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Let us assume that $NP = DSPACE(n)$, then arrive at a violation of the space hierarchy theorem (i.e. a contradiction).

Let us define an arbitrary language L from $DSPACE(n^2)$. So, there exists a deterministic Turing machine M that decides L in $O(n^2)$ space (which decides it in at most $O\left(2^{(n^2)}\right)$ time).

Then let us define a language $L_{pad} = \left\{< x, 1^{|x|^2} > s.t. x \in L\right\}$. We now create a deterministic Turing machine N to decide whether an input $y \in L_{pad}$:

> Does there exists a string z such that $y =< z, 1^{|z|^2} >$? If not, then outright reject. If there is, then run $M(z)$ for $2^{|z|^2}$ steps and output that answer.

Clearly, the space is linear in $|y|$ because the input to L_{pad} is $|z| + |z|^2$ in length and the space of the algorithm is $|z|^2$, which is $linear(|z| + |z|^2)$. Since N runs in $DSPACE(n)$, $L_{pad} \in NP$ because of our assumption that $NP = DSPACE(n)$.

Now we want to show that $L \in DSPACE(n)$ also. We use L_{pad} and N to do this. Given an input x, pad it with $|x|^2$ number of 1's to create y, then run $N(y)$. The padding algorithm is obviously deterministic polynomial time because $|x|^2$ is $polynomial(|x|)$. And N runs in NP time. Since $P \subset NP$, L is thus decidable in NP time, which makes it decidable in $DSPACE(n)$. But L is an arbitrary language from $DSPACE(n^2)$, violating the space hierarchy theorem, because n is $o(n^2)$.