Homework 3 Question 3

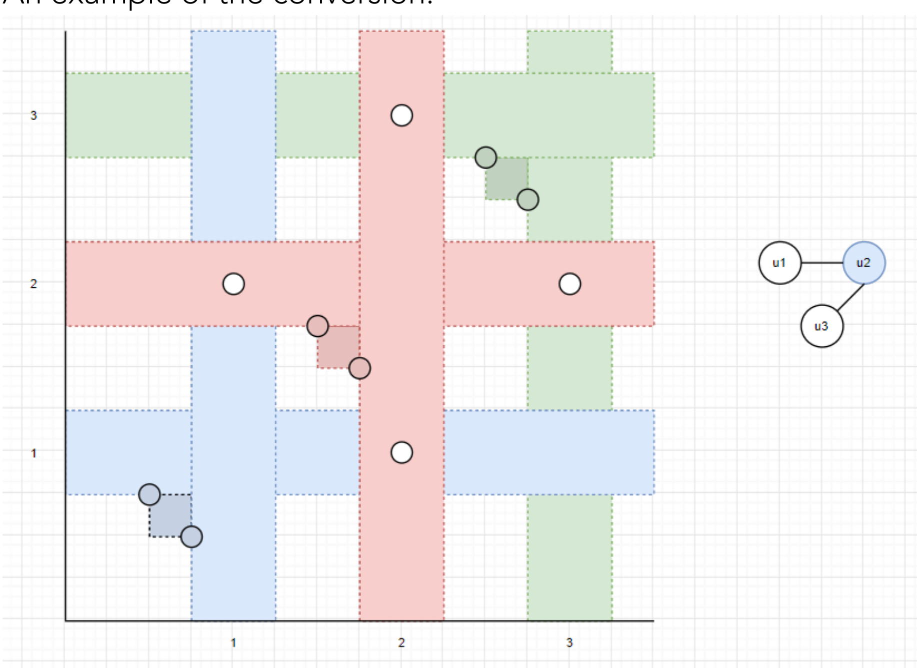
To prove this problem is NP complete, we need to prove it's NP and NP-hard. It's obviously NP because given a certificate of the set of rectangles, it's polynomial time to verify whether their union covers all the dots.

Now to prove it's NP-hard, we will reduce in polynomial time from the vertex cover problem (which is known to be NP-hard) and convert the answer to SET COVER RECTANGLES back in polynomial time. In the vertex cover problem, we are given a graph and want to know whether there's k' vertices that cover all the edges in the graph. We convert this graph to the points and rectangles needed for SET COVER RECTANGLES as such:

1. Write the adjacency matrix representation of the graph (0 where there's no edge and 1 where there's an edge)
2. Convert all 1's to a dot
3. Add two dots next to each position on the diagonal of the matrix, one inside the column, one inside the row
4. For each row, add a rectangle that covers the whole row; for each column, add a rectangle that covers the whole column
5. For each pair of dots on the diagonal, add a rectangle that covers them

The answer to "are there k' vertices in the graph of n vertices that cover all the edges?" is the answer to "are there $k' + n$ rectangles whose union covers all the dots?"

An example of the conversion:



Now we need to prove both directions

1) If there are k' vertices that cover all edges, then there are $k' + n$ rectangles whose union covers all points.
   To cover all the points, we need to cover all the points who were 1's in the adjacency matrix and also the pairs points added to each position along the diagonal. The k' points that are the vertex cover will be converted to their corresponding $2k'$ rectangles, where k' of them are the column rectangles and the other k' of them are the row rectangles. This already covers $2k'$ of the diagonal dots, now we need to cover the remaining $2n - 2k'$ diagonal dots, which will be covered by $n - k'$ rectangles. Altogether, this is $2k' + n - k' = k' + n$ rectangles.

2) If there are $k' + n$ rectangles whose union covers all points, then there are k' vertices that cover all edges.
   To cover a pair of points on the diagonal, we need to choose either only the square that covers both of them, or only both the column and row rectangles, or only one of the column/row rectangles and the square, or all three. Let's analyze each case

   1. **Choosing only the square that covers both of them**: if we choose the (i, i) square and only that square that covers the pair of points on the diagonal for vertex i, then we do not include that vertex in our vertex cover
   2. **Choosing only both the column and row rectangles**: if we choose the rectangles and only the rectangles covering column i and row i, then we include vertex i in our vertex cover
   3. **Choosing only one of the column/row rectangles and also the square**: if we choose the (i, i) square that covers the pair of points on the diagonal for vertex i and also either the column i or row i rectangle, then we include vertex i in our vertex cover
   4. **Choosing all three**: if we choose the (i, i) square, the i-th column, and the i-th row, we include vertex i in our vertex cover

   This will give us a vertex cover in the graph because in an adjacency matrix, each pair of points $(i, j), (j, i)$ represents an edge between vertices i and j, so hitting all the 1's in an adjacency matrix means every edge is covered. The vertex cover is of size $\leq k'$ because for each of the four cases listed above, we pick either 1, 2, or 3 of the corresponding rectangles for each node, and there are $n + k'$ rectangles in the solution, so there are at most k' nodes in the vertex cover.