SIGMA 202 B

Estimation spectrale non paramétrique

A-1

rappels: $\gamma(t,s) = E((X_t - \mu_t)(X_s - \mu_s)^*)$

Stationnaire 2nd ordre

$$\gamma(t,s) = \gamma(t-s)$$

$$\gamma(k) = E((X_t - \mu)(X_{t-k} - \mu)^*)$$

$$\gamma(-k) = \gamma(k)^*$$

Mesure spectrale

$$\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi hf} \, \nu(df) \quad \forall h \in \mathbb{Z}$$

$\{\gamma(k)\}$ positif (Herglotz) $\Leftrightarrow$ $S_{XX}(f) = \sum_h \gamma(h) e^{-j2\pi f h} \geq 0 \quad \forall f \in [-\frac{1}{2}, \frac{1}{2}[$

Filtrage : $Y_t = \sum_{k \in \mathbb{Z}} \psi_k X_{t-k}$ $\quad \sum |\psi_k|^2 < \infty$

$$\gamma_{YY}(h) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \psi_j \psi_k^* \gamma_{XX}(h+k-j)$$

mesure spectrale $\quad S_{YY}(f) = |\Psi(e^{-j2\pi f})|^2 S_{XX}(f)$

$$\Psi(e^{-j2\pi f}) = \sum_k \psi_k e^{-j2\pi f k}$$

Periodogramme : $\hat{S}_{P,XX}(f) = \frac{1}{N} \left| \sum_{t=1}^{N} X_t e^{-j2\pi f t} \right|^2$

Corrélogramme

A-2

$$\hat{S}_{C,XX}(f) = \sum_{k=-N+1}^{N-1} \hat{r}(k) e^{-j2\pi f k}$$

estimation non biaisée :

$$\hat{r}(k) = \frac{1}{N-k} \sum_{t=k+1}^{N} X_t X_{t-k}^*$$

estimation biaisée :

$$\hat{r}(k) = \frac{1}{N} \sum_{t=k+1}^{N} X_t X_{t-k}^* \quad \leftarrow \text{OK car } \varphi_c(f) \geq 0$$

$$\hat{S}_{P,XX}(f) = \hat{S}_{C,XX}(f) \quad !$$

Dem :

(1) $$Z_t = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_k e_{t-k} \qquad e_t = \text{bruit blanc.}$$

filtre $$x(f) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} x_k e^{-j2\pi f k} \Longrightarrow |x(f)|^2 = \hat{S}_{P,XX}(f)$$

$$S_{ee}(f) = 1 \Longrightarrow \varphi_Z(f) = |x(f)|^2 \Rightarrow \varphi_Z(f) = \hat{S}_{P,XX}(f).$$

(2) $$r_Z(k) = E(Z_t Z_{t-k}^*)$$

$$= \frac{1}{N} \sum_{p=1}^{N} \sum_{s=1}^{N} x_p x_s^* \underbrace{E(e_{t-p}\, e_{t-k-s}^*)}_{\delta(p,\, k+s)}$$

$$= \frac{1}{N} \sum_{p=k+1}^{N} x_p x_{p-k}^*$$

$$= \begin{cases} \hat{r}(k) & k = 0..N-1 \\ 0 & k \geq N \end{cases}$$

$$\varphi_Z(f) = \sum_{k=-N+1}^{N-1} \hat{r}(k) e^{-j2\pi f k} = \hat{S}_{C,XX}(f)$$

$$\Rightarrow \hat{S}_{P,XX}(f) = \hat{S}_{C,XX}(f)$$

A-3

$$MSE = E(|\hat{a}-a|^2) = E(|\hat{a} - E(\hat{a}) + E(\hat{a}) - a|^2)$$
$$= E(|\hat{a} - E(\hat{a})|^2) + E(|E(\hat{a}) - a|^2)$$
$$+ 2\,\mathrm{Re}\big(\underbrace{E(\hat{a} - E(\hat{a}))}_{=0}{}^{*}(E(\hat{a}) - a)\big)$$
$$= \mathrm{var}(a) + |\mathrm{biais}(a)|^2$$

Biais du périodogramme.

$$E(\hat{S}_{p,xx}(f)) = E(\hat{S}_{c,xx}(f)) = \sum_{k=-N+1}^{N-1} E(\hat{r}(k))\, e^{-j2\pi f k}.$$

$$E(\hat{r}(k)) = \left(1 - \frac{k}{N}\right) r_k$$
$$E(\hat{r}(-k)) = \left(1 - \frac{k}{N}\right) r_{-k}$$

$$E(\hat{S}_{p,xx}(f)) = \sum_{k=-(N-1)}^{N-1} \underbrace{\left(1 - \frac{|k|}{N}\right)} r(k)\, e^{-j2\pi f k}$$

$$\text{definir} \Rightarrow W_B(k) = \begin{cases} 1 - \frac{|k|}{N} & k \in [-N+1, N-1] \\ 0 & \text{si } |k| \geq N \end{cases}$$

$$E(\hat{S}_{p,xx}(f)) = \sum_{k=\infty}^{\infty} W_B(k)\, r(k)\, e^{-j2\pi f k}$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{xx}(g)\, W_B(f-g)\, dg$$

$$\boxed{\lim_{N\to\infty} E(\hat{S}_{p,xx}(f)) = \varphi(f)}$$

$$W_B(f) = \sum_{k=-N+1}^{N-1} \frac{N - |k|}{N}\, e^{-j2\pi f k}$$
$$= \frac{1}{N} \sum_{t=1}^{N} \sum_{s=1}^{N} e^{-j2\pi f (t-s)}$$
$$= \frac{1}{N} \left|\sum_{t=1}^{N} e^{-j2\pi f t}\right|^2 = \frac{1}{N} \left|\sum_{t=1}^{N} \left(e^{-j2\pi f}\right)^t\right|^2$$
$$= \frac{1}{N} \left|\frac{e^{j2\pi f N} - 1}{e^{j2\pi f} - 1}\right|^2 = \frac{1}{N} \left|\frac{e^{-j2\pi f\frac{N}{2}} - e^{-j2\pi f\frac{N}{2}}}{e^{j2\pi f/2} - e^{j2\pi f/2}}\right|^2$$
$$= \frac{1}{N} \frac{\sin(\pi f N)}{\sin(\pi f)}$$

(noyau de Fejer)

A-4

2 effets :

traînage ou dispersion (smearing)

perte de resolution. $\rightarrow$ limite en $\frac{1}{N}$

Ne pas confondre avec la précision (zero padding)

Variance du périodogramme.

BB complexe (ou circulaire)

$$\begin{cases} E(e_t e_s^*) = \sigma^2 \delta_{t,s} \\ E(e_t e_s) = 0 \quad \forall t \, \forall s. \end{cases}$$

$$\Leftrightarrow \begin{cases} E(Re(e_t) Re(e_s)) = \frac{1}{2}\sigma^2 \delta_{t,s} \\ E(Im(e_t) Im(e_s)) = \frac{1}{2}\sigma^2 \delta_{t,s} \\ E(Re(e_t) Im(e_s)) = 0 \end{cases}$$

Variance de $P_{ee}(f)$ $\quad$ BB $\Rightarrow S_{ee}(f) = \sigma^2$

$$\lim_{N\to\infty} E\left([\hat{S}_{P,ee}(f) - S_{ee}(f)][\hat{S}_{P,ee}(g) - S_{ee}(g)] \right) = \begin{cases} S_{ee}(f)^2 & f=g \\ 0 & f\neq g \end{cases}$$

(rappel : $\lim_{N\to\infty} E(P_{ee}(f)) = S_{ee}(f)$)

Montrons que

$$E(\hat{S}_{P,ee}(f)\hat{S}_{P,ee}(g)) = S_{ee}(f) S_{ee}(g) + S_{ee}(f)^2 \delta_{f,g}.$$

Lemme : (Gaussiennes centrées).

$$E(abcd) = E(ab)E(cd) + E(ac)E(bd) + E(ad)E(bc) - 2\,E(a)E(b)E(c)E(d)$$

$$E(\varphi_p(f)\varphi_p(g)) = \frac{1}{N^2} \sum_{t=1}^{N}\sum_{s=1}^{N}\sum_{p=1}^{N}\sum_{m=1}^{N} E(e_t e_s^* e_p e_m^*) . e^{-j2\pi f(t-s)} e^{-j2\pi f(p-m)}$$

$$E(e_t e_s^* e_p e_m^*) = E(e_t e_s^*)E(e_p e_m^*) + \cancel{E(e_t e_p)E(e_s e_m)^*} + E(e_t e_m^*)E(e_s^* e_p)$$
$$= \sigma^4 (\delta_{t,s}\delta_{p,m} + \delta_{t,m}\delta_{s,p})$$

(A.5)

$$E(\hat{S}_{p,ee}(f)\hat{S}_{p,ee}(g)) = \frac{\sigma^4}{N^2}\sum_{t=1}^{N}\sum_{s=1}^{N}\sum_{p=1}^{N}\sum_{m=1}^{N}(\delta_{t,s}\delta_{p,m}+\delta_{t,m}\delta_{s,p})\, e^{-j2\pi f(t-s)}\, e^{-j2\pi g(p-m)}$$

$$= \sigma^4 + \frac{\sigma^4}{N^2}\left|\sum_{t=1}^{N} e^{-j2\pi(f-g)t}\right|^2$$

$$= \sigma^4 + \sigma^4\left|\frac{\sin \pi(f-g)N}{N\sin \pi(f-g)}\right|^2$$

$$\lim_{N\to\infty} E(\hat{S}_{p,ee}(f)\hat{S}_{p,ee}(g)) = \sigma^4 + \sigma^4\,\delta_{f,g} \qquad \text{QED}$$

La relation est vraie aussi pour bruit blanc filtré.
on ne fait pas la preuve.

BLACKMAN – TUKEY

$$\hat{S}_{BT,XX}(f) = \sum_{k=-(M-1)}^{M-1} w(k)\,\gamma(k)\,e^{-j2\pi fk}$$

$w(k) = w(-k)$ $\qquad$ $w(k)=0 \quad \forall k \geq M$

$$W(f) = \sum_{t=-\infty}^{\infty} w_k e^{-j2\pi fk} = \sum_{t=-(M-1)}^{M-1} w_k e^{-j2\pi fk} \qquad W(f)\in\mathbb{R}\,!$$

$$\hat{S}_{BT,XX}(f) = \int_{-1/2}^{1/2}\hat{S}_{P,XX}(g)\,W(f-g)\,dg$$

moyenne pondérée locale.

Si $\{w(k)\}$ est positive semi définie $\Longrightarrow$ $\hat{S}_{B,XX}(f) \geq 0 \ \forall f$

(sans demonstration)

(Le produit d'Hadamard de 2 matrices définies positives est déf. positive)

BARTLETT

découpe N échant. en L sous-éch de taille $M = \frac{N}{L}$

$$\hat{S}_{B,xx}(f) = \frac{1}{L} \sum_{j=1}^{L} \frac{1}{M} \left| \sum_{i=1}^{M} \tilde{X}_{i,t} \, e^{-j2\pi f t} \right|^2$$

$$\tilde{X}_{i,t} = X_{(i-1)M+t} \quad \begin{cases} t \in [1, M] \\ j \in [1, L] \end{cases}$$

résolution réduite en $\frac{1}{M}$.

WELCH

$$\tilde{X}_{i,t} = X_{(i-1)K+t} \quad \begin{cases} t \in [1, M] \\ j \in [1, S] \end{cases}$$

si $K = M \Rightarrow$ Bartlett $S = L = \frac{N}{M}$

recommandé : $K = \frac{M}{2}$ $S = \frac{2M}{N}$

$$\hat{S}_{W,xx}(f) = \frac{1}{S} \sum_{j=1}^{S} \hat{S}^{(i)}_{P,xx}(f) \qquad S$$

avec $\hat{S}^{(i)}_{P,xx} = \frac{1}{P} \times \frac{1}{M} \left| \sum_{t=1}^{M} v(t) \, \tilde{X}_{i,t} \, e^{-j2\pi f t} \right|^2$

$P = \frac{1}{N} \sum_{t=1}^{M} |v(t)|^2 \longrightarrow$ pour normaliser chaque périodogramme.

DANIEL

(lissage fréquentiel

$$\hat{S}_{D,xx}(f) = \frac{1}{2J+1} \sum_{j=-J}^{J} \hat{S}_{P,xx}\left(f + \frac{j}{\tilde{N}}\right) \quad \begin{cases} \tilde{N} = N \text{ sans padding} \\ \tilde{N} > N \text{ si padding} \end{cases}$$

C'est un cas particulier de Blackman Tukey

$$W(f) = \begin{cases} 1/\beta & \text{si } f \in [-\beta, \beta] \\ 0 & \text{sinon.} \end{cases} \qquad \beta = \frac{2J}{\tilde{N}}$$