Reduction Algorithm

Given an SAT problem with n variables and a formula, turn all $\overline{x_i}$ in the formula into a new variable y_i in a new formula, which is not negated. Also, add n new clauses $(x_i \vee y_i)$ for all $i \in \{1,2, \dots n\}$, where $y_i = \overline{x_i}$. Run the imaginary Monotone SAT code on this new formula with n as k. The answer from this is the answer for SAT. For example:

$$(x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2}) \wedge (\overline{x_3} \vee x_1) \wedge (\overline{x_1} \vee x_3)$$

Will be reduced to

$$(x_1 \vee x_2 \vee x_3) \wedge (y_1 \vee y_2) \wedge (y_3 \vee x_1) \wedge (y_1 \vee x_3) \wedge (x_1 \vee y_1) \wedge (x_2 \vee y_2) \wedge (x_3 \vee y_3), \quad k = 3$$

This reduction algorithm is polynomial because you're just going through the entire formula and creating a new, non-negated variable for each negated variable (linear, actually). Transforming the output is constant time.

Proof of Correctness

We need to prove both of the two following claims:

1. if the answer to $MonotoneSAT(formula', k)$ is yes, then the answer to $SAT(formula)$ is also yes.
2. if the answer to $MonotoneSAT(formula', k)$ is no, then the answer to $SAT(formula)$ is also no.

1. Suppose $MonotoneSAT(formula', k)$ returns yes, then $formula'$ is satisfiable with at most k variables set to true. We added $k = n$ new clauses. So, $MonotoneSAT(formula', k)$ returning yes means the k new clauses are all true, meaning at least one variable in each of those k disjunctions are true; and since we have the constraint of setting at most k variables to true, exactly one variable from each of these k new clauses is true. This satisfies that for any i, x_i and $\overline{x_i}$ cannot both be true in the original problem. The remaining clauses are essentially the original problem. Thus, $SAT(formula)$ also returns yes.
2. We will prove the contrapositive of the 2nd claim: if the answer to $SAT(formula)$ is yes, then the answer to $MonotoneSAT(formula', k)$ is also yes. Since there is a one-to-one correspondence from all the variables $a \in formula$ to all variables $b \in formula'$ (injection from $formula$ variables to $formula'$ variables), we copy over a's truth value to b's truth value: set all $y_i = \neg x_i$, and $x_i = x_i$. The k new clauses are all obviously true, and the remaining clauses are essentially the original problem, hence also true. Thus, $MonotoneSAT(formula', k)$ also returns yes.