HW10

Yichen Dong

November 8, 2018

Problem 1

We have to show that all combinations of inner products are orthagonal, that is, that they equal 0. So

$\langle 1, \cos(nx) \rangle = 0$

$\langle 1, \sin(nx) \rangle = 0$

$\langle \cos(nx), \cos(mx) \rangle = 0$

$\langle \sin(nx), \sin(mx) \rangle = 0$

$\langle \cos(nx), \sin(nx) \rangle = 0$

Since $w(x) = 1$, each of these is just the integral of the products,

$$\langle 1, \cos(nx) \rangle = \int_0^{2\pi} 1 \cdot \cos(nx)\, dx = \frac{1}{n} \sin(nx) \Big|_0^{2\pi}$$

Since n is an integer, $\frac{1}{n}\sin(2n\pi)$ is alway a multiple of 2π, and $\sin(0) = 0$ and $\sin(2n\pi) = 0$, so

$\langle 1, \cos(nx) \rangle = 0$

$$\langle 1, \sin(nx) \rangle = \int_0^{2\pi} 1 \cdot \sin(nx)\, dx = -\frac{1}{n}\cos(nx)\Big|_0^{2\pi} = -\frac{1}{n} + \frac{1}{n} = 0$$

$$\langle \cos(nx), \cos(mx) \rangle = \int_0^{2\pi} \cos(nx)\cos(mx)\, dx = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta))$$

$\alpha = \cos(mx)$, $\beta = \cos(nx)$

$$= \frac{1}{2}\int \cos(x(m+n))\, dx + \frac{1}{2}\int \cos(x(m-n))\, dx$$

$u = x(m+n)$ $\qquad$ $u = x(m-n)$

$du = (m+n)\, dx$

$$= \frac{1}{2\cdot(m+n)} \cdot \sin(x(m+n))\Big|_0^{2\pi} + \frac{1}{2(m-n)} \cdot \sin(x(m-n))\Big|_0^{2\pi}$$

Since $m+n$ and $m-n$ are integers, they will be a multiple of 2π.

$$= 0 - 0 - \left(-\frac{1}{2(m+n)} \cdot 1 - \frac{1}{2(m-n)} \cdot 1\right) = 0$$

$\langle \sin(nx), \sin(mx) \rangle = \frac{1}{2} \int \cos(x(m-n)) - \cos(x(m+n))\, dx$

$u = x(m-n) \qquad u = x(m+n)$

$= \frac{1}{2(m-n)} \left(\sin(x(m-n)) \right) \Big|_0^{2\pi} - \frac{1}{2(m-n)} \left(\sin(x(m+n)) \right) \Big|_0^{2\pi}$

$= 0 - 0 + 0 - 0 = 0$

$\langle \cos(nx), \sin(nx) \rangle = \int \sin(nx) \cos(nx)\, dx$

$u = \sin(nx)$

$du = \frac{1}{n} \cos(nx)\, dx$

$= \frac{1}{n} \int u\, du$

$= \frac{u^2}{2n} = \frac{\sin^2(nx)}{2n} \Big|_0^{2\pi} = 0 - 0 = 0$

Problem 2

Linear independence is defined as

$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \vec{0}$ only if $a_i = 0$.

If we replace the $\vec{v}_i$ with $q_i(x)$, we will get a set of orthogonal functions

If we take the inner product of that set, with any $q_i(x)$ of the set

$\langle a_1 q_1(x) + a_2 q_2(x) + \dots + a_n q_n(x), q_i(x) \rangle$

$= \langle 0, q_i(x) \rangle = 0$, since the left hand side evaluates to 0.

but, $\langle a_1 q_1(x) + a_2 q_2(x) + \dots + a_n q_n(x), q_i(x) \rangle$

can also be written as

$\langle a_1 q_1(x), q_i(x) \rangle + \langle a_2 q_2(x), q_i(x) \rangle + \dots + \langle a_i q_i(x), q_i(x) \rangle + \dots$
$+ \langle a_n q_n(x), q_i(x) \rangle = 0$

Since everything except $\langle a_i q_i(x), q_i(x) \rangle$ evaluates to 0, we have $\langle a_i q_i(x), q_i(x) \rangle$ which is >0 unless $a_i = 0$. We can repeat this for $i = 1, 2, \dots, n$ to shown that $\sum_{i=n}^{n} a_i = 0$ for our original equation to be true.

b. $IV(\hat{f}) = \int_D V(\hat{f}(x))\,dx = \int_D E((\hat{f}(x) - E(\hat{f}(x)))^2)\,dx$

$ISB(\hat{f}) = \int_D (E(\hat{f}(x)) - f(x))^2\,dx$

$IMSE(\hat{f}) = \int_D E((\hat{f}(x) - f(x))^2)\,dx = IV(\hat{f}) + ISB(\hat{f})$

$Bias(\hat{f}(x)) = E[\hat{f}(x)] - f(x)$

$V[\hat{f}(x)] = E[(\hat{f}(x) - E[\hat{f}(x)])^2]$

$MSE[\hat{f}(x)] = E[(\hat{f}(x) - f(x))^2]$

$= V(\hat{f}(x)) + Bias(\hat{f}(x))^2$

$$IMSE(\hat{f}) = \int_D MSE[\hat{f}(x)]\,dx$$

$$= \int_D V(\hat{f}(x)) + Bias(\hat{f}(x))^2\,dx$$

$$= IV(\hat{f}(x)) + \int_D Bias(\hat{f}(x))^2\,dx$$

$$= IV(\hat{f}) + \underbrace{\int_D (E[\hat{f}(x)] - f(x))^2\,dx}$$

$$= IV(\hat{f}) + ISB(\hat{f})$$

Problem 3

$$\left\|\sum_{k=1}^{m} q_k\right\|^2 = \sum_{k=1}^{m} \|q_k\|^2$$

$$= \|q_1+q_2+\dots+q_m\|^2 = \sum_{k=1}^{m} \|q_k\|^2$$

Since $\|\cdot\| = L_2$ norm,

$$\langle (q_1+q_2+\dots+q_m), (q_1+q_2+\dots+q_m) \rangle^{\frac{1}{2}\cdot 2} = \langle \sqrt{q_1^2}, \sqrt{q_1^2}\rangle + \langle \sqrt{q_2^2}, \sqrt{q_2^2}\rangle + \dots + \langle \sqrt{q_m^2}, \sqrt{q_m^2}\rangle$$

$$= \int (q_1+q_2+\dots+q_m)(q_1+q_2+\dots+q_m)\,dx = \int q_1^2\,dx + \int q_2^2\,dx + \dots + \int q_m^2\,dx$$

$$= \int q_1^2 + q_1 q_2 + \dots + q_m^2\,dx =$$

$$= \int q_1^2\,dx + \int q_1 q_2\,dx + \dots + \int q_m^2\,dx =$$

However, since we know $\int q_i(x) q_j(x) \overset{1 \text{ in this case}}{w(x)}\,dx = 0$ if $i \neq j$, the $\int q_1 q_2\,dx$, etc., that are not q_i^2 become 0, and we are left with

$$= \int q_1^2\,dx + \int q_2^2\,dx + \dots + \int q_m^2\,dx = \int q_1^2\,dx + \int q_2^2\,dx + \dots + \int q_m^2\,dx$$

and thus we satisfy the proof.

I don't believe this will hold for a general L_p norm. If L_3, then we have

$$\left(\int_D (q_1+q_2+\dots+q_m)^3\, dx\right)^{\frac{1}{3}} \neq \left(\int q_1^3\, dx\right)^{\frac{1}{3}} + \left(\int q_2^3 dx\right)^{\frac{1}{3}} + \dots + \left(q_m^3\, dx\right)^{\frac{1}{3}}$$

and the relationship does not hold as well.

Problem 4

Chebyshev:

$T_0(x) = 1 \qquad T_1(x) = x$

$T_2(x) = 2x^2 - 1 \qquad T_3(x) = 4x^3 - 3x$

$q_1 = 1$
$q_2 = x$
$q_3 = x^2$
$q_4 = x^3$

$\tilde{q}_1 = q_1 = 1$

$u = 1 - x^2$
$du = -2x\,dx$
$\rightarrow -\frac{1}{2}\int \frac{1}{\sqrt{u}}du$

$$\tilde{q}_2 = q_2 - \frac{\langle \tilde{q}_1, q_2 \rangle}{\langle \tilde{q}_1, \tilde{q}_1 \rangle}\tilde{q}_1 = q_2 - \frac{\int_{-1}^{1} x(1-x^2)^{-\frac{1}{2}}dx}{\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}dx} \cdot 1$$

$\rightarrow \sin^{-1}(x)$

$$= q_2 - \frac{\left[-\frac{1}{2}\cdot 2\sqrt{1-x^2}\right]_{-1}^{1} \rightarrow 0}{\left[\sin^{-1}(x)\right]_{-1}^{1} \rightarrow = \pi} = q_2 = x$$

$$\hat{q}_3 = q_3 - \frac{\langle \tilde{q}_1, q_3 \rangle}{\langle \hat{q}_1, \hat{q}_1 \rangle}\hat{q}_1 - \frac{\langle \tilde{q}_2, q_3 \rangle}{\langle \tilde{q}_2, \tilde{q}_2 \rangle}\tilde{q}_2$$

$$= x^2 - \frac{\int_{-1}^{1} x^2(1-x^2)^{-\frac{1}{2}}dx}{\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}dx} - \frac{\int_{-1}^{1} x^3(1-x^2)^{-\frac{1}{2}}dx}{\int_{-1}^{1} x^2(1-x^2)^{-\frac{1}{2}}dx}\cdot x$$

$\int_{-1}^{1} x^2(1-x^2)^{-\frac{1}{2}}dx = \int \sin^2(u)\,du$ if $x = \sin(u)$, $dx = \cos(u)\,du$

$\sin^2(u) = \frac{1}{2} - \frac{1}{2}\cos(2u)$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \cos(u)$

and $u = \sin^{-1}(x)$

$\int \sin^2(u) = \int \frac{1}{2}du - \int \frac{1}{2}\cos(2u)\,du$

$= \frac{u}{2} - \frac{1}{4}\sin(2u) = \frac{u}{2} - \frac{1}{2}\sin(u)\cos(u)$

$= \frac{u}{2} - \frac{1}{2}\sin(u)\sqrt{1-\sin^2(u)} = \frac{\sin^{-1}(x)}{2} - \frac{1}{2}x\sqrt{1-x^2}\Big|_{-1}^{1}$

$= \frac{\pi}{2}$

$\int_{-1}^{1} x^3 (1-x^2)^{-\frac{1}{2}} dx = 0$, since $\frac{x^3}{\sqrt{1-x^2}}$ is an odd function and -1 to 1 is symmetric

Then,

$$\hat{q}_3 = x^2 - \frac{\frac{\pi}{2}}{\pi} - 0 = x^2 - \frac{1}{2}$$

Normalize $T_2(1) = 1$, Then we have $T_2(x) = 2x^2 - 1$

$$\hat{q}_4 = x^3 - \frac{\langle \hat{q}_1, q_4 \rangle}{\langle \hat{q}_1, \hat{q}_1 \rangle} \hat{q}_1 - \frac{\langle \hat{q}_2, q_4 \rangle}{\langle \hat{q}_2, \hat{q}_2 \rangle} \hat{q}_2 - \frac{\langle \hat{q}_3, q_4 \rangle}{\langle \hat{q}_3, \hat{q}_3 \rangle} \cdot \hat{q}_3$$

$$= x^3 - \frac{\int_{-1}^{1} x^3 (1-x^2)^{-\frac{1}{2}} dx}{\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} dx} - \frac{\int_{-1}^{1} x^4 (1-x^2)^{-\frac{1}{2}} dx}{\int_{-1}^{1} x^2 (1-x^2)^{-\frac{1}{2}}} \cdot x$$

$$- \left. \frac{\int_{-1}^{1} (x^5 - \frac{1}{2}x^3)(1-x^2)^{-\frac{1}{2}} dx}{\int_{-1}^{1} (x^2 - \frac{1}{2})(x^2 - \frac{1}{2})(1-x^2)^{-\frac{1}{2}} dx} \right\} 0$$ because odd and symmetric

$$= x^3 - 0 - \frac{\frac{3\pi}{8}}{\frac{\pi}{2}} = x^3 - \frac{3}{4}x$$

Normalize $T_3(1) = 1$ $= 4x^3 - 3x$

Problem 4b

```r
library(dplyr)

##
## Attaching package: 'dplyr'

## The following objects are masked from 'package:stats':
##
##     filter, lag

## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union

ortho = scan("Orthogonal.txt")
q1 = function(x){1}
q2 = function(x){x}
q3 = function(x){2*x^2-1}
q4 = function(x){4*x^3-3*x}
g_x = function(x){dnorm(x,0,.3)}
n = length(ortho)

c_1 = 1/n*sum(q1(ortho)*g_x(ortho))
c_2 = 1/n*sum(q2(ortho)*g_x(ortho))
c_3 = 1/n*sum(q3(ortho)*g_x(ortho))
c_4 = 1/n*sum(q4(ortho)*g_x(ortho))

c_1

## [1] 0.9410636

c_2

## [1] -0.00415153

c_3

## [1] -0.8583737

c_4

## [1] 0.01297196

sum(c_1*q1(ortho))

## [1] 0.9410636

sum(c_2*q2(ortho))

## [1] 0.01014201
```

```r
sum(c_3*q3(ortho))
```

```
## [1] 700.3602
```

```r
sum(c_4*q4(ortho))
```

```
## [1] 0.1002212
```

$$\hat{f}(x) = \frac{1}{1000}(.94 \cdot 1 \cdot 1 + .01x + 700(2x^2-1) + .10(4x^3-3x))$$

$$= .0004x^3 + 1.4x^2 - .00029x - .699059$$

```r
f_hat_x = function(x){.0004*x^3+1.4*x^2-.00029*x-.699059}
ortho = as.data.frame(ortho)
ortho = ortho %>%
  mutate(f_hat = f_hat_x(ortho),
         dnorm = dnorm(ortho,0,.3))
```

I definitely don't think I did this right. I might have misunderstood what q(x) meant or what g(x) is supposed to be, because my answers do not make any sense whatsoever.

Problem 5

Properties of cubic splines:

① $S_{i-1}(t_i) = y_i = S_i(t_i)$

② $S'_{i-1}(t_i) = S'_i(t_i)$

③ $S''_{i-1}(t_i) = S''_i(t_i) = z_i$

$$S_{i-1}(0) = (0+1) + (0+1)^3 = 4 + (0-1) + (0-1)^3 = S_i$$
$$= 1 + 1 = 4 - 1 - 1 = 2 \checkmark \text{ Satisfies ①}$$

$$S'_{i-1} = 3(x+1)^2 + 1 \quad , \quad S'_i = 3(x-1)^2 + 1$$

$$S'_{i-1}(0) = 3 \cdot 1^2 + 1 = 4 \quad , \quad S'_i = 3(-1)^2 + 1 = 4$$

$$S'_{i-1}(0) = S'_i(0) \checkmark \text{ Satisfies ②}$$

$$S''_{i-1} = 6(x+1) \quad , \quad S''_i = 6(x-1)$$

$$S''_{i-1}(0) = 6 \quad , \quad S''_i(0) = -6 \quad , \times \text{ Does not satisfy ③}$$

For a natural spline, we should also check whether $z_0 = z_n = 0$, which in this case is

$$S''_0(-1) \quad , \quad S''_1(1) = 0$$

$$S''_0(-1) = 6(-1+1) = 0 \quad , \quad S''_1(1) = 6(1-1) = 0 \checkmark$$

Satisfies the condition of a "natural" cubic spline b.

$$f(x)\begin{cases} S_0(x) & x\in(1,\ 2) \\ S_1(x) & x\in(2,\ 3) \\ S_2(x) & x\in(3,\ 4)\end{cases}$$

$t_0=1$, $t_1=2$, $t_2=3$, $t_3=4$

where $S_0''(1)=S_2''(4)=0=z_0=Z_3$

$y_0=1$, $y_1=\frac{1}{2}$, $y_2=\frac{1}{3}$, $y_3=\frac{1}{4}$

$f(1)=1,\ f(2)=\frac{1}{2},\ f(3)=\frac{1}{3},\ f(4)=\frac{1}{4}$

$S_0''(2)=z_1,\ S''(3)=z_2$

$$h_{i-1}\cdot z_{i-1}+2(h_i+h_{i-1})z_i+h_iz_{i+1}=\frac{6}{h_i}(y_{i+1}-y_i)-\frac{6}{h_{i-1}}(y_i-y_{i-1})$$

for $i=1$

$$h_0\cdot z_0+2(h_1+h_0)\cdot z_1+h_1\cdot z_2=\frac{6}{h_1}(y_2-y_1)-\frac{6}{h_0}(y_1-y_0)$$

$\boxed{h_0=h_1=h_2=1 \text{ (since all } t_i \text{ and } t_{i+1} \text{ are 1 apart)}}$

$$1\cdot 0+2(1+1)\cdot z_1+1\cdot z_2=\frac{6}{1}\left(\frac{1}{3}-\frac{1}{2}\right)-\frac{6}{1}\left(\frac{1}{2}-1\right)$$

$$=4z_1+z_2=2$$

for $i=2$

$$h_1\cdot z_1+2(h_2+h_1)\cdot z_2+h_2\cdot z_3=\frac{6}{h_2}(y_3-y_2)-\frac{6}{h_1}(y_2-y_1)$$

$$=z_1+4z_2+0=6\left(\frac{1}{4}-\frac{1}{3}\right)-6\left(\frac{1}{3}-\frac{1}{2}\right)=\frac{1}{2}$$

$4z_1+z_2=2 \rightarrow z_2=2-4z_1$

$z_1+4z_2=\frac{1}{2} \rightarrow z_1+4(2-4z_1)=\frac{1}{2} \rightarrow 15z_1=\frac{15}{2},$

$z_1=\frac{1}{2}$

$\hookrightarrow 4(\frac{1}{2})+z_2=2 \rightarrow z_2=0$

So we have

$z_0 = 0,\ z_1 = \frac{1}{2},\ z_2 = 0,\ z_3 = 0$

Then we know

$$S_i(x) = \frac{z_i}{6h_i}(t_{i+1}-x)^3 + \frac{z_{i+1}}{6h_i}(x-t_i)^3 + \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}\right)(x-t_i)$$
$$+ \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right)(t_{i+1}-x)$$

So,

$$S_0(x) = \frac{0}{6}(2-x)^3 + \frac{\frac{1}{2}}{6}(x-1)^3 + \left(\frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{6}\right)(x-1)$$
$$+ (1-0)(2-x)$$
$$= \frac{(x-1)^3}{12} + \frac{5}{12}x - \frac{5}{12} + 2 - x = \frac{(x-1)^3}{12} - \frac{7}{12}x - \frac{19}{12}$$

$$S_1(x) = \frac{1}{12}(3-x)^3 + 0 + \left(\frac{1}{3} - 0\right)(x-2)$$
$$+ \left(\frac{1}{2} - \frac{1}{12}\right)(3-x)$$
$$= \frac{1}{12}(3-x)^3 + \frac{1}{3}x - \frac{2}{3} + \frac{5}{4} - \frac{5}{12}x$$
$$= \frac{1}{12}(3-x)^3 - \frac{1}{12}x + \frac{7}{15}$$

$$S_2(x) = 0 + 0 + \left(\frac{1}{4} - 0\right)(x-3) + \left(\frac{1}{3} - 0\right)(4-x)$$
$$= \frac{1}{4}x - \frac{3}{4} + \frac{4}{3} - \frac{1}{3x} = -\frac{1}{12}x + \frac{7}{12}$$