Collaborated with yc2454 (Yalu Cai)

a. Let us create a partial function f_{halt} that is defined on all inputs, and outputs either 0 or 1 on each input. Then let us create a set S of non-trivial partial functions, where the only function in it is f_{halt}. Assume for the sake of contradiction that there exists a Turing machine $HaltChecker$ that decides whether a Turing machine M halts on input x, then there exists a Turing machine M for which $HaltChecker$ computes f_{halt}, because $HaltChecker$ outputs either 0 or 1 on a machine and its input. But Rice's Theorem states that there doesn't exist such a Turing machine that can decide whether an input machine computes a non-trivial set of partial functions, hence reaching a contradiction. The halting problem is thus uncomputable.
b. We assume that f_S is computable and reach the contradiction that this would allow the halting problem to be computable. Given a Turing machine M and its input x, the halting problem is concerned with whether M halts on x. Let a set S be a non-trivial set of partial functions. We make S only have one partial function $f_{\emptyset}$, which is undefined for every input. We make a new Turing machine M_b that loops infinitely for every input other than x and just simulates M on x for x, i.e. M_b's pseudocode looks like this:

```
M_b(input){
    if (input!=x){
        loop infinitely;
    }
    else{
        return M(x);
    }
}
```

If there were a Turing machine that computes f_S, then we would know whether M_b halts on x, and thus whether M halts on x. Because if f_S outputs 1 on input M_b, then we know M_b doesn't halt on x because that means M_b computes $f_{\emptyset}$, which is undefined for x. The complement of S (let's call it S') is also a non-trivial set of partial functions. If there were a Turing machine that computes $f_{S'}$ and outputs 1 on input M_b, then we know M_b (and thus, M) halts on x, because then that means there's a function in S' that is undefined for every input other than x and is defined for x, and has output $M_b(x)$. Either of these Turing machines that decide f_S or $f_{S'}$ is a decider for the halting problem, which we know is incomputable. Rice's theorem is thus proven.