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1. Let us negate the goal and assume the negated goal is true to prove the contrapositive, which is logically equivalent to the original statement. The negated goal would be “ an NP-complete problem that has a algorithm.” We assume this to prove that SAT will have a algorithm. Since any NP-complete problem can be reduced to any other NP-complete problem, SAT can be reduced to in polynomial time, and so we could solve SAT through such a reduction. The total time associated with such a reduction plus the time to solve is , where means polynomial time in , which we can express as or , where and are natural numbers. We now need to show is . is anyway, so we can ignore it, and is still , so we only need to show is . The two sides have the same base –– 2, so we only **need to show is .**

If a function is , that by definition means , which means . Hence, . To show that is , we need to show . We already showed that as approaches infinity, the numerator approaches ; and it’s obvious that the denominator approaches infinity, so the whole thing approaches . is thus , and so is . So has a algorithm. The contrapositive is thus proven. QED.

1. We can solve by solving the part of it through brute force. For sake of simplicity, let , then . Brute forcing is of runtime . To show that has a algorithm, we need to show is , which through definition of little o notation means we need to show , such that , . Substituting in we get that **we need to show, such that ,** . Now take logs of both sides:

Observe that because grows faster than , so such a that we mentioned. Hence, has a algorithm.

1. Let us assume the negated goal to prove the contrapositive, i.e. . This is equivalent to (SAT has aalgorithm, which means we have to prove both and . For the second one, we will prove the original direction

Claim 1: ( )

Proof: If , it means there’s a polynomial time algorithm to solve , i.e. an algorithm where is a natural number and is the length of the formula. We can solve problems using this algorithm by padding with the correct number of ’s, which is a polynomial time reduction. The size of the corresponding formula for an formula of size is . Then we use the supposed algorithm, and the runtime on this will be for some natural number . We want to show this is . To show this, we need to show such that . Take the log of both sides:

Since grows faster than and overtakes it at , we can ignore the lone because it will become insignificant, i.e. is :

is a constant so we can ignore it:

is a constant, so we can ignore that too:

This is true because the limit of this as approaches infinity is , as grows faster than . QED.

Claim 2: ( )

Proof: Using what we proved in the previous two parts of the question, we know that if doesn’t have any algorithms, then no NP-complete problem has a algorithm (part 1), but *does* have a algorithm, which means it is not NP-complete. QED.