Preliminary draft

Calculus Ia: Limits and differentiation

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Any errors that remain are the author's sole responsibility.

Colophon
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Don't just read it; fight it! Ask your own question, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

– Paul Halmos

Preface

This lecture note is as boring as it gets, since it tries to be politically correct and does not contain:

1. Your own effort in trying new things in mathematics;
2. Your own taste of what is beautiful and what is not;
3. Your happiness in understanding a concept or finding a proof;
4. Your failures and experiences to refine your future choices;
5. Your interaction and teamwork with your friends;
6. Your lunch hours, roadtrips, family, dreams, and drunken moments (some by alcohol, some by art, some by people);
7. Your splendid life with all the possibilities ahead.

The goal of this lecture note is to simply provide some remainders in case one needs them. Like a photo album. In some sense, the primary goal would be for you to understand some mathematical concepts, and gradually you should be able to express your ideas in mathematical terms with ease.

It is often asked about a reference book for this course. I don't want to recommend anything in particular, but I would recommend to have at least one "classical" textbook at hand, preferably with detailed solutions to the exercises. Try the exercise yourself first, and when you really get stuck, read the solution, then try the exercise again several days later.

Instead I could recommand some "casual" books:

1. «Gödel, Escher, Bach: An Eternal Golden Braid», Hofstadter, Basic Books.
2. «Proofs from THE BOOK», Aigner-Ziegler, Springer.[1]
3. «How to solve it», Pólya, Princeton University Press.
4. «Flatland: A Romance of Many Dimensions», Abbott, Seeley & Co.

1: This one is not that casual…!

Have fun!

Yichao Huang

P.S. Although this note can be publicly distributed and reused, it is not my intention. It thus contains many personal touches and certainly does not stand the test of time.

Contents

List of Figures

A "gentle" introduction to the language of mathematics 1

Since this notes is written during the Corona time, let us start by reviewing some common misunderstandings.

Do the following sentences convey the same message?[1]

1. The government has no recommandation for wearing masks.
2. The government does not recommand wearing masks.
3. The government recommand against wearing masks.

1: Does "The government now recommands wearing masks" implies "The government recommanded against wearing masks before"?

1.1 Mathematical symbols

Some symbols are specific to mathematics, such as

$$\begin{array}{cl} \forall & \text{for all} \\ \exists & \text{there exist(s)} \\ \vee & \text{or} \\ \wedge & \text{and} \\ \infty & \text{infinity} \end{array}$$

and many more.[2]

One uses these symbols to write statements in mathematics. For example, one can write

$$\forall x \in \mathbb{R}, (x^2 - 1 \geq 0) \vee (x^3 + 1 \geq 0).$$

This reads (from left to right!) "for all real number x, (we have) $x^2 - 1 \geq 0$ or $x^3 + 1 \geq 0$". In practice, the symbol $\vee$ is not that often used, and one encounters more often

$$\forall x \in \mathbb{R}, (x^2 - 1 \geq 0) \text{ or } (x^3 + 1 \geq 0).$$

2: In short, this course BSMA1002 is about the symbol $'$ and the next course BSMA1003 is about the symbol $\int$.

Notice that the meaning of the word "or" is inexclusive.[3] Also, notice that this phrase is not true, but that is not the point here.

One can "operate" on statements. For example, with the **negation** symbol

$$\neg \quad \text{not}$$

one can "negate" a statement:

$$\neg \left(\forall x \in \mathbb{R}, (x^2 - 1 \geq 0) \text{ or } (x^3 + 1 \geq 0)\right).$$

What does it mean? How do one write it in plain language?

(Before moving on, one could first to come up with a personal attempt. The goal is not to succeed at the first try, but to, *inter alia*, figure out some patterns and be aware of the possible difficulties.)

3: **Exclusive or** or **exclusive disjunction** is a logical operation that outputs true only when inputs differ (one is true, the other is false). In logic, **or** by itself means the **inclusive or**, distinguished from an exclusive or, which is false when both of its arguments are true, while an "or" is true in that case. In sum, $A \vee B$ is true if A is true, or if B is true, or if both A and B are true.

The general rule for negating a statement is the following: (tbc).[4]

4: Don't try to remember these sentences, but rather, do some examples and **understand the principle behind it**.

For the example above, an equivalent way of writing the statement

$$\neg\left(\forall x \in \mathbb{R}, (x^2 - 1 \geq 0) \text{ or } (x^3 + 1 \geq 0)\right)$$

is

$$\exists x \in \mathbb{R}, \neg(x^2 - 1 \geq 0) \text{ and } \neg(x^3 + 1 \geq 0).$$

And if we really want to get rid of the negation symbol, we can also write it as

$$\exists x \in \mathbb{R}, (x^2 - 1 < 0) \text{ and } (x^3 + 1 < 0).$$

Remark 1.1.1 In practice, this means that if P is some property and if one wants to **disprove** a statement of type "for all x, P is true" ($\forall x, P$), one should show the existence of some x such that P is false ($\exists x, \neg P$). Showing that such x exists can be done by explicit construction ("pulling a rabbit out of a hat"), or by abstraction (without necessarily knowing all the properties of such x).

Implications are highly frequent statements in mathematics.[5] If P and Q are two properties (or two statements), the symbol[6]

$$\Longrightarrow \quad \text{implies}$$

used in the following statement

$$P \Longrightarrow Q$$

means "If P is true, then Q is true". It does not give information on Q if P is false.

5: Which phrase is an implication in the classical **syllogism** "All men are mortal. Socrates is a man. Therefore, Socrates is mortal."?

6: In some Finnish textbooks it is written $\longrightarrow$. Different people use different notations, but usually they look similar and understandable by context.

One can draw a **truth table** to understand better the symbol $\Longrightarrow$. In the table, 1 means "true" and 0 means "false", and I leave you to figure out the rest.

(tbc)

It is quite useful to realize that the implication symbol can be replaced by other symbols before. At first sight this mights seem strange, but let us draw the table of truth for

$$(\neg P) \vee Q$$

and compare it to the table before:

(tbc)

Equivalence of two statements P and Q, with the symbol

$$P \equiv Q$$

should be understood as two implications:

$$P \Longrightarrow Q \quad \text{and} Q \Longrightarrow P.$$

It simply says that they are either both true or both false.[7]

7: Try to draw the truth table (on a paper, on an electronic device or in your head)!

Proposition 1.1.1 *The following statements are equivalent:*[8]

1.

$$P \Longrightarrow Q.$$

2.

$$(\neg P) \vee Q.$$

8: Personal dedication to my undergrad teacher Mr. Mohan: "LASSE" (Les assertions suivantes sont équivalentes).

Proposition 1.1.2 *The following statements are equivalent:*[9]

1.

$$P \equiv Q.$$

2.

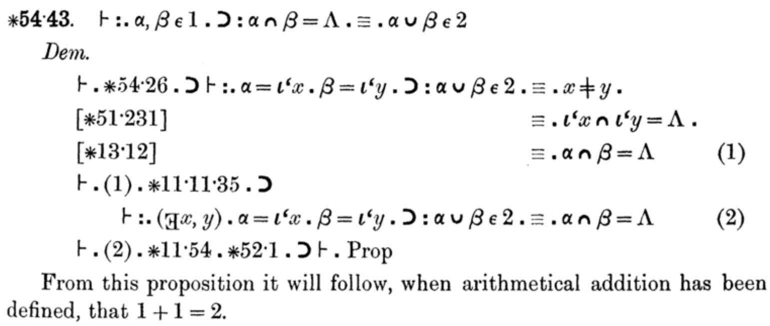
$$(P \Longrightarrow Q) \wedge (Q \Longrightarrow P).$$

It is a little self-referencing if you take it as the definition of equivalence!

9: So, what does "$((\neg P) \vee Q) \wedge ((\neg Q) \vee P)$" mean?

Proof by contradiction is also commonly used in mathematics (and in everyday life).[10] The principle can be summarized as the following: (tbc)

10: Proof by contradiction is formulated as $P \equiv P \vee \bot \equiv \neg(\neg P) \vee \bot \equiv \neg P \longrightarrow \bot$, where $\bot$ is a logical contradiction or a false statement (a statement which true value is false). If $\bot$ is reached via $\neg P$ via a valid logic, then $\neg P \longrightarrow \bot$ is proved as true so P is proved as true. I didn't bother to read the above phrases myself (I copied it from Wikipedia), since in practice, one should seize the **idea** (which I think you all have it naturally) rather than relying on formal manipulation of symbols. It is up to you to find out what is the best way to understand a new concept!



$*54{\cdot}43.\ \vdash :. \alpha, \beta \in 1 . \supset : \alpha \cap \beta = \Lambda . \equiv . \alpha \cup \beta \in 2$

Dem.

$\vdash . *54{\cdot}26 . \supset \vdash :. \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . x \neq y .$

$[*51{\cdot}231] \qquad \equiv . \iota`x \cap \iota`y = \Lambda .$

$[*13{\cdot}12] \qquad \equiv . \alpha \cap \beta = \Lambda \qquad (1)$

$\vdash . (1) . *11{\cdot}11{\cdot}35 . \supset$

$\vdash :. (\exists x, y) . \alpha = \iota`x . \beta = \iota`y . \supset : \alpha \cup \beta \in 2 . \equiv . \alpha \cap \beta = \Lambda \qquad (2)$

$\vdash . (2) . *11{\cdot}54 . *52{\cdot}1 . \supset \vdash .$ Prop

From this proposition it will follow, when arithmetical addition has been defined, that $1 + 1 = 2$.

Figure 1.1: Whitehead and Russell proving $1 + 1 = 2$. Full story here.

Remark 1.1.2 Of course, mathematicians (or scientists) never write with symbols only, unless you are a hardcore logician. You will soon know where to draw the line: the above is just a showcase of the mathematical rigor.

One of the advantages of mathematics compared to other science, is that (almost) all proofs are reproducible and can be checked.[11] It is a good way to train your critical thinking skills: by doing mathematics (the right way!), you are living one of the rare moments where you can distinguish completely right from wrong and form a clear judgement.

11: Even this one.

1.2 Mathematical induction

One of the early difficulties of transitioning into a good undergrad student is to write mathematical sound and concise proofs. We have already seen what is proof by contradiction; let us review another classical proof technique: **proof by induction**.

Here is a learning technique: you can **start by an example before reading the theoretical descriptions**. So let us search "proof by induction" on the internet, go to the Wikipedia page, and check out the following example:

Example 1.2.1 (Sum of consecutive natural numbers) ...

[Week I]
What are...sets?

Sets: definitions and properties 2

2.1 Sets, elements, subsets

2.2 Union, intersection, cardinality

2.3 Finite sets and a taste of combinatorics

2.4 Story time: some paradoxes

(tbc)

For a more elaborated logic paradox of the same flavor, check out the poem on the door of Åsa Hirvonen (last retrieved: August 2020).

Infimum and supremum | 3

"In mathematics, a small positive infinitesimal quantity, usually denoted ϵ, whose limit is usually taken as $\epsilon \longrightarrow 0$."

– Wolfram MathWorld.

All symbols are created equal, but some symbols are more equal than others. You can write $y = f(x)$ or $b = f(a)$ or $v = f(u)$ or $s = f(t)$, but at least in this course, we reserve the notation ϵ (and later δ) for special purposes.

3.1 Relations and ordering

…

3.2 An epsilon of room

3.3 Development: the Archimedean property

[Week II]
What are...sequences?

4 Sequences: definitions and properties

[Week III&IV]
What are...functions?

Functions: definitions and properties

5

In high school, most functions are given in the form of a formula:

$$y = f(x) = x^2 + 1.$$

However, in full generality, a function is defined in a more abstract way. The essential idea is to associate an element to a given element (in the example above, for each real number x, we associate the real number $y = x^2 + 1$). The abstract definition has many advantages and covers more situations, for example, we will see that a sequence can be seen as a function (from a set of integers $\mathbb{Z}_{n\geq 0}$ to the set of real numbers $\mathbb{R}$).

5.1 Notations

(tbc)

Sometimes, by abuse of notation,[1]

1: Sometimes it means "The author is tired."

Example 5.1.1 The Möbius function $\mu : \mathbb{Z}_{>0} \longrightarrow \{-1, 0, 1\}$ is defined depending on the factorization of a positive integer into prime factors. For any positive integer $n > 0$, the value of $\mu(n)$ is defined in the following way:

1. $\mu(n) = 1$ if n is a **square-free** positive integer with an **even** number of prime factors;
2. $\mu(n) = -1$ if n is a **square-free** positive integer with an **odd** number of prime factors;
3. $\mu(n) = 0$ if n has a squared prime factor.

The Möbius function μ has an alternative definition in terms of roots of unity (if you are interested, take a look at the extra exercise A.1.)[2]

2: Using the Möbius function μ, one defines the Mertens function $M : \mathbb{R} \longrightarrow \mathbb{R}$, $M(x) = \sum_{n\in\mathbb{Z}_{>0}, n\leq x} \mu(n)$. It is unknown whether $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for all $\epsilon > 0$.

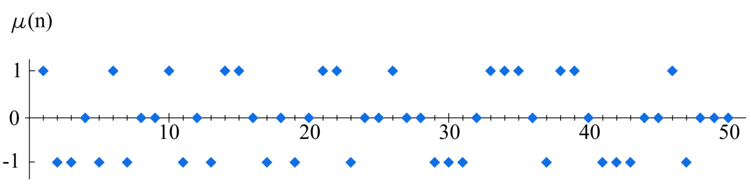


Figure 5.1: First values of the Möbius function μ.

5.2 Injections, surjections, bijections

5.3 Story time: “Je le vois, mais je ne crois pas”

5.4 Development: Bendixon-Bernstein-Borel-Cantor-Dedekind-Schröder-Zermelo theorem

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

6 Limits and continuity of functions

6.1 Bolzano's theorem

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

6.2 Story time: the origins of rigorous Calculus

Stories of this type are best summerized in SMBC.[1]

1: SMBC=Saturday Morning Breakfast Cereal. Similar comics: xkcd, phdcomics, abstruse goose.

6.3 Development: uniform continuity

[Week V&VI]
What are...derivatives?

7 Calculations of derivatives

Differentiation: rigorous definition | 8

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

8.1 Rolle’s theorem

8.2 Story time: the Weierstrass function

8.3 Development: Hölder continuity

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[Week VII&Beyond] Elementary functions and inequalities

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Appendix

Some more exercises | A

A.1 Root of unity

A.2 Constant-recursive sequence

...

Notation

The next list describes several symbols that will be later used within the body of the document.

c	Speed of light in a vacuum inertial frame
h	Planck constant

Greek Letters with Pronounciation

Character	Name	Character	Name
α	alpha *AL-fuh*	ν	nu *NEW*
β	beta *BAY-tuh*	ξ, Ξ	xi *KSIGH*
γ, Γ	gamma *GAM-muh*	o	omicron *OM-uh-CRON*
δ, Δ	delta *DEL-tuh*	π, Π	pi *PIE*
ϵ	epsilon *EP-suh-lon*	ρ	rho *ROW*
ζ	zeta *ZAY-tuh*	σ, Σ	sigma *SIG-muh*
η	eta *AY-tuh*	τ	tau *TOW (as in cow)*
θ, Θ	theta *THAY-tuh*	υ, Υ	upsilon *OOP-suh-LON*
ι	iota *eye-OH-tuh*	ϕ, Φ	phi *FEE, or FI (as in hi)*
κ	kappa *KAP-uh*	χ	chi *KI (as in hi)*
λ, Λ	lambda *LAM-duh*	ψ, Ψ	psi *SIGH, or PSIGH*
μ	mu *MEW*	ω, Ω	omega *oh-MAY-guh*

Capitals shown are the ones that differ from Roman capitals.