a. According to Piazza post @1180, this means we have to prove $\forall\ 0 < \gamma < 2$, there exists a weighted vertex cover problem whose linear programming solution $LP < \frac{1}{\gamma} OPT$, where OPT is the optimal (minimum) weight.
Let's consider a graph G with n vertices. Let there be an edge between every pair of vertices in G. Let the weight $w_i = 1$ for each vertex. It's obvious that the optimal weighted vertex cover is $n - 1$ by choosing any $n - 1$ vertices (the complement set of vertices of a graph's vertex cover is an independent set, and all possible independent sets of such a graph as we described have cardinality 1 because every pair of vertices is connected by an edge). The optimal linear programming solution LP has $x_i = \frac{1}{2}$ for every vertex, and so $LP = \vec{w} \cdot \vec{x} = \frac{n}{2}$ (it's impossible for LP to be less than $\frac{n}{2}$ because then the average x_i would be less than $\frac{1}{2}$, and that cannot satisfy the constraint $x_u + x_v \geq 1\ \forall$ edges (u, v)).
If we graph $f(n) = \frac{\frac{1}{2}\lceil n\rceil}{\lceil n\rceil - 1}$ (we put x in the ceiling function because number of vertices is always a natural number), we see that $f(n)$ starts from 1 and asymptotically approaches $\frac{1}{2}$. As γ approaches 2, $LP < \frac{1}{\gamma} OPT$ is satisfied as n approaches infinity. When $\gamma \ll 2$, $\frac{1}{\gamma} > \frac{1}{2}$, so a smaller n could satisfy $LP < \frac{1}{\gamma} OPT$. Choose an n whose $f(x)$ is less than $\frac{1}{\gamma}$. Thus, $\forall\ 0 < \gamma < 2$, there exists a weighted vertex cover problem whose linear programming solution is less than $\frac{1}{\gamma} OPT$.

b. We formulate this into a linear programming problem.
Objective function to minimize: $\vec{w} \cdot \vec{x}$
Constraints: $0 \leq x_i \leq 1\ \forall\ i \in V$

$$\sum_{i \in H} x_i \geq 1\ \forall\ H$$

After we obtain $\vec{x}$, round all $\vec{x}_i \geq \frac{1}{c}$ to 1 and all other $\vec{x}_j$'s to 0. If and only if $\vec{x}_i == 1$ is $\vec{x}_i \in S$.

Proof of correctness:

1) We want to show that this indeed produces a hitting set S
This problem is essentially vertex cover on a hypergraph. An edge in a hypergraph has multiple vertices, and as long as one vertex on an edge is hit, the entire edge is covered. Our constraints ensure that at least one $x_i \in H$ is greater than or equal to $\frac{1}{|H|}$, which in turn is greater than or equal to $\frac{1}{c}$ because $|H| \leq c$, and so that x_i would be rounded up to 1 to be included in S.
2) We want to show this is a c-approximation.

$$\sum_{v \in S} w_v \leq c \sum_{v \in S} w_v x_v \leq c \sum_{v} w_v x_v \leq c\ OPT$$

The first inequality is because $x_v \geq \frac{1}{c}\ \forall\ v \in S$. The second inequality is because now it also includes $v \notin S$. The third inequality is because $\vec{w} \cdot \vec{x} \leq OPT$, because $\vec{x}$ found by our algorithm can be fractions while the solution OPT found by integer programming is all integers. QED.

Runtime Analysis: we're taught linear programming is **polynomial** time.