moreau Decomposition:

$\vec{x} = \mathrm{Prox}_c f(x) + \mathrm{Prox}_{c^{-1}} f^*(c^{-1}x)\cdot c$

$f(x) = \|x\|_2$, $f^*(y) = \begin{cases} 0, & \|y\|_2 \le 1 \\ +\infty, & \|y\|_2 > 1 \end{cases}$

$\mathrm{Env}_{\frac{1}{c}} f^*(\frac{x}{c}) = \inf\limits_{w}\{f(w) + \frac{c}{2}\|w - \frac{x}{c}\|^2\}$

$= \inf\limits_{\|w\|_2 \le 1}\{\frac{c}{2}\|w-\frac{x}{c}\|^2\} = \begin{cases} 0, & \|x\|_2 \le c \\ \frac{c}{2}(\|\frac{x}{c}\|_2^2 - 1), & \|x\|_2 > c \end{cases}$

$\therefore \mathrm{Prox}_{\frac{1}{c}} f^*(\frac{x}{c}) = \begin{cases} \frac{\vec{x}}{c}, & \|x\|_2 \le c \\ \frac{x}{\|x\|_2}, & \|x\|_2 > c \end{cases}$

$\therefore \mathrm{Prox}_c f(x) = \vec{x} - c\cdot \mathrm{Prox}_{\frac{1}{c}} f^*(\frac{x}{c})$

$$= \begin{cases} 0, & \|x\|_2 \le c \\ \vec{x} - \frac{c}{\|x\|_2}\vec{x}, & \|x\|_2 > c \end{cases}$$

写为 c_0, 与 c 区分

(b) $\mathrm{Prox}_c f(x) = \underset{\vec{y}}{\mathrm{argmin}} \left\{ \frac{1}{2} y^T A y + b^T y + c_0 + \frac{1}{2c}(y-x)^T(y-x) \right\}$

$= \underset{\vec{y}}{\mathrm{argmin}} \left\{ \frac{1}{2} y^T (A + \frac{1}{c} I) y + (\vec{b} - \frac{1}{c}\vec{x}) y + C_0 + \frac{1}{2c} x^T x \right\}$

其存在最小值，当且仅当 $A + \frac{1}{c} I \succeq 0$、当此条件成立时，

$\mathrm{Prox}_c f(x) = y$，其中 y 满足 $(A + \frac{1}{c} I)\vec{y} = \frac{1}{c}\vec{x} - \vec{b}$

若有 $A + \frac{1}{c} I \succ 0$，则有：$\mathrm{Prox}_c f(x) = (A + \frac{1}{c} I)^{-1} (\frac{1}{c}\vec{x} - \vec{b})$

(c) $\mathrm{Prox}_c g(x) = \underset{\vec{w}}{\mathrm{argmin}} \left\{ g(\vec{w}) + \frac{1}{2c}(w-x)^2 \right\}$，认为 $x \geq 0$.

$= \underset{w \geq 0}{\mathrm{argmin}} \left\{ \lambda w^3 + \frac{1}{2c}(w-x)^2 \right\}$，当 $\lambda < 0$ 时其不存在.

当 $\lambda = 0$ 时，$\mathrm{Prox}_c g(x) = x$。

当 $\lambda > 0$ 时，$\frac{d}{dw}\left(\lambda w^3 + \frac{1}{2c}(w-x)^2\right) = 0 \Rightarrow \mathrm{Prox}_c g(x) = \frac{-1 + \sqrt{1 + 12\lambda c x}}{6\lambda c}$

2、对于 $d(x)$，由前知 $d(x)$ is convex，$\therefore \frac{1}{2} d^2(x)$ is convex

$\mathrm{Prox}_c d(x) = \underset{w}{\mathrm{argmin}} \left\{ d(w) + \frac{1}{2c} \|w - x\|_2^2 \right\}$

1. $x \in C$，则 $\partial d(w) = 0$，$\mathrm{Prox}_c d(x) = \vec{x}$.

2. $x \notin C$，则令 $w_0 = \underset{\tilde{w} \in C}{\mathrm{argmin}} \|\tilde{w} - w\|_2$，易知 w_0 唯一.

则 $d(w) = \|w - w_0\|_2$ 且 $d(w + t \cdot (w - w_0)) = \|w - w_0\|_2 + t\|w - w_0\|$

(讲义结论) 故易知，$\partial d(w) = \frac{\vec{w} - \vec{w}_0}{\|w - w_0\|_2} = (\vec{w} - P_C(\vec{w})) / \|\vec{w} - P_C(\vec{w})\|_2$

$\therefore \frac{\vec{w} - P_C(w)}{\|w - P_C(w)\|} + \frac{1}{c}(\vec{w} - \vec{x}) = 0 \Rightarrow w = P_C(x) + t \cdot (x - P_C(x))$ $(t \geq 0.)$

则 $d(w) + \frac{1}{2c}\|w - x\|_2^2 = \left[t\|x - P_C(x)\|_2 + \frac{\|x - P_C(x)\|_2^2}{2c}(t-1)^2 \right]$

① $d(x) \leq c$，则 $t^* = 0$，$\mathrm{Prox}_c d(x) = P_C(x)$.

② $d(x) > c$，则 $t^* = 1 - \frac{c}{\|x - P_C(x)\|_2}$，$\mathrm{Prox}_c d(x) = P_C(x) + \left(1 - \frac{c}{\|x - P_C(x)\|_2}\right) \cdot (x - P_C(x))$

故综上，$\mathrm{Prox}_{c}\,d(x) = \begin{cases} P_c(x), & d(x) \le c \\ x + c \cdot \dfrac{P_c(x) - x}{\|P_c(x) - x\|_2}, & d(x) > c. \end{cases}$

对于 $\frac{1}{2}d_C^2$，其为:

$$\mathrm{prox}_c \tfrac{1}{2}d_C^2 = \underset{u}{\mathrm{argmin}}\left\{\frac{1}{2}\cdot d_C^2(u) + \frac{1}{2c}\|u-x\|^2\right\}$$

$$= \underset{u}{\mathrm{argmin}} \inf_{v\in C}\left\{\frac{1}{2}\|u-v\|^2 + \frac{1}{2c}\|u-x\|^2\right\}$$

$$= \underset{u}{\mathrm{argmin}} \inf_{v\in C}\left\{\frac{1}{2}\left((1+\frac{1}{c})u^Tu - 2(v^T+\frac{x^T}{c})u\right)\right\}$$

则 u 与 v 满足：$u = \frac{c}{c+1}v + \frac{1}{c+1}x$

又有：$v = d_C(u)$，即 $v = \underset{v\in C}{\mathrm{argmin}}\|u-v\|_2$

则 $v = \underset{v\in C}{\mathrm{argmin}}\left\|\frac{c}{c+1}v + \frac{1}{c+1}x - v\right\|_2 = \underset{v\in C}{\mathrm{argmin}}\frac{1}{c+1}\|x-v\|$

故：$v = d_C(x)$

故有：$\mathrm{prox}_c \frac{1}{2}d^2(x) = \frac{c}{c+1}d_C(x) + \frac{1}{c+1}x$

3. (a) $\mathrm{Prox}_c f(x) = \underset{w}{\mathrm{argmin}} \{ g(w) + a^T w + \frac{1}{2c} \|w - x\|_2^2 \}$

(常数项舍去)

$= \underset{w}{\mathrm{argmin}} \{ g(w) + a^T w + \frac{1}{2c} (w^T w - 2x^T w) \}$

$= \underset{w}{\mathrm{argmin}} \{ g(w) + \frac{1}{2c} (w^T w - 2(x - ac)^T w) \}$

$= \underset{w}{\mathrm{argmin}} \{ g(w) + \frac{1}{2c} \|w - (x - ca)\|_2^2 \}$

$= \mathrm{Prox}_c\, g(x - ca)$

(b) $\mathrm{Prox}_c f(x) = \underset{w}{\mathrm{argmin}} \{ g(w) + \frac{1}{2\mu} \|w - a\|^2 + \frac{1}{2c} \|w - x\|_2^2 \}$ (舍去常数项)

$= \underset{w}{\mathrm{argmin}} \{ g(w) + (\frac{1}{2\mu} + \frac{1}{2c}) w^T w - (\frac{1}{\mu} a + \frac{1}{c} x)^T w \}$

$= \underset{w}{\mathrm{argmin}} \{ g(w) + \frac{\mu^{-1} + c^{-1}}{2} \|w - \frac{1}{\mu^{-1} + c^{-1}} (\frac{a}{\mu} + \frac{x}{c})\|_2^2 \}$

$= \mathrm{Prox}_\lambda\, g(\lambda(\frac{x}{c} + \frac{a}{\mu}))$, $\lambda^{-1} = \mu^{-1} + c^{-1}$.

4.

$$x_k = \arg\min_x \left\{ g(x) + \frac{1}{2\eta} \| x - y_k + \eta \nabla f(y_k) \|_2^2 \right\}$$

$\therefore$ 有:

$$g(x_k) + \frac{1}{2\eta} \| x_k - y_k + \eta \nabla f(y_k) \|_2^2 \le g(y_k) + \frac{1}{2\eta} \| \eta \nabla f(y_k) \|_2^2$$

移项,得

$$g(x_k) \le g(y_k) - \langle \nabla f(y_k), x_k - y_k \rangle - \frac{1}{2\eta} \| x_k - y_k \|^2$$

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5、令 $f(x) = \begin{cases} 0, & x \in L \\ +\infty, & \text{otherwise} \end{cases}$

则 $f^*(y) = \sup_x \{ y^T x - f(x) \} = \sup_{x \in L} y^T x$

若 $y^T x \neq 0$，则 $\lim_{t \to +\infty} y^T(tx)$ 与 $\lim_{t \to -\infty} y^T(tx)$ 中一定有一个为 $+\infty$

$\therefore y \in L^\perp$， $\therefore f^*(y) = \begin{cases} 0, & y \in L^\perp \\ +\infty, & \text{otherwise} \end{cases}$

$\text{dom} f^* = L^\perp$

故 $P_L(x) = \text{Prox}_c f(x)$ ，$P_{L^\perp}(x) = \text{Prox}_c f^*(x)$ ，$\forall c$.

由 Moreau 分解，$x = P_c f(x) + c P_{c^{-1}} f^*(c^{-1} x)$. 取 $c=1$，

有 $x = P_L(x) + P_{L^\perp}(x)$