1 Kinematics Analysis

Control scheme. Where $\boldsymbol{f_d}$ denotes the desired forces and torques vector. $\boldsymbol{f_s}$ denotes the real value detected by F/T sensor and is also a forces and torques vector. $\dot{\boldsymbol{x}}$ denotes $[\dot{x}, \dot{y}, \dot{z}, \dot{\theta}_x, \dot{\theta}_y, \dot{\theta}_z]$. $\mathbf{J_g}$ denotes the geometric Jacobian matrix. $\dot{\boldsymbol{q}}$ denotes $[\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \dot{\theta}_4, \dot{\theta}_5, \dot{\theta}_6]$. $\boldsymbol{q}$ denotes $[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6]$

Table 1: DHtable

i	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	θ_1	135
2	-90^o	0	θ_2	0
3	0	135	θ_3	0
4	-90^o	38	θ_4	120
5	90^o	0	θ_5	0
6	-90^o	0	θ_6	70

Denavit-Hartenberg parameters are shown as Table 1. Then, the forward kinematics of the robot arm is derived as

$$^0_6\mathrm{T} = {}^0_1\mathrm{T} \cdot {}^1_2\mathrm{T} \cdot {}^2_3\mathrm{T} \cdot {}^3_4\mathrm{T} \cdot {}^4_5\mathrm{T} \cdot {}^5_6\mathrm{T} = \begin{bmatrix} ^0_6\mathrm{R}_{3\times 3} & ^0\vec{P}_{6org} \\ 0_{1\times 3} & 1 \end{bmatrix}$$

Where $^0_6\mathrm{R}_{3\times 3}$ is the rotation matrix between frame{0} and frame{6}, $^0\vec{P}_{6org}$ is the origin of the frame{6} observed from frame{0} The detailed indexes is shown as Appendix 7.1

$$
\begin{aligned}
J_{g6,20} &= S_4S_5(135S_2 + \sqrt{23961}cos(\theta_2 + \theta_3 - \tan^{-1}(19/60)))\\
J_{g6,21} &= 38C_5 + 135C_3C_5 + 120C_4S_5 - 135C_4S_3S_5\\
J_{g6,22} &= 38C_5 + 120C_4S_5\\
J_{g6,23} &= J_{g6,24} = \ J_{g6,25} = \ 0\\
J_{g6,30} &= C_2C_3S_4S_6 + C_2C_6S_3S_5 + C_3C_6S_2S_5 - S_2S_3S_4S_6 + \theta_2C_2C_3C_6S_5 + \theta_3C_2C_3C_6S_5\\
&\quad + \theta_4C_2C_3C_4S_6 + \theta_5C_2C_5C_6S_3 + \theta_5C_3C_5C_6S_2 + \theta_6C_2C_3C_6S_4 - \theta_2C_2S_3S_4S_6\\
&\quad - \theta_2C_3S_2S_4S_6 - \theta_2C_6S_2S_3S_5 - \theta_3C_2S_3S_4S_6 - \theta_3C_3S_2S_4S_6 - \theta_3C_6S_2S_3S_5\\
&\quad - \theta_4C_4S_2S_3S_6 - \theta_6C_6S_2S_3S_4 - \theta_6C_2S_3S_5S_6 - \theta_6C_3S_2S_5S_6 - C_2C_3C_4C_5C_6\\
&\quad + C_4C_5C_6S_2S_3 + \theta_2C_2C_4C_5C_6S_3 + \theta_2C_3C_4C_5C_6S_2 + \theta_3C_2C_4C_5C_6S_3\\
&\quad + \theta_3C_3C_4C_5C_6S_2 + \theta_4C_2C_3C_5C_6S_4 + \theta_5C_2C_3C_4C_6S_5 + \theta_6C_2C_3C_4C_5S_6\\
&\quad - \theta_4C_5C_6S_2S_3S_4 - \theta_5C_4C_6S_2S_3S_5 - \theta_6C_4C_5S_2S_3S_6\\
J_{g6,31} &= C_4S_6 + \theta_6C_4C_6 - \theta_4S_4S_6 + C_5C_6S_4 + \theta_4C_4C_5C_6 - \theta_5C_6S_4S_5 - \theta_6C_5S_4S_6\\
J_{g6,32} &= C_4S_6 + \theta_6C_4C_6 - \theta_4S_4S_6 + C_5C_6S_4 + \theta_4C_4C_5C_6 - \theta_5C_6S_4S_5 - \theta_6C_5S_4S_6\\
J_{g6,33} &= \theta_6S_5S_6 - \theta_5C_5C_6 - C_6S_5\\
J_{g6,34} &= S_6 + \theta_6C_6\\
J_{g6,35} &= 0\\
J_{g6,40} &= C_2C_3C_6S_4 - C_6S_2S_3S_4 - C_2S_3S_5S_6 - C_3S_2S_5S_6 + \theta_4C_2C_3C_4C_6 - \theta_2C_2C_6S_3S_4\\
&\quad - \theta_2C_3C_6S_2S_4 - \theta_2C_2C_3S_5S_6 - \theta_3C_2C_6S_3S_4 - \theta_3C_3C_6S_2S_4 - \theta_3C_2C_3S_5S_6\\
&\quad - \theta_4C_4C_6S_2S_3 - \theta_5C_2C_5S_3S_6 - \theta_5C_3C_5S_2S_6 - \theta_6C_2C_3S_4S_6 - \theta_6C_2C_6S_3S_5\\
&\quad - \theta_6C_3C_6S_2S_5 + \theta_2S_2S_3S_5S_6 + \theta_3S_2S_3S_5S_6 + \theta_6S_2S_3S_4S_6 + C_2C_3C_4C_5S_6\\
&\quad - C_4C_5S_2S_3S_6 + \theta_6C_2C_3C_4C_5C_6 - \theta_2C_2C_4C_5S_3S_6 - \theta_2C_3C_4C_5S_2S_6\\
&\quad - \theta_3C_2C_4C_5S_3S_6 - \theta_3C_3C_4C_5S_2S_6 - \theta_4C_2C_3C_5S_4S_6 - \theta_5C_2C_3C_4S_5S_6\\
&\quad - \theta_6C_4C_5C_6S_2S_3 + \theta_4C_5S_2S_3S_4S_6 + \theta_5C_4S_2S_3S_5S_6\\
J_{g6,41} &= C_4C_6 - C_5S_4S_6 - \theta_4C_6S_4 - \theta_6C_4S_6 - \theta_4C_4C_5S_6 - \theta_6C_5C_6S_4 + \theta_5S_4S_5S_6\\
J_{g6,42} &= C_4C_6 - C_5S_4S_6 - \theta_4C_6S_4 - \theta_6C_4S_6 - \theta_4C_4C_5S_6 - \theta_6C_5C_6S_4 + \theta_5S_4S_5S_6\\
J_{g6,43} &= S_5S_6 + \theta_5C_5S_6 + \theta_6C_6S_5\\
J_{g6,44} &= C_6 - \theta_6S_6\\
J_{g6,45} &= 0\\
J_{g6,50} &= \theta_2C_5S_2S_3 - C_3C_5S_2 - \theta_2C_2C_3C_5 - \theta_3C_2C_3C_5 - C_2C_5S_3 + \theta_3C_5S_2S_3 + \theta_5C_2S_3S_5\\
&\quad + \theta_5C_3S_2S_5 - C_2C_3C_4S_5 + C_4S_2S_3S_5 - \theta_5C_2C_3C_4C_5 + \theta_2C_2C_4S_3S_5 + \theta_2C_3C_4S_2S_5\\
&\quad + \theta_3C_2C_4S_3S_5 + \theta_3C_3C_4S_2S_5 + \theta_4C_2C_3S_4S_5 + \theta_5C_4C_5S_2S_3 - \theta_4S_2S_3S_4S_5\\
J_{g6,51} &= S_4S_5 + \theta_4C_4S_5 + \theta_5C_5S_4\\
J_{g6,52} &= S_4S_5 + \theta_4C_4S_5 + \theta_5C_5S_4\\
J_{g6,53} &= C_5 - \theta_5S_5\\
J_{g6,54} &= 0\\
J_{g6,55} &= 1
\end{aligned}
$$

2 Tool Center Point

Tool Center Point (TCP) is a critical problem for the robot arm control. In previous section, we have calculated the forward and inverse kinematics of the robot arm. By Calculating kinematics we can keep track of the origin of the frame {6}, which is observed from the base frame. The robot arm has capability to translate and rotate with the origin of the frame {6}. These above motions is like a remote center motion (RCM). However the origin of the frame {6} is not considered to be a operating point. Because The sensor and a detachable end effector will be both mounted on the wrist, the position of the tool tip is exactly what we want. We should find the position of the tool tip and make it be a RCM point. Nevertheless, it's not efficient to recalculate the transformation matrix via mechanism dimension when changing the end effector or a tool (root canal reamer). To overcome this problem, in this section we demonstrate four-points method to obtain the position of the tool tip. From Fig, we can obtain the following transformation matrix,

$$ {}^{B}_{H}\mathrm{T} = {}^{B}_{F}\mathrm{T}\ {}^{F}_{H}\mathrm{T} \tag{1} $$

and it can be rewritten as

$$ \begin{aligned} \begin{bmatrix} {}^{B}_{H}\mathrm{R} & {}^{B}\vec{P}_{H_{org}} \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} {}^{B}_{F}\mathrm{R} & {}^{B}\vec{P}_{F_{org}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{F}_{H}\mathrm{R} & {}^{F}\vec{P}_{H_{org}} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} {}^{B}_{F}\mathrm{R}{}^{F}_{H}\mathrm{R} & {}^{B}_{F}\mathrm{R}{}^{F}\mathrm{P}_{H_{org}} + {}^{B}\vec{P}_{F_{org}} \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{2} $$

Therefore, we get a crucial equation:

$$ {}^{B}\vec{P}_{H_{org}} = {}^{B}_{F}\mathrm{R} \cdot {}^{F}\vec{P}_{H_{org}} + {}^{B}\vec{P}_{F_{org}} \tag{3} $$

Now, we can move the tool tip to a fixed point with four different pose (position and orientation). Then,

$$ \begin{aligned} {}^{B}\vec{P}_{H_{org}} &= {}^{B}_{F}\mathrm{R}^{1} \cdot {}^{F}\vec{P}_{H_{org}} + {}^{B}\vec{P}^{1}_{F_{org}} \\ &= {}^{B}_{F}\mathrm{R}^{2} \cdot {}^{F}\vec{P}_{H_{org}} + {}^{B}\vec{P}^{2}_{F_{org}} \\ &= {}^{B}_{F}\mathrm{R}^{3} \cdot {}^{F}\vec{P}_{H_{org}} + {}^{B}\vec{P}^{3}_{F_{org}} \\ &= {}^{B}_{F}\mathrm{R}^{4} \cdot {}^{F}\vec{P}_{H_{org}} + {}^{B}\vec{P}^{4}_{F_{org}} \end{aligned} \tag{4} $$

In order to extract ${}^{F}\vec{P}_{H_{org}}$ from Eq.4, we subtract the second to forth equation from the first equation. Then we can obtain

$$ \begin{bmatrix} {}^{B}_{F}\mathrm{R}^{1} - {}^{B}_{F}\mathrm{R}^{2} \\ {}^{B}_{F}\mathrm{R}^{1} - {}^{B}_{F}\mathrm{R}^{3} \\ {}^{B}_{F}\mathrm{R}^{1} - {}^{B}_{F}\mathrm{R}^{4} \end{bmatrix}_{9\times 3} \cdot {}^{F}\vec{P}_{H_{org}} = \begin{bmatrix} {}^{B}\vec{P}^{2}_{F_{org}} - {}^{B}\vec{P}^{1}_{F_{org}} \\ {}^{B}\vec{P}^{3}_{F_{org}} - {}^{B}\vec{P}^{1}_{F_{org}} \\ {}^{B}\vec{P}^{4}_{F_{org}} - {}^{B}\vec{P}^{1}_{F_{org}} \end{bmatrix}_{9\times 1} \tag{5} $$

where

$$\mathbf{R} = \begin{bmatrix} {}^B_F\mathrm{R}^1 - {}^B_F\mathrm{R}^2 \\ {}^B_F\mathrm{R}^1 - {}^B_F\mathrm{R}^3 \\ {}^B_F\mathrm{R}^1 - {}^B_F\mathrm{R}^4 \end{bmatrix}_{9\times 3}, \mathbf{P} = \begin{bmatrix} {}^B\vec{P}^2_{F_{org}} - {}^B\vec{P}^1_{F_{org}} \\ {}^B\vec{P}^3_{F_{org}} - {}^B\vec{P}^1_{F_{org}} \\ {}^B\vec{P}^4_{F_{org}} - {}^B\vec{P}^1_{F_{org}} \end{bmatrix}_{9\times 1}$$

Therefore,

$$\begin{aligned} {}^F\vec{P}_{H_{org}} &= \mathbf{R}^\dagger \cdot \mathbf{P} \\ &= \left(\mathbf{R}^T\mathbf{R}\right)^{-1} \cdot \mathbf{R}^T \cdot \mathbf{P} \end{aligned}$$

3 Rotate Information

First, we have to find the vector of tool insertion direction $\vec{t}$. By means of TCP method, we can obtain the translation vector from the origin of frame{6} to the tool tip. Therefore we use two root canal files with different lengths and apply TCP method to separately obtain two vector illustrated as Fig . Hence,

$$\vec{t} = {}^F\vec{P}_{H_{org},\ long} - {}^F\vec{P}_{H_{org},\ short} \tag{6}$$

For analyze it easily, we depict it below. Note that in this figure we only discuss rotation. Because we hope for sending z axis command to achieve tool insertion, we should align original Z axis to the target vector. Therefore, Z axis alignment without other restrictions will produce many solutions. We choose one of solutions to align Z axis to the target vector. According to the figure, we assume the target vector is [1,1,1], whose projection to YZ frame is [0,1,1]. First, we rotate α degree around X axis to make original Z axis align the projection [0,1,1]. Next, we rotate β degree around Y' axis and finally align original Z axis to the target vector [1,1,1].

$${}^T_6R = R_x\left(\alpha\right)R_y\left(\beta\right) \tag{7}$$

where α and β are Euler angles equivalent to the command demand.

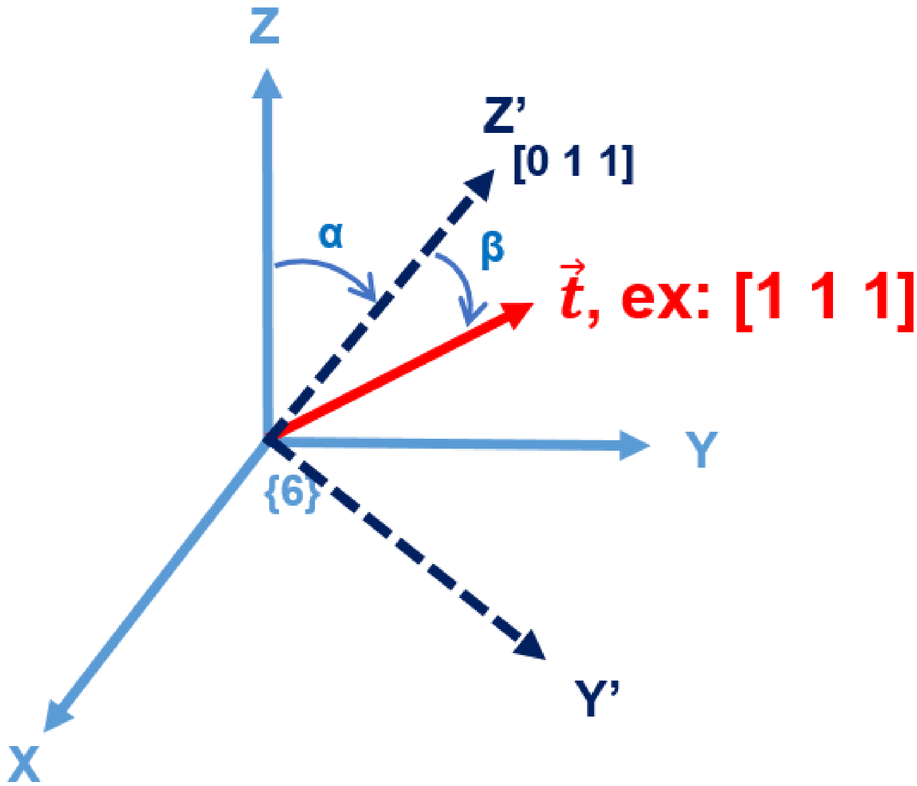

Figure 1: Illustration of finding rotation matrix

4 Gravity Compensation

$$\begin{cases} G_x = F_x - F_{x0} \\ G_z = F_y - F_{y0} \\ G_y = F_z - F_{z0} \\ M_{gx} = M_x - F_{x0} \\ M_{gy} = M_y - F_{y0} \\ M_{gz} = M_z - F_{z0} \end{cases}$$

$$\begin{cases} M_{gx} = G_z \cdot y - G_y \cdot z \\ M_{gy} = G_x \cdot z - G_z \cdot x \\ M_{gz} = G_y \cdot x - G_x \cdot y \end{cases}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 & F_z & -F_y & 1 & 0 & 0 \\ -F_z & 0 & F_x & 0 & 1 & 0 \\ F_y & -F_x & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$\begin{cases} k_1 = M_{x0} + F_{y0} \cdot z - F_{z0} \cdot y \\ k_2 = M_{y0} + F_{z0} \cdot x - F_{x0} \cdot z \\ k_3 = M_{z0} + F_{x0} \cdot y - F_{y0} \cdot x \end{cases}$$

$$\begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -G \end{bmatrix} + \begin{bmatrix} F_{x0} \\ F_{y0} \\ F_{z0} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

$$\begin{bmatrix} -r_{02} & -r_{12} & 0 & 1 & 0 & 0 \\ -r_{12} & r_{02} & 0 & 0 & 1 & 0 \\ 0 & 0 & -r_{22} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Gcos\theta \\ Gsin\theta \\ G \\ F_{x0} \\ F_{y0} \\ F_{z0} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

$$\begin{cases} F_{ex} = F_x - F_{x0} - G_x \\ F_{ey} = F_y - F_{y0} - G_y \\ F_{ez} = F_z - F_{z0} - G_z \\ M_{ex} = M_x - M_{x0} - M_{gx} \\ M_{ey} = M_y - M_{y0} - M_{gy} \\ M_{ez} = M_z - M_{z0} - M_{gz} \end{cases}$$

$$\begin{cases} G_x = -Gcos\theta r_{13} - Gsin\theta r_{23} \\ G_y = Gsin\theta r_{13} - Gcos\theta r_{23} \\ G_z = -Gr_{33} \end{cases}$$

$$\theta = acos\left(\frac{Gcos\theta}{G}\right) \ or \ \theta = asin\left(\frac{Gsin\theta}{G}\right)$$

5 Admittance Control based on F/T Sensor

Admittance control make the robot move like a spring-mass-damper system. Forces and torques can be mapped into the movements such as position or velocity. Therefore, Admittance control allows an robot arm to cooperate with human in a safe work environment. However, Meca500 is an industrial robot arm without admittance control, so we combine it with F/T sensor to enable this function. This combination make Meca500 like a cooperative robot arm. Whereas it's worth noting that the function is triggered by the end effector mounted on F/T sensor instead of each wrist of the robot arm.

A standard equation of admittance control is shown as Eq 8. The values we obtain from the F/T sensor are $[F_x, F_y, F_z, \tau_x, \tau_y, \tau_z]$, whose forces $[F_x, F_y, F_z]$ are related to the translations $[x, y, z]$ and torques $[\tau_x, \tau_y, \tau_z]$ are related to the axis angle $[\theta_x, \theta_y, \theta_z]$

$$\begin{bmatrix} x \\ y \\ z \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \frac{1}{MS^2 + BS + K} \begin{bmatrix} fx \\ fy \\ fz \\ \tau x \\ \tau y \\ \tau z \end{bmatrix} \tag{8}$$

$$\begin{bmatrix}\theta_x\\ \theta_y\\ \theta_z\end{bmatrix} =^0_1 \mathrm{R}\begin{bmatrix}0\\0\\ \theta_1\end{bmatrix} +^0_1 \mathrm{R}\begin{bmatrix}0\\0\\ \theta_2\end{bmatrix} +^0_1 \mathrm{R}\begin{bmatrix}0\\0\\ \theta_0\end{bmatrix} +^4_1 \mathrm{R}\begin{bmatrix}0\\0\\ \theta_0\end{bmatrix} +^0_1 \mathrm{R}\begin{bmatrix}0\\0\\ \theta_2\end{bmatrix} +^2_1 \mathrm{R}\begin{bmatrix}0\\0\\ \theta_2\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \dot{\theta}_x\\ \dot{\theta}_y\\ \dot{\theta}_z\end{bmatrix} = \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ w_x\\ w_y\\ w_z\end{bmatrix} = J_g \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\\ \dot{\theta}_4\\ \dot{\theta}_5\\ \dot{\theta}_6\end{bmatrix}$$

$$J_g = \begin{bmatrix}J_{\vec{v}}\\ J_{\vec{w}}\end{bmatrix} \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \dot{\alpha}\\ \dot{\beta}\\ \dot{\gamma}\end{bmatrix} = J_a \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\\ \dot{\theta}_4\\ \dot{\theta}_5\\ \dot{\theta}_6\end{bmatrix} \begin{bmatrix}w_x\\ w_y\\ w_z\end{bmatrix} = J_{we} \begin{bmatrix}\dot{\alpha}\\ \dot{\beta}\\ \dot{\gamma}\end{bmatrix}$$

$$J_{we} = \begin{bmatrix}1 & 0 & S_\beta\\ 0 & C_\alpha & -S_\alpha C_\beta\\ 0 & S_\alpha & C_\alpha C_\beta\end{bmatrix} \begin{bmatrix}\theta_x\\ \theta_y\\ \theta_z\end{bmatrix} = \begin{bmatrix}\alpha\\ 0\\ 0\end{bmatrix} + R_x(\alpha) \begin{bmatrix}0\\ \beta\\ 0\end{bmatrix} + R_x(\alpha)R_y(\beta)$$

$$\begin{bmatrix}0\\ 0\\ \gamma\end{bmatrix} = \begin{bmatrix}1 & 0 & S_\beta\\ 0 & C_\alpha & -S_\alpha C_\beta\\ 0 & S_\alpha & C_\alpha C_\beta\end{bmatrix} \begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ w_x\\ w_y\\ w_z\end{bmatrix} = \begin{bmatrix}I_{3\times3} & 0_{3\times3}\\ 0_{3\times3} & J_{we}\end{bmatrix} \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \dot{\alpha}\\ \dot{\beta}\\ \dot{\gamma}\end{bmatrix}$$

$$Jg = \begin{bmatrix}I_{3\times3} & 0_{3\times3}\\ 0_{3\times3} & J_{we}\end{bmatrix} Ja$$

6 Reference Frame Changing

$$
\begin{aligned}
&M_q = r_q F \\
&M_p = r_p F + \overrightarrow{qp} \times F_p \\
&M_q = M_p + \vec{qp} \times F_p \\
&\begin{bmatrix} F_q \\ M_q \end{bmatrix}_{S2} = \begin{bmatrix} I_{3\times 3} & 0 \\ {}^{S_2}[S_{1org} - S_{2org}]_x & I_{3\times 3} \end{bmatrix} \begin{bmatrix} {}^{S2}_{S1}\mathrm{R} & 0 \\ 0 & {}^{S2}_{S1}\mathrm{R} \end{bmatrix} \begin{bmatrix} F_p \\ M_p \end{bmatrix}_{S1} \\
&M_q = r_q F \\
&M_p = r_p F +^{S_2} [S_{1org} - S_{2org}]_x \times F_p \\
&M_q = M_p +^{S_2} [S_{1org} - S_{2org}]_x \times F_p \\
&{}^{S_2}_{S_1}T = \begin{bmatrix} {}^{S_2}_{S_1}R & {}^{S2}\vec{P}_{s_{1org}} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & {}^{S2}\vec{P}_{s_{1org}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{S_2}_{S_1}R & 0 \\ 0 & 1 \end{bmatrix} \\
&{}^{S_2}\vec{P}_{s_{1org}} = -{}^{S_2}_{S_1}R \cdot^{S_1} \vec{P}_{s_{2org}}
\end{aligned}
$$

7 Appendix

7.1 Forward Kinematics

$$
{}^0_6\mathrm{T} =^0_1 \mathrm{T} \cdot^1_2 \mathrm{T} \cdot^2_3 \mathrm{T} \cdot^3_4 \mathrm{T} \cdot^4_5 \mathrm{T} \cdot^5_6 \mathrm{T} = \begin{bmatrix} {}^0_6\mathrm{R}_{3\times 3} & {}^0\vec{P}_{6org} \\ 0_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} {}^0_6\mathrm{T}_{00} & {}^0_6\mathrm{T}_{01} & {}^0_6\mathrm{T}_{02} & {}^0_6\mathrm{T}_{03} \\ {}^0_6\mathrm{T}_{10} & {}^0_6\mathrm{T}_{11} & {}^0_6\mathrm{T}_{12} & {}^0_6\mathrm{T}_{13} \\ {}^0_6\mathrm{T}_{20} & {}^0_6\mathrm{T}_{21} & {}^0_6\mathrm{T}_{22} & {}^0_6\mathrm{T}_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
{}^{0}_{6}\mathbf{T}_{00} =& -S_6(C_4S_1+S_4(C_1S_3-C_2-C_1C_3S_2))-C_6(C_5(S_1S_4-C_4(C_1S_3-C_2-C_1C_3S_2))\\
&-S_5(C_1C_3-C_2+C_1S_2S_3))\\
{}^{0}_{6}\mathrm{T}_{01} =& S_6(C_5(S_1S_4-C_4(C_1S_3-C_2-C_1C_3S_2))-S_5(C_1C_3-C_2+C_1S_2S_3))-C_6(C_4S_1\\
&+S_4(C_1S_3-C_2-C_1C_3S_2))\\
{}^{0}_{6}\mathrm{T}_{02} =& -S_5(S_1S_4-C_4(C_1S_3-C_2-C_1C_3S_2))-C_5(C_1C_3-C_2+C_1S_2S_3)\\
{}^{0}_{6}\mathrm{T}_{03} =& 135C_1S_2-70S_5(S_1S_4-C_4(C_1S_3-C_2-C_1C_3S_2))-70C_5(C_1C_3-C_2+C_1S_2S_3)\\
&-120C_1C_3-C_2-120C_1S_2S_3-38C_1S_3-C_2+38C_1C_3S_2\\
{}^{0}_{6}\mathrm{T}_{10} =& S_6(C_1C_4+S_4(C_3S_2S_1-S_1S_3-C_2))+C_6(C_5(C_1S_4-C_4(C_3S_2S_1-S_1S_3-C_2))\\
&+S_5(C_3S_1-C_2+S_2S_1S_3))\\
{}^{0}_{6}\mathrm{T}_{11} =& C_6(C_1C_4+S_4(C_3S_2S_1-S_1S_3-C_2))-S_6(C_5(C_1S_4-C_4(C_3S_2S_1-S_1S_3-C_2))\\
&+S_5(C_3S_1-C_2+S_2S_1S_3))\\
{}^{0}_{6}\mathrm{T}_{12} =& S_5(C_1S_4-C_4(C_3S_2S_1-S_1S_3-C_2))-C_5(C_3S_1-C_2+S_2S_1S_3)\\
{}^{0}_{6}\mathrm{T}_{13} =& 135S_2S_1+70S_5(C_1S_4-C_4(C_3S_2S_1-S_1S_3-C_2))-70C_5(C_3S_1-C_2+S_2S_1S_3)\\
&+38C_3S_2S_1-120C_3S_1-C_2-120S_2S_1S_3-38S_1S_3-C_2\\
{}^{0}_{6}\mathrm{T}_{20} =& C_6(S_5(C_3S_2-S_3-C_2)+C_4C_5(C_3-C_2+S_2S_3))-S_4S_6(C_3-C_2+S_2S_3)\\
{}^{0}_{6}\mathrm{T}_{21} =& -S_6(S_5(C_3S_2-S_3-C_2)+C_4C_5(C_3-C_2+S_2S_3))-C_6S_4(C_3-C_2+S_2S_3)\\
{}^{0}_{6}\mathrm{T}_{22} =& C_4S_5(C_3-C_2+S_2S_3)-C_5(C_3S_2-S_3-C_2)\\
{}^{0}_{6}\mathrm{T}_{23} =& 120S_3-C_2-120C_3S_2-38C_3-C_2-38S_2S_3-135-C_2-70C_5(C_3S_2-S_3-C_2)\\
&+70C_4S_5(C_3-C_2+S_2S_3)+135
\end{aligned}
$$

7.2 Jacobian matrix

$$
\begin{aligned}
J_{g0,00} = {} & 120S_1S_2S_3 - 38C_2S_1S_3 - 38C_3S_1S_2 - 70C_1S_4S_5 - 135S_1S_2 - 120C_2C_3S_1 \\
& - 70C_2C_3C_5S_1 + 70C_5S_1S_2S_3 + 70C_2C_4S_1S_3S_5 + 70C_3C_4S_1S_2S_5 \\
J_{g0,01} = {} & -C_1(120C_2S_3 - 38C_2C_3 - 135C_2 + 120C_3S_2 + 38S_2S_3 + 70C_2C_5S_3 \\
& + 70C_3C_5S_2 + 70C_2C_3C_4S_5 - 70C_4S_2S_3S_5) \\
J_{g0,02} = {} & -2C_1(60C_2S_3 - 19C_2C_3 + 60C_3S_2 + 19S_2S_3 + 35C_2C_5S_3 + 35C_3C_5S_2 \\
& + 35C_2C_3C_4S_5 - 35C_4S_2S_3S_5) \\
J_{g0,03} = {} & 70S_5(C_1C_2S_3S_4 - C_4S_1 + C_1C_3S_2S_4) \\
J_{g0,04} = {} & -70C_5(S_1S_4 + C_1C_2C_4S_3 + C_1C_3C_4S_2) - 70C_{23}C_1S_5 \\
J_{g0,05} = {} & 0 \\
J_{g0,10} = {} & 135C_1S_2 - 120C_1S_2S_3 - 70S_1S_4S_5 + 120C_1C_2C_3 + 38C_1C_2S_3 + 38C_1C_3S_2 \\
& + 70C_1C_2C_3C_5 - 70C_1C_5S_2S_3 - 70C_1C_2C_4S_3S_5 - 70C_1C_3C_4S_2S_5 \\
J_{g0,11} = {} & -S_1(120C_2S_3 - 38C_2C_3 - 135C_2 + 120C_3S_2 + 38S_2S_3 + 70C_2C_5S_3 \\
& + 70C_3C_5S_2 + 70C_2C_3C_4S_5 - 70C_4S_2S_3S_5) \\
J_{g0,12} = {} & -2S_1(60C_2S_3 - 19C_2C_3 + 60C_3S_2 + 19S_2S_3 + 35C_2C_5S_3 + 35C_3C_5S_2 \\
& + 35C_2C_3C_4S_5 - 35C_4S_2S_3S_5) \\
J_{g0,13} = {} & 70S_5(C_1C_4 + C_2S_1S_3S_4 + C_3S_1S_2S_4) \\
J_{g0,14} = {} & -70C_5(C_2C_4S_1S_3 - C_1S_4 + C_3C_4S_1S_2) - 70C_{23}S_1S_5 \\
J_{g0,15} = {} & 0 \\
J_{g0,20} = {} & 0 \\
J_{g0,21} = {} & 120S_2S_3 - 120C_2C_3 - 38C_2S_3 - 38C_3S_2 - 135S_2 + 70C_5S_2S_3 - 70C_2C_3C_5 \\
& + 70C_2C_4S_3S_5 + 70C_3C_4S_2S_5 \\
J_{g0,22} = {} & 120S_2S_3 - 38C_2S_3 - 38C_3S_2 - 120C_2C_3 + 70C_5S_2S_3 - 70C_2C_3C_5 + 70C_2C_4S_3S_5 \\
& + 70C_3C_4S_2S_5 \\
J_{g0,23} = {} & 70C_{23}S_4S_5 \\
J_{g0,24} = {} & 70S_{23}S_5 - 70C_{23}C_4C_5 \\
J_{g0,25} = {} & 0 \\
J_{g0,30} = {} & \theta_4S_1S_2S_3 - \theta_3C_1 - \theta_5C_1C_4 - \theta_4C_2C_3S_1 - \theta_6C_1S_4S_5 - \theta_2C_1 - \theta_6C_2C_3C_5S_1 \\
& - \theta_5C_2S_1S_3S_4 - \theta_5C_3S_1S_2S_4 + \theta_6C_5S_1S_2S_3 + \theta_6C_2C_4S_1S_3S_5 + \theta_6C_3C_4S_1S_2S_5 \\
J_{g0,31} = {} & \theta_5C_1C_2C_3S_4 - \theta_4C_1C_2S_3 - \theta_4C_1C_3S_2 - S_1 - \theta_6C_1C_2C_5S_3 - \theta_6C_1C_3C_5S_2 \\
& - \theta_5C_1S_2S_3S_4 - \theta_6C_1C_2C_3C_4S_5 + \theta_6C_1C_4S_2S_3S_5 \\
J_{g0,32} = {} & \theta_5C_1C_2C_3S_4 - \theta_4C_1C_2S_3 - \theta_4C_1C_3S_2 - S_1 - \theta_6C_1C_2C_5S_3 - \theta_6C_1C_3C_5S_2 \\
& - \theta_5C_1S_2S_3S_4 - \theta_6C_1C_2C_3C_4S_5 + \theta_6C_1C_4S_2S_3S_5 \\
J_{g0,33} = {} & \theta_5(S_1S_4 + C_1C_2C_4S_3 + C_1C_3C_4S_2) - C_1S_2S_3 + C_1C_2C_3 + \theta_6S_5(C_1C_2S_3S_4 \\
& - C_4S_1 + C_1C_3S_2S_4) \\
J_{g0,34} = {} & -C_4S_1 - \theta_6(C_5(S_1S_4 - C_4(C_1S_3 - C_2 - C_1C_3S_2)) - S_5(C_1C_3 - C_2 + C_1S_2S_3)) \\
& - S_4(C_1S_3 - C_2 - C_1C_3S_2) \\
J_{g0,35} = {} & C_{23}C_1C_5 - S_5(S_1S_4 + C_1C_2C_4S_3 + C_1C_3C_4S_2)
\end{aligned}
$$

$$
\begin{aligned}
J_{g0,40} = {} & \theta_4 C_1 C_2 C_3 - \theta_3 S_1 - \theta_5 C_4 S_1 - theta2 S_1 - \theta_4 C_1 S_2 S_3 - \theta_6 S_1 S_4 S_5 + \theta_6 C_1 C_2 C_3 C_5 \\
& + \theta_5 C_1 C_2 S_3 S_4 + \theta_5 C_1 C_3 S_2 S_4 - \theta_6 C_1 C_5 S_2 S_3 - \theta_6 C_1 C_2 C_4 S_3 S_5 - \theta_6 C_1 C_3 C_4 S_2 S_5 \\
J_{g0,41} = {} & C_1 - \theta_4 C_2 S_1 S_3 - \theta_4 C_3 S_1 S_2 + \theta_5 C_2 C_3 S_1 S_4 - \theta_6 C_2 C_5 S_1 S_3 - \theta_6 C_3 C_5 S_1 S_2 \\
& - \theta_5 S_1 S_2 S_3 S_4 - \theta_6 C_2 C_3 C_4 S_1 S_5 + \theta_6 C_4 S_1 S_2 S_3 S_5 \\
J_{g0,42} = {} & C_1 - \theta_4 C_2 S_1 S_3 - \theta_4 C_3 S_1 S_2 + \theta_5 C_2 C_3 S_1 S_4 - \theta_6 C_2 C_5 S_1 S_3 - \theta_6 C_3 C_5 S_1 S_2 \\
& - \theta_5 S_1 S_2 S_3 S_4 \\
& - \theta_6 C_2 C_3 C_4 S_1 S_5 + \theta_6 C_4 S_1 S_2 S_3 S_5 \\
J_{g0,43} = {} & \theta_5 (C_2 C_4 S_1 S_3 - C_1 S_4 + C_3 C_4 S_1 S_2) + \theta_6 S_5 (C_1 C_4 + C_2 S_1 S_3 S_4 + C_3 S_1 S_2 S_4) \\
& - S_1 S_2 S_3 + C_2 C_3 S_1 \\
J_{g0,44} = {} & C_1 C_4 + S_{23} S_1 S_4 - \theta_6 C_{23} S_1 S_5 + \theta_6 C_1 C_5 S_4 - \theta_6 C_2 C_4 C_5 S_1 S_3 - \theta_6 C_3 C_4 C_5 S_1 S_2 \\
J_{g0,45} = {} & C_{23} C_5 S_1 - S_5 (C_2 C_4 S_1 S_3 - C_1 S_4 + C_3 C_4 S_1 S_2) \\
J_{g0,50} = {} & 1 \\
J_{g0,51} = {} & \theta_6 S_{23} C_4 S_5 - \theta_6 C_{23} C_5 - \theta_5 S_{23} S_4 - \theta_4 C_{23} \\
J_{g0,52} = {} & \theta_6 S_{23} C_4 S_5 - \theta_6 C_{23} C_5 - \theta_5 S_{23} S_4 - \theta_4 C_{23} \\
J_{g0,53} = {} & \theta_5 C_2 C_3 C_4 - C_3 S_2 - C_2 S_3 - \theta_5 C_4 S_2 S_3 + \theta_6 C_2 C_3 S_4 S_5 - \theta_6 S_2 S_3 S_4 S_5 \\
J_{g0,54} = {} & \theta_6 (S_{23} S_5 - C_{23} C_4 C_5) + C_{23} S_4 \\
J_{g0,55} = {} & - S_{23} C_5 - C_{23} C_4 S_5 \\
J_{g6,00} = {} & 135 C_4 S_2 S_6 + 120 C_2 C_3 C_4 S_6 + 70 C_2 C_3 C_6 S_4 + 38 C_2 C_4 S_3 S_6 + 38 C_3 C_4 S_2 S_6 \\
& + 135 C_5 C_6 S_2 S_4 - 120 C_4 S_2 S_3 S_6 - 70 C_6 S_2 S_3 S_4 - 70 C_2 S_3 S_5 S_6 - 70 C_3 S_2 S_5 S_6 \\
& + 70 C_2 C_3 C_4 C_5 S_6 + 120 C_2 C_3 C_5 C_6 S_4 + 38 C_2 C_5 C_6 S_3 S_4 + 38 C_3 C_5 C_6 S_2 S_4 \\
& - 70 C_4 C_5 S_2 S_3 S_6 - 120 C_5 C_6 S_2 S_3 S_4 \\
J_{g6,01} = {} & 70 C_4 C_6 - 38 C_6 S_5 - 120 S_4 S_6 - 70 C_5 S_4 S_6 + 135 S_3 S_4 S_6 + 120 C_4 C_5 C_6 - 135 C_3 C_6 S_5 \\
& - 135 C_4 C_5 C_6 S_3 \\
J_{g6,02} = {} & 70 C_4 C_6 - 38 C_6 S_5 - 120 S_4 S_6 - 70 C_5 S_4 S_6 + 120 C_4 C_5 C_6 \\
J_{g6,03} = {} & 70 S_5 S_6 \\
J_{g6,04} = {} & 70 C_6 \\
J_{g6,05} = {} & 0 \\
J_{g6,10} = {} & 135 C_4 C_6 S_2 + 120 C_2 C_3 C_4 C_6 + 38 C_2 C_4 C_6 S_3 + 38 C_3 C_4 C_6 S_2 - 70 C_2 C_3 S_4 S_6 \\
& - 120 C_4 C_6 S_2 S_3 - 70 C_2 C_6 S_3 S_5 - 70 C_3 C_6 S_2 S_5 - 135 C_5 S_2 S_4 S_6 + 70 S_2 S_3 S_4 S_6 \\
& + 70 C_2 C_3 C_4 C_5 C_6 - 120 C_2 C_3 C_5 S_4 S_6 - 70 C_4 C_5 C_6 S_2 S_3 - 38 C_2 C_5 S_3 S_4 S_6 \\
& - 38 C_3 C_5 S_2 S_4 S_6 + 120 C_5 S_2 S_3 S_4 S_6 \\
J_{g6,11} = {} & 38 S_5 S_6 - 120 C_6 S_4 - 70 C_4 S_6 + 135 C_6 S_3 S_4 + 135 C_3 S_5 S_6 - 120 C_4 C_5 S_6 - 70 C_5 C_6 S_4 \\
& + 135 C_4 C_5 S_3 S_6 \\
J_{g6,12} = {} & 38 S_5 S_6 - 120 C_6 S_4 - 70 C_4 S_6 - 120 C_4 C_5 S_6 - 70 C_5 C_6 S_4 \\
J_{g6,13} = {} & 70 C_6 S_5 \\
J_{g6,14} = {} & - 70 S_6 \\
J_{g6,15} = {} & 0
\end{aligned}
$$