High-Effort Crowds: Limited Liability via Tournaments

We consider how to optimally incentivize crowdsourcing agents to exert a desired level of effort when agents endogenously choose their effort from a continuum. We cast it as a principal-agent problem where the principal first applies a performance measurement to (noisily) evaluate agents' effort and then reward agents using a rank-order contest that satisfies limited liability (all payments are non-negative). In an idealized setting where the performance measurements are assumed to have Gaussian noise, we 1) solve the optimal rank-order payments for risk-neutral, risk-averse and loss-averse agents while considering individual rationality, and 2) identify a sufficient statistic of the quality of the performance measurements.

We further empirically evaluate two types of commonly used performance measurements: spot-checking and peer prediction mechanisms. Our real-data estimated agent-based model experiments generalize our theoretical results beyond the Gaussian assumption and suggest which performance measurements incentivize effort with the lowest cost. Moreover, we evaluate the robustness of performance measurements against agents' strategic reporting.

1 INTRODUCTION

Crowdsourcing, on platforms like Amazon Mechanical Turk, suffers from incentive problems. The requesters would like to pay the workers to incentivize high effort. However, workers can increase their payments by spending less time on each task and answering more tasks, which could wastefully spend the requesters' budgets. At the extreme, which has been extensively studied [4, 26], workers may answer with little effort or even randomly.

In settings with continuous effort, it matters not just whether workers exert effort, but how much effort they exert. There may be lackadaisical workers, who provide mediocre-effort work. For example, while labeling tweets for content moderation, people can report whatever is in their minds after reading the first sentence instead of carefully reading the whole tweet, or they can work on a fraction of tweets while skipping the rest. In these and many other cases, effort is not simple binary, but measured on a continuum. Evidence suggests that lackadaisical behaviors may be ubiquitous on crowdsourcing systems. In one study, 46% of Mechanical Turk workers failed at least one of the validity checks which was twice the percentage in student groups [2].

Eliciting effort from such crowdsourcing workers requires dealing with two problems: how to evaluate agents' effort given their reports on the assigned tasks, and how to reward agents for their effort given the potentially noisy measurement of agents' effort.

Performance Measurements. At the heart of the first problem is that the requester only receives a noisy measurement of the agent's effort, called a *performance score*. The noise may be due to a relatively small number of tasks each agent answers or a lack of ground truth upon which to base the agent's payment. Both are common in the crowdsourcing setting. The performance score is derived from a *performance measurement* which assesses the agent's effort based on her accomplished tasks.

Spot-checking mechanisms and peer prediction mechanisms are two candidates of the performance measurement. The former score agents based on their performances on a subset of the tasks with known ground truth while the latter score each agent according to the correlations in her reports and her peers' reports which works in the absence of ground truth.

Agents can act strategically in two ways. First, they can choose the amount of effort, and second they can deliberately misreport in an attempt to increase their performance score. Theoretically, spot-checking [11, 28] and peer prediction [4, 19, 26] mechanisms are designed to guarantee truthfulness: truthfully reporting (without manipulation) is an equilibrium. While intuitively they measure the quality of reports, and thus promote effort, a more precise characterization of this has only been put forth in the binary effort setting.

Payment Functions. To elicit a desired effort from agents in the continuous effort setting, the requester must carefully choose a *payment function* to transform the performance scores to the final payments. Existing literature on spot-checking and peer prediction derive payments by linearly rescaling the performance scores because this preserves the theoretically guarantees of truthfulness (agents cannot gain by manipulation). However, the linear payment functions typically violate limited liability—they rely on negative payments. Additionally requiring the payments to be non-negative may lead to extravagant payments. For example, say a desired equilibrium (e.g. all agents working with full effort) pays an agent \$10 in expectation, a possible deviation (e.g. working with 90% of effort) pays her \$9.9 in expectation, and the additional effort required to play the desired equilibrium over the alternative strategy costs the agent \$1 worth of effort. Now, because of limited liability, the payment function cannot subtract a constant (\$9.9) from the payments, and is forced to rescale the payment by 10 to compensate the cost of effort and pay each agent \$100. Such a problem is even troublesome for performance measurements whose scores are unbounded below.

We consider payment functions that satisfy limited liability which is always preferred and often required in realistic settings. Specifically, we consider the *rank-order payment function (RO-payment function)* where an agent's payment only depends on the rank of her score among the other agents. Rank-order payment functions are broadly studied as tournaments [6, 7, 13] and contests [9, 16]. Such a technique is desirable for its simplicity, which can be implemented without any foreknowledge of how the scores relate to effort. Furthermore, RO-payment functions trivially bound the ex-post budget. However, a challenge of using RO-payment functions is that it is not clear whether their nonlinear translations of performance scores into payments preserve the truthfulness guarantees of the performance measurements.

The Principal-Agent Problem. If we focus on eliciting effort and ignore, for now, the concern of truthfulness, we face a principal-agent problem. As a running example, suppose a principal wants to recover the ground truth of a batch of tasks using the collected labels from a group of homogeneous agents, who have the same utility function and information structure.[1] The principal first commits to a *payment mechanism* which consists of a performance measurement and a RO-payment function. Next, agents best-response the payment mechanism by exerting the same effort that forms a symmetric equilibrium, i.e. no unilateral deviation in effort can increase an agent's expected utility. The problem of the principal is to optimize the payment mechanism so that a goal effort can be elicited in the symmetric equilibrium with the minimum cost of budget.

Given the difficulty of the optimization problem for an arbitrary performance measurement, in the first part of this paper, we optimize the payment mechanism within the space of idealized performance measurements. That is, we assume the noise of the performance measurement follows the Gaussian distribution whose mean and standard deviation are functions of agents' effort.[2] With this assumption, the optimization can be naturally divided into two parts.

First, given an idealized performance measurement, we optimize the RO-payment function. In the economics literature, a related problem is referred to as a tournament. For example, one of the most well-known results shows that winner-take-all is optimal if both the principal and the agents are risk-neutral in a setting where individual rationality (IR) is not considered [8, 21]. Instead, in our setting, we require IR to be a constraint of optimization and observe that the optimal RO-payment function is more inclusive (rewards more agents) when IR is binding. We emphasize that not considering IR will lead to unreasonable solutions in a lot of cases: when the number of agents is large, the goal effort is high and/or agents' cost of effort is not very convex. All are common in the crowdsourcing setting. We further consider risk-averse and loss-averse agents and observe that these settings both result in more inclusive optimal RO-payment functions as compared to those for risk-neutral agents.

Second, under the optimal RO-payment function, we optimize the idealized performance measurement. We derive a sufficient statistic called the *sensitivity* that captures how much payment is required to elicit a certain performance. At a high level, a performance measurement with higher sensitivity is more accurate (has lower variance) and is more sensitive to changes in effort.

Realistic Performance Measurement. In reality, however, our aforementioned solution faces two problems. First, performance measurements like spot-checking and peer prediction mechanisms are not idealized and for some of them, the performance scores are not well approximated by Gaussian noise. To this end, we would like to verify the ability of sensitivity to predict the superiority of

[1] Although not without loss of generality, homogeneous agents are widely assumed in the principle-agent works [9, 23]. The selection process could result in increased homogeneity among agents' background. Furthermore, agents are homogeneous while dealing with objective tasks with low dependence on experience.

[2] This implicitly assumes that agents are reporting truthfully.

different performance measurements beyond the idealized setting and evaluate them for practical use.

Second, the payment mechanism is optimized while assuming agents are truthful. However, mechanisms should be developed to deal with agents who may strategically manipulate their reports. Because RO-payments are nonlinear, this may be an issue even with provably truthful performance measurements. Therefore, we would like to verify the robustness of the truth-telling equilibrium of different performance measurements under the optimal RO-payment function.

We conduct agent-based model (ABM) experiments with parameters estimated using two real-world crowdsourcing datasets. Our empirical results confirm that sensitivity is effective in predicting the efficacy of a performance measurement in reducing the principal's cost even when the Gaussian model is not a good fit. Then, we use our ABM to evaluate a set of commonly used spot-checking and peer prediction mechanisms using two desiderata: how effective is it in reducing the cost of the principal while eliciting a goal effort, and how robust is it against manipulation.

We summarize our contributions as follows.

- We propose a two-stage approach to develop payment mechanisms with limited liability which turns the problem of incentivize crowdousrcing workers for a desired effort into a principal-agent problem;
- We solve the principal's optimization problem under the solution concept of symmetric equilibrium and within the space of all idealized performance measurements and rank-order payment functions;
- Our solutions consider individual rationality which fill the gap of previous literature in tournament.
- We identify a sufficient statistic of the efficacy of the performance measurement in eliciting a goal effort at low cost, which provides a new dimension for evaluating the spot-checking and peer prediction mechanisms.
- We conduct agent-based model experiments to generalize our theory beyond the idealized setting and provide insights on which performance measurement to use in practice.

2 RELATED WORK

Tournament Design. The most relevant related works lie in the economics literature on tournaments. In line with our results, the WTA mechanism is proven to be optimal for neutral agents in small tournaments with symmetrically distributed noise [21], and in arbitrarily-sized tournament when the noise has increasing hazard rate [8]. The follow-up work [7] shows in the tournament setting that the equilibrium effort decreases as the noise of the effort measurement becomes more dispersed, in the sense of the dispersive order. In our situation with Gaussian noise, where the mean of the performance score is a function of effort, we show that it is the ratio of the derivative of the mean to the variance of the noise that affects the principal's utility.

Green and Stokey [13] compare tournaments with independent contracts which pay agents based on their numerical outputs rather than the ranking of the outputs. In their model where the outputs of agents depend not only on their effort but also on an unknown common shock, they show that if there is no common shock, the independent contracts dominant tournaments. However, if the distribution of the common shock is sufficiently diffuse, tournaments dominant independent contracts.

For risk-averse agents, Krishna and Morgan [21] show that the optimal RO-payment function is WTA when there are $n \leq 3$ risk-averse agents, and should pay the agent ranks in the second place positively when $n = 4$. Kalra and Shi [16] show that, for arbitrary number of agents, the more risk-averse the agents are, the larger the number of agents should be rewarded with a focus on logistic and uniform noise distributions. Drugov and Ryvkin [6] generalize their results by considering more general noise distributions and non-separable preferences.

We note that in the tournament literature, the IR constraint is buried into the sufficient conditions for the existence of pure strategy symmetric equilibrium. However, what the optimal payment function is while considering IR remains unknown. This problem is essential in our crowdsourcing setting where IR is likely to be binding.

Crowdsourcing And The Principal-Agent Problem. Additional literature considers the crowdsoucing problem from the principal-agent perspective. Ho et al. [15] model the crowdsourcing process as a multi-round principal-agent problem. Instead of equilibrium analysis, they solve the problem with multi-armed bandit algorithms. Ghosh and Hummel [12] consider agents with heterogeneous ability and endogenous effort and focus on analyzing the cases when the optimal contract supports an equilibrium that favors the principal's utility. The main difference is that they do not consider the payments to the agents as a cost of the principal (e.g. the payments are unredeamable points) which is not the case in our setting where crowd works are compensated with money. Easley and Ghosh [9] consider a crowdsourcing model where agents are strategic in deciding whether to participant in a task. Like [13], they focus on when the principal should apply an output-independent contract or a winner-take-all contest, which is shown to depend on the agents' behaviour models.

Spot-Checking And Peer Prediction. Literature on spot-checking and peer prediction focuses on designing truthful mechanisms (e.g. whether agents can benefit by manipulating their reports) mostly in the binary-effort case [4, 11, 20, 26]. Kong and Schoenebeck [18] consider a discrete hierarchical effort model where choosing higher effort is more informative but more costly. With assumptions, the maximum effort is proven to be elicitable and payments are optimized using a linear program that requires detailed knowledge of agent costs and quality.

In general, this work deals with payments by rescaling them to be large enough to motivate the agents. For example, say a desired equilibrium (e.g. working with full effort) pays \$10 in expectation, an alternative strategy profile (e.g. working with 90% effort) pays \$9.9 in expectation and the additional effort required to play the desired equilibrium over the alternative equilibrium costs the agent \$1. Then the mechanism can rescale the payments with an affine map so that the desired equilibrium pays \$10 in expectation and an alternative strategy profile pays \$0.

However, there are two large problems with this solution. First, it assumes a binary (or at least discrete) levels of effort so there is a non-trivial gap between the payment of the desired and alternative equilibria. Second, it assumes that negative payments are possible.[3] Neither of these is true in the setting we consider. Additionally, even an affine rescaling will not necessarily preserve expectations, and thus truthfulness, if agents are risk or loss adverse.

Our approach diverges sharply from previous peer prediction work which focuses nearly entirely on strategic considerations where linear rescaling is the only known technique available. Instead, we separate the agent choices of how much effort to exert from how honestly to report. This allows us to use a principal-agent framework to study how to elicit effort. In general, we obtain a weaker truthfulness guarantee, which is derived from empirical results showing that a litany of strategies do not work. Of course, it is possible that some strategy we failed to consider does work. However, our truthfulness results are slightly stronger in several ways as well. First, we need not rely on the number of agents going to infinity, but can run tests in finite settings (note that some mechanisms that are provably truthful in the limit of a large number of agents and conceivably allow beneficial

[3]If all original payment are guaranteed to be positive—as they are in some mechanisms—then if the mapping is restricted to be linear, instead of affine, no negative payments will be incurred. However, this can lead to extravagant payments. In our aforementioned example that payment of the desirable equilibrium would need to be rescaled to \$100.

manipulations in the finite settings we consider). Additionally, some of the mechanisms do not work for all prior distributions, and we can empirically test if they work for the prior distributions that we learn from data.

3 MODEL

Throughout the paper, we use capital notations denote random variables while lowercase notations denote their realizations. Bold notations denote vectors or matrices.

3.1 Crowdsourcing

A principal (requester) has a set of m tasks $[m] = \{1, 2, \ldots, m\}$. Each task $j \in [m]$ has a ground truth $y_j \in \mathcal{Y}$—that the principal would like to recover—which was sampled from a prior distribution $w \in \Delta_{\mathcal{Y}}$, where $\mathcal{Y}$ is a discrete set and $\Delta_{\mathcal{Y}}$ is the set of all possible distributions over $\mathcal{Y}$. To this end, each task is assigned to n_0 agents and each agent i is assigned a subset of tasks $A_i \subseteq [m]$. Let m_a denote the maximum number of tasks assigned to any agent. Then $|A_i| \leq m_a$ for every agent i. This implies a lower bound on the number of agents: $n \geq \lceil m \cdot n_0/m_a \rceil$.

Effort and cost. Agents are strategic in choosing an effort level. Let $e_i \in [0, 1]$ denote the effort chosen by agent i. Let $c(e)$ be a non-negative, increasing and convex cost function.

Signals and reports. Each agent i working on an assigned task j, receives a signal denoted $X_{i,j} \in \mathcal{X}$, where $\mathcal{X}$ is the signal space. We assume that $0 \notin \mathcal{X}$ and let $X_{i,j} = 0$ for any $j \notin A_i$. For tasks $j \in A_i$, $X_{i,j}$ are i.i.d. sampled from a distribution that depends only the ground truth y_j and agent i's effort level e_i.

Let Γ_{work} and Γ_{shirk} be $|\mathcal{Y}|$ by $|\mathcal{X}|$ matrices, where, for $y \in \mathcal{Y}$ and $s \in \mathcal{X}$, the y, s entry of Γ_{work} and Γ_{shirk} denotes the probability that an agent who puts in full effort and no effort, respectively, will receive a signal s when the ground truth is y.

Given e_i, agent i's signal $X_{i,j}$ for the jth task where the ground truth is y_j will be sampled according to the y_jth row of

$$e_i\Gamma_{\text{work}} + (1 - e_i)\Gamma_{\text{shirk}}.$$

We will let Γ_{shirk} be uniform in each column. This setup is a modified version of the Dawid-Skene (DS) model [5] where we have added effort.

We use $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ to denote the signal and report profiles of all agents respectively. Note that $\hat{\boldsymbol{x}}$ is not necessarily equal to $\boldsymbol{x}$ for strategic agents. For now, we assume all agents report truthfully, so that $\hat{\boldsymbol{x}} = \boldsymbol{x}$. Strategic reporting is discussed in Section 3.3.

Mechanism. Given $\hat{\boldsymbol{x}}$, a payment mechanism $\mathcal{M} : (\{0\} \cup \mathcal{X})^{n\times m} \to \mathbb{R}^n_{\geq 0}$ pays each agent i a non-negative payment t_i. We decompose the payment mechanism into two parts (Fig. 1). First, we apply a performance measurement $\psi : (\{0\} \cup \mathcal{X})^{n\times m} \to \mathbb{R}^n$ on agents' reports that outputs a (possibly negative and random) score $s_i = \psi(\hat{\boldsymbol{x}})_i$ for each i. In our experiments, we focus on two sets of the performance measurements: spot-checking and peer prediction, which will be discussed later.

Second, we apply a rank-order payment function that pays $\hat{t}_j$ to the j'th ranked agent according to performance score. WLOG, suppose $s_1 \geq s_2 \geq \cdots \geq s_n$. Then, agent i's payment is $t_i = \hat{t}_i$.

Definition 3.1. We call a RO-payment function increasing if $\hat{t}_j \geq \hat{t}_k$ if $j \leq k$.

3.2 The Principal-Agent Model

We seek a payment mechanism that maximizes the principal's payoff in the symmetric equilibrium. Now, we model this crowdsourcing problem as a principal-agent problem.

First, the principal assigns the tasks to agents and commits to a payment mechanism consisting of a performance measurement ψ and a RO-payment function $\hat{\boldsymbol{t}}$. Then, the agents respond by working

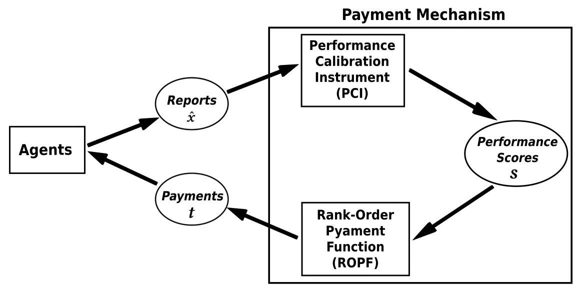


Fig. 1. Components of a payment mechanism: a performance measurement and a RO-payment function.

with some effort that maximize their expected utility. Intuitively, the effort affects the distribution of an agent's performance score and then affects the distribution of ranking as well as the payment. In this paper, we consider three utility functions:

$$u_a(t_i, e_i) = \begin{cases} t_i - c(e_i) & \text{for neutral agents,} \\ t_i - c(e_i) - \rho \cdot (c(e_i) - t_i)^+ & \text{for loss-averse agents,} \\ r_a(t_i) - c(e_i) & \text{for risk-averse agents.} \end{cases}$$

Here, for loss-averse agents[4], $(x)^+$ equals x for $x \geq 0$ and 0 otherwise, and ρ is a non-negative loss-aversion factor. For risk-averse agents, r_a is a non-negative, concave and differentiable function with $r_a(0) = 0$ and $r'_a(0) < \infty$. Loss-adverse agents incur additional loss when they are not compensated for the work they expended (e.g. when they are stiffed). Risk-adverse agents proportionally value moderate rewards more the high rewards.

Combined, we often use the following utility function for simplicity,

$$u_a(t_i, e_i) = r_a(t_i) - c(e_i) - \rho \cdot (c(e_i) - t_i)^+. \tag{1}$$

We focus on the solution concept of symmetric equilibrium. That is, all agents exerting effort ξ is an equilibrium if any unilateral deviation will decrease the expected utility, i.e. $\mathbb{E}[u_a(t_i(e_i, \xi), e_i)] \leq \mathbb{E}[u_a(t_i(\xi, \xi), \xi)]$ for any $e_i \in [0, 1]$, where $t_i(e_i, \xi)$ is a random payment function on agent i's effort and all the other agents' effort $e_k = \xi$ for any $k \neq i$.

The problem of the principal is then to optimize the payment mechanism such that a goal effort ξ can be incentivized in the symmetric equilibrium with the minimum payment.[5] Additionally requiring the payment to satisfy limited liability (LL) and individual rationality (IR) leads us to the principal's constraint optimization problem.

To get a sense for the IR constraint, consider the following situation: The principal would like 10 agents to each exert \$10 of effort. However, the principal implements a RO-payment function (e.g. the winner-take-all) which induces a symmetric equilibrium where the agents each contribute \$10 of effort but only pays the top agent \$65. This is not IR because each agent is only paid \$6.50 in expectation, but requires \$10 of effort. However, simply increasing the payment of the top agent,

[4]Note that we mainly consider the 1-order loss-aversion. We briefly discuss the case of higher order loss-aversion, i.e. $u_a(t_i, e_i) = t_i - c(e_i) - \rho \cdot ((c(e_i) - t_i)^+)^r$ for $r > 1$, in Section 4.1.2.

[5]We note that in reality, the optimization problem of the principal can be much harder. The principal can optimize over the space of the parameters of the crowdsourcing system such as the number of agents, the number of tasks each agent answers and the goal effort. However, the optimization over these parameters requires a finer grind model of the principal's utility, i.e. how does the principal evaluate the contributions from agents, which is beyond the interests of this paper. Therefore, we simply assume the principal fixes these parameters which fixes the (expected) reward of agents' contribution and he tries to optimize the payment mechanism to incentivize the goal effort with minimum cost. We will show, later in this paper, how the goal effort affects the principal's decision.

changes the effort in equilibrium. So a different payment structure is needed to ensure the original equilibrium.

The Gaussian assumptions. However, the optimization problem over the space of all performance measurements is still too hard to analyze. To make it theoretically tractable, as commonly assumed in principal-agent literature, we apply the Gaussian noise assumption. Again, let e_i be agent i's effort and ξ be all the other agents' effort.

Assumption 3.1. *We assume the agent i's performance score S_i follows the Gaussian distribution with p.d.f. $g^{(i)}_{e_i,\xi}$ and c.d.f. $G^{(i)}_{e_i,\xi}$, where the mean $\mu(e_i,\xi)$ and standard deviation $\sigma(e_i,\xi)$ are functions of agents' effort. Furthermore, let $g^{(-i)}_{e_i,\xi}$ and $G^{(-i)}_{e_i,\xi}$ be the same notations for all the other agents' score distribution under the same effort profile. We assume μ and σ to be differentiable.*

Assumption 3.2. *We assume the distribution $g^{(-i)}_{e_i,\xi}$ is independent of e_i.*

Assumption 3.2 implies that any unilateral deviation $e_i \in [0,1]$ from a symmetric effort profile where all agents' effort is ξ will not change other agents' score distribution. This implies $g^{(-i)}_{e_i,\xi} = g^{(i)}_{\xi,\xi}$. This assumption is intuitively true for spot-checking mechanisms where agents' performance scores are independent conditioned on the ground truth, and for peer prediction mechanisms when the number of agents is large. In Section 5.2, we show that for peer prediction mechanisms with a reasonably large n, this assumption approximately holds with a small deviation $|e_i - \xi|$, which is all we ask in our theory. For simplicity, through out the paper, we use $g_{e_i,\xi}$ to denote agent i's score distribution and $g_{\xi,\xi}$ to denote other agents' score distribution.

We additionally make the following assumption which, at a high level, guarantees that a unilateral deviation to a higher effort does not harm the agent's expected performance score.[6]

Assumption 3.3. *Fixing ξ, let $\mu'_\xi(e_i) = \frac{\partial \mu(e_i,\xi)}{\partial e_i}$ be the derivative of $\mu(e_i,\xi)$ over e_i as a function of e_i and $\sigma'_\xi(e_i)$ is the similar notation for the standard deviation. We assume $\mu'_\xi(e_i) + \sigma'_\xi(e_i) \geq 0$ for any $e_i, \xi \in [0,1]$.*

Definition 3.2. We call a performance measurement an idealized performance measurement if it can generate performance scores that satisfy Assumption 3.1, 3.2 and 3.3.

3.3 Strategic Reports

We consider strategic reporting in Section 6.2 and define the terminologies here. For agent i, given her signal X_i, let $\hat{X}_i$ be the report of agent i. Her strategy σ_i is a random mapping from X_i to $\hat{X}_i$. As a common assumption in the peer prediction literature [1, 4], we assume agents' strategies are task-independent, which implies that agent i will first choose a $\pi_i : \mathcal{X} \to \Delta_{\mathcal{X}}$, then draws $\hat{X}_i$ from the distribution $\pi_i(X_i)$ as her reports. For example, agents can combine signal 3 and 4 to 4 by reporting 4 whenever seeing a signal 3.

4 OPTIMIZING PAYMENT MECHANISM IN THE IDEALIZED SETTING

This section answers the question of how to pay agents optimally for a desired effort level in the idealized setting. The optimization consists of two parts: optimizing RO-payment function while fixing any idealized performance measurement, and optimizing idealized performance measurement given the optimal RO-payment function. For the former, we observe that the optimal RO-payment function is increasing for all agent utility models that we considered, and both risk/loss-aversion and individual rationality will make the optimal RO-payment function more inclusive which

[6] In our experiments, we observe that $\sigma'_\xi(e_i)$ is insignificant compared with $\mu'_\xi(e_i)$.

rewards a larger number of agents. For the latter, we identify the sufficient statistic of a good performance measurement called the sensitivity. We show that a performance measurement with higher sensitivity can incentivize the same effort level in the symmetric equilibrium with a lower total payment.

4.1 Optimizing The Rank-Order Payment Function

We first rewrite the principal's problem given a performance measurement ψ. Suppose all the agents except i exert an effort ξ. Then, given ψ, agent i knows the probability that she ends up with each rank j when her effort is e_i, which is denoted as $p_j(e_i, \xi)$. Recall that by Assumption 3.1, $G_{e_i,\xi}$ is the c.d.f. of the score distribution of agent i; by Assumption 3.2, $G_{\xi,\xi}$ is the c.d.f. of the score distribution of all the other agents. Then, this probability is given by

$$p_j(e_i, \xi) = \binom{n-1}{j-1} \int_{-\infty}^{\infty} G_{\xi,\xi}(x)^{n-j} \left[1 - G_{\xi,\xi}(x)\right]^{j-1} dG_{e_i,\xi}(x). \tag{2}$$

We then can write agent i's expected utility under the RO-payment function $\hat{\boldsymbol{t}}$ as

$$\mathbb{E}[U_a(e_i, \xi)] = \sum_{j=1}^{n} p_j(e_i, \xi) u_a\left(\hat{t}_j, e_i\right), \tag{3}$$

where U_a denotes the random variable of the agent's utility and u_a is agent's utility function defined in Eq. (1).

Maximizing the expected utility w.r.t. e_i then leads to the first order constraint (FOC) which is a necessary condition of symmetric equilibrium. For sufficiency, additional conditions on the distribution of the performance score and the agents' cost function are required. For example, it is shown that when the distribution of the noise (in our case, this is the Gaussian) is "dispersed enough", the existence of symmetric equilibrium is guaranteed [22]. Again, in our theory sections, we assume this is true, while we empirically verify this assumption in Section 5.2 under the performance measurements and cost functions that we consider. For now, we assume FOC is also sufficient for symmetric equilibrium.

Let $p'_j(\xi) = \frac{\partial p_j(e_i,\xi)}{\partial e_i}\Big|_{e_i=\xi}$ denote the derivative of the probability an agent ends up with rank j w.r.t. a unilateral deviation in effort when all agents' effort is ξ, and let $c'(\xi)$ denote the derivative of the cost. Also, note that $p_j(\xi, \xi) = \frac{1}{n}$ for any j due to symmetry. Now, given n and ξ, we formally write down the principal's problem.

$$\min_{\hat{\boldsymbol{t}}} \quad \sum_{j=1}^{n} \hat{t}_j \tag{4}$$

$$\text{s.t.} \quad \hat{\boldsymbol{t}} \geq 0 \quad (LL), \qquad \frac{1}{n}\sum_{j=1}^{n} u_a\left(\hat{t}_j, \xi\right) \geq 0 \quad (IR), \qquad \sum_{j=1}^{n} p'_j(\xi) u_a\left(\hat{t}_j, \xi\right) = 0 \quad (FOC).$$

Before we present our results, we present the following lemma that is essential for future proofs.

Lemma 4.1. *Fixing $\xi \in [0, 1]$, if $n \to \infty$, $p'_j(\xi)$ is decreasing in j for any $1 \leq j \leq n$.*

We leave the proof in Appendix A. Lemma 4.1 shows that after convergence, a small unilateral deviation results in a probability of ranking that is monotone decreasing in j. The key of the proof lies in the fact that after convergence, $p'_j(\xi)$ can be approximated with some form of the quantile function of Gaussian, which is known to be the inverse error function. Then, with the monotonicity of the inverse error function, we complete the proof.

Next, we present the optimal RO-payment functions for the principal's problem under the three utility models.

4.1.1 Neutral Agents. Now suppose agents are neutral, i.e. $u_a(t_i, e_i) = t_i - c(e_i)$. We have the following results.

Proposition 4.2. *Suppose $n \longrightarrow \infty$, $\xi \in [0, 1]$ and agents are neutral.*

(1) ***IR is not binding:*** *If $\lim_{n\to\infty} \frac{c'(\xi)}{np'_1(\xi)} \geq c(\xi)$, the optimal RO-payment function is winner-take-all, i.e. $\hat{t}_1 = \frac{c'(\xi)}{p'_1(\xi)}$ is the reward to the top one agent and $\hat{t}_j = 0$ for $1 < j \leq n$;*

(2) ***IR is binding:*** *Otherwise, the optimal RO-payment function is not unique and can be achieved by a threshold function that rewards the top $\hat{n}$ agents equally, i.e. $\hat{t}_j = \frac{n}{\hat{n}}c(\xi)$ for $1 \leq j \leq \hat{n}$ and 0 otherwise. The threshold $\hat{n}$ is determined by $\frac{n}{\hat{n}}\sum_{j=1}^{\hat{n}} p'_j(\xi)c(\xi) = c'(\xi)$.*

The proof is deferred to Appendix B.1. As a sketch, the proposition holds because by Lemma 4.1, $p'_j(\xi)$ is decreasing in j. This implies that if IR is not binding, when we take the gradient of the total payment in Eq. (4) w.r.t. each $\hat{t}_j$, the gradient reaches its maximum when $j = 1$. Thus, the most payment-saving RO-payment function is to put all of the budget on $\hat{t}_1$ to maximize the gain of any unilateral deviation to a higher effort.

It is worth noting that except for the extreme cases where $\frac{c'(\xi)}{p'_1(\xi)} \longrightarrow \infty$, Proposition 4.2 implies that IR is always binding when $n \longrightarrow \infty$. However, we emphasize that the condition $n \longrightarrow \infty$ in Proposition 4.2 (as well as the following propositions) is only needed because Lemma 4.1, which is used in the proof, requires it. In Section 5.2, we empirically show that Lemma 4.1 still holds for a reasonably large group size, e.g. $n = 50$, which makes the condition in Proposition 4.2 less extreme in this case.

We further note that when agents are neutral and IR is binding, the minimum total payment equals the total cost $nc(\xi)$. Furthermore, more than one RO-payment function can achieve the optimum. We provide a threshold function as one of the solutions.

4.1.2 Loss-averse Agents. Suppose agents are loss-averse, i.e. $u_a(t_i, e_i) = t_i - c(e_i) - \rho \cdot (c(e_i) - t_i)^+$. The expression of the optimal RO-payment function becomes more complicated in the loss-aversion case. Here, we present an informal version of our results while leaving the precise version in Appendix C.1.

Proposition 4.3. *(Informal) Suppose $n \longrightarrow \infty$, $\xi \in [0, 1]$ and agents are loss-averse.*

(1) ***IR is not binding:*** *The optimal RO-payment function pays 1) 0 to the bottom agents with ranking $j > \bar{n}$, 2) $c(\xi)$ to the intermediate agents with ranking $1 < j \leq \bar{n}$, and 3) $\hat{t}_1 > c(\xi)$ to the top one agent. Here, the threshold $\bar{n} \leq \frac{n}{2}$ is determined by $(1 + \rho)p'_{\bar{n}}(\xi) = p'_1(\xi)$;*

(2) ***IR is binding:*** *The optimal RO-payment function follows the same structure as the case of IR not binding, but with a threshold $\hat{n} \geq \bar{n}$.*

The proof is shown in Appendix C.1. As a sketch, note that the gradient of the total payment w.r.t. $\hat{t}_j$ is maximized at $j = 1$ only when $\hat{t}_1 \leq c(\xi)$. When $\hat{t}_1 > c(\xi)$ the gradient is discounted with a factor $\frac{1}{1+\rho}$. Therefore, with the decreasing property of $p_j(\xi', \xi)$ in j, the optimal RO-payment function will "fill in" $\hat{t}_j$ to $c(\xi)$ in the increasing order of j until some $\bar{n}$ such that the discounted gradient w.r.t. $\hat{t}_1$ is larger the undiscounted gradient w.r.t. $\hat{t}_{\bar{n}}$. Then, the rest budget is put on $\hat{t}_1$.

Proposition 4.3 shows that for loss-averse agents, the optimal RO-payment function has three levels of payments: the bottom agents are paid zero; intermediate agents receive the baseline payment that equals to their cost; the top one agent gets a bonus that is larger than her cost. We call this type of RO-payment function the *winner-take-more* payment function. Perhaps interestingly, winner-take-more takes a similar form of the baseline-bonus payment scheme which tends to perform well in real-world scenarios [14]. We show the optimal RO-payment functions in Fig. 2 to better illustrate our ideas.

To better illustrate our results, we introduce the inclusiveness of a (monotone) RO-payment function.

Definition 4.4. Given a monotone RO-payment function such that $\hat{t}_j \geq \hat{t}_k$ if $j \leq k$, the inclusiveness of such a RO-payment function is defined as the number of agents who receive non-zero payments, denoted as n^I. We call RO-payment function A is (weakly) more inclusive than RO-payment function B if its n^I_A is no less than n^I_B.

For example, $n^I = 1$ for WTA, and $n^I = \bar{n}$ and $n^I = \hat{n}$ in the case of loss-averse agents with IR not binding and binding respectively. Now, we show that n^I is increasing as agents become more and more loss-averse.

Corollary 4.5. *Suppose $n \to \infty$ and agents are loss-averse. The inclusiveness of the optimal RO-payment function n^I is (weakly) increasing in ρ.*

Remark. Our results for loss-averse agents rely on the first-order loss-aversion model. The study of higher-order loss-aversion is beyond the scope of this paper, i.e. $u_a(t_i, e_i) = t_i - c(e_i) - \rho \cdot ((c(e_i) - t_i)^+)^r$ for $r > 1$. Our conjecture is that in this case, the optimal RO-payment function will no longer pay a fraction of agents constantly, but pay agents decreasingly w.r.t. their ranking.

4.1.3 Risk-averse Agents. Now, suppose agents are risk-averse, i.e. $u_a(t_i, e_i) = r_a(t_i) - c(e_i)$. Again, we only show the informal version of our proposition.

Proposition 4.6. *(Informal) Suppose $n \to \infty$ and agents are risk-averse.*

*(1) **IR is not binding:** The optimal RO-payment function pays zero to the bottom agents with ranking $j > \bar{n}$, and the remaining agents strictly decreasing in their ranking, i.e. $\hat{t}_j > \hat{t}_k$ if $j < k \leq \bar{n}$, with $\bar{n} \leq \frac{n}{2}$ determined by r_a and $\boldsymbol{p}'(\xi)$.*

*(2) **IR is binding:** The optimal RO-payment function follows the same structure as the case of IR not binding, but with a threshold $\hat{n} \geq \bar{n}$.*

The precise version of this proposition and its proof are provided in Appendix D.1. As shown in Fig. 2, Proposition 4.3 illustrates that the optimal RO-payment function for risk-averse agents also only rewards the agents whose ranking is above some threshold, and the rewards are decreasing in the ranking j. As a proof sketch, this is because if the principal keep increasing the payment for the top one agent, the gradient of the total payment w.r.t. $\hat{t}_1$ is discounted greater and greater due to the concavity of the reward function. Therefore, after $\hat{t}_1$ becomes large enough, the principal is better off to increase $\hat{t}_2$, $\hat{t}_3$ and so on from zero to something instead of keep increasing $\hat{t}_1$.

The proposition shows that, in line with the results of neutral and loss-averse agents, the payment function becomes more inclusive when IR is binding compared with the case where IR is ignored.

We further point it out that the same result in Corollary 4.5 does not generalize to the risk-averse agents: more risk-aversion does not always imply a more inclusive RO-payment function.[7] We provide a formal illustration in Appendix D.2. Our intuition is that this can happen because of the flexibility of the model of risk-averse agents, i.e. $r_a(x)$ can be "more concave" in a lot of ways.

To sum up, we show that the optimal RO-payment functions for three common models of agent utility are all increasing payment schemes. Furthermore, inclusive payments are likely to be referred for two reasons. On one hand, IR requires the optimal RO-payment function to be more inclusive

[7] Actually, we show that "more risk-averse" leads to more inclusive optimal RO-payment function when IR is not binding, but both it is not true when IR is binding.

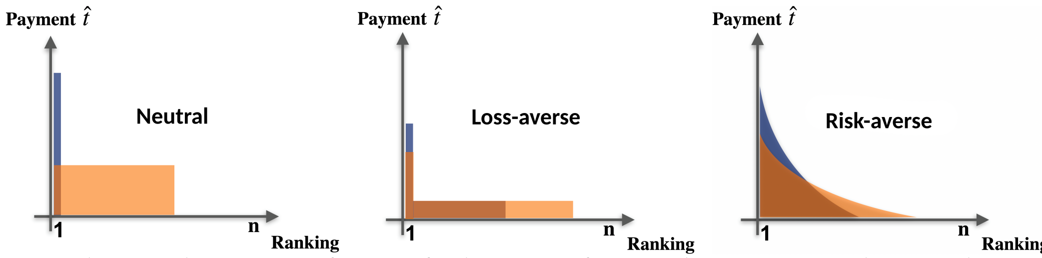


Fig. 2. The optimal RO-payment functions for three types of agents. Blue payments are the cases where IR is ignored, and orange payments are the cases where IR is considered. All payment schemes are increasing.

(compared with the case where IR is ignored) so as to guarantee the existence of equilibrium. On the other hand, when agents have "fairness" concerns, e.g. they are risk or loss-averse, more inclusive RO-payment functions are optimal (compared with the neutral case, where WTA is optimal). In Section 5.3, we empirically show how the inclusiveness of the optimal RO-payment functions interacts with the cost function and equilibrium effort.

4.2 Optimizing The Performance Measurement

A performance measurement can affect principal's optimal utility by affecting $p_j(e_i, \xi)$. In the idealized setting, assuming all the other agents but i exert effort ξ, every performance measurement maps agent i's effort e_i to a Gaussian distribution of her performance score with mean and std functions of e_i. We denote these two functions as $\mu(e_i, \xi)$ and $\sigma(e_i, \xi)$ respectively. Therefore, in the principal-agent problem that we care about, $\mu(e_i, \xi)$ and $\sigma(e_i, \xi)$ determine the performance of a performance measurement. Let $\mu'_\xi(\xi)$ be the derivative of μ w.r.t. e_i when $e_i = \xi$, and $\sigma'_\xi(\xi)$ be the same notation for σ. For any RO-payment function that is increasing (Definition 3.1), we identify the following sufficient statistic of the performance of a performance measurement, called the *sensitivity*.

Definition 4.7. Suppose Assumption 3.1 holds. The sensitivity of a performance measurement whose performance score distribution has mean $\mu(e_i, \xi)$ and standard deviation $\sigma(e_i, \xi)$ is defined as $\delta(\xi) = \frac{\mu'_\xi(\xi)+\sigma'_\xi(\xi)}{\sigma_\xi(\xi)}$.

From this definition, the sensitivity of a performance measurement is defined under the symmetric equilibrium concept and depends on the effort in the symmetric equilibrium. At a high level, a performance measurement is more sensitive if scores it generates are more sensitive in effort change and has high accuracy. Also note that $\delta(\xi) \geq 0$ by Assumption 3.2.

Proposition 4.8. *Suppose Assumption 3.1 and 3.2 hold. Let δ be the sensitivity of the performance measurement and let $\hat{\boldsymbol{t}}$ be any RO-payment function that is increasing. Then, fixing any $\xi \in [0, 1]$, the minimum total payment $\sum_{j=1}^{n} \hat{t}_j$ is (weakly) decreasing in $\delta(\xi)$.*

The proof is shown in Appendix E. At a high level, the intuition is that if an agent slightly increases her effort, it becomes easier for her to be ranked in higher places. This effect is amplified by a performance measurement with higher sensitivity. Therefore, with a more sensitive performance measurement, the first order constraint in Eq. (4) can be satisfied with lower payment. Because both of the other constraints are independent of performance measurements, we can conclude that higher sensitivity implies (at least weakly) lower payment from the principal.

Now, we have optimized the performance measurement and the RO-payment function separately. The following corollary fits the optimization results together.

Corollary 4.9. *Fixing a goal effort, let ψ' be a performance measurement with higher sensitivity than ψ. Let $\hat{\boldsymbol{t}}'$ and $\hat{\boldsymbol{t}}$ be their corresponding optimal RO-payment functions respectively. Then the payment mechanism consisted of ψ' and $\hat{\boldsymbol{t}}'$ has lower minimal total payment than the payment mechanism consisted of ψ and $\hat{\boldsymbol{t}}$ in the symmetric equilibrium.*

The proof is straightforward by comparing three payment mechanisms: mechanism 1 is consisted of ψ' and $\hat{\boldsymbol{t}}'$, mechanism 2 is consisted of ψ' and $\hat{\boldsymbol{t}}$ and mechanism 3 is consisted of ψ and $\hat{\boldsymbol{t}}$. First, by our results in Section 4.1, both $\hat{\boldsymbol{t}}'$ and $\hat{\boldsymbol{t}}$ are increasing. Then, by Proposition 4.8, mechanism 2 should be cheaper to implement than mechanism 3. Furthermore, mechanism 1 must be cheaper than mechanism 2 because $\hat{\boldsymbol{t}}'$ is the optimal RO-payment function for ψ' which completes the proof.

As a summary, Proposition 4.8 shows that if the RO-payment function is increasing, which is exactly the case for any agent utility model that we consider, the sensitivity of a performance measurement is a sufficient statistic of its performance in our principal agent problem. Then, Corollary 4.9 implies that to optimize a payment mechanism, one should focus on maximizing the sensitivity of a performance measurement and apply the corresponding optimal RO-payment function.

5 INCLUSIVENESS OF THE OPTIMAL RANK-ORDER PAYMENT FUNCTION

In this section, we generalize our theoretical results about the optimal RO-payment function beyond the idealized setting with our agent-based model. We first setup our agent-based model experiments and then use them to justify our assumptions in the sense that our theoretical results are empirically shown to hold even when the assumptions are violated. Furthermore, we use our agent-based model to explore the relationship between the inclusiveness of the optimal RO-payment function and the goal effort, agents' cost functions and agents' utility models. Finally, we include a discussion about a variance of our principal model: the fairness-seeking principal. We show that when the principal cares about the fairness of the payment, he can increase the inclusiveness of the RO-payment function and trade-off the total payment for the fairness of the payment.

5.1 Experiment Setup

5.1.1 Datasets. We use two crowdsourcing datasets to estimate the prior of ground truth w and agents' signal matrix Γ, called world 1 ($W1$) [3] and world 2 ($W2$) [27] respectively.

World 1 has a signal space of size 5 and a binary ground truth, say $\{1, 2\}$. Agents are asked to grade the synthetic accessibility of compounds with scores 1 to 5, where 1 indicates inappropriate to be synthesized and 5 stands for appropriate. Scores in between lower the confidence of the grading. The binary ground truth indicates whether a compound is appropriate or inappropriate. The dataset includes the assessments of 100 compounds (tasks) from 18 agents. World 2 has an identical signal space and ground truth space of size 4 (actually the size is 5, but we ignore the rarest one which occurs 9 out of 300 times). The dataset contains 6000 classifications of the sentiment of 300 tweets (tasks) provided by 110 workers. The estimated parameters for $W1$ and $W2$ are:

$$w_1 = \begin{bmatrix} 0.613 & 0.387 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0.684 & 0.221 & 0.032 & 0.037 & 0.026 \\ 0.092 & 0.191 & 0.050 & 0.200 & 0.467 \end{bmatrix};$$

$$w_2 = \begin{bmatrix} 0.196 & 0.241 & 0.247 & 0.316 \end{bmatrix}, \Gamma_2 = \begin{bmatrix} 0.770 & 0.122 & 0.084 & 0.024 \\ 0.091 & 0.735 & 0.130 & 0.044 \\ 0.033 & 0.062 & 0.866 & 0.039 \\ 0.068 & 0.164 & 0.099 & 0.669 \end{bmatrix}.$$

Note that we use the estimated confusion matrices as the underlying full-effort-working matrices Γ_{work}, which assumes the real-world agents are exerting full effort. Obviously, this is an under-estimation of Γ_{work} since the real-world agents' effort should be smaller than 1. Since the agents' effort cannot be directly observed, it is impossible to estimate the Γ_{work} with no bias. However, this is not a big concern since none of our experimental results depends on the correct estimation of

Γ_{work}. Furthermore, the experiments are run with two different world models to show the robustness of our results.

5.1.2 The Performance Measurements. We implement two types of performance measurement: spot-checking and peer prediction mechanisms.

Spot-checking. Let p_c be the probability of spot-checking, i.e. the principal has the access to the ground truth of $n_c = p_c \cdot n$ randomly sampled tasks. We consider two spot-checking mechanisms in our experiments. First, a straightforward idea is to set the performance score to be the accuracy of the each agent's reports on the spot-check questions. We denote this performance measurement as **SC-Acc**.

Alternatively, we apply the mechanism considered in [11], which is inspired by the Dasgupta-Ghosh mechanism [4]. We denote this performance measurement as **SC-DG**. Given an agent's reports and a set of spot-checking questions with the ground truth, SC-DG randomly chooses a common task (bonus task) and two distinct tasks (penalty tasks). Then, the agent is scored 1 if her report on the bonus task agrees with the ground truth, and scored −1 if agreeing on the penalty tasks. The final score of the agent is the average score after repeated sampling.

Peer prediction. We consider five types of commonly used peer prediction mechanisms. The idea of peer prediction is to score each agent using some form of the correlation between her reports and her peers' reports.

First, we implement the naive idea of paying an agent 1 if her report on a random task agrees with a random peer's report on the same task, and paying 0 otherwise. This performance measurement is called the output agreement mechanism (**OA**) as discussed in [10].

Second, in the same paper, Faltings et al. [10] propose the peer truth serum (**PTS**) mechanism. The only difference between PTS and OA is that the payment is proportional to $\frac{1}{R(x)}$, where R is a public distribution of reports and x is the report of the pair of agents on the random task. We implement PTS by setting R to be the empirical distribution of all the other agents' reports other than i while computing agent i's payment.

Third, we consider the matrix f-mutual information mechanism (**f-MMI**). Inspired by Kong and Schoenebeck [20], **f-MMI** scores each agent using the estimation of the f-mutual information between her reports and her peer's report, where f can be any convex function.

With attention to detail, the f-MMI uses the empirical distributions to estimate the mutual information. We can simply estimate the empirical distributions between two agents' reports, i.e. $\widetilde{P}_{\hat{X}_i,\hat{X}_j}$ for joint distribution and $\widetilde{P}_{\hat{X}_i}$ for marginal distribution. Then, the MI between reports $\hat{X}_i$ and $\hat{X}_j$ can be estimated,

$$\widetilde{MI}_{i,j}^{f-MMI} = \sum_{x,y} \widetilde{P}_{\hat{X}_i,\hat{X}_j}(x,y) f\left(\frac{\widetilde{P}_{\hat{X}_i}(x)\widetilde{P}_{\hat{X}_j}(y)}{\widetilde{P}_{\hat{X}_i,\hat{X}_j}(x,y)}\right). \tag{5}$$

The matrix mutual information mechanism then scores each agent i the average of the estimated MI between i and each of her peer. To speed up the mechanism, instead of pairing agent i with each of her peer j, we simply learn the empirical distributions of the reports on each task of all agents but i. This can be seen as a "virtual agent" reporting based on the empirical distributions of all agents but i. Then, we learn the joint distribution as well as the mutual information between agent i's reports and this virtual agent's reports.

Forth, we implement the pairing f-mutual information mechanism (**f-PMI**) [24]. Similar to SC-DG, f-PMI randomly samples the bonus and penalty tasks and scores each agent based on whether her reports agree with the "ground truth" on the three tasks. The main difference is that instead of using the ground truth, the f-PMI learns a soft predictor on each task using all the other agents' reports. Then, a f-mutual information is estimated for each agent using the soft-predictor and the agent's reports. Note that the f-PMI contains the well known DG mechanism [4] and CA mechanism [26] as special case when f is $f(x) = \frac{1}{2}|x-1|$.

In a detailed manner, Schoenebeck and Yu [24] provide an alternative way to estimate the MI. Specifically, the quotient of the joint distribution between $\hat{X}_i$ and $\hat{X}_{-i}$ and the product of the marginal can be written as

$$\frac{P_{\hat{X}_i,\hat{X}_{-i}}(\hat{x}_i,\hat{x}_{-i})}{P_{\hat{X}_i}(\hat{x}_i)P_{\hat{X}_{-i}}(\hat{x}_{-i})} = \frac{P_{\hat{X}_i|\hat{X}_{-i}}(\hat{x}_i|\hat{x}_{-i})}{P_{\hat{X}_i}(\hat{x}_i)}. \tag{6}$$

The denominator can be empirically estimated. While the numerator is a soft-predictor, which, given the reports of all agents except i on a particular task j, produces a forecast of agent i's report on the same task in the form of a distribution. In our experiments, we set the soft-predictor for agent i's report on task j as the empirical distribution of all the other agents' reports on the same task.

For both f-MMI and f-PMI, we consider four types of commonly used f-divergence for the MMI and PMI mechanisms, as shown in Table 1.

Finally, we implement the determinant mutual information mechanism (**DMI**) Kong [17]. Kong [17] generalizes the Shannon mutual information to the determinant mutual information. Specifically, for a pair of agents i and j, the set of the commonly answered tasks is divided into two disjoint subsets A and B. Again, we empirically estimate the joint distribution with reports in A and B respectively, and score agent i the product of the determinants of these two estimated joint distribution matrices and take average over all the other agents.

Table 1. Four f-divergences

f-divergence (short name)	$f(a)$
Total variation distance (TVD)	$\frac{1}{2}\|a-1\|$
KL-divergence (KL)	$a\log a$
Pearson χ^2 (Sqr)	$(a-1)^2$
Squared Hellinger (Hlg)	$\left(1-\sqrt{a}\right)^2$

5.1.3 Parameters And Methods. Now, we introduce the parameter setting of our agent-based model and how we use it to visualize the optimal RO-payment functions.

Unless otherwise specified, we set the number of tasks to be $m = 1000$ with each agent answering $m_a = 100$ tasks. Every task is assigned to (at least) $n_0 = 5$ agents and there are $n = 52$ agents in total.[8] We consider various commonly used cost functions including the polynomial cost $c(e) = e^r$ and the exponential cost $c(e) = \exp(r \cdot e)$. When dealing with loss-averse agents, ρ is set to be 0.5. For SC, we vary the spot-checking probability from 0.1 to 0.3. For peer prediction, we use four types of commonly used f-mutual information with f listed in Table 1.

Given any of the performance measurements, we estimate the distributions of the performance score of an agent before and after a unilateral deviation of effort $\xi + \Delta e$ for $\xi \in [0, 1)$. To do so, we first sample the effort of all agents ξ from 0 to 0.99 with step size 0.01. Fixing each of the ξ, we simulate the report matrix $\boldsymbol{x}$ when all agents are exerting effort ξ and input $\boldsymbol{x}$ to the performance measurement which gives us n samples of the performance score before deviation. Then, let one of the n agents deviates to $\xi + \Delta e$ with $\Delta e = 0.01$ for estimation. Repeating the above process will

[8]The number of agents $n > 50$ is to guarantee that each task is assigned with at least $n_0 = 5$ agents and tasks are assigned to agents randomly.

give us one sample of the score after deviation. We then repeat the process and generate 5000 samples for each of the cases which gives us an estimation of the score distributions before and after the unilateral deviation. We fit the Guassian model to the generated samples and use our theory to predict the optimal RO-payment functions under different parameter settings. Finally, the probability $p(\xi + \Delta e, \xi, j)$ can be estimated for any $1 \leq j \leq n$, and the optimal RO-payment function can be developed based on our results from Section 4.1. Note that when n is finite, there may not be integer solutions for the thresholds $\bar{n}$ and $\hat{n}$ in our propositions, and thus the thresholds are solved with approximation, i.e. the integers that are closest to the real solutions.

When comparing the performances of different performance measurements in Section 6, we use a non-parametric method (KDE) to estimate the performance score distributions to verify the robustness of our results beyond the Gaussian assumption.

5.2 Assumption Justifications

In our theory section, we made several assumptions to support our theoretical analysis. Now, we justify the assumptions with ABM experiments and show that the assumptions are (approximately) capturing the real problem correctly.

First, in Assumption 3.1 and 3.3, we assume the distribution of agents' performance score follows the Gaussian distribution and any unilateral deviation to a slightly higher effort leads to an increase in the mean of the score. Note that for any performance measurement with bounded scores, the distribution of the performance score is clearly not Gaussian. However, our experiments show that the Gaussian distribution can fit the performance score distributions of all of the considered performance measurements with exceptions of DMI, KL-PMI and Hlg-PMI whose performance score distributions tend to be heavy-tailed. Furthermore, for those performance measurements whose performance score distributions cannot be well fitted by Gaussian, we use KDE to estimate the performance score distributions. In Section 6.1.2, we show that the sensitivity is still a good predictor of the performance of a performance measurement even when the performance scores it generates can not be well fitted with Gaussian.

Second, while applying a peer prediction mechanism, if an agent i deviates from e_i to a slightly higher effort $e_i + \Delta e$, the distribution of other agents' performance scores can be changed since the they depend on the correlations between agents' reports. This obviously violates Assumption 3.2 especially when n is not large enough. For validity check, we estimate this change of distribution and confirm that it is trivial even when n is not significantly large, e.g. $n = 50$. Actually, in Section 6.2, we show that this effect is small not only when the agent deviates to a slightly larger effort, but even when the agent can deviate to arbitrary untruthful strategies.

Third, as a commonly used approach, we assume the sufficiency of the first order constraint (FOC). However, for the optimal RO-payment functions that we considered, the performance measurements that we implemented, and the commonly considered convex cost functions, e.g. the polynomial and exponential cost, we observe that the expected utility Eq. (3) is concave w.r.t. e_i for a large range of ξ. Therefore, there exists a unique e_i that maximizes each agent's expected utility which implies that FOC is sufficient for the existence as well as the uniqueness of the symmetric equilibrium.

Finally, although n is assumed to be sufficiently large to prove Lemma 4.1, we show that the lemma still holds for all the performance measurements that we considered when $n = 50$ which is a reasonable number of agents in the crowdsourcing setting. That's to say, all of our propositions hold even when n is not large enough.

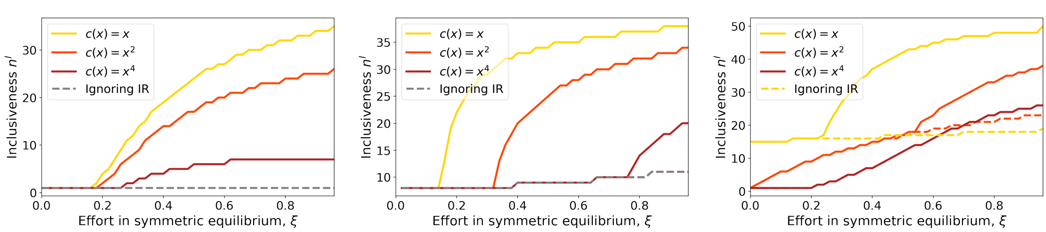
5.3 Visualizing The Inclusiveness

In this part, our goal is to see how ξ and the cost function affects the inclusiveness of the optimal RO-payment functions, which indicates the number of agents that receive non-zero payments.

Fixing a performance measurement, we visualize the inclusiveness of the optimal RO-payment function versus ξ in Fig. 3 for neutral, loss-averse and risk-averse agents respectively. Our first observation is that IR is likely to be binding when the cost function is "less convex" and the effort ξ is high. This observation is in line with our theory as whether IR is binding depends on the ratio $\frac{c'(\xi)}{c(\xi)}$. To see this, in Proposition 4.2 for example, the condition for IR is not binding can be rewritten as $\frac{c'(\xi)}{c(\xi)} > np_1'(\xi)$. For most commonly used cost functions, e.g. quadratic or higher order power functions and exponential functions, this ratio is likely to be larger with more convex functions and ξ is approaching to 1. In both cases, compensating agents' cost requires more payment than maintaining the equilibrium.

Second, if we look at the lines with the same color, comparing with the neutral agents, the optimal RO-payment function is more inclusive, when agents are loss and risk-averse.

To sum up, the main takeaway is that when the cost function is less convex, or the goal effort is high, or the number of agents is large (all common in the crowdsourcing setting), IR is likely to be binding and more inclusive payment functions are required (compared with the case when IR is ignored). Furthermore, the optimal RO-payment function is more inclusive in the case of both loss-averse and risk-averse agents (compared with the neutral agents).



(a) Neutral agents. (b) Loss-averse agents. (c) Risk-averse agents.

Fig. 3. The inclusiveness of the optimal RO-payment functions under different agent utilities as a function of the symmetric equilibrium effort ξ with different cost functions. The solid curves represent the inclusiveness of the optimal RO-payment functions when considering IR while the dashed curves show the inclusiveness when IR is ignored. Note that for (a) and (b), the optimal RO-payment function does not depend on the cost functions and is all represented by the same grey dashed curve. In this example, we apply the SC-Acc mechanism with spot-checking probability 0.25. For (b), $\rho = 0.5$ and for (c), $r_a(t) = \log(t + 1)$.

5.4 The Fairness-Seeking Principal

Here, we provide a variance of our standard principal model. We have been focusing on the risk-neutral principal who aims to minimize the total payment given a fixed goal effort level. However, the principal may want to pay the agents with surplus to trade-off the efficiency and the fairness of the payments for various reasons. For example, the principal wants to reduce the variance of the payments even though he has to pay more due to an intrinsic notion of fairness, social pressure, or the low participation rate. We model the fairness-seeking principal with a penalty term in their utility function, $\Theta(\hat{\boldsymbol{t}})$. In this way, the principal aims to solve the same problem in (4) but to minimize the linear combination of the total payment and the fairness cost, i.e.

$$\min_{\hat{\boldsymbol{t}}} \quad \sum_{j=1}^{n} \hat{t}_j + \lambda \cdot \Theta(\hat{\boldsymbol{t}}).$$

To simplify the analysis and provide intuitions, we consider two examples of the penalty functions while assuming the agents are neutral. First, similar to the loss-aversion case, let $\Theta(\hat{\boldsymbol{t}}) = \sum_j (c - \hat{t}_j)^+$ where c is a positive constant, e.g. $c = c(\xi)$ with a goal effort ξ. This example models the fact that the principal wants to pay the agents some money to overcome their cost of effort. With the same arguments as in Appendix C.1, one can easily verify that the optimal RO-payment function in this case is similar to the loss-averse agents case as shown in Fig. 2. In the optimal RO-payment function, the principal pays a fraction of agents who are ranked higher than some threshold c and the top one agent more than c. The only difference lies in the threshold. For example, when IR is not binding, the optimal threshold $\bar{n}$ satisfies $p'_{\bar{n}+1}(\xi) = (1-\lambda)p'_1(\xi)$ instead of $(1+\rho)p'_{\bar{n}+1}(\xi) = p'_1(\xi)$ in Proposition 4.3.

For another example, let $\Theta(\hat{\boldsymbol{t}}) = \frac{1}{n}\sum_{j=1}^{n}(\hat{t}_j - \bar{t})^2$, where $\bar{t} = \frac{1}{n}\sum_{j=1}^{n}\hat{t}_j$. In this case, the principal aims to reduce the the variance of the payments. Since the LL, IR and FOC constraints are linear in $\hat{\boldsymbol{t}}$, the principal's problem is convex which can be numerically solved. We thus use our ABM to learn the probability $p'_j(\xi)$ and then explore the trade-off between the efficiency, i.e. the total payment, and the fairness, i.e. the variance of the payments.

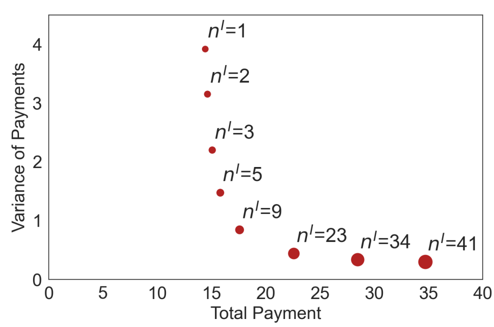


Fig. 4. The trade-off between the efficiency and the fairness of the payments with λ varying from 0 to 1. In this example, we learn $p'_j(\xi)$ from the dataset of $W1$ with performance measurement the SC-Acc and $\xi = 0.8$. The cost function is $c(x) = x^8$.

In Fig. 4, we visualize the principal's trade-off between lowering the variance and saving the budget. The principal can trade-off the fairness and the efficiency by implementing a more inclusive mechanism which pays the agents more money than their cost of effort and use the surplus to reduce the variance. The trade-off depends on the setting, i.e. the cost function, goal effort and the confusion mapping. However, our numerical results suggest that the principal can greatly lower the variance without too much more cost of budget. For example, the variance can be reduced by 50% with only a 7% increase in the total payment in the example of Fig. 4.

6 EVALUATING REALISTIC PERFORMANCE MEASUREMENTS

In the idealized setting, we have reduced the optimization of performance measurement to the problem of maximizing the sensitivity of a performance measurement. However, in reality, we are facing at least two problems.

On on hand, we cannot arbitrarily increase the sensitivity of a performance measurement. Instead, we are given several tools that can serve as candidates of performance measurements, namely, spot-checking and peer prediction mechanisms. We then ask: which mechanism has the highest sensitivity and how their sensitivities change with the setting, i.e. the goal effort of the principal.

On the other hand, we cannot naively assume agents are honest in reporting their signals in realistic settings. Agents can strategically report and try to game the mechanism for extra utility.

As mentioned, truth telling can guaranteed to be an equilibrium by the carefully designed spot-checking and peer prediction mechanisms. However, the nonlinear rank-order payments that we apply do not preserve the truthful property. Therefore, in Section 6.2, we empirically verify and compare the strategy robustness of the performance measurements considered in this paper. One additional result suggests that the more inclusive the RO-payment function is, the more robust the payment mechanism is against strategic reporting.

6.1 The Sensitivity of Performance Measurements

Now, we visualize and compare the sensitivity of the performance measurements that we consider. To verify the effectiveness of using sensitivity to predict the superiority of different performance measurements, we further use KDE to estimate the distributions of the performance scores and visualize the optimal payment. The ranking of the total payments of different performance measurements is in line with the ranking of their sensitivities which verify the effectiveness of sensitivity.

6.1.1 Experiment Setup. We apply the same method to generate samples of the performance scores for each of the performance measurements introduced in Section 5.1.2. With these samples, we first fit them with Gaussian distributions and estimate the means and std. We smooth the curves by fitting the means and std with cubic functions. Finally, we can compute the sensitivity for each performance measurement.

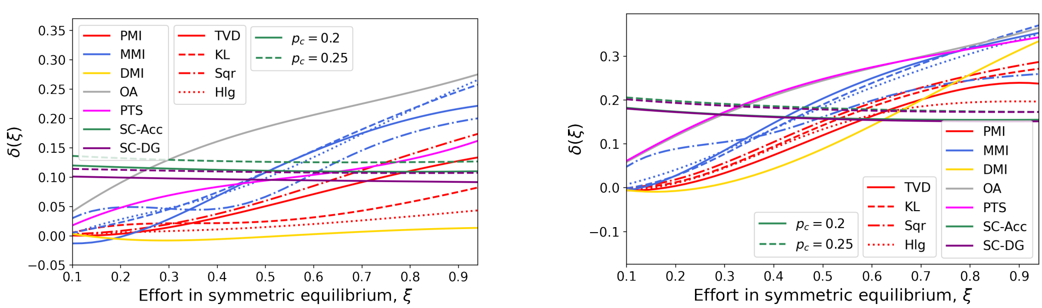
In addition, we use KDE to estimate the performance score distributions to go beyond the Gaussian assumption. We apply Scott's rule for the bandwidth selection function [25], where the bandwidth is $n^{\frac{1}{5}}$ for 1d samples where n is the number of samples. Given the non-parametric estimation of the performance score, we then can estimate p_j, the probability of an agent being ranked in the j's place with a unilateral deviation in her effort. Next, given the cost function and parameters for loss/risk-averse agents, we can compute the optimal RO-payment function shown in Section 4.1, which gives us the optimal total payment for each of the performance measurements.

6.1.2 Results. In Fig. 5, we show the sensitivity versus the goal effort for different performance measurements where higher curves are preferred. Note that there are cases where $\delta(\xi)$ can be negative. When $\delta(\xi)$ is very close to zero and the effort change is small, the estimate may veer negative due to the limited number of samples. In Fig. 6 we further show the minimum payments of different performance measurements where the performance score distributions are estimated via KDE. Here, performance measurements with lower curves are preferred. Our results are summarized below:

(1) Our first observation is that the spot-checking mechanisms have a relatively consistent sensitivity, while most of the peer prediction mechanisms have increasing δ. This is because when the effort is large, the reports of agents have less variance. Consequently, the peer prediction mechanisms can estimate the correlations between agents' report more accurately which benefits $\delta(\xi)$. However, since the performance score of the spot-checking mechanism does not depend on the peers' reports and effort, $\delta(\xi)$ of the spot-checking mechanisms is more consistent for different ξ.
(2) Not surprisingly, the $\delta(\xi)$ of the spot-checking mechanism is increasing as the spot-checking probability increases, which leads a more accurate estimation of agents' effort. Under the same condition, the simple idea of the SC-Acc slightly outperforms the SC-DG.
(3) We observe that the the performances of peer prediction mechanisms have a large variance. Perhaps surprisingly, OA, a simple scoring rule has the highest sensitivity. The f-MMI mechanisms also do a good job in both $W1$ and $W2$, while the f-PMI mechanisms are less sensitive. The DMI is unlikely to work well consistently.

(4) Last but not least, our results with the KDE method verify the effectiveness of sensitivity. Fig. 6 shows that the optimal payments (when IR is ignored, which does not change the relative order of the curves) for different performance measurements under $W1$ approximately follow the analogous orders as the Gaussian model's predictions in Fig. 5 (a). To see this, for example, note that in Fig. 5 (a), the higher the curve, the better the performance measurement is; while in Fig. 6, the lower the curve is, the better the performance measurement is. The ranking of the performances of performance measurements are approximately consistent in both figures.

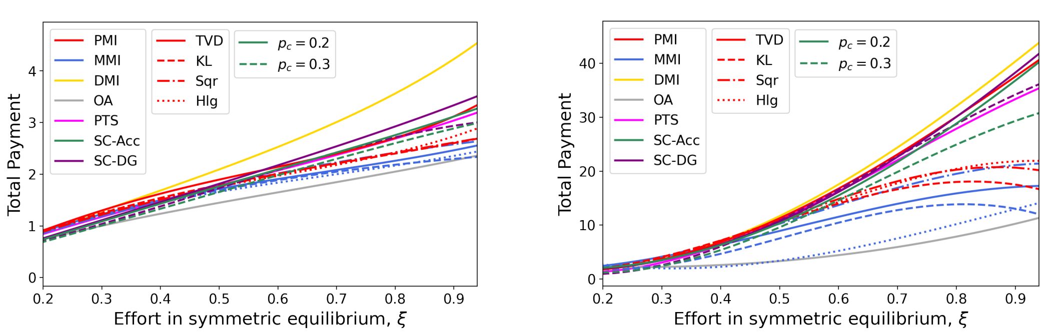
In summary, when the goal effort is low, our experiments suggest the use of spot-checking. When the required effort is large, applying a peer prediction based performance measurement can be cheaper. For the purpose of saving budget, if the agents are assumed to be reporting truthfully, the best performing performance measurements are SC-Acc for the spot-checking based performance measurement and OA, Hlg-MMI or KL-MMI for peer prediction based performance measurement.



(a) $W1$.

(b) $W2$.

Fig. 5. The ratio $\eta(\xi)$ versus ξ for different performance measurements in (a) $W1$ and (b) $W2$. Here, the performance score distributions are estimated with Gaussian models.



(a) $W1$, Neutral agents.

(b) $W1$, Loss-averse agents.

Fig. 6. The optimal payments of different performance measurements in $W1$ with (a) neutral agents and (b) loss-averse agents with $\rho = 0.5$. Here, the performance score distributions are estimated with KDE and the cost function is set to be $c(x) = x^2$. Note that for presentation convenience,[9] we ignore the IR constraint in these two examples which, if is binding, will not change the relative order of different performance measurements. Furthermore, in (a), when IR is binding IR, the payments are identical everywhere.

6.2 The Strategy Robustness of Performance Measurements

We know that applying a carefully designed RO-payment function on the performance scores can help with reducing the total payment. However, the truthfulness guarantees of the original performance measurements do not hold when the payment function is the non-linear RO-payment function. The concerns come from two aspects. First, even though any unilateral deviation will decrease the expectation of the performance score, it is not necessarily true that it will also lower the ranking. This is because one may change the performance score distributions of other agents if the peer prediction mechanisms are applied as performance measurements. Second, because the RO-payment function is non-linear, a decrease in the expectation of the performance score does not transform directly to a decrease in the payment. For example, a lower mean but higher variance performance score distribution may bring higher probability of winning the first prize. This concern is universal to not only the peer prediction but also the spot-checking based performance measurements.

In this section, we show that the second concern troubles the incentive robustness of some of the performance measurements while the first concern is not salient in our tests. We than empirically show and compare the incentive robustness of different performance measurements under various settings.

6.2.1 Experiment Setup. We apply the same method to generate samples as discussed in Section 5.1.3. Specifically, fixing an effort ξ, we generate the samples of the performance scores of three cases: 1) all agents are truthful; 2) an agent i's performance score when she deviates to an untruthful strategy π; 3) and other agents' performance scores when agent i deviates. Then, we fit the samples to the Gaussian models. We name the estimated distributions of the three cases $\hat{g}_1$, $\hat{g}_2$ and $\hat{g}_3$, with the corresponding means and standard deviations μ_i and σ_i for $i = 1, 2, 3$ respectively.

For the first concern, we simply have to compare $\hat{g}_1$ and $\hat{g}_2$ while varying the number of agents $n \in \{10, 20, \ldots, 50\}$ and the goal effort $\xi \in \{0.4, 0.6, 0.8, 1\}$ for each of the peer prediction performance measurements.

For the second concern, we have two steps. First, we normalize $\hat{g}_2$ and $\hat{g}_3$ such that $\hat{g}_3$ is a standard Gaussian. This can be done with the following formula: $\mu_2' = \frac{\mu_2 - \mu_3}{\sigma_3}$ and $\sigma_2' = \frac{\sigma_2}{\sigma_3}$. Then, given a RO-payment function, we compute a threshold on the standard deviation $\sigma_t(\mu)$ such that if a unilateral deviation changes the mean by μ, i.e. $\mu_2' = \mu$, at least a std of $\sigma_t(\mu)$ is needed to achieve a non-negative gain in payments. We do this for the most exclusive payment function, i.e. WTA, and a more inclusive payment function, i.e. the optimal RO-payment function for loss-averse agents which is the winner-take-more (WTM), as two examples. For WTA, the value of the top payment does not affect $\sigma_t(\mu)$. For WTM, the optimal $\hat{n}$ of the WTM payment function depends on the effort ξ and how loss-averse the agent are, and thus the curve $\sigma_t(\mu)$ also depends on ξ and ρ.

In the second step, we estimate the μ_i and σ_i for $i = 2, 3$ for each of the performance measurements under $\xi \in \{0.1, 0.4, 0.7, 1\}$ and sampling the untruthful strategies π from a large set of untruthful strategies. Specifically, we sample all combinations of the ways to merge the signals with 5 signals in total for $W1$ and 4 signals in total for $W2$. That's to say, while seeing a signal x, the agents report $\pi(x) \neq x$ with some probability (fixed at 0.5 in our experiments).[10] For example, three types of the strategies in $W1$ can be (1) mapping signal $4 \to 5$, (2) mapping $1 \to 2$ and $5 \to 4$ and (3) mapping $x \to x - 1$ for $x \in \{2, 3, 4, 5\}$.

Again, we fix the number of agents as $n = 52$, the sampling size as 5000 in our experiments unless otherwise specified.

[10]The main reason that we consider mixed strategy is to avoid missing signals in agents' reports, which trivially results in all zero scores and thus zero variance for DMI.

6.2.2 Results. For the first concern, we observe that the unilateral deviation to untruthful strategies can lower the mean and the std of the other agents' performance score distributions. However, this influence is small when n is relatively large, e.g. $n \geq 30$. Our results imply that this concern alone, cannot trouble the incentive robustness of the performance measurements.

For the second concern, we first empirically notice that the required std to achieve non-negative gain after deviation appears to be a threshold function. That is, if the difference between the mean of the Gaussian after and before the deviation is μ, the deviating agent is better-off if and only if the new std (after normalization) is at least $\sigma_t(\mu)$. With this observation, we show the threshold curves of the winner-take-all and winner-take-more payment functions in Fig. 7. Note that the threshold curve of the WTM depends on ξ and ρ. Here, we only show one example of the curve, where the main takeaway is that a more inclusive RO-payment function is more tolerant of deviations that have a negative gain in the mean but a higher variance of the performance score, but is less tolerant for deviations that have positive gains. When the effort ξ is relatively high, our results show that the former is likely to be the case.

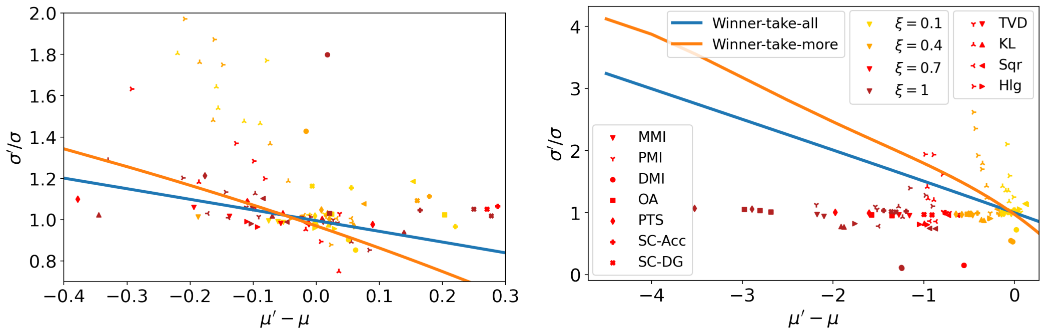
The incentive robustness of different performance measurements is presented in Fig. 7 with each dot represents a performance measurement under a specific ξ and an untruthful strategy π. Here, the y-axis is the std and the x-axis is the gain in the mean of the performance score distribution after deviation (normalized such that the original distribution is the standard Gaussian). Given the very large strategy space, we only show a few "harmful" strategies under each setting in the sense that they have large gains in the mean and high standard deviations. We summarize our results as follows.

(1) In general, higher effort in equilibrium increases the incentive robustness of the performance measurements, especially for peer prediction performance measurements. This is illustrated in Fig. 7 (b) where dots with deeper color have lower gain in the means. Intuitively, this is because a higher effort can help the peer prediction based performance measurements to distinguish the untruthful deviations.
(2) The pairing mutual information mechanism (PMI), although it successfully decreasing the mean of the performance scores after deviation, greatly increases the variance. Thus, untruthful deviations are encouraged by PMI in our scenario where payments are rank-based.
(3) The output agreement mechanism (OA) does a bad job in $W1$ where the strategies that map a less common signal to a more common signal can bring positive gains in the performance scores. This is because OA is not designed to have the truthful guarantee.
(4) In $W1$, where the ground truth space does not equal to the signal space, spot-checking performance measurements do not promise truthfulness. This is because when comparing their reports to the ground truth, the agents are ambivalent between reporting 4 and 5 which both maps to the same ground truth. Therefore, they are better-off merging these two signals to the one that has the higher probability of agreeing with the ground truth.

Overall, the performance measurements that never locate above the curves (when effort is reasonably high) and are thus claimed to be empirically robust, are TVD-MMI and Hlg-MMI.

6.3 Discussion of The Best Performance Measurement

We saw that the output agreement mechanism (OA) and the matrix f-mutual information mechanism (f-MMI) have high $\delta(\xi)$ in general. Thus, assuming agents report truthfully, these mechanisms can elicit high effort for the least cost. However, just as our experiments have shown, OA is not designed to induce truthful equilibrium. Thus, under the OA, agents have the incentive to manipulate their reports. In contrast, the truth-telling equilibrium of Hlg-MMI did appear robust.



(a) $W1$

(b) $W2$

Fig. 7. The standard deviation versus gain in the mean of the performance score distributions after unilateral deviations for different performance measurements. The thresholds on the standard deviations under the WTA and the WTM payment functions are plotted as curves: dots below the curves correspond to the cases that such deviations cannot bring positive payment gain. Specifically, the WTM curve we provide here as an example pays the top one agent $\hat{t}_1 = 11c$ and agent j $\hat{t}_j = c$ for $2 \leq j \leq 40$ (with $n = 50$ agents in total), while the bottom 10 agents are paid 0. We use colors to represent different effort level and the marker shapes to represent different performance measurements.

We therefore, in practice, to reward crowdsourincing, recommend using the matrix mutual information mechanism with the squared Hellinger Distance.

7 CONCLUSION AND FUTURE WORK

We propose a two-stage payment mechanism to incentivize crowdsourcing workers which turns the design of crowdsourcing mechanisms with continuous-effort agents into a principal-agent problem. Centered around the two components of the payment mechanism: the performance measurement and the rank order payment function, we develop both theoretical and empirical analysis and obtain the following results (contributions): For the optimal rank order payment functions:

- We solve the optimal RO-payment functions while taking care of the individual rational (IR) constraint and show that the IR constraint results in more inclusive RO-payment functions;
- We additionally show that fairness-seeking agents (risk-averse and loss-averse agents) and principals are all reasons for people to prefer inclusive RO-payment functions;
- We observe that inclusive RO-payment functions are more robust against strategic reporting.

For the best performance measurement:

- We prove that the sensitivity of a performance measurement is the sufficient statistic of its superiority in the idealized setting;
- We empirically verify the effectiveness of sensitivity beyond the Gaussian assumptions;
- Our agent-based model experiments suggest that the matrix mutual information mechanism with the squared Hellinger Distance is a good performance measurement both in reducing the cost of principal and in being robust against strategic reporting.

Several promising future directions exist. First, heterogeneous agents that have different cost functions and confusion matrices could serve as a potential generalization of this paper, where the asymmetric or mixed strategy equilibrium should be considered. Second, we focus on rank-based payments in this paper, but our insights might be generalized to other contracts, e.g. the independent contract[13]? Finally, although RO-payment functions do not require much information from the principal, they do require some. In particular, at the desired effort, the agents' cost and its derivative, and agents' signal distributions must be estimated. How can these parameters be learned by the principal and how robust are mechanisms to mispecifications of these parameters?

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A PROOF OF LEMMA 4.1

Proof. Fixing ξ, we simply let $g_e \sim N(\mu(e,\xi), \sigma(e,\xi))$ be the p.d.f. of the scores when agent i's effort is e and all the other agents' effort is ξ, and let G_e be the c.d.f.. Let S be a random variable with p.d.f. g_e. Let $q_e(p)$ be the quantile function of S such that $\int_{-\infty}^{q(p)} g_e(x)dx = p$.

Because $p_j(\xi,\xi) = \frac{1}{n}$, it's equivalent to show that $p_j(\xi',\xi)$ is decreasing in j, where $\xi' = \xi + \Delta e$. Note that $p_j(\xi',\xi)$ is the jth order statics, which concentrates to its expectation when n is sufficiently large. Therefore, $p_j(\xi',\xi)$ can be approximated by the quantile function, i.e. $p_j(\xi',\xi) = G_{\xi'}\left(q_\xi\left(1-\frac{j}{n}\right)\right) - G_{\xi'}\left(q_\xi\left(1-\frac{j+1}{n}\right)\right)$. Let $\mu = \mu(\xi,\xi)$ and $\Delta\mu = \mu(\xi',\xi) - \mu(\xi,\xi)$. Let σ and $\Delta\sigma$ be the similar notations for std. Note that $\Delta e \to 0$ implies $\Delta\mu \to 0$ and $\Delta\sigma \to 0$ since $\mu(e)$ and $\sigma(e)$ is differentiable (Assumption 3.1).

We first prove the following intermediate step.

Lemma A.1. $G_{\xi'}(x) \approx G_\xi(x) - (\Delta\mu + \Delta\sigma) g_\xi(x)$.

Proof.

$$
\begin{aligned}
G_{\xi'}(x) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu-\Delta\mu}{\sigma+\Delta\sigma}\right)^2} ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{s-\mu}{\sigma} - \frac{\Delta\mu}{\sigma}\right)^2} ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{s-\mu}{\sigma}\right)^2 + \left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{(s-\mu)}{\sigma}\frac{\Delta\mu}{\sigma}} ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{s-\mu}{\sigma}\right)^2} \left(1 + \left(1-\frac{\Delta\sigma}{\sigma}\right)\frac{(s-\mu)}{\sigma}\frac{\Delta\mu}{\sigma}\right) ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2} \left(1 + \frac{(s-\mu)}{\sigma}\frac{\Delta\sigma}{\sigma}\right)\left(1 + \frac{(s-\mu)}{\sigma}\frac{\Delta\mu}{\sigma}\right) ds \\
&\approx \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2} \left(1 + \frac{\Delta\sigma + \Delta\mu}{\sigma}\frac{(s-\mu)}{\sigma}\right) ds \\
&= G_\xi(x) - (\Delta\mu + \Delta\sigma) g_\xi(x)
\end{aligned}
$$

□

Then,

$$
\begin{aligned}
p_j(\xi',\xi) &= G_{\xi'}\left(q_\xi\left(1-\frac{j}{n}\right)\right) - G_{\xi'}\left(q_\xi\left(1-\frac{j+1}{n}\right)\right) \\
&\approx \frac{1}{n} + (\Delta\mu + \Delta\sigma)\left(g_\xi\left(q_\xi\left(1-\frac{j+1}{n}\right)\right) - g_\xi\left(q_\xi\left(1-\frac{j}{n}\right)\right)\right)
\end{aligned} \tag{7}
$$

By assumption 3.3, $(\Delta\mu + \Delta\sigma)$ is positive. Then, it's sufficient to show $g_\xi\left(q_\xi\left(1-\frac{j+1}{n}\right)\right) - g_\xi\left(q_\xi\left(1-\frac{j}{n}\right)\right)$ is decreasing in j. To make our life easier, we consider this in the continuous scale. Let $p = 1 - \frac{j+1}{n}$ and $\Delta p = \frac{1}{n}$. Then, let $f(p) = g_\xi(q_\xi(p)) - g_\xi(q_\xi(p+\Delta p))$ with $p \in (0,1)$. We want to show that $f(p)$ is increasing in p.

First note that $\int_{-\infty}^{q_\xi(p)} g_\xi(x)dx = p$. Taking the derivative of p of both sides, we have $g_\xi(q_\xi(p)) = q_e'(p)^{-1}$. Thus, we want to show that $f(p) = q_e'(p)^{-1} - q_e'(p+\Delta p)^{-1}$ is increasing in p.

It is well know that the quantile of the Gaussian distribution can be represented by the inverse error function, i.e. $q(p) = \sqrt{2}\sigma \cdot \mathrm{erf}^{-1}(2p-1) + \mu$ for a Gaussian with mean μ and std σ, where erf^{-1}

is the inverse error function. Furthermore, we know the derivative of the inverse error function is $\frac{d}{dx}\operatorname{erf}^{-1}(x) = \frac{1}{2}\sqrt{\pi}e^{\left(\operatorname{erf}^{-1}(x)\right)^2}$. Combining these,

$$\begin{aligned}\frac{\partial}{\partial p}f(p) &= \frac{1}{\sqrt{2\pi}\sigma}\cdot\frac{\partial}{\partial p}\left(e^{-\left(\operatorname{erf}^{-1}(2p-1)\right)^2} - e^{-\left(\operatorname{erf}^{-1}(2(p+\Delta p)-1)\right)^2}\right)\\ &= \frac{\sqrt{2}}{\sigma}\left(-\operatorname{erf}^{-1}(2p-1) + \operatorname{erf}^{-1}(2(p+\Delta p)-1)\right)\end{aligned}$$

Because $\operatorname{erf}^{-1}(x)$ is increasing in x, we know $\frac{d}{dp}f(p)$ is positive which completes the proof. □

B PROOF OF THE OPTIMAL RANK ORDER PAYMENT FUNCTION FOR NEUTRAL AGENTS

B.1 Proof of Proposition 4.2

Proof. We start with solving the principal's optimization problem 4. Given that the agents are neutral we can write down the Lagrange and the KKT conditions as:

$$L(\hat{\boldsymbol{t}}, \alpha, \beta, \gamma) = \sum_{j}^{n}\hat{t}_j - \sum_{j}^{n}\alpha_j\hat{t}_j + \beta c(\xi) - \frac{\beta}{n}\sum_{j=1}^{n}\hat{t}_j - \gamma\sum_{j=1}^{n}p'_j(\xi)\cdot\hat{t}_i + \gamma c'(\xi).$$

① $\alpha_j = 1 - \frac{\beta}{n} - \gamma\cdot p'_j(\xi)$ for any $j \in [n]$;
② $\alpha_j\hat{t}_j = 0$ for any $j \in [n]$;
③ $\beta\cdot\left(c(\xi) - \frac{1}{n}\sum_{j=1}^{n}\hat{t}_j\right) = 0$;
④ $\sum_{j=1}^{n}p'_j(\xi)\cdot\hat{t}_j = c'(\xi)$;
⑤ $\boldsymbol{\alpha}, \beta \geq 0$;
⑥ $-\hat{\boldsymbol{t}}, (c(\xi) - \frac{1}{n}\sum_{j=1}^{n}\hat{t}_j) \leq 0$.

Let $\omega(\xi) = c'(\xi)/p'_1(\xi)$. Now, we show that if IR is not binding, the solution to this problem is $\hat{t}_1 = \omega(\xi)$ and $\hat{t}_j = 0$ for any $j > 1$. IR is not binding impels $\beta = 0$ (condition ③). Then, we look at condition ①. Note that $\alpha_j \geq 0$ for any j and at least one of the α_j is equal to zero. Otherwise $\hat{t}_j = 0$ for any j (condition ②), and condition ④ is violated. There are two possible cases: if $\gamma < 0$, $\alpha_j = 0$ if and only if $p'_j(\xi)$ reaches its minimum; If $\gamma > 0$, $\alpha_i = 0$ if and only if $p'_j(\xi)$ reaches its maximum. (Note that $\gamma = 0$ is trivially infeasible.)

In lemma 4.1, we show that $p'_j(\xi)$ is decreasing in j given a fixed ξ. This property implies that the first case, i.e. $\gamma < 0$, is not feasible. Because $p'_j(\xi)$ reaches its minimum when $j = n$. However, if $\alpha_n = 0$ and $\hat{t}_n > 0$, condition ④ is violated given that c is increasing (RHS of ④ is positive) and $p'_j(\xi) < 0$ (LHS of ④ is negative). Therefore, the only possible solution is $\alpha_1 = 0$ and $\hat{t}_1 > 0$. By condition ④, $\hat{t}_1 = \omega(\xi)$ as $\Delta e \to 0^+$.

The above argument assumes IR is not binding, when is true when $\omega(\xi) \geq n\cdot c(\xi)$ or equivalently, $\eta(\xi) \geq n\cdot p'_j(\xi)$. If $\eta(\xi) < n\cdot p'_i(\xi)$, IR is binding, which implies that $\sum_{j=1}^{n}\hat{t}_j = n\cdot c(\xi)$. Any RO-payment function that satisfies FOC and makes IR binding are optimal. If we apply a threshold RO-payment function that pays agent j $\hat{t}_j = \tau$ if $1 \leq j \leq \hat{n}$, we completes the proof by solving for $\hat{n}$ and τ. □

C PROOF OF THE OPTIMAL RANK ORDER PAYMENT FUNCTION FOR LOSS-AVERSE AGENTS

C.1 Proof of Proposition 4.3

The precise version of Proposition 4.3 is shown here. For simplification, let $\eta(\xi) = \frac{c'(\xi)}{c(\xi)}$. Then, let

$$H(\xi, k) = \left(\left(1 + \rho \frac{n-k}{n}\right) \eta(\xi) - \sum_{j=2}^{k} p'_j(\xi) + \rho \sum_{j=k+1}^{n} p'_j(\xi) \right),$$

and let $L(k) = (1+\rho)(n-k) + 1$, we have the following results.

Proposition C.1. *Suppose $n \to \infty$, $\xi \in [0, 1]$ and agents are loss-averse.*

*(1) **IR is not binding:** If $H(\xi, \bar{n}) \geq L(\bar{n}) \cdot p'_1(\xi)$, the optimal RO-payment function satisfies $\hat{t}_1 = H(\xi, \bar{n}) \cdot c(\xi)/p'_1(\xi)$, $\hat{t}_j = c(\xi)$ for $1 < j \leq \bar{n}$ and $\hat{t}_j = 0$ for $\bar{n} < j \leq n$, with threshold $\bar{n}$ such that $(1+\rho)p'_{\bar{n}}(\xi) = p'_1(\xi)$;*

*(2) **IR is binding:** Otherwise, the optimal RO-payment function makes IR binding and pays fewer agents 0, where $\hat{t}_1 = H(\xi, \hat{n}) \cdot c(\xi)/p'_1(\xi)$, $\hat{t}_j = c(\xi)$ for $1 < j \leq \hat{n}$ and $\hat{t}_j = 0$ for $\hat{n} < j \leq n$, with threshold $\hat{n}$ such that $H(\xi, \hat{n}) = L(\hat{n}) \cdot p'_1(\xi)$.*

Proof. Given the indifferentiability of the loss-averse utility, instead of using KKT conditions, we provide a more intuitive proof. As usual, we first ignore the IR constraint. Then the goal of the principal is to satisfied FOC with the minimum payments. Thus, starting with the all-zero payment, he will pay agents with the largest marginal return until FOC is satisfied. The marginal return of paying an agent with ranking j is

$$d\hat{t}_j \begin{cases} = (1+\rho)p'_j(\xi) & \text{if } \hat{t}_j < c(\xi), \\ \in [p'_j(\xi), (1+\rho)p'_j(\xi)] & \text{if } \hat{t}_j = c(\xi), \\ = p'_j(\xi) & \text{if } \hat{t}_j > c(\xi). \end{cases}$$

Then, by Lemma 4.1, the optimal RO-payment function pays each agent j their cost $c(\xi)$ in the order of their ranking until some $\bar{n}$ such that the principal is marginally better off to pay the top one agent more than $c(\xi)$ rather than paying the $\bar{n}+1$ agent anything positive. The threshold $\bar{n}$ therefore satisfies $(1+\rho)p'_{\bar{n}}(\xi) = p'_1(\xi)$. Thus, the optimal RO-payment function is $\hat{t}_j = c(\xi)$ for $1 < j \leq \bar{n}$, $\hat{t}_j = 0$ for $\bar{n} < j \leq n$ and $\hat{t}_1$ such that FOC is satisfied. This gives us

$$\hat{t}_1 = \left(\left(1 + \rho \frac{n-k}{n}\right) c'(\xi) - c(\xi) \sum_{j=2}^{k} p'_j(\xi) + \rho c(\xi) \sum_{j=k+1}^{n} p'_j(\xi) \right) / p'_1(\xi).$$

Let $H(\xi, k) = \left(\left(1 + \rho \frac{n-k}{n}\right) \eta(\xi) - \sum_{j=2}^{k} p'_j(\xi) - \rho \sum_{j=k+1}^{n} p'_j(\xi) \right)$, then $\hat{t}_1 = H(\xi, \bar{n}) \cdot \frac{c(\xi)}{p'_1(\xi)}$. The condition for this to be true relies on IR being satisfied, i.e. $\hat{t}_1 + (\bar{n}-1)c(\xi) \geq nc(\xi) + \rho(n-\bar{n})c(\xi)$. Let $L(k) = (1+\rho)(n-k)+1$. The condition becomes $H(\xi, \bar{n}) \geq L(\bar{n}) \cdot p'_1(\xi)$.

When IR is binding, i.e. $H(\xi, \bar{n}) < L(\bar{n}) \cdot p'_1(\xi)$, the payments satisfy $\sum_{j=1}^{n} \hat{t}_j = nc(\xi) + \sum_{j=1}^{n} \rho(c(\xi) - \hat{t}_j)^+$. Then, the goal is to minimize $\sum_{j=1}^{n} \rho(c(\xi) - \hat{t}_j)^+$, i.e. to overcome as more agents' cost as possible. With the same argument, the optimal ORPF pays agents with ranking smaller than some threshold $\hat{n}$ their cost and pay the top one agent $\hat{t}_1$ such that FOC is satisfied and IR is binding. This gives us $\hat{t}_1 = H(\xi, \hat{n}) \cdot \frac{c(\xi)}{p'_1(\xi)}$ and $\hat{n}$ such that $H(\xi, \hat{n}) = L(\hat{n}) \cdot p'_1(\xi)$. Note that $\hat{n} < n$ because when $\hat{n} = n$, IR is satisfied because everyone is paid her cost but FOC can never be satisfied because there is no incentive to exert higher effort.

Finally, we complete the proof by showing $\hat{n} \geq \bar{n}$. Note that $H(\xi, \bar{n}) < L(\bar{n}) \cdot p_1'(\xi)$ but $H(\xi, \hat{n}) = L(\hat{n}) \cdot p_1'(\xi)$. We only have to show that the marginal return of increasing k is positive for function $H(\xi, k) - L(k) \cdot p_1'(\xi)$. We have that the marginal return is $(1+\rho)p_1'(\xi) - (1+\rho)p_k'(\xi) \geq 0$, which completes the proof.

□

C.2 Proof of Corollary 4.5

Proof. The proof is straightforward. With Proposition C.1, when IR is not binding, $n^I = \bar{n}$ which is determined by $(1+\rho)p_{\bar{n}}'(\xi) = p_1'(\xi)$. Because $p_j'(\xi)$ is decreasing in j by Lemma 4.1, $\bar{n}$ is increasing in ρ.

When IR is binding, $n^I = \hat{n}$ determined by $H(\xi, \hat{n}) = L(\hat{n}) \cdot p_1'(\xi)$. If we can show that $H(\xi, k) - L(k) \cdot p_1'(\xi)$ is decreasing in ρ, we can complete the proof because we know that $H(\xi, k) - L(k) \cdot p_1'(\xi)$ is increasing in k. It turns out the derivative of this term w.r.t. ρ is $\frac{n-k}{n}\eta(\xi) - \sum_{j=k+1}^{n} p_j'(\xi) - (n-k)p_1'(\xi)$. Because $\sum_{j=k+1}^{n} p_j'(\xi) \leq 0$ for any k, and $\frac{\eta(\xi)}{n} < p_1'(\xi)$ when $n \to \infty$, the derivative is negative and we complete the proof.

□

D PROOF OF THE OPTIMAL RANK ORDER PAYMENT FUNCTION FOR RISK-AVERSE AGENTS

D.1 Proof of Proposition 4.6

The precise version of Proposition 4.6 is shown here. Let $\phi(x) = r_a^{-1}(x)$ be the inverse of the reward function, and ϕ' be the derivative. Let $v(j, k, \beta, \xi) = (\phi')^{-1}\left(\left(\phi'(0) - \frac{\beta}{n}\right) \cdot \frac{p_j'(\xi)}{p_{k+1}'(\xi)} + \frac{\beta}{n}\right)$.

Proposition D.1. *Suppose $n \to \infty$ and agents are risk-averse.*

*(1) **IR is not binding:** If $\sum_{j=1}^{\bar{n}} v(j, \bar{n}, 0, \xi) \geq n \cdot c(\xi)$, the optimal RO-payment function satisfies $r_a(\hat{t}_j) = v(j, \bar{n}, 0, \xi)$ for $1 \leq j \leq \bar{n}$ and $\hat{t}_j = 0$ otherwise, with $\bar{n} \leq \frac{n}{2}$ determined by the FOC constraint, i.e. $\sum_{j=1}^{\bar{n}} p_j'(\xi) \cdot v(j, \bar{n}, 0, \xi) = c'(\xi)$;*

*(2) **IR is binding:** Otherwise, the optimal RO-payment function satisfies $r_a(\hat{t}_j) = v(j, \hat{n}, \beta, \xi)$ for $1 \leq j \leq \hat{n}$ and $\hat{t}_j = 0$ otherwise, with $\hat{n} \geq \bar{n}$ and β determined by the FOC and IR constraints.*

Proof. Because $\phi(x) = r_a^{-1}(x)$ is a differentiable convex function, the problem is a convex optimization problem. We can rewrite the principal's problem in terms of $r_j = r_a(\hat{t}_j)$ and write down the Lagrange and the KKT conditions.

$$L(\boldsymbol{r}, \alpha, \beta, \gamma) = \sum_{j}^{n} \phi(r_j) - \sum_{j}^{n} \alpha_j r_j + \beta c(\xi) - \frac{\beta}{n} \sum_{j=1}^{n} r_j - \gamma \sum_{j=1}^{n} p_j'(\xi) \cdot r_i + \gamma c'(\xi).$$

① $\alpha_j = \phi'(r_j) - \frac{\beta}{n} - \gamma \cdot p_j'(\xi)$ for any $j \in [n]$;
② $\alpha_j r_j = 0$ for any $j \in [n]$;
③ $\beta \cdot \left(c(\xi) - \frac{1}{n}\sum_{j=1}^{n} r_j\right) = 0$;
④ $\sum_{j=1}^{n} p_j'(\xi) \cdot r_j = c'(\xi)$;
⑤ $\boldsymbol{\alpha}, \beta \geq 0$;
⑥ $-\boldsymbol{r}, (c(\xi) - \frac{1}{n}\sum_{j=1}^{n} r_j) \leq 0$.

Again, we start with the case where IR is not binding and $\beta = 0$. Thus, by ①, $\alpha_j = \phi'(r_j) - \gamma \cdot p_j'(\xi)$. Whenever $\hat{t}_j > 0$, $r_j > 0$ and $\alpha_j = 0$. By Lemma 4.1, $p_j'(\xi)$ is decreasing in j, and for the same reason in Appendix B.1, $\gamma > 0$. Therefore, the optimal payment scheme takes a threshold form for some threshold $\bar{n}$ where $\hat{t}_j > 0$ for $1 \le j \le \bar{n}$ and $\hat{t}_j = 0$ otherwise. Furthermore, the payments satisfy that $\frac{\phi'(r_1)}{p'_1(\xi)} = \frac{\phi'(r_2)}{p'_2(\xi)} = \cdots = \frac{\phi'(0)}{p'_{\bar{n}+1}(\xi)}$, or alternatively $r_j = (\phi')^{-1}\left(\phi'(0) \cdot \frac{p'_j(\xi)}{p'_{\bar{n}+1}(\xi)}\right)$. Note that because $r'_a(0) < \infty$, $\phi'(0) > 0$ and the solution is feasible. Then, to find the threshold $\bar{n}$, we can simply solve the FOC constraint, i.e. $\sum_{j=1}^{\bar{n}} p'_j(\xi) \cdot r_j = c'(\xi)$. The solution does not take a clean closed-form, but we know that $\bar{n} \le \frac{n}{2}$ because $p'_j(\xi) \le 0$ when $j \ge \frac{n}{2}$ (Eq. (7)), in which case $\alpha_j > 0$ for sure.

When IR is binding and $\beta > 0$, the same arguments still hold and $r_j = (\phi')^{-1}\left(\left(\phi'(0) - \frac{\beta}{n}\right) \cdot \frac{p'_j(\xi)}{p'_{\hat{n}+1}(\xi)}\right)$. Again, by soling IR is binding and FOC is satisfied, we have solutions for β and $\hat{n}$. Furthermore, we know that while fixing any ξ, in the case where IR is considered, the threshold $\hat{n}$ is no less than $\bar{n}$ which is the threshold when IR is not considered. First, if $\phi'(0) - \frac{\beta}{n} < 0$, $p'_{\hat{n}+1}(\xi) < 0$ and $\hat{n} \ge \frac{n}{2} \ge \bar{n}$. Second, if $\phi'(0) - \frac{\beta}{n} \ge 0$, suppose $\bar{n} > \hat{n}$. Every r_j in the IR-binding case is smaller than the case where IR is not binding. Consequently, ④ is violated which implies that $\bar{n} \le \hat{n}$.

□

D.2 Risk-aversion And Inclusiveness

Now, we show that more risk-averse agents does not imply more an inclusive optimal RO-payment function.

Corollary D.2. *Suppose $n \to \infty$ and agents are risk-averse. Let r_{a1} and r_{a2} be two concave reward functions of agents such that $\frac{\phi'_1(x)}{\phi'_1(0)} > \frac{\phi'_2(x)}{\phi'_2(0)}$ for any $x > 0$, where ϕ'_1 and ϕ'_2 are the derivative of the inverse of r_{a1} and r_{a2} respectively. Then, if IR is not binding, the optimal RO-payment function when agents have reward function r_{a1} is more inclusive than the case of r_{a2}. However, if IR is not binding, both cases are possible.*

Proof. First, we show that if IR is not binding, the RO-payment function is more inclusive as $\frac{\phi'(x)}{\phi'(0)}$ becomes larger for any $x > 0$. By Proposition D.1, when IR is not binding, the optimal RO-payment function is determined by

$$\frac{\phi'(r_a(\hat{t}_j))}{\phi'(0)} = \frac{p'_j(\xi)}{p'_{\bar{n}+1}(\xi)}. \tag{8}$$

Suppose $\frac{\phi'_1(x)}{\phi'_1(0)} > \frac{\phi'_2(x)}{\phi'_2(0)}$ for any $x > 0$, but $\hat{\boldsymbol{t}}_1$ is more exclusive than $\hat{\boldsymbol{t}}_2$, i.e. $\bar{n}_1 < \bar{n}_2$. Then, for any $j \le \bar{n}$, $\frac{p'_j(\xi)}{p'_{\bar{n}_1+1}(\xi)} < \frac{p'_j(\xi)}{p'_{\bar{n}_2+1}(\xi)}$ due to Lemma 4.1. As a result, to satisfy eq. (8), $r_{a1}(\hat{t}_{1,j}) < r_{a2}(\hat{t}_{2,j})$ for any $j \le \bar{n}_1 \le \bar{n}_2$. However, one of the payments, $\hat{\boldsymbol{t}}_1$ or $\hat{\boldsymbol{t}}_2$ must violate IR, which implies $\sum_{j=1}^{\bar{n}} r_a(\hat{t}_j) = c(\xi)$, because $\sum_{j=1}^{\bar{n}_1} r_a(\hat{t}_{1,j}) < \sum_{j=1}^{\bar{n}_2} r_a(\hat{t}_{2,j})$. Therefore, $\hat{\boldsymbol{t}}_1$ must be at least as inclusive as $\hat{\boldsymbol{t}}_2$.

Second, we show that this pattern does not generally hold when IR is binding. Now, the optimal RO-payment function must satisfy

$$\frac{\phi'(r_a(\hat{t}_j)) - \frac{\beta}{n}}{\phi'(0) - \frac{\beta}{n}} = \frac{p'_j(\xi)}{p'_{\hat{n}+1}(\xi)}, \tag{9}$$

with $\beta > 0$. On one hand, the optimal RO-payment function can be more exclusive as $\frac{\phi'(x)}{\phi'(0)}$ increasing. Again, suppose $\frac{\phi'_1(x)}{\phi'_1(0)} > \frac{\phi'_2(x)}{\phi'_2(0)}$ for any $x > 0$ and $\phi'_1(0) = \phi'_2(0)$. In this case,

$$\frac{\phi'_1(x) - \frac{\beta}{n}}{\phi'_1(0) - \frac{\beta}{n}} - \frac{\phi'_2(x) - \frac{\beta}{n}}{\phi'_2(0) - \frac{\beta}{n}} = \frac{\phi'_1(x) - \phi'_2(x)}{\phi'_1(0) - \frac{\beta}{n}} > 0.$$

This implies that if $\phi_1'(0) - \frac{\beta}{n} > 0$, the same arguments in the IR not binding case still hold and $\hat{\boldsymbol{t}}_1$ must be at least as inclusive as $\hat{\boldsymbol{t}}_2$.

On the other hand, $\hat{\boldsymbol{t}}_1$ can be more exclusive when $\frac{\phi_1'(x)}{\phi_1'(0)} > \frac{\phi_2'(x)}{\phi_2'(0)}$ for any $x > 0$. Consider the case where $\phi_1'(x) - \phi_2'(x) > \phi_1'(0) - \phi_2'(0)$, $0 < \phi_2'(0) < \phi_1'(0) < \frac{\beta}{n}$. In this case,

$$\begin{aligned}
\frac{\phi_1'(x) - \frac{\beta}{n}}{\phi_1'(0) - \frac{\beta}{n}} - \frac{\phi_2'(x) - \frac{\beta}{n}}{\phi_2'(0) - \frac{\beta}{n}} &= \frac{\phi_1'(x)\phi_2'(0) - \phi_2'(x)\phi_1'(0) + \frac{\beta}{n} \cdot (\phi_2'(x) - \phi_1'(x) + \phi_1'(0) - \phi_2'(0))}{(\phi_1'(0) - \frac{\beta}{n}) \cdot (\phi_2'(0) - \frac{\beta}{n})} \\
&< \frac{\phi_1'(x)\phi_2'(0) - \phi_2'(x)\phi_1'(0) + \phi_2'(0) \cdot (\phi_2'(x) - \phi_1'(x) + \phi_1'(0) - \phi_2'(0))}{(\phi_1'(0) - \frac{\beta}{n}) \cdot (\phi_2'(0) - \frac{\beta}{n})} \\
&= \frac{(\phi_1'(0) - \phi_2'(0)) \cdot (\phi_2'(0) - \phi_2'(x))}{(\phi_1'(0) - \frac{\beta}{n}) \cdot (\phi_2'(0) - \frac{\beta}{n})} \\
&\leq 0.
\end{aligned}$$

This implies that when agents become more risk-averse, i.e. $\frac{\phi_1'(x)}{\phi_1'(0)} > \frac{\phi_2'(x)}{\phi_2'(0)}$ for any $x > 0$, the LHS of eq. (9) becomes smaller. Now, suppose $\hat{n}_1 > \hat{n}_2$. We have $p'_{\hat{n}_1+1} < p'_{\hat{n}_2+1} < 0$, and thus $\frac{p_j'(\xi)}{p'_{\hat{n}_1+1}(\xi)} > \frac{p_j'(\xi)}{p'_{\hat{n}_2+1}(\xi)}$ for any $j \leq \hat{n}_2$. As a result, to satisfy eq. (9), $r_{a1}(\hat{t}_{1,j}) > r_{a2}(\hat{t}_{2,j})$ for any $j \leq \hat{n}_2$. Again, this violates the IR constraint for the same reason in the IR not binding case, which implies $\hat{n}_1 \leq \hat{n}_2$.

□

E PROOF OF PROPOSITION 4.8

Proof. While fixing ξ and ξ', we view $p_j(\xi', \xi)$ as a function of $\mu(\xi)$ and $\sigma(\xi)$, denoted as $p_j(\mu, \sigma, \xi', \xi)$.

The intuition of the proof is that suppose $\hat{\boldsymbol{t}}^*$ is the optimal RO-payment function when performance measurement Ψ is applied. Now, fixing ξ, if $\delta(\xi)$ increases, we show that the FOC constraint is easier to be satisfied, i.e. FOC can be satisfied with strictly lower total payment. This implies that with a performance measurement that has higher sensitivity, the principal is at least not worse-off. To see this, when IR is not binding, it is straightforward that the principal can reduce the payments to satisfy FOC without violating IR and LL. When IR is not binding, the principal can reduce $\hat{t}_1$ by ϵ_1 and increase $\hat{t}_n$ by $\epsilon_n \leq \epsilon_1$ such that FOC is satisfied and IR is still binding.[11]

With this intuition, our goal is to show that FOC can be satisfied with strictly lower payment as δ increases. Let $\lambda_j = r_a\left(\hat{t}_j\right) - \rho(c(\xi') - \hat{t}_j)^+$. Note that the FOC constraint says that $\sum_{j=1}^n (p_j(\mu, \sigma, \xi', \xi) - \frac{1}{n}) \cdot \lambda_j = c'(\xi)$. Because the only term that depends on $\delta(\xi)$ is $p_j(\mu, \sigma, \xi', \xi)$. The rest of the proof can be summarized in Lemma E.1, which shows that the left-hand-side of the FOC constraint is increasing in δ while fixing the payment, or equivalently, FOC can be satisfied with lower payment as δ increases.

We then complete the proof by showing $\lambda_j = r_a\left(\hat{t}_j\right) - \rho(c(\xi') - \hat{t}_j)^+$ is decreasing in j under the optimal RO-payment function for any type of agents. This can be proven by showing that the optimal RO-payment function is monotone decreasing with $\hat{t}_j \leq \hat{t}_k$ if $j \geq k$. According to Proposition 4.2, C.1 and D.1, this is exactly the case.

□

[11]For neutral agents, $\epsilon_n = \epsilon_1$, in which case the principal is equivalent. For risk/loss-averse agents, $\epsilon_n < \epsilon_1$ because such agents prefer more inclusive payments, in which case the principal is better-off.

Lemma E.1. *For any $\xi \in [0,1]$, $\sum_{j=1}^{n} p_j(\mu,\sigma,\xi',\xi)\cdot\lambda_j$ is increasing in $\delta(\xi) = \frac{\mu'(\xi)+\sigma'(\xi)}{\sigma(\xi)}$ if $0 < \lambda_j \leq \lambda_k$ for any $1 \leq k \leq j \leq n$.*

Proof. Let μ, σ, $\Delta\mu$ and $\Delta\sigma$ be the same definitions as in Appendix B.1. With the same approach in Lemma A.1, we can rewrite the probability $p(\mu,\sigma,\xi',\xi,j)$ as Let μ and σ be the mean and std of the Gaussian score distribution when agent i's effort is ξ (her peers' effort is ξ as well). Let agent i deviate to effort $\xi' = \xi + \Delta e$ for $\Delta e \to 0$. The corresponding changes in the mean and std of the Gaussian are denoted as $\Delta\mu$ and $\Delta\sigma$ respectively. Let $G_0(x)$ be the c.d.f. of the standard Gaussian distribution. With the same approach in Lemma A.1, we can rewrite the probability $p(\mu,\sigma,\xi',\xi,j)$ as

$$
\begin{aligned}
p_j(\mu,\sigma,\xi',\xi) &= \int_{-\infty}^{\infty} g(\xi+\Delta e, x)\,(G(\xi,x))^{n-j}\,(1-G(\xi,x))^{j-1}\,dx \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu-\Delta\mu}{\sigma+\Delta\sigma}\right)^2}\left(\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2}ds\right)^{n-j}\left(1-\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2}ds\right)^{j-1}dx \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu-\Delta\mu}{\sigma+\Delta\sigma}\right)^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{1}{2}y^2}dy\right)^{n-j}\left(1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{1}{2}y^2}dy\right)^{j-1}dx \\
&\qquad\qquad (y = \tfrac{s-\mu}{\sigma}) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\left(1-\frac{\Delta\sigma}{\sigma}\right)z-\frac{\Delta\mu}{\sigma}\right)^2}(G_0(z))^{n-j}\,(1-G_0(z))^{j-1}\,dz \qquad (z = \tfrac{x-\mu}{\sigma}) \\
&\approx \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}z^2}\left(1+\frac{\Delta\sigma+\Delta\mu}{\sigma}z\right)(G_0(z))^{n-j}\,(1-G_0(z))^{j-1}\,dz
\end{aligned}
$$

Let $\delta = \frac{\Delta\mu+\Delta\sigma}{\sigma}$. We have,

$$
\begin{aligned}
\sum_{j=1}^{n}\lambda_j\frac{\partial\Delta p_j(\mu,\sigma,\xi',\xi)}{\partial\delta} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ze^{-\frac{1}{2}z^2}\sum_{j=1}^{n}\lambda_j\left((G_0(z))^{n-j}\,(1-G_0(z))^{j-1}\right)dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} ze^{-\frac{1}{2}z^2}\sum_{j=1}^{n}(\lambda_j-\lambda_{n-j+1})\left((G_0(z))^{n-j}\,(1-G_0(z))^{j-1}\right)dz \\
&\geq 0
\end{aligned}
$$

□