Integer Programming

Video: 8A-8C

Reading: Rader, Deterministic Operations Research: Models and Methods in Linear Optimization, 2013, Chapter 13 – Sections 13.1 and 13.2.

https://catalyst.library.jhu.edu/catalog/bib_4838331

Consider the following problem:

An advanced civilization Q is monitoring the evolution of intelligent life on other planets. Currently, seven of these planets are in the most dangerous time of their history when the inhabitants have the technology to start a nuclear war, but don't have the technology to stop it.

Q can help any planet to avoid a nuclear war by arriving there, breaking into computer networks, analyzing the data and modifying it a bit to change the history and avert the war. However, Q's resources are limited and, with so many planets being in danger at the same time, Q must prioritize.

The table below shows the resources and personnel needed for each rescue mission

Planet	Spaceships	Fuel (in Q's volume units)	Pilots	Data Scientists	Mathematicians
1	1	3.4	2	5	1
2	1	1.5	2	4	0
3 (Earth)	1	6.3	2	3	2
4	1	3.2	2	1	6
5	2	7.8	3	12	1
6	1	1.3	2	11	3
7	1	8.7	2	7	2

If it's not possible to save all the planets, Q's strategy is maximizing the sum of the ages of the saved life forms.

Planet	Time for life to evolve (in Q's time units)
1	12
2	32
3 (Earth)	90
4	95
5	82
6	18
7	15

The available resources are 6 spaceships, 30 units of fuel, 12 pilots, 25 data scientists and 10 mathematicians. Will Earth be saved?

Let $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 be 1 if the corresponding planet is saved and 0 otherwise. The ILP (integer linear program) to be solved by civilization Q is

$$
\begin{aligned}
&\min \quad f(x) = -\left(12x_1 + 32x_2 + 90x_3 + 95x_4 + 82x_5 + 18x_6 + 15x_7\right) \\
&s.t. \\
&x_1 + x_2 + x_3 + x_4 + 2x_5 + x_6 + x_7 \le 6 \\
&3.4x_1 + 1.5x_2 + 6.3x_3 + 3.2x_4 + 7.8x_5 + 1.3x_6 + 8.7x_7 \le 30 \\
&2x_1 + 2x_2 + 2x_3 + 2x_4 + 3x_5 + 2x_6 + 2x_7 \le 12 \\
&5x_1 + 4x_2 + 3x_3 + 1x_4 + 12x_5 + 11x_6 + 7x_7 \le 25 \\
&x_1 + 2x_3 + 6x_4 + x_5 + 3x_6 + 2x_7 \le 10 \\
&x_1, x_2, x_3, x_4, x_5, x_6, x_7 \in \{0,1\}
\end{aligned}
$$

Let's begin with solving an LP, which is a relaxation of the ILP, with $0 \le x_1, x_2, x_3, x_4, x_5, x_6, x_7 \le 1$ instead of $x_1, x_2, x_3, x_4, x_5, x_6, x_7 \in \{0,1\}$.

Applying MATLAB's linprog(), an optimal solution to the LP happens to be integer, (1?, 2?, 3?, 4?, 5?, 6?, 7?) and the optimal value of the objective function is 8?.

Question 1: Replace the 1?, 2?, 3?, 4?, 5?, 6? and 7? with the correct numbers. 1, 1, 1, 1, 1, 0, 0

Question 2: Replace the 8? with the correct number. -311

The original ILP is the LP plus the integrality constraints. Additional constraints can only increase the optimal value of the objective function (in the case of minimization), so the answer to question 1 is also the optimal solution to the ILP.

In this example, the solution of the LP is integer right away. Some classes of LPs have integer optimal solutions, or an optimal solution may happen to be integer due to the values of the parameters. Also, an LP can have both integer and non-integer optimal solutions.

Question 3: Can the optimal solution to the LP returned by linprog() with the default options become non integer if one of the time for life to evolve parameters (12, 32, 90, 95, 82, 18, 15) is changed, but remains positive integer?

- yes
- no

However, in most cases, an optimal solution to an LP relaxation of an ILP is not integer and we need to add something else to solve the ILP.

Branch and Bound Algorithm

Video: 8D-8I

Reading: Rader, Deterministic Operations Research: Models and Methods in Linear Optimization, 2013, Chapter 14 – Sections 14.1 and 14.2.

https://catalyst.library.jhu.edu/catalog/bib_4838331

Consider another problem:

Four of the saved planets require further investments into their development. Civilization Q decides to send their philosophers to live and publish their work on these planets in order to speed up the development of humane societies. Each Q's philosopher brings the time of the emergence of humane society closer by the amount of time shown in the table:

Planet	Advance made by one Q's philosopher, in Q's time units	Time to humane society without Q's involvement, in Q's time units

1	0.2	0.7
2	0.5	0.8
3 (Earth)	0.4	0.9
4	0.1	0.3

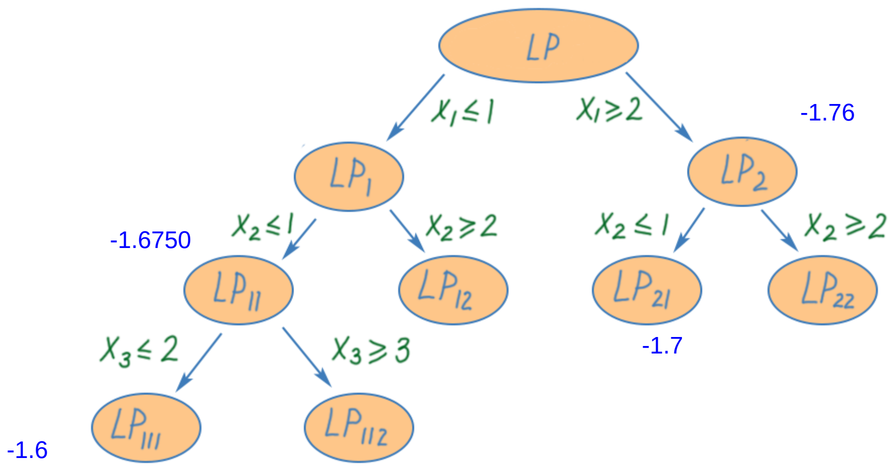
5 philosophers have volunteered, and the goal is to maximize the total amount of time saved for the civilizations. However, a philosopher suffers when their potential is unrealized, so they refuse to go to a planet where the society needs less than what they can potentially provide. So, for example, no more than three philosophers will agree to go to planet 1.

How many philosophers will go to each planet?

Let x_1, x_2, x_3 and x_4 be the numbers of philosophers arriving at each planet. The ILP is

$$
\begin{aligned}
&\min \quad f(x) = -\left(0.2x_1 + 0.5x_2 + 0.4x_3 + 0.1x_4\right) \\
&\text{s.t.} \\
&x_1 + x_2 + x_3 + x_4 \le 5 \\
&0.2x_1 \le 0.7 \\
&0.5x_2 \le 0.8 \\
&0.4x_3 \le 0.9 \\
&0.1x_4 \le 0.3 \\
&x_1, x_2, x_3, x_4 \ge 0 \\
&x_1, x_2, x_3, x_4 \in \mathbb{Z}
\end{aligned}
$$

Solving the LP relaxation of this ILP without the $x_1, x_2, x_3, x_4 \in \mathbb{Z}$ constraint (denoted by "LP" in the next figure and in the questions), we get an optimal solution with x_1^* between 1 and 2. In the ILP, x_1^* is integer, so we won't filter out an optimal solution to the ILP if we add the constraint "$x_1 \le 1$ or $x_1 \ge 2$" to the relaxed LP. However, adding an "or" constraint converts an LP into an NLP, so, instead, we split the LP into two LPs (denoted by "LP_1" and "LP_2"), one with the $x_1 \le 1$ constraint and the other one with the $x_1 \ge 2$ constraint.



Question 4: The optimal value of the objective function in LP is -1.9300

Question 5: The optimal value of the objective function in LP_1 $f_{LP1}(x^*)$ is (please enter 0 if infeasible) -1.9150

Question 6: (continued from the previous question) $f_{LP11}(x^*) = ?$ (please enter 0 if infeasible) -1.6750

Question 7: (continued from the previous question) $f_{LP111}(x^*) = ?$ (please enter 0 if infeasible) -1.6

Question 8: (continued from the previous question) $f_{LP112}(x^*) = ?$ (please enter 0 if infeasible) 0

Question 9: (continued from the previous question) $f_{LP12}(x^*) = ?$ (please enter 0 if infeasible) 0

Question 10: (continued from the previous question) $f_{LP2}(x^*) = ?$ (please enter 0 if infeasible) -1.76

Question 11: (continued from the previous question) $f_{LP21}(x^*) = ?$ (please enter 0 if infeasible) -1.7

Question 12: (continued from the previous question) $f_{LP22}(x^*) = ?$ (please enter 0 if infeasible) 0

The optimal solution to the ILP is (1?, 2?, 3?, 4?).

Question 13: Replace the 1?, 2?, 3? and 4? with the correct numbers. 2,1,2,0

The answer to the previous question can be verified with intlinprog() in MATLAB.

Above, we processed nodes in depth-first order. However, the branch and bound algorithm can also use breadth-first (upper levels of the tree are processed first) or best-first (nodes with the best values of the objective function are processed first) or any other order.

Question 14: If LP_{21} is processed before LP_{11} in the above example, is it possible that LP_{11} will need to be solved to solve the ILP?

- yes
- no

Question 15: If LP_{21} is processed before LP_{111} in the above example, is it possible that LP_{111} will need to be solved to solve the ILP?

- yes
- no

Question 16: Which algorithm results in a smaller number of LPs solved in the above example, depth-first or breadth-first?

- depth-first
- breadth-first
- the number of LPs solved is the same with both algorithms in this example

Question 17: Which algorithm results in a smaller number of LPs solved in the above example, breadth-first or best-first?

- breadth-first
- best-first
- the number of LPs solved is the same with both algorithms in this example

The branch and bound approach can be used to solve non-linear integer problems as well.

Lagrangian Relaxation for Integer Programming

Above, we used linear relaxation to solve an ILP. That is, some constraints from the ILP were removed, converting it into an LP. An optimal solution to the LP may or may not be an optimal solution to the ILP, but the optimal value of the objective function in LP gives the lower bound for the objective function in ILP (in case of a minimization problem).

As with linear relaxion, Lagrangian relaxation provides a lower bound and, sometimes, even the optimal solution. With Lagrangian relaxation, instead of removing an integrality constraint, we move an inequality or equality constraint into the objective function.

In our philosophers' example,

$$\min \quad f(x) = -\left(0.2x_1 + 0.5x_2 + 0.4x_3 + 0.1x_4\right)$$

s.t.

$x_1 + x_2 + x_3 + x_4 \leq 5$

$0.2x_1 \leq 0.7$

$0.5x_2 \leq 0.8$

$0.4x_3 \leq 0.9$

$0.1x_4 \leq 0.3$

$x_1, x_2, x_3, x_4 \geq 0$

$x_1, x_2, x_3, x_4 \in \mathbb{Z}$

the most inconvenient inequality constraint is $x_1 + x_2 + x_3 + x_4 \leq 5$. So, let's move $x_1 + x_2 + x_3 + x_4 \leq 5$ to the objective function in the form of $\sup_{\mu \geq 0} \mu\left(x_1 + x_2 + x_3 + x_4 - 5\right)$.

Question 18: If the $x_1 + x_2 + x_3 + x_4 \leq 5$ constraint is satisfied, $\sup_{\mu \geq 0} \mu\left(x_1 + x_2 + x_3 + x_4 - 5\right)$ is

- $-\infty$
- $+\infty$
- 0
- none of the above

Question 19: If the $x_1 + x_2 + x_3 + x_4 \leq 5$ constraint is not satisfied, $\sup_{\mu \geq 0} \mu\left(x_1 + x_2 + x_3 + x_4 - 5\right)$ is

- $-\infty$
- $+\infty$
- 0
- none of the above

Therefore, the optimal solution to

$$\min_x \sup_{\mu \geq 0} -\left(0.2x_1 + 0.5x_2 + 0.4x_3 + 0.1x_4\right) + \mu\left(x_1 + x_2 + x_3 + x_4 - 5\right)$$

s.t.

$0.2x_1 \leq 0.7$

$0.5x_2 \leq 0.8$

$0.4x_3 \leq 0.9$ (1)

$0.1x_4 \leq 0.3$

$x_1, x_2, x_3, x_4 \geq 0$

$x_1, x_2, x_3, x_4 \in \mathbb{Z}$

will be the optimal solution to the original philosophers' ILP.

Recall from module 5 that the Lagrangian is defined as $L(x,\mu) = f(x) + \mu g(x),\ \mu \geq 0$ for $\min f(x)$ s.t. $g(x) \leq 0$, so let's denote

$$L(x,\mu) = -\left(0.2x_1 + 0.5x_2 + 0.4x_3 + 0.1x_4\right) + \mu\left(x_1 + x_2 + x_3 + x_4 - 5\right)$$

According to the minimax theorem, $\sup_y \inf_x f(x,y) = \inf_x \sup_y f(x,y)$, so

$$\min_x \sup_{\mu\geq 0} L(x,\mu)$$

is equivalent to

$$\max_{\mu\geq 0} \inf_x L(x,\mu).$$

In other words, $\inf_x L(x,\mu)$ with any specific μ is a lower bound on the value of the primal objective function, $\sup_{\mu\geq 0} L(x,\mu)$. Therefore,

$$\begin{aligned}
&\max_{\mu\geq 0} \inf_x L(x,\mu) \\
&\text{s.t.} \\
&0.2x_1 \leq 0.7 \\
&0.5x_2 \leq 0.8 \\
&0.4x_3 \leq 0.9 \\
&0.1x_4 \leq 0.3 \\
&x_1, x_2, x_3, x_4 \geq 0 \\
&x_1, x_2, x_3, x_4 \in \mathbb{Z}
\end{aligned} \qquad (2)$$

is the Lagrangian dual of the original philosophers' problem and

$$\begin{aligned}
&\min_x L(x,\mu) \\
&\text{s.t.} \\
&0.2x_1 \leq 0.7 \\
&0.5x_2 \leq 0.8 \\
&0.4x_3 \leq 0.9 \\
&0.1x_4 \leq 0.3 \\
&x_1, x_2, x_3, x_4 \geq 0 \\
&x_1, x_2, x_3, x_4 \in \mathbb{Z}
\end{aligned} \qquad (3)$$

is a *Lagrangian relaxation* of that problem.

Let's rearrange the terms in $L(x, \mu)$

$$L(x, \mu) = (\mu - 0.2)x_1 + (\mu - 0.5)x_2 + (\mu - 0.4)x_3 + (\mu - 0.1)x_4 - 5\mu$$

and solve (2).

The solution is based on solving Lagrangian relaxations with various values of μ and then taking the maximum of the objective function value over μ.

For $0 \le \mu \le 0.1$, the coefficients of x_1, x_2, x_3 and x_4 are all negative or zero, so, to minimize, we chose the maximum possible integers for x_1, x_2, x_3 and x_4, which are 3, 1, 2 and 3, correspondingly.

Question 20: For $0 \le \mu \le 0.1$, $\sup_{\mu} \inf_{x} L(x, \mu) =$ -1.8

Question 21: For $0.1 < \mu \le 0.2$, $\sup_{\mu} \inf_{x} L(x, \mu) =$ -1.7

Question 22: For $0.2 < \mu \le 0.4$, $\sup_{\mu} \inf_{x} L(x, \mu) =$ -1.7

Question 23: For $0.4 < \mu \le 0.5$, $\sup_{\mu} \inf_{x} L(x, \mu) =$ -2.1

Question 24: For $\mu > 0.5$, $\sup_{\mu} \inf_{x} L(x, \mu) =$ -2.5

Question 25: The max of $\inf_{x} L(x, \mu)$ over all non-negative μ is achieved at $\mu =$ .2

Next, we substitute μ in the relaxation program by the answer to the previous question.

Question 26: How many optimal solutions does this relaxation program have? 3

One of these solutions corresponds to an optimal solution to the original ILP. We solved it!

Question 27: Use a pen and paper and the Lagrangian dual to solve

$$\max x_1 + 2x_2 + 3x_3$$

s.t.

$$x_1 - 5x_2 - 3x_3 \le 1$$
$$x_1 + x_2 + x_3 \le 100$$
$$x_1, x_2, x_3 \ge 0$$
$$x_1, x_2, x_3 \in \mathbb{Z}$$

The optimal solution is (x_1, x_2, x_3) with x_1= 0

Question 28: (continued from the previous question) x_2= 0

Question 29: (continued from the previous question) x_3= 100

Lagrangian Duality, Continued

In a linear case, the Lagrangian dual is the LP dual as defined in module 3.

Recall from module 3 that the

$$\begin{array}{l} \min c^T x \\ Ax \ge b \\ x \ge 0 \end{array} \quad \text{and} \quad \begin{array}{l} \max b^T y \\ A^T y \le c \\ y \ge 0 \end{array} \quad \text{are LP duals to each other.}$$

Let's find the Lagrangian dual to $\begin{array}{l} \min c^T x \\ Ax \ge b \\ x \ge 0 \end{array}$.

If all the inequality constraints are moved into the objective function, and μ is a column vector of the constraint multipliers then the Lagrangian dual of

$$\begin{array}{l} \min c^T x \\ Ax \ge b \\ x \ge 0 \end{array} \quad \text{is} \quad \begin{array}{l} \max\limits_{\mu \ge 0} \inf\limits_{x} c^T x + \mu^T (b - Ax) \\ x \ge 0 \end{array} \quad \text{or} \quad \begin{array}{l} \max\limits_{\mu} \inf\limits_{x} b^T \mu + c^T x - \mu^T Ax \\ x \ge 0, \mu \ge 0 \end{array}$$

$\inf\limits_{x} b^T \mu + c^T x - \mu^T Ax$ is $-\infty$ if any of the components of $c^T - \mu^T A$ are negative or $b^T \mu$ otherwise.

$c^T - \mu^T A \ge 0$ is equivalent to $A^T \mu \le c$, so we have

$$\max_{\mu} b^T \mu$$
$$A^T \mu \le c$$
$$\mu \ge 0$$

Using the above approach, we can find a dual to any LPs. For example, the dual to

$$\begin{array}{l} \min c^T x \\ Ax \geq b \end{array} \quad \text{is} \quad \begin{array}{l} \max b^T y \\ A^T y 1? c \\ y 2? 0 \end{array}$$

and the dual to
$$\begin{array}{l} \min c^T x \\ Ax \leq b \end{array} \quad \text{is} \quad \begin{array}{l} \max b^T y \\ A^T y 3? c \\ y 4? 0 \end{array}.$$

Question 30: Replace the 1? with the correct expression.

- ≤
- =
- ≥
- none of the above

Question 31: Replace the 2? with the correct expression.

- ≤
- =
- ≥
- none of the above

Question 32: Replace the 3? with the correct expression.

- ≤
- =
- ≥
- none of the above

Question 33: Replace the 4? with the correct expression.

- ≤
- =
- ≥
- none of the above