

国家税务总局徐州市税务局稽查局

Convex Opt Homework 4

$$1. \quad \|x\|_M^* = \sup_x \{ x^T y \mid y^T M y \leq 1 \}$$

$$= \sup_x \{ (M^{-\frac{1}{2}} x)^T (M^{\frac{1}{2}} y) \mid (M^{\frac{1}{2}} y)^T (M^{\frac{1}{2}} y) \leq 1 \}$$

$$\text{令 } M^{\frac{1}{2}} y = z$$

$$= \sup_x \{ \langle z, M^{-\frac{1}{2}} x \rangle \mid z^T z \leq 1 \}$$

$$= \|M^{-\frac{1}{2}} x\|_2 = \sqrt{x^T M^{-1} x} = \|x\|_{M^{-1}}$$

$$\text{故 } \|\cdot\|_M^* = \|\cdot\|_{M^{-1}}$$

$$2. \quad \text{对于 } (\vec{x}, \vec{y}), \text{ 定义内积 } \langle (\vec{x}_1, \vec{y}_1), (\vec{x}_2, \vec{y}_2) \rangle = x_1^T x_2 + y_1^T y_2$$

$$(\text{或看作 } \begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}), \text{ 记 } \|\begin{bmatrix} \vec{x} \\ \vec{y} \end{bmatrix}\| = \sqrt{\alpha \|\vec{x}\|_A^2 + \|\vec{y}\|_B^2}$$

$$\text{则 } \|\begin{bmatrix} \vec{x}_0 \\ \vec{y}_0 \end{bmatrix}\|^* = \sup_{x, y} \{ x_0^T x + y_0^T y \mid \sqrt{\alpha \|\vec{x}\|_A^2 + \|\vec{y}\|_B^2} \leq 1 \}$$

$$\text{考虑问题: } \begin{aligned} & \max_{x, y} x_0^T x + y_0^T y \\ & \text{s.t. } \alpha \|\vec{x}\|_A^2 + \|\vec{y}\|_B^2 \leq 1 \end{aligned}$$

由此优化问题的结构特殊性, 可转化为 two-step optimization (dynamic programming)

Step 1: $\max_{\lambda, \mu} f(\lambda) + g(\mu)$ Step 2: $f(\lambda) = \max_x x_0^T x, \text{ s.t. } \|\vec{x}\|_A^2 \leq \lambda$

其中,

$g(\mu) = \max_y y_0^T y, \text{ s.t. } \|\vec{y}\|_B^2 \leq \mu$

s.t. $\alpha \lambda + \mu = 1$

$$\text{由定义, } f(\lambda) = \sqrt{\lambda} \|\vec{x}_0\|_A^*, \quad g(\mu) = \sqrt{\mu} \|\vec{y}_0\|_B^*$$

$$\text{故原问题转化为 } \begin{aligned} & \max_{\lambda, \mu} \sqrt{\lambda} \|\vec{x}_0\|_A^* + \sqrt{\mu} \|\vec{y}_0\|_B^* \\ & \text{s.t. } \alpha \lambda + \mu = 1 \end{aligned}$$

$$\text{令 } \lambda = \frac{\sin^2 \theta}{\alpha}, \quad \mu = \cos^2 \theta. \text{ 则得其最大值为 } \sqrt{\frac{1}{\alpha} \|\vec{x}_0\|_A^{*2} + \|\vec{y}_0\|_B^{*2}}$$

$$\text{故: } \|\begin{bmatrix} \vec{x}_0 \\ \vec{y}_0 \end{bmatrix}\|^* = \sqrt{\frac{1}{\alpha} \|\vec{x}_0\|_A^{*2} + \|\vec{y}_0\|_B^{*2}}$$

$$3. \lambda_{\max} \left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 9 \end{bmatrix} \right) = 14.1216, \quad \lambda_{\min} \left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 9 \end{bmatrix} \right) = 0.4749$$

$$\text{故 } \text{cond} \left(\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 9 \end{bmatrix} \right) = \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right| = 31.55$$

$$4. \langle y, AX \rangle = y(X_{11} + X_{12} - X_{31} + 2X_{33}) \\ = y \cdot \text{tr}(M^T X), \quad M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \text{tr}(yM)^T X$$

$$= \langle yM, X \rangle$$

$$\text{故 } A^*(y) = yM = \begin{bmatrix} y & y & 0 \\ 0 & 0 & 0 \\ -y & 0 & 2y \end{bmatrix}$$

$$5. \text{ 对于 } Z \in \mathbb{R}^{3 \times 2}$$

$$\langle Z, A(X) \rangle = \text{tr}(Z^T A(X))$$

$$= X_{11}Z_{11} + X_{12}Z_{12} + X_{13}Z_{21} + X_{21}Z_{22}$$

$$+ X_{22}Z_{31} + X_{23}Z_{32}$$

$$\text{若令 } B(Z) = \begin{bmatrix} Z_{11} & Z_{12} & Z_{21} \\ Z_{22} & Z_{31} & Z_{32} \end{bmatrix}$$

$$\text{则 } \langle Z, A(X) \rangle = \text{tr}(Z^T A(X))$$

$$= \text{tr}(B(Z)^T \cdot X)$$

$$= \langle B(Z), X \rangle$$

$$\text{故 } A^*(Z) = B(Z) = \begin{bmatrix} Z_{11} & Z_{12} & Z_{21} \\ Z_{22} & Z_{31} & Z_{32} \end{bmatrix}$$