1. (a) 其 feasible region 为: $\mathcal{F}=\{x_1^3=x_2 \mid x_1\geq 0, x_2\geq 0\}\cup\{x_1\leq 0, x_2\geq 0\}$

而 $x_1^3-x_2$ 不为 convex, 故不满足 SCQ.

(b) KKT 条件为:

$$L = x_1^2+x_2^2-\lambda_1 x_2+\lambda_2(x_1^3-x_2)+\lambda_3 x_1^3(x_2-x_1^3)$$

$$\begin{cases} \partial L/\partial x_1 = 0 \\ \partial L/\partial x_2 = 0 \\ \lambda_1,\lambda_2,\lambda_3 \geq 0 \\ \lambda_i g_i = 0 \end{cases}$$

解得 KKT 为: $(0,0,\lambda_1,\lambda_2,\lambda_3)$, $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}_+$

在 $(0,0)$ 处, $C_l=\left\{\begin{bmatrix} d_1\\ d_2\end{bmatrix} \,\middle|\, d_2\geq 0\right\}$, $C_t = C_l\cup\left\{\vec{d} \,\middle|\, \begin{matrix} d_2=0\\ d_1\geq 0\end{matrix}\right\}$

$C_l\neq C_t$, $C_l^*\neq C_t^*$, 故 ACQ, GCQ 不成立.

(c) $$g(\lambda_1,\lambda_2,\lambda_3)=\min_{x}\ x_1^2+x_2^2-\lambda_1x_2+\lambda_2x_1^3-\lambda_2x_2+\lambda_3x_1^3x_2-\lambda_3x_1^{3+3}$$

$$=\min_{\vec{x}}\left\{-\lambda_3x_1^6+(\lambda_3x_2+\lambda_2)x_1^3+x_1^2+x_2^2-\lambda_1x_2-\lambda_2x_2\right\}$$

$$=\min_{\vec{x}}\left(x_2^2+(\lambda_3x_1^3-\lambda_1-\lambda_2)x_2-\lambda_3x_1^6+\lambda_2x_1^3+x_1^2\right)$$

$$=\min_{x_1}\left\{-\frac{1}{4}(\lambda_3x_1^3-\lambda_1-\lambda_2)^2-\lambda_3x_1^6+\lambda_2x_1^3+x_1^2\right\}$$

若 $\lambda_2\neq 0$, $\lambda_3\neq 0$, 则 $g(\lambda)\to-\infty$ when $x_1\to\infty$

$\therefore$ $$g(\lambda)=\begin{cases}-\frac{1}{4}\lambda_1^2, & \lambda_2=\lambda_3=0\\ -\infty, & \text{otherwise.}\end{cases}$$

$\therefore$ Dual problem:

$$\max\ -\frac{1}{4}\lambda_1^2$$
$$\text{s.t. } \lambda_1\geq 0,\ \lambda_2=\lambda_3=0$$

2. $\min \quad (x_1-2)^2+(x_2-3)^2$

$$\text{s.t.} \quad g_1(x)=x_1+x_2\le 0$$
$$g_2(x)=x_1^2-4\le 0$$

(a) g_1, g_2 为 Convex, 且对于 $\begin{bmatrix}-1\\0\end{bmatrix}$, $g_1(\begin{bmatrix}-1\\0\end{bmatrix})=-1<0$, $g_2(\begin{bmatrix}-1\\0\end{bmatrix})=-3<0$

∴满足 SCQ

(b) Lagrangian: $L=(x_1-2)^2+(x_2-3)^2+\lambda_1(x_1+x_2)+\lambda_2(x_1^{\cancel{4}\,2}-4)$

① none is active:

$$\begin{cases}\partial L/\partial x_1=0 & ①\\ \partial L/\partial x_2=0 & ②\\ \lambda_1=\lambda_2=0 & ③\\ x_1+x_2<0,\ x_1^{\cancel{4}\,2}-4<0\end{cases}$$

①~③解得 $x_1=2,\ x_2=3$

不满足 $x_1+x_2<0$

② the first is active:

$$\begin{cases}\partial L/\partial x_1=0\\ \partial L/\partial x_2=0\\ \lambda_1>0,\ \lambda_2=0\\ x_1+x_2=0\\ x_1^2-4<0\end{cases}$$ 解得 $$\begin{cases}x=-\frac{1}{2}\\ y=\frac{1}{2}\\ \lambda_1=5.\end{cases}$$ ，满足.

③ the second one is active:

$$\begin{cases}\partial L/\partial x_1=0 & ①\\ \partial L/\partial x_2=0 & ②\\ \lambda_1=0,\ \lambda_2>0\\ x_1+x_2<0\\ x_1^2+4=0 & ④\end{cases} \Longrightarrow$$ ①②④解得 $$\begin{cases}x_1=2\\ x_2=3,\end{cases}$$ 不满足 $x_1+x_2<0$

综上, KKT 为: $(x_1,x_2,\lambda_1,\lambda_2)=(-\frac{1}{2},\frac{1}{2},5,0)$

(c) $g(\lambda_1,\lambda_2)=\min\limits_{x}\ (x_1-2)^2+(x_2-3)^2+\lambda_1(x_1+x_2)+\lambda_2(x_1^2-4)\quad(\lambda_1,\lambda_2\geq 0)$

$$=\min_{x}\ (1+\lambda_2)x_1^2+(\lambda_1-4)x_1+4+x_2^2+(\lambda_1-6)x_2+9-4\lambda_2$$

$$=13-4\lambda_2-\frac{(\lambda_1-4)^2}{4(1+\lambda_2)}-\frac{(\lambda_1-6)^2}{4}$$

∴ dual problem 为:

$$\max_{\lambda_1,\lambda_2}\quad 13-4\lambda_2-\frac{(\lambda_1-4)^2}{4(1+\lambda_2)}-\frac{(\lambda_1-6)^2}{4}$$

$$\text{s.t.}\quad \lambda_1\geq 0,\ \lambda_2\geq 0.$$

Convex Opt

3、

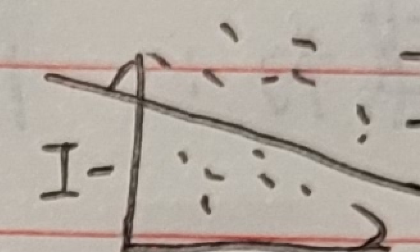
$$\min_{w,\beta} \quad \frac{1}{2}\|w\|^2$$
$$\text{s.t.} \quad y_i(\langle w, x_i\rangle+\beta)\geq 1$$

(a) SCQ：$\exists\, w,\beta$ s.t. $(i=1,\cdots,m)$
$$y_i(w^T x_i+\beta)>1$$

SVM 中，$y_i\in\{1,-1\}$，则意味着

$w^T x_i+\beta>1,\ i\in I_+$ ； $w^T x_i+\beta<-1,\ i\in I_-$

即：$\exists\ f(\vec{x})=\check{w}^T x+\check{\beta}$, s.t. $\begin{cases} f(x_i)>0, & i\in I_+ \\ f(x_i)<0, & i\in I_- \end{cases}$

即 样本点可被一个 hyperplane 严格分开，

(b) dual problem：

即：$\max_{\lambda_i}\left\{\min_{w,\beta} \frac{1}{2}\|w\|^2-\sum\lambda_i\left(y_i(w^T x_i+\beta)-1\right)\right\}$
s.t. $\lambda_i\geq 0,\ i=1,\cdots,m$

$$\min_{w,\beta} \frac{1}{2}\|w\|^2 - w^T\cdot\sum\lambda_i y_i\vec{x}_i - \sum\beta\lambda_i y_i + \sum\lambda_i$$

$$= \min_{\beta} -\frac{1}{2}\left\|\sum\lambda_i y_i\vec{x}_i\right\|_2^2 - \sum\beta\lambda_i y_i + \sum\lambda_i$$

则其存在不为 $-\infty$ 的极小值 $\Leftrightarrow \sum\lambda_i y_i=0$

∴ dual problem 为：

$$\max_{\vec{\lambda}} \quad -\frac{1}{2}\left\|\sum\lambda_i y_i\vec{x}_i\right\|_2^2 + \sum\lambda_i$$
$$\text{s.t.} \quad \begin{cases} \lambda_i\geq 0, & i=1,\cdots,m \\ \sum_{i=1}^{m}\lambda_i y_i=0 \end{cases}$$

4、(a) $g(\lambda) = \min\limits_{x} \; x^2+1+\lambda(x-2)(x-4)$, $(\lambda \geq 0)$

$= \min\limits_{x} \; (1+\lambda)x^2 - 6\lambda x + 8\lambda + 1$

$= 8\lambda+1 + \dfrac{(6\lambda)^2}{4(1+\lambda)} = 8\lambda+1+\dfrac{9\lambda^2}{1+\lambda}$

$\therefore$ Dual problem:

$\max\limits_{\lambda} \; 8\lambda+1+\dfrac{9\lambda^2}{1+\lambda}$

s.t. $\lambda \geq 0$

(b) $g(\lambda) = \min\limits_{x} c^T x + f(x)\cdot\lambda$, $\lambda \geq 0$

$= -\max\limits_{x} \lambda \cdot \left((-\frac{1}{\lambda}c)^T x - f(\vec{x}) \right)$

$= \begin{cases} -\infty, & -\frac{1}{\lambda}c \notin \operatorname{dom} f^* \\ -\lambda f^*(-\frac{c}{\lambda}), & -\frac{1}{\lambda} \in \operatorname{dom} f^* \end{cases}$

$\therefore$ 其为： $\max \; -\lambda f^*(-\frac{c}{\lambda})$

s.t. $\lambda \geq 0, -\frac{1}{\lambda}c \in \operatorname{dom} f^*$

(c) $\min\limits_{x,y} \; \frac{1}{n}\sum \phi_i(y_i) + \frac{\mu}{2}\|x\|^2$

s.t. $y_i = a_i^T x_i$

$g(\lambda) = \min\limits_{x,y} \; \frac{1}{n}\sum \phi_i(y_i) + \frac{\mu}{2}\|x\|^2 + \sum \lambda_i (y_i - a_i^T x)$

$= \min\limits_{x,y} \; \sum \frac{\phi_i(y_i)}{n} + \lambda_i y_i + \frac{\mu}{2}\|x\|^2 + (-\sum \lambda_i a_i)^T x$

$= \begin{cases} -\infty, & \text{otherwise} \\ -\frac{1}{n}\sum \phi_i^*(-n\lambda_i) - \frac{2}{\mu}\|\sum \lambda_i a_i\|_*^2, & -n\lambda_i \in \operatorname{dom} \phi_i^*, \forall i \end{cases}$

$\therefore$ dual problem:

$\max\limits_{\lambda} \; -\frac{1}{n}\sum \phi_i^*(-n\lambda_i) - \frac{2}{\mu}\|\sum \lambda_i a_i\|_*^2$

s.t. $-n\lambda_i \in \operatorname{dom} \phi_i^*, \forall i.$

5. KKT 为：$\begin{cases} x_1^2+x_2^2+x_3^2=1 \\ (-3+\lambda)x_1+1=0 \\ (1+\lambda)x_2+1=0 \\ (2+\lambda)x_3+1=0 \end{cases} \Rightarrow \frac{1}{(-3+\lambda)^2}+\frac{1}{(1+\lambda)^2}+\frac{1}{(2+\lambda)^2}=1$

借助工具(python)解得：KKT 点有：

$$\begin{cases} \vec{x}=(0.16,0.47,-0.87),\ \vec{\lambda}=-3.15 \\ \vec{x}=(0.36,-0.82,0.45),\ \vec{\lambda}=0.22 \\ \vec{x}=(0.90,-0.35,0.26),\ \vec{\lambda}=1.89 \\ \vec{x}=(-0.97,-0.20,0.17),\ \vec{\lambda}=4.04 \quad (*) \end{cases}$$

一一验证，得 $(*)$ 为最大值对应的 KKT point.

而 dual problem 为：

$$\max_{\lambda} -\frac{1}{\lambda-3}-\frac{1}{\lambda+1}-\frac{1}{\lambda+2}-\lambda \quad , \ s.t.\ \lambda>3$$

解得 $\lambda^*=4.04$. 而 $x^*=\arg\min_x f(x)+\sum\lambda_i^* f_i(x)$，解得 $x^*=\begin{bmatrix}-0.97\\-0.20\\0.17\end{bmatrix}$

发现与原问题极值相同，即 $p^*=d^*$，有 strong duality.