High-Effort Crowds: Limited Liability via Tournaments

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November 29, 2022

Abstract

Consider the crowdsourcing setting where agents are strategic in both exerting a proper level of effort and manipulating their reports in response to the assigned tasks. Casting it as a principal-agent problem, we investigate the design of payment mechanisms that satisfy the limited liability (all payments are non-negative) to incentivize truthful reports at a desired effort level with a small cost of budget. The payment mechanism is composed of a *performance measurement* which noisily evaluates agents' effort based on their reports, and a *payment function* which transforms the scores output by the performance measurement as the payments.

Under this framework, we first observe that although the linear payment function preserves the truthfulness of the performance measurement (e.g. peer prediction mechanisms or spot-checking mechanisms), it is very inefficient in eliciting a goal effort. We thus suggest applying a rank-order payment function (tournament) on agents' scores, and analytically optimize it in an idealized setting where the performance measurements are assumed to have Gaussian noise. We further identify a sufficient statistic of the quality of the performance measurements, called the sensitivity, which serves as a new dimension of evaluation. Finally, we show that adding a common shock to agents' scores can preserve the truthfulness of the performance measurements under the non-linear winner-take-all payment function.

Our real-data estimated agent-based model experiments reinforce our theoretical results and show that the proposed mechanism can greatly reduce the payment of effort elicitation while preserving the truthfulness of the performance measurement. In addition, we empirically evaluate several commonly considered performance measurements in terms of their sensitivities and strategic robustness.

1 Introduction

Crowdsourcing, on platforms like Amazon Mechanical Turk, suffers from incentive problems. The requesters would like to pay the workers to incentivize effortful reports. However, workers can increase their payments by spending less time on each task and answering more tasks, which could wastefully spend the requesters' budgets. At the extreme, which has been extensively studied [4, 26], workers may answer with little effort or even randomly.

In settings with continuous effort, it matters not just whether workers exert effort, but how much effort they exert. There may be lackadaisical workers, who provide mediocre-effort work. For example, while labeling tweets for content moderation, people can report whatever is in their minds after reading the first sentence instead of carefully reading the whole tweet, or they can work on a fraction of tweets while skipping the rest. In these and many other cases, effort is not simply binary, but measured on a continuum. Evidence suggests that lackadaisical behaviors may be ubiquitous on crowdsourcing systems. In one study, 46% of Mechanical Turk workers failed at least one of the validity checks which was twice the percentage in student groups [2].

The real problem is more complicated as agents can be strategic not only in exerting different levels of effort but also in misreporting their true answers in an attempt to increase their rewards. Truth-telling is desired as it is the cornerstone of collecting informative and high-quality data via crowdsourcing.

In this paper, we aim to provide a framework of designing payment mechanisms that can incentivize truthful reports at a desired effort level with a small cost of budget.[1] Eliciting effortful and truthful reports in such a crowdsourcing setting requires dealing with two problems: how to evaluate the quality of agents' reports, and how to reward agents given the potentially noisy evaluations.

Performance Measurements. Towards the first problem, previous literature on spot-checking mechanisms [12, 28] and peer prediction mechanisms [4, 26, 20] has provided us with powerful tools. The former score agents based on their performances on a subset of the tasks with known ground truth while the latter score each agent according to the correlations in her reports and her peers' reports which works in the absence of ground truth. Collectively, we call these mechanisms the *performance measurements* which map agents' reports to their *performance scores*.

At a high level, the performance measurements evaluate the quality of agents' reports based on how informative they are. Therefore, on one hand, the performance score can be used to guarantee truthfulness: truthfully reporting is an equilibrium where any unilateral manipulation will decrease the expected score.[2] On the other hand, the performance score serves as a noisy measurement of the agent's effort, and thus can be used to promote effort.[3] However, a more precise characterization of the effort elicitation has only been put forth in the binary effort setting and with no consideration of the cost of the requesters' budgets.

Limited Liability and Payment Functions. To effectively elicit a desired effort from agents in the continuous effort setting, the requester must carefully choose a *payment function* to transform the performance scores to the final payments. Existing literature on spot-checking and peer prediction suggests applying a linear payment function which derive payments by linearly rescaling the performance scores. Although the linear payment function trivially preserves the truthfulness of the performance measurements, it typically violate limited liability—they rely on negative payments. Additionally requiring the payments to be non-negative may lead to extravagant payments. For example, say a desired equilibrium (e.g. all agents working with full effort) pays an agent \$10 in expectation, a possible deviation (e.g. working with 90% of effort) pays her \$9.9 in expectation, and (due to the variance of payments) the minimum payment is 0 in both cases. Suppose exerting full effort costs the agent \$1 worth of effort more than exerting 90% of effort. Thus, to maintain the desired equilibrium, the payment function has to rescale the payments to compensate this \$1 difference in the cost of effort. However, because of limited liability, the linear payment function cannot subtract a constant (\$9.9) from the payments, and is forced to rescale the payment by 10 which pays each agent \$100 (10 times the actual cost). Such a problem is even troublesome for performance measurements whose scores are unbounded below.[4]

We consider payment functions that satisfy limited liability which is always preferred and often required in realistic settings. Specifically, we consider the *rank-order payment function (RO-payment function)* where an agent's payment only depends on the rank of her score among the other agents. Rank-order payments are broadly studied as tournaments [6, 7, 14] and contests [9, 17]. Such a technique is preferred especially in the peer prediction setting where an agent's performance score depends on others' reports, which makes it unfair to base the payments on the absolute values of the performance scores. Furthermore, RO-payment functions are easier to implement[5] and trivially bound the ex-post budget. However, a concern of using RO-payment functions is that their nonlinear translations of performance scores into payments do not, in general, preserve the truthfulness guarantees of the performance measurements.

The Principal-Agent Problem If we focus on eliciting effort and ignore, for now, the concern of truthfulness, we face a principal-agent problem. As a running example, suppose a principal wants to recover the ground truth of a batch of tasks using the collected labels from a group of homogeneous agents, who have the same utility function and information structure.[6] The principal first commits to a *payment mechanism* which

[1] As we will see, sometimes eliciting a goal effort in an equilibrium (discouraging deviations) can be much more costly than just compensating agents' cost of effort.

[2] More precisely, for spot-checking mechanisms, truthful reporting is the best-response; while for peer prediction mechanisms, all agents reporting truthfully is an equilibrium.

[3] The noise may be due to a relatively small number of tasks each agent answers or a lack of ground truth upon which to base the agent's payment. Both are common in the crowdsourcing setting.

[4] There does not exist a linear payment function to guarantee limited liability if the performance score can be negative infinity.

[5] as sometimes it is much easier to evaluate the relative values rather than the absolute values of the contributions

[6] Although not without loss of generality, homogeneous agents are widely assumed in the principle-agent works [9, 24]. The selection process could result in increased homogeneity among agents' background. Furthermore, agents are homogeneous while

consists of a performance measurement and a RO-payment function. Next, agents best-response the payment mechanism by exerting the same effort that forms a symmetric equilibrium, i.e. no unilateral deviation in effort can increase an agent's expected utility. The problem of the principal is to optimize the payment mechanism so that a goal effort can be elicited in the symmetric equilibrium with the minimum cost of budget.

To put forth theoretical analysis, we optimize the payment mechanism in an idealized setting where the noise of the performance measurement follows the Gaussian distribution whose mean and standard deviation are functions of agents' effort.[7] First, given an idealized performance measurement, we provide the analytical solutions of the optimal RO-payment functions for three types of agents: risk/loss-neutral, risk-averse and loss-averse. While the similar problem has been studied as tournaments in the economics literature, our results fill the gap by considering individual rationality (IR) to be a hard constraint of the optimization problem. Requiring the function to pay agents at least their cost of effort, we observe that IR, while binding, results in optimal RO-payment functions that are more inclusive (reward more agents).

Second, under the optimal RO-payment function, we optimize the idealized performance measurement. We derive a sufficient statistic called the *sensitivity* which serves as a new dimension of evaluating a performance measurement in terms of how much payment is required to elicit a certain effort. At a high level, a performance measurement with higher sensitivity is more accurate (has lower variance) and is more sensitive to changes in effort.

The effectiveness of using rank-order payment functions to elicit effort is confirmed by our real-data estimated agent-based models (ABMs). Our ABM experiments show that given a performance measurement, the optimized RO-payment function can greatly reduce the payment spent on eliciting a goal effort compared with the linear payment function.

Truthfulness Under The Rank-Order Payment Function Now, we know that RO-payment functions are very effective in effort elicitation. The question is therefore whether we can obtain truthfulness at the same time. In this paper, we provide a positive answer. Consider the winner-take-all tournament[8]as an example. The concern is that the agent may misreport to increase the variance of her performance score, which helps her improve the chance of winning the reward, even at a cost of the expected score. A similar issue also appears and has been dealt with in the forecast competition setting [11]. Inspired by their solution, we show that adding a common shock (a Gaussian noise with identical standard deviation) to all agents' performance scores can preserve the truthfulness of the performance measurement. Although the common shock may decrease the sensitivity of the performance measurement, we empirically show that such an idea can give us a truthful payment mechanism that is still much more effective in eliciting effort than linear payments.

Evaluating Realistic Performance Measurements Our paper puts forward a new dimension of evaluating performance measurements: how effective are they to elicit a desired effort with a low cost. We show that under the Gaussian assumption[9], two properties determine the effectiveness of a performance measurement: 1) the sensitivity and 2) how robust it is to prevent untruthful deviations from increasing the variance of the performance score, called the *variational robustness*. We further empirically evaluate several commonly considered spot-checking and peer prediction mechanisms in terms of these two properties. Our ABM results provide suggestions for which mechanism to use in real crowdsourcing settings.

We summarize our contributions as follows.

- We propose a two-stage framework to develop payment mechanisms with limited liability which turns the problem of incentivize crowdousrcing workers for a desired effort into a principal-agent problem;
- We analytically solve the principal's optimization problem of the rank-order payment function under the constraint of individual rationality which fill the gap of previous literature in tournament;
- We identify a sufficient statistic of the effectiveness of the performance measurement in effort elicitation, which provides a new dimension for evaluating the spot-checking and peer prediction mechanisms;

dealing with objective tasks with low dependence on experience.

[7]This implicitly assumes that agents are reporting truthfully.

[8]which is proven to be the optimal RO-payment function when agents are risk/loss-neutral and IR is not binding

[9]which is verified by our ABM to be a reasonable fit for most of the performance measurements

- We propose the idea of adding a common shock to preserve the truthfulness of the performance measurement under the non-linear rank-order payment function;
- We conduct agent-based model experiments to evaluate the performance measurements which provides insights on which one to use in practice.

2 Related Work

Tournament Design. The most relevant related works lie in the economics literature on tournaments. In line with our results, the WTA mechanism is proven to be optimal for neutral agents in small tournaments with symmetrically distributed noise [22], and in arbitrarily-sized tournament when the noise has increasing hazard rate [8]. The follow-up work [7] shows in the tournament setting that the equilibrium effort decreases as the noise of the effort measurement becomes more dispersed, in the sense of the dispersive order. In our situation with Gaussian noise, where the mean of the performance score is a function of effort, we show that it is the ratio of the derivative of the mean to the variance of the noise that affects the principal's utility.

Green and Stokey [14] compare tournaments with independent contracts which pay agents based on their numerical outputs rather than the ranking of the outputs. In their model where the outputs of agents depend not only on their effort but also on an unknown common shock, they show that if there is no common shock, the independent contracts dominant tournaments. However, if the distribution of the common shock is sufficiently diffuse, tournaments dominant independent contracts.

For risk-averse agents, Krishna and Morgan [22] show that the optimal RO-payment function is WTA when there are $n \leq 3$ risk-averse agents, and should pay the agent ranks in the second place positively when $n = 4$. Kalra and Shi [17] show that, for arbitrary number of agents, the more risk-averse the agents are, the larger the number of agents should be rewarded with a focus on logistic and uniform noise distributions. Drugov and Ryvkin [6] generalize their results by considering more general noise distributions and non-separable preferences.

We note that in the tournament literature, the IR constraint is buried into the sufficient conditions for the existence of pure strategy symmetric equilibrium. However, what the optimal payment function is while considering IR remains unknown. This problem is essential in our crowdsourcing setting where IR is likely to be binding.

Crowdsourcing And The Principal-Agent Problem. Additional literature considers the crowdsoucing problem from the principal-agent perspective. Ho et al. [16] model the crowdsourcing process as a multi-round principal-agent problem. Instead of equilibrium analysis, they solve the problem with multi-armed bandit algorithms. Ghosh and Hummel [13] consider agents with heterogeneous ability and endogenous effort and focus on analyzing the cases when the optimal contract supports an equilibrium that favors the principal's utility. The main difference is that they do not consider the payments to the agents as a cost of the principal (e.g. the payments are unredeamable points) which is not the case in our setting where crowd works are compensated with money. Easley and Ghosh [9] consider a crowdsourcing model where agents are strategic in deciding whether to participant in a task. Like [14], they focus on when the principal should apply an output-independent contract or a winner-take-all tournament, which is shown to depend on the agents' behaviour models.

Spot-Checking And Peer Prediction. Literature on spot-checking and peer prediction focuses on designing truthful mechanisms (e.g. whether agents can benefit by manipulating their reports) mostly in the binary-effort case [12, 4, 26, 21]. Kong and Schoenebeck [19] consider a discrete hierarchical effort model where choosing higher effort is more informative but more costly. With assumptions, the maximum effort is proven to be elicitable and payments are optimized using a linear program that requires detailed knowledge of agent costs and quality.

Our approach diverges sharply from previous peer prediction work which focuses nearly entirely on strategic considerations where linear rescaling is the only known technique available. Instead, we separate the agent choices of how much effort to exert from how honestly to report. This allows us to use a principal-agent framework to study how to elicit effort. In general, we obtain a weaker truthfulness guarantee, which is derived from empirical results showing that a litany of strategies do not work. Of course, it is possible that some strategy we failed to consider does work. However, our truthfulness results are slightly stronger in several ways as well. First, we need not rely on the number of agents going to infinity, but can run tests in finite settings (note that some mechanisms that are provably truthful in the limit of a large number of agents and conceivably allow beneficial manipulations in the finite settings we consider). Additionally, some of the mechanisms do not work for all prior distributions, and we can empirically test if they work for the prior distributions that we learn from data.

3 Model

Throughout the paper, we use capital notations denote random variables while lowercase notations denote their realizations. Bold notations denote vectors or matrices.

3.1 Crowdsourcing

A principal (requester) has a set of m tasks $[m] = \{1, 2, \ldots, m\}$. Each task $j \in [m]$ has a ground truth $y_j \in \mathcal{Y}$—that the principal would like to recover—which was sampled from a prior distribution $w \in \Delta_{\mathcal{Y}}$, where $\mathcal{Y}$ is a discrete set and $\Delta_{\mathcal{Y}}$ is the set of all possible distributions over $\mathcal{Y}$. To this end, each task is assigned to n_0 agents and each agent i is assigned a subset of tasks $A_i \subseteq [m]$. Let m_a denote the maximum number of tasks assigned to any agent. Then $|A_i| \leq m_a$ for every agent i. This implies a lower bound on the number of agents: $n \geq \lceil m \cdot n_0 / m_a \rceil$.

Effort and cost. Agents are strategic in choosing an effort level. Let $e_i \in [0, 1]$ denote the effort chosen by agent i. Let $c(e)$ be a non-negative, increasing and convex cost function.

Signals and reports. Each agent i working on an assigned task j, receives a signal denoted $X_{i,j} \in \mathcal{X}$, where $\mathcal{X}$ is the signal space. We assume that $0 \notin \mathcal{X}$ and let $X_{i,j} = 0$ for any $j \notin A_i$. For tasks $j \in A_i$, $X_{i,j}$ are i.i.d. sampled from a distribution that depends only the ground truth y_j and agent i's effort level e_i.

Let Γ_{work} and Γ_{shirk} be $|\mathcal{Y}|$ by $|\mathcal{X}|$ matrices, where, for $y \in \mathcal{Y}$ and $s \in \mathcal{X}$, the y, s entry of Γ_{work} and Γ_{shirk} denotes the probability that an agent who puts in full effort and no effort, respectively, will receive a signal s when the ground truth is y.

Given e_i, agent i's signal $X_{i,j}$ for the jth task where the ground truth is y_j will be sampled according to the y_jth row of

$$e_i \Gamma_{\text{work}} + (1 - e_i) \Gamma_{\text{shirk}}.$$

We will let Γ_{shirk} be uniform in each column. This setup is a modified version of the Dawid-Skene (DS) model [5] where we have added effort.

We use $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ to denote the signal and report profiles of all agents respectively. Note that $\hat{\boldsymbol{x}}$ is not necessarily equal to $\boldsymbol{x}$ for strategic agents. For now, we assume all agents report truthfully, so that $\hat{\boldsymbol{x}} = \boldsymbol{x}$. Strategic reporting is discussed in Section 3.3.

Mechanism. Given $\hat{\boldsymbol{x}}$, a payment mechanism $\mathcal{M} : (\{0\} \cup \mathcal{X})^{n \times m} \to \mathbb{R}^n_{\geq 0}$ pays each agent i a non-negative payment t_i. We decompose the payment mechanism into two parts (Fig. 1). First, we apply a performance measurement $\psi : (\{0\} \cup \mathcal{X})^{n \times m} \to \mathbb{R}^n$ on agents' reports that outputs a (possibly negative and random) score $s_i = \psi(\hat{\boldsymbol{x}})_i$ for each i. In our experiments, we focus on two sets of the performance measurements: spot-checking and peer prediction, which will be discussed later.

Second, we apply a rank-order payment function that pays $\hat{t}_j$ to the j'th ranked agent according to her performance score. WLOG, suppose $s_1 \geq s_2 \geq \cdots \geq s_n$. Then, agent i's payment is $t_i = \hat{t}_i$.

As a comparison, in Section 6.4, we additionally consider applying a linear payment function that rewards each agent i a linear transformation of her performance score, i.e. $t_i = a s_i + b$ where a and b are constants. Because the linear payment function has only two parameters to tune, it is much less flexible and thus less effective in incentivizing agents' effort.

Definition 3.1. We call a RO-payment function increasing if $\hat{t}_j \geq \hat{t}_k$ if $j \leq k$.

3.2 The Principal-Agent Model

We seek a payment mechanism that maximizes the principal's payoff in the symmetric equilibrium. Now, assuming agents report their signals truthfully, we model this crowdsourcing problem as a principal-agent problem.

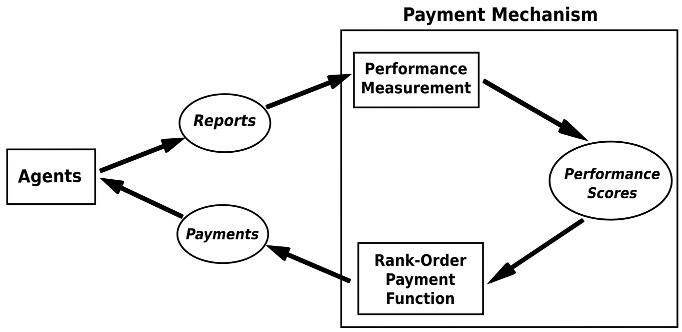


Figure 1: Components of a payment mechanism: a performance measurement and a RO-payment function.

First, the principal assigns the tasks to agents and commits to a payment mechanism consisting of a performance measurement ψ and a RO-payment function $\hat{\boldsymbol{t}}$. Then, the agents respond by working with some effort that maximize their expected utility. Intuitively, the effort affects the distribution of an agent's performance score and then affects the distribution of ranking as well as the payment. In this paper, we consider three utility functions:

$$u_a(t_i, e_i) = \begin{cases} t_i - c(e_i) & \text{for neutral agents,} \\ t_i - c(e_i) - \rho \cdot (c(e_i) - t_i)^+ & \text{for loss-averse agents,} \\ r_a(t_i) - c(e_i) & \text{for risk-averse agents.} \end{cases}$$

Here, for loss-averse agents[10], $(x)^+$ equals x for $x \geq 0$ and 0 otherwise, and ρ is a non-negative loss-aversion factor. For risk-averse agents, r_a is a non-negative, concave and differentiable function with $r_a(0) = 0$ and $r_a'(0) < \infty$. Loss-adverse agents incur additional loss when they are not compensated for the work they expended (e.g. when they are stiffed). Risk-adverse agents proportionally value moderate rewards more the high rewards.

Combined, we often use the following utility function for simplicity,

$$u_a(t_i, e_i) = r_a(t_i) - c(e_i) - \rho \cdot (c(e_i) - t_i)^+. \tag{1}$$

We focus on the solution concept of symmetric equilibrium. That is, all agents exerting effort ξ is an equilibrium if any unilateral deviation will decrease the expected utility, i.e. $\mathbb{E}[u_a(t_i(e_i, \xi), e_i)] \leq \mathbb{E}[u_a(t_i(\xi, \xi), \xi)]$ for any $e_i \in [0, 1]$, where $t_i(e_i, \xi)$ is a random payment function on agent i's effort and all the other agents' effort $e_k = \xi$ for any $k \neq i$.

The problem of the principal is then to optimize the payment mechanism such that a goal effort ξ can be incentivized in the symmetric equilibrium with the minimum payment.[11] Additionally requiring the payment to satisfy limited liability (LL) and individual rationality (IR) leads us to the principal's constraint optimization problem. In Appendix F, we further provide a variant of the principal's model, where the principal also cares about the fairness of the rank-order payments, e.g. reducing the variance of the payments.

To get a sense for the IR constraint, consider the following situation: The principal would like 10 agents to each exert \$10 of effort. However, the principal implements a RO-payment function (e.g. the winner-take-all) which induces a symmetric equilibrium where the agents each contribute \$10 of effort but only pays the top agent \$65. This is not IR because each agent is only paid \$6.50 in expectation, but requires \$10 of effort. However, simply increasing the payment of the top agent, changes the effort in equilibrium. So a different payment structure is needed to ensure the original equilibrium.

[10]Note that we mainly consider the 1-order loss-aversion. We briefly discuss the case of higher order loss-aversion, i.e. $u_a(t_i, e_i) = t_i - c(e_i) - \rho \cdot \left((c(e_i) - t_i)^+\right)^r$ for $r > 1$, in Section 4.1.2.

[11]We note that in reality, the optimization problem of the principal can be much harder. The principal can optimize over the space of the parameters of the crowdsourcing system such as the number of agents, the number of tasks each agent answers and the goal effort. However, the optimization over these parameters requires a finer grind model of the principal's utility, i.e. how does the principal evaluate the contributions from agents, which is beyond the interests of this paper. Therefore, we simply assume the principal fixes these parameters which fixes the (expected) reward of agents' contribution and he tries to optimize the payment mechanism to incentivize the goal effort with minimum cost. We will show, later in this paper, how the goal effort affects the principal's decision.

The Gaussian assumptions. However, the optimization problem over the space of all performance measurements is still too hard to analyze. [12] To make it theoretically tractable, as commonly assumed in principal-agent literature, we apply the Gaussian noise assumption. Again, let e_i be agent i's effort and ξ be all the other agents' effort.

Assumption 3.1. *We assume the agent i's performance score S_i follows the Gaussian distribution with p.d.f. $g^{(i)}_{e_i,\xi}$ and c.d.f. $G^{(i)}_{e_i,\xi}$, where the mean $\mu(e_i,\xi)$ and standard deviation $\sigma(e_i,\xi)$ are functions of agents' effort. Furthermore, let $g^{(-i)}_{e_i,\xi}$ and $G^{(-i)}_{e_i,\xi}$ be the same notations for all the other agents' score distribution under the same effort profile. We assume μ and σ to be differentiable.*

Assumption 3.2. *We assume the distribution $g^{(-i)}_{e_i,\xi}$ is independent of e_i.*

Assumption 3.2 implies that any unilateral deviation $e_i \in [0,1]$ from a symmetric effort profile where all agents' effort is ξ will not change other agents' score distribution. This implies $g^{(-i)}_{e_i,\xi} = g^{(i)}_{\xi,\xi}$. This assumption is intuitively true for spot-checking mechanisms where agents' performance scores are independent conditioned on the ground truth, and for peer prediction mechanisms when the number of agents is large. In Section 6.2, we show that for peer prediction mechanisms with a reasonably large n, this assumption approximately holds with a small deviation $|e_i - \xi|$, which is all we ask in our theory. For simplicity, through out the paper, we use $g_{e_i,\xi}$ to denote agent i's score distribution and $g_{\xi,\xi}$ to denote other agents' score distribution.

We additionally make the following assumption which, at a high level, guarantees that a unilateral deviation to a higher effort does not harm the agent's expected performance score.[13]

Assumption 3.3. *Fixing ξ, let $\mu'_\xi(e_i) = \frac{\partial \mu(e_i,\xi)}{\partial e_i}$ be the derivative of $\mu(e_i,\xi)$ over e_i as a function of e_i and $\sigma'_\xi(e_i)$ is the similar notation for the standard deviation. We assume $\mu'_\xi(e_i) \geq 0$ and furthermore, $\mu'_\xi(e_i) + \sigma'_\xi(e_i) \geq 0$ for any $e_i, \xi \in [0,1]$.*

In our agent-based model simulations, we observe that the derivative of the standard deviation is in the second-order compared with the derivative of the mean. Thus, Assumption 3.3 is approximately saying that μ is increasing in e_i.

Definition 3.2. We call a performance measurement an idealized performance measurement if it can generate performance scores that satisfy Assumption 3.1, 3.2 and 3.3.

3.3 Strategic Reports

We consider strategic reporting in Section 5 and define the terminologies here. While considering strategic reporting, we fix all agents effort and thus omit the notations that indicate agents effort. Then, for agent i, given her signal X_i, let $\hat{X}_i$ be the report of agent i. Her strategy θ_i is a random mapping from X_i to $\hat{X}_i$. Let Θ be the common strategy space. As a common assumption in the peer prediction literature [4, 1], we assume agents' strategies are task-independent, which implies that agent i will first choose a $\pi_i : \mathcal{X} \to \Delta_{\mathcal{X}}$, then draws $\hat{X}_i$ from the distribution $\pi_i(X_i)$ as her reports. For example, agents can combine signal 3 and 4 to 4 by reporting 4 whenever seeing a signal 3. Specifically, we use τ_i to denote the truth-telling strategy, i.e. $\tau(X) = X$.

Previous literature has provided a large number of choices of performance measurements (including but not limited to peer prediction mechanisms and spot-checking mechanisms) that are truthful in the sense of Bayesian Nash equilibrium. We can write the expected performance score as a function of the agents' strategy profile, i.e.

$$\mathbb{E}[S_i(\theta_i, \boldsymbol{\theta_{-i}})] = \mathbb{E}_{\boldsymbol{X},\boldsymbol{\theta}}\left[\psi(\hat{\boldsymbol{X}})_i\right],$$

where ψ is the performance measurement.

Definition 3.3. We call a performance measurement truthful if no unilateral deviation from a strategy profile where all the agents report truthfully can increase the expected performance score, i.e. $\mathbb{E}[S_i(\theta_i, \boldsymbol{\tau_{-i}})] \leq \mathbb{E}[S_i(\tau_i, \boldsymbol{\tau_{-i}})]$ for any agent i and any strategy $\theta_i \in \Theta$.

[12]The main difficulty is that, in general, we don't analytically know what's the distribution of the performance scores output by a performance measurement, which usually has no closed-form.

[13]In our experiments, we observe that $\sigma'_\xi(e_i)$ is insignificant compared with $\mu'_\xi(e_i)$.

We can define the truthfulness of a payment mechanism in similar sense. Let $p_j(\theta_i, \boldsymbol{\theta_{-i}})$ be the probability that the indicative agent's performance score is ranked j. Then, the expected payment under a strategy profile $\boldsymbol{\theta}$ can be written as

$$\mathbb{E}[t_i(\theta_i, \boldsymbol{\theta_{-i}})] = \sum_{j=1}^{n} p_j(\theta_i, \boldsymbol{\theta_{-i}})\hat{t}_j.$$

Definition 3.4. We call a payment mechanism truthful if no unilateral deviation from a strategy profile where all the agents report truthfully can increase the expected payment, i.e.

$$\mathbb{E}[t_i(\theta_i, \boldsymbol{\tau_{-i}})] \leq \mathbb{E}[t_i(\tau_i, \boldsymbol{\tau_{-i}})],$$

for any agent i and any strategy $\theta_i \in \Theta$.

Note that the linear payment function trivially transfers the truthfulness of the performance measurement to the truthfulness of the payment mechanism. However, because the rank-order payment function is non-linear, this property does not generally hold. We will study this issue in depth in Section 5.

Again, to theoretically track the problem, we adopt the following two assumptions which are analogous to Assumption 3.1 and 3.2 with respect to agents' reporting strategies.

Assumption 3.4. *We assume the agent i's performance score S_i follows the Gaussian distribution with p.d.f. $g_{\boldsymbol{\theta}}^{(i)}$ and c.d.f. $G_{\boldsymbol{\theta}}^{(i)}$, where the mean $\mu_i(\boldsymbol{\theta})$ and standard deviation $\sigma_i(\boldsymbol{\theta})$ are functions of agents' strategy profile.*

Assumption 3.5. *Let $\boldsymbol{\theta}$ be the initial strategy profile. Suppose agent i unilaterally deviates to an arbitrary strategy θ_i'. Let the corresponding change in the mean of the score distribution of an agent $j \in [n]$ be $\Delta\mu_j(\theta_i', \boldsymbol{\theta}) = \mu_j(\theta_i', \boldsymbol{\theta_{-i}}) - \mu_j(\boldsymbol{\theta})$. We assume $|\Delta\mu_i(\theta_i', \boldsymbol{\theta})| \geq \Delta\mu_j(\theta_i', \boldsymbol{\theta})$ for any $j \neq i$ and $\theta_i' \in \Theta$.*

Assumption 3.5 says that if one agent unilaterally changes her reporting strategy, she will change the mean of her own score more than the mean of any other agent's score.[14] Intuitively, this assumption holds for any spot-checking mechanisms and for any peer prediction mechanisms when the number of agents is relatively large.

4 Optimizing Payment Mechanism In The Idealized Setting

This section answers the question of how to reward agents optimally for a desired effort level in the idealized setting. The optimization consists of two parts: optimizing the rank-order payment function while fixing any idealized performance measurement, and optimizing the idealized performance measurement given the optimal RO-payment function. For the former, we observe that the optimal RO-payment function is increasing for all agent utility models that we considered, and both risk/loss-aversion and individual rationality will make the optimal RO-payment function more inclusive which rewards a larger number of agents. For the latter, we identify the sufficient statistic of a good performance measurement called the sensitivity. We show that a performance measurement with higher sensitivity can incentivize the same effort level in the symmetric equilibrium with a lower total payment.

4.1 Optimizing The Rank-Order Payment Function

We first rewrite the principal's problem given a performance measurement ψ. Suppose all the agents except i exert an effort ξ. Then, given ψ, agent i knows the probability that she ends up with each rank j when her effort is e_i, which is denoted as $p_j(e_i, \xi)$. Recall that by Assumption 3.1, $G_{e_i,\xi}$ is the c.d.f. of the score distribution of agent i; by Assumption 3.2, $G_{\xi,\xi}$ is the c.d.f. of the score distribution of all the other agents. Then, this probability is given by

$$p_j(e_i, \xi) = \tbinom{n-1}{j-1} \int_{-\infty}^{\infty} G_{\xi,\xi}(x)^{n-j} \left[1 - G_{\xi,\xi}(x)\right]^{j-1} dG_{e_i,\xi}(x). \tag{2}$$

[14]Note that this assumption is weaker than Assumption 3.2 - the analogue assumption for agents' effort strategy - as the latter requires all the other agents' score distributions stay unchanged if there is an unilateral deviation in effort.

We then can write agent i's expected utility under the RO-payment function $\hat{\boldsymbol{t}}$ as

$$\mathbb{E}[U_a(e_i,\xi)] = \sum_{j=1}^{n} p_j(e_i,\xi) u_a\left(\hat{t}_j, e_i\right), \tag{3}$$

where U_a denotes the random variable of the agent's utility and u_a is agent's utility function defined in Eq. (1).

Maximizing the expected utility w.r.t. e_i then leads to the first order constraint (FOC) which is a necessary condition of symmetric equilibrium. For sufficiency, additional conditions on the distribution of the performance score and the agents' cost function are required. For example, it is shown that when the distribution of the noise (in our case, this is the Gaussian) is "dispersed enough", the existence of symmetric equilibrium is guaranteed [23]. Again, in our theory sections, we assume this is true, while we empirically verify this assumption in Section 6.2 under the performance measurements and cost functions that we consider. For now, we assume FOC is also sufficient for symmetric equilibrium.

Let $p'_j(\xi) = \frac{\partial p_j(e_i,\xi)}{\partial e_i}\big|_{e_i=\xi}$ denote the derivative of the probability an agent ends up with rank j w.r.t. a unilateral deviation in effort when all agents' effort is ξ, and let $c'(\xi)$ denote the derivative of the cost. Also, note that $p_j(\xi,\xi) = \frac{1}{n}$ for any j due to symmetry. Now, given n and ξ, we formally write down the principal's problem.

$$\begin{aligned} \min_{\hat{\boldsymbol{t}}} \quad & \sum_{j=1}^{n} \hat{t}_j \\ \text{s.t.} \quad & \hat{\boldsymbol{t}} \geq 0 \quad (LL), \qquad \frac{1}{n}\sum_{j=1}^{n} u_a\left(\hat{t}_j,\xi\right) \geq 0 \quad (IR), \qquad \sum_{j=1}^{n} p'_j(\xi) u_a\left(\hat{t}_j,\xi\right) = 0 \quad (FOC). \end{aligned} \tag{4}$$

Before we present our results, we present the following lemma that is essential for future proofs.

Lemma 4.1. *Fixing $\xi \in [0,1]$, if $n \to \infty$, $p'_j(\xi)$ is decreasing in j for any $1 \leq j \leq n$.*

We leave the proof in Appendix A. Theorem 4.1 shows that after convergence, a small unilateral deviation results in a probability of ranking that is monotone decreasing in j. The key of the proof lies in the fact that after convergence, $p'_j(\xi)$ can be approximated with some form of the quantile function of Gaussian, which is known to be the inverse error function. Then, with the monotonicity of the inverse error function, we complete the proof.

Next, we present the optimal RO-payment functions for the principal's problem under the three utility models.

4.1.1 Neutral Agents

Now suppose agents are neutral, i.e. $u_a(t_i, e_i) = t_i - c(e_i)$. We have the following results.

Proposition 4.2. *Suppose $n \to \infty$, $\xi \in [0,1]$ and agents are neutral.*

1. ***IR is not binding:*** *If $\lim_{n\to\infty} \frac{c'(\xi)}{np'_1(\xi)} \geq c(\xi)$, the optimal RO-payment function is winner-take-all, i.e. $\hat{t}_1 = \frac{c'(\xi)}{p'_1(\xi)}$ is the reward to the top one agent and $\hat{t}_j = 0$ for $1 < j \leq n$;*
2. ***IR is binding:*** *Otherwise, the optimal RO-payment function is not unique and can be achieved by a threshold function that rewards the top $\hat{n}$ agents equally, i.e. $\hat{t}_j = \frac{n}{\hat{n}} c(\xi)$ for $1 \leq j \leq \hat{n}$ and 0 otherwise. The threshold $\hat{n}$ is determined by $\frac{n}{\hat{n}} \sum_{j=1}^{\hat{n}} p'_j(\xi) c(\xi) = c'(\xi)$.*

The proof is deferred to Appendix B.1. As a sketch, the proposition holds because by Theorem 4.1, $p'_j(\xi)$ is decreasing in j. This implies that if IR is not binding, when we take the gradient of the total payment in Eq. (4) w.r.t. each $\hat{t}_j$, the gradient reaches its maximum when $j = 1$. Thus, the most payment-saving RO-payment function is to put all of the budget on $\hat{t}_1$ to maximize the gain of any unilateral deviation to a higher effort.

It is worth noting that except for the extreme cases where $\frac{c'(\xi)}{p_1'(\xi)} \to \infty$, Theorem 4.2 implies that IR is always binding when $n \to \infty$. However, we emphasize that the condition $n \to \infty$ in Theorem 4.2 (as well as the following propositions) is only needed because Theorem 4.1, which is used in the proof, requires it. In Section 6.2, we empirically show that Theorem 4.1 still holds for a reasonably large group size, e.g. $n = 50$, which makes the condition in Theorem 4.2 less extreme in this case.

We further note that when agents are neutral and IR is binding, the minimum total payment equals the total cost $nc(\xi)$. Furthermore, more than one RO-payment function can achieve the optimum. We provide a threshold function as one of the solutions.

4.1.2 Loss-averse Agents

Suppose agents are loss-averse, i.e. $u_a(t_i, e_i) = t_i - c(e_i) - \rho \cdot (c(e_i) - t_i)^+$. The expression of the optimal RO-payment function becomes more complicated in the loss-aversion case. Here, we present an informal version of our results while leaving the precise version in Appendix C.1.

Proposition 4.3. *(Informal) Suppose $n \to \infty$, $\xi \in [0, 1]$ and agents are loss-averse.*

1. ***IR is not binding:*** *The optimal RO-payment function pays 1) 0 to the bottom agents with ranking $j > \bar{n}$, 2) $c(\xi)$ to the intermediate agents with ranking $1 < j \le \bar{n}$, and 3) $\hat{t}_1 > c(\xi)$ to the top one agent. Here, the threshold $\bar{n} \le \frac{n}{2}$ is determined by $(1 + \rho)p'_{\bar{n}}(\xi) = p'_1(\xi)$;*

2. ***IR is binding:*** *The optimal RO-payment function follows the same structure as the case of IR not binding, but with a threshold $\hat{n} \ge \bar{n}$.*

The proof is shown in Appendix C.1. As a sketch, note that the gradient of the total payment w.r.t. $\hat{t}_j$ is maximized at $j = 1$ only when $\hat{t}_1 \le c(\xi)$. When $\hat{t}_1 > c(\xi)$ the gradient is discounted with a factor $\frac{1}{1+\rho}$. Therefore, with the decreasing property of $p_j(\xi', \xi)$ in j, the optimal RO-payment function will "fill in" $\hat{t}_j$ to $c(\xi)$ in the increasing order of j until some $\bar{n}$ such that the discounted gradient w.r.t. $\hat{t}_1$ is larger the undiscounted gradient w.r.t. $\hat{t}_{\bar{n}}$. Then, the rest budget is put on $\hat{t}_1$.

Theorem 4.3 shows that for loss-averse agents, the optimal RO-payment function has three levels of payments: the bottom agents are paid zero; intermediate agents receive the baseline payment that equals to their cost; the top one agent gets a bonus that is larger than her cost. We call this type of RO-payment function the *winner-take-more* payment function. Perhaps interestingly, winner-take-more takes a similar form of the baseline-bonus payment scheme which tends to perform well in real-world scenarios [15]. We show the optimal RO-payment functions in Fig. 2 to better illustrate our ideas.

To better illustrate our results, we introduce the inclusiveness of a (monotone) RO-payment function.

Definition 4.4. Given a monotone RO-payment function such that $\hat{t}_j \ge \hat{t}_k$ if $j \le k$, the *inclusiveness* of such a RO-payment function is defined as the number of agents who receive non-zero payments, denoted as n^I. We call RO-payment function A is (weakly) more inclusive than RO-payment function B if its n^I_A is no less than n^I_B.

For example, $n^I = 1$ for the winner-take-all tournament, and $n^I = \bar{n}$ and $n^I = \hat{n}$ in the case of loss-averse agents with IR not binding and binding respectively. Now, we show that n^I is increasing as agents become more and more loss-averse.

Corollary 4.5. *Suppose $n \to \infty$ and agents are loss-averse. The inclusiveness of the optimal RO-payment function n^I is (weakly) increasing in ρ.*

Remark. Our results for loss-averse agents rely on the first-order loss-aversion model. The study of higher-order loss-aversion is beyond the scope of this paper, i.e. $u_a(t_i, e_i) = t_i - c(e_i) - \rho \cdot ((c(e_i) - t_i)^+)^r$ for $r > 1$. Our conjecture is that in this case, the optimal RO-payment function will no longer pay a fraction of agents constantly, but pay agents decreasingly w.r.t. their ranking.

4.1.3 Risk-averse Agents

Now, suppose agents are risk-averse, i.e. $u_a(t_i, e_i) = r_a(t_i) - c(e_i)$. Again, we only show the informal version of our proposition.

Proposition 4.6. *(Informal) Suppose $n \to \infty$ and agents are risk-averse.*

1. ***IR is not binding:*** *The optimal RO-payment function pays zero to the bottom agents with ranking $j > \bar{n}$, and the remaining agents strictly decreasing in their ranking, i.e. $\hat{t}_j > \hat{t}_k$ if $j < k \leq \bar{n}$, with $\bar{n} \leq \frac{n}{2}$ determined by r_a and $\boldsymbol{p}'(\xi)$.*

2. ***IR is binding:*** *The optimal RO-payment function follows the same structure as the case of IR not binding, but with a threshold $\hat{n} \geq \bar{n}$.*

The precise version of this proposition and its proof are provided in Appendix D.1. As shown in Fig. 2, Theorem 4.3 illustrates that the optimal RO-payment function for risk-averse agents also only rewards the agents whose ranking is above some threshold, and the rewards are decreasing in the ranking j. As a proof sketch, this is because if the principal keep increasing the payment for the top one agent, the gradient of the total payment w.r.t. $\hat{t}_1$ is discounted greater and greater due to the concavity of the reward function. Therefore, after $\hat{t}_1$ becomes large enough, the principal is better off to increase $\hat{t}_2$, $\hat{t}_3$ and so on from zero to something instead of keep increasing $\hat{t}_1$.

The proposition shows that, in line with the results of neutral and loss-averse agents, the payment function becomes more inclusive when IR is binding compared with the case where IR is ignored.

We further point it out that the same result in Theorem 4.5 does not generalize to the risk-averse agents: more risk-aversion does not always imply a more inclusive RO-payment function.[15] We provide a formal illustration in Appendix D.2. Our intuition is that this can happen because of the flexibility of the model of risk-averse agents, i.e. $r_a(x)$ can be "more concave" in a lot of ways.

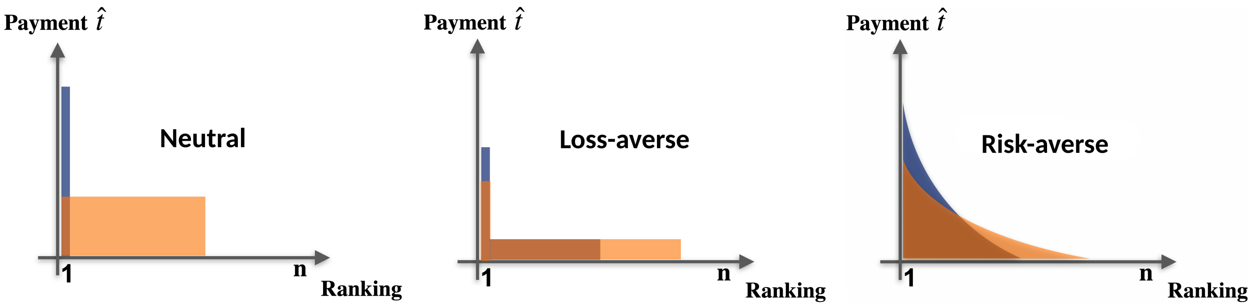


Figure 2: The optimal RO-payment functions for three types of agents. Blue payments are the cases where IR is ignored, and orange payments are the cases where IR is considered. All payment schemes are increasing.

To sum up, we show that the optimal RO-payment functions for three common models of agent utility are all increasing payment schemes. Furthermore, inclusive payments are likely to be referred for two reasons. On one hand, IR requires the optimal RO-payment function to be more inclusive (compared with the case where IR is ignored) so as to guarantee the existence of equilibrium. On the other hand, when agents have "fairness" concerns, e.g. they are risk or loss-averse, more inclusive RO-payment functions are optimal (compared with the neutral case, where WTA is optimal). In Section 6.3, we empirically show how the inclusiveness of the optimal RO-payment functions interacts with the cost function and equilibrium effort.

4.2 Optimizing The Performance Measurement

A performance measurement can affect principal's optimal utility by affecting $p_j(e_i, \xi)$. In the idealized setting, assuming all the other agents but i exert effort ξ, every performance measurement maps agent i's effort e_i to a Gaussian distribution of her performance score with mean and std functions of e_i. We denote these two functions as $\mu(e_i, \xi)$ and $\sigma(e_i, \xi)$ respectively. Therefore, in the principal-agent problem that we care about,

[15] Actually, we show that "more risk-averse" leads to more inclusive optimal RO-payment function when IR is not binding, but both it is not true when IR is binding.

$\mu(e_i, \xi)$ and $\sigma(e_i, \xi)$ determine how food a performance measurement is. Let $\mu'_\xi(\xi)$ be the derivative of μ w.r.t. e_i when $e_i = \xi$, and $\sigma'_\xi(\xi)$ be the same notation for σ. For any RO-payment function that is increasing (Theorem 3.1), we identify the following sufficient statistic of the performance of a performance measurement, called the *sensitivity*.

Definition 4.7. Suppose Assumption 3.1 holds. The sensitivity of a performance measurement whose performance score distribution has mean $\mu(e_i, \xi)$ and standard deviation $\sigma(e_i, \xi)$ is defined as $\delta(\xi) = \frac{\mu'_\xi(\xi)+\sigma'_\xi(\xi)}{\sigma_\xi(\xi)}$.

From this definition, the sensitivity of a performance measurement is defined under the symmetric equilibrium concept and depends on the effort in the symmetric equilibrium. At a high level, a performance measurement is more sensitive if it can generate scores that are more sensitive in effort change and have high accuracy. Also note that $\delta(\xi) \geq 0$ by Assumption 3.2.

Proposition 4.8. *Suppose Assumption 3.1 and 3.2 hold. Let δ be the sensitivity of the performance measurement and let $\hat{\boldsymbol{t}}$ be any RO-payment function that is increasing. Then, fixing any $\xi \in [0, 1]$, the minimum total payment $\sum_{j=1}^{n} \hat{t}_j$ is (weakly) decreasing in $\delta(\xi)$.*

The proof is shown in Appendix E. At a high level, the intuition is that if an agent slightly increases her effort, it becomes easier for her to be ranked in higher places. This effect is amplified by a performance measurement with higher sensitivity. Therefore, with a more sensitive performance measurement, the first order constraint in Eq. (4) can be satisfied with lower payment. Because both of the other constraints are independent of performance measurements, we can conclude that higher sensitivity implies (at least weakly) lower payment from the principal.

Now, we have optimized the performance measurement and the RO-payment function separately. The following corollary fits the optimization results together.

Corollary 4.9. *Fixing a goal effort, let ψ' be a performance measurement with higher sensitivity than ψ. Let $\hat{\boldsymbol{t}}'$ and $\hat{\boldsymbol{t}}$ be their corresponding optimal RO-payment functions respectively. Then the payment mechanism consisted of ψ' and $\hat{\boldsymbol{t}}'$ has lower minimal total payment than the payment mechanism consisted of ψ and $\hat{\boldsymbol{t}}$ in the symmetric equilibrium.*

The proof is straightforward by comparing three payment mechanisms: mechanism 1 is consisted of ψ' and $\hat{\boldsymbol{t}}'$, mechanism 2 is consisted of ψ' and $\hat{\boldsymbol{t}}$ and mechanism 3 is consisted of ψ and $\hat{\boldsymbol{t}}$. First, by our results in Section 4.1, both $\hat{\boldsymbol{t}}'$ and $\hat{\boldsymbol{t}}$ are increasing. Then, by Theorem 4.8, mechanism 2 should be cheaper to implement than mechanism 3. Furthermore, mechanism 1 must be cheaper than mechanism 2 because $\hat{\boldsymbol{t}}'$ is the optimal RO-payment function for ψ' which completes the proof.

As a summary, Theorem 4.8 shows that if the RO-payment function is increasing, which is exactly the case for any agent utility model that we consider, the sensitivity of a performance measurement is a sufficient statistic of its performance in our principal agent problem. Then, Theorem 4.9 implies that to optimize a payment mechanism, one should focus on maximizing the sensitivity of a performance measurement and apply the corresponding optimal RO-payment function.

5 Truthful Winner-Take-All Tournament

So far, we have shown how to optimize a payment mechanism to incentivize a desired effort with the minimum payment. This section focuses on dealing with the concern of truthfulness caused by the non-linear rank-order payment function. The question is, given a truthful performance measurement, can we obtain a truthful payment mechanism?

The answer is positive. In this section, we fix agents' effort and consider the winner-take-all tournament. We show that we can always add a large artificial Gaussian noise to agents' performance scores such that they are worse-off by playing any untruthful reporting strategy.

5.1 High Variance Benefits Deviations

We start by understanding why a truthful performance measurement does not imply a truthful payment mechanism under the tournament setting. Recall that a truthful performance measurement can guarantee that any untruthful strategy will decrease the expected performance score. However, under the non-linear rank-order payment function, it matters not only the expected score, but also the distribution of the score. For example, if an untruthful strategy can increase the variance of the performance score, while it decreases the expected score, such a strategy can potentially help the agent to be ranked in the first place and beat the others in the winner-take-all tournament. We formally write down this observation in the following lemma.

Lemma 5.1. *Suppose Assumption 3.4 and 3.5 hold. Suppose there is a truthful performance measurement under which, without loss of generality, the score distribution of truth telling is a standard Gaussian denoted as g_0, and the score distribution of an untruthful strategy is g_θ whose mean is $-\mu < 0$ and standard deviation is σ. Suppose one agent deviates from the truth-telling strategy profile by playing strategy θ. Then, under the winner-take-all tournament with $n \geq 2$ agents, the expected payment of the deviating agent is decreasing in μ and increasing in σ.*

Proof. Under the winner-take-all tournament, to reason about an agent's expected payment, we only have to focus on the probability of being ranked in the first place. We denote this probability as $p_1(\mu, \sigma)$ as functions of μ and σ. Then, the proof follows by showing the first-order derivative of p_1 with respect to μ and σ are negative and positive respectively.

$$\begin{aligned}
p_1(\mu, \sigma) &= \int_{-\infty}^{+\infty} g_\theta(x) G_0(x)^{n-1} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x+\mu}{\sigma}\right)^2} G_0(x)^{n-1} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} G_0(\sigma y - \mu)^{n-1} dy
\end{aligned}$$

$$\begin{aligned}
\frac{\partial p_1}{\partial \mu} &= -\frac{n-1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}y^2} G_0(\sigma y - \mu)^{n-2} g_0(\sigma y - \mu) dy \\
&< 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial p_1}{\partial \sigma} &= \frac{n-1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y e^{-\frac{1}{2}y^2} G_0(\sigma y - \mu)^{n-2} g_0(\sigma y - \mu) dy \\
&= \frac{n-1}{\sqrt{2\pi}} \int_{0}^{+\infty} y e^{-\frac{1}{2}y^2} \left(g_0(\sigma y - \mu) G_0(\sigma y - \mu)^{n-2} - g_0(-\sigma y - \mu) G_0(-\sigma y - \mu)^{n-2} \right) dy \\
&\geq \frac{n-1}{\sqrt{2\pi}} \int_{0}^{+\infty} y e^{-\frac{1}{2}y^2} g_0(\sigma y + \mu) \left((G_0(\sigma y - \mu)^{n-2} - G_0(-\sigma y - \mu)^{n-2} \right) dy \\
&> 0.
\end{aligned}$$

□

Lemma 5.1 shows that it is beneficial to enlarge the variance of the performance score distribution in the tournament setting. Based on the lemma, we can immediately obtain two results which are summarized in the following two propositions. On one hand, we show that any untruthful strategy that does not increase the variance of the performance score will never bother the truthfulness. On the other hand, we show that for any untruthful strategy, if it can bring a large enough increase in the variance of the performance score, it will be favoured by the agents.

Proposition 5.2. *Suppose Assumption 3.4 and 3.5 hold. Given a truthful performance measurement, for any unilateral untruthful deviation from the truth-telling strategy profile, if it (weakly) decreases the variance of the performance score, it will never make (all three types of) the deviating agent better-off.*

Proof. We know that under a truthful performance measurement, any untruthful deviation will decrease the expected performance score. Then, by Lemma 5.1, the smaller the variance is, the smaller the probability of winning the first prize is, and thus the smaller the utility of such a deviation is. Then, the proposition follows because even when an untruthful strategy has the same variance as truth-telling, it has a (strictly) smaller expected score, and thus leads to a smaller probability of winning the first prize. □

Proposition 5.3. *Suppose Assumption 3.4 and 3.5 hold. Suppose there is a truthful performance measurement under which the score distribution of truth-telling is the standard Gaussian, and for an unilateral untruthful deviation θ the mean and standard deviation of the score distribution are $-\mu \leq 0$ and σ_θ respectively. Then, under the winner-take-all tournament with $n \geq 3$ agents, for any $\mu \geq 0$, there exists a threshold $\bar{\sigma}$ such that for any $\sigma > \bar{\sigma}$, it is better-off for (all three types of) the agent to deviate.*

Proof. Under the winner-take-all tournament, the proposition is equivalent to show that such a deviation will increase p_1 the probability of being ranked at the first place compared with the truth-telling strategy profile, where the probability is $\frac{1}{n}$ due to symmetry.

By Lemma 5.1, we know that p_1 is monotone increasing in σ while fixing μ. Then, we only have to show that $p_1 > \frac{1}{n}$ when $\sigma \to \infty$. This is straightforward because when σ is large enough, the Gaussian distribution will converge to a uniform distribution, which means p_1 will converge to $\frac{1}{2} > \frac{1}{n}$ when $n \geq 3$. □

5.2 Common Shock Helps Truthfulness

We propose a solution to the above problem. By Proposition 5.2, we only have to deal with the untruthful strategies that increase the variance of the performance score. The main idea is to dilute the difference between the variances of two Gaussian distributions. To do so, we can manually add a (diffuse) common shock to every agent's performance score and then apply the rank-order payment function. We wrap this idea into a modified payment mechanism called the *strategy-robust* payment mechanism.

A strategy-robust payment mechanism is a payment mechanism with an additional intermediate step. That is, after applying a performance measurement which turns agents' reports into their performance scores, the strategy-robust payment mechanism will independently sample an term from Gaussian distribution g_ϵ with zero mean and a standard deviation of σ_ϵ for each agent, denoted as ϵ_i. Then, the new performance score of agent i is $s_i' = s_i + \epsilon_i$. Finally, the mechanism will reward agents by applying a rank-order payment function on the new performance scores.

Proposition 5.4. *Suppose there is a truthful performance measurement under which the original score distribution of truth-telling to be a Gaussian distribution with zero mean and a standard deviation of σ_τ, and the mean and standard deviation of the score distribution of an unilateral untruthful deviation θ are $-\mu \leq 0$ and $\sigma_\theta > \sigma_\tau$ respectively. Then, under the winner-take-all tournament with $n \geq 2$ agents, there exists a $\bar{\sigma}_\epsilon$ such that if $\sigma_\epsilon > \bar{\sigma}_\epsilon$, the strategy-robust payment mechanism is truthful.*

Proof. The proof follows by showing that after adding a large enough common shock, any untruthful strategy will decrease the probability that the new performance score is ranked in the first place.

First, note that the sum of two Gaussian variables also follows the Gaussian distribution. We then know that when agents are truthful the new performance score follows the Gaussian distribution with zero mean and a standard deviation of $\sigma_\tau' = \sqrt{\sigma_\tau^2 + \sigma_\epsilon^2}$, and when an agent unilateral deviates to an untruthful strategy θ the new score follows the Gaussian distribution with a mean of $-\mu$ and a standard deviation of $\sigma_\theta' = \sqrt{\sigma_\theta^2 + \sigma_\epsilon^2}$. Then, we can write down the probability of winning.

$$
\begin{aligned}
p_1 &= \int_{-\infty}^{+\infty} g_\theta(x) G_\tau(x)^{n-1} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma'_\theta} e^{-\frac{1}{2}\left(\frac{x+\mu}{\sigma'_\theta}\right)^2} \left(\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma'_\tau} e^{-\frac{1}{2}\left(\frac{y}{\sigma'_\tau}\right)^2} dy\right)^{n-1} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma'_\theta} e^{-\frac{1}{2}\left(\frac{x+\mu}{\sigma'_\theta}\right)^2} \left(\int_{-\infty}^{\frac{x}{\sigma'_\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt\right)^{n-1} dx && (t = \tfrac{y}{\sigma'_\tau}) \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \left(\int_{-\infty}^{\frac{\sigma'_\theta}{\sigma'_\tau}z-\frac{\mu}{\sigma'_\tau}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt\right)^{n-1} dz && (z = \tfrac{x+mu}{\sigma'_\theta}) \\
&= \int_{-\infty}^{+\infty} g_0(x) G_0\left(\frac{\sigma'_\theta}{\sigma'_\tau}x - \frac{\mu}{\sigma'_\tau}\right)^{n-1} dx,
\end{aligned}
$$

where g_0 is the p.d.f. of the standard Gaussian. We want to show that if σ_ϵ is large enough, p_1 is smaller than $\frac{1}{n}$, which is the probability of winning while being truthful in the symmetric equilibrium.

$$
\begin{aligned}
p_1 - \frac{1}{n} &= \int_{-\infty}^{+\infty} g_0(x) \left(G_0\left(\frac{\sigma'_\theta}{\sigma'_\tau}x - \frac{\mu}{\sigma'_\tau}\right)^{n-1} - G_0(x)^{n-1}\right) dx \\
&= \int_{-\infty}^{+\infty} g_0(z + \frac{\mu}{\sigma'_\theta - \sigma'_\tau}) \left(G_0\left(\frac{\sigma'_\theta}{\sigma'_\tau}z + \frac{\mu}{\sigma'_\theta - \sigma'_\tau}\right)^{n-1} - G_0\left(z + \frac{\mu}{\sigma'_\theta - \sigma'_\tau}\right)^{n-1}\right) dz \quad (z = x - \tfrac{\mu}{\sigma'_\theta - \sigma'_\tau}) \\
&\approx \int_{-\infty}^{+\infty} g_0(z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}) \left(G_0\left(\frac{\sigma'_\theta}{\sigma'_\tau}z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}\right)^{n-1} - G_0\left(z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}\right)^{n-1}\right) dz \qquad (\sigma_\epsilon \gg \sigma_\theta) \\
&= \int_{0}^{+\infty} g_0(z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}) \left(G_0\left(\frac{\sigma'_\theta}{\sigma'_\tau}z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}\right)^{n-1} - G_0\left(z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}\right)^{n-1}\right) dz \\
&\quad + \int_{0}^{+\infty} g_0(z - \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}) \left(G_0\left(-\frac{\sigma'_\theta}{\sigma'_\tau}z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}\right)^{n-1} - G_0\left(-z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}\right)^{n-1}\right) dz \\
&\approx \int_{0}^{+\infty} g_0(z - \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}) \left(G_0\left(-\frac{\sigma'_\theta}{\sigma'_\tau}z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}\right)^{n-1} - G_0\left(-z + \frac{2\sigma_\epsilon\mu}{\sigma_\theta^2 - \sigma_\tau^2}\right)^{n-1}\right) dz \\
&< 0.
\end{aligned}
$$

Therefore, when the error distribution is diffuse enough, any untruthful deviation will never be preferred by the agent under the winner-take-all tournament, which completes the proof. □

At a high level, the proof works because adding a large common shock weakens the advantage of the untruthful deviation (having a lager variance) while it enlarges the disadvantage (having a smaller expected score). Although the idea of the strategy-robust payment mechanism works well empirically (as we will see in Section 6.4), we point out two limitations of the proof. First, the proof works only for winner-take-all tournament. The winner-take-all tournament has a property that it is always strictly better to have a larger performance score. However, for example, if the rank-order payment function also rewards the agent ranked on the second place, agents may benefit from a relatively small performance score, as she has to win the prize by allowing someone beat her in the tournament. Without this monotonicity, the same proof idea fails. However, we note that empirically, the idea works well for the other rank-order payment functions considered in this paper.

Second, there is no closed-formed solution of the threshold $\bar{\sigma_\epsilon}$. That means, the common shock may have to be very large to guarantee the truthfulness of the strategy-robust payment mechanism. As we will see in the next section, this is a bad news. Fortunately, we empirically show that we do not have to add a common shock with a variance of infinity in practice. A reasonably small noise (e.g. the standard deviation of the common shock is five times of the standard deviation of the original performance score) can guarantee truthfulness in many cases.

We further emphasize that adding a common shock to the performance score will not change any results in Section 4 as all the proofs trivially generalize.

5.3 Common Shock Harms Sensitivity

We will show that there is a tradeoff between the truthfulness and the sensitivity. Adding a large common shock to the performance score will help the truthfulness of the strategy-robust payment mechanism, but will decrease the sensitivity of the original performance measurement. Intuitively, as the sensitivity $\delta = \frac{\mu'+\sigma'}{\sigma}$ is inversely proportional to the variance of the performance score, the added common shock will decrease the sensitivity and thus increases the total payment to incentivize a desired effort.

Proposition 5.5. *Let σ be the standard deviation of the original performance score and let σ_ϵ be the standard deviation of the common shock. If $\sigma_\epsilon \geq \sqrt{3}\sigma$, the sensitivity of the performance measurement (after adding the common shock) is decreasing in σ_ϵ.*

Proof. Let σ be the standard deviation of the original performance score, and let $\Delta\mu$ and $\Delta\sigma$ be the change of mean and standard deviation after an agent slightly increases her effort by Δe respectively. By Assumption 3.3, $\Delta_m u \geq 0$ and $\Delta_m u + \Delta\sigma \geq 0$. Then, by definition,

$$\delta = \frac{\Delta\mu + \sqrt{(\sigma+\Delta\sigma)^2+\sigma_\epsilon^2} - \sqrt{\sigma^2+\sigma_\epsilon^2}}{\Delta e\sqrt{\sigma^2+\sigma_\epsilon^2}}.$$

Taking the first-order derivative over σ_ϵ,

$$\frac{\partial\delta}{\partial\sigma_\epsilon} = \frac{-\sigma_\epsilon\left(\sqrt{(\sigma+\Delta\sigma)^2+\sigma_\epsilon^2}\Delta\mu + (\sigma+\Delta\sigma)^2 - \sigma^2\right)}{\Delta e\left((\sigma+\Delta\sigma)^2+\sigma_\epsilon^2\right)^{\frac{1}{2}}\left(\sigma^2+\sigma_\epsilon^2\right)^{\frac{3}{2}}}.$$

Whether the derivative $\frac{\partial\delta}{\partial\sigma_\epsilon}$ is negative or not entirely depends on the numerator. We then let

$$h(\sigma_\epsilon) = -\left(\sqrt{(\sigma+\Delta\sigma)^2+\sigma_\epsilon^2}\Delta\mu + (\sigma+\Delta\sigma)^2 - \sigma^2\right).$$

We want to show $h(\sigma_\epsilon) \leq 0$.

$$\begin{aligned} h(\sigma_\epsilon) &\approx -\left(\sqrt{\sigma^2+\sigma_\epsilon^2}\Delta\mu + 2\sigma\Delta\sigma\right) && (\Delta\sigma \ll \sigma) \\ &\leq -2\sigma\left(\Delta\mu + \Delta\sigma\right) && (\Delta\mu \geq 0 \text{ and } \sigma_\epsilon \geq \sqrt{3}\sigma.) \\ &\leq 0. && (\Delta_m u + \Delta\sigma \geq 0) \end{aligned}$$

Proposition 5.5 suggests that there is a tradeoff between using the common shock to guarantee the truthfulness and applying the rank-order payment function to efficiently incentivize a desired effort. Adding a larger common shock increases the robustness of the mechanism against strategy reporting but may require in a larger payment to elicit a goal effort. Therefore, while guaranteeing the truthfulness, we want the variance of the common shock to be as small as possible.

□

5.4 Variational Robustness

Our discussions lead to a new aspect of the strategic robustness of the performance measurement. Under the tournament setting, a truthful performance measurement, which can always punish a unilateral untruthful deviation by decreasing its expected score, is not enough. A robust performance measurement should not only be truthful but also prevent untruthful strategies from increasing the variance of the performance measurement. We name this property of a performance measurement the *variational robustness*, as we will define more formally soon.

Under the tournament setting, the variational robustness is an important property which is related to both the truthfulness of the payment mechanism and how well the payment mechanism can elicit a goal effort at a low cost. This is because, as shown in Section 5.3, to guarantee the truthfulness of the payment mechanism, we have to add a common shock on the performance score which decreases the sensitivity of the performance measurement. Therefore, for a performance measurement that is less variationally robust, it has to scarify more of its sensitivity to achieve the truthfulness of the corresponding strategy-robust payment mechanism.

Definition 5.6. Given a performance measurement ψ that is truthful, let σ_τ be the standard deviation of the score distribution at the truth-telling strategy profile while fixing an effort ξ. Let σ_ϵ be the minimum standard deviation of the common shock that makes the strategy-robust payment mechanism which is consisted of ψ and the winner-take-all RO-payment function truthful. Then, the variational robustness (at the fixed effort ξ) of ψ is defined as $\vartheta_\psi = \frac{\sigma_\tau}{\sqrt{\sigma_\tau^2+\sigma_\epsilon^2}}$.

We measure the variational robustness as ϑ, which takes its value from $(0, 1]$. A ϑ_ψ of 1 implies that under such the performance measurement ψ, no unilateral untruthful deviation can be beneficial from the payment mechanism that is consisted of ψ and the winner-take-all RO-payment function even without adding the common shock. We also note that although ϑ is defined for truthful performance measurement under the winner-take-all RO-payment function, it can be generalized to untruthful performance measurement and other monotone RO-payment functions in a straightforward way. We will empirically evaluate this property of several commonly considered performance measurements in Section 7.2.

6 How Effective is The Rank-Order Payment Function: ABM Experiments

In this section, we compare the rank-order payment function with the linear payment function with respect to the minimum payment to elicit a goal effort. We will show that even after adding a common shock to guarantee truthfulness, the optimized rank-order payment function is still much more effective in eliciting effort than the linear payment function, which guarantees truthfulness without the common shock.

This section is structured as follows. First, we setup our agent-based model experiments and then use them to justify our assumptions: we will show that either the assumptions empirically holds or the theoretical results still holds even when the assumptions are not perfectly satisfied. Next, we provide a discussion on the inclusiveness of the optimal RO-payment function to see how it depends on the the goal effort, agents' cost functions and utility models. Finally, we present a comparison between the minimum payments of the optimal RO-payment functions and that of the linear payment functions.

6.1 Experiment Setup

6.1.1 Datasets

We use two crowdsourcing datasets to estimate the prior of ground truth w and agents' signal matrix Γ, called world 1 ($W1$) [3] and world 2 ($W2$) [27] respectively.

World 1 has a signal space of size 5 and a binary ground truth, say $\{1, 2\}$. Agents are asked to grade the synthetic accessibility of compounds with scores 1 to 5, where 1 indicates inappropriate to be synthesized and 5 stands for appropriate. Scores in between lower the confidence of the grading. The binary ground truth indicates whether a compound is appropriate or inappropriate. The dataset includes the assessments of 100 compounds (tasks) from 18 agents. World 2 has an identical signal space and ground truth space of size 4 (actually the size is 5, but we ignore the rarest one which occurs 9 out of 300 times). The dataset contains 6000 classifications of the sentiment of 300 tweets (tasks) provided by 110 workers. The estimated parameters for $W1$ and $W2$ are:

$$w_1 = \begin{bmatrix} 0.613 & 0.387 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0.684 & 0.221 & 0.032 & 0.037 & 0.026 \\ 0.092 & 0.191 & 0.050 & 0.200 & 0.467 \end{bmatrix};$$

$$w_2 = \begin{bmatrix} 0.196 & 0.241 & 0.247 & 0.316 \end{bmatrix}, \Gamma_2 = \begin{bmatrix} 0.770 & 0.122 & 0.084 & 0.024 \\ 0.091 & 0.735 & 0.130 & 0.044 \\ 0.033 & 0.062 & 0.866 & 0.039 \\ 0.068 & 0.164 & 0.099 & 0.669 \end{bmatrix}.$$

Note that we use the estimated confusion matrices as the underlying full-effort-working matrices Γ_{work}, which assumes the real-world agents are exerting full effort. Obviously, this is an under-estimation of Γ_{work} since the real-world agents' effort should be smaller than 1. Since the agents' effort cannot be directly observed, it is impossible to estimate the Γ_{work} with no bias. However, this is not a big concern since none of our experimental results depends on the correct estimation of Γ_{work}. Furthermore, the experiments are run with two different world models to show the robustness of our results.

6.1.2 The Performance Measurements

We implement two types of performance measurement: spot-checking and peer prediction mechanisms.

Spot-checking. Let p_c be the probability of spot-checking, i.e. the principal has the access to the ground truth of $n_c = p_c \cdot n$ randomly sampled tasks. We consider two spot-checking mechanisms in our experiments. First, a straightforward idea is to set the performance score to be the accuracy of the each agent's reports on the spot-check questions. We denote this performance measurement as **SC-Acc**.

Alternatively, we apply the mechanism considered in [12], which is inspired by the Dasgupta-Ghosh mechanism [4]. We denote this performance measurement as **SC-DG**. Given an agent's reports and a set of spot-checking questions with the ground truth, SC-DG randomly chooses a common task (bonus task) and two distinct tasks (penalty tasks). Then, the agent is scored 1 if her report on the bonus task agrees with the ground truth, and scored -1 if agreeing on the penalty tasks. The final score of the agent is the average score after repeated sampling.

Peer prediction. We consider five types of commonly used peer prediction mechanisms. The idea of peer prediction is to score each agent using some form of the correlation between her reports and her peers' reports.

First, we implement the naive idea of paying an agent 1 if her report on a random task agrees with a random peer's report on the same task, and paying 0 otherwise. This performance measurement is called the output agreement mechanism (**OA**) as discussed in [10].

Second, in the same paper, Faltings et al. [10] propose the peer truth serum (**PTS**) mechanism. The only difference between PTS and OA is that the payment is proportional to $\frac{1}{R(x)}$, where R is a public distribution of reports and x is the report of the pair of agents on the random task. We implement PTS by setting R to be the empirical distribution of all the other agents' reports other than i while computing agent i's payment.

Third, we consider the matrix f-mutual information mechanism (***f*-MMI**). Inspired by Kong and Schoenebeck [21], ***f*-MMI** scores each agent using the estimation of the f-mutual information between her reports and her peer's report, where f can be any convex function.

With attention to detail, the f-MMI uses the empirical distributions to estimate the mutual information. We can simply estimate the empirical distributions between two agents' reports, i.e. $\widetilde{P}_{\hat{X}_i,\hat{X}_j}$ for joint distribution and $\widetilde{P}_{\hat{X}_i}$ for marginal distribution. Then, the MI between reports $\hat{X}_i$ and $\hat{X}_j$ can be estimated,

$$\widetilde{MI}_{i,j}^{f-MMI} = \sum_{x,y} \widetilde{P}_{\hat{X}_i,\hat{X}_j}(x,y) f\left(\frac{\widetilde{P}_{\hat{X}_i}(x)\widetilde{P}_{\hat{X}_j}(y)}{\widetilde{P}_{\hat{X}_i,\hat{X}_j}(x,y)}\right). \tag{5}$$

The matrix mutual information mechanism then scores each agent i the average of the estimated MI between i and each of her peer. To speed up the mechanism, instead of pairing agent i with each of her peer j, we simply learn the empirical distributions of the reports on each task of all agents but i. This can be seen as a "virtual agent" reporting based on the empirical distributions of all agents but i. Then, we learn the joint distribution as well as the mutual information between agent i's reports and this virtual agent's reports.

Forth, we implement the pairing f-mutual information mechanism (***f*-PMI**) [25]. Similar to SC-DG, f-PMI randomly samples the bonus and penalty tasks and scores each agent based on whether her reports agree with the "ground truth" on the three tasks. The main difference is that instead of using the ground truth, the f-PMI learns a soft predictor on each task using all the other agents' reports. Then, a f-mutual information is estimated for each agent using the soft-predictor and the agent's reports. Note that the f-PMI contains the well known DG mechanism [4] and CA mechanism [26] as special case when f is $f(x) = \frac{1}{2}|x-1|$.

In a detailed manner, Schoenebeck and Yu [25] provide an alternative way to estimate the MI. Specifically, the quotient of the joint distribution between $\hat{X}_i$ and $\hat{X}_{-i}$ and the product of the marginal can be written as

$$\frac{P_{\hat{X}_i,\hat{X}_{-i}}(\hat{x}_i,\hat{x}_{-i})}{P_{\hat{X}_i}(\hat{x}_i)P_{\hat{X}_{-i}}(\hat{x}_{-i})} = \frac{P_{\hat{X}_i|\hat{X}_{-i}}(\hat{x}_i|\hat{x}_{-i})}{P_{\hat{X}_i}(\hat{x}_i)}. \tag{6}$$

The denominator can be empirically estimated. While the numerator is a soft-predictor, which, given the reports of all agents except i on a particular task j, produces a forecast of agent i's report on the same task in the form of a distribution. In our experiments, we set the soft-predictor for agent i's report on task j as the empirical distribution of all the other agents' reports on the same task.

For both f-MMI and f-PMI, we consider four types of commonly used f-divergence for the MMI and PMI mechanisms, as shown in Table 1.

Finally, we implement the determinant mutual information mechanism (**DMI**) Kong [18]. Kong [18] generalizes the Shannon mutual information to the determinant mutual information. Specifically, for a pair of agents i and j, the set of the commonly answered tasks is divided into two disjoint subsets A and B. Again, we empirically estimate the joint distribution with reports in A and B respectively, and score agent i the product of the determinants of these two estimated joint distribution matrices and take average over all the other agents.

Table 1: Four f-divergences

f-divergence (short name)	$f(a)$
Total variation distance (TVD)	$\frac{1}{2}\|a-1\|$
KL-divergence (KL)	$a\log a$
Pearson χ^2 (Sqr)	$(a-1)^2$
Squared Hellinger (Hlg)	$(1-\sqrt{a})^2$

6.1.3 Parameters And Methods

Now, we introduce the parameter setting of our agent-based model and how we use it to visualize the optimal payments.

Unless otherwise specified, we set the number of tasks to be $m = 1000$ with each agent answering $m_a = 100$ tasks. Every task is assigned to (at least) $n_0 = 5$ agents and there are $n = 52$ agents in total.[16] We consider various commonly used cost functions including the polynomial cost $c(e) = e^r$ and the exponential cost $c(e) = \exp(r \cdot e)$. When dealing with loss-averse agents, ρ is set to be 0.5. For SC, we vary the spot-checking probability from 0.1 to 0.3. For peer prediction, we use four types of commonly used f-mutual information with f listed in Table 1.

Given any of the performance measurements, we estimate the distributions of the performance score of an agent before and after a unilateral deviation of effort $\xi + \Delta e$ for $\xi \in [0, 1)$. To do so, we first sample the effort of all agents ξ from 0 to 0.99 with step size 0.01. Fixing each of the ξ, we simulate the report matrix $\boldsymbol{x}$ when all agents are exerting effort ξ and input $\boldsymbol{x}$ to the performance measurement which gives us n samples of the performance score before deviation. Then, let one of the n agents deviates to $\xi + \Delta e$ with $\Delta e = 0.01$ for estimation. Repeating the above process will give us one sample of the score after deviation. We then repeat the process and generate 5000 samples for each of the cases which gives us an estimation of the score distributions before and after the unilateral deviation. Finally, we use the generated samples to fit the Gaussian model. By estimating the mean and the standard deviation, we then can estimate the probability $p(\xi + \Delta e, \xi, j)$ for any $1 \leq j \leq n$, and the optimal RO-payment function can be developed based on our results from Section 4.1.[17]

While considering agents' reporting strategies, we generate samples and estimate the Gaussian model in the same way. In particular, fixing an effort ξ, we generate the samples of the performance scores of three

[16] The number of agents $n > 50$ is to guarantee that each task is assigned with at least $n_0 = 5$ agents and tasks are assigned to agents randomly.

[17] Note that when n is finite, there may not be integer solutions for the thresholds $\bar{n}$ and $\hat{n}$ in our propositions, and thus the thresholds are rounded to the closest integers.

cases: 1) all agents are truthful; 2) an agent i's performance score when she deviates to an untruthful strategy π; 3) and other agents' performance scores when agent i deviates.[18] Then, we fit the samples to the Gaussian models. We name the estimated distributions of the three cases $\hat{g}_1$, $\hat{g}_2$ and $\hat{g}_3$, with the corresponding means and standard deviations μ_i and θ_i for $i = 1, 2, 3$ respectively.

For the space of untruthful strategies, we heuristically choose a set of strategies that merge a signal to another. Specifically, while seeing a signal s, the agents report $\pi(s) \neq s$ with some probability (fixed at 0.5 in our experiments).[19] For example, three types of the strategies in $W1$ can be (1) mapping signal $4 \to 5$, (2) mapping $1 \to 2$ and $5 \to 4$ and (3) mapping $x \to x - 1$ for $x \in \{2, 3, 4, 5\}$.

6.2 Assumption Justifications

In our theory section, we made several assumptions to support our theoretical analysis. Now, we justify the assumptions with ABM experiments and show that the assumptions are (approximately) capturing the real problem correctly.

First, in Assumption 3.1 and 3.3, we assume the distribution of agents' performance score follows the Gaussian distribution and any unilateral deviation to a slightly higher effort leads to an increase in the mean of the score. Note that for any performance measurement with bounded scores, the distribution of the performance score is clearly not Gaussian. However, our experiments show that the Gaussian distribution can fit the performance score distributions of all of the considered performance measurements well with exceptions of DMI, KL-PMI and Hlg-PMI whose performance score distributions tend to be heavy-tailed. However, we note that these mechanisms are unlikely to perform well serving as performance measurements due to their high variance in the performance score.

Second, while applying a peer prediction mechanism, if an agent i unilaterally deviates by exerting a slightly different effort $e_i + \Delta e$ or playing an untruthful strategy, the distribution of other agents' performance scores can be changed, because the performance scores depend on the correlations between agents' reports. This obviously violates Assumption 3.2 and 3.5 especially when n is not large enough. For validity check, we estimate this change of distribution and confirm that it is trivial even when n is not significantly large, e.g. $n = 50$.

Third, as a commonly used approach, we assume the sufficiency of the first order constraint (FOC). However, for the optimal RO-payment functions that we considered, the performance measurements that we implemented, and the convex cost functions including the polynomial and exponential cost functions, we observe that the expected utility (Eq. (3)) is concave w.r.t. e_i in our setting. Therefore, there exists a unique e_i that maximizes each agent's expected utility which implies that FOC is sufficient for the existence as well as the uniqueness of the symmetric equilibrium.

Finally, although n is assumed to be sufficiently large to prove Theorem 4.1, we show that the lemma still holds for all the performance measurements that we considered when $n = 50$ which is a reasonable number of agents in the crowdsourcing setting. That's to say, all of our propositions hold even when n is not large enough.

6.3 Visualizing The Inclusiveness

To gain a better understanding of the optimal rank-order payment functions, we study how the goal effort ξ and the cost function c affect the inclusiveness of the optimal RO-payment functions, which indicates the number of agents that receive non-zero payments.

Fixing a performance measurement, we visualize the inclusiveness of the optimal RO-payment function versus ξ in Fig. 3 for neutral, loss-averse and risk-averse agents respectively. Our first observation is that IR is likely to be binding when the cost function is "less convex" and the effort ξ is high. This observation is in line with our theory as whether IR is binding depends on the ratio $\frac{c'(\xi)}{c(\xi)}$. To see this, in Theorem 4.2 for example, the condition for IR is not binding can be rewritten as $\frac{c'(\xi)}{c(\xi)} > np_1'(\xi)$. For most commonly used cost functions, e.g. quadratic or higher order power functions and exponential functions, this ratio is likely to

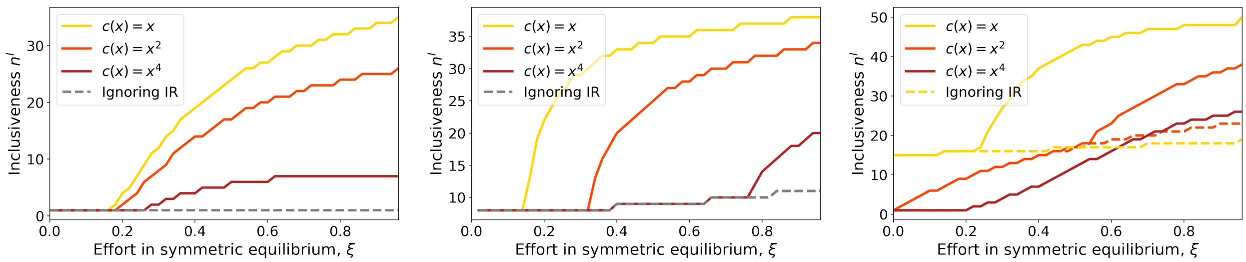
[18] As we will see in Section 6.2, the score distribution of case 3) is almost identical to that of case 1). In the following sections, we thus use the estimated score distribution of case 1) for the truth-telling distribution.

[19] The main reason that we consider mixed strategy is to avoid missing signals in agents' reports, which may greatly decrease the variance of performance scores. For example, it trivially results in all zero scores and thus zero variance for DMI.

be larger with more convex functions and ξ is approaching to 1. In both cases, compensating agents' cost requires more payment than maintaining the equilibrium.

Second, if we look at the lines with the same color, comparing with the neutral agents, the optimal RO-payment function is more inclusive, when agents are loss and risk-averse.

To sum up, the main takeaway is that when the cost function is less convex, or the goal effort is high, or the number of agents is large (all are common in the crowdsourcing setting), IR is likely to be binding and more inclusive payment functions are required (compared with the case when IR is ignored). Furthermore, the optimal RO-payment function is more inclusive in the case of both loss-averse and risk-averse agents (compared with the neutral agents).



(a) Neutral agents. (b) Loss-averse agents. (c) Risk-averse agents.

Figure 3: The inclusiveness of the optimal RO-payment functions under different agent utilities as a function of the symmetric equilibrium effort ξ with different cost functions. The solid curves represent the inclusiveness of the optimal RO-payment functions when considering IR while the dashed curves show the inclusiveness when IR is ignored. Note that for (a) and (b), the optimal RO-payment function does not depend on the cost functions and is all represented by the same grey dashed curve. In this example, we apply the SC-Acc mechanism with spot-checking probability 0.25. For (b), $\rho = 0.5$ and for (c), $r_a(t) = \log(t + 1)$.

6.4 Rank-Order VS Linear Payment Functions

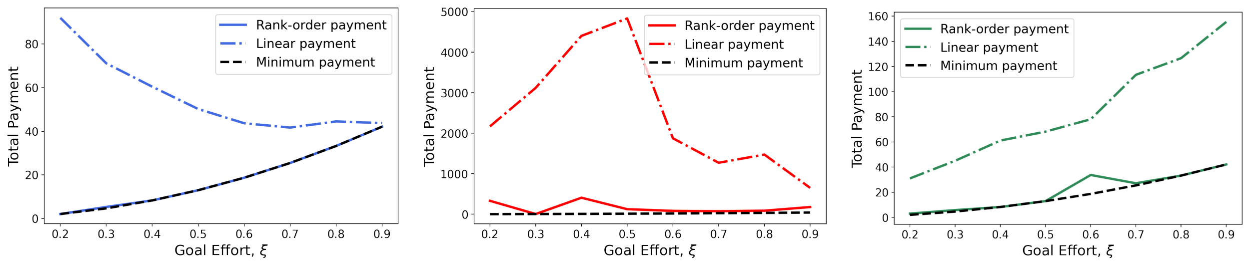
Recall that although the linear payment function can guarantee the truthfulness of the payment mechanism, they are not flexible to optimize and may experience enormous cost of budget in many cases. Here, we provide a more fine-grained comparison between the payment of the linear function and the payment of the rank-order function.

Parameters of The Linear Payment Function. Recall that a linear payment function rewards an agent $t_i = a \cdot s_i + b$ where s_i is the performance score and a and b are constant. However, for those performance measurements whose performance scores are unbounded below, i.e. negative infinite scores appear with positive probabilities, there do not exist constant factors to guarantee limited liability. Therefore, we first make the following modification to the linear payment function. Instead of considering the factor b as a constant, we treat it as a variable. That is, the linear payment function first sets the factor a to satisfy the first order condition: let $dc(\xi)$ be the derivative of the cost function at the goal effort ξ, and $ds(\xi)$ be the derivative of the expected performance score w.r.t. an agent's effort when her effort and all the other agents' effort are ξ. Then, given a and all agents' performance scores $\boldsymbol{s}$, the linear function computes the factor $\tilde{b}$ such that the minimum payment is equal to zero. This modification makes $\tilde{b}$ a variable that depends on agents' reporting strategies and thus does not technically preserve the truthfulness of the performance measurement.[20] However, because if b is a constant, to guarantee limited liability, it must be the case that $b \geq \tilde{b}$. Therefore, the modified linear payment function serves as a lower bound of the real total payment. We will show that even compared with the lower bound, the RO-payment function induces much smaller payments.

Adding Common Shock of The RO-payment function. Next, we apply the idea of the strategy-robust payment mechanism to make rank-order payments truthful. For every goal effort and every untruthful deviation, we compute the minimum variance of the common shock that can guarantee truthfulness. Then, for every goal effort, we pick the largest required common shock which guarantees that no unilateral deviation (in the strategy space that we considered) will result in a larger expected payment. The added common shock

[20]While it is theoretically possible in some cases, to game such a modified linear payment function with a unilateral untruthful deviation is considerably hard. An agent can benefit with an untruthful strategy only if she can reduce the bottom agent's score more than the decrease in her own score, which almost never happens when the number of agents is large.

enlarges the variance of the score distributions which decreases $\boldsymbol{p}'(\xi)$ the derivative of the ranking probabilities and thus increases the minimum payment. Our comparison is between the modified linear payment function discussed above and the RO-payment function after adding the common shock.



(a) Matrix mutual information mechanism with the Hellinger divergence (Hlg-MMI). (b) Pairing mutual information mechanism with the Hellinger divergence (Hlg-PMI). (c) Spot-checking with accuracy score (SC-Acc).

Figure 4: The comparison between the total payments of the modified linear payment functions and the optimal rank-order payment functions after adding the common shock. All three examples are in the case of risk-neutral agents (and thus the corresponding optimal RO-payment function when IR is not binding is winner-take-all by Theorem 4.2), and use the cost function of $c(e) = e^2$. The minimum payment curve corresponds to the case where IR is binding everywhere, i.e. the black-dashed curves are identical in three examples where the formula is $n \cdot c(\xi)$.

Results. Fig. 4 show three examples of the total payments of the modified linear payment function and the optimal RO-payment function which are winner-take-all. Our first observation is that the linear payment functions experience much larger payments almost for every goal effort (even when what is shown are lower bounds of the payments of the real linear payment functions). This is especially true for the performance measurements whose scores are unbounded below (e.g. the KL-PMI shown in Fig. 4 (b)), where the payments are hundreds or even thousands times of the payment achieved with the RO-payment functions.

The second takeaway from these examples is that the RO-payment function is very effective in eliciting the goal effort. This can be observed from the figures where the solid curves are almost identical to the black dashed curved.[21]

We note that Fig. 4 is based on the winner-take-all tournament. Similar pattern can be observed while considering more inclusive RO-payment functions. Furthermore, we obverse that a more inclusive RO-payment function is more robust against strategic reporting. That is, with a more inclusive payment function, the deviating agent needs a larger increase in the variance of the performance score to benefit, which results in a smaller required common shock.

Our observations here suggest that linear payment functions are not practical in realistic settings. We need more efficient payment functions to map the performance scores to the payments, and the rank-order payment function is a good choice.

7 Evaluating Realistic Performance Measurements

In the idealized setting, we have reduced the optimization of performance measurement to the problem of maximizing the sensitivity of a performance measurement. However, in reality, we are facing at least two problems.

On one hand, we cannot arbitrarily increase the sensitivity of a performance measurement. Instead, we are given several choices of performance measurements, namely, the spot-checking and the peer prediction mechanisms. We then ask: which mechanism has the highest sensitivity and how their sensitivities change with the setting, i.e. the goal effort of the principal.

On the other hand, achieving a high sensitivity is only part of the problem. To obtain truthfulness under the rank-order payment function, we have to add a common shock if the increase in the variance of the score distribution can be beneficial for some untruthful strategies. As shown in Section 5.3, such a common shock may harm the sensitivity. Therefore, the "strategy-robustness" of a performance measurement serves as a second important property of our problem.

[21]With an exception of the pairing mechanism, which is less robust against strategic reporting and thus requires a larger common shock to maintain truthfulness.

In this section, we use our agent-based model to empirically evaluate several commonly considered performance measurements as introduced in Section 6.1.2 with respect to the above two properties. Based on the comparison, we then suggest the best mechanism(s) for use.

7.1 The Sensitivity of Performance Measurements

Now, we visualize and compare the sensitivity of the performance measurements.

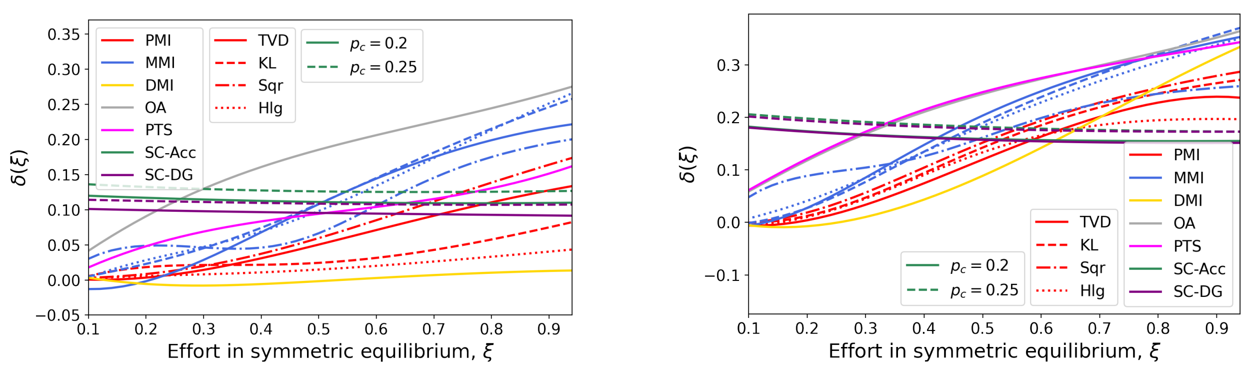
In Fig. 5, we show the (smoothed) sensitivity versus the goal effort for different performance measurements where higher curves are preferred. Note that there are cases where $\delta(\xi)$ can be negative. Our results are summarized below:

1. Our first observation is that the spot-checking mechanisms have a relatively consistent sensitivity, while most of the peer prediction mechanisms have increasing δ. This is because when the effort is large, the reports of agents have less variance. Consequently, the peer prediction mechanisms can estimate the correlations between agents' report more accurately which benefits $\delta(\xi)$. However, since the performance score of the spot-checking mechanism does not depend on the peers' reports and effort, $\delta(\xi)$ of the spot-checking mechanisms is more consistent for different ξ.

2. Not surprisingly, the $\delta(\xi)$ of the spot-checking mechanism is increasing as the spot-checking probability increases, which leads a more accurate estimation of agents' effort. Under the same condition, there is no significant difference between SC-Acc and SC-DG.

3. We observe that the the performances of peer prediction mechanisms have a large variance. If we focus on the high-effort range with $\xi \geq 0.6$, perhaps surprisingly, OA, a simple scoring rule has the highest sensitivity. The f-MMI mechanisms also do a good job in both $W1$ and $W2$, while the f-PMI mechanisms are less sensitive.

In summary, when the goal effort is low, our experiments suggest the use of spot-checking. When the required effort is large, applying a peer prediction based performance measurement can be cheaper. For the purpose of saving budget, if the agents are assumed to be reporting truthfully, the best performing performance measurements are SC-Acc for the spot-checking based performance measurement and OA, Hlg-MMI or KL-MMI for peer prediction based performance measurement.



(a) $W1$. (b) $W2$.

Figure 5: The ratio $\eta(\xi)$ versus ξ for different performance measurements in (a) $W1$ and (b) $W2$. Here, the performance score distributions are estimated with Gaussian models.

7.2 The Variational Robustness of Performance Measurements

Recall that both the sensitivity and the variational robustness of a performance measurement will affect its effectiveness of eliciting a goal effort. Using the same method in Section 6.1.3, we can estimate the variational robustness of different performance measurements. The two dimensions are shown in the scatter plot Fig. 6. For performance that are not theoretically truthful, we ignore the cases where untruthful deviations can bring positive gain in the expected score (which can neither be solved by applying the linear payment function) and only consider applying the common shock to deal with the untruthful strategies that decrease the expected score but increase the variance.

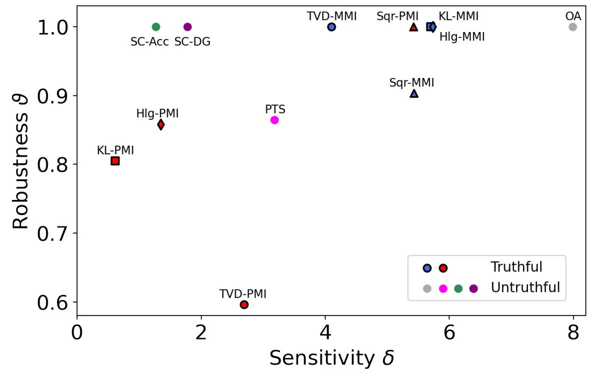


Figure 6: The sensitivity and the variational robustness of difference performance measurements in $W1$. The goal effort is fixed at $\xi = 0.8$. Performance measurements that are theoretically truthful have markers with black edges while those are not truthful have no edge.[22]

Although the absolute values of the variational robustness of performance measurements depend on the goal effort ξ and the RO-payment function, the following patterns generally hold. First, the higher the goal effort is, the more robust the performance measurements are, especially for peer prediction mechanisms. Intuitively, this is because a higher effort implies more information in agents' reports which can help the performance measurements to distinguish the untruthful deviations.

Second, in general, the pairing mutual information mechanisms (PMI) are not variationally robust, while the matrix mutual information mechanisms (MMI) especially KL-MMI and Hlg-MMI are consistently robust for high effort, i.e. $\xi \geq 0.5$. The output agreement mechanism (OA), though it is not truthful, is variationally robust for high effort.

Combining the comparisons on the sensitivity and the variational robustness, we saw that the matrix mutual information mechanisms KL-MMI and Hlg-MMI have both high sensitivity and high variational robustness and are (approximately) truthful. We therefore, in practice, to reward crowdsourincing, recommend using the matrix mutual information mechanism with the squared Hellinger Distance.

8 Conclusion And Future Work

We propose a two-stage payment mechanism to incentivize crowdsourcing workers which turns the design of crowdsourcing mechanisms with continuous-effort agents into a principal-agent problem. Centered around the two components of the payment mechanism: the performance measurement and the rank order payment function, we develop both theoretical and empirical analysis and obtain the following results (contributions): For the optimal rank order payment functions:

- We solve the optimal RO-payment functions while taking care of the individual rational (IR) constraint and show that the IR constraint results in more inclusive RO-payment functions;
- We additionally show that fairness-seeking agents (risk-averse and loss-averse agents) and principals are all reasons for people to prefer inclusive RO-payment functions;
- We observe that inclusive RO-payment functions are more robust against strategic reporting.

For the best performance measurement:

- We prove that the sensitivity of a performance measurement is the sufficient statistic of its superiority in the idealized setting;

[22]Note that the matrix mutual information mechanisms (MMI) are actually approximately truthful (a slightly weaker version of truthfulness) and the error vanishes as m the number of tasks is large enough.

- We empirically verify the effectiveness of sensitivity beyond the Gaussian assumptions;
- Our agent-based model experiments suggest that the matrix mutual information mechanism with the squared Hellinger Distance is a good performance measurement both in reducing the cost of principal and in being robust against strategic reporting.

Several promising future directions exist. First, heterogeneous agents that have different cost functions and confusion matrices could serve as a potential generalization of this paper, where the asymmetric or mixed strategy equilibrium should be considered. Second, we focus on rank-based payments in this paper, but our insights might be generalized to other contracts, e.g. the independent contract[14]? Finally, although RO-payment functions do not require much information from the principal, they do require some. In particular, at the desired effort, the agents' cost and its derivative, and agents' signal distributions must be estimated. How can these parameters be learned by the principal and how robust are mechanisms to mispecifications of these parameters?

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A Proof of Lemma 4.1

Proof. Fixing ξ, we simply let $g_e \sim N(\mu(e,\xi), \sigma(e,\xi))$ be the p.d.f. of the scores when agent i's effort is e and all the other agents' effort is ξ, and let G_e be the c.d.f.. Let S be a random variable with p.d.f. g_e. Let $q_e(p)$ be the quantile function of S such that $\int_{-\infty}^{q(p)} g_e(x)dx = p$.

Because $p_j(\xi,\xi) = \frac{1}{n}$, it's equivalent to show that $p_j(\xi',\xi)$ is decreasing in j, where $\xi' = \xi + \Delta e$. Note that $p_j(\xi',\xi)$ is the jth order statics, which concentrates to its expectation when n is sufficiently large. Therefore, $p_j(\xi',\xi)$ can be approximated by the quantile function, i.e. $p_j(\xi',\xi) = G_{\xi'}\left(q_\xi\left(1-\frac{j}{n}\right)\right) - G_{\xi'}\left(q_\xi\left(1-\frac{j+1}{n}\right)\right)$. Let $\mu = \mu(\xi,\xi)$ and $\Delta\mu = \mu(\xi',\xi) - \mu(\xi,\xi)$. Let σ and $\Delta\sigma$ be the similar notations for std. Note that $\Delta e \to 0$ implies $\Delta\mu \to 0$ and $\Delta\sigma \to 0$ since $\mu(e)$ and $\sigma(e)$ is differentiable (Assumption 3.1).

We first prove the following intermediate step.

Lemma A.1. *$G_{\xi'}(x) \approx G_\xi(x) - (\Delta\mu + \Delta\sigma)g_\xi(x)$.*

Proof.

$$\begin{aligned}
G_{\xi'}(x) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu-\Delta\mu}{\sigma+\Delta\sigma}\right)^2} ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{s-\mu}{\sigma} - \frac{\Delta\mu}{\sigma}\right)^2} ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{s-\mu}{\sigma}\right)^2 + \left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{(s-\mu)}{\sigma}\frac{\Delta\mu}{\sigma}} ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{s-\mu}{\sigma}\right)^2} \left(1 + \left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{(s-\mu)}{\sigma}\frac{\Delta\mu}{\sigma}\right) ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2} \left(1 + \frac{(s-\mu)}{\sigma}\frac{\Delta\sigma}{\sigma}\right)\left(1 + \frac{(s-\mu)}{\sigma}\frac{\Delta\mu}{\sigma}\right) ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2} \left(1 + \frac{\Delta\sigma+\Delta\mu}{\sigma}\frac{(s-\mu)}{\sigma}\right) ds \\
&= G_\xi(x) - (\Delta\mu + \Delta\sigma)g_\xi(x)
\end{aligned}$$

□

Then,

$$\begin{aligned}
p_j(\xi',\xi) &= G_{\xi'}\left(q_\xi\left(1-\frac{j}{n}\right)\right) - G_{\xi'}\left(q_\xi\left(1-\frac{j+1}{n}\right)\right) \\
&\approx \frac{1}{n} + (\Delta\mu+\Delta\sigma)\left(g_\xi\left(q_\xi\left(1-\frac{j+1}{n}\right)\right) - g_\xi\left(q_\xi\left(1-\frac{j}{n}\right)\right)\right) \qquad (7)
\end{aligned}$$

By assumption 3.3, $(\Delta\mu + \Delta\sigma)$ is positive. Then, it's sufficient to show $g_\xi\left(q_\xi\left(1-\frac{j+1}{n}\right)\right) - g_\xi\left(q_\xi\left(1-\frac{j}{n}\right)\right)$ is decreasing in j. To make our life easier, we consider this in the continuous scale. Let $p = 1 - \frac{j+1}{n}$ and $\Delta p = \frac{1}{n}$. Then, let $f(p) = g_\xi(q_\xi(p)) - g_\xi(q_\xi(p+\Delta p))$ with $p \in (0,1)$. We want to show that $f(p)$ is increasing in p.

First note that $\int_{-\infty}^{q_\xi(p)} g_\xi(x)dx = p$. Taking the derivative of p of both sides, we have $g_\xi(q_\xi(p)) = q_e'(p)^{-1}$. Thus, we want to show that $f(p) = q_e'(p)^{-1} - q_e'(p+\Delta p)^{-1}$ is increasing in p.

It is well know that the quantile of the Gaussian distribution can be represented by the inverse error function, i.e. $q(p) = \sqrt{2}\sigma \cdot \operatorname{erf}^{-1}(2p-1) + \mu$ for a Gaussian with mean μ and std σ, where erf^{-1} is the inverse error function. Furthermore, we know the derivative of the inverse error function is $\frac{d}{dx}\operatorname{erf}^{-1}(x) = \frac{1}{2}\sqrt{\pi}e^{\left(\operatorname{erf}^{-1}(x)\right)^2}$. Combining these,

$$\begin{aligned}
\frac{\partial}{\partial p}f(p) &= \frac{1}{\sqrt{2\pi}\sigma} \cdot \frac{\partial}{\partial p}\left(e^{-\left(\operatorname{erf}^{-1}(2p-1)\right)^2} - e^{-\left(\operatorname{erf}^{-1}(2(p+\Delta p)-1)\right)^2}\right) \\
&= \frac{\sqrt{2}}{\sigma}\left(-\operatorname{erf}^{-1}(2p-1) + \operatorname{erf}^{-1}(2(p+\Delta p)-1)\right)
\end{aligned}$$

Because $\text{erf}^{-1}(x)$ is increasing in x, we know $\frac{d}{dp}f(p)$ is positive which completes the proof.

□

B Proof Of The Optimal Rank Order Payment Function For Neutral Agents

B.1 Proof of Proposition 4.2

Proof. We start with solving the principal's optimization problem 4. Given that the agents are neutral we can write down the Lagrange and the KKT conditions as:

$$L(\hat{\boldsymbol{t}}, \alpha, \beta, \gamma) = \sum_{j}^{n} \hat{t}_j - \sum_{j}^{n} \alpha_j \hat{t}_j + \beta c(\xi) - \frac{\beta}{n} \sum_{j=1}^{n} \hat{t}_j - \gamma \sum_{j=1}^{n} p'_j(\xi) \cdot \hat{t}_i + \gamma c'(\xi).$$

① $\alpha_j = 1 - \frac{\beta}{n} - \gamma \cdot p'_j(\xi)$ for any $j \in [n]$;

② $\alpha_j \hat{t}_j = 0$ for any $j \in [n]$;

③ $\beta \cdot \left(c(\xi) - \frac{1}{n} \sum_{j=1}^{n} \hat{t}_j\right) = 0$;

④ $\sum_{j=1}^{n} p'_j(\xi) \cdot \hat{t}_j = c'(\xi)$;

⑤ $\boldsymbol{\alpha}, \beta \geq 0$;

⑥ $-\hat{\boldsymbol{t}}, (c(\xi) - \frac{1}{n} \sum_{j=1}^{n} \hat{t}_j) \leq 0$.

Let $\omega(\xi) = c'(\xi)/p'_1(\xi)$. Now, we show that if IR is not binding, the solution to this problem is $\hat{t}_1 = \omega(\xi)$ and $\hat{t}_j = 0$ for any $j > 1$. IR is not binding impels $\beta = 0$ (condition ③). Then, we look at condition ①. Note that $\alpha_j \geq 0$ for any j and at least one of the α_j is equal to zero. Otherwise $\hat{t}_j = 0$ for any j (condition ②), and condition ④ is violated. There are two possible cases: if $\gamma < 0$, $\alpha_j = 0$ if and only if $p'_j(\xi)$ reaches its minimum; If $\gamma > 0$, $\alpha_i = 0$ if and only if $p'_j(\xi)$ reaches its maximum. (Note that $\gamma = 0$ is trivially infeasible.)

In lemma 4.1, we show that $p'_j(\xi)$ is decreasing in j given a fixed ξ. This property implies that the first case, i.e. $\gamma < 0$, is not feasible. Because $p'_j(\xi)$ reaches its minimum when $j = n$. However, if $\alpha_n = 0$ and $\hat{t}_n > 0$, condition ④ is violated given that c is increasing (RHS of ④ is positive) and $p'_j(\xi) < 0$ (LHS of ④ is negative). Therefore, the only possible solution is $\alpha_1 = 0$ and $\hat{t}_1 > 0$. By condition ④, $\hat{t}_1 = \omega(\xi)$ as $\Delta e \to 0^+$.

The above argument assumes IR is not binding, when is true when $\omega(\xi) \geq n \cdot c(\xi)$ or equivalently, $\eta(\xi) \geq n \cdot p'_j(\xi)$. If $\eta(\xi) < n \cdot p'_i(\xi)$, IR is binding, which implies that $\sum_{j=1}^{n} \hat{t}_j = n \cdot c(\xi)$. Any RO-payment function that satisfies FOC and makes IR binding are optimal. If we apply a threshold RO-payment function that pays agent j $\hat{t}_j = \tau$ if $1 \leq j \leq \hat{n}$, we completes the proof by solving for $\hat{n}$ and τ. □

C Proof Of The Optimal Rank Order Payment Function For Loss-Averse Agents

C.1 Proof of Proposition 4.3

The precise version of Theorem 4.3 is shown here. For simplification, let $\eta(\xi) = \frac{c'(\xi)}{c(\xi)}$. Then, let

$$H(\xi, k) = \left(\left(1 + \rho \frac{n-k}{n}\right) \eta(\xi) - \sum_{j=2}^{k} p'_j(\xi) + \rho \sum_{j=k+1}^{n} p'_j(\xi)\right),$$

and let $L(k) = (1+\rho)(n-k) + 1$, we have the following results.

Proposition C.1. *Suppose $n \to \infty$, $\xi \in [0,1]$ and agents are loss-averse.*

1. ***IR is not binding:*** *If $H(\xi,\bar{n}) \geq L(\bar{n}) \cdot p_1'(\xi)$, the optimal RO-payment function satisfies $\hat{t}_1 = H(\xi,\bar{n}) \cdot c(\xi)/p_1'(\xi)$, $\hat{t}_j = c(\xi)$ for $1 < j \leq \bar{n}$ and $\hat{t}_j = 0$ for $\bar{n} < j \leq n$, with threshold $\bar{n}$ such that $(1+\rho)p_{\bar{n}}'(\xi) = p_1'(\xi)$;*
2. ***IR is binding:*** *Otherwise, the optimal RO-payment function makes IR binding and pays fewer agents 0, where $\hat{t}_1 = H(\xi,\hat{n}) \cdot c(\xi)/p_1'(\xi)$, $\hat{t}_j = c(\xi)$ for $1 < j \leq \hat{n}$ and $\hat{t}_j = 0$ for $\hat{n} < j \leq n$, with threshold $\hat{n}$ such that $H(\xi,\hat{n}) = L(\hat{n}) \cdot p_1'(\xi)$.*

Proof. Given the indifferentiability of the loss-averse utility, instead of using KKT conditions, we provide a more intuitive proof. As usual, we first ignore the IR constraint. Then the goal of the principal is to satisfied FOC with the minimum payments. Thus, starting with the all-zero payment, he will pay agents with the largest marginal return until FOC is satisfied. The marginal return of paying an agent with ranking j is

$$d\hat{t}_j \begin{cases} = (1+\rho)p_j'(\xi) & \text{if } \hat{t}_j < c(\xi), \\ \in [p_j'(\xi), (1+\rho)p_j'(\xi)] & \text{if } \hat{t}_j = c(\xi), \\ = p_j'(\xi) & \text{if } \hat{t}_j > c(\xi). \end{cases}$$

Then, by Lemma 4.1, the optimal RO-payment function pays each agent j their cost $c(\xi)$ in the order of their ranking until some $\bar{n}$ such that the principal is marginally better off to pay the top one agent more than $c(\xi)$ rather than paying the $\bar{n}+1$ agent anything positive. The threshold $\bar{n}$ therefore satisfies $(1+\rho)p_{\bar{n}}'(\xi) = p_1'(\xi)$. Thus, the optimal RO-payment function is $\hat{t}_j = c(\xi)$ for $1 < j \leq \bar{n}$, $\hat{t}_j = 0$ for $\bar{n} < j \leq n$ and $\hat{t}_1$ such that FOC is satisfied. This gives us

$$\hat{t}_1 = \left(\left(1 + \rho\frac{n-k}{n}\right) c'(\xi) - c(\xi)\sum_{j=2}^{k} p_j'(\xi) + \rho c(\xi) \sum_{j=k+1}^{n} p_j'(\xi) \right) / p_1'(\xi).$$

Let $H(\xi,k) = \left(\left(1+\rho\frac{n-k}{n}\right)\eta(\xi) - \sum_{j=2}^{k} p_j'(\xi) - \rho\sum_{j=k+1}^{n} p_j'(\xi) \right)$, then $\hat{t}_1 = H(\xi,\bar{n}) \cdot \frac{c(\xi)}{p_1'(\xi)}$. The condition for this to be true relies on IR being satisfied, i.e. $\hat{t}_1 + (\bar{n}-1)c(\xi) \geq nc(\xi) + \rho(n-\bar{n})c(\xi)$. Let $L(k) = (1+\rho)(n-k)+1$. The condition becomes $H(\xi,\bar{n}) \geq L(\bar{n}) \cdot p_1'(\xi)$.

When IR is binding, i.e. $H(\xi,\bar{n}) < L(\bar{n}) \cdot p_1'(\xi)$, the payments satisfy $\sum_{j=1}^{n} \hat{t}_j = nc(\xi) + \sum_{j=1}^{n} \rho(c(\xi) - \hat{t}_j)^+$. Then, the goal is to minimize $\sum_{j=1}^{n} \rho(c(\xi) - \hat{t}_j)^+$, i.e. to overcome as more agents' cost as possible. With the same argument, the optimal ORPF pays agents with ranking smaller than some threshold $\hat{n}$ their cost and pay the top one agent $\hat{t}_1$ such that FOC is satisfied and IR is binding. This gives us $\hat{t}_1 = H(\xi,\hat{n}) \cdot \frac{c(\xi)}{p_1'(\xi)}$ and $\hat{n}$ such that $H(\xi,\hat{n}) = L(\hat{n}) \cdot p_1'(\xi)$. Note that $\hat{n} < n$ because when $\hat{n} = n$, IR is satisfied because everyone is paid her cost but FOC can never be satisfied because there is no incentive to exert higher effort.

Finally, we complete the proof by showing $\hat{n} \geq \bar{n}$. Note that $H(\xi,\bar{n}) < L(\bar{n}) \cdot p_1'(\xi)$ but $H(\xi,\hat{n}) = L(\hat{n}) \cdot p_1'(\xi)$. We only have to show that the marginal return of increasing k is positive for function $H(\xi,k) - L(k) \cdot p_1'(\xi)$. We have that the marginal return is $(1+\rho)p_1'(\xi) - (1+\rho)p_k'(\xi) \geq 0$, which completes the proof.

□

C.2 Proof of Corollary 4.5

Proof. The proof is straightforward. With Proposition C.1, when IR is not binding, $n^I = \bar{n}$ which is determined by $(1+\rho)p_{\bar{n}}'(\xi) = p_1'(\xi)$. Because $p_j'(\xi)$ is decreasing in j by Lemma 4.1, $\bar{n}$ is increasing in ρ.

When IR is binding, $n^I = \hat{n}$ determined by $H(\xi,\hat{n}) = L(\hat{n}) \cdot p_1'(\xi)$. If we can show that $H(\xi,k) - L(k) \cdot p_1'(\xi)$ is decreasing in ρ, we can complete the proof because we know that $H(\xi,k) - L(k) \cdot p_1'(\xi)$ is increasing in k. It turns out the derivative of this term w.r.t. ρ is $\frac{n-k}{n}\eta(\xi) - \sum_{j=k+1}^{n} p_j'(\xi) - (n-k)p_1'(\xi)$. Because $\sum_{j=k+1}^{n} p_j'(\xi) \leq 0$ for any k, and $\frac{\eta(\xi)}{n} < p_1'(\xi)$ when $n \to \infty$, the derivative is negative and we complete the proof.

□

D Proof Of The Optimal Rank Order Payment Function For Risk-Averse Agents

D.1 Proof of Proposition 4.6

The precise version of Theorem 4.6 is shown here. Let $\phi(x) = r_a^{-1}(x)$ be the inverse of the reward function, and ϕ' be the derivative. Let $\upsilon(j, k, \beta, \xi) = (\phi')^{-1}\left(\left(\phi'(0) - \frac{\beta}{n}\right) \cdot \frac{p'_j(\xi)}{p'_{k+1}(\xi)} + \frac{\beta}{n}\right)$.

Proposition D.1. *Suppose $n \to \infty$ and agents are risk-averse.*

1. ***IR is not binding:*** *If $\sum_{j=1}^{\bar{n}} \upsilon(j, \bar{n}, 0, \xi) \geq n \cdot c(\xi)$, the optimal RO-payment function satisfies $r_a(\hat{t}_j) = \upsilon(j, \bar{n}, 0, \xi)$ for $1 \leq j \leq \bar{n}$ and $\hat{t}_j = 0$ otherwise, with $\bar{n} \leq \frac{n}{2}$ determined by the FOC constraint, i.e. $\sum_{j=1}^{\bar{n}} p'_j(\xi) \cdot \upsilon(j, \bar{n}, 0, \xi) = c'(\xi)$;*

2. ***IR is binding:*** *Otherwise, the optimal RO-payment function satisfies $r_a(\hat{t}_j) = \upsilon(j, \hat{n}, \beta, \xi)$ for $1 \leq j \leq \hat{n}$ and $\hat{t}_j = 0$ otherwise, with $\hat{n} \geq \bar{n}$ and β determined by the FOC and IR constraints.*

Proof. Because $\phi(x) = r_a^{-1}(x)$ is a differentiable convex function, the problem is a convex optimization problem. We can rewrite the principal's problem in terms of $r_j = r_a(\hat{t}_j)$ and write down the Lagrange and the KKT conditions.

$$L(\boldsymbol{r}, \alpha, \beta, \gamma) = \sum_{j}^{n} \phi(r_j) - \sum_{j}^{n} \alpha_j r_j + \beta c(\xi) - \frac{\beta}{n} \sum_{j=1}^{n} r_j - \gamma \sum_{j=1}^{n} p'_j(\xi) \cdot r_i + \gamma c'(\xi).$$

① $\alpha_j = \phi'(r_j) - \frac{\beta}{n} - \gamma \cdot p'_j(\xi)$ for any $j \in [n]$;

② $\alpha_j r_j = 0$ for any $j \in [n]$;

③ $\beta \cdot \left(c(\xi) - \frac{1}{n}\sum_{j=1}^{n} r_j\right) = 0$;

④ $\sum_{j=1}^{n} p'_j(\xi) \cdot r_j = c'(\xi)$;

⑤ $\boldsymbol{\alpha}, \beta \geq 0$;

⑥ $-\boldsymbol{r}, (c(\xi) - \frac{1}{n}\sum_{j=1}^{n} r_j) \leq 0$.

Again, we start with the case where IR is not binding and $\beta = 0$. Thus, by ①, $\alpha_j = \phi'(r_j) - \gamma \cdot p'_j(\xi)$. Whenever $\hat{t}_j > 0$, $r_j > 0$ and $\alpha_j = 0$. By Lemma 4.1, $p'_j(\xi)$ is decreasing in j, and for the same reason in Appendix B.1, $\gamma > 0$. Therefore, the optimal payment scheme takes a threshold form for some threshold $\bar{n}$ where $\hat{t}_j > 0$ for $1 \leq j \leq \bar{n}$ and $\hat{t}_j = 0$ otherwise. Furthermore, the payments satisfy that $\frac{\phi'(r_1)}{p'_1(\xi)} = \frac{\phi'(r_2)}{p'_2(\xi)} = \cdots = \frac{\phi'(0)}{p'_{\bar{n}+1}(\xi)}$, or alternatively $r_j = (\phi')^{-1}\left(\phi'(0) \cdot \frac{p'_j(\xi)}{p'_{\bar{n}+1}(\xi)}\right)$. Note that because $r'_a(0) < \infty$, $\phi'(0) > 0$ and the solution is feasible. Then, to find the threshold $\bar{n}$, we can simply solve the FOC constraint, i.e. $\sum_{j=1}^{\bar{n}} p'_j(\xi) \cdot r_j = c'(\xi)$. The solution does not take a clean closed-form, but we know that $\bar{n} \leq \frac{n}{2}$ because $p'_j(\xi) \leq 0$ when $j \geq \frac{n}{2}$ (Eq. (7)), in which case $\alpha_j > 0$ for sure.

When IR is binding and $\beta > 0$, the same arguments still hold and $r_j = (\phi')^{-1}\left(\left(\phi'(0) - \frac{\beta}{n}\right) \cdot \frac{p'_j(\xi)}{p'_{\hat{n}+1}(\xi)}\right)$. Again, by soling IR is binding and FOC is satisfied, we have solutions for β and $\hat{n}$. Furthermore, we know that while fixing any ξ, in the case where IR is considered, the threshold $\hat{n}$ is no less than $\bar{n}$ which is the threshold when IR is not considered. First, if $\phi'(0) - \frac{\beta}{n} < 0$, $p'_{\hat{n}+1}(\xi) < 0$ and $\hat{n} \geq \frac{n}{2} \geq \bar{n}$. Second, if $\phi'(0) - \frac{\beta}{n} \geq 0$, suppose $\bar{n} > \hat{n}$. Every r_j in the IR-binding case is smaller than the case where IR is not binding. Consequently, ④ is violated which implies that $\bar{n} \leq \hat{n}$.

□

D.2 Risk-aversion And Inclusiveness

Now, we show that more risk-averse agents does not imply more an inclusive optimal RO-payment function.

Corollary D.2. *Suppose $n \to \infty$ and agents are risk-averse. Let r_{a1} and r_{a2} be two concave reward functions of agents such that $\frac{\phi_1'(x)}{\phi_1'(0)} > \frac{\phi_2'(x)}{\phi_2'(0)}$ for any $x > 0$, where ϕ_1' and ϕ_2' are the derivative of the inverse of r_{a1} and r_{a2} respectively. Then, if IR is not binding, the optimal RO-payment function when agents have reward function r_{a1} is more inclusive than the case of r_{a2}. However, if IR is not binding, both cases are possible.*

Proof. First, we show that if IR is not binding, the RO-payment function is more inclusive as $\frac{\phi'(x)}{\phi'(0)}$ becomes larger for any $x > 0$. By Proposition D.1, when IR is not binding, the optimal RO-payment function is determined by

$$\frac{\phi'(r_a(\hat{t}_j))}{\phi'(0)} = \frac{p_j'(\xi)}{p_{\bar{n}+1}'(\xi)}. \tag{8}$$

Suppose $\frac{\phi_1'(x)}{\phi_1'(0)} > \frac{\phi_2'(x)}{\phi_2'(0)}$ for any $x > 0$, but $\hat{\boldsymbol{t}}_1$ is more exclusive than $\hat{\boldsymbol{t}}_2$, i.e. $\bar{n}_1 < \bar{n}_2$. Then, for any $j \leq \bar{n}$, $\frac{p_j'(\xi)}{p_{\bar{n}_1+1}'(\xi)} < \frac{p_j'(\xi)}{p_{\bar{n}_2+1}'(\xi)}$ due to Lemma 4.1. As a result, to satisfy eq. (8), $r_{a1}(\hat{t}_{1,j}) < r_{a2}(\hat{t}_{2,j})$ for any $j \leq \bar{n}_1 \leq \bar{n}_2$. However, one of the payments, $\hat{\boldsymbol{t}}_1$ or $\hat{\boldsymbol{t}}_2$ must violate IR, which implies $\sum_{j=1}^{\bar{n}} r_a(\hat{t}_j) = c(\xi)$, because $\sum_{j=1}^{\bar{n}_1} r_a(\hat{t}_{1,j}) < \sum_{j=1}^{\bar{n}_2} r_a(\hat{t}_{2,j})$. Therefore, $\hat{\boldsymbol{t}}_1$ must be at least as inclusive as $\hat{\boldsymbol{t}}_2$.

Second, we show that this pattern does not generally hold when IR is binding. Now, the optimal RO-payment function must satisfy

$$\frac{\phi'(r_a(\hat{t}_j)) - \frac{\beta}{n}}{\phi'(0) - \frac{\beta}{n}} = \frac{p_j'(\xi)}{p_{\hat{n}+1}'(\xi)}, \tag{9}$$

with $\beta > 0$. On one hand, the optimal RO-payment function can be more exclusive as $\frac{\phi'(x)}{\phi'(0)}$ increasing. Again, suppose $\frac{\phi_1'(x)}{\phi_1'(0)} > \frac{\phi_2'(x)}{\phi_2'(0)}$ for any $x > 0$ and $\phi_1'(0) = \phi_2'(0)$. In this case,

$$\frac{\phi_1'(x) - \frac{\beta}{n}}{\phi_1'(0) - \frac{\beta}{n}} - \frac{\phi_2'(x) - \frac{\beta}{n}}{\phi_2'(0) - \frac{\beta}{n}} = \frac{\phi_1'(x) - \phi_2'(x)}{\phi_1'(0) - \frac{\beta}{n}} > 0.$$

This implies that if $\phi_1'(0) - \frac{\beta}{n} > 0$, the same arguments in the IR not binding case still hold and $\hat{\boldsymbol{t}}_1$ must be at least as inclusive as $\hat{\boldsymbol{t}}_2$.

On the other hand, $\hat{\boldsymbol{t}}_1$ can be more exclusive when $\frac{\phi_1'(x)}{\phi_1'(0)} > \frac{\phi_2'(x)}{\phi_2'(0)}$ for any $x > 0$. Consider the case where $\phi_1'(x) - \phi_2'(x) > \phi_1'(0) - \phi_2'(0)$, $0 < \phi_2'(0) < \phi_1'(0) < \frac{\beta}{n}$. In this case,

$$\begin{aligned}
\frac{\phi_1'(x) - \frac{\beta}{n}}{\phi_1'(0) - \frac{\beta}{n}} - \frac{\phi_2'(x) - \frac{\beta}{n}}{\phi_2'(0) - \frac{\beta}{n}} &= \frac{\phi_1'(x)\phi_2'(0) - \phi_2'(x)\phi_1'(0) + \frac{\beta}{n} \cdot (\phi_2'(x) - \phi_1'(x) + \phi_1'(0) - \phi_2'(0))}{(\phi_1'(0) - \frac{\beta}{n}) \cdot (\phi_2'(0) - \frac{\beta}{n})} \\
&< \frac{\phi_1'(x)\phi_2'(0) - \phi_2'(x)\phi_1'(0) + \phi_2'(0) \cdot (\phi_2'(x) - \phi_1'(x) + \phi_1'(0) - \phi_2'(0))}{(\phi_1'(0) - \frac{\beta}{n}) \cdot (\phi_2'(0) - \frac{\beta}{n})} \\
&= \frac{(\phi_1'(0) - \phi_2'(0)) \cdot (\phi_2'(0) - \phi_2'(x))}{(\phi_1'(0) - \frac{\beta}{n}) \cdot (\phi_2'(0) - \frac{\beta}{n})} \\
&\leq 0.
\end{aligned}$$

This implies that when agents become more risk-averse, i.e. $\frac{\phi_1'(x)}{\phi_1'(0)} > \frac{\phi_2'(x)}{\phi_2'(0)}$ for any $x > 0$, the LHS of eq. (9) becomes smaller. Now, suppose $\hat{n}_1 > \hat{n}_2$. We have $p_{\hat{n}_1+1}' < p_{\hat{n}_2+1}' < 0$, and thus $\frac{p_j'(\xi)}{p_{\hat{n}_1+1}'(\xi)} > \frac{p_j'(\xi)}{p_{\hat{n}_2+1}'(\xi)}$ for any $j \leq \hat{n}_2$. As a result, to satisfy eq. (9), $r_{a1}(\hat{t}_{1,j}) > r_{a2}(\hat{t}_{2,j})$ for any $j \leq \hat{n}_2$. Again, this violates the IR constraint for the same reason in the IR not binding case, which implies $\hat{n}_1 \leq \hat{n}_2$.

□

E Proof of Proposition 4.8

Proof. While fixing ξ and ξ', we view $p_j(\xi', \xi)$ as a function of $\mu(\xi)$ and $\sigma(\xi)$, denoted as $p_j(\mu, \sigma, \xi', \xi)$.

The intuition of the proof is that suppose $\hat{\boldsymbol{t}}^*$ is the optimal RO-payment function when performance measurement Ψ is applied. Now, fixing ξ, if $\delta(\xi)$ increases, we show that the FOC constraint is easier to be satisfied, i.e. FOC can be satisfied with strictly lower total payment. This implies that with a performance measurement that has higher sensitivity, the principal is at least not worse-off. To see this, when IR is not binding, it is straightforward that the principal can reduce the payments to satisfy FOC without violating IR and LL. When IR is not binding, the principal can reduce $\hat{t}_1$ by ϵ_1 and increase $\hat{t}_n$ by $\epsilon_n \leq \epsilon_1$ such that FOC is satisfied and IR is still binding.[23]

With this intuition, our goal is to show that FOC can be satisfied with strictly lower payment as δ increases. Let $\lambda_j = r_a\left(\hat{t}_j\right) - \rho(c(\xi') - \hat{t}_j)^+$. Note that the FOC constraint says that $\sum_{j=1}^{n}(p_j(\mu, \sigma, \xi', \xi) - \frac{1}{n}) \cdot \lambda_j = c'(\xi)$. Because the only term that depends on $\delta(\xi)$ is $p_j(\mu, \sigma, \xi', \xi)$. The rest of the proof can be summarized in Lemma E.1, which shows that the left-hand-side of the FOC constraint is increasing in δ while fixing the payment, or equivalently, FOC can be satisfied with lower payment as δ increases.

We then complete the proof by showing $\lambda_j = r_a\left(\hat{t}_j\right) - \rho(c(\xi') - \hat{t}_j)^+$ is decreasing in j under the optimal RO-payment function for any type of agents. This can be proven by showing that the optimal RO-payment function is monotone decreasing with $\hat{t}_j \leq \hat{t}_k$ if $j \geq k$. According to Proposition 4.2, C.1 and D.1, this is exactly the case.

$\square$

Lemma E.1. *For any $\xi \in [0,1]$, $\sum_{j=1}^{n} p_j(\mu, \sigma, \xi', \xi) \cdot \lambda_j$ is increasing in $\delta(\xi) = \frac{\mu'(\xi)+\sigma'(\xi)}{\sigma(\xi)}$ if $0 < \lambda_j \leq \lambda_k$ for any $1 \leq k \leq j \leq n$.*

Proof. Let μ, σ, $\Delta\mu$ and $\Delta\sigma$ be the same definitions as in Appendix B.1. With the same approach in Lemma A.1, we can rewrite the probability $p(\mu, \sigma, \xi', \xi, j)$ as Let μ and σ be the mean and std of the Gaussian score distribution when agent i's effort is ξ (her peers' effort is ξ as well). Let agent i deviate to effort $\xi' = \xi + \Delta e$ for $\Delta e \to 0$. The corresponding changes in the mean and std of the Gaussian are denoted as $\Delta\mu$ and $\Delta\sigma$ respectively. Let $G_0(x)$ be the c.d.f. of the standard Gaussian distribution. With the same approach in Lemma A.1, we can rewrite the probability $p(\mu, \sigma, \xi', \xi, j)$ as

$$
\begin{aligned}
p_j(\mu, \sigma, \xi', \xi) &= \int_{-\infty}^{\infty} g(\xi + \Delta e, x) \binom{n-1}{j-1} \left(G(\xi, x)\right)^{n-j} \left(1 - G(\xi, x)\right)^{j-1} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu-\Delta\mu}{\sigma+\Delta\sigma}\right)^2} \binom{n-1}{j-1} \left(\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2} ds\right)^{n-j} \left(1 - \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2} ds\right)^{j-1} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu-\Delta\mu}{\sigma+\Delta\sigma}\right)^2} \binom{n-1}{j-1} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{1}{2}y^2} dy\right)^{n-j} \left(1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{1}{2}y^2} dy\right)^{j-1} dx \\
&\qquad\qquad (y = \tfrac{s-\mu}{\sigma}) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\left(1-\frac{\Delta\sigma}{\sigma}\right)z - \frac{\Delta\mu}{\sigma}\right)^2} \binom{n-1}{j-1} \left(G_0(z)\right)^{n-j} \left(1 - G_0(z)\right)^{j-1} dz \qquad (z = \tfrac{x-\mu}{\sigma}) \\
&\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} \left(1 + \frac{\Delta\sigma + \Delta\mu}{\sigma} z\right) \binom{n-1}{j-1} \left(G_0(z)\right)^{n-j} \left(1 - G_0(z)\right)^{j-1} dz
\end{aligned}
$$

Let $\delta = \frac{\Delta\mu+\Delta\sigma}{\sigma}$. We have,

[23] For neutral agents, $\epsilon_n = \epsilon_1$, in which case the principal is equivalent. For risk/loss-averse agents, $\epsilon_n < \epsilon_1$ because such agents prefer more inclusive payments, in which case the principal is better-off.

$$
\begin{aligned}
&\sum_{j=1}^{n}\lambda_j\frac{\partial \Delta p_j}{\partial \delta}\\
=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ze^{-\frac{1}{2}z^2}\sum_{j=1}^{n}\lambda_j\tbinom{n-1}{j-1}\left((G_0(z))^{n-j}\left(1-G_0(z)\right)^{j-1}\right)dz\\
=&\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} ze^{-\frac{1}{2}z^2}\sum_{j=1}^{n}\lambda_j\tbinom{n-1}{j-1}\left((G_0(z))^{n-j}\left(1-G_0(z)\right)^{j-1}\right)dz\\
&-\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} ze^{-\frac{1}{2}z^2}\sum_{j=1}^{n}\lambda_j\tbinom{n-1}{j-1}\left((G_0(-z))^{n-j}\left(1-G_0(-z)\right)^{j-1}\right)dz\\
=&\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} ze^{-\frac{1}{2}z^2}\sum_{j=1}^{n}(\lambda_j-\lambda_{n-j+1})\tbinom{n-1}{j-1}\left((G_0(z))^{n-j}\left(1-G_0(z)\right)^{j-1}\right)dz\\
=&\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} ze^{-\frac{1}{2}z^2}\frac{1}{2}\left(\sum_{j=1}^{n}(\lambda_j-\lambda_{n-j+1})\tbinom{n-1}{j-1}\left((G_0(z))^{n-j}\left(1-G_0(z)\right)^{j-1}\right)\right.\\
&\left.+\sum_{k=1}^{n}(\lambda_{n-k+1}-\lambda_k)\tbinom{n-1}{n-k}\left((G_0(z))^{k-1}\left(1-G_0(z)\right)^{n-k}\right)\right)dz\\
=&\frac{1}{2\sqrt{2\pi}}\int_{0}^{\infty} ze^{-\frac{1}{2}z^2}\left(\sum_{j=1}^{n}\tbinom{n-1}{j-1}(\lambda_j-\lambda_{n-j+1})\left(G_0(z)\right)^{n-j}\left(1-G_0(z)\right)^{j-1}-\left(G_0(z)\right)^{j-1}\left(1-G_0(z)\right)^{n-j}\right)dz\\
\geq&0.
\end{aligned}
$$

The last inequality holds because when $z > 0$, the two terms within the summation are positive or negative or zero simultaneously, which results in a product that is always non-negative.

$\square$

F The Fairness-Seeking Principal

Here, we provide a variance of our standard principal model. We have been focusing on the risk-neutral principal who aims to minimize the total payment given a fixed goal effort level. However, the principal may want to pay the agents with surplus to trade-off the efficiency and the fairness of the payments for various reasons. For example, the principal wants to reduce the variance of the payments even though he has to pay more due to an intrinsic notion of fairness, social pressure, or the low participation rate. We model the fairness-seeking principal with a penalty term in their utility function, $\Theta(\hat{\boldsymbol{t}})$. In this way, the principal aims to solve the same problem in (4) but to minimize the linear combination of the total payment and the fairness cost, i.e.

$$
\min_{\hat{\boldsymbol{t}}} \quad \sum_{j=1}^{n}\hat{t}_j+\lambda\cdot\Theta(\hat{\boldsymbol{t}}).
$$

To simplify the analysis and provide intuitions, we consider two examples of the penalty functions while assuming the agents are neutral. First, similar to the loss-aversion case, let $\Theta(\hat{\boldsymbol{t}})=\sum_j(c-\hat{t}_j)^+$ where c is a positive constant, e.g. $c=c(\xi)$ with a goal effort ξ. This example models the fact that the principal wants to pay the agents some money to overcome their cost of effort. With the same arguments as in Appendix C.1, one can easily verify that the optimal RO-payment function in this case is similar to the loss-averse agents case as shown in Fig. 2. In the optimal RO-payment function, the principal pays a fraction of agents who are ranked higher than some threshold c and the top one agent more than c. The only difference lies in the threshold. For example, when IR is not binding, the optimal threshold $\bar{n}$ satisfies $p'_{\bar{n}+1}(\xi)=(1-\lambda)p'_1(\xi)$ instead of $(1+\rho)p'_{\bar{n}+1}(\xi)=p'_1(\xi)$ in Proposition 4.3.

For another example, let $\Theta(\hat{\boldsymbol{t}}) = \frac{1}{n}\sum_{j=1}^{n}(\hat{t}_j - \bar{t})^2$, where $\bar{t} = \frac{1}{n}\sum_{j=1}^{n}\hat{t}_j$. In this case, the principal aims to reduce the the variance of the payments. Since the LL, IR and FOC constraints are linear in $\hat{\boldsymbol{t}}$, the principal's problem is convex which can be numerically solved. We thus use our ABM to learn the probability $p'_j(\xi)$ and then explore the trade-off between the efficiency, i.e. the total payment, and the fairness, i.e. the variance of the payments.

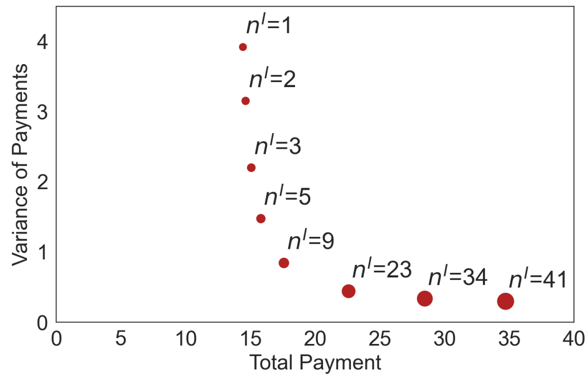

Figure 7: The trade-off between the efficiency and the fairness of the payments with λ varying from 0 to 1. In this example, we learn $p'_j(\xi)$ from the dataset of $W1$ with performance measurement the SC-Acc and $\xi = 0.8$. The cost function is $c(x) = x^8$.

In Fig. 7, we visualize the principal's trade-off between lowering the variance and saving the budget. The principal can trade-off the fairness and the efficiency by implementing a more inclusive mechanism which pays the agents more money than their cost of effort and use the surplus to reduce the variance. The trade-off depends on the setting, i.e. the cost function, goal effort and the confusion mapping. However, our numerical results suggest that the principal can greatly lower the variance without too much more cost of budget. For example, the variance can be reduced by 50% with only a 7% increase in the total payment in the example of Fig. 7.