1. [10 points] Chapter 1, Problem 1.5

For and , the Fourier coefficient of on is given by

Equation

If we expand that, it is equal to

Given any non-empty the number of times (the parity) is across all is exactly , because for each , will be and an equal number of times across all . To put this mathematically,

Equation

If ’s magnitude exceeds , that means the magnitude of the summation needs to exceed , meaning one of two things needs to happen:

1. there needs to be more than ’s where : specifically, at least
2. there needs to be more than ’s where : specifically, at least

(this number was derived from the system of equations , , where is the number of where for option and for option )

It is possible to have satisfy one of these for one , but neither of these conditions can be satisfied across two ’s, because for each stays the same for all ’s, so that means the corresponding ’s of the said ’s in question need to stay the same, and because we said earlier half of each ’s ’s is , this would force the two ’s to have the same ’s for all ’swhich is impossible. And satisfying either of these dissatisfies .

For the case of when , we need at least of ’s to be equal to for the Fourier constant to be greater than in magnitude, because . In this case, no other could satisfy it because of Equation 2.

QED.

This is not true for any with . For example:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |

As you can see, , and

1. [15 points] Chapter 1, Problem 1.9
2. Let’s examine how to get the indicator function for first.

It’s easy to see .

Now let’s see

We see that .

And so on.

Basically, when , have the term ; when , have the term . To put this into purely mathematical terms,

So, you get

That is a real multilinear polynomial. When you expand that all out, you can see that you can also write it in the form of

1. Let’s define For the numberis in Thus, is a Boolean function; it computes the AND of the bits . The multilinear polynomial 1.11 given in this problem shows that any can be represented as a linear combination of AND functions over .

The set of all functions forms a vector space , since we can pointwise add two functions and we can multiply a function by a real scalar. The vector space is -dimensional. Here we illustrate the multilinear expansion of from this perspective:

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | again |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

More generally, the multilinear polynomial 1.11 given in this problem shows that every function in is a linear combination of the functions; i.e. the AND functions are a spanning set for . Since the number of AND functions is , we can deduce that they are in fact a linearly independent basis for . In particular this justifies the uniqueness of equation 1.11.

1. Let’s recall what we did in part a:

All the coefficients are integers because we established in part a that : it’s obvious that for this expression, since , each term’s coefficient must be either or . When is multiplied with , which is either or , then each coefficient would be either , or . And because there are ’s such that , the coefficients would be integers in the range .

1. The mapping from to , and from to , and vice versa can be modelled by these equations:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  | equation |  |  | equation |
|  |  |  |  |  |  |
|  |  |  |  |

Expressing in terms of , we are translating from the domain to the domain (i.e. to in the above chart). So it would be

To express in terms of , we need to translate the other way (i.e. from to in the above chart):

1. [15 points] Chapter 1, Problem 1.1

1. , ,
2. The indicator function is the product we gave in part c, but should skip over because doesn’t matter now (it could be either or ):