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Application of singular value decomposition to vertical seismic profiling

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ABSTRACT

An essential part of the interpretation of vertical seismic profiles (VSP) is the separation of the upgoing and downgoing waves. This paper presents a new approach which is based on the decomposition of time-shifted VSP sections into eigenimages, using singular value decomposition (SVD). The first few eigenimages of the time-shifted VSP section contain the contributions of the horizontally aligned downgoing waves. The last few eigenimages contain the contribution of uncorrelated noise components. The separated upgoing waves are recovered as a partial sum of the eigenimages.

Important aspects of this approach are that regular sampling of the recording levels is not required, that the first-break times need not be measured with extreme accuracy, that noise rejection may be automatically included in the processing, and that eigenimages or sums of eigenimages which may be computed as part of the approach can provide important additional information.

INTRODUCTION

The use of vertical seismic profiling (VSP) has become an important tool in seismic surveys. An essential part of VSP interpretation, due to the relatively small amplitudes of the upgoing waves, is the separation of the upgoing and the downgoing wave components. This is of particular importance when VSP records are compared to conventional seismic profiles recorded on the surface.

A common approach to the separation of the two wave components, owing to their different apparent phase velocities, is velocity filtering in the frequency-wavenumber domain. This method was introduced by Treitel et al. (1967) and recently applied by Suprajitno and Greenhalgh (1985). One shortcoming of this approach is the requirement of regular sampling in both the time and space domains. Other methods which have been suggested are the subtraction technique of Hardage (1983), which uses median filtering and requires accurate determination of the time of first breaks, and an optimal least-squares method based on the theoretical formulation of up and downgoing waves in a stratified medium (Seeman and Horowicz, 1983).

This paper presents a different approach which does not require regularly sampled data or acoustic impedance information nor particular accuracy in the times of the first breaks. Our method is based on the singular value decomposition (SVD) of the time-shifted VSP data matrix and is particularly interesting because the SVD processed section is not only free from the strong downgoing wave component but has an increased signal-to-noise ratio for the upgoing wave component.

THE SVD APPROACH

Recently a seismic processing technique based on optimal extraction of multichannel information has been introduced. This approach is known in the literature either as the Karhunen-Loève or as the principal component transformation. The Karhunen-Loève method has been applied in several different ways, and excellent treatments are presented by Jones (1985) and Hemon and Mace (1978). The SVD approach used in the present paper is another way of viewing the Karhunen-Loève reconstruction technique, and we briefly derive the relationship between the Karhunen-Loève (or principal component) reconstruction approach and SVD in the Appendix. In spite of the fact that SVD is generally encountered as a powerful decomposition in matrix computations (Klema and Lamb, 1980; Ursin and Zheng, 1985), we prefer the SVD point of view because of the equivalent description used in image processing (Andrews and Hunt, 1977). We believe that the de-

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composition of the input data matrix into eigenimages, which is the SVD description, is a particularly illuminating one.

Let $\underset{\sim}{X}$ be a seismic data matrix which contains M traces each with N data points (generally $M < N$), i.e.,

$$\underset{\sim}{X} = \{x_{ij}\}, \quad i = 1, 2, \ldots, M; \quad j = 1, 2, \ldots, N. \tag{1}$$

The SVD of $\underset{\sim}{X}$ is given by

$$\underset{\sim}{X} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T, \tag{2}$$

where superscript T indicates transpose, r is the rank of $\underset{\sim}{X}$, $\mathbf{u}_i$ is the ith eigenvector of $\underset{\sim}{X}\underset{\sim}{X}^T$, $\mathbf{v}_i$ is the ith eigenvector of $\underset{\sim}{X}^T\underset{\sim}{X}$, and σ_i is the ith singular value of $\underset{\sim}{X}$. The singular values σ_i can be shown to be the positive square roots of the eigenvalues of the covariance matrices $\underset{\sim}{X}\underset{\sim}{X}^T$ or $\underset{\sim}{X}^T\underset{\sim}{X}$ (Lanczos, 1961); the eigenvalues are always real and positive due to the positive definite nature of covariance matrices. In equation (2) the factor $\mathbf{u}_i\mathbf{v}_i^T$ is an $(M \times N)$ matrix of unitary rank which is called the ith eigenimage of $\underset{\sim}{X}$ (Andrews and Hunt, 1977). Owing to the orthogonality of the eigenvectors, eigenimages form an orthogonal basis for the representation of $\underset{\sim}{X}$. As can be seen from the form of equation (2), the contribution to the construction of $\underset{\sim}{X}$ of the eigenimage associated with a given singular value is proportional to that singular value's magnitude. Since the singular values are always ordered in decreasing magnitude, the greatest contributions in the representation of $\underset{\sim}{X}$ are contained in the first few eigenimages.

Suppose that $\underset{\sim}{X}$ represents a seismic section and that all M traces are linearly independent, i.e., no trace may be represented in terms of a linear combination of the other $M - 1$ traces. In this case $\underset{\sim}{X}$ is of full rank M and all the σ_i are different from zero. Hence, the perfect reconstruction of $\underset{\sim}{X}$ requires all eigenimages. On the other hand, in the case where all M traces are equal to within a scale factor, all traces are linearly dependent; $\underset{\sim}{X}$ is of rank one and may be perfectly represented by the first eigenimage $\sigma_1\mathbf{u}_1\mathbf{v}_1^T$. In the general case, depending on the linear dependence which exists among the traces, $\underset{\sim}{X}$ may be reconstructed from only a few of the first eigenimages. In this case, the data may be considered to be composed of traces which show a high degree of trace-to-trace correlation. If only p, $p < r$, eigenimages are used to approximate $\underset{\sim}{X}$, a reconstruction error ε is given by

$$\varepsilon = \sum_{k=p+1}^{r} \sigma_k^2.$$

We can now define band-pass $\underset{\sim}{X}_{BP}$, low-pass $\underset{\sim}{X}_{LP}$, and high-pass $\underset{\sim}{X}_{HP}$ SVD images in terms of the ranges of singular values used. The band-passed image is reconstructed by rejecting highly correlated as well as highly uncorrelated traces and is given by

$$\underset{\sim}{X}_{BP} = \sum_{i=p}^{q} \sigma_i \mathbf{u}_i \mathbf{v}_i^T, \qquad 1 < p \le q < r. \tag{3}$$

The summation for $\underset{\sim}{X}_{LP}$ is from $i = 1$ to $p - 1$ and for $\underset{\sim}{X}_{HP}$ from $i = q + 1$ to r. It may be simply shown that the percentage of energy which is contained in a reconstructed image $\underset{\sim}{X}_{BP}$ is given by E, where

$$E = \frac{\sum_{i=p}^{q} \sigma_i^2}{\sum_{i=1}^{r} \sigma_i^2}. \tag{4}$$

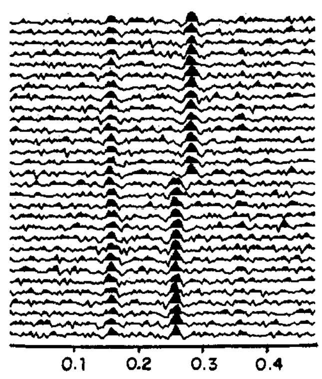
The choice of p and q depends on the relative magnitudes of the singular values, which are a function of the input data; we have determined these parameters from a plot of the eigenvalues λ_i given by $\lambda_i = \sigma_i^2$ as a function of the index i. This is reasonable given the form of equation (4). The shape of this plot depends on the trace-to-trace correlation in the data. In certain cases, an abrupt change in the eigenvalues is easily recognized. In other cases, the change in eigenvalue magnitude is more gradual and care must be exercised in the choice of the appropriate index value. We will illustrate this point with synthetic and real data examples.

Prior to the application of SVD to VSP, we illustrate the use of SVD in seismic problems with a very simple example. The example is not for the VSP problem at hand but serves to demonstrate very clearly the concepts of eigenimages which we have discussed above. Figure 1a represents a synthetic 30 trace seismic section showing three reflectors, one of which contains a fault. The section has been corrupted with additive pseudo-white noise with standard deviation of 20 percent of the maximum amplitude. Figure 1b shows the variation of the relative magnitudes of the eigenvalues. In this particular case Figure 1b shows that the signal portion of $\underset{\sim}{X}$ is contained in only the first two eigenimages. For purposes of illustration, Figure 2 shows certain eigenimages and sums of eigenimages. We note in particular that the second eigenimage shows the signature of the faulted reflector and the highly correlated horizontal information appears in the first eigenimage. The large improvement in signal-to-noise (S/N) ratio in $\underset{\sim}{X}_{LP}$ shown in Figure 2d represents the sum of only the first two eigenimages. Also no distortion in pulse shape has occurred. Figure 2b represents $\underset{\sim}{X}_{BP}$, given in equation (3), which in this particular case is composed of only the second eigenimage.

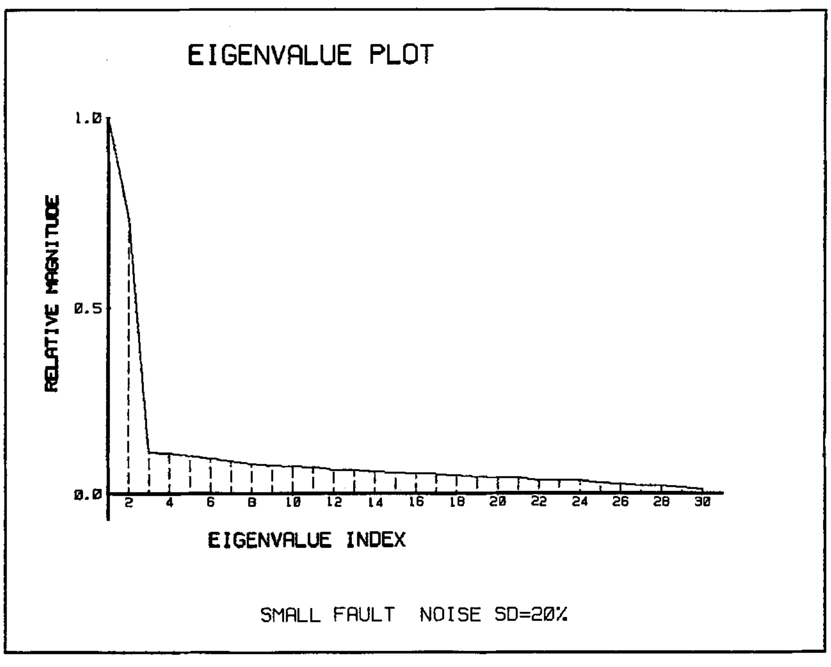
APPLICATION TO VSP

Our principal goal in VSP processing is the separation of the up and downgoing wave components. A subsidiary goal is to improve, where possible, the S/N ratio in the extracted upgoing signal. A necessary first step in the SVD approach is the picking of first breaks in the strong downgoing wave component. This may be accomplished visually or by any "picking" method available. Small errors in first break times do not have significant influence on the results obtained with SVD processing, since eigenimages which are associated with first break variations may always be included in the reconstruction. We specify four stages in the SVD processing of VSP data: preprocessing, static time shifting, SVD filtering, and stacking.

In the preprocessing stage we apply the usual gain correction and apply band-pass filtering if required (Hardage, 1983). Next, the time of first break of each trace is determined and subsequently the traces are aligned by shifting each one appropriately. The SVD filtering stage requires an examination of the eigenvalue magnitudes associated with the shifted data matrix; with the appropriate choice of p and q, the output data matrix is computed as the band-passed sum of eigenimages expressed in equation (3). The final stage consists of



(a)



(b)

FIG. 1. (a) A synthetic 30 trace seismic section showing three reflectors, the second reflector containing a fault. Additive white noise is 20 percent of the maximum signal amplitude. (b) Plot of the relative magnitude of the eigenvalues associated with data in (a).

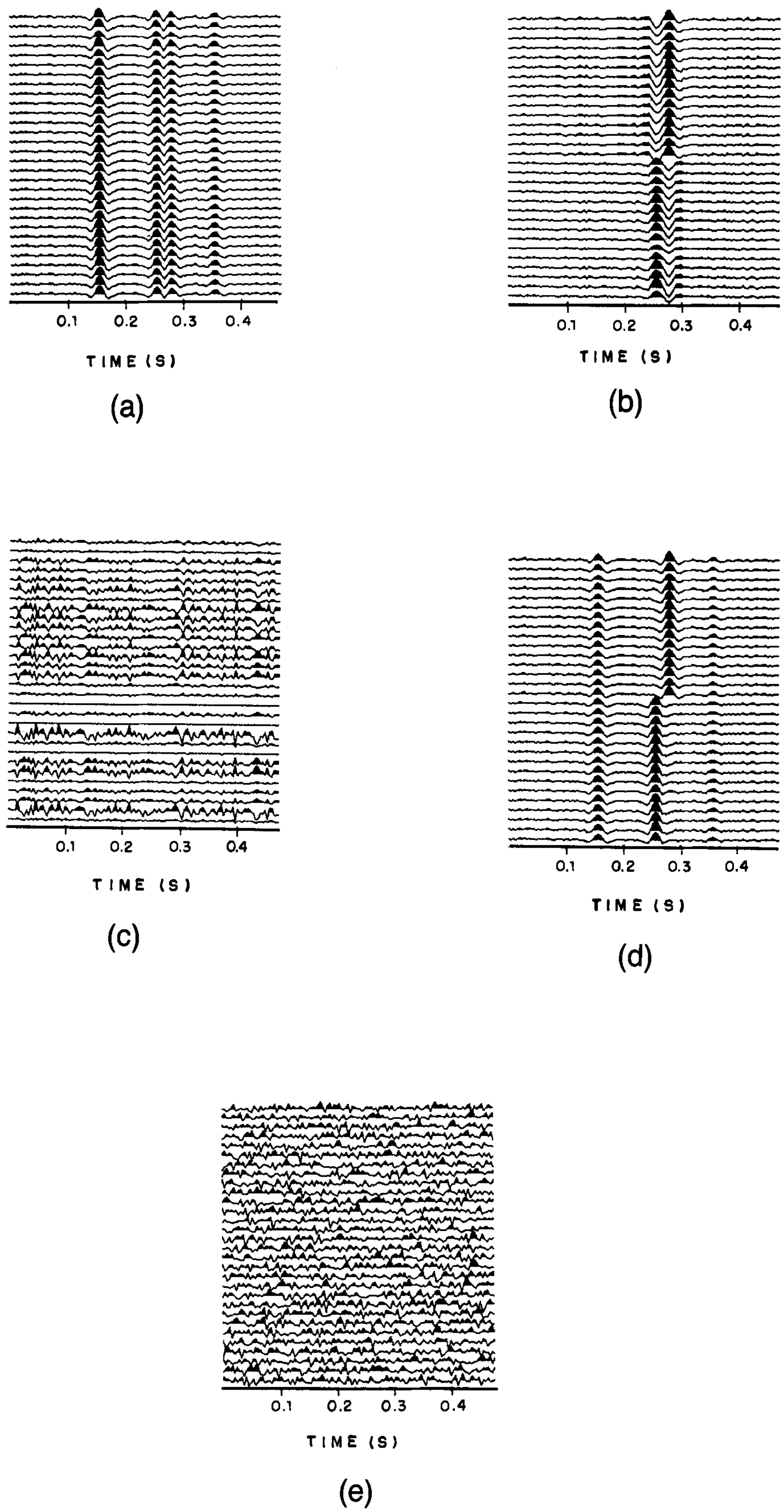
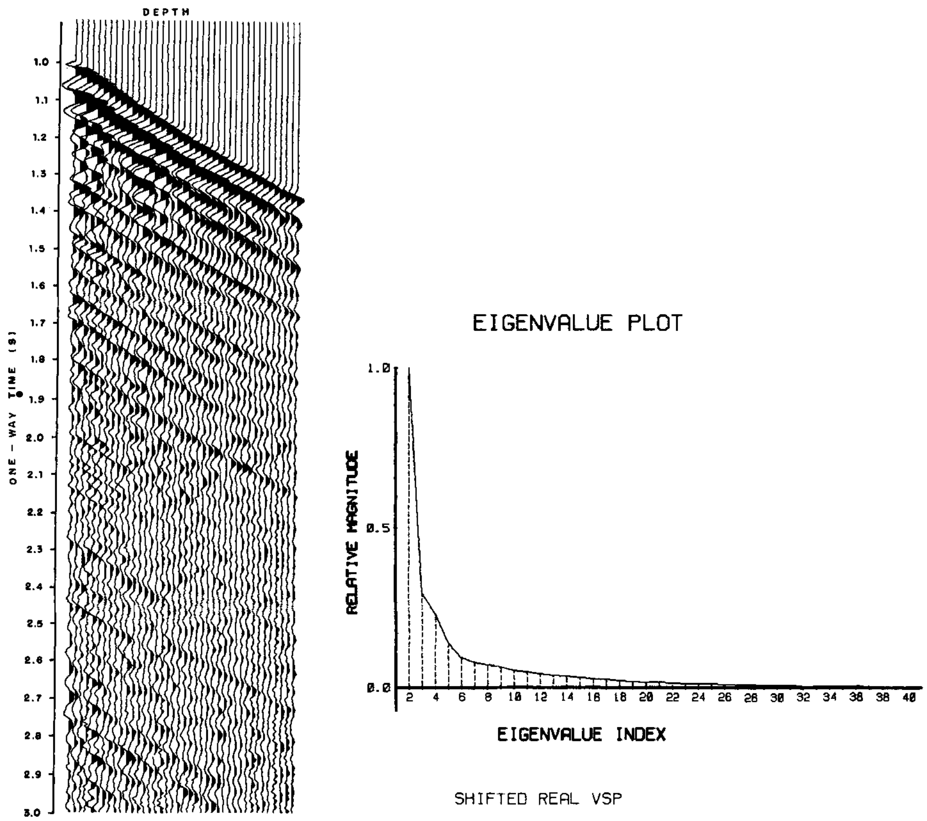
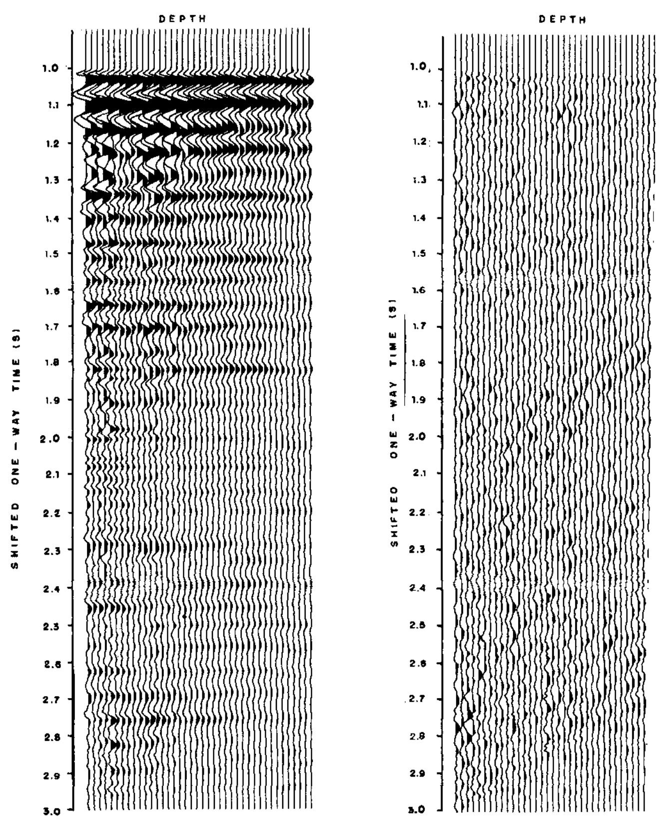


FIG. 2. (a) The first eigenimage of the data in Figure 1a. (b) The second eigenimage of the data in Figure 1a. (c) The third eigenimage of the data in Figure 1a. (d) $\mathbf{X}_{LP}$, $p = 3$, associated with Figure 1a. (e) $\mathbf{X}_{HP}$, $q = 2$, associated with Figure 1a.



(a) (b)

FIGS. 3a, 3b. (a) 40 trace VSP section showing data from the Campos Basin in Brazil. (b) A plot of the relative magnitude of the eigenvalues of the data in (a) time-shifted relative to the two first traces, with the first eigenvalue set to zero.



(c) (d)

FIGS. 3c, 3d. (c) $\underset{\sim}{\mathbf{X}}_{LP}$, $p = 6$, associated with the time-shifted input data. (d) $\underset{\sim}{\mathbf{X}}_{BP}$, $p = 6$ and $q = 28$, associated with the time-shifted input data.

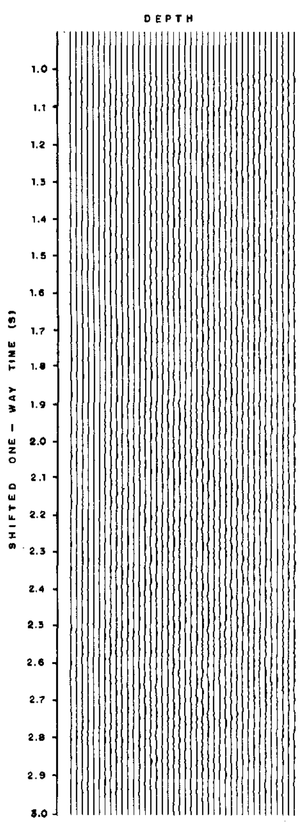
time shifting the recovered upgoing waves to their original positions, transferring these components into two-way traveltime using the computed first breaks, and stacking to produce a final trace (Seeman and Horowicz, 1983).

REAL DATA EXAMPLE

The VSP data considered here were recorded in the Campos Basin, Rio de Janeiro, in Brazil using an air-gun source. Figure 3a shows the input data, which consist of 40 traces with regular spatial sampling. First breaks used in the processing were picked manually. Figure 3a demonstrates very clearly the dominance of the downgoing wave component in VSP sections. Figure 3c shows the separated, time-shifted downgoing waves, which represent the sum of the first five eigenimages. This number was chosen by examination of the eigenvalues in Figure 3b. The separated upgoing waves are shown in Figure 3d and represent a band-passed image with $p = 6$ to $q = 28$. The rejected noise component is shown in Figure 3e and represents 0.7 percent of the total input energy and 7.4 percent of the energy in the separated upgoing wave component. The final upgoing wave components in the two-way traveltime domain are shown in Figure 3f and a stacked trace repeated ten times as before is shown in Figure 3g.

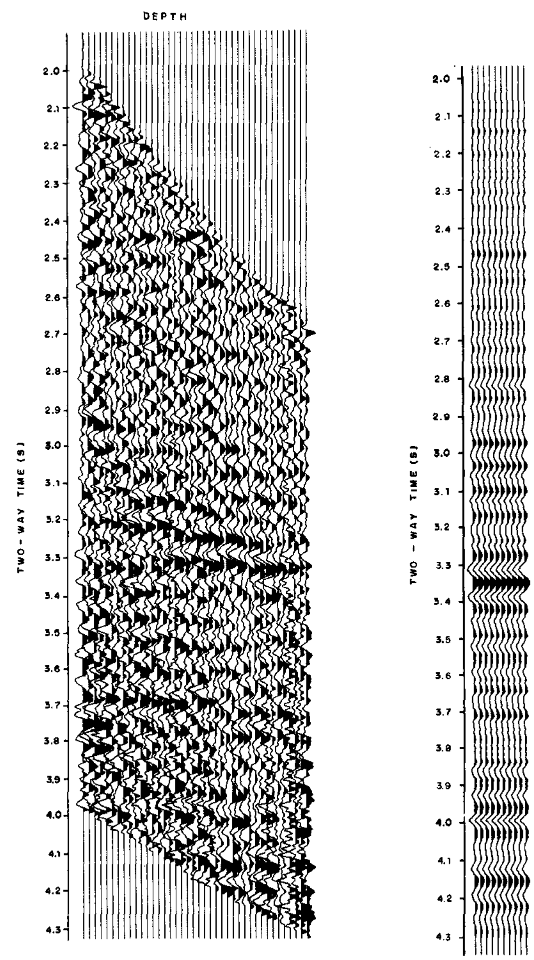
A visual analysis of the data in Figure 3f shows upgoing energy. Since other events are also present, application of SVD low-pass filtering to this section is immediately suggested. The plot of the relative magnitude of the eigenvalues, which is shown in Figure 3k, clearly demonstrates the importance of the first few eigenimages to the coherent upgoing signal. Interestingly the first eigenimage, Figure 3h, shows some potentially important characteristics. We observe that some of the traces in this image, for example trace 30, are virtually zero. Examination of the corresponding trace in Figure 3f shows an absence of upgoing energy. It appears that such traces in the first eigenimage may be used for data quality control.

We now construct a "best" upgoing wave section by rejecting uncorrelated energy. Figure 3i illustrates $\underset{\sim}{X}_{LP}$ with $p = 6$. This figure should be compared with Figure 3f. The increase in signal-to-noise energy is clear. Figure 3j shows a stack of data in Figure 3i repeated ten times for comparison with Figure 3g.



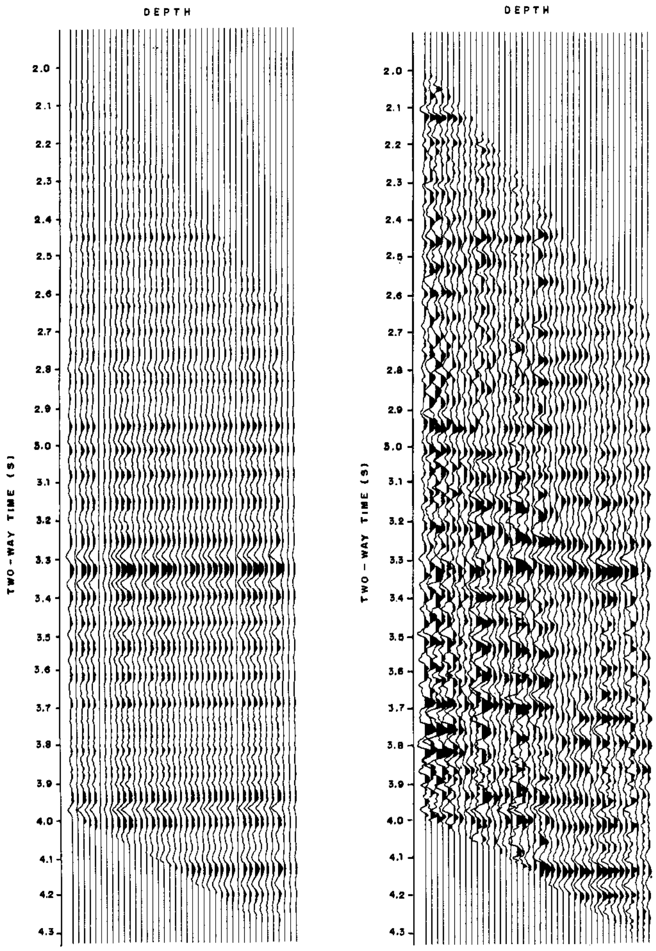
(e)

FIG. 3e. $\underset{\sim}{X}_{HP}$, $q = 28$, associated with the time-shifted input data.



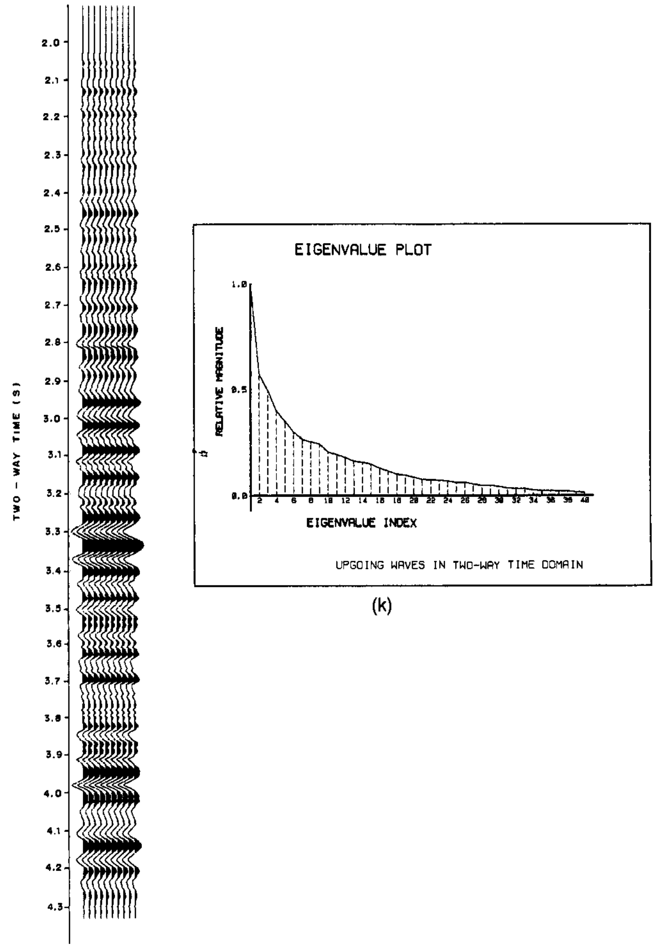
(f) (g)

FIGS. 3f, 3g. (f) The upgoing wave components transformed to the two-way traveltime domain. (g) The stacked upgoing wave trace repeated ten times.



(h) (i)

FIGS. 3h, 3i. (h) The first eigenimage associated with the data in (f). (i) $\underset{\sim}{X}_{LP}$, $p = 6$, associated with the data in (f).



(j) (k)

FIGS. 3j, 3k. (j) The stacked upgoing wave trace of the data in (f) after new SVD low-pass filtering. (k) A plot of the relative magnitude of the eigenvalues of the data in (f).

CONCLUSIONS

We have presented a new method of separating upgoing and downgoing waves in VSP sections. This approach offers the advantages that sampling between recording levels need not be regular and, as determined from tests on synthetic data examples, small errors in the first-break arrivals are not particularly significant. Further, noise rejection is a natural part of the processing scheme. We have found by application that the separation of the required upgoing wave component is performed efficiently in the presence of a downgoing wave component having an order of magnitude greater amplitude. The proposed processing scheme is easy to implement and the eigenimages represent a very useful and interesting decomposition of the input data. The choice of the band-pass indices, p and q, depend, naturally, on the data themselves. However, by means of examining both the eigenvalue magnitude variation and the reconstructed images, an optimal choice can be made. Finally, we would like to mention that the SVD approach presented here can be readily extended into the complex domain. This extension would allow the separation of components which may have suffered phase shifts for various reasons.

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APPENDIX

RELATIONSHIP BETWEEN SVD AND KARHUNEN-LOÈVE IMAGE RECONSTRUCTIONS

The SVD of a matrix $\underset{\sim}{X}$ is written in matrix form as

$$\underset{\sim}{X} = \underset{\sim}{U}\underset{\sim}{\Sigma}\underset{\sim}{V}^T, \tag{A-1}$$

where

- $\underset{\sim}{X}$ is an $(M \times N)$ matrix,
- $\underset{\sim}{U}$ is an $(M \times M)$ matrix, the columns of which are composed of the eigenvectors of $\underset{\sim}{X}\underset{\sim}{X}^T$,
- $\underset{\sim}{V}$ is an $(N \times N)$ matrix, the columns of which are composed of the eigenvectors of $\underset{\sim}{X}^T\underset{\sim}{X}$,

and

- $\underset{\sim}{\Sigma}$ is an $(M \times N)$ diagonal matrix with entries $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_M$, where σ_i is the ith singular value of $\underset{\sim}{X}$.

Let $\underset{\sim}{C}$ be the $(M \times M)$ covariance matrix of the input data computed as $\underset{\sim}{C} = \underset{\sim}{X}\underset{\sim}{X}^T$. Then we may write

$$\underset{\sim}{C} = \underset{\sim}{U}\underset{\sim}{\Lambda}\underset{\sim}{U}^T, \tag{A-2}$$

where $\underset{\sim}{\Lambda}$ is an $(M \times M)$ diagonal matrix composed of the eigenvalues of $\underset{\sim}{C}$.

The Karhunen-Loeve or principal component matrix of $\underset{\sim}{X}$ is given by $\underset{\sim}{Y}$, an $(M \times N)$ matrix, where

$$\underset{\sim}{Y} = \underset{\sim}{U}^T\underset{\sim}{X}. \tag{A-3}$$

Substituting equation (A-1) into equation (A-3), we obtain

$$\begin{aligned}\underset{\sim}{Y} &= \underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{\Sigma}\underset{\sim}{V}^T \\ &= \underset{\sim}{\Sigma}\underset{\sim}{V}^T.\end{aligned} \tag{A-4}$$

The principal components may be viewed as the inner product of the eigenvectors of $\underset{\sim}{X}\underset{\sim}{X}^T$ with the data, or as the weighted eigenvectors of $\underset{\sim}{X}^T\underset{\sim}{X}$.

Misadjustment reconstruction from the principal components is obtained from equation (A-3) as

$$\underset{\sim}{X} = \underset{\sim}{U}\underset{\sim}{Y} = \underset{\sim}{U}\underset{\sim}{\Sigma}\underset{\sim}{V}^T. \tag{A-5}$$

The computation of the SVD filtered data $\underset{\sim}{X}_{BP}$ can be more efficiently performed by computing the principal eigenvectors of either $\underset{\sim}{X}\underset{\sim}{X}^T$ or $\underset{\sim}{X}^T\underset{\sim}{X}$ depending on whether $M \leq N$ or $M > N$, respectively. Let $\underset{\sim}{U}_{BP}$ and $\underset{\sim}{V}_{BP}$ represent $\underset{\sim}{U}$ and $\underset{\sim}{V}$ with the first $(p - 1)$ columns and the last $(r - q)$ columns set to zero. Then we compute $\underset{\sim}{X}_{BP}$ as

$$\underset{\sim}{X}_{BP} = \underset{\sim}{U}_{BP}\underset{\sim}{U}_{BP}^T\underset{\sim}{X} = \underset{\sim}{X}\underset{\sim}{V}_{BP}\underset{\sim}{V}_{BP}^T. \tag{A-6}$$

The above development may be repeated for the complex case by replacing T, the matrix transpose, with H, the complex transpose.