Collaborated with yc2454 (Yalu Cai)

a. Assume for the sake of contradiction that the halting problem is computable, so that given a machine M and its input x, we know whether M halts on x (1 for yes, 0 for no). If M does halt on x, then let S be the set of all trivial functions f where $f(x) = 1$; similarly, if M does not halt on x, then let S be the set of all trivial functions f where $f(x) = 0$. Then in either these cases, f_S would be computable, and the answer would be yes, the given machine M computes one of the functions in S. But Rice's theorem says f_S is incomputable, reaching a contradiction. Thus, the halting problem is incomputable.
b. We assume that f_S is computable and reach the contradiction that this would allow the halting problem to be computable. Given a Turing machine M and its input x, the halting problem is concerned with whether M halts on x. Let a set S be a non-trivial set of partial functions. We make S only have one partial function $f_\emptyset$, which is undefined for every input. We make a new Turing machine M_b that loops infinitely for every input other than x and just simulates M on x for x, i.e. M_b's pseudocode looks like this:

```
M_b(input){
    if (input!=x){
        loop infinitely;
    }
    else{
        return M(x);
    }
}
```

If there were a Turing machine that computes f_S, then we would know whether M_b halts on x, and thus whether M halts on x. Because if f_S outputs 1 on input M_b, then we know M_b doesn't halt on x because that means M_b computes $f_\emptyset$, which is undefined for x. The complement of S (let's call it S') is also a non-trivial set of partial functions. If there were a Turing machine that computes $f_{S'}$ and outputs 1 on input M_b, then we know M_b (and thus, M) halts on x, because then that means there's a function in S' that is undefined for every input other than x and is defined for x, and has output $M_b(x)$. Either of these Turing machines that decide f_S or $f_{S'}$ is a decider for the halting problem, which we know is incomputable. Rice's theorem is thus proven.