Marble in a bowl

1 Solution 1: Newton's 2nd Law

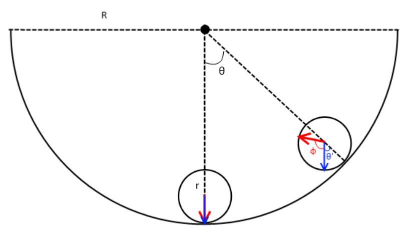

Figure 1: Rotation

By Newton's Second Law,

$$\begin{aligned} ma_{tan} &= mg\sin(\theta) - f \\ &= -m(R-r)\ddot{\theta} \end{aligned} \tag{1.1}$$

The marble rotates by ϕ from a stationary frame, due to friction.

$$\tau_\phi = \frac{2}{5}mr^2\ddot{\phi} \tag{1.2}$$

$$= rf \tag{1.3}$$

Figure 1 shows the rotation of the marble. The blue arrow represents the frame reference and remains vertical. The red arrow represents a reference on the marble. We can think of it like using a marker to draw a line on the marble. This helps us visualise ϕ.

The arc-length covered on the bowl is

$$R\theta = -r(\theta' + \phi) \tag{1.4}$$

Note that ϕ and θ' are in the opposite direcction of θ. We can also see that the relation will extend to angular acceleration, and thus

$$\begin{aligned} R\ddot{\theta} &= -r(\ddot{\theta}' + \ddot{\phi}) \\ \ddot{\theta}' &= -\ddot{\theta} \\ \ddot{\phi} &= -\frac{R-r}{r}\ddot{\theta} \end{aligned} \tag{1.5}$$

Making use of 1.2, 1.3 and 1.5, we get

$$f = -\frac{2}{5}m(R-r)\ddot{\theta} \tag{1.6}$$

Combine 1.6 and 1.1, applying the small angle approximation ($\sin\theta \approx \theta$), and then making use of the SHM equation, we should obtain

$$\omega = \sqrt{\frac{5g}{7(R-r)}} \tag{1.7}$$

1.1 Comments

A common mistake here is to assume $R\theta = -r\phi$, which is incorrect from our frame of reference using newton's laws as we have shown above.

One must also be very careful about the sign convention used here, as there are a few different angles and rotations to consider.

2 Solution 2: 2 Rotations

In certain questions, using a system of 2 rotations might give us a simpler solution. We try that here.

We first take the reference to be the center of the bowl following the marble, so that the marble without extra rotations will rotate about an axis in the center of the bowl, i.e. the marble always has the same point of contact on the bowl. Using the parrallel axis theorem

$$\begin{aligned} I_\theta &= I_\phi + m(R-r)^2 \\ &= \frac{2}{5}mr^2 + m(R-r)^2 \end{aligned} \tag{2.1}$$

This is the moment of inertia for our first rotation, giving us

$$I_\theta\ddot{\theta} = Rf - (R-r)mg\sin\theta \tag{2.2}$$

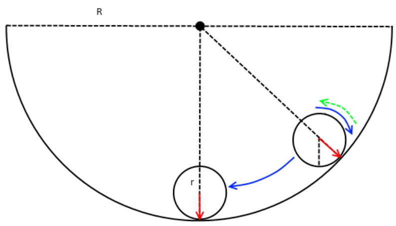

Figure 2: Added rotation

However, out first rotation also includes a rotation for the marble about its center, as seen from the blue and red arrow below.

This means that for the same second rotation of the marble about its center, there will be a ficticuous torque in the other direction. This torque is in the same direction as the second rotation but opposite to the first.

$$\begin{aligned}\tau_f &= -mr^2\ddot{\theta} \\ \tau_\phi &= \frac{2}{5}mr^2\ddot{\phi} \\ &= rf + \tau_f \\ &= rf - mr^2\ddot{\theta} \qquad (2.3)\end{aligned}$$

Using 1.5, and then combining f into 2.2

$$f = -\frac{2}{5}m(R-2r)\ddot{\theta} \qquad (2.4)$$

$$\ddot{\theta} + \frac{5g}{7(R-r)}\sin\theta = 0$$

$$\omega = \sqrt{\frac{5g}{7(R-r)}} \qquad (2.5)$$

2.1 Comments

In this attempt, we see that instead of a simpler, we had made it worse. Though there are some questions where using a system of two rotations would simplfy things, this is certainly not one of them. Overall, it would have been better to stick to using newton's laws for this question.

3 Solution 3: Lagrangian (with calculus)

The formula for Lagrangian is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = 0 \tag{3.1}$$

$$L = T - V \tag{3.2}$$

$$v = (R - r)\dot{\theta} \tag{3.3}$$

$$\dot{\phi} = -\frac{R-r}{r}\dot{\theta} \tag{3.4}$$

For T, the kinetic energy of the system,

$$\begin{aligned} T &= \frac{1}{2}mv^2 + \frac{1}{2}I_o\dot{\phi}^2 \\ &= \frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{2} \times \frac{2}{5}mr^2\frac{(R-r)^2}{r^2}\dot{\theta}^2 \\ &= \frac{7}{10}m(R-r)^2\dot{\theta}^2 \end{aligned} \tag{3.5}$$

For V, the potential energy of the system,

$$V = mg(R-r)(1-\cos\theta) \tag{3.6}$$

The lagrangian L is

$$\begin{aligned} L &= T - V \\ &= \frac{7}{10}m(R-r)^2\dot{\theta}^2 - mg(R-r)(1-\cos\theta) \\ &= m(R-r)[\frac{7}{10}(R-r)\dot{\theta}^2 - g(1-\cos\theta)] \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) &= 2 \times \frac{7}{10}m(R-r)^2\dot{\theta} \\ &= \frac{7}{5}m(R-r)^2\ddot{\theta} \end{aligned} \tag{3.8}$$

$$\frac{\partial L}{\partial \theta} = -mg(R-r)\sin\theta \tag{3.9}$$

Substituting into 3.1, one obtains

$$\frac{7}{5}m(R-r)^2\ddot{\theta} + -mg(R-r)\sin\theta = 0 \tag{3.10}$$

yielding ω to be

$$\omega = \sqrt{\frac{5g}{7(R-r)}} \tag{3.11}$$

3.1 Comments

The main advantage of this method is that the sign convention is not very important, and is useful if you find it difficult to visualise the different rotations.

4 Conclusion

The key takeaway will be to always understand the assumptions you make while answering the question. Are what conditions are your assumptions valid? In this situation are the conditions satisfied?

Futhermore, we must also be aware about how the assumptions we made at the start will affect the rest of the solution. For example, for solution 2, the frame we assumed at the start caused an additional ficticious force later on.