(1) [10 points] Chapter 1, Problem 1.5

For $f: \{-1,1\}^n \to \mathbb{R}$ and $S \subseteq [n]$, the Fourier coefficient of f on S is given by

Equation 1

$$\hat{f}(S) = < f, \mathrm{X}_S >$$

If we expand that, it is equal to

$$\frac{1}{N} \sum_{\vec{x} \sim \{-1,1\}^n} f(\vec{x}) \mathrm{X}_S(\vec{x})$$

Given any non-empty $S \subseteq [n]$, the number of times X_S (the parity) is -1 across all $\vec{x} \sim \{-1,1\}^n$ is exactly $\frac{N}{2}$, because for each $i \in S$, x_i will be -1 and 1 an equal number of times across all $\vec{x}$. To put this mathematically,

Equation 2

$$\forall S \subseteq [n] \; such \; that \; |S| > 0, \sum_{\vec{x} \sim \{-1,1\}^n} \mathrm{X}_S(x) = 0$$

If $\hat{f}(S)$'s magnitude exceeds $\frac{1}{2}$, that means the magnitude of the summation needs to exceed $\frac{N}{2}$, meaning one of two things needs to happen:

1. there needs to be more than $\frac{N}{2}$ $\vec{x}$'s where $f(\vec{x}) = \mathrm{X}_S(\vec{x})$: specifically, at least $\left\lceil \frac{3N+2}{2} \right\rceil$
2. there needs to be more than $\frac{N}{2}$ $\vec{x}$'s where $f(\vec{x}) \neq \mathrm{X}_S(\vec{x})$: specifically, at least $\left\lceil \frac{3N+2}{2} \right\rceil$

   (this number was derived from the system of equations $x - y > \frac{N}{2}$, $x + y = N$, where x is the number of $\vec{x}'s$ where $f(\vec{x}) = \mathrm{X}_S(\vec{x})$ for option 1 and $f(\vec{x}) \neq \mathrm{X}_S(\vec{x})$ for option 2)

It is possible to have $f(\vec{x})$ satisfy one of these for one S, but neither of these conditions can be satisfied across two S's, because $f(\vec{x})$ for each $\vec{x}$ stays the same for all S's, so that means the corresponding $\mathrm{X}_S(\vec{x})$'s of the said $\vec{x}$'s in question need to stay the same, and because we said earlier half of each S's $\mathrm{X}_S(\vec{x})$'s is -1, this would force the two S's to have the same $\mathrm{X}_s(\vec{x})$'s for all $\vec{x}$'s, which is impossible. And satisfying either of these dissatisfies $S = \emptyset$.

For the case of when $S = \emptyset$, we need at least $\left\lceil \frac{3N+2}{2} \right\rceil$ of $f(\vec{x})$'s to be equal to 1 for the Fourier constant to be greater than $\frac{1}{2}$ in magnitude, because $X_\emptyset(\vec{x}) = 1$. In this case, no other S could satisfy it because of Equation 2.

QED.

This is not true for any $f: \{-1,1\}^n \to \mathbb{R}$ with $\|f\|_2 = 1$. For example:

$f(\vec{x})$	x_1	x_2	$\mathrm{X}_S(\{x_1\})$	$\mathrm{X}_S(\{x_2\})$	$\mathrm{X}_S(\{x_1, x_2\})$	$\mathrm{X}_\emptyset$
$\sqrt{1.5}$	1	-1	1	-1	-1	1
$\sqrt{1.5}$	1	1	1	1	1	1
$\sqrt{0.5}$	-1	-1	-1	-1	1	1
$\sqrt{0.5}$	-1	1	-1	1	-1	1

As you can see, $\left|\hat{f}(\{x_1\})\right| > \frac{1}{2}$, and $\left|\hat{f}(\mathrm{X}_\emptyset)\right| > \frac{1}{2}$

(2) [15 points] Chapter 1, Problem 1.9

a. Let's examine how to get the indicator function P_{a_1,a_2} for $f: \{0,1\}^2 \to \{0,1\}$ first.

$$P_{1,1}(x_1, x_2)\begin{cases} 1 \; if \; x_1 = 1 \; and \; x_2 = 1 \\ 0 \; otherwise \end{cases}$$

It's easy to see $P_{1,1}(x_1, x_2) = x_1 x_2$.

Now let's see $P_{0,1}$

$$P_{0,1}(x_1, x_2)\begin{cases} 1 \; if \; x_1 = 0 \; and \; x_2 = 1 \\ 0 \; otherwise \end{cases}$$

We see that $P_{0,1}(x_1, x_2) = (1 - x_1)x_2$.

And so on.

Basically, when $a_i = 1$, have the term x_i; when $a_i = 0$, have the term $1 - x_i$. To put this into purely mathematical terms,

$$P_{a_1,\dots,a_n}(x_1,\dots,x_n) = \prod_{i=1}^{n}[(2a_i - 1)x_i + (-a_i + 1)]$$

So, you get

$$f(\vec{x}) = \sum_{a_1,\dots a_n \in \{0,1\}} P_{a_1,\dots a_n}(\vec{x}) f(a_1,\dots a_n)$$

That is a real multilinear polynomial. When you expand that all out, you can see that you can also write it in the form of

$$f(\vec{x}) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$$

b. Let's define $\Psi_S = \prod_{i \in S} x_i$ For $\vec{x} \in \{0,1\}^n$, the number $\Psi_S = \prod_{i \in S} x_i$ is in $\{0,1\}$. Thus, $\Psi_S : \{0,1\}^n \to \{0,1\}$ is a Boolean function; it computes the AND of the bits $(x_i)_{i \in S}$. The multilinear polynomial 1.11 given in this problem shows that any f can be represented as a linear combination of AND functions over $\{0,1\}$.

The set of all functions $f : \{0,1\}^n \to \mathbb{R}$ forms a vector space V, since we can pointwise add two functions and we can multiply a function by a real scalar. The vector space V is 2^n-dimensional. Here we illustrate the multilinear expansion of OR_2 from this perspective:

x_1	x_2	$x_1 \vee x_2$	P_{a_1,a_2}	P_{a_1,a_2} again	$\Psi_\emptyset$	$\Psi_{\{x_1\}}$	$\Psi_{\{x_2\}}$	$\Psi_{\{x_1,x_2\}}$
0	1	1	$(1 - x_1)x_2$	$-x_1x_2 + x_2$	1	0	1	0
0	0	0	$(1 - x_1)(1 - x_2)$	$x_1x_2 - x_2 - x_1 + 1$	1	0	0	0
1	0	1	$x_1(1 - x_2)$	$-x_1x_2 + x_1$	1	1	0	0
1	1	1	x_1x_2	x_1x_2	1	1	1	1

$$f(\vec{x}) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i$$

$$= 1 * (-x_1x_2 + x_2) + 0 * (x_1x_2 - x_2 - x_1 + 1) + 1 * (-x_1x_2 + x_1) + 1(x_1x_2)$$

$$= (0 * 1) + (1 * x_1) + (1 * x_2) + (-1 * x_1x_2)$$

$$OR_2 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix} = 0\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\\1\end{bmatrix} + 1\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + (-1)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

More generally, the multilinear polynomial 1.11 given in this problem shows that every function $f: \{0,1\}^n \to \mathbb{R}$ in V is a linear combination of the AND functions; i.e. the AND functions are a spanning set for V. Since the number of AND functions is $2^n = \dim V$, we can deduce that they are in fact a linearly independent basis for V. In particular this justifies the uniqueness of equation 1.11.

c. Let's recall what we did in part a:

$$f(\vec{x}) = \sum_{a_1,\ldots a_n \in \{0,1\}} P_{a_1,\ldots a_n}(\vec{x}) f(a_1, \ldots a_n)$$

All the coefficients are integers because we established in part a that $P_{a_1,\ldots,a_n}(x_1, \ldots, x_n) = \prod_{i=1}^n [(2a_i - 1)x_i + (-a_i + 1)]$: it's obvious that for this expression, since $a_i, x_i \in \{-1,1\}$, each term's coefficient must be either -1 or 1. When $P_{a_1,\ldots,a_n}(x_1, \ldots, x_n)$ is multiplied with f, which is either 0 or 1, then each coefficient would be either $-1, 0$, or 1. And because there are 2^n S's such that $S \subseteq [n]$, the coefficients would be integers in the range $[-2^n, 2^n]$.

d. The mapping from 1 to 0, and from -1 to 1, and vice versa can be modelled by these equations:

a	b	equation	b	a	equation
1	0	$b = -\frac{1}{2}a + \frac{1}{2}$	0	1	$a = -2b + 1$
-1	1		1	-1	

Expressing $q(x)$ in terms of $p(x)$, we are translating from the $\{-1,1\}$ domain to the $\{1,0\}$ domain (i.e. a to b in the above chart). So it would be

$$\frac{1}{2} + \frac{1}{2}p(\overrightarrow{x'})$$

To express $\overrightarrow{x'}$ in terms of $\vec{x}$, we need to translate the other way (i.e. from b to a in the above chart):

$$\frac{1}{2} + \frac{1}{2}p(1 - 2x_1, \ldots, 1 - 2x_n)$$

(3) [15 points] Chapter 1, Problem 1.1

a. $\frac{1}{2}x_1x_2 + \frac{1}{2}x_2 + \frac{1}{2}x_1 - \frac{1}{2}$

b. $min_3: -\frac{3}{4} + \frac{1}{4}x_1x_2x_3 + \frac{1}{4}x_1x_2 + \frac{1}{4}x_2x_3 + \frac{1}{4}x_1x_3 + \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3$

$max_3: \frac{3}{4} + \frac{1}{4}x_1x_2x_3 - \frac{1}{4}x_1x_2 - \frac{1}{4}x_2x_3 - \frac{1}{4}x_1x_3 + \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3$

c. $\boxed{\prod_{i=1}^{n} \frac{1+a_ix_i}{2}}$ (not sure if you want another format but $\boxed{\frac{1}{2^n}\sum_{S\subseteq[n]} a_i^{|S|} \cdot \mathrm{X}_S}$ works too.) In the following parts of the question I wrote all the indicator functions as the first one.

d. $\phi_{\{a\}} = \frac{1_{\{a\}}}{\boldsymbol{E}[1_{\{a\}}]}$, $\boldsymbol{E}[1_{\{a\}}] = 1 \times \boldsymbol{P}(\vec{x} = \vec{a}) + 0 \times \boldsymbol{P}(\vec{x} \neq \vec{a}) = \boldsymbol{P}(\vec{x} = \vec{a}) = \frac{1}{2^n}$,

$$\phi_{\{a\}} = 2^n \times 1_{\{a\}} = \boxed{2^n \prod_{i=1}^{n} \frac{1 + a_ix_i}{2}}$$

e. The indicator function is the product we gave in part c, but should skip over i because x_i doesn't matter now (it could be either -1 or 1):

$$1_{\{a,a+e_i\}} = \prod_{j=1}^{i-1} \frac{1 + a_jx_j}{2} \times \prod_{j=i+1}^{n} \frac{1 + a_jx_j}{2}$$

$$\boldsymbol{E}[1_{\{a,a+e_i\}}] = 1 \times \boldsymbol{P}(\vec{x} = \vec{a}) + 0 \times \boldsymbol{P}(\vec{x} \neq \vec{a}) = \boldsymbol{P}(\vec{x} = \vec{a}) = \frac{1}{2^{n-1}}$$

$$\boxed{\phi_{\{a,a+e_i\}} = 2^{n-1} \times \prod_{j=1}^{i-1} \frac{1 + a_jx_j}{2} \times \prod_{j=i+1}^{n} \frac{1 + a_jx_j}{2}}$$

f. The indicator function is still the same. But the expectation $\boldsymbol{E}(\vec{x}) = \prod_{i=1}^{n} \boldsymbol{E}(x_i) = \rho^n$ because each x_i is independent.

$$\boxed{\frac{1}{\rho^n} \prod_{i=1}^{n} \frac{1 + a_ix_i}{2}}$$

g. $\forall i \in [n], x_iy_i \in \{1,-1\}$.

Let's look at the case when n is even. There are an even number of x_iy_i's, let's say that $n = 2k$. Of these $2k$ products, j of them have $f(\vec{x}, \vec{y}) = 1$ and $2k - j$ of them have $f(\vec{x}, \vec{y}) = -1$. So, $\vec{x} \cdot \vec{y} = 1 \times j + (-1) \times (2k - j) = 2j - 2k$, which is an even number. -1 raised to an even number is 1.

Now let's look at the case when n is odd. So now let's have $n = 2k + 1$. Again, j of them have $f(\vec{x}, \vec{y}) = 1$ and $2k - j$ of them have $f(\vec{x}, \vec{y}) = -1$. So, $\vec{x} \cdot \vec{y} = 1 \times j + (-1) \times (2k + 1 - j) = 2j - 2k - 1$, which is an odd number. -1

raised to an even number is -1. We have thus proven that whenever n is an even number, $f(\vec{x},\vec{y}) = 1$, and whenever n is an odd number, $f(\vec{x},\vec{y}) = -1$.

$$\boxed{f(\vec{x},\vec{y}) = (-1)^n}$$

I'm not sure if that's what you want because it doesn't have any products or summations or whatever but it's just as simple as that.

h. When we convert 1 to 0 and -1 to 1, this functional is basically

$$\bigwedge_{i=1}^{n} x_i \vee \bigwedge_{i=1}^{n} -x_i$$

i.e. AND($\vec{x}$) OR AND($-\vec{x}$)

The conversion we just mentioned is like this:

$$\frac{1-x_i}{2}$$

And then the two AND's are

$$\prod_{i=1}^{n} \frac{1-x_i}{2}, \quad \prod_{i=1}^{n} \frac{1+x_i}{2}$$

The multilinear polynomial for OR in the $\{0,1\}$ domain is

$$-x_1x_2 + x_1 + x_2$$

So wrapping it all up, the answer is

$$-\prod_{i=1}^{n} \frac{1-x_i}{2} \times \prod_{i=1}^{n} \frac{1+x_i}{2} + \prod_{i=1}^{n} \frac{1-x_i}{2} + \prod_{i=1}^{n} \frac{1+x_i}{2}$$

$$= \boxed{-\prod_{i=1}^{n} \frac{1-x_i^2}{4} + \prod_{i=1}^{n} \frac{1-x_i}{2} + \prod_{i=1}^{n} \frac{1+x_i}{2}}$$

Or if you want this in terms of sums instead:

$$\boxed{\left(-\frac{1}{4^n} \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{j \in S} x_j^2\right) + \left(\frac{1}{2^n} \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{j \in S} x_j\right) + \left(\frac{1}{2^n} \sum_{S \subseteq [n]} \prod_{j \in S} x_j\right)}$$

i. This is just the opposite of part h. If the answer to part h is h, the answer to this would be $\boxed{1-h}$.

j. $\frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_1x_2 + \frac{1}{2}x_1x_3$

k. That means if there's three or zero 1's, or, all 1's or all -1's. This is the same as part h, but for $n = 3$.

$$\boxed{\frac{1}{4} + \frac{1}{4}x_1x_2 + \frac{1}{4}x_1x_3 + \frac{1}{4}x_2x_3}$$

l. $\frac{3}{4} - \frac{1}{4}x_1 - \frac{1}{4}x_2x_3 - \frac{1}{4}x_1x_2x_3$

m. $-\frac{1}{2}x_1x_2 + \frac{1}{2}x_1x_4 - \frac{1}{2}x_2x_3 - \frac{1}{2}x_3x_4$

n. $\frac{1}{4}x_1x_4x_5 + \frac{1}{4}x_1x_5x_6 + \frac{1}{4}x_1x_2x_3 + \frac{1}{4}x_1x_2x_4 + \frac{1}{4}x_1x_3x_6 + \frac{1}{4}x_2x_4x_6 + \frac{1}{4}x_2x_5x_6 + \frac{1}{4}x_2x_3x_5 + \frac{1}{4}x_3x_4x_6 + \frac{1}{4}x_3x_4x_5 - x_1 - x_2 - x_3 - x_4 - x_5 - x_6$

o.

p. Maj_5: $\frac{3}{8}\left(\sum_{i\in[5]} x_i\right) - \frac{1}{8}\left(\sum_{i\neq j\neq k\in[5]} x_ix_jx_k\right) + \frac{3}{8}(x_1x_2x_3x_4x_5))$

Maj_7: $\frac{5}{16}\left(\sum_{i\in[7]} x_i\right) - \frac{1}{16}\left(\sum_{i\neq j\neq k\in[7]} x_ix_jx_k\right) + \frac{1}{16}\left(\sum_{i\neq j\neq k\neq l\neq m\in[7]} x_ix_jx_kx_lx_m\right) - \frac{5}{16}(x_1x_2x_3x_4x_5x_6x_7)$

q. There are $\binom{n}{2}$ products x_ix_j. Let's say exactly m of the n x_i's are -1, then there are $m(n-m)$ products x_ix_j that equal -1, and the remaining $\binom{n}{2} - m(n-m)$ products that equal 1. This means

$$\sum_{1\le i<j\le n} x_ix_j = 1\left(\binom{n}{2} - m(n-m)\right) + (-1)\big(m(n-m)\big)$$
$$= \frac{n(n-1)}{2} - 2m(n-m)$$

We can determine that $2m(n-m)$ is even, but what about $\frac{n(n-1)}{2}$? We know $n(n-1)$ is always even. We can see that if $\frac{n(n-1)}{2}$ is even, then $\sum_{1\le i<j\le n} x_ix_j$ is even, making $\mathrm{X}\left(\sum_{1\le i<j\le n} x_ix_j\right) = 1$. If $\frac{n(n-1)}{2}$ is odd, then $\sum_{1\le i<j\le n} x_ix_j$ is odd, making $\mathrm{X}\left(\sum_{1\le i<j\le n} x_ix_j\right) = -1$. So, in fact,

$$\mathrm{X}\left(\sum_{1\le i<j\le n} x_ix_j\right) = \begin{cases} 1 \text{ if} \frac{n(n-1)}{2}\%2 = 0 \\ -1 \text{ otherwise} \end{cases}$$

Which is

$$\boxed{\mathrm{X}\left(\sum_{1\le i<j\le n} x_ix_j\right) = -1^{\frac{n(n-1)}{2}} = \mathrm{X}\left(\frac{n(n-1)}{2}\right)}$$